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Soft cooperation systems and games

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ABSTRACT
A cooperative game for a set of agents establishes a fair allocation of the profit obtained for their cooperation. In order to obtain this allocation a characteristic function is known. It establishes the profit of each coalition of agents if this coalition decides to act alone. Originally players are considered symmetric and then the allocation only depends on the characteristic function, this paper is about cooperative games with an asymmetric set of agents. We introduced cooperative games with a soft set of agents which explains those parameters determining the asymmetry among them in the cooperation. Now the characteristic function is defined not over the coalitions but over the soft coalitions, namely the profit depends not only on the formed coalition but also on the attributes considered for the players in the coalition. The best known of the allocation rules for cooperative games is the Shapley value. We propose a Shapley kind solution for soft games.

KEYWORDS
game theory; soft sets; Shapley value; cooperation systems

1. Introduction

In a cooperative situation for a finite set of agents (players) with a common utility we must look for a fair allocation of this profit among them. Nowadays cooperative games are being used in numerous fields of engineering and artificial intelligence in order to solve this kind of situations. A cooperative game, see Driessen (1988), is defined by a characteristic function which establishes the worth of each subset of players (coalition). Using this information, a payoff vector containing the chosen allocation of the profit is obtained. A value for cooperative games is a function determining a payoff vector for each cooperative game. The most known value is the Shapley value (Shapley 1953). But there are a lot of values more in the literature and, in order to decide which is the most appropriate for our situation each value is provided with a set of reasonable conditions (axioms).

Originally, the characteristic function is the only information about the players that we use to formulate a value, and then players are considered symmetric in their actions and relations. But in real life cooperative situations are more complex. Any additional information about the players must change the formulation of the values. So, several works in recent year have developed different versions of cooperative games with

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additional information about the players: coalition structures (Aumann and Dreze 1974), a priori unions (Owen 1977), communication structures (Myerson 1977), conference structures (Myerson 1980), permission structures (Gilles et al. 1992), fuzzy communication structures (Jiménez-Losada et al. 2010), fuzzy authorization structures (Gallardo et al. 2015), proximity relations (Fernández et al. 2016), fuzzy settings (Borkotokey and Mesiar 2013) or fuzzy restrictions (Gallardo et al. 2017). Some of these studies evidenced the importance of the structures of sets in the analysis of the restrictions or modifications in the game. So several known set systems were used in this sense: convex geometries (Bilbao and Edelman 2000), antimatroids (Algaba et al. 2004) or matroids (Bilbao et al. 2001). Also, new particular partial ordered sets have been introduced to describe determined situations: augmenting and decreasing systems (Ordóñez and Jiménez-Losada 2017) or embedded coalition structures (Alonso-Meijide et al. 2017).

In this paper we introduce a new model to study asymmetric problems in cooperative games. Molodtsov (1999) initiated the concept of soft set theory as a new mathematical tool. He also proposed the application of soft sets to game theory defining soft games but in normal form. During the last years a lot of papers about soft sets and operations have been published (Maji et al. 2003; Yang 2008; Ali et al. 2009; Ge and Yang 2011; Sezgin and Atag 2011; Zhu and Wen 2013). We propose an application of soft sets to describe coalitions with information about the distribution of the tasks among the agents. The asymmetry properties among the players in a cooperative game are the parameters to define a soft set of players. We consider then a characteristic function over the soft subsets of players, the soft coalitions. The meaning of this worth for a soft coalition is the profit obtained for the coalition depending on the parameters satisfied by the players in the coalition.

The next section is dedicated to the preliminaries about cooperative games and soft sets. In section 3 we introduce soft coalitions and we analyze the structure of a soft coalition system for a game. Soft cooperative games are defined in section 4. Section 5 shows a version of Shapley value for soft cooperative games. Finally in section 6 we provide this Shapley solution with an axiomatization.

2. Preliminaries

2.1. *Posets*

A finite *poset* (partial ordered set) is a pair $(K, <)$ where K is a finite set and $<$ is a partial order relation over K. The *bottoms* of the poset are the minimal elements in the order and the *tops* are the maximal ones. Let $x, y \in K$ with $x < y$ in the poset $(K, <)$. The *interval* $[x, y]$ is

$$[x, y] = \{z \in K : x \leq z \leq y\}. \tag{1}$$

We say y covers x if $x < y$ and $[x, y] = \{x, y\}$. The *Hasse diagram* of the poset is the graph whose vertices are the elements in K, whose links are the cover relations (we have a link between two elements if one of them covers the other one) and if $x < y$ then y is drawn above x. A (maximal saturated) *chain* in the poset is a sequence of elements $\{x_1, \ldots, x_p\}$ in K such that x_1 is a bottom, x_p is a top and x_k covers x_{k-1} for all $k = 2, \ldots, p$. Namely, a chain is a line in the Hasse diagram from a bottom to a top. A poset is named *graded* if all the chains have the same number of elements.

The power set of K is the family of subsets of K and it is denoted by 2^K. The poset $(2^K, \subset)$ using the inclusion of sets as order is a boolean algebra.

2.2. *Cooperative games*

We follow Driessen (1988) for this section. A *cooperative game with transferable utility* (game) is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is a real function over the power set of N with $v(\emptyset) = 0$. The empty set is actually a degenerate idea of coalition and it is included only as a technical instrument. The elements of $N = \{1, 2, ..., n\}$ are called *players*, the subsets $S \subseteq N$ *coalitions* and $v(S)$ is the *worth* of S. We suppose fixed the set of players N in all the paper and then we identify the game (N, v) with the characteristic function v. The family of games with N as set of players is denoted as $\mathcal{G}^N$. Let $v \in \mathcal{G}^N$. Game v is *superadditive* if for two disjoint coalitions $S, T \subseteq N$, $S \cap T = \emptyset$, $v(S \cup T) \geq v(S) + v(T)$. If we interpret the worths in the characteristic function v as benefits then the above condition implies an incentive to cooperate. The *marginal contribution* of player i in v for coalition S not containing i is the number $v(S \cup \{i\}) - v(S)$. A player $i \in N$ is a *null player* in v if all her marginal contributions are zero, i.e $v(S \cup \{i\}) = v(S)$ for all $S \subsetneq N \setminus \{i\}$. Two players $i, j \in N$ are *symmetric* in v if their marginal contributions to coalitions not containing both are equal, i.e $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

