SIAM J. CONTROL OPTIM.
Vol. 61, No. 2, pp. 484–510

GLOBAL CONTROLLABILITY OF THE BOUSSINESQ SYSTEM WITH NAVIER-SLIP-WITH-FRICTION AND ROBIN BOUNDARY CONDITIONS*

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Abstract. In this paper, we deal with the global exact controllability to the trajectories of the Boussinesq system posed in 2D or 3D smooth bounded domains. The velocity field of the fluid must satisfy a Navier-slip-with-friction boundary condition, and a Robin boundary condition is imposed to the temperature. We assume that one can act on the velocity and the temperature on a small part of the boundary. For the proof, we first transform the boundary control problem into a distributed control problem. Then, we prove a global approximate controllability result by adapting the strategy of Coron, Marbach, and Sueur [*J. Eur. Math. Soc.* (*JEMS*), 22 (2020), pp. 1625–1673]; this relies on the controllability properties of the inviscid Boussinesq system and the analysis of appropriate asymptotic boundary layer expansions. Finally, we conclude with a local controllability result; as in many other cases, this can be established as a consequence of the null controllability of a linearized system through a fixed-point argument. Our contribution can be viewed as an extension of the results in [*J. Eur. Math. Soc.* (*JEMS*), 22 (2020), pp. 1625–1673], where thermal effects were not considered. Thus, we prove that the ideas behind the controllability properties of the Euler system and the well-prepared dissipation technique can be adapted to the present situation. Furthermore, we cover all the classical boundary conditions for the temperature, that is, those of the Robin, Neumann, and Dirichlet kinds.

Key words. Boussinesq system, Navier-slip-with-friction boundary conditions, global controllability, boundary layers, global Carleman inequalities

MSC codes. 35Q35, 76D55, 93B05, 93C10

DOI. 10.1137/21M1425566

1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) be a smooth bounded domain with $\Gamma := \partial\Omega$, and let $\Gamma_c \subset \Gamma$ be a nonempty open subset which intersects all connected components of Γ. It will be said that Γ_c is the control boundary. Let us set

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*Received by the editors June 9, 2021; accepted for publication (in revised form) September 16, 2022; published electronically March 24, 2023.
https://doi.org/10.1137/21M1425566

Funding: The second and fifth authors were partially supported by grant PID2020-114976GB-I00 funded by MCIN/AEI/10.13039/501100011033. The fifth author was partially supported by grant IJC2018–037863-I funded by MCIN/AEI/10.13039/501100011033. The first author was partially supported by CNPq-Brazil, by grant 2019/0014 of the Paraíba State Research Foundation (FAPESQ), and the project CAPES-MathAmSud: ACIPDE. The third author was partially supported by the project TRECOS ANR-20-CE40-0009 funded by the ANR (2021–2024). The fourth author was partially supported by CNPq-Brazil.

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$$H_c := \{u \in L^2(\Omega)^n :\ \mathrm{div}\ u = 0 \text{ in } \Omega,\ u\cdot\nu = 0 \text{ on } \Gamma\backslash\Gamma_c\},$$

where $\nu = \nu(x)$ is the outward unit normal vector to Ω at the points $x \in \Gamma$. Here, the equality $u\cdot\nu = 0$ on $\Gamma\backslash\Gamma_c$ must be understood in the following sense:

$$\langle u\cdot\nu, g\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0\ \forall g \in H^{1/2}(\Gamma) \text{ with } g \equiv 0 \text{ on } \Gamma_c.$$

For a given vector field f, we denote by $[f]_{tan}$, $D(f)$, and $N(f)$ the tangential part of f, the deformation tensor, and the tangential Navier boundary operator, respectively, given as follows:

$$(1)\qquad [f]_{tan} := f - (f\cdot\nu)\nu, \quad D(f) := \frac{1}{2}\left(\nabla f + \nabla f^t\right), \quad N(f) := [D(f)\nu + Mf]_{tan}.$$

Here and henceforth, it is assumed that $M = M(t,x)$ is a smooth, symmetric matrix-valued function. It will be called the *friction matrix* and will be viewed as a measure of the boundary rugosity. We will also set

$$R(\theta) := \frac{\partial\theta}{\partial\nu} + m\theta,$$

where $m = m(t,x)$ is another smooth function, again related to the properties of the boundary, known as the *heat transfer coefficient*.

Let $T > 0$ be a final time. We will consider the (incomplete) Boussinesq system

$$(2)\qquad \begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = \theta e_n, \quad \mathrm{div}\ u = 0 & \text{in} \quad (0,T)\times\Omega, \\ \partial_t\theta - \Delta\theta + u\cdot\nabla\theta = 0 & \text{in} \quad (0,T)\times\Omega, \\ u\cdot\nu = 0, \quad N(u) = 0, \quad R(\theta) = 0 & \text{on} \quad (0,T)\times(\Gamma\setminus\Gamma_c), \\ u(0,\cdot) = u_0, \quad \theta(0,\cdot) = \theta_0 & \text{in} \quad \Omega, \end{cases}$$

where the functions u, θ, and p must be respectively viewed as the velocity field, the temperature, and the pressure of a viscous Newtonian fluid subject to thermal effects, and e_n is the nth vector of the canonical basis of $\mathbb{R}^n$. Regarded as a control system, we will interpret that the state is (u,θ) and the control is the lateral trace of (u,θ) on $(0,T)\times\Gamma_c$.

