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Exact controllability to the trajectories of the one-phase Stefan problem

Jon Asier Bárcena-Petisco[*] Enrique Fernández-Cara[†] Diego A. Souza[‡]

Abstract

This paper deals with the boundary exact controllability to the trajectories of the one-phase Stefan problem in one spatial dimension. This is a free-boundary problem that models solidification and melting processes. We prove the local exact controllability to (smooth) trajectories. To this purpose, we first reformulate the problem as the local null controllability of a coupled PDE-ODE system with distributed controls. Then, a new Carleman inequality for the adjoint of the linearized PDE-ODE system, coupled on the boundary through nonlocal in space and memory terms, is presented. This leads to the null controllability of an appropriate linear system. Finally, the result is obtained via local inversion, by using Liusternik-Graves' Theorem. As a byproduct of our approach, we find that some parabolic equations which contains memory terms located on the boundary are null-controllable.

Keywords: Free-boundary problems, one-phase Stefan problem, exact controllability to the trajectories, global Carleman inequalities, Inverse Function Theorem.

Mathematics Subject Classification: 35R35, 80A22, 93B05, 93C20

1 Introduction

Melting and soldification phenomena occur in many situations in nature and industry, from freezing of polar ice sheets to the continuous casting of steel, see for instance [LSTY83]. The related thermodynamical model of liquid-solid phase transition possesses a classical mathematical formulation: the *Stefan problem*, named after the work of the Slovene physicist and mathematician *Josef Stefan*. The model involves a moving free boundary, i.e. the spatial physical domain is time-dependent and the liquid-solid interface is unknown.

In the Stefan problem, the dynamics of the interface is governed by the heat flux induced by melting or solidification. In other words, its time-evolution must be modeled by a nonlinear ODE.

Among other situations, Stefan problems have also been employed to model the evolution of tumor growth processes [FR99] and the diffusion of information in online social networks [LLW13]. Other applications can be found in [Dav01], [Mei11] and [AS18].

For the sake of completeness, we will give a short description of the mathematical formulation of the Stefan problem. A detailed presentation is given for instance in [Gup03].

*Dep. of Math., Univ. of the Basque Country UPV/EHU, Barrio Sarriena s/n, 48940 Leioa, Spain. E-mail: jonasier.barcena@ehu.eus
†Dep. EDAN and IMUS, Univ. of Sevilla and UFPB, Aptdo. 1160, 41080 Sevilla, Spain. E-mail: cara@us.es
‡Dep. EDAN and IMUS, Univ. of Sevilla, Aptdo. 1160, 41080 Sevilla, Spain. E-mail: desouza@us.es

Let $T \in \mathbb{R}_{>0}$ be given. At each time $t \in [0,T]$, the material domain is separated in two parts: the set $(0,\ell(t))$ (the liquid phase domain) and the set $(\ell(t),+\infty)$ (the solid phase domain). Here, $\ell = \ell(t)$ indicates the position of the interface; it must satisfy $\ell(0) = \ell_0$ and $\ell(t) \in (\ell_*,+\infty)$ at least for all small times, where ℓ_0 and ℓ_* are given and $\ell_0 > \ell_* > 0$. Here and henceforth, for any $\ell \in C^0([0,T];\mathbb{R}_{>0})$, we set

$$Q_\ell := \{(x,t) : t \in (0,T), \quad x \in (0,\ell(t))\} \ \text{ and } \ H^{1,2}(Q_\ell) := \{u \in L^2(Q_\ell) : u_x, u_{xx}, u_t \in L^2(Q_\ell)\}.$$

This paper deals with the controllability properties of the following *one-phase Stefan problem*:

$$\begin{cases} u_t - u_{xx} = 0 & \text{in} \quad Q_\ell, \\ u(0,t) = v(t), \quad u(\ell(t),t) = 0 & \text{in} \quad (0,T), \\ \beta \ell_t(t) = -u_x(\ell(t),t) & \text{in} \quad (0,T), \\ u(x,0) = u_0(x) & \text{in} \quad (0,\ell_0), \\ \ell(0) = \ell_0. \end{cases} \tag{1.1}$$

Here, β is the so called Stefan number (a positive constant) and the initial state $u_0 \in H^1(0,\ell_0)$ satisfies $u_0(x) \geq 0$ for all $x \in [0,\ell_0]$ and $u_0(\ell_0) = 0$. The functions $u = u(x,t)$ and $v = v(t)$ may be respectively viewed as the temperature of the liquid phase and the imposed temperature on the left boundary. In (1.1), v is the *control* (devised for heating or freezing the liquid) and (u,ℓ) is the *state*. See Figure 1 for a solution.

In this paper, the main goal is to prove the local exact controllability of (1.1) to the (smooth) trajectories at time $T > 0$. By definition, a trajectory of (1.1) is a triplet $(\bar u, \bar\ell, \bar v)$ belonging to $H^{1,2}(Q_{\bar\ell}) \times H^1(0,T) \times H^{3/4}(0,T)$ satisfying

$$\begin{cases} \bar u_t - \bar u_{xx} = 0 & \text{in} \quad Q_{\bar\ell}, \\ \bar u(0,t) = \bar v(t), \quad u(\bar\ell(t),t) = 0 & \text{in} \quad (0,T), \\ \beta \bar\ell_t(t) = -\bar u_x(\bar\ell(t),t) & \text{in} \quad (0,T), \\ \bar u(x,0) = \bar u_0(x) & \text{in} \quad (0,\bar\ell_0), \\ \bar\ell(0) = \bar\ell_0, \end{cases} \tag{1.2}$$

where $\bar\ell_0 > \ell_*$, $\bar\ell(t) \in (\ell_*,+\infty)$ and $\bar v(t) > 0$ for all $t \in [0,T]$, $\bar u_0 \in H^1(0,\bar\ell_0)$, $\bar u_0(x) \geq 0$ for all $x \in [0,\ell_0]$, with $\bar u_0(\bar\ell_0) = 0$, and the compatibility condition $\bar u_0(0) = \bar v(0)$ holds.

We will denote by $\mathcal{T}$ the space of triplets $(\bar u, \bar\ell, \bar v) \in H^{1,2}(Q_{\bar\ell}) \times H^1(0,T) \times H^{3/4}(0,T)$ such that the function $(y,t) \mapsto \bar u(y\bar\ell(t),t)$ belongs to $W^{1,\infty}(0,T;H^1(0,1))$ and $\bar\ell \in W^{1,\infty}(0,T)$.

The assumption $\overline{v}(t) > 0$ has a physical meaning: we assume that, on the left of the fixed boundary, we have a liquid phase domain and, accordingly, the temperature is strictly positive. If we started from an uncontrolled solution for which $\overline{v}(t)$ is not ≥ 0, it would become difficult to identify liquid and solid regions.

Note that the solutions of (1.2) remain positive when the initial value and the boundary value $\bar v$ are non-negative and not identically zero in view of the weak and strong maximum principles for parabolic equations, see for instance [Nir53, **?**].

Our main result is the following:

Theorem 1.1. *Let $(\bar u, \bar\ell, \bar v)$ be a trajectory of (1.1) with $(\bar u, \bar\ell) \in \mathcal{T}$ and $\bar v(t) > 0$ for all $t \in [0,T]$. Then, there exists $\delta > 0$ with the following property: for any $\ell_0 \in (\ell_*,+\infty)$ and any $u_0 \in H^1(0,\ell_0)$ with $u_0(\ell_0) = 0$ and $u_0(x) \geq 0$ for all $x \in [0,\ell_0]$ satisfying*

$$|\ell_0 - \bar\ell_0| + \|u_0(\cdot\,\ell_0) - \bar u_0(\cdot\,\bar\ell_0)\|_{H^1(0,1)} \leq \delta, \tag{1.3}$$

there exists a nonnegative control $v \in H^{3/4}(0,T)$ such that the associated state (u,ℓ), with

$$v(0) = u_0(0), \quad u \in H^{1,2}(Q_\ell), \ \ \ell \in H^1(0,T) \ \text{ and } \ \ell(t) \in (\ell_*,+\infty) \ \ \forall t \in [0,T],$$

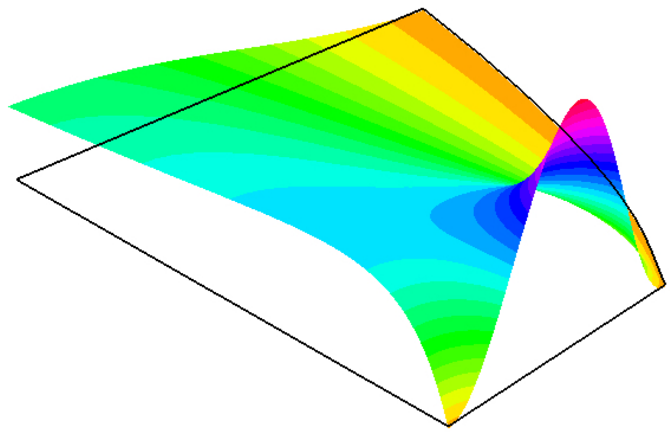

Figure 1: A solution to the free-boundary problem. Space (resp. time) goes from left to right (resp. from bottom to top).

satisfies

$$\ell(T) = \bar\ell(T) \quad \textit{and} \quad u(\cdot\,,T) = \bar u(\cdot\,,T) \quad \textit{in} \quad (0,\bar\ell(T)). \tag{1.4}$$

In order to simplify the notation, we will assume form now on that $\beta = 1$.

Remark 1.2. We will see in Section 2 that, in order to reformulate (1.1), (1.4) in a fixed cylindrical parabolic domain, it must be ensured that the interfaces $x = \bar\ell(t)$ and $x = \ell(t)$ remain far from the left boundary $x = 0$. This justifies the assumptions in Theorem 1.1 on the initial data, $\bar u_0$ and $\bar\ell_0$. It is reasonable to expect such a behavior of the free boundary provided u_0 and ℓ_0 are close enough respectively to $\bar u_0$ and $\bar\ell_0$. □

Remark 1.3. A careful inspection of the proof shows that Theorem 1.1 also holds if we just assume that $\bar v$ is nonnegative and not identical to zero.

Let us mention some previous works on the control of (1.1) and other similar models.

The analysis of the controllability properties of linear and nonlinear parabolic PDEs in cylindrical domains is nowadays classical in control theory; some relevant contributions are [FR71, LR95, FPZ95, FI96, FCZ00] and the references therein. On the other hand, the study of the controllability and stabilizability properties of free-boundary problems for PDEs has not been explored too much, although some important results have been obtained recently.

Regarding the one-phase Stefan problem, we can mention [DPRM03], where the trace is controlled using power-series method, [FCLdM16], where local null controllability is proved, [FCdS17a], where the result is extended to systems with semilinearities involving zero order terms, [KDK18], where controllability is obtained by a backstepping technique and [WLL22], where the result is generalized to quasi-linear parabolic equations. Regarding two-phase models, still fewer results have been proved. Let us highlight [KK20], where exponential stability is obtained with the help of backstepping transformation, [AFCS22], where null controllability is obtained and [GM23], where the control is established in the case of a periodic box.