A *payoff vector* for a game $v \in \mathcal{G}^N$ is any $x \in \mathbb{R}^N$ where, for each player $i \in N$, number x_i represents the payment of i owing to his cooperation possibilities. Given x a payoff vector and S a coalition, we use the notation $x(S) = \sum_{i \in S} x_i$. An *imputation* for the game v is an efficient payoff vector satisfying the individual rationality principle, $x(N) = v(N)$ and $x_i \geq v(\{i\})$ for all player $i \in N$. A *value* for cooperative games over N is a function $\psi : \mathcal{G}^N \to \mathbb{R}^N$ which assigns to each game $v \in \mathcal{G}^N$ a payoff vector $\psi(v) \in \mathbb{R}^N$. The element to measure the contribution of a player i in a coalition S with $i \notin S$ for a game v is the *marginal contribution* given by $v(S \cup \{i\}) - v(S)$. The *Shapley value* (Shapley 1953) is for each game $v \in \mathcal{G}^N$ and any player $i \in N$

$$sh_i(v) = \sum_{\{S \subseteq N : i \notin S\}} \gamma_S^N \left[v(S \cup \{i\}) - v(S) \right], \tag{2}$$

where

$$\gamma_S^N = \frac{(n - |S| - 1)!(|S|)!}{n!} \tag{3}$$

is the probability of getting the marginal contribution between S and $S \cup \{i\}$ in the boolean algebra 2^N, namely the quotient between the number of chains containing this link and the total number of chains. If $v \in \mathcal{G}^N$ is superadditive then $sh(v)$ is an imputation. Formula (2) is the most known of the Shapley value in the sense that it permits to reduce the calculations only to the marginal contributions. This value is the only one over $\mathcal{G}^N$ satisfying the following axioms.
(S1) *Efficiency.* If $v \in \mathcal{G}^N$ it holds $\sum_{i \in N} sh_i(v) = v(N)$.
(S2) *Additivity.* It is an additive function, that is if $v_1, v_2 \in \mathcal{G}^N$ we have the equality $sh(v_1 + v_2) = sh(v_1) + sh(v_2)$.
(S3) *Null player axiom.* If $i \in N$ is a null player for the game v then $sh_i(v) = 0$.
(S4) *Equal treatment axiom.* If i, j are symmetric for v then $sh_i(v) = sh_j(v)$.

2.3. *Soft sets*

We follow Maji et al. (2003) for this subsection. Consider two non-empty finite sets, one of them U of elements and the another one E of parameters to be issued to the elements in U.

A *soft set* over U is a pair (F, A) where $A \subseteq E$, and $F : A \to 2^U$. For each $e \in A$ the set $F(e)$ is named the *e-approximation* of the soft set. Soft sets are defined in Molodtsov (1999). The *support* of (F, A) is

$$A_F = \{e \in A : F(e) \neq \emptyset\}. \tag{4}$$

If $A \subseteq E$ then we use $(\emptyset, A)$ as the *relative null soft set* to A namely $\emptyset(e) = \emptyset$ for every $e \in A$. The *relative whole soft set* to A is (U, A) where $U(e) = U$ for each $e \in A$.

Now we consider two different soft sets (F, A) and (G, B). The first soft set is a *soft subset* of the second one, $(F, A) \subseteq (G, B)$, if $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$. If $B \subseteq A$ then (F, B) is the soft subset of (F, A) given by the restriction of F to B, called the *restriction* of (F, A) to B. The (extended) *union* of both of the soft sets is defined as $(F, A) \cup (G, B) = (H, A \cup B))$ with

$$H(e) = \begin{cases} F(e) \cup G(e), & \text{if } e \in A \cap B \\ F(e), & \text{if } e \in A \setminus B \\ G(e), & \text{if } e \in B \setminus A. \end{cases}$$

The (restricted) *intersection* of both soft sets when $A \cap B \neq \emptyset$ is $(F, A) \cap (G, B) = (H, A \cap B)$ with $H(e) = F(e) \cap G(e)$ for all $e \in A \cap B$. There exist another different concepts for union and also for intersection, the restricted union and the extended intersection. This paper can be developed doing the same things but using these other definitions.

Although the definition of soft set (F, A) seems to suppose that $A \neq \emptyset$, we also consider a technical soft set for the situation without parameters denoted as $(\emptyset, \emptyset)$. This new soft set is a soft subset of any soft set in U. Moreover, using $(\emptyset, \emptyset)$ the intersection can be introduced also for softs sets with disjoint sets of parameters. Let $(F, A), (G, B)$ be soft sets, we define $(F, A) \cap (G, B) = (\emptyset, \emptyset)$ if $A \cap B = \emptyset$.

3. Soft coalition systems.

We consider a fixed finite set of players N in a cooperative situation and also a universal set of parameters E which define how the players are organized to carry out the cooperation: issues, actions or tasks aimed at getting a common profit. If we take a soft set of N, (F_0, A_0), we are determining the parameters needed for the cooperation, A_0, and which members of N are able to make it, F_0. If $e \notin A_0$ then players have decided not to take into account parameter e, for instance if e is a determined task then it is not made. If $e \in A_0$ and $F(e) = S \neq \emptyset$ then this parameter is associated to coalition S, so if e is a task we understand that this task is made by the players in S. Finally if $e \in A_0$ but $F(e) = \emptyset$ then we suppose that e is associated to an external element, the task is made by other figures beyond N. The external elements are not players because they are out of the game in the sense that their payoffs are fixed.

Definition 3.1. A soft set of players is any soft set (F_0, A_0) over N with parameters in E such that $A_0 \neq \emptyset$ and there is at least one $e \in A_0$ with $F_0(e) \neq \emptyset$.

If a soft set of players is formed, players get the profit of the cooperation and they pay the externalities. Now they have to allocate the rest of the profit among them. Coalitions are the instruments of the players to bargain their payoffs. But the externalities defined in the soft set of players are not negotiable. So the idea of coalition is not exactly any soft subset of the soft set of players as we could expect[1], it must be a soft subset of the soft set of players but also must accept the externalities already decided.

Definition 3.2. Let (F_0, A_0) be a soft set of players over N with parameters in E. The soft set (F, A) is a soft coalition of (F_0, A_0) if it is a soft subset of (F_0, A_0) and $A_0 \backslash (A_0)_{F_0} \subseteq A \backslash A_F$. The family of soft coalitions of (F_0, A_0) is denoted as $SC(F_0, A_0)$.

Hence, a soft coalition of a soft set of players (F_0, A_0) is a soft subset of (F_0, A_0) assuming the externalities of (F_0, A_0). The simplification of the soft cooperation to a crisp one is done in the following sense. The *crisp coalition* of $(F, A) \in SC(F_0, A_0)$ is

$$S_F^A = \bigcup_{e \in A} F(e). \tag{5}$$

We take particularly $S_\emptyset^\emptyset = \emptyset$. Observe that we always suppose $S_{F_0}^{A_0} \neq \emptyset$. The concept of soft subset of (F, A) assumes that for any parameter $e \in A \setminus A_F$ the restriction $(F, A \backslash \{e\})$ is a soft subset. But in our situation we think that they are not comparable. So we introduce a particular partial order relation among the soft coalitions.

Definition 3.3. Let (F_0, A_0) be a soft set of players. A soft coalition $(F, A) \in SC(F_0, A_0)$ is smaller than another $(G, B) \in SC(F_0, A_0)$ if $(F, A) \in SC(G, B)$. This relation is denoted as $(F, A) \sqsubseteq (G, B)$.