1.1. Main result. Let us introduce the notation

$$X_T(\Omega) := [C_w^0([0,T];H_c) \cap L^2(0,T;H^1(\Omega)^n)] \times [C_w^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))].$$

Here, for any Banach space B, $C_w^0([0,T];B)$ denotes the space of weakly continuous B-valued functions, that is, the functions $\phi : [0,T] \mapsto B$ such that $t \in [0,T] \to \langle\psi,\phi(t)\rangle_{B',B}$ is continuous for every $\psi \in B'$.

We have the following result.

THEOREM 1.1. *Let $T > 0$ be a positive time, let $(u_0,\theta_0) \in H_c \times L^2(\Omega)$ be a given initial state, and let $(\overline{u},\overline{\theta}) \in X_T(\Omega)$ be a weak trajectory of* (2). *Then, there exists a controlled weak solution to* (2) *in $X_T(\Omega)$ satisfying*

$$(3)\qquad (u,\theta)(T,\cdot) = \left(\overline{u},\overline{\theta}\right)(T,\cdot).$$

Remark 1.1. For the precise notions of weak trajectory and controlled weak solution, see Definition 2.1 below. Essentially, what we require of $(\bar{u},\bar{\theta})$ and (u,θ) is to belong to $X_T(\Omega)$ and satisfy (2) in the weak (distributional) sense.

Remark 1.2. In Theorem 1.1, we do not indicate explicitly which are the controls. As already said, once the controlled solution is found, the associated control is the lateral trace of the solution on $(0, T) \times \Gamma_c$.

Remark 1.3. Theorem 1.1 is stated as an existence result. The lack of uniqueness is for two main reasons. First, there exist many controls that drive the solution to (2) to the desired trajectory. Second, even if we select a criterion in order to fix the control without ambiguity, it is obviously unknown whether the associated state is unique in the 3D case (in two dimensions, it is known that the corresponding weak solution is unique; see, for instance, [2, 26] for the Navier–Stokes case).

1.2. Bibliographical comments. We now recall some existing results in the literature related to Theorem 1.1.

There are several papers where the controllability properties of the Boussinesq equations are investigated. Most of them are local results covering boundary conditions of various kinds. For instance, in [14] the local exact boundary controllability to the trajectories was obtained with boundary controls acting over the whole boundary; in [15], the exact controllability with distributed controls and periodic boundary conditions was analyzed; in [19], the author proved the local exact controllability to the trajectories with Dirichlet boundary conditions; this situation is also handled with a reduced number of controls in [10, 17]. For nonviscous Boussinesq fluids, this subject has been investigated by Fernández-Cara, Santos, and Souza [11].

On the other hand, the literature on the Navier–Stokes and Boussinesq equations with Navier-slip boundary conditions is scarce. Let us recall some controllability results obtained for the Navier–Stokes system: in [7], a small-time global result for the 2D equations has been proved where the exact controllability can be achieved in the interior of the spatial domain; the residual boundary layers are apparently too strong to be handled satisfactorily during the control design strategy. Guerrero proved in [18] the local exact controllability to the trajectories with general nonlinear Navier boundary conditions. Finally, the small-time global exact controllability with Navier-slip-with-friction boundary conditions towards weak trajectories was proved in [8] by Coron, Marbach and Sueur; this article provides a positive answer to the famous open question by J.-L. Lions concerning global null controllability of the Navier–Stokes equations when the boundary conditions are of this kind. Recently, in [25], this result was extended from Leray weak controlled solutions to the case of smooth controlled solutions. For what concerns the Boussinesq system with Navier-slip boundary conditions, see [23, 29] for some local results.

1.3. Strategy of the proof and plan of the paper. Let us briefly indicate the main ideas and results needed for the proof of Theorem 1.1.

In section 2, we will reduce the task to the solution of a *distributed controllability* problem by applying a classical domain extension technique. Then, we will limit our considerations to *smooth initial data* by using the smoothing effect of the uncontrolled Boussinesq system.

In section 3, starting from sufficiently smooth initial data, we prove a *global approximate controllability* result by adapting the strategy introduced by Coron, Marbach, and Sueur in [8] in the Navier–Stokes case.

In section 4, we prove a *local controllability result.* Here, we use an appropriate Carleman inequality for the adjoint of a linearized system (which leads to the null control of a related linearized system) and a fixed-point strategy.

In section 5, we combine all these arguments and achieve the proof.

2. Domain extension and smoothing effect.

2.1. Domain extension. Let us consider a smooth extended bounded domain $\mathcal{O}$ such that $\Omega\cup\Gamma_c\subset\mathcal{O}$ and $\Gamma\setminus\Gamma_c\subset\Gamma_{\mathcal{O}}:=\partial\mathcal{O}$. In what follows, if there is no ambiguity, we will also denote by $\nu(x)$ the outward unit normal vector to $\mathcal{O}$ at the points $x\in\partial\mathcal{O}$.