On the other hand, the local null control of 1D fluid-structure systems has been studied using similar techniques: see [DFC05], where we find boundary controls at both sides of the solid and [LTT13], with just one boundary control. Similarly, in [GZ21], the authors study the controllability of free-boundary systems for the viscous Burgers equation in a domain with one moving endpoint. It is also worth mentioning [CMRT15], where the velocity of the fluid is controlled to zero, the position of the particle is controlled to a given target and the smallness assumption is removed, at the price of allowing T to be large enough.

Null controllability problems for fluid-structure systems in multi-dimensional domain have also been considered, for example, in [IT07]. Regarding non-parabolic equations, the exact controllability to constant trajectories is analyzed in [GKS20]; see also [ABHK18] for a study of the controllability of water waves, governed by the Euler equations.

Finally, in view of the statement of our main result in this paper, it is worth mentioning some recent contributions dealing with controllability properties under a positivity constraint. Thus, see [LTZ17] and [PZ18] respectively for linear and semilinear heat equations and [MTZ19, BWZ20, LM21] for the control of population dynamics of several species, the fractional Laplacian and reaction-diffusion systems, respectively.

In this paper, we will be concerned with a somewhat different situation, which leads to several new difficulties. Let us give more details:

- To our knowledge, our result is the first one concerning the exact control to the trajectories in the context of a parabolic system where the spatial domain changes with time. Up to now, the available results have dealt with null controllability (or exact controllability to constant trajectories).

  Actually, in the context of Stefan problems, the physical meaning of the solutions found in previous works is limited due to the fact that the controlled solutions do not necessarily preserve positivity. Our result is a breakthrough in that direction, because our solutions preserve positivity, and thus have a proper physical meaning.

- In fact, we control both components of the state: the final temperature and the final position of the liquid-solid interface. Obviously, this brings an additional difficulty to the proof of Theorem 1.1.

- After a suitable change of variable and some additional arguments, it will be seen that the free-boundary control problem is equivalent to the null controllability of a nonlinear parabolic PDE-ODE system, which can be viewed as a nonlinear parabolic equation with nonlocal in space and also nonlocal in time (memory-like) terms on the boundary.

  To establish this property, we will use two main tools: a new global Carleman inequality (where the weights are chosen to deal satisfactorily with the boundary terms) and Lyusternik–Graves' Inverse Function Theorem.

  Hence, we are able to establish a Carleman inequality for a system that has nonlocal terms on the boundary condition. To our knowledge, this is a novelty in the literature. It remains to see how much this result can be generalized and whether it can be useful to get observability and/or controllability properties for other problems.

Remark 1.4. Whether or not global inversion is also possible (which would provide global in time control) is obviously a very interesting open question. However, taking into account the kind of estimates we would need, it does not seem an easy task.

Remark 1.5. The success of this technique opens new possibilities. First, for similar two-phase Stefan and 1D fluid-structure problems. But, going beyond, we plan to explore other problems where similar ideas may be used. Among them, we have Stefan problems with radial symmetry, on star-shaped sets or even under more general conditions. Our Carleman estimate also deserves deep analysis and maybe can give ideas to control linear (or semilinear) PDE's in higher dimensions with "special" memory terms: on the boundary, supported in space in a interior compact set, etc. □

Remark 1.6. Let us also observe that the method used to prove local exact controllability leads in a natural way to several iterative algorithms that can be used to compute numerical approximations. For instance, after an appropriate reformulation of the control problem, quasi-Newton methods are in order. This issue will be the goal of a forthcoming paper. □

The rest of this paper is organized as follows.

In Section 2, we will reformulate the free-boundary problem as a nonlinear parabolic system in a cylindrical domain and we will establish some well-posedness results. In Section 3 we prove a new Carleman inequality. In Section 4, we will first prove the null controllability of a linearized PDE-ODE system and then we will give the proof of Theorem 1.1.

2 Preliminaries

2.1 Reformulation of the free-boundary problem in a cylindrical domain

In order to study the controllability of (1.1), it is useful to present a reformulation as a nonlinear parabolic equation in a cylindrical domain.

More precisely, let us set

$$p(y,t) := u\left(y\ell(t),t\right) \quad \text{and} \quad q(t) := \ell(t)^2$$

for $(y,t) \in Q_1 := (0,1) \times (0,T)$. After the transformation $(u,\ell) \to (p,q)$, recalling that we have taken $\beta = 1$, (1.1) reads

$$\begin{cases} qp_t - p_{yy} + yp_y(1,\cdot)p_y = 0 & \text{in} \quad Q_1, \\ p(0,\cdot) = v, \quad p(1,\cdot) = 0 & \text{in} \quad (0,T), \\ p(\cdot\,,0) = p_0 & \text{in} \quad (0,1), \\ q_t + 2p_y(1,\cdot) = 0 & \text{in} \quad (0,T), \\ q(0) = q_0, \end{cases} \tag{2.1}$$

where $q_0 := \ell_0^2$ and $p_0(y) := u_0(y\ell_0)$ in $(0,1)$.

Remark 2.1. By introducing the square of $\ell(t)$, the Stefan condition on the interface becomes a linear constraint on q_t and $p_y(1\,,\cdot)$. Otherwise, we would have

$$\ell_t(t) = -\frac{1}{\ell(t)}p_y(1,t).$$

Since ℓ has a strictly positive lower bound ℓ_*, squaring is a diffeomorphism. □

With a similar change of variables, (1.2) is transformed into

$$\begin{cases} \bar{q}\bar{p}_t - \bar{p}_{yy} + y\bar{p}_y(1,\cdot)\bar{p}_y = 0 & \text{in} \quad Q_1, \\ \bar{p}(0,\cdot) = \bar{v}, \quad \bar{p}(1,\cdot) = 0 & \text{in} \quad (0,T), \\ \bar{p}(\cdot\,,0) = \bar{p}_0 & \text{in} \quad (0,1), \\ \bar{q}_t + 2\bar{p}_y(1,\cdot) = 0 & \text{in} \quad (0,T), \\ \bar{q}(0) = \bar{q}_0, \end{cases} \tag{2.2}$$

where $\bar{p}_0(y) := \bar{u}_0(y\bar{\ell}_0)$, $\bar{q}_0 := \bar{\ell}_0^2$ and $\bar{p}(y,t) = \bar{u}\left(\bar{\ell}(t)y,t\right)$ and $\bar{q}(t) := \bar{\ell}(t)^2$ for $(y,t) \in Q_1$. Note that, by assumption, $\bar{q}(t) \in (q_*,+\infty)$ for all $t \in [0,T]$ with $q_* = \ell_*^2$.

Thus, to prove that (1.1) is locally exactly controllable to the trajectory $(\bar{u},\bar{\ell})$ is equivalent to prove that (2.1) is locally exactly controllable to $(\bar{p},\bar{q})$. Consequently, Theorem 1.1 will be a direct consequence of the following result:

Proposition 2.2. *Let $(\bar p,\bar q,\bar v)\in[W^{1,\infty}(0,T;H^1(0,1))\cap H^{1,2}(Q_1)]\times W^{1,\infty}(0,T)\times H^{3/4}(0,T)$ satisfying (2.2), with $\bar v(t)>0$ for all $t\in[0,T]$. Then, there exists $\delta>0$ with the following property: for any $p_0\in H^1(0,1)$ with $p_0(1)=0$ and any $q_0\in(q_*,+\infty)$ satisfying*

$$|q_0-\bar q_0|+\|p_0-\bar p_0\|_{H^1(0,1)}\le\delta,$$

there exists a nonnegative control $v\in H^{3/4}(0,T)$ such that the associated solution (p,q) to (2.1), with

$$v(0)=p_0(0),\ \ p\in H^{1,2}(Q_1),\ \ q\in H^1(0,T)\quad\text{and}\quad q(t)\in(q_*,+\infty)\ \ \forall t\in[0,T],$$

satisfies

$$q(T)=\bar q(T)\quad\text{and}\quad p(\cdot\,,T)=\bar p(\cdot\,,T)\quad\text{in}\quad(0,1).$$

2.2 Reformulation as a null controllability problem

Now, we will reformulate the desired control property as a null controllability requirement.

To do this, let us introduce the change of variable $(z,h)=(p-\bar p,(q-\bar q)/2)$. Then, the local exact controllability to the trajectories for (2.1) is reduced to the local null controllability of the following system, where we have denoted again x the spatial variable:

$$\begin{cases}\bar q z_t-z_{xx}+x\bar p_x(1,\cdot)z_x+x\bar p_x z_x(1\,,\cdot)+2\bar p_t h+2hz_t+xz_x(1,\cdot)z_x=0 & \text{in}\quad Q_1,\\ z(0,\cdot)=\hat v,\quad z(1,\cdot)=0 & \text{in}\quad (0,T),\\ z(\cdot\,,0)=z_0 & \text{in}\quad (0,1),\\ h_t+z_x(1,\cdot)=0 & \text{in}\quad (0,T),\\ h(0)=h_0, & \end{cases}\tag{2.3}$$

where $z_0:=p_0-\bar p_0$, $h_0:=(q_0-\bar q_0)/2$, $\hat v=v-\bar v$ and $2h(t)+\bar q(t)\in(q_*,+\infty)$ for all $t\in[0,T]$.

Here, we have used (2.2) to simplify some terms.

Consequently, Proposition 2.2 is obviously equivalent to the following result:

Proposition 2.3. *Let $(\bar p,\bar q,\bar v)\in[W^{1,\infty}(0,T;H^1(0,1))\cap H^{1,2}(Q_1)]\times W^{1,\infty}(0,T)\times H^{3/4}(0,T)$ satisfying (2.2), with $\bar v(t)>0$ for all $t\in[0,T]$. There exists $\delta>0$ with the following property: for any $p_0\in H^1(0,1)$ with $p_0(1)=0$ and any $q_0\in(q_*,+\infty)$ satisfying*

$$|q_0-\bar q_0|+\|p_0-\bar p_0\|_{H^1_0(0,1)}\le\delta,$$

there exists a nonnegative control $v\in H^{3/4}(0,T)$ such that the associated solution (z,h) to (2.3), where we have taken $z_0:=p_0-\bar p_0$, $h_0:=(q_0-\bar q_0)/2$ and $\hat v=v-\bar v$, with

$$\hat v(0)=z_0(0),\ \ z\in H^{1,2}(Q_1),\ \ h\in H^1(0,T)\quad\text{and}\quad 2h(t)+\bar q(t)\in(q_*,+\infty)\ \ \forall t\in[0,T],$$

satisfies

$$h(T)=0\quad\text{and}\quad z(\cdot\,,T)=0\quad\text{in}\quad(0,1).$$

2.3 Reformulation as a distributed control problem

Let us establish a result similar to Proposition 2.3 for a distributed control system.