The family of coalitions with the inclusion relation is a boolean algebra in the classical situation but not in our version. Suppose $N = \{1, 2, 3\}$ and $E = \{e_1, e_2, e_3\}$. We consider the soft set of players (F_0, A_0) given by $A_0 = E$ and $F_0(e_1) = \{1, 2\}$, $F_0(e_2) = \{2, 3\}$, $F_0(e_3) = \emptyset$. So parameter e_3 is satisfied by an externality (for instance $(\emptyset, \emptyset)$ is not a soft coalition in this case). The Hasse diagram of the poset of soft coalitions $SC(F_0, A_0)$ with the relation in Definition 3.3 is shown in Figure 1. Each triple in the picture represents a soft coalition (F, A) with the image of the parameters by F ordered. We use a star to indicate that the parameter is not considered. For instance $(*, \{2, 3\}, \emptyset)$ represents $(F, \{e_2, e_3\})$ with $F(e_2) = \{2, 3\}$ and $F(e_3) = \emptyset$. The reader can see on the picture that if we do not modify the concept of soft subset to soft coalition, it will be impossible to calculate some marginal contributions. For instance, $(\{2\}, \{3\}, \emptyset)$ only covers one soft subset without player 3, $(\{2\}, \emptyset, \emptyset)$, but there are two soft coalitions covered by it, $(\{2\}, \emptyset, \emptyset)$ and $(\{2\}, *, \emptyset)$. Using the concept of soft subset we cannot calculate the marginal contribution of player 3 in front of $(\{2\}, *, \emptyset)$. Now player 3 can measure his contribution to $(\{2\}, \{3\}, \emptyset)$ taking into account both of the options for player 2 into the soft coalition, or parameter e_1 is not assigned or it is assigned to an externality, $(\{2\}, \emptyset, \emptyset)$ and $(\{2\}, *, \emptyset)$.

[1] In the classical cooperative game theory all the subsets of the set of players are coalitions.

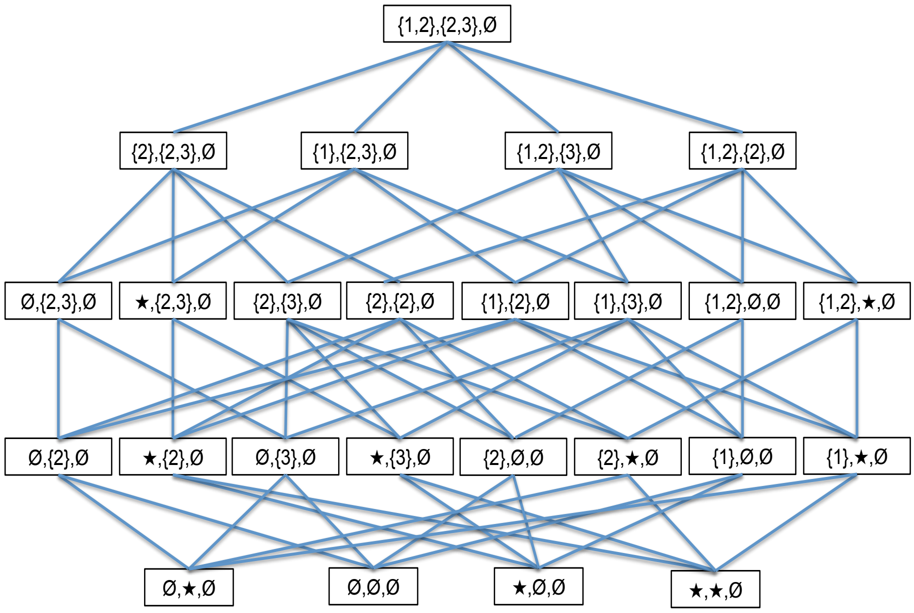

Figure 1. Soft coalition system.

If $(F_0.A_0)$ is a soft set of players over N with parameters E then its *assignment set* is

$$M(F_0, A_0) = \{(i, e) : e \in A_0, i \in F_0(e)\}. \tag{6}$$

Associated to a given subset of assignments $K \subseteq M(F_0, A_0)$ we define a particular soft set (F_K, B_K) with

$$B_K = \{e \in E : \exists i \in N \text{ with } (i, e) \in K\} \text{ and,} \tag{7}$$

$$F_K(e) = \{i \in N : (i, e) \in K\} \, \forall e \in B_K. \tag{8}$$

Proposition 3.4. *Let (F_0, A_0) be a soft set of players. The poset $(SC(F_0, A_0), \sqsubset)$ satisfies the following properties.*

(a) The top is (F_0, A_0) and the set of bottoms is

$$\Phi(F_0, A_0) = \{(\emptyset, A) : A_0 \setminus (A_0)_{F_0} \subseteq A \subseteq A_0\}.$$

(b) For each $(\emptyset, A) \in \Phi(F_0, A_0)$ the interval $[(\emptyset, A), (F_0, A_0)]$ is equivalent to the boolean algebra $2^{M(F_0,A_0)}$.

(c) If $K \subseteq M(F_0, A_0)$ then $(F_K, B_K) \cup (\emptyset, A) \in [(\emptyset, A), (F_0, A_0)]$ with $(\emptyset, A) \in \Phi(F_0, A_0)$ is the only one in the interval identified with K.

(d) It is a $(\sum_{e \in A_0} |F_0(e)|)$-graded poset and its number of chains is $2^{|(A_0)_{F_0}|} \left(\sum_{e \in A_0} |F_0(e)|\right)!$.

(e) For any $(F, A) \in SC(F_0, A_0)$, the number of chains[2] from the bottom to (F, A)

[2]The number of chains in the subposet $SC(F, A)$.

is $2^{|A_F|}\left(\sum_{e\in A}|F(e)|\right)!$ *and the number of chains*[3] *from* (F,A) *to the top is*

$$\left(\sum_{e\in A}|F_0(e)\setminus F(e)|+\sum_{e\in A_0\setminus A}|F_0(e)|\right)!.$$

Proof. All the statements can be proved jointly. It is evident since definition of soft coalition that (F_0,A_0) is the top of the structure. Let $(\emptyset,A)\in\Phi(F_0,A_0)$. Suppose $(G,B)\in SC(F_0,A_0)$ with $(G,B)\sqsubseteq(\emptyset,A)$. We have $(G,B)\in SC(\emptyset,A)$. As $(G,B)\subseteq(\emptyset,A)$ then $B\subseteq A$ and $G(e)=\emptyset$ for all $e\in B$. As also $A\subseteq B$ then $(G,B)=(\emptyset,A)$.

Consider a soft coalition in the bottom, $(\emptyset,A)\in\Phi(F_0,A_0)$. If $(G,B)\in[(\emptyset,A),(F_0,A_0)]$ then $K_{(G,B)}=\{(i,e):e\in B,i\in G(e)\}\subseteq M(F_0,A_0)$. Moreover, if $(G',B')\neq(G,B)$ satisfies $K_{(G,B)}=K_{(G',B')}$ then there exists $e\in B\setminus B'$ (or $e\in B'\setminus B$) such that $G(e)=\emptyset$ (or $G'(e)=\emptyset$). But as $(\emptyset,A)\subseteq(G',B')$ then $e\notin A$ and as $(\emptyset,A)\sqsubseteq(G,B)$ then $e\in A$. Now we take $K\subseteq M$. We can define the soft coalition $(F_K,B_K)\cup(\emptyset,A)\in[(\emptyset,A),(F_0,A_0)]$ where $B_K=\{e\in E:\exists i\in N \text{ with } (i,e)\in K\}$ and for every $e\in B_K$, $F_K(e)=\{i\in N:(i,e)\in K\}$. So $2^{M(F_0,A_0)}$ is identified one to one with the interval $[(\emptyset,A),(F_0,A_0)]$ for all $(\emptyset,A)\in\Phi(F_0,A_0)$. The other statements follow from the above identification. We have the cardinalities $|\Phi(F_0,A_0)|=2^{|(A_0)_{F_0}|}$ and $|M(F_0,A_0)|=\sum_{e\in A_0}|F_0(e)|$. As each chain ends in only one bottom then the number of chains is

$$|M(F_0,A_0)|!\,|\Phi(F_0,A_0)|=\left(\sum_{e\in A_0}|F_0(e)|\right)!\,2^{|(A_0)_{F_0}|}.$$

Obviously if we consider the subposet from the bottom to (F,A) then we get $SC(F,A)$ and the number of chains follows from the before reasoning using that

$$\Phi(F,A)=\{(\emptyset,B):A\setminus A_F\subseteq B\subseteq A\}.$$

Finally, observe that we can identify the interval $[(F,A),(F_0,A_0)]$ with the boolean algebra 2^M where

$$M=\{(i,e):e\in A,i\in F_0(e)\setminus F(e)\}\cup\{(i,e):e\in A_0\setminus A,i\in F_0(e)\}.$$

□

4. Soft games.

We introduce in this section the concept of cooperative soft game.