Let us introduce the following notations:

$$\mathcal{O}_T:=(0,T)\times\mathcal{O}\quad\text{and}\quad \varLambda_T:=(0,T)\times\partial\mathcal{O}.$$

In what follows, we will assume that M and m are extended to $[0,T]\times\partial\mathcal{O}$ as smooth functions in such a way that M is symmetric on $(0,T)\times\partial\mathcal{O}$. This will allow us to speak of $N(u)$ and $R(\theta)$ on $\varLambda_T$.

In general, the notation will be abridged. For instance, if $u\in H^2(\mathcal{O})^n$ and $\theta\in H^1(\mathcal{O})$, $\|(u,\theta)\|_{H^2\times H^1}$ will stand for the norm of (u,θ) in the space $H^2(\mathcal{O})^n\times H^1(\mathcal{O})$. The scalar product and norm in L^2 spaces will be respectively denoted by $(\cdot\,,\cdot)$ and $\|\cdot\|$. The symbol C will stand for a generic positive constant.

We will need the space

$$H:=\{u\in L^2(\mathcal{O})^n:\ \operatorname{div}\,u=0\text{ in }\mathcal{O},\quad u\cdot\nu=0\text{ on }\partial\mathcal{O}\}.$$

The following proposition enables us to extend the initial conditions to the whole domain $\mathcal{O}$:

PROPOSITION 2.1. *Let $(u_0,\theta_0)\in H_c\times L^2(\Omega)$ be given. Then, there exist $(u_*,\theta_*)\in L^2(\mathcal{O})^{n+1}$ and $\sigma_*\in C^\infty(\mathcal{O})$ with* Supp $\sigma_*\subset\mathcal{O}\setminus\overline{\Omega}$ *such that*

$$\begin{array}{c} u_*=u_0\quad\textit{and}\quad\theta_*=\theta_0\ \textit{in}\ \Omega,\qquad \operatorname{div}\,u_*=\sigma_*\ \textit{in}\ \mathcal{O},\qquad u_*\cdot\nu=0\ \textit{on}\ \partial\mathcal{O},\\ \|u_*\|+\|\sigma_*\|\le C\|u_0\|\quad\textit{and}\quad\|\theta_*\|\le C\|\theta_0\|.\end{array}\tag{4}$$

Proof. Let $\theta_*\in L^2(\mathcal{O})$ be the extension by zero of θ_0 to the whole domain $\mathcal{O}$. Then, we have

$$\|\theta_*\|\le\|\theta_0\|.$$

Next, in order to find an appropriate extension of u_0, we first note that the space

$$\mathscr{H}_c:=\{\phi\in C^1(\overline{\Omega};\mathbb{R}^n):\ \operatorname{div}\,\phi=0\text{ in }\Omega,\quad\phi\cdot\nu=0\text{ on }\Gamma\backslash\Gamma_c\}$$

is dense in H_c. Let us put $\Gamma_c=\cup_{i=1}^k\Gamma_c^i$, where the Γ_c^i denote the intersections of Γ_c with the connected components of Γ, and let $(\mathcal{O}\setminus\overline{\Omega})^i$ denote the subset of $\mathcal{O}\setminus\overline{\Omega}$ for which $\partial(\mathcal{O}\setminus\overline{\Omega})^i\cap\partial\Omega=\Gamma_c^i$. Also, let $\omega^i\subset\subset(\mathcal{O}\setminus\overline{\Omega})^i$ be a nonempty open subset, and let $\sigma_*^i\in C_c^\infty(\omega^i)$ be given with

$$\int_{(\mathcal{O}\backslash\overline{\Omega})^i}\sigma_*^i=1\quad\text{for}\quad i=1,\dots,k.$$

Let us assume that $u_0\in H_c$, and let $(u_{0,m})_{m\ge1}$ be a sequence in $\mathscr{H}_c$ with $u_{0,m}\to u_0$ in H_c. For every $i\in\{1,\dots,k\}$ and $m\ge1$, the following nonhomogeneous elliptic problem admits a unique solution $w_m^i\in H^1((\mathcal{O}\backslash\overline{\Omega})^i)$:

$$\begin{cases} -\Delta w_m^i=-a_m^i\sigma_*^i & \text{in}\quad(\mathcal{O}\backslash\overline{\Omega})^i,\quad \displaystyle\int_{(\mathcal{O}\backslash\overline{\Omega})^i}w_m^i=0,\\ \dfrac{\partial w_m^i}{\partial\nu}=u_{0,m}\cdot\nu & \text{on}\quad\Gamma_c^i,\\ \dfrac{\partial w_m^i}{\partial\nu}=0 & \text{on}\quad\partial(\mathcal{O}\backslash\overline{\Omega})^i\setminus\Gamma_c^i,\end{cases}$$

where $a_m^i := \int_{\Gamma_c} (u_{0,m} \cdot \nu)\, d\Gamma$. It is clear that, for every $i \in \{1, \dots, k\}$, a_m^i converges to some $a^i \in \mathbb{R}$ and w_m^i converges to some $w^i \in H^1((\mathcal{O}\backslash\overline{\Omega})^i)$ as $m \to +\infty$.