Thus, let us set

$$Q := (-1,1) \times (0,T) \quad \text{and} \quad H_0^{1,2}(Q) := \{z \in H^{1,2}(Q) : z(-1,\cdot) = z(1,\cdot) = 0 \ \text{ in } \ (0,T)\}$$

and let us consider a non-empty open set $\omega \subset\subset (-1,0)$. The following holds:

Proposition 2.4. *Assume that $(\bar{p},\bar{q}) \in [W^{1,\infty}(0,T;H^1(-1,1)) \cap H_0^{1,2}(Q)] \times W^{1,\infty}(0,T)$, with $\bar{q}(t) \in (q_*,+\infty)$ for all $t \in [0,T]$. There exists $\delta > 0$ with the following property: for any $z_0 \in H_0^1(-1,1)$ and any $h_0 \in \mathbb{R}$ satisfying*

$$|h_0| + \|z_0\|_{H_0^1(-1,1)} \le \delta,$$

there exists a control $w \in L^2(\omega \times (0,T))$ such that the associated solutions to the system

$$\begin{cases} \bar{q}z_t - z_{xx} + x\bar{p}_x(1,\cdot)z_x + x\bar{p}_x z_x(1\,,\cdot) + 2\bar{p}_t h + 2hz_t + xz_x(1,\cdot)z_x = w1_\omega & \textit{in} \quad Q, \\ z(-1,\cdot) = 0, \quad z(1,\cdot) = 0 & \textit{in} \quad (0,T), \\ z(\cdot\,,0) = z_0 & \textit{in} \quad (-1,1), \\ h_t + z_x(1,\cdot) = 0 & \textit{in} \quad (0,T), \\ h(0) = h_0 \end{cases} \tag{2.4}$$

with $(z,h) \in H_0^{1,2}(Q) \times H^1(0,T)$ and $\|(z,h)\|_{H_0^{1,2}(Q)\times H^1(0,T)} \le C\|(z_0,h_0)\|_{H_0^1(-1,1)\times\mathbb{R}}$, satisfies

$$h(T) = 0 \quad \textit{and} \quad z(\cdot\,,T) = 0 \quad \textit{in} \quad (-1,1),$$

for some constant $C > 0$.

The proof of Proposition 2.4 will be given in Section 4.2. The main reason to consider this extended problem is that the boundary controls obtained with the help of Carleman estimates are not sufficiently regular for our purposes; in principle, they are just $L^2(0,T)$, while we need at least $H^{3/4}(0,T)$ controls. With distributed controls, local parabolic results can be used easily to improve the regularity of the control.

Obviously, Proposition 2.3 follows from Proposition 2.4 by restricting to Q_1 and accepting that the boundary control $\hat{v} = \hat{v}(t)$ is just the trace of z at $x = 0$. In particular, the control v we are searching for will be the sum of the traces of z and $\bar{p}$. Accordingly, we will have $v(0) = u_0(0)$.

Also, note that we can take δ small enough to have $2h(t) + \bar{q}(t) \in (q_*,+\infty)$ for all $t \in [0,T]$. Since $\bar{v}(t) > 0$ for all $t \in [0,T]$, in view of the bounds for the solution (z,h), by taking δ sufficiently small, we can ensure that $v := \hat{v} + \bar{v} > 0$.

2.4 Linearization

Now, our aim is to linearize (2.4) in a neighborhood of $(0,0)$ and analyze the null controllability properties of the resulting system. Thus, let us consider the non-homogeneous linear system

$$\begin{cases} \bar{q}z_t - z_{xx} + x\bar{p}_x(1,\cdot)z_x + x\bar{p}_x z_x(1\,,\cdot) + 2\bar{p}_t h = f_1 + w1_\omega & \text{in} \quad Q, \\ z(-1,\cdot) = 0, \quad z(1,\cdot) = 0 & \text{in} \quad (0,T), \\ z(\cdot\,,0) = z_0 & \text{in} \quad (-1,1), \\ h_t + z_x(1,\cdot) = f_2 & \text{in} \quad (0,T), \\ h(0) = h_0, \end{cases} \tag{2.5}$$

where f_1 and f_2 belong to appropriate spaces of functions that decay exponentially as $t \to T^-$ and will be made precise below.

In order to prove the null controllability of (2.5), we are going to follow the Hilbert Uniqueness Method (see [Lio88]). Accordingly, we will first deduce an observability inequality for the adjoint of (2.5), which is the following:

$$\begin{cases} -\bar{q}\varphi_t - \varphi_{xx} - x\bar{p}_x(1,\cdot)\varphi_x + \bar{p}_x(1,\cdot)\varphi = g_1 & \text{in} \quad Q, \\ \varphi(-1,\cdot) = 0, \quad \varphi(1,\cdot) = \gamma + \displaystyle\int_{-1}^{1} x\bar{p}_x(x,\cdot)\varphi(x,\cdot)\,dx & \text{in} \quad (0,T), \\ \varphi(\cdot\,,T) = \varphi_T & \text{in} \quad (-1,1), \\ \gamma_t = \displaystyle\int_{-1}^{1} 2\bar{p}_t(x,\cdot)\varphi(x,\cdot)\,dx + g_2 & \text{in} \quad (0,T), \\ \gamma(T) = \gamma_T. \end{cases} \tag{2.6}$$

It is worth mentioning that, in [GZ21], the authors point out that the exact controllability to the trajectories for the free-boundary viscous Burgers equation is an open problem. They also linearize that problem and compute its adjoint system (which is similar to (2.6)).

2.5 Well-posedness of the adjoint system

Henceforth, we will denote by $(\cdot\,,\cdot)_2$ the usual scalar product in $L^2(-1,1)$ and $\|\cdot\|_2$ will stand for the associated norm.

For clarity, we will provisionally change (2.6) by a similar in time system with general coefficients:

$$\begin{cases} -\bar{q}(t)\varphi_t - \varphi_{xx} - a\varphi_x - b\varphi = f & \text{in} \quad Q, \\ \varphi(-1,\cdot) = 0, \quad \varphi(1,t) = \gamma(t) + (N(\cdot\,,t),\varphi(\cdot\,,t))_2 & \text{in} \quad (0,T), \\ \varphi(\cdot\,,T) = \varphi_T & \text{in} \quad (-1,1), \\ \gamma'(t) = (R(\cdot\,,t),\varphi(\cdot\,,t))_2 + g(t) & \text{in} \quad (0,T), \\ \gamma(T) = \gamma_T. \end{cases} \tag{2.7}$$

Note that the boundary condition on φ at $x = 1$ involves γ (that is essentially a primitive in time of a spatial integral of φ) and an additional spatial integral of φ. Thus, in this system, we find nonlocal in space and nonlocal in time (that is, memory-like) boundary terms.

The following result holds:

Proposition 2.5. *Let us assume that $R \in L^2(Q)$, $N \in H^1(0,T;L^2(-1,1))$, a, $b \in L^2(0,T;L^\infty(-1,1))$ and $\bar{q} \in C^0([0,T])$ with $\bar{q}(t) \in (q_*,+\infty)$ for all $t \in [0,T]$. Let $f \in L^2(Q)$, $g \in L^2(0,T)$, $\varphi_T \in H^1(-1,1)$ and $\gamma_T \in \mathbb{R}$ be given and assume that*

$$\varphi_T(-1) = 0 \;\; \text{and} \;\; \varphi_T(1) = \gamma_T + (N(\cdot\,,T),\varphi_T)_2. \tag{2.8}$$

Then, there exists a unique strong solution in $H^{1,2}(Q)\times H^1(0,T)$ to (2.7) such that the following estimate holds:

$$\|\varphi\|^2_{H^{1,2}(Q)} + \|\gamma\|^2_{H^1(0,T)} \le e^{C(1+T)}\left(\|f\|^2_{L^2(Q)} + \|g\|^2_{L^2(0,T)} + \|\varphi_0\|^2_{H^1(0,1)} + |\gamma_0|^2\right),$$

where C is a positive constant depending on a, b, R, N and $\bar{q}$.

The proof is standard. The main ideas can be sketched as follows:

- For the existence, we introduce the Hilbert space

$$\mathcal{B} = H^{3/4}(0,T;L^2(-1,1)) \times H^{3/4}(0,T)$$

and the mapping $\Lambda : \mathcal{B} \times [0,1] \mapsto \mathcal{B}$, given by $\Lambda((\hat\varphi,\hat\gamma),\sigma) = (\varphi,\gamma)$ if and only if (φ,γ) is the unique solution to

$$\begin{cases} -\bar q(t)\varphi_t - \varphi_{xx} - a\varphi_x - b\varphi = \sigma f & \text{in} \quad Q, \\ \varphi(-1,\cdot) = 0, \quad \varphi(1,t) = \sigma\left(\hat\gamma(t) + (N(\cdot\,,t),\hat\varphi(\cdot\,,t))_2\right) & \text{in} \quad (0,T), \\ \varphi(T) = \sigma\varphi_T & \text{in} \quad (-1,1), \\ \gamma'(t) = (R(\cdot\,,t),\varphi(\cdot\,,t))_2 + \sigma g(t) & \text{in} \quad (0,T), \\ \gamma(T) = \sigma\gamma_T. \end{cases} \tag{2.9}$$

We check that Λ is well-defined and, also, that (φ,γ) solves (2.7) if and only if $\Lambda((\varphi,\gamma),1) = (\varphi,\gamma)$. But it is not difficult to see that Λ satisfies all the assumptions of the Leray-Schauder's Fixed-Point Principle (see for instance [Zei86]). To this end, it suffices to take into account the usual energy and boundary estimates satisfied by the solution to (2.9). Consequently, (2.7) is solvable.

- For the uniqueness, we assume that two solutions (φ_1,γ_1) and (φ_2,γ_2) exist and we consider the system satisfied by $(\Phi,\Gamma) := (\varphi_1-\varphi_2,\gamma_1-\gamma_2)$.

  Then, from energy and boundary estimates and Gronwall's Lemma, we deduce at once that $(\Phi,\Gamma) = (0,0)$.

Remark 2.6. In order to guarantee that Λ satisfies the hypotheses of the Leray-Schauder's Principle, we need N in $H^1(0,T;L^2(-1,1))$. Indeed, if we multiply the PDE in (2.7) by φ_t and integrate by parts, we get the boundary integral

$$\int_0^T \varphi_x(1,t)\varphi_t(1,t)\,dt.$$

This can be bounded by differentiating in time the identity $\varphi(1,t) = \gamma(t) + (N(\cdot\,,t),\varphi(\cdot\,,t))_2$, which is possible if N is as above. Whether this result may be true under weaker regularity assumption is an open question.

2.6 Well-posedness of the linearized system

The aim of this section is to prove the existence and uniqueness of a global solution to (2.5).

For convenience, we will establish the result for a similar system, where (again) we have introduced general coefficients.