Definition 4.1. A soft game over N is a triple (F_0,A_0,v) where (F_0,A_0) is a soft set of players and $v:SC(F_0,A_0)\to\mathbb{R}$ with $v(\emptyset,A)=0$ if $(\emptyset,A)\in\Phi(F_0,A_0)$. The set of soft games over N is denoted as $\mathcal{SG}^N$.

The worth of a soft coalition (F,A) in a soft game is understood as the profit obtained by the involved players S_F^A taking into account that parameters in A are

[3] The number of chains in the subposet $[(F,A),(F_0,A_0)]$.

satisfied and after paying the externalities. Therefore it is logical to take $v = 0$ in $\Phi(F_0, A_0)$.

A soft game over N fixes first the soft set of players formed and also the coalition into N (its crisp coalition) decided by the players. This coalition has a determined configuration with regard to the parameters in order to get a determined profit, including if they consider to satisfy several parameters by external actions. And second, the game establishes a characteristic function over the soft coalitions of this soft set of players. So, the problem in a soft game is how to allocate the worth of the soft set of players, and the worths of other soft coalitions are only the instrument to get the allocation vector.

Usually games are considered superadditive, then we introduce this concept for soft games.

Definition 4.2. Let $(F_0, A_0, v) \in \mathcal{SG}^N$ be a soft game. The game is superadditive if for all $(F, A), (G, B) \in SC(F_0, A_0)$ with $(F, A) \cap (G, B) \in \Phi(F_0, A_0)$ it holds

$$v(F, A) + v(G, B) \leq v\left((F, A) \cup (G, B)\right).$$

A game $(F_0, A_0, v) \in \mathcal{SG}^N$ determines a worth for each soft coalition (F, A) including those with parameters out of the support (with $A \setminus A_F \neq \emptyset$). In some situations these distinctions between not being in the support or being empty cannot be interesting. We introduce the following concept.

Definition 4.3. Let (F_0, A_0) be a soft set of players. A soft coalition (F, A) is full if $A = A_F \cup (A_0 \setminus (A_0)_{F_0})$. $SC_{\text{full}}(F_0, A_0)$ represents the family of full soft coalitions. A soft game (F_0, A_0, v) is full if $v(F, A) = v(F, A_F \cup (A_0 \setminus (A_0)_{F_0}))$ for all $(F, A) \in SC(F_0, A_0)$. The family of full games is denoted as $\mathcal{FSG}^N$.

If the soft game is full[4] then the only interesting soft coalitions are the full ones. Hence, we can consider the game restricted to the full coalitions. The concept of full can be extended to soft sets in general, a soft set (F, A) is full if $A_F = A$. Observe that full soft set coincides with the concept of soft set studied in Zhu and Wen (2013).

Finally we can see classical cooperative games as a special family of soft games.

Definition 4.4. A soft game $(F_0, A_0, v) \in \mathcal{SG}^N$ is crisp if $v(F, A) = v(G, B)$ for all pair of soft coalitions $(F, A), (G, B) \in SC(F_0, A_0)$ with $S_F^A = S_G^B$.

Each crisp soft game $(F_0, A_0, v) \in \mathcal{SG}^N$ is identified with the classical game $w^{(F_0, A_0, v)} \in \mathcal{G}^{S_{F_0}^{A_0}}$ defined as $w^{(F_0, A_0, v)}(S) = v(F, A)$ for any $(F, A) \in SC(F_0, A_0)$ with $S_F^A = S$ (we can always find one). Hence we can consider the family of crisp soft games as the set of all the classical games defined over coalitions in N, we use

$$\mathcal{G}_*^N = \bigcup_{\{S \subseteq N : S \neq \emptyset\}} \mathcal{G}^S. \tag{9}$$

Of course $\mathcal{G}_*^N \subseteq \mathcal{FSG}^N$.

Let $i \in N$ be a player. We define the i-soft set relative to $A \subseteq E$ as (F_i, A) with

[4]Full soft sets and therefore full soft games can be introduced in another equivalence way. Let (F_0, A_0) be a soft set of players. A soft coalition (F, A) is spanning if $A = A_0$. A soft game (F_0, A_0, v) is spanning if $v(F, A) = v((F, A) \cup (\emptyset, A_0))$ for all $(F, A) \in SC(F_0, A_0)$. It is easy to test that (F_0, A_0, v) is full if and only if it is spanning.

$F_i(e) = \{i\}$ for all $e \in A$. Observe that if $(F_i, A) \in SC(F_0, A_0)$ for any soft set of players (F_0, A_0) then it must be $A \subseteq A_0$ and $i \in F_0(e)$ for all $e \in A$.

First we introduce the concept of payoff vector for a soft game.

Definition 4.5. Let $(F_0, A_0, v) \in \mathcal{SG}^N$. A soft payoff vector for the game (F_0, A_0, v) is $x = [x^e]_{e\in(A_0)_{F_0}}$ with $x^e \in \mathbb{R}^{F_0(e)}$. A soft imputation of the game is a soft payoff vector x satisfying

- $\sum_{e\in(A_0)_{F_0}} x^e(F_0(e)) = v(F_0, A_0)$, and
- for each $e \in A_0$ and $i \in F_0(A_0)$, it holds $x_i^e \geq v\left((F_i, \{e\}) \cup (\emptyset, A_0 \setminus (A_0)_{F_0})\right)$.

The first condition means efficiency, an imputation is an allocation of the worth of the soft set of players. The second one is the individual rationality principle accommodated to soft games: the payoff of a player for each one of the parameters assigned to him is at least the worth of the minimal option of cooperation for this player with this parameter, namely only this parameter is satisfied by him and keeping the initial externalities from the soft set of players. Given a soft payoff vector x we obtain a crisp payoff vector as $\overline{x} \in \mathbb{R}^N$ with

$$\overline{x}_i = \sum_{\{e\in A_0:i\in F_0(e)\}} x_i^e \tag{10}$$

Hence, if $i \notin S_{F_0}^{A_0}$ then $\overline{x}_i = 0$.

We introduce finally soft payoff functions and crisp payoff functions.