Let us set

$$u_* := \begin{cases} u_0 & \text{in} \quad \Omega, \\ \nabla w^i & \text{in} \quad (\mathcal{O}\backslash\overline{\Omega})^i \quad \text{for} \quad i = 1, \dots, k. \end{cases}$$

It is then clear that $u_* \in L^2(\mathcal{O})^n$, $\operatorname{div}\, u_* = \sigma_*$ in $\mathcal{O}$ for some $\sigma_* \in C_c^\infty(\mathcal{O}\backslash\overline{\Omega})$, and $u_* \cdot \nu = 0$ on $\partial\mathcal{O}$. On the other hand, we see that, by construction, (4) is satisfied. ☐

Let us introduce the following notation:

$$W_T(\mathcal{O}) := [C_w^0([0,T]; L^2(\mathcal{O})^n) \cap L^2(0,T; H^1(\mathcal{O})^n)] \\ \times [C_w^0([0,T]; L^2(\mathcal{O})) \cap L^2(0,T; H^1(\mathcal{O}))].$$

The notion of solution used throughout the paper is the following.

DEFINITION 2.1. *Let $T > 0$ be a positive time, and let $(u_0, \theta_0) \in H_c \times L^2(\Omega)$ be given. It will be said that $(u, \theta) \in X_T(\Omega)$ is a controlled weak trajectory of* (2) *with initial condition (u_0, θ_0) if (u, θ) is the restriction to $(0,T) \times \Omega$ of a weak Leray solution, still denoted by (u, θ), in the space $W_T(\mathcal{O})$, to the nonlinear system*

$$(5) \qquad \begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = \theta e_n + v, \quad \operatorname{div}\, u = \sigma & \text{in} \quad \mathcal{O}_T, \\ \partial_t \theta - \Delta \theta + u \cdot \nabla \theta = w & \text{in} \quad \mathcal{O}_T, \\ u \cdot \nu = 0, \quad N(u) = 0, \quad R(\theta) = 0 & \text{on} \quad \Lambda_T, \\ u(0, \cdot) = u_*, \quad \theta(0, \cdot) = \theta_* & \text{in} \quad \mathcal{O}, \end{cases}$$

where

- *$v \in C^0([0,T]; H^1(\mathcal{O})^n) \cap H^1(0,T; L^2(\mathcal{O})^n)$, $w \in C^0([0,T]; H^1(\mathcal{O})) \cap H^1(0,T; L^2(\mathcal{O}))$, and $\sigma \in C^\infty(\mathcal{O}_T)$ are supported by $(0,T) \times (\overline{\mathcal{O}} \setminus \overline{\Omega})$, and*
- *(u_*, θ_*) is an extension of (u_0, θ_0) furnished by Proposition* 2.1*, satisfying* $\operatorname{div}\, u_* = \sigma(0, \cdot)$.

Let us recall an existence result of weak solution to (5); it is taken from [27] (see Proposition 3.7 in that reference) and see also [4, Proposition 2.2].

PROPOSITION 2.2. *Let us assume that $T > 0$ and v, σ, w, and (u_*, θ_*) are as in Definition* 2.1*. Then there exists at least one weak Leray solution (u, θ) to* (5)*.*

2.2. Smoothing effect of the uncontrolled Boussinesq system. The goal of this section is to show that, starting from L^2 initial data, at small time the solution is smooth. For convenience, this property will be stated as follows.

LEMMA 2.1. *Let us assume that $T > 0$ and $(\overline{u}, \overline{\theta}) \in C^\infty(\overline{\mathcal{O}}_T)^{n+1}$ is such that $\operatorname{div}\, \overline{u} = 0$ in $\mathcal{O}_T$ and $\overline{u} \cdot \nu = 0$ on Λ_T. Then, there exists a smooth function $\Psi_T : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\Psi_T(0) = 0$ such that, for any $(r_*, q_*) \in H \times L^2(\mathcal{O})$ and any weak Leray solution $(r, q) \in W_T(\mathcal{O})$ to*

$$(6) \quad \begin{cases} \partial_t r - \Delta r + (r \cdot \nabla) r + (\overline{u} \cdot \nabla) r + (r \cdot \nabla) \overline{u} + \nabla \pi = q e_n, \quad \operatorname{div}\, r = 0 & \text{in} \quad \mathcal{O}_T, \\ \partial_t q - \Delta q + (r + \overline{u}) \cdot \nabla q + r \cdot \nabla \overline{\theta} = 0 & \text{in} \quad \mathcal{O}_T, \\ r \cdot \nu = 0, \quad N(r) = 0, \quad R(q) = 0 & \text{on} \quad \Lambda_T, \\ r(0, \cdot) = r_*, \quad q(0, \cdot) = q_* & \text{in} \quad \mathcal{O}, \end{cases}$$

the following property holds:

$$\exists\, t_0 \in [0,T]; \quad \|(r,q)(t_0, \cdot)\|_{H^3 \times H^3} \leq \Psi_T\left(\|(r_*, q_*)\|\right).$$

The proof of this lemma is quite classical but, for completeness, will be given in Appendix A.

3. Approximate controllability problem.

In this section, we prove an approximate controllability result starting from sufficiently smooth initial data.