More precisely, the following result holds:

Proposition 2.7. *Assume that (a,R,N) belongs to the space $L^2(0,T;L^\infty(-1,1))\times L^2(Q)\times L^\infty(0,T;L^2(-1,1))$ and $\bar q \in W^{1,\infty}(0,T)$, with $\bar q(t) \in (q_*,+\infty)$ for all $t\in[0,T]$. Let $F\in L^2(Q)$, $G\in L^2(0,T)$, $z_0 \in H^1_0(-1,1)$ and $h_0\in\mathbb{R}$ be given. There exists a unique strong solution in $H^{1,2}_0(Q)\times H^1(0,T)$ to the system*

$$\begin{cases} \bar q(t)z_t - z_{xx} + az_x + Rh + Nz_x(1,\cdot) = F & \textit{in} \quad Q, \\ z(-1,\cdot) = 0, \quad z(1,\cdot) = 0 & \textit{in} \quad (0,T), \\ z(\cdot\,,0) = z_0 & \textit{in} \quad (-1,1), \\ h_t + z_x(1,\cdot) = G & \textit{in} \quad (0,T), \\ h(0) = h_0, \end{cases} \tag{2.10}$$

such that the following inequality holds:

$$\|z\|^2_{H^{1,2}_0(Q)} + \|h\|^2_{H^1(0,T)} \le e^{C(1+T)}\left(\|F\|^2_{L^2(Q)} + \|G\|^2_{L^2(0,T)} + \|z_0\|^2_{H^1_0(-1,1)} + |h_0|^2\right), \tag{2.11}$$

where C is a positive constant depending on a, R, N and $\bar q$.

Again, the proof is standard and we will only give a brief sketch.

For the existence, we can follow the Faedo-Galerkin strategy.

Thus, for instance with the help of the "special" basis of $H_0^1(-1,1)$ formed by the eigenfunctions of the Dirichlet Laplacian operator, we can easily introduce a sequence of Galerkin approximations $(z_n, h_n) : [0,T] \mapsto H_0^1(-1,1) \times \mathbb{R}$. Then, from the usual energy estimates and Gronwall's Lemma, it is not difficult to see that the (z_n, h_n) are uniformly bounded in the spaces indicated in (2.11). Consequently, convergent subsequences can be extracted. Following standard well known arguments, it can be deduced that any associated limit is a strong solution to (2.10).

The uniqueness of the solution is an almost direct consequence of the energy estimates and, once more, Gronwall's Lemma.

At this point, we will introduce the definition of solution by transposition to (2.10):

Definition 2.8. It will be said that $(z,h) \in L^2(Q) \times L^2(0,T)$ is a solution by transposition to (2.10) if

$$\iint_Q z(x,t) f(x,t)\,dx\,dt + \int_0^T h(t) g(t)\,dt = M(f,g) \quad \forall (f,g) \in L^2(Q) \times L^2(0,T), \tag{2.12}$$

where the linear form M on $L^2(Q) \times L^2(0,T)$ is given by

$$M(f,g) := \iint_Q F(x,t)\varphi(x,t)\,dx\,dt + \bar{q}(0)(z_0, \varphi(\cdot\,,0))_2 + h_0\gamma(0) + \int_0^T G(t)\gamma(t)\,dt$$

and (φ,γ) is the unique strong solution to

$$\begin{cases} -(\bar{q}\varphi)_t - \varphi_{xx} - (a\varphi)_x = f & \text{in} \quad Q, \\ \varphi(-1,\cdot) = 0, \quad \varphi(1,t) = \gamma(t) + (N(\cdot\,,t), \varphi(\cdot\,,t))_2 & \text{in} \quad (0,T), \\ \varphi(\cdot\,,T) = 0 & \text{in} \quad (-1,1), \\ \gamma'(t) = (R(\cdot\,,t), \varphi(\cdot\,,t))_2 + g & \text{in} \quad (0,T), \\ \gamma(T) = 0. \end{cases} \tag{2.13}$$

Since the boundary and final conditions in (2.13) satisfy the appropiate compatibility conditions (2.8), Proposition 2.5 guarantees the existence and uniqueness of a strong solution to (2.13). Consequently, Definition 2.8 makes sense.

Proposition 2.9. *Let the assumptions in Proposition 2.7 be satisfied. Suppose that $a \in L^2(0,T;W^{1,\infty}(-1,1))$ and $N \in H^1(0,T;L^2(-1,1))$. Then, there exists a unique solution by transposition to (2.10).*

Proof. Note that M is a continuous linear form on $L^2(Q) \times L^2(0,T)$ in view of Proposition 2.5. Therefore, we deduce from Riesz Representation Theorem that there exists exactly one solution by transposition to (2.10). □

Note that strong solutions to (2.10) are solutions by transposition.

3 A new Carleman estimate

With the purpose of studying the observability of (2.6), we will establish a new Carleman estimate. First, let us recall the definitions of several classical weights, frequently used in this framework, see [FI96].

Let ω_0 be a non-empty open set, with $\omega_0 \subset\subset \omega$ and let be a function η in $C^2([-1,1])$ satisfying

$$\eta > 0 \quad \text{in} \quad [-1,1], \qquad \min_{x\in[-1,1]\setminus\omega_0} |\eta_x(x)| > 0, \quad \eta(-1) = \eta(1) = \min_{x\in[-1,1]} \eta(x). \tag{3.1}$$

We associate to η the following weights:

$$\begin{aligned}
\alpha(x,t) &:= \frac{e^{2\lambda m\|\eta\|_\infty} - e^{\lambda(m\|\eta\|_\infty+\eta(x))}}{t(T-t)} && \forall (x,t)\in Q,\\
\xi(x,t) &:= \frac{e^{\lambda(m\|\eta\|_\infty+\eta(x))}}{t(T-t)} && \forall (x,t)\in Q,\\
\hat{\alpha}(t) &:= \max_{x\in[-1,1]} \alpha(x,t) = \alpha(1,t) = \alpha(-1,t) && \forall t\in(0,T),\\
\hat{\xi}(t) &:= \min_{x\in[-1,1]} \xi(x,t) = \xi(1,t) = \xi(-1,t) && \forall t\in(0,T),
\end{aligned}$$

where $m > 1$ and λ is a sufficiently large positive constant (to be chosen later).

We will present and prove a global Carleman inequality that holds for the solutions to a simplified version of (2.7). It will be later extended to the solutions to (2.7) and, consequently, to the adjoint states in (2.6).

Theorem 3.1. *Let us assume that $R \in L^\infty(0,T;L^2(-1,1))$, $N \in W^{1,\infty}(0,T;L^2(-1,1))$ and $d \in C^1([0,T])$ with $d(t) > d_* > 0$ for all $t \in [0,T]$. There exist constants $\lambda_0 \geq 1$, $s_0 \geq 1$ and $C_0 > 0$ such that, for any $\lambda \geq \lambda_0$, any $s \geq s_0(T+T^2)$, any $(\psi_T, \gamma_T) \in H^1(-1,1)\times\mathbb{R}$ satisfying (2.8) and any source terms $f \in L^2(Q)$ and $g \in L^2(0,T)$, the strong solution to*

$$\begin{cases}
\psi_t + d(t)\psi_{xx} = f & \text{in} \quad Q,\\
\psi(-1,\cdot) = 0, \quad \psi(1,t) = \gamma(t) + (N(\cdot,t),\psi(\cdot,t))_2 & \text{in} \quad (0,T),\\
\psi(\cdot,T) = \psi_T & \text{in} \quad (-1,1)\\
\gamma_t(t) - (R(\cdot,t),\psi(\cdot,t))_2 = g & \text{in} \quad (0,T),\\
\gamma(T) = \gamma_T
\end{cases} \tag{3.2}$$

satisfies

$$\begin{aligned}
&\iint_Q \left[(s\xi)^{-1}(|\psi_{xx}|^2 + |\psi_t|^2) + \lambda^2(s\xi)|\psi_x|^2 + \lambda^4(s\xi)^3|\psi^2\right] e^{-2s\alpha}\,dx\,dt\\
&\quad + \int_0^T \left[\lambda^3(s\hat{\xi})^3|\psi(1,t)|^2 + \lambda(s\hat{\xi})(|\psi_x(-1,t)|^2 + |\psi_x(1,t)|^2)\right] e^{-2s\hat{\alpha}}\,dt\\
&\quad \leq C_0\left(s^3\lambda^4\int_0^T\!\!\int_\omega \xi^3|\psi|^2 e^{-2s\alpha}\,dx\,dt + \iint_Q |f|^2 e^{-2s\alpha}\,dx\,dt + \int_0^T |g|^2 e^{-2s\hat{\alpha}}\,dt\right).
\end{aligned} \tag{3.3}$$

Proof. As already mentioned, this Carleman inequality is new. It is one of the main contributions in the paper.

The main difficulty to overcome is that we have to deal with nonlocal terms on the boundary, both in the space and time variables. In order to deal with them, we will use that time derivatives do not exhibit nonlocal behavior in time and, then, that the nonlocal in space terms are written on the boundary, at $x = 1$, just where $-\alpha$ and ξ attain their minima.

For brevity, the Lebesgue integration elements dx and dt will be omitted. On the other hand, $(\cdot\,,\cdot)$ and $\|\cdot\|$ will stand for the usual scalar product and norm in $L^2(Q)$.

We start by noting that

$$\begin{aligned}
&\alpha_x = -\lambda\xi\eta_x, \qquad \alpha_{xx} = -\lambda^2\xi\eta_x^2 - \lambda\xi\eta_{xx},\\
&\alpha_t = -\xi^2\left[e^{-2\lambda\eta} - e^{-\lambda(m\|\eta\|_\infty+\eta)}\right](T-2t), \qquad \alpha_{xt} = \lambda\xi^2\eta_x e^{-\lambda(m\|\eta\|_\infty+\eta)}(T-2t),\\
&\alpha_{tt} = 2\xi^2\left[e^{-2\lambda\eta} - e^{-\lambda(m\|\eta\|_\infty+\eta)}\right] + 2(T-2t)^2\xi^3\left[e^{-\lambda(m\|\eta\|_\infty+3\eta)} - e^{-2\lambda(m\|\eta\|_\infty+\eta)}\right].
\end{aligned} \tag{3.4}$$

It follows that there exists $C > 0$ such that, for sufficiently large λ at any $(x,t) \in \overline{Q}$, one has

$$|\alpha_t| \le CT\xi^2, \quad |\alpha_{xt}| \le T\lambda\xi^2, \quad |\alpha_{tt}| \le C(\xi^2 + T^2\xi^3) \le \; CT^2\xi^3. \tag{3.5}$$

Let us set $w := e^{-s\alpha}\psi$. We observe that $w(-1,\cdot) = 0$ and, from the definitions of α and w, we get:

$$\lim_{t\to 0^+} t^{-2}(T-t)^{-2}w(\cdot\,,t) = \lim_{t\to T^-} t^{-2}(T-t)^{-2}w(\cdot\,,t) = 0 \quad \text{and} \quad w_x(\cdot\,,T) = w_x(\cdot\,,0) = 0.$$

Let us introduce the partial differential operator $P := \partial_t + d\partial_{xx}$. Then

$$e^{-s\alpha}f = e^{-s\alpha}P(e^{s\alpha}w) = P_e\, w + P_k\, w,$$

where $P_e w := dw_{xx} + (s\alpha_t + s^2 d\alpha_x^2)w$ and $P_k w := w_t + 2sd\alpha_x w_x + sd\alpha_{xx}w$ are the self-adjoint and skew-adjoint parts of P. It follows that

$$P_e w + (P_k w - sd\alpha_{xx}w) = e^{-s\alpha}f - sd\alpha_{xx}w \tag{3.6}$$

and, consequently,

$$\|e^{-s\alpha}f - sd\alpha_{xx}w\|^2 = \|P_e w\|^2 + \|P_k w - sd\alpha_{xx}w\|^2 + 2(P_e w, P_k w - sd\alpha_{xx}w). \tag{3.7}$$