Definition 4.6. A soft payoff function for soft games over N is a function Y over $\mathcal{SG}^N$ such that for all $(F_0, A_0, v) \in \mathcal{SG}^N$ the image $Y(F_0, A_0, v) = [Y^e(F_0, A_0, v)]_{e\in(A_0)_{F_0}}$ is a soft payoff vector. The crisp payoff function associated to Y is defined for each $(F_0, A_0, v) \in \mathcal{SG}^N$ as $\overline{Y}(F_0, A_0, v)$.

5. The Shapley payoff function for soft games

Next we determine a soft payoff function in the Shapley sense following Bilbao and Edelman (2000) and Grabisch and Lange (2007). Let $(F_0, A_0, v) \in \mathcal{SG}^N$ be a soft game. Let $e \in (A_0)_{F_0}$ and $i \in F_0(e)$. If $(F, A) \in SC(F_0, A_0)$ with $i \notin F(e)$ then the *e-marginal contribution* of player i to (F, A) is

$$D_{(i,e)}^v(F, A) = v((F, A) \cup (F_i, \{e\})) - v(F, A). \tag{11}$$

We need to determine the probability of obtaining each e-marginal contribution of player i in our structure of soft coalitions. Figure 2 shows the idea, we look for the number of chains containing the link $(F, A), ((F, A) \cup (F_i, \{e\})$, namely the number of chains from the bottom to (F, A) multiplied by the number of chains from $(F, A) \cup (F_i, \{e\})$ to (F_0, A_0). Using Proposition 3.4(e) this number is

$$2^{|A_F|}\left(\sum_{e'\in A} |F(e')|\right)!\left(\sum_{e'\in A} |F_0(e') \setminus F(e')| + \sum_{e'\in A_0\setminus A} |F_0(e')| - 1\right)!,$$

and so, the probability is the quotient between this number and the total number of chains (see Proposition 3.4(d)).

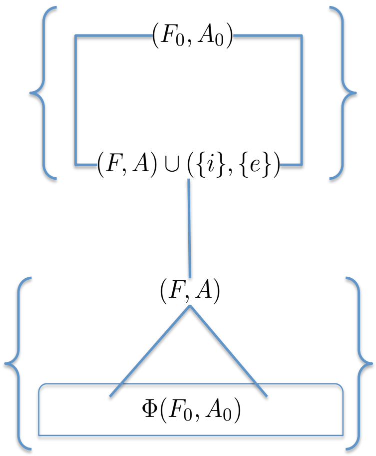

Figure 2. Chains for each marginal contribution.

Definition 5.1. Let $(F_0, A_0, v) \in \mathcal{SG}^N$. For each $e \in A_0$ and $i \in F_0(e)$ the Shapley soft payoff function is defined by

$$Sh_i^e(F_0, A_0, v) = \sum_{\{(F,A) \in SC(F_0,A_0): i \notin F(e)\}} \Gamma_{(F,A)}^{(F_0,A_0)} D_{(i,e)}^v(F, A),$$

where

$$\Gamma_{(F,A)}^{(F_0,A_0)} = \frac{\left(\sum_{e' \in A} |F(e')|\right)! \left(\sum_{e' \in A} |F_0(e') \setminus F(e')| + \sum_{e' \in A_0 \setminus A} |F_0(e')| - 1\right)!}{2^{|(A_0)_{F_0} \setminus A_F|} \left(\sum_{e' \in A_0} |F_0(e')|\right)!}.$$

The Shapley crisp payoff vector is

$$\overline{Sh}_i(F_0, A_0, v) = \sum_{\{e \in A_0 : i \in F_0(e)\}} Sh_i^e(F_0, A_0, v).$$

Example 1. Let $N = \{1, 2, 3\}$ and $E = \{e_1, e_2, e_3\}$. Consider the soft game $(F_0, A_0, v) \in \mathcal{SG}^N$ with $A_0 = E$ and $F_0(e_1) = \{1, 2\}$, $F_0(e_2) = \{2, 3\}$ and $F_0(e_3) = \emptyset$. Parameter e_3 is not negotiable because players have decided to use for it an externality. The poset of soft coalitions is in Figure 1. Next table shows the characteristic function v of the game (following the special notation of the soft sets explained for Figure 1).

$\{1,2\},\{2,3\},\emptyset$	$\{2\},\{2,3\},\emptyset$	$\{1\},\{2,3\},\emptyset$	$\{1,2\},\{3\},\emptyset$	$\{1,2\},\{2\},\emptyset$	$\emptyset,\{2,3\},\emptyset$
100	50	50	60	60	30
$*,\{2,3\},\emptyset$	$\{2\},\{3\},\emptyset$	$\{2\},\{2\},\emptyset$	$\{1\},\{2\},\emptyset$	$\{1\},\{3\},\emptyset$	$\{1,2\},\emptyset,\emptyset$
0	20	20	20	20	30
$\{1,2\},*,\emptyset$	$\emptyset,\{2\},\emptyset$	$*,\{2\},\emptyset$	$\emptyset,\{3\},\emptyset$	$*,\{3\},\emptyset$	$\{2\},\emptyset,\emptyset$
20	0	0	10	0	10
$\{2\},*,\emptyset$	$\{1\},\emptyset,\emptyset$	$\{1\},*,\emptyset$			
10	0	10			

Table 1. Worths of the characteristic function v.

Obviously $v(F,A)=0$ if $(F,A)\in\Phi(F_0,A_0)$. The reader can test that this game is superadditive. Observe that, in this game, parameter e_1 is crucial, namely if parameter e_1 is not satisfied then there is not any profit. But it can be satisfied by an externality. Sometimes satisfying a parameter by an externality is better than not satisfying it (for instance $(\{1,2\},\emptyset,\emptyset)$ is better than $(\{1,2\},*,\emptyset)$), but sometimes in the opposite way (for instance $(\{1\},\emptyset,\emptyset)$ is worse than $(\{1\},*,\emptyset)$). The goal of the problem is to allocate the worth

$$v(\{1,2\},\{2,3\},\emptyset)=100.$$

Following the above definition,

$$\begin{aligned} Sh_1^{e_1}(F_0,A_0,v) &= 27.5,\ Sh_2^{e_1}(F_0,A_0,v)=29.1666,\\ Sh_2^{e_2}(F_0,A_0,v) &= 20.8333,\ Sh_3^{e_2}(F_0,A_0,v)=22.5. \end{aligned}$$

Finally, $\overline{Sh}(F_0,A_0,v)=(27.5,50,22.5)$.

The Shapley soft payoff function can be formulated as a sum of usual Shapley values. This new formula will be the main tool for the proofs of the next results.