PROPOSITION 3.1. *Let us assume that $T > 0$, and let $(\overline{u}, \overline{\theta})$, $\overline{v}$, $\overline{w}$, and $\overline{\sigma}$ be as in Definition* 2.1. *Suppose that $(\overline{u}, \overline{\theta})$ is, together with $\overline{p}$, an associated solution, and assume that the triplet $(\overline{u}, \overline{p}, \overline{\theta})$ belongs to $C^\infty(\overline{\mathcal{O}}_T; \mathbb{R}^{n+2})$. Let $(u_*, \theta_*) \in [H^3(\mathcal{O})^n \cap H] \times H^3(\mathcal{O})$ be an initial state. Then, for any $\delta > 0$, there exist regular controls v, w, and σ, again supported in $\overline{\mathcal{O}} \setminus \overline{\Omega}$, and an associated weak solution to* (5) *satisfying*

$$\|(u, \theta)(T, \cdot) - (\overline{u}, \overline{\theta})(T, \cdot)\| \le \delta.$$

For the proof, we will follow the strategy introduced by Coron, Marbach, and Sueur in [8]. Let us explain how it works.

First, a change of scale associated to a small parameter $\varepsilon > 0$ is introduced and (5) is transformed into a Boussinesq system with small viscosity ε that must be controlled in the (long) time interval $[0, T/\varepsilon]$, starting from a small initial state; see (7). The advantage of this scaling is that we can benefit from the nonlinear terms $(u \cdot \nabla)u$ and $u \cdot \nabla\theta$.

Formally, by taking $\varepsilon = 0$, we obtain the inviscid Boussinesq system; see (11). For this hyperbolic system, we can construct a very particular nontrivial trajectory that connects $(0, 0) \in \mathbb{R}^{n+1}$ to itself and sends any particle outside the physical domain before the final time T. By linearizing the inviscid Boussinesq system around the previous trajectory, we obtain a new hyperbolic linear system that is small-time globally null-controllable (actually, what we do is apply the so-called *return method*, due to Coron; see [5]; note that the linearization around the trivial state leads to a noncontrollable system).

In the particular case of a "special slip" boundary condition for the velocity and a Neumann boundary condition for the temperature, that is, with M such that $[\nabla \times u]_{\tan} = 0$ on Λ_T and $m \equiv 0$, we immediately conclude by estimating the remainder terms. We do not need to use the long interval time $[0, T/\varepsilon]$ to control in this case, since the solution is already small at intermediate times $T_* \in (0, T/\varepsilon)$.

Unfortunately, in the general case, a boundary layer appears. This phenomenon was already taken into account in [12, 22] for the Navier–Stokes PDEs. Thus, we have to introduce some corrector terms in the asymptotic expansion of the solution depending on ε, in order to estimate the residual layers. It is found that the boundary layer decays but not enough. Hence, the corrector is not sufficiently small at the final time T/ε and we still cannot conclude.

In order to overcome this difficulty, we adapt the *well-prepared dissipation method,* introduced by Marbach in [28]. Here, the idea is to design a control strategy that reinforces the action of the natural dissipation of the boundary layer after an intermediate time. A desired small state is obtained at final time, and we can finally achieve the proof.

In what follows, we will frequently need vector functions $(u, p, \theta, v, w, \sigma)$ representing adequate states (u, p, θ), controls (v, w), and auxiliary functions σ, corresponding to some linear or nonlinear systems. In all cases, it will be implicitly assumed that v, w, and σ vanish outside $[0, T] \times (\overline{\mathcal{O}} \setminus \overline{\Omega})$.

3.1. Time scaling.

Let us introduce

$$\begin{cases} u^\varepsilon(t, x) := \varepsilon u(\varepsilon t, x), & p^\varepsilon(t, x) := \varepsilon^2 p(\varepsilon t, x), \quad \theta^\varepsilon(t, x) := \varepsilon^2 \theta(\varepsilon t, x), \\ v^\varepsilon(t, x) := \varepsilon^2 v(\varepsilon t, x), & w^\varepsilon(t, x) := \varepsilon^3 w(\varepsilon t, x), \quad \sigma^\varepsilon(t, x) := \varepsilon \sigma(\varepsilon t, x). \end{cases}$$

In these new variables, (5) reads as

$$
\begin{cases}
\partial_t u^\varepsilon - \varepsilon\Delta u^\varepsilon + (u^\varepsilon\cdot\nabla)u^\varepsilon + \nabla p^\varepsilon = \theta^\varepsilon e_n + v^\varepsilon, \quad \operatorname{div} u^\varepsilon = \sigma^\varepsilon & \text{in} \quad (0,T/\varepsilon)\times\mathcal{O},\\
\partial_t\theta^\varepsilon - \varepsilon\Delta\theta^\varepsilon + u^\varepsilon\cdot\nabla\theta^\varepsilon = w^\varepsilon & \text{in} \quad (0,T/\varepsilon)\times\mathcal{O},\\
u^\varepsilon\cdot\nu = 0, \quad N(u^\varepsilon)=0, \quad R(\theta^\varepsilon)=0 & \text{on} \quad (0,T/\varepsilon)\times\partial\mathcal{O},\\
u^\varepsilon(0,\cdot) = \varepsilon u_*, \quad \theta^\varepsilon(0,\cdot)=\varepsilon^2\theta_* & \text{in} \quad \mathcal{O}.
\end{cases} \tag{7}
$$

Thus, we work along a large time interval $[0,T/\varepsilon]$, starting from the small initial data $(\varepsilon u_*, \varepsilon^2\theta_*)$. The counterpart is the small viscosity. Accordingly, (7) must be viewed as a singular perturbation of a nonlinear inviscid system.