The rest of the proof is devoted to analyzing the term $(P_e w, P_k w - sd\alpha_{xx}w)$. From the above definition of the operators P_e and P_k, it follows that

$$\begin{aligned} 2(P_e w, P_k w - sd\alpha_{xx}w) = \;& 2\,(dw_{xx}, w_t) + 2\,(dw_{xx}, 2sd\alpha_x w_x) \\ & + 2\left(s\alpha_t w + s^2 d\alpha_x^2 w, w_t\right) + 2\left(s\alpha_t w + s^2 d\alpha_x^2 w, 2sd\alpha_x w_x\right) \\ & =: \; I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.8}$$

For the first integral term I_1, we integrate by parts in space and obtain that

$$\begin{aligned} I_1 =& \; -2\iint_Q dw_x w_{xt} + 2\int_0^T \left[dw_t w_x\right]_{x=-1}^{x=1} \\ =& \iint_Q d_t w_x^2 - \int_{-1}^1 \left[dw_x^2\right]_{t=0}^{t=T} + 2\int_0^T \left[dw_t w_x\right]_{x=-1}^{x=1}. \end{aligned} \tag{3.9}$$

For the second one, we integrate again by parts in space and deduce that

$$I_2 = \; -2s\iint_Q d^2\alpha_{xx}|w_x|^2 + 2s\int_0^T \left[d^2\alpha_x|w_x|^2\right]_{x=-1}^{x=1}. \tag{3.10}$$

For the third term, we integrate by parts in time. The following is found:

$$I_3 = \; -s\iint_Q \alpha_{tt}|w|^2 - s^2\iint_Q (d\alpha_x^2)_t|w|^2 + \int_{-1}^1 \left[(s\alpha_t + s^2 d\alpha_x^2)|w|^2\right]_{t=0}^{t=T}. \tag{3.11}$$

Then, for the fourth term, we see that

$$I_4 = -\iint_Q d\left(2s^2(\alpha_t\alpha_x)_x + 6s^3 d\alpha_x^2\alpha_{xx}\right)|w|^2 + 2\int_0^T \left[d(s^2\alpha_t\alpha_x + s^3 d\alpha_x^3)|w|^2\right]_{x=-1}^{x=1}. \tag{3.12}$$

Hence, from (3.8)-(3.12), we get:

$$
\begin{aligned}
2(P_e w, P_k w - sd\alpha_{xx} w) = & \iint_Q (-2sd^2\alpha_{xx} + d_t)|w_x|^2 \\
& + \iint_Q \left(-s\alpha_{tt} - s^2(d\alpha_x^2)_t - 2s^2 d(\alpha_x\alpha_t)_x - 6s^3 d^2\alpha_x^2\alpha_{xx}\right)|w|^2 \\
& + \int_0^T \left[2dw_t w_x + 2sd^2\alpha_x|w_x|^2 + 2s^2 d\alpha_x(\alpha_t + sd\alpha_x^2)|w|^2\right]_{x=-1}^{x=1} \\
& - \int_{-1}^1 \left[dw_x^2 - \left(s\alpha_t + s^2 d\alpha_x^2\right)|w|^2\right]_{t=0}^{t=T} \\
= & \; I_{D1} + I_{D2} + I_{BS} + I_{BT},
\end{aligned}
$$

where I_{D1} and I_{D2} (resp. I_{BS} and I_{BT}) correspond to distributed (resp. boundary and initial and final) terms.

Obviously, $I_{BT} = 0$. Let us estimate the distributed terms. Thanks to (3.4) and from (3.1), we have

$$
\begin{aligned}
I_{D1} &= 2s\lambda^2 \iint_Q d^2\eta_x^2\xi|w_x|^2 + 2s\lambda \iint_Q d^2\eta_{xx}\xi|w_x|^2 + \iint_Q d_t|w_x|^2 \\
&\geq Cs\lambda^2 \iint_Q \xi|w_x|^2 - Cs\lambda^2 \int_0^T\!\!\int_{\omega_0} \xi|w_x|^2 - C\left(s\lambda \iint_Q \xi|w_x|^2 + \iint_Q |w_x|^2\right).
\end{aligned}
$$

Hence, using the fact that $(s\xi)^{-1} \leq 1/(4s_0)$ and $\lambda \geq \lambda_0$ and taking s_0 and λ_0 large enough, we obtain:

$$
Cs\lambda^2 \int_0^T\!\!\int_{\omega_0} \xi|w_x|^2 + I_{D1} \geq Cs\lambda^2 \iint_Q \xi|w_x|^2. \tag{3.13}
$$

In order to estimate for I_{D2}, we use (3.1), (3.4), (3.5) and also that $s \geq s_0(T+T^2)$ and $\lambda \geq \lambda_0$. This gives:

$$
Cs^3\lambda^4 \int_0^T\!\!\int_{\omega_0} \xi^3|w|^2 + I_{D2} \geq Cs^3\lambda^4 \iint_Q \xi^3|w|^2. \tag{3.14}
$$

Finally, let us estimate I_{BS}. Recalling that $w(-1,\cdot) = 0$ in $(0,T)$, we deduce that:

$$
\begin{aligned}
I_{BS} &= 2s^2 \int_0^T d\alpha_x(\alpha_t + sd\alpha_x^2)|w|^2\big|_{x=1} + 2s\int_0^T \left[d^2\alpha_x|w_x|^2\right]_{x=-1}^{x=1} + 2\int_0^T dw_t w_x\big|_{x=1} \\
&=: I_{BS1} + I_{BS2} + I_{BS3}.
\end{aligned} \tag{3.15}
$$

Thanks to (3.4), (3.5) and the fact that $s \geq s_0(T+T^2)$ and $w_t = -s\alpha_t w + e^{-s\alpha}\psi_t$, we see that

$$
\begin{aligned}
I_{BS1} &\geq -2s^3\lambda^3 \int_0^T d^2\eta_x^3\hat{\xi}^3|w|^2\big|_{x=1} - Cs^3\lambda \int_0^T \hat{\xi}^3|w|^2\big|_{x=1}, \\
I_{BS2} &= -2s\lambda \int_0^T \left[d^2\eta_x\hat{\xi}|w_x|^2\right]_{x=-1}^{x=1}, \\
I_{BS3} &\geq 2\int_0^T d\psi_t w_x e^{-s\hat{\alpha}}\big|_{x=1} - Cs^3 \int_0^T \hat{\xi}^3|w|^2\big|_{x=1} - Cs\int_0^T \hat{\xi}|w_x|^2\big|_{x=1}.
\end{aligned}
$$

Using again (3.1), that $(s\xi)^{-1} \leq 1/(4s_0)$ and the inequality $\lambda \geq \lambda_0$, taking s_0 and λ_0 large enough and recalling the Cauchy-Schwarz inequality, we find from the previous estimate that

$$
I_{BS} \geq C^{-1}\int_0^T \left(s^3\lambda^3\hat{\xi}^3|w|^2 + s\lambda\hat{\xi}|w_x|^2\right)\big|_{x=1} + C^{-1}s\lambda\int_0^T \hat{\xi}|w_x|^2\big|_{x=-1} - C\int_0^T (s\lambda\hat{\xi})^{-1}e^{-2s\hat{\alpha}}|\psi_t|^2\big|_{x=1}. \tag{3.16}
$$

From (3.7), (3.13), (3.14) and (3.16) and the facts that $(s\xi)^{-1} \leq 1/(4s_0)$ and $\lambda \geq \lambda_0$, taking s_0 and λ_0 large enough, we conclude that

$$\begin{aligned}
&\|P_e w\|^2 + \|P_k w - sd\alpha_{xx} w\|^2 \\
&\quad + s^3\lambda^4 \iint_Q \xi^3|w|^2 + s\lambda^2 \iint_Q \xi|w_x|^2 + \int_0^T \left(s^3\lambda^3\hat{\xi}^3|w|^2 + s\lambda\hat{\xi}|w_x|^2\right)\big|_{x=1} + s\lambda \int_0^T \hat{\xi}|w_x|^2\big|_{x=-1} \\
&\quad \leq C\Big(\|e^{-s\alpha} f\|_2^2 + s^3\lambda^4 \int_0^T\!\!\int_{\omega_0} \xi^3|w|^2 + s\lambda^2 \int_0^T\!\!\int_{\omega_0} \xi|w_x|^2 + \int_0^T (s\lambda\hat{\xi})^{-1} e^{-2s\hat{\alpha}}|\psi_t|^2\big|_{x=1}\Big).
\end{aligned} \tag{3.17}$$

Now, using that $P_e w = w_{xx} + (s\alpha_t + s^2\alpha_x^2)w$, we get:

$$\begin{aligned}
s^{-1} \iint_Q \xi^{-1}|w_{xx}|^2 &= s^{-1} \iint_Q \xi^{-1}|P_e w - (s\alpha_t + s^2\alpha_x^2)w|^2 \\
&\leq Cs^{-1} \iint_Q \xi^{-1}\left(|P_e w|^2 + s^2\lambda^2\xi^4|w|^2 + s^4\lambda^4\xi^4|w|^2\right) \\
&\leq C\left(s^{-1} \iint_Q \xi^{-1}|P_e w|^2 + \iint_Q s^3\lambda^4\xi^3|w|^2\right).
\end{aligned} \tag{3.18}$$

We can do the same for $P_k w - sd\alpha_{xx} w = w_t + 2s\alpha_x w_x$. Then,

$$\begin{aligned}
s^{-1} \iint_Q \xi^{-1}|w_t|^2 &= s^{-1} \iint_Q \xi^{-1}|(P_k w - sd\alpha_{xx} w) - 2s\alpha_x w_x|^2 \\
&\leq Cs^{-1} \iint_Q \xi^{-1}\left(|P_k w - sd\alpha_{xx} w|^2 + s^2\lambda^2\xi^2|w_x|^2\right) \\
&\leq C\left(s^{-1} \iint_Q \xi^{-1}|P_k w - sd\alpha_{xx} w|^2 + \iint_Q s\lambda^2\xi|w_x|^2\right).
\end{aligned} \tag{3.19}$$

From (3.17), (3.18) and (3.19), by introducing a cut-off function to estimate the local gradient integral and performing the usual integration by parts, the following holds

$$\begin{aligned}
&\iint_Q s^{-1}\xi^{-1}(|w_t|^2 + |w_{xx}|^2) + \iint_Q s\lambda^2\xi|w_x|^2 + s^3\lambda^4 \iint_Q \xi^3|w|^2 \\
&\quad + s\lambda \int_0^T \hat{\xi}|w_x|^2\big|_{x=-1} + \int_0^T \left(s^3\lambda^3\hat{\xi}^3|w|^2 + s\lambda\hat{\xi}|w_x|^2\right)\big|_{x=1} \\
&\quad \leq C\left(\|e^{-s\alpha} f\|_{L^2(Q)}^2 + s^3\lambda^4 \int_0^T\!\!\int_\omega \xi^3|w|^2 + s^{-1}\lambda^{-1} \int_0^T \xi^{-1} e^{-2s\alpha}|\psi_t|^2\big|_{x=1}\right).
\end{aligned} \tag{3.20}$$