Lemma 5.2. *Let $(F_0,A_0,v)\in\mathcal{SG}^N$, $e\in A_0$ and $i\in F_0(e)$. The Shapley soft payoff function satisfies*

$$Sh_i^e(F_0,A_0,v)=\frac{1}{2^{|(A_0)_{F_0}|}}\sum_{A_0\setminus(A_0)_{F_0}\subseteq A\subseteq A_0} sh_{(i,e)}(w^A_{(F_0,A_0,v)}),$$

where $w^A_{(F_0,A_0,v)}\in\mathcal{G}^{M(F_0,A_0)}$ is defined for each $K\subseteq M(F_0,A_0)$ by

$$w^A_{(F_0,A_0,v)}(K)=v\left((F_K,B_K)\cup(\emptyset,A)\right).$$

Proof. Let $(F,A)\in SC(F_0,A_0)$ be a soft coalition with $i\notin F(e)$. Following Proposition 3.4 we identify it with $K\subseteq M(F_0,A_0)$ satisfying $(i,e)\notin K$. There are $2^{|A_F|}$ sets of parameters $A_0\setminus(A_0)_{F_0}\subseteq A'\subseteq A_0$ such that $(F_K,B_K)\cup(\emptyset,A')=(F,A)$, one for each bottom contained in (F,A). So,

$$\begin{aligned} Sh_i^e(F_0,A_0,v) &= \frac{1}{2^{|(A_0)_{F_0}|}}\sum_{A_0\setminus(A_0)_{F_0}\subseteq A'\subseteq A_0}\ \sum_{\{K\subseteq M(F_0,A_0):(i,e)\notin K\}}\gamma_K^{M(F_0,A_0)}\cdot\\ &\quad\cdot\left[w^{A'}_{(F_0,A_0,v)}(K\cup\{(i,e)\})-w^{A'}_{(F_0,A_0,v)}(K)\right], \end{aligned}$$

because since (3) and Proposition 3.4

$$\begin{aligned} \gamma_K^{M(F_0,A_0)} &= \frac{|K|!(|M(F_0,A_0)|-|K|-1)!}{|M(F_0,A_0)|!}\\ &= \frac{\left(\sum_{e'\in A}|F(e')|\right)!\left(\sum_{e'\in A}|F_0(e')\setminus F(e')|+\sum_{e'\in A_0\setminus A}|F_0(e')|-1\right)!}{\left(\sum_{e'\in A_0}|F_0(e')|\right)!}. \end{aligned}$$

The proof finishes using (2) □

Example 2. Consider the soft game in Example 1. In our example $M(F_0, A_0) = \{(1, e_1), (2, e_1), (2, e_2), (3, e_2)\}$, and $(A_0)_{F_0} = \{e_1, e_2\}$. There are four games in the formula of Lemma 5.2, one for each $\{e_3\} \subseteq A \subseteq E$. For instance, if we take $A = \{e_3\}$ then game $\left(M(F_0, A_0), w^A_{(F_0,A_0,v)}\right)$ is obtained in Table 2.

$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{b,c\}$
10	10	0	0	20	20	20	20
$\{b,d\}$	$\{c,d\}$	$\{a,b,c\}$	$\{a,b,d\}$	$\{a,c,d\}$	$\{b,c,d\}$	$\{abcd\}$	
20	20	60	60	50	50	100	

Table 2. Game $w^{\{e_3\}}_{(F_0,A_0,v)}$ taking $a = (1, e_1)$, $b = (2, e_1)$, $c = (2, e_2)$, $d = (3, e_2)$.

Similar to the classical theory the soft superadditivity implies that the soft Shapley payoff vector is a soft imputation.

Proposition 5.3. *If $(F_0, A_0, v) \in \mathcal{SG}^N$ is soft superadditive then $Sh(F_0, A_0, v)$ is a soft imputation.*

Proof. Suppose $(F_0, A_0, v) \in \mathcal{SG}^N$ soft superadditive. The Shapley soft payoff is efficient. Let $e \in A_0$ and $i \in F_0(e)$. We prove that for all $A_0 \setminus (A_0)_{F_0} \subseteq A \subseteq A_0$ the game $w^A_{(F_0,A_0,v)}$ is superadditive. Let $K, K' \subseteq M(F_0, A_0)$ be coalitions with $K \cap K' = \emptyset$. We have $B_{K\cup K'} = B_K \cup B_{K'}$ and $F_{K\cup K'}(e') = F_K(e') \cup F_{K'}(e')$ for all $e' \in B_{K\cup K'}$, thus

$$(F_{K\cup K'}, B_{K\cup K'}) \cup (\emptyset, A) = [(F_K, B_K) \cup (\emptyset, A)] \cup [(F_{K'}, B_{K'}) \cup (\emptyset, A)],$$

with $[(F_K, B_K) \cup (\emptyset, A)] \cap [(F_{K'}, B_{K'}) \cup (\emptyset, A)] \in \Phi(F_0, A_0)$. As (F_0, A_0, v) is superadditive then

$$\begin{aligned} w^A_{(F_0,A_0,v)}(K \cup K') &= v((F_{K\cup K'}, B_{K\cup K'}) \cup (\emptyset, A)) \\ &\geq v((F_K, A_K) \cup (\emptyset, A)) + v((F_{K'}, A_{K'}) \cup (\emptyset, A)) \\ &= w^A_{(F_0,A_0,v)}(K) + w^A_{(F_0,A_0,v)}(K'). \end{aligned}$$

So, the Shapley value of each of these games is an imputation and we get

$$sh_{(i,e)}\left(w^A_{(F_0,A_0,v)}\right) \geq w^A_{(F_0,A_0,v)}(i, e).$$

Lemma 5.2 implies that

$$Sh^e_i(F_0, A_0, v) \geq \frac{1}{2^{|(A_0)_{F_0}|}} \sum_{A_0\setminus(A_0)_{F_0}\subseteq A\subseteq A_0} v((F_i, \{e\}) \cup (\emptyset, A)).$$

Finally, we apply again soft superadditivity, so

$$\begin{aligned} v((F_i, \{e\}) \cup (\emptyset, A)) &\geq v((F_i, \{e\}) \cup (\emptyset, A_0 \setminus (A_0)_{F_0})) + v(\emptyset, A \setminus (A_0 \setminus (A_0)_{F_0})) \\ &= v((F_i, \{e\}) \cup (\emptyset, A_0 \setminus (A_0)_{F_0})). \end{aligned}$$

□

Next we see what happens if we take a particular kind of soft game, full or crisp. In the full case we obtain a more simple formula using only the full soft coalitions. For the crisp games we observe that our value coincides with the classical Shapley value of the associate crisp game.

Proposition 5.4. *Let $(F_0, A_0, v) \in \mathcal{SG}^N$ be a soft game.*

(a) If the game is full then

$$Sh_i^e(F_0, A_0, v) = \sum_{\{(F,A)\in SC_{\text{full}}(F_0,A_0): i\notin F(e)\}} \Gamma_{(F,A)}^{(F_0,A_0)} D_{(i,e)}^v(F, A),$$

where

$$\Gamma_{(F,A)}^{(F_0,A_0)} = \frac{\left(\sum_{e\in A} |F(e)|\right)! \left(\sum_{e\in A} |F_0(e) \setminus F(e)| - 1\right)!}{\left(\sum_{e\in A_0} |F_0(e)|\right)!}.$$

(b) If the game is crisp then

$$\overline{Sh}_i(F_0, A_0, v) = sh_i(w^{(F_0,A_0,v)}).$$

Proof. Suppose first that (F_0, A_0, v) is full. For each bottom we have the same structure (Proposition 3.4) and also for full soft games we have the same marginal contributions. So, following Lemma 5.2 all the classical crisp games over $M(F_0, A_0)$ are the same, namely for all $A_0 \setminus (A_0)_{F_0} \subseteq A \subseteq A_0$ we get then $w^A_{(F_0,A_0,v)} = w^{A_0}_{(F_0,A_0,v)}$.