In order to prove Proposition 3.1, it is sufficient to check that

$$\|u^\varepsilon(T/\varepsilon,\cdot) - \varepsilon\overline{u}(T,\cdot)\| = o(\varepsilon) \quad \text{and} \quad \left\|\theta^\varepsilon(T/\varepsilon,\cdot) - \varepsilon^2\overline{\theta}(T,\cdot)\right\| = o(\varepsilon^2).$$

3.2. A special slip boundary condition. In this section, we consider a special situation where the fluid perfectly slips and the proof of Proposition 3.1 is much simpler (there is no boundary layer). For the moment, we will assume that the target trajectory is zero, i.e., $(\overline{u},\overline{p},\overline{\theta},\overline{v},\overline{w},\overline{\sigma})\equiv 0$, and we will try to control (7) during the time interval $[0,T]$ instead of $[0,T/\varepsilon]$. The goal is to prove

$$\|u^\varepsilon(T,\cdot)\| = o(\varepsilon) \quad \text{and} \quad \|\theta^\varepsilon(T,\cdot)\| = o(\varepsilon^2). \tag{8}$$

Thus, let us assume that the friction coefficient M is the *Weingarten map* (or shape operator) M_w. Thanks to [8, Lemma 1], on the uncontrolled boundary one has zero normal velocity and zero tangential vorticity, that is,

$$u\cdot\nu = 0 \quad \text{and} \quad [\nabla\times u]_{tan} = 0 \quad \text{on} \quad \Lambda_T. \tag{9}$$

3.2.1. Ansatz with no correction term. Let us introduce an asymptotic expansion of the solution to (7):

$$
\begin{cases}
u^\varepsilon = u^0 + \varepsilon u^1 + \varepsilon r^\varepsilon, & p^\varepsilon = p^0 + \varepsilon p^1 + \varepsilon\pi^\varepsilon, & \theta^\varepsilon = \theta^0 + \varepsilon^2\theta^1 + \varepsilon^2 q^\varepsilon,\\
v^\varepsilon = v^0 + \varepsilon v^1, & w^\varepsilon = w^0 + \varepsilon^2 w^1, & \sigma^\varepsilon = \sigma^0.
\end{cases} \tag{10}
$$

There is some intuition behind (10). The first term $(u^0,p^0,\theta^0,v^0,w^0,\sigma^0)$ is the solution to an inviscid system; take $\varepsilon=0$ in (7). It models a reference trajectory around which we linearize the original system, exactly as is done when applying Coron's return method; see [5]. It will be chosen in such a way that the associated flow flushes the particles out of the physical domain before $t=T$; see (13) below for a more precise explanation. The second term $(u^1,p^1,\theta^1,v^1,w^1)$ takes into account the initial data (u_*,θ_*) and will be controlled to zero in the physical domain Ω; see Lemma 3.2 below. Then, $(r^\varepsilon,\pi^\varepsilon,q^\varepsilon)$ contains higher order terms; see (19). At the end, using (10), what we have to prove is that $\|(r^\varepsilon,q^\varepsilon)(T,\cdot)\| = o(1)$, in order to be able to conclude (8).

3.2.2. Inviscid flow. By taking $\varepsilon=0$ in (7), we obtain the following system:

$$
\begin{cases}
\partial_t u^0 + (u^0\cdot\nabla)u^0 + \nabla p^0 = \theta^0 e_n + v^0, \quad \operatorname{div} u^0 = \sigma^0 & \text{in} \quad \mathcal{O}_T,\\
\partial_t\theta^0 + u^0\cdot\nabla\theta^0 = w^0 & \text{in} \quad \mathcal{O}_T,\\
u^0\cdot\nu = 0 & \text{on} \quad \Lambda_T,\\
u^0(0,\cdot) = u^0(T,\cdot) = 0, \quad \theta^0(0,\cdot) = \theta^0(T,\cdot) = 0 & \text{in} \quad \mathcal{O},
\end{cases} \tag{11}
$$

where v^0, w^0, and σ^0 are spatially supported in $\overline{\mathcal{O}}\setminus\overline{\Omega}$.

Let us introduce the flow function $\Phi^0 := \Phi^0(s;t,x)$ associated to u^0. That is, for any (t,x), $\Phi^0(\cdot\,;t,x)$ solves the ODE problem

$$\left\{\begin{array}{lcl} \partial_s \Phi^0(s;t,x) & = & u^0(s,\Phi^0(s;t,x)), \\ \Phi^0(s;t,x)\big|_{s=t} & = & x. \end{array}\right. \tag{12}$$

Then, we look for trajectories such that

$$\forall x \in \overline{\mathcal{O}},\ \exists\ t_x \in (0,T) \text{ such that } \Phi^0(t_x;0,x) \in \overline{\mathcal{O}} \setminus \overline{\Omega}. \tag{13}$$

This property is obvious for the points x already located in $\overline{\mathcal{O}} \setminus \overline{\Omega}$. For the points $x \in \overline{\Omega}$, we use the following result, whose proof can be found in [6, 8] in the 2D case and [8, 16] in the 3D case:

LEMMA 3.1. *There exists a nonzero solution to* (11) $(u^0,p^0,\theta^0,v^0,w^0,\sigma^0) \in C^\infty([0,T]\times\overline{\mathcal{O}};\mathbb{R}^{2n+4})$ *such that the associated flow* Φ^0, *defined in* (12), *satisfies* (13). *Moreover, we can choose* u^0, θ^0, *and* w^0 *such that*

$$\theta^0 = w^0 = 0 \quad \textit{and} \quad \nabla\times u^0 = 0 \quad \textit{in} \quad [0,T]\times\overline{\mathcal{O}} \tag{14}$$

and u^0,p^0,v^0, *and* σ^0 *are compactly supported in time in* $(0,T)$.