Observe that $w_x\big|_{x=-1} = e^{-s\hat{\alpha}}\psi_x\big|_{x=-1}$, since $\psi(-1,\cdot) = 0$ and $w_x\big|_{x=1} = e^{-s\hat{\alpha}}\psi_x\big|_{x=1} + s\lambda\hat{\xi}\eta_x w\big|_{x=1}$. Thus, we can come back to ψ and deduce that

$$\begin{aligned}
I(s,\lambda,\psi) := &\iint_Q e^{-2s\alpha}\left[(s\xi)^{-1}(|\psi_t|^2 + |\psi_{xx}|^2) + s\lambda^2\xi|\psi_x|^2 + s^3\lambda^4\xi^3|\psi|^2\right] + s^3\lambda^3 \int_0^T e^{-2s\hat{\alpha}}\hat{\xi}^3|\psi|^2\big|_{x=1} \\
&+ s\lambda \int_0^T e^{-2s\hat{\alpha}}\hat{\xi}|\psi_x|^2\big|_{x=-1} + s\lambda \int_0^T e^{-2s\hat{\alpha}}\hat{\xi}|\psi_x|^2\big|_{x=1} \\
&\leq C\left(\iint_Q e^{-2s\alpha}|f|^2 + s^3\lambda^4 \int_0^T\!\!\int_\omega e^{-2s\alpha}\xi^3|\psi|^2 + s^{-1}\lambda^{-1} \int_0^T \hat{\xi}^{-1} e^{-2s\hat{\alpha}}|\psi_t|^2\big|_{x=1}\right).
\end{aligned} \tag{3.21}$$

To conclude the proof, we have to eliminate the last term in (3.21). Using $(3.2)_{3,5}$, we find that

$$\psi_t\big|_{x=1} = (R(\cdot\,,t) + N_t(\cdot\,,t), \psi(\cdot\,,t))_2 + (N(\cdot\,,t), \psi_t(\cdot\,,t))_2 + g.$$

Then, since $R \in L^\infty(0,T;L^2(-1,1))$ and $N \in W^{1,\infty}(0,T;L^2(-1,1))$, performing some immediate estimates, we obtain:

$$\int_0^T (s\lambda\hat{\xi})^{-1} e^{-2s\hat{\alpha}} |\psi_t|^2\big|_{x=1} \le C \iint_Q (s\lambda\hat{\xi})^{-1} e^{-2s\hat{\alpha}} (|\psi|^2 + |\psi_t|^2) + \int_0^T (s\lambda\hat{\xi})^{-1} e^{-2s\hat{\alpha}} |g|^2. \tag{3.22}$$

Note that $\hat{\xi}(t)^{-1} e^{-2s\hat{\alpha}(t)} \le \xi(x,t)^{-1} e^{-2s\alpha(x,t)}$ for all $(x,t) \in Q$. Accordingly, we have from (3.22) that

$$\begin{aligned} s^{-1}\lambda^{-1} \int_0^T \hat{\xi}^{-1} e^{-2s\hat{\alpha}} |\psi_t|^2\big|_{x=1} &\le C s^{-1}\lambda^{-1} \iint_Q \xi^{-1} e^{-2s\alpha} (|\psi|^2 + |\psi_t|^2) + s^{-1}\lambda^{-1} \int_0^T \hat{\xi}^{-1} e^{-2s\hat{\alpha}} |g|^2 \\ &\le C\lambda_0^{-1} s^{-1} \iint_Q \xi^{-1} e^{-2s\alpha} |\psi_t|^2 + \frac{C}{256\, s_0^4 \lambda_0^5} s^3 \lambda^4 \iint_Q \xi^3 e^{-2s\alpha} |\psi|^2 \\ &\quad + s^{-1}\lambda^{-1} \int_0^T \hat{\xi}^{-1} e^{-2s\hat{\alpha}} |g|^2. \end{aligned}$$

This can be used together with (3.21) for s_0 and λ_0 large enough. As a result, we get (3.3) and the proof is done. $\square$

Note that, in view of $(3.2)_3$ and $(3.2)_5$, we can also include weighted integrals of γ and γ_t in the left hand side of (3.3).

Let us now present a suitable Carleman inequality for the solutions to a properly chosen adjoint system. This will imply the null controllability of the linearized system (2.5) (see Proposition 4.1 below).

The following holds:

Corollary 3.2. *Assume that $(\bar{p}, \bar{q})$ belongs to $[W^{1,\infty}(0,T;H^1(-1,1)) \cap H_0^{1,2}(Q)] \times W^{1,\infty}(0,T)$, with $\bar{q}(t) \in (q_*, +\infty)$ for all $t \in [0,T]$. There exist constants $\lambda_0 \ge 1$, $s_0 \ge 1$ and $C_0 > 0$ such that, for any $\lambda \ge \lambda_0$, any $s \ge s_0(T + T^2)$, any $\varphi_T \in H^1(-1,1)$ any $\gamma_T \in \mathbb{R}$ with*

$$\varphi_T(-1) = 0 \quad \textit{and} \quad \varphi_T(1) = 2\gamma_T + \int_{-1}^1 \bar{p}_x(x,T) x \varphi_T(x)\, dx \tag{3.23}$$

and any right hand sides $g_1 \in L^2(Q)$ and $g_2 \in L^2(0,T)$, the strong solution to (2.6) satisfies:

$$\begin{aligned} &\iint_Q \left[(s\xi)^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \lambda^2 (s\xi)|\varphi_x|^2 + \lambda^4 (s\xi)^3 |\varphi|^2\right] e^{-2s\alpha}\, dx\, dt \\ &\quad + \int_0^T \left[|\gamma_t|^2 + \lambda(s\hat{\xi})\left(|\varphi_x(-1,t)|^2 + |\varphi_x(1,t)|^2\right) + \lambda^3 (s\hat{\xi})^3 \left(|\varphi(1,t)|^2 + |\gamma|^2\right)\right] e^{-2s\hat{\alpha}}\, dt \\ &\quad \le C_0 \left(\iint_Q |g_1|^2 e^{-2s\alpha}\, dx\, dt + \int_0^T |g_2|^2 e^{-2s\hat{\alpha}}\, dt + s^3\lambda^4 \int_0^T\!\!\int_\omega \xi^3 |\varphi|^2 e^{-2s\alpha}\, dx\, dt \right). \end{aligned} \tag{3.24}$$

The proof is easy. Indeed, let us apply Theorem 3.1 with

$$d = \frac{1}{\bar{q}}, \quad f = -\frac{1}{\bar{q}}\left[g_1 + \bar{p}_x(1,\cdot)(x\varphi_x - \varphi)\right], \quad N(x,t) = x\bar{p}_x(x,t), \quad R = 2\bar{p}_t \quad \text{and} \quad g = g_2.$$

Then, one has

$$\begin{aligned} &\iint_Q \left[(s\xi)^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \lambda^2 (s\xi)|\varphi_x|^2 + \lambda^4 (s\xi)^3 |\varphi|^2\right] e^{-2s\alpha}\, dx\, dt \\ &\quad + \int_0^T \left[|\gamma_t|^2 + \lambda(s\hat{\xi})\left(|\varphi_x(-1,t)|^2 + |\varphi_x(1,t)|^2\right) + \lambda^3 (s\hat{\xi})^3 \left(|\gamma|^2 + |\varphi(1,t)|^2\right)\right] e^{-2s\hat{\alpha}}\, dt \\ &\quad \le C_0 \left(s^3\lambda^4 \int_0^T\!\!\int_\omega \xi^3 |\varphi|^2 e^{-2s\alpha}\, dx\, dt + \iint_Q |f|^2 e^{-2s\alpha}\, dx\, dt + \int_0^T |g|^2 e^{-2s\hat{\alpha}}\, dt \right). \end{aligned}$$

But it is clear that the lower order terms in f can be absorbed and this yields (3.24).

We will also need a second Carleman inequality for the solution to (2.6) with weights that do not vanish at $t=0$. More precisely, let the function $r=r(t)$ be given by $r(t)=T^2/4$ in $[0,T/2]$ and $r(t)=t(T-t)$ in $[T/2,T]$ and set $D_1 := (-1,1)\times(0,T/2)$, $D_2 := (-1,1)\times(T/2,T)$,

$$\zeta(x,t) := \frac{e^{2\lambda m\|\eta\|_\infty} - e^{\lambda(m\|\eta\|_\infty+\eta(x))}}{r(t)} \quad \text{and} \quad \mu(x,t) := \frac{e^{\lambda(m\|\eta\|_\infty+\eta(x))}}{r(t)} \quad \forall (x,t)\in Q, \tag{3.25}$$

where η is given in (3.1) and $m>1$. Let us also introduce the functions

$$\hat\zeta(t) := \max_{x\in[-1,1]} \zeta(x,t),\ \hat\mu(t) := \min_{x\in[-1,1]} \mu(x,t),\ \zeta^*(t) := \min_{x\in[-1,1]} \zeta(x,t),\ \mu^*(t) := \max_{x\in[-1,1]} \mu(x,t) \quad \forall t\in(0,T)$$

and

$$\rho_0(t) := e^{s\zeta^*(t)},\ \rho_1(t) := e^{s\hat\zeta(t)},\ \rho_2(t) := \mu^*(t)^{-3/2} e^{s\zeta^*(t)},\ \rho_3(t) := e^{s\hat\zeta(t)}\hat\mu(t)^{-3/2} \ \text{ and } \ \rho_4(t) := \rho_3(t)^{1/2}.$$

Remark 3.3. Note that $e^{s\hat\zeta}$ and $e^{s\zeta^*}$ (resp. $\hat\mu$ and μ^*) blow up exponentially (resp. polynomially) as $t\to T^-$.□

Remark 3.4. It is not difficult to deduce the following:

- Since $\rho_4^{-1}\in L^\infty(0,T)$, we have that $\rho_4\rho_3^{-1} = \rho_4^{-1}\in L^\infty(0,T)$.
- If we take λ_0 large enough, for instance $\lambda_0 \ge (\log 2)/[(m-1)\|\eta\|_\infty]$ and $\lambda\ge\lambda_0$, then $e^{\lambda m\|\eta\|_\infty} - 2e^{\lambda\|\eta\|_\infty} + e^{\lambda\eta(1)} > 0$ and, therefore, $\rho_4\rho_2^{-1}\in L^\infty(0,T)$.
- Since $\rho_{4,t} := e^{s\hat\zeta/2}(\frac{s}{2}\hat\mu^{-3/4}\hat\zeta_t - \frac{3}{4}\hat\mu^{-7/4}\hat\mu_t)$, by taking λ_0 large enough and $\lambda\ge\lambda_0$, we also have $\rho_{4,t}\rho_0^{-1}\in L^\infty(0,T)$. □