Take now (F_0, A_0, v) crisp. If (F_0, A_0, v) is crisp then it is also full, then for all $i \in S^{A_0}_{F_0}$ we have

$$\overline{Sh}_i(F_0, A_0, v) = \sum_{\{e\in A_0: i\in F_0(e)\}} Sh_i^e(F_0, A_0, v) = \sum_{\{e\in A_0: i\in F_0(e)\}} sh_{(i,e)}(w^{A_0}_{(F_0,A_0,v)}).$$

But in each chain into the interval $[(\emptyset, A_0), (F_0, A_0)]$ only the first pair (i, e) obtains a non-zero contribution from the definition of crisp game. For this game we can identify each chain in the interval with a chain in $2^{S^{A_0}_{F_0}}$. Moreover, as the interval is a boolean algebra there is the same quantity of chains in the interval defining the same order for the players. Thus, adding all the parameters for player i we get his non-null marginal contribution in all the chains, so

$$\sum_{\{e\in A_0: i\in F_0(e)\}} sh_{(i,e)}(w^{A_0}_{(F_0,A_0,v)}) = sh_i\left(w^{(F_0,A_0,v)}\right).$$

□

6. Axioms for the Shapley soft payoff function

Next we introduce axioms in the Shapley way for our payoff function over N. Let Y be a soft payoff function. We suppose that we look for efficient soft payoff vectors.

Soft efficiency. For all $(F_0, A_0, v) \in \mathcal{SG}^N$ it holds

$$\sum_{e \in (A_0)_{F_0}} \sum_{i \in F_0(e)} Y_i^e(F_0, A_0, v) = v(F_0, A_0).$$

Let $(F_0, A_0, v) \in \mathcal{SG}^N$. A player $i \in S_{F_0}^{A_0}$ is *e-null* for the soft game with $e \in A_0$ such that $i \in F(e)$ if $v((F, A) \cup (F_i, \{e\})) = v(F, A)$ for all $(F, A) \in SC(F_0, A_0)$.

Soft null player axiom. If $i \in N$ is an e-null player for all $(F_0, A_0, v) \in \mathcal{SG}^N$ then $Y_i^e(F_0, A_0, v) = 0$.

Let $(F_0, A_0, v) \in \mathcal{SG}^N$. Player i in e, with $i \in F_0(e)$, is *replaceable* by player j in e', with $j \in F_0(e')$, for the soft game if for all $(F, A) \in SC(F_0, A_0)$, with $i \notin F(e)$ and $j \notin F(e')$ we have $v((F, A) \cup (F_i, \{e\})) = v((F, A) \cup (F_j, \{e'\}))$. Agent i making task e instead of agent j making task e' gets the same profit for any organization.

Soft equal treatment axiom. If i in e is replaceable by j in e' for the soft game (F_0, A_0, v) then $Y_i^e(F_0, A_0, v) = Y_j^{e'}(F_0, A_0, v)$.

Observe that players i, j can be the same in the above axioms, in that case we are talking about player i changing tasks.

Finally we consider additivity as in the classical theory of the Shapley value.

Additivity. If $(F_0, A_0, v_1), (F_0, A_0, v_2)$ are soft games over N with the same soft set of players then for all $e \in A_0$ and $i \in F_0(e)$

$$Y_i^e(F_0, A_0, v_1 + v_2) = Y_i^e(F_0, A_0, v_1) + Y_i^e(F_0, A_0, v_2).$$

Next lemma obtains a formula of the soft Shapley value by a family of classical games. This formula will be the main tool for the proofs.

Theorem 6.1. *The Shapley soft payoff function Sh is the only soft payoff function satisfying soft efficiency, soft null player axiom, soft equal treatment axiom and additivity.*

Proof. First we prove that the Shapley soft payoff function satisfies the axioms.
EFFICIENCY. Let $(F_0, A_0, v) \in \mathcal{SG}^N$. For each $(\emptyset, A) \in \Phi(F_0, A_0)$ we have

$$\sum_{(i,e) \in M(F_0, A_0)} sh_{(i,e)}(w^A_{(F_0,A_0,v)}) = w^A_{(F_0,A_0,v)}(M(F_0, A_0)) = v(F_0, A_0).$$

But then by Proposition 5.2 and the efficiency of the Shapley value (S1) we get

$$\begin{aligned}
\sum_{e \in (A_0)_{F_0}} \sum_{i \in F_0(e)} Sh_i^e(F_0, A_0, v) &= \sum_{e \in (A_0)_{F_0}} \sum_{i \in F_0(e)} \frac{1}{2^{|(A_0)_{F_0}|}} \sum_{A_0 \setminus (A_0)_{F_0} \subseteq A \subseteq A_0} sh_{(i,e)}(w^A_{(F_0,A_0,v)}) \\
&= \frac{1}{2^{|(A_0)_{F_0}|}} \sum_{A_0 \setminus (A_0)_{F_0} \subseteq A \subseteq A_0} \sum_{(i,e) \in M(F_0,A_0)} sh_{(i,e)}(w^A_{(F_0,A_0,v)}) \\
&= \frac{1}{2^{|(A_0)_{F_0}|}} \sum_{A_0 \setminus (A_0)_{F_0} \subseteq A \subseteq A_0} v(F_0, A_0) = v(F_0, A_0).
\end{aligned}$$

Soft Null Player Axiom. Suppose i is an e-null player for the soft game (F_0, A_0, v). We will prove that for any $A_0 \setminus (A_0)_{F_0} \subseteq A \subseteq A_0$ the pair (i, e) is a null player for the game $w^A_{(F_0,A_0,v)}$. Let $K \subseteq M(F_0, A_0)$ be a coalition with $(i, e) \notin K$. Observe that $(F_{K\cup\{(i,e)\}}, B_{K\cup\{(i,e)\}}) = (F_K, B_K) \cup (F_i, \{e\})$ since Proposition 3.4. Hence we get

$$\begin{aligned} w^A_{(F_0,A_0,v)}(K \cup \{(i, e)\}) &= v\left((F_{K\cup\{(i,e)\}}, B_{K\cup\{(i,e)\}}) \cup (\emptyset, A)\right) \\ &= v\left((F_K, B_K) \cup (\emptyset, A) \cup (F_i, \{e\})\right) = v\left((F_K, B_K) \cup (\emptyset, A)\right) \\ &= w^A_{(F_0,A_0,v)}(K). \end{aligned}$$

As the Shapley value satisfies the null player axiom (S3) $sh_{(i,e)}(w^A_{(F_0,A_0,v)}) = 0$ for all every set $A_0 \setminus (A_0)_{F_0} \subseteq A \subseteq A_0$. Proposition 5.2 implies that $Sh^e_i(F_0, A_0, v) = 0$.
Soft equal treatment axiom. Consider i in e replaceable by j in e' for the soft game (F_0, A_0, v). It is easy to see that $(i, e), (j, e')$ are symmetric for $w^A_{(F_0,A_0,v)}$ because if $K \subseteq M(F_0, A_0) \setminus \{(i, e), (j, e')\}$ then, following an above reasoning,

$$\begin{aligned} w^A_{(F_0,A_0,v)}(K \cup \{(i, e)\}) &= v\left((F_K, A_K) \cup (F_i, \{e\}) \cup (\emptyset, A)\right) \\ &= v\left((F_K, A_K) \cup (F_j, \{e'\}) \cup (\emptyset, A)\right) \\ &= w^A_{(F_0,A_0,v)}(K \cup \{(j, e')\}), \end{aligned}$$

for all $A_0 \setminus (A_0)_{F_0} \subseteq A \subseteq A_0$. As the Shapley value satisfies the equal treatment axiom (S4) we have by Proposition 5.2

$$\begin{aligned} Sh^e_i(F_0, A_0, v) &= \frac{1}{2^{|(A_0)_{F_0}|}} \sum_{A_0\setminus(A_0)_{F_0}\subseteq A\subseteq A_0} sh_{(i,e)}(w^A_{(F_0,A_0,v)}) \\ &= \frac{1}{2^{|(A_0)_{F_0}|}} \sum_{A_0\setminus(A_0)_{F_0}\subseteq A\subseteq A_0} sh_{(j,e')}(w^A_{(F_0,A_0,v)}) = Sh^{e'}_j(F_0, A_0, v). \end{aligned}$$