Note that, in the proof of this result, the assumption that Γ_c intersects all connected components of Γ must be used.

In what follows, when needed, it will be assumed that $(u^0,p^0,\theta^0,v^0,w^0,\sigma^0)$ has been extended by zero after time T.

3.2.3. Flushing. In accordance with Lemma 3.1, we take $\theta^0 = w^0 = 0$ in (10). Let (u^1,θ^1) be the solution to the linear problem

$$\left\{\begin{array}{lll} \partial_t u^1 + (u^0\cdot\nabla)u^1 + (u^1\cdot\nabla)u^0 + \nabla p^1 = \Delta u^0 + v^1, \quad \operatorname{div} u^1 = 0 & \text{in} & \mathcal{O}_T, \\ \partial_t \theta^1 + u^0\cdot\nabla\theta^1 = w^1 & \text{in} & \mathcal{O}_T, \\ u^1\cdot\nu = 0 & \text{on} & \mathit{\Lambda}_T, \\ u^1(0,\cdot) = u_*, \quad \theta^1(0,\cdot) = \theta_* & \text{in} & \mathcal{O}, \end{array}\right. \tag{15}$$

where v^1 and w^1 are forcing terms, spatially supported in $\overline{\mathcal{O}} \setminus \overline{\Omega}$. Thanks to (14), we have $\Delta u^0 = \nabla(\operatorname{div} u^0) + \nabla\times(\nabla\times u^0) = \nabla\sigma^0$. Thus, this term can be absorbed by v^1. Of course, (15) is a linear uncoupled system.

LEMMA 3.2. *Let us assume that* $(u_*,\theta_*) \in [H^3(\mathcal{O})^n \cap H] \times H^3(\mathcal{O})$. *There exist forcing terms*

$$\begin{array}{l} v^1 \in C^1([0,T];H^1(\mathcal{O})^n) \cap C^0([0,T];H^2(\mathcal{O})^n), \\ w^1 \in C^1([0,T];H^2(\mathcal{O})) \cap C^0([0,T];H^3(\mathcal{O})) \end{array} \tag{16}$$

with

$$\operatorname{Supp}\,(v^1,w^1) \subset\subset [0,T]\times\overline{\mathcal{O}}\setminus\overline{\Omega}, \tag{17}$$

such that the associated solution (u^1,θ^1) *to* (15) *satisfies* $(u^1,\theta^1)(T,\cdot) = (0,0)$ *in* Ω. *Moreover,* u^1 *belongs to* $C^0([0,T];H^2(\mathcal{O})^n)\cap L^\infty(0,T;H^3(\mathcal{O})^n)$ *and a similar property holds for* θ^1.

Proof. First, note that the result for u^1 is proved in [8, Lemma 3]. Second, for θ^1 we have a similar situation and we can apply the same arguments. For completeness, let us sketch the main ideas.

We will use the smooth partition of unity $\{\eta_\ell : 1 \le \ell \le L\}$ defined in [8, Appendix A], which is related to Φ^0 as follows: thanks to (13), we can find $\gamma > 0$ and open balls B_ℓ for $1 \le \ell \le L$ covering $\overline{\mathcal{O}}$ with the following property:

$$\left\{ \begin{array}{l} \forall \ell,\ \exists t_\ell \in (\gamma, T-\gamma),\ \exists m_\ell \in \{1,\dots,\mathcal{M}\} \text{ such that} \\ \Phi^0(s;0,B_\ell) \subset Q_{m_\ell}\, \forall s \in (t_\ell - \gamma, t_\ell + \gamma), \end{array} \right. \tag{18}$$

where the Q_{m_ℓ} are squares (or cubes) that never intersect $\overline{\Omega}$ that cover a compact set K in $\overline{\mathcal{O}}$ such that $K \cap \Omega = \emptyset$ and

$$\forall x \in \overline{\mathcal{O}},\ \exists\ t_x \in (0,T) \text{ such that } \Phi^0(t_x;0,x) \in K$$

and $\mathcal{M} \in \mathbb{N}$ is the number of such a cubes; hence, every ball spends a positive amount of time within a given square (cube) where we can use a localized control to act on the θ^1. Here, it is assumed that the η_ℓ satisfy $0 \le \eta_\ell \le 1$, $\sum \eta_\ell \equiv 1$ and Supp $(\eta_\ell) \subset B_\ell$.