Corollary 3.5. *Let the assumptions in Corollary 3.2 be satisfied. There exist constants $\lambda_1\ge1$, $s_1\ge1$ and $C_1>0$ such that, for any $\lambda\ge\lambda_1$, any $s\ge s_1(T+T^2)$, any $\varphi_T\in H^1(-1,1)$ any $\gamma_T\in\mathbb{R}$ satisfying (3.23) and any right hand sides $g_1\in L^2(Q)$ and $g_2\in L^2(0,T)$, the unique strong solution to (2.6) satisfies:*

$$\begin{aligned}
&\int_0^T \left[|\gamma_t|^2 + \hat\mu\left(|\varphi_x(-1,t)|^2 + |\varphi_x(1,t)|^2\right) + \hat\mu^3\left(|\gamma|^2 + |\varphi(1,t)|^2\right)\right] e^{-2s\hat\zeta}\,dt \\
&+ \iint_Q \left[\mu^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \mu|\varphi_x|^2 + \mu^3|\varphi|^2\right] e^{-2s\zeta}\,dx\,dt + \|\varphi(\cdot,0)\|^2_{H^1(-1,1)} + |\gamma(0)|^2 \\
&\le C_2\left(\iint_Q |g_1|^2 e^{-2s\zeta^*}\,dx\,dt + \int_0^T |g_2|^2 e^{-2s\hat\zeta}\,dt + \int_0^T\!\!\int_\omega (\mu^*)^3|\varphi|^2 e^{-2s\zeta^*}\,dx\,dt\right).
\end{aligned} \tag{3.26}$$

Proof. It suffices to start from (3.24) and split the left hand side in two parts, respectively corresponding to the restrictions of φ to D_1 and D_2 and the corresponding restrictions of γ to $(0,T/2)$ and $(T/2,T)$.

Let us start by proving the following estimate for the solution to (2.6):

$$\begin{aligned}
&\|\gamma\|^2_{H^1(0,T/2)} + \|\varphi\|^2_{L^2(0,T/2;H^2(-1,1))} + \|\varphi_t\|^2_{L^2(D_1)} \\
&\quad\le\ e^{C(1+T)}\left(\|(g_1,g_2)\|^2_{L^2(0,3T/4;L^2(-1,1))\times L^2(0,3T/4)}\right. \\
&\quad\left. + \frac{1}{T^2}\|(\varphi,\gamma)\|^2_{L^2(T/2,3T/4;L^2(-1,1))\times L^2(T/2,3T/4)}\right).
\end{aligned} \tag{3.27}$$

To do that, let us introduce a function $\kappa \in C^1([0,T])$ with $\kappa \equiv 1$ in $[0,T/2]$, $\kappa \equiv 0$ in $[3T/4,T]$ and $|\kappa'| \le C/T$ for some $C>0$. Using classical energy estimates for the system satisfied by $(\kappa\varphi,\kappa\gamma)$, we get

$$\|\kappa\gamma\|^2_{H^1(0,T)} + \|\kappa\varphi\|^2_{H^{1,2}(Q)} \le e^{C(1+T)}\left(\|(\kappa g_1,\kappa g_2)\|^2_{L^2(Q)\times L^2(0,T)} + \|(\kappa'\varphi,\kappa'\gamma)\|^2_{L^2(Q)\times L^2(0,T)}\right),$$

which leads to (3.27).

Since the weights are bounded from above and from below, using (3.27) we obtain a first estimate in D_1:

$$\begin{aligned}
&\int_0^{T/2} \left[|\gamma_t|^2 + \hat\mu\left(|\varphi_x(-1,t)|^2 + |\varphi_x(1,t)|^2\right) + \hat\mu^3\left(|\gamma|^2 + |\varphi(1,t)|^2\right)\right] e^{-2s\hat\zeta}\,dt\\
&\iint_{D_1} \left[\mu^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \mu|\varphi_x|^2 + \mu^3|\varphi|^2\right] e^{-2s\zeta}\,dx\,dt + |\gamma(0)|^2 + \|\varphi(\cdot\,,0)\|^2_{H^1(-1,1)}\\
&\le C\left[\int_0^{3T/4}\left(\int_{-1}^1 |g_1|^2 e^{-2s\zeta}\,dx + |g_2|^2 e^{-2s\hat\zeta}\right)dt\right.\\
&\qquad \left. + \int_{T/2}^{3T/4}\left(\int_{-1}^1 \lambda^4(s\mu)^3|\varphi|^2 e^{-2s\zeta}\,dx + \lambda^3(s\hat\mu)^3|\gamma|^2 e^{-2s\hat\zeta}\right)dt\right],
\end{aligned}\tag{3.28}$$

where C is a positive constant depending on s, λ and T.

On the other hand, since $\alpha = \zeta$ and $\xi = \mu$ in D_2, thanks to Corollary 3.2 we have:

$$\begin{aligned}
&\iint_{D_2} \left[(s\mu)^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \lambda^2(s\mu)|\varphi_x|^2 + \lambda^4(s\mu)^3|\varphi|^2\right] e^{-2s\zeta}\,dx\,dt\\
&\quad + \int_{T/2}^{T} \left[|\gamma_t|^2 + \hat\mu\left(|\varphi_x(-1,t)|^2 + |\varphi_x(1,t)|^2\right) + \hat\mu^3\left(|\gamma|^2 + |\varphi(1,t)|^2\right)\right] e^{-2s\hat\zeta}\,dt\\
&\quad \le C_0\left(\iint_Q |g_1|^2 e^{-2s\alpha} + \int_0^T |g_2|^2 e^{-2s\hat\alpha} + s^3\lambda^4\int_0^T\!\!\int_\omega \xi^3|\varphi|^2 e^{-2s\alpha}\right).
\end{aligned}$$

From the definition of ζ, μ and $\hat\zeta$, we deduce that the last right hand side can be replaced by

$$C(T,s,\lambda)\left(\iint_Q |g_1|^2 e^{-2s\zeta}\,dx\,dt + \int_0^T |g_2|^2 e^{-2s\hat\zeta}\,dt + \int_0^T\!\!\int_\omega \mu^3|\varphi|^2 e^{-2s\zeta}\,dx\,dt\right),$$

and this, in view of (3.28), leads to (3.26). □

4 Exact controllability to the trajectories

This section is devoted to prove the null controllability of the linear system (2.5) and the local null controllability of the nonlinear PDE-ODE system (2.4).

4.1 Controllability of the linearized problem

In the sequel, we will take $\lambda = \lambda_1$ and $s = s_1$ (the constants furnished by Corollary 3.5) and we will use the notation

$$C^k_\rho([0,T];B) := \{v : \rho v \in C^k([0,T];B)\} \quad\text{and}\quad W^{r,p}_\rho(0,T;B) := \{v : \rho v \in W^{k,r}(0,T;B)\}.$$

Here, it is assumed that B is a Banach space, $\rho : [0,T) \mapsto \mathbb{R}$ is a positive measurable function, $k \in \mathbb{N}$, $r \in \mathbb{R}_{\geq 0}$ and $p \in [1,+\infty]$. Accordingly, we set

$$\|v\|_{C^k_\rho([0,T];B)} := \|\rho v\|_{C^k([0,T];B)} \quad \text{and} \quad \|v\|_{W^{r,p}_\rho(0,T;B)} := \|\rho v\|_{W^{r,p}(0,T;B)}.$$

In particular, when $B = \mathbb{R}$, we simply write $C^k_\rho([0,T])$ and $W^{r,p}_\rho(0,T)$; when $p = 2$, we use the notation $H^r(0,T;B) := W^{r,2}(0,T;B)$ and $H^r(0,T) := W^{r,2}(0,T)$.

We will also need the spaces $Z(\rho) := H^{1,2}_\rho(Q) := \{v : \rho v \in H^{1,2}(Q)\}$ and $Z_0(\rho) := H^{1,2}_{0,\rho}(Q) := \{v : \rho v \in H^{1,2}_0(Q)\}$, endowed with the norm $\|v\|_{Z(\rho)} := \|\rho v\|_{H^{1,2}(Q)}$.

Let us introduce the linear operators

$$\mathcal{L}_1(z,h) := \bar{q} z_t - z_{xx} + x\bar{p}_x(1,\cdot) z_x + x\bar{p}_x z_x(1\,,\cdot) + 2\bar{p}_t h \quad \text{and} \quad \mathcal{L}_2(z,h) := h_t + z_x(1,\cdot) \tag{4.1}$$

and the space E, given by

$$\begin{aligned} E :=\{(z,h,w) \in L^2_{\rho_0}(Q) \times L^2_{\rho_1}(0,T) \times L^2_{\rho_2}(\omega \times (0,T)) :& \\ \mathcal{L}_1(z,h) - w 1_\omega \in L^2_{\rho_3}(Q), \quad \mathcal{L}_2(z,h) \in L^2_{\rho_3}(0,T)& \\ z \in Z_0(\rho_4), \quad h \in H^1_{\rho_4}(0,T)\}.& \end{aligned} \tag{4.2}$$

It is clear that E is a Hilbert space for the norm $\|\cdot\|_E$, where

$$\begin{aligned} \|(z,h,w)\|_E^2 := &\|(z,h,w1_\omega)\|^2_{L^2_{\rho_0}(Q)\times L^2_{\rho_1}(0,T)\times L^2_{\rho_2}(Q)} \\ &+ \|\mathcal{L}_1(z,h) - w1_\omega\|^2_{L^2_{\rho_3}(Q)} \\ &+ \|\mathcal{L}_2(z,h)\|^2_{L^2_{\rho_3}(0,T)} + \|h\|^2_{H^1_{\rho_4}(0,T)} + \|z\|^2_{Z(\rho_4)}. \end{aligned}$$

The null controllability of the linearized system is guaranteed by the following result:

Proposition 4.1. *Assume that $(f_1, f_2) \in L^2_{\rho_3}(Q) \times L^2_{\rho_3}(0,T)$ and $(z_0, h_0) \in H^1_0(-1,1) \times \mathbb{R}$. Then, there exists a solution to (2.5) satisfying $(z,h) \in E$.*

Since the weights in the definition of E grow exponentially as $t \to T$, any triplet $(z,h,w) \in E$ satisfies $z(\cdot\,,T) = 0$, $h(T) = 0$ and $w(\cdot\,,T) = 0$. In particular, thanks to Proposition 4.1, one easily deduces that (2.5) is null-controllable.