Additivity. We take $w^A_{(F_0,A_0,v_1)}, w^A_{(F_0,A_0,v_2)}$ for each $A_0 \setminus (A_0)_{F_0} \subseteq A \subseteq A_0$. Also we consider $w^A_{(F_0,A_0,v_1+v_2)}$. It holds

$$w^A_{(F_0,A_0,v_1+v_2)} = w^A_{(F_0,A_0,v_1)} + w^A_{(F_0,A_0,v_2)},$$

and then the axiom follows from the additivity of the Shapley value (S2),

$$\begin{aligned} Sh(F_0, A_0, v_1) + Sh(F_0, A_0, v_2) &= \frac{1}{2^{(A_0)_{F_0}}} \sum_{A_0\setminus(A_0)_{F_0}\subseteq A\subseteq A_0} sh(w^A_{(F_0,A_0,v_1)}) + w^A_{(F_0,A_0,v_2)} \\ &= \frac{1}{2^{(A_0)_{F_0}}} \sum_{A_0\setminus(A_0)_{F_0}\subseteq A\subseteq A_0} sh(w^A_{(F_0,A_0,v_1+v_2)}) \\ &= Sh(F_0, A_0, v_1 + v_2). \end{aligned}$$

Now we suppose Y a soft payoff function satisfying the four axioms. Let (F_0, A_0) be a soft set of players. For each $(G, B) \in SC(F_0, A_0)$ we introduce the functions $\delta_{(G,B)}$,

$\xi_{(G,B)}$ as

$$\delta_{(G,B)}(F,A) = \begin{cases} 1, & \text{if } (G,B) = (F,A) \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } \xi_{(G,B)}(F,A) = \begin{cases} 1, & \text{if } (G,B) \sqsubseteq (F,A) \\ 0, & \text{otherwise} \end{cases}$$

for all $(F,A) \in SC(F_0, A_0)$. Of course, we have $\xi_{(G,B)} = \sum_{(G,B)\sqsubseteq(F,A)} \delta_{(F,A)}$ and the Möbius inversion formula (Stanley 1986) applied to the poset $SC(F_0, A_0)$ implies that there exist numbers $\mu((G,B),(F,A))$ (and it is possible to calculate them) satisfying

$$\delta_{(G,B)} = \sum_{(G,B)\sqsubseteq(F,A)} \mu((G,B),(F,A))\xi_{(F,A)}.$$

For each $(G,B) \in SC(F_0, A_0) \setminus \Phi(F_0, A_0)$ we can define the soft game $(F_0, A_0, \delta_{(G,B)})$, $(F_0, A_0, \xi_{(G,B)})$. If $(F_0, A_0, v) \in \mathcal{SG}^N$ then we find a number $r_{(G,B)}$ for every $(G,B) \in SC(F_0, A_0) \setminus \Phi(F_0, A_0)$ such that

$$v = \sum_{(G,B)\in SC(F_0,A_0)\setminus\Phi(F_0,A_0)} v(G,B)\delta_{(G,B)} = \sum_{(G,B)\in SC(F_0,A_0)\setminus\Phi(F_0,A_0)} r_{(G,B)}\xi_{(G,B)}.$$

Function Y satisfies additivity then

$$Y(F_0, A_0, v) = \sum_{(G,B)\in SC(F_0,A_0)\setminus\Phi(F_0,A_0)} Y(r_{(G,B)}\xi_{(G,B)}).$$

Observe that if $e \in A_0$ and $i \in F_0(e) \setminus G(e)$ or $e \notin B$ then i is e-null player for $r_{(G,B)}\xi_{(G,B)}$, thus $Y_i^e(F_0, A_0, r_{(G,B)}\xi_{(G,B)}) = 0$ by the soft null player axiom. If $e, e' \in B$ and $i \in G(e)$, $j \in G(e')$ then player i in e is replaceable by j in e' for $r_{(G,B)}\xi_{(G,B)}$, because for each $(F,A) \in SC(F_0, A_0)$ with $i \notin F(e)$ and $j \notin F(e')$

$$r_{(G,B)}\xi_{(G,B)}((F,A) \cup (F_i, \{e\})) = 0 = r_{(G,B)}\xi_{(G,B)}((F,A) \cup (\{j\}, \{e'\})).$$

The soft equal treatment axiom implies that

$$Y_i^e(F_0, A_0, r_{(G,B)}\xi_{(G,B)}) = Y_j^{e'}(F_0, A_0, r_{(G,B)}\xi_{(G,B)}) = J.$$

Finally if we apply soft efficiency we obtain

$$\begin{aligned} \sum_{e\in A_0}\sum_{i\in F_0(e)} Y_i^e(F_0, A_0, r_{(G,B)}\xi_{(G,B)}) &= \sum_{e\in B}\sum_{i\in G(e)} Y_i^e(F_0, A_0, r_{(G,B)}\xi_{(G,B)}) \\ &= J\left(\sum_{e\in B} |G(e)|\right) \\ &= r_{(G,B)}\xi_{(G,B)}(F_0, A_0) = r_{(G,B)}. \end{aligned}$$

Hence there is a unique worth for J. □

The above axiomatization implies another one for the Shapley crisp function. Now we think of a crisp payoff function $\overline{Y} : \mathcal{SG}^N \to \mathbb{R}^N$. Suppose $(F_0, A_0, v) \in \mathcal{SG}^N$.

Efficiency is the same than (S1) of the Shapley value,

$$\sum_{i\in N} \overline{Y}_i(F_0, A_0, v) = v(F_0, A_0).$$

A player i is a *crisp null player* in the above soft game if i is an e-null player for all $e \in A_0$ with $i \in F_0(e)$. Similarly, players $i, j \in N$ are crisp symmetric if for all e, e' with $i \in F_0(e)$ and $j \in F_0(e)$ we have that i en e is replaceable by j en e'.

Theorem 6.2. *The Shapley crisp payoff function Sh is the only crisp payoff function satisfying efficiency, crisp null player axiom, crisp equal treatment axiom and additivity.*

7. Conclusions

The paper introduced cooperative games with soft cooperation which permits to study new situations of asymmetric players in games. An analysis of the soft coalition structure was done. We also studied a soft Shapley value for these games.

This paper pretends to be only a first introduction of the soft set theory in cooperative games. We think that there is an immense task to accomplish: crisp values, the core, stability, convexity, other solution concepts...

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