Proof. Let us consider the following subspace of $H^{1,2}(Q) \times H^1(0,T)$:

$$P_0 := \{(\varphi,\gamma) \in H^{1,2}(Q) \times H^1(0,T) : \varphi(\cdot\,,-1) = 0, \quad \varphi(1,\cdot) - \gamma - \int_{-1}^1 \bar{p}_x(x,\cdot) x \varphi(x,\cdot)\,dx = 0 \ \text{ in } \ (0,T)\}.$$

Let $\mathcal{A} : P_0 \times P_0 \mapsto \mathbb{R}$ be the bilinear form

$$\mathcal{A}((\hat\varphi,\hat\gamma),(\varphi,\gamma)) := \int_0^T\!\!\int_\omega \rho_2^{-2} \hat\varphi \varphi\,dx\,dt + \iint_Q \rho_0^{-2} \mathcal{L}_1^*(\hat\varphi,\hat\gamma) \mathcal{L}_1^*(\varphi,\gamma)\,dx\,dt + \int_0^T \rho_1^{-2} \mathcal{L}_2^*(\hat\varphi,\hat\gamma)\mathcal{L}_2^*(\varphi,\gamma)\,dt$$

and let $\mathcal{F} : P_0 \mapsto \mathbb{R}$ be the linear form

$$\mathcal{F}(\varphi,\gamma) := \bar{q}(0) \int_0^1 z_0(x) \cdot \varphi(x,0)\,dx + h_0 \gamma(0) + \iint_Q f_1 \varphi\,dx\,dt + \int_0^T f_2 \gamma\,dt,$$

where

$$\mathcal{L}_1^*(\phi,\gamma) := -\bar{q}\varphi_t - \varphi_{xx} - x\bar{p}_x(1,\cdot)\varphi_x + \bar{p}_x(1,\cdot)\varphi \quad \text{and} \quad \mathcal{L}_2^*(\phi,\gamma) := \gamma_t - \int_{-1}^1 2\overline{p}_t(x,\cdot)\varphi(x,\cdot)\,dx.$$

Note that the observability inequality (3.26) holds for every $(\phi,\kappa)\in P_0$. Consequently, $\mathcal{A}(\cdot\,,\cdot)$ is a scalar product in P_0 and there exists $C>0$ such that, for all $(\varphi,\gamma)\in P_0$, the following estimate holds:

$$|\mathcal{F}(\varphi,\gamma)|\le C\left(\|z_0\|_{L^2(-1,1)}+|h_0|+\|f_1\|_{L^2_{\rho_3}(Q)}+\|f_2\|_{L^2_{\rho_3}(0,T)}\right)\sqrt{\mathcal{A}((\varphi,\gamma),(\varphi,\gamma))}.$$

In the sequel, we will denote by P the completion of P_0 for the scalar product $\mathcal{A}$. We will still denote by $\mathcal{A}$ and $\mathcal{F}$ the corresponding continuous extensions. Note that P can be identified with the Hilbert space

$$\begin{aligned}\{(\varphi,\gamma)\in L^2_{loc}(Q_T)\times L^2_{loc}(0,T):\ &\mathcal{A}((\varphi,\gamma),(\varphi,\gamma))<+\infty,\\ &\varphi|_{\{-1\}\times(0,T)}=0,\quad \varphi(1,\cdot)-\gamma-\int_{-1}^{1}\bar p_x(x,\cdot)x\varphi(x,\cdot)\,dx=0 \text{ in } (0,T),\\ &(\varphi,\gamma)\ \text{ satisfies }\ (3.26)\}.\end{aligned}$$

From the Lax-Milgram Theorem, there exists a unique $(\hat\varphi,\hat\gamma)$ satisfying

$$\mathcal{A}((\hat\varphi,\hat\gamma),(\varphi,\gamma))=\mathcal{F}(\varphi,\gamma)\quad\forall(\varphi,\gamma)\in P,\quad(\hat\varphi,\hat\gamma)\in P. \tag{4.3}$$

Let us introduce $(\hat z,\hat h,\hat w)$, with

$$(\hat z,\hat h):=(\rho_0^{-2}\mathcal{L}_1^*(\hat\varphi,\hat\gamma),\rho_1^{-2}\mathcal{L}_2^*(\hat\varphi,\hat\gamma)),\quad \hat w=-\rho_2^{-2}\hat\varphi 1_\omega.$$

From (4.3), we get:

$$\iint_Q\rho_0^2|\hat z|^2\,dx\,dt+\int_0^T\rho_1^2|\hat h|^2\,dt+\int_0^T\!\!\int_\omega\rho_2^2|\hat w|^2\,dx\,dt=\mathcal{A}((\hat\varphi,\hat\gamma),(\hat\varphi,\hat\gamma))=\mathcal{F}(\hat\varphi,\hat\gamma).$$

Therefore, taking into account the continuity of $\mathcal{F}$, we have:

$$\iint_Q\rho_0^2|\hat z|^2\,dx\,dt+\int_0^T\rho_1^2|\hat h|^2\,dt+\int_0^T\!\!\int_\omega\rho_2^2|\hat w|^2\,dx\,dt\le C\left(\|z_0\|^2_{L^2(-1,1)}+|h_0|^2+\|f_1\|^2_{L^2_{\rho_3}(Q)}+\|f_2\|^2_{L^2_{\rho_3}(0,T)}\right). \tag{4.4}$$

Note that, in particular, $(\hat z,\hat h,\hat w)\in L^2(Q)\times L^2(0,T)\times L^2(\omega\times(0,T))$. Then, from (4.3), we see that $(\hat z,\hat h)$ is the unique solution by transposition of (2.5) with $w=\hat w$, see Proposition 2.9. Thanks to the fact that the z_0, $\hat w$, f_1 and f_2 are sufficiently regular, Proposition 2.7 guarantees that $(\hat z,\hat h)$ is indeed the strong solution to (2.5) in $H_0^{1,2}(Q)\times H^1(0,T)$.

Let us finally prove that $(\hat z,\hat h,\hat w)\in E$.

Using (2.5) and (4.4), we can easily check that $\hat z\in L^2_{\rho_0}(Q)$, $\hat h\in L^2_{\rho_1}(0,T)$, $\hat w\in L^2_{\rho_2}(\omega\times(0,T))$, $\mathcal{L}_1(\hat z,\hat h)-\hat w 1_\omega\in L^2_{\rho_3}(Q)$ and $\mathcal{L}_2(\hat z,\hat h)\in L^2_{\rho_3}(0,T)$.

It remains to check that $\hat h\in H^1_{\rho_4}(0,T)$ and $\hat z\in Z_0(\rho_4)$. With that purpose, we define $\tilde z=\rho_4\hat z$ and $\tilde h=\rho_4\hat h$. Then, $(\tilde z,\tilde h)$ is the solution to the system:

$$\begin{cases}\mathcal{L}_1(\tilde z,\tilde h)=(\rho_4\rho_3^{-1})\rho_3 f_1+(\rho_4\rho_2^{-1})\rho_2 w1_\omega+(\rho_{4,t}\rho_0^{-1})\rho_0\hat z & \text{in}\quad Q,\\ \tilde z(-1,\cdot)=0 & \text{in}\quad (0,T),\\ \tilde z(1,\cdot)=0 & \text{in}\quad (0,T),\\ \tilde z(\cdot\,,0)=\rho_4(0)z_0 & \text{in}\quad (-1,1),\\ \mathcal{L}_2(\tilde z,\tilde h)=\rho_4 f_2+\rho_{4,t}h & \text{in}\quad (0,T),\\ \tilde h(0)=\rho_4(0)h_0. & \end{cases} \tag{4.5}$$

Consequently, thanks to Remark 3.4 and Proposition 2.7, we obtain the desired estimates and $(z,h,w)\in E$, as desired. $\square$

4.2 Controllability of the nonlinear system

We now prove the controllability of (2.4) by applying a local inversion theorem.

More precisely, we are going to use the following result, whose proof can be found for instance in [ATF87, Chapter 2, p. 107]:

Theorem 4.2 (Liusternik-Graves' Theorem)**.** *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces. Let $b_{1,0} \in \mathcal{B}_1$ be given, let $\Lambda : \mathcal{B}_1 \mapsto \mathcal{B}_2$ be of class C^1 (meaning that it possesses the Fréchet derivative at each point $b_1 \in \mathcal{B}_1$ and the mapping $b_1 \mapsto \Lambda'(b_1)$ is continuous for the uniform topology of bounded operators) in a neighborhood of $b_{1,0}$ and set $b_{2,0} := \Lambda(b_{1,0})$. Assume that $\Lambda'(b_{1,0}) : \mathcal{B}_1 \mapsto \mathcal{B}_2$ is surjective. Then, there exists $\delta > 0$ such that, for every $b_2 \in \mathcal{B}_2$ satisfying $\|b_2 - b_{2,0}\|_{\mathcal{B}_2} \leq \delta$, there exists at least one solution $b_1 \in \mathcal{B}_1$ to the equation $\Lambda(b_1) = b_2$.*

Recalling the notations introduced in (4.1) and (4.2), we shall apply this result with $\mathcal{B}_1 = E$, $\mathcal{B}_2 = F_1 \times F_2$ and

$$\Lambda(z, h, w) = (\mathcal{L}_1(z, h) - w1_\omega + 2hz_t + xz_x(1, \cdot)z_x,\ \mathcal{L}_2(z, h),\ z(\cdot\,, 0),\ h(0)) \tag{4.6}$$

for every $(z, h, w) \in E$. Here, we have introduced the Hilbert spaces $F_1 := L^2_{\rho_3}(Q) \times L^2_{\rho_3}(0, T)$ for the right hand sides and $F_2 := H^1_0(-1, 1) \times \mathbb{R}$ for the initial conditions.

Since Λ contains linear and bilinear terms, thanks to the definition of E it is not difficult to check that Λ is continuous. Indeed, we only have to prove that the $L^2_{\rho_3}(Q)$-valued bilinear form

$$((z_1, h_1, w_2), (z_2, h_2, w_2)) \mapsto 2h_1z_{2,t} + xz_{1,x}(1, \cdot)z_{2,x}$$

is bounded. This is true because $h_1 \in H^1_{\rho_4}(0, T)$ and $z_1, z_2 \in Z_0(\rho_4)$ and, in particular, we have $\rho_4h_1 \in H^1(0, T)$, $\rho_4z_{2,t} \in L^2(Q)$, $z_{1,x}(1, \cdot) \in L^2(0, T)$ and $\rho_4z_2 \in C^0([0, T]; H^1_0(-1, 1))$.

Therefore, $\Lambda \in C^1(\mathcal{B}_1; \mathcal{B}_2)$.

On the other hand, note that $\Lambda'(0, 0, 0) : \mathcal{B}_1 \mapsto \mathcal{B}_2$ is given by

$$\Lambda'(0, 0, 0)(z, h, v) = (\mathcal{L}_1(z, h, w), \mathcal{L}_2(z, h, w), z(\cdot\,, 0), h(0)) \quad \forall (z, h, v) \in \mathcal{B}_1.$$

In view of the null controllability result for (2.5) given in Proposition 4.1, $\Lambda'(0, 0, 0)$ is surjective.

Consequently, we can apply Theorem 4.2 with these data and the proof of Proposition 2.4 is achieved.

Indeed, as a consequence of Theorem 4.2 we see that, for any sufficiently small $(z_0, h_0) \in H^1_0(-1, 1) \times \mathbb{R}$, there exists $(z, h, w) \in E$ with $\Lambda(z, h, w) = (0, 0, z_0, h_0)$. In view of (4.6), this means that (z, h, w) is a solution to (2.3) with $z(\cdot\,, T) = 0$ and $h(T) = 0$.

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