Let us introduce a smooth function $\varrho : \mathbb{R} \mapsto [0,1]$, with $\varrho = 1$ on $(-\infty, -\gamma)$ and $\varrho = 0$ on $(\gamma, +\infty)$. For each ℓ, consider the solution $\overline{\theta}_\ell$ to

$$\left\{ \begin{array}{lll} \partial_t \overline{\theta}_\ell + u^0 \cdot \nabla \overline{\theta}_\ell = 0 & \text{in} & (0,T) \times \mathcal{O}, \\ \overline{\theta}_\ell(0,\cdot) = \eta_\ell \theta_* & \text{in} & \mathcal{O}, \end{array} \right.$$

and set $\theta_\ell(t,x) := \varrho(t - t_\ell)\overline{\theta}_\ell(t,x)$. Since $\varrho(T - t_\ell) = 0$ and $\varrho(-t_\ell) = 1$, θ_ℓ solves the linear problem

$$\left\{ \begin{array}{lll} \partial_t \theta_\ell + u^0 \cdot \nabla \theta_\ell = w_\ell & \text{in} & (0,T) \times \mathcal{O}, \\ \theta_\ell(0,\cdot) = \eta_\ell \theta_*, \quad \theta_\ell(T,\cdot) = 0 & \text{in} & \mathcal{O}, \end{array} \right.$$

where $w_\ell(t,x) := \varrho_t(t - t_\ell)\overline{\theta}_\ell$. Since ϱ_t vanishes outside $(-\gamma,\gamma)$, one has (18), and η_ℓ is compactly supported in B_ℓ, it is easy to see that w_ℓ is supported in $[0,T] \times Q_{m_\ell}$.

At this point, we take $\theta^1 := \sum_\ell \theta_\ell$ and $w^1 := \sum_\ell w_\ell$ and we see that the second PDE and the second initial condition in (15) are satisfied. Thanks to this explicit construction, the spatial regularity of w^1 and $\overline{\theta}_\ell$ are the same. Therefore, $w^1 \in C^1([0,T], H^2(\mathcal{O})) \cap C^0([0,T], H^3(\mathcal{O}))$. The fact that θ^1 belongs to $C^0([0,T]; H^2(\mathcal{O})) \cap L^\infty(0,T; H^3(\mathcal{O}))$ readily comes from the fact that the θ_ℓ satisfy the same. This ends the proof. ∎

Lemma 3.2 is a null controllability result. Thanks to the linearity and reversibility of (15), it leads to an exact controllability result:

LEMMA 3.3. *Let us assume that* $(u_*, \theta_*), (u_T, \theta_T) \in [H^3(\mathcal{O})^n \cap H] \times H^3(\mathcal{O})$. *Then, there exist* v^1 *and* w^1 *as in* (16) *and* (17) *such that the associated solution to* (15) *satisfies* $(u^1, \theta^1)(T,\cdot) = (u_T, \theta_T)$. *Moreover,* u^1 *belongs to* $C^0([0,T]; H^2(\mathcal{O})^n) \cap L^\infty(0,T; H^3(\mathcal{O})^n)$ *and a similar property holds for* θ^1.

3.2.4. Equations and estimates for the remainder. The equations for r^ε, π^ε, and q^ε in the extended domain $\mathcal{O}_T$ are

(19)

$$\begin{cases} \partial_t r^\varepsilon - \varepsilon \Delta r^\varepsilon + (u^\varepsilon \cdot \nabla) r^\varepsilon + \nabla \pi^\varepsilon = f^\varepsilon - A^\varepsilon r^\varepsilon + \varepsilon q^\varepsilon e_n + \varepsilon \theta^1 e_n, \quad \text{div } r^\varepsilon = 0 & \text{in} \quad \mathcal{O}_T, \\ \partial_t q^\varepsilon - \varepsilon \Delta q^\varepsilon + u^\varepsilon \cdot \nabla q^\varepsilon = h^\varepsilon - B^\varepsilon r^\varepsilon & \text{in} \quad \mathcal{O}_T, \\ r^\varepsilon \cdot \nu = 0, \quad [\nabla \times r^\varepsilon]_{tan} = -[\nabla \times u^1]_{tan}, \quad R(q^\varepsilon) = -R(\theta^1) & \text{on} \quad \Lambda_T, \\ r^\varepsilon(0,\cdot) = 0, \quad q^\varepsilon(0,\cdot) = 0 & \text{in} \quad \mathcal{O}, \end{cases}$$

where we have introduced

$$\begin{aligned} f^\varepsilon &:= \varepsilon\Delta u^1 - \varepsilon(u^1\cdot\nabla)u^1, A^\varepsilon r^\varepsilon := (r^\varepsilon\cdot\nabla)(u^0+\varepsilon u^1),\\ h^\varepsilon &:= \varepsilon\Delta\theta^1 - \varepsilon u^1\cdot\nabla\theta^1, \qquad B^\varepsilon r^\varepsilon := \varepsilon r^\varepsilon\cdot\nabla\theta^1. \end{aligned}$$

We can establish energy estimates for the remainder by multiplying $(19)_1$ by r^ε and $(19)_2$ by q^ε. Indeed, after integration by parts, and thanks to the interpolation inequality in [2, Theorem III.2.36]), we easily obtain the following estimates:

$$\begin{aligned} \left|\int_{\partial\mathcal{O}} q^\varepsilon R(\theta^1)\,d\Gamma\right| &\le \|q^\varepsilon\|_{L^2(\partial\mathcal{O})}\left\|R(\theta^1)\right\|_{L^2(\partial\mathcal{O})} \le C\|q^\varepsilon\|_{H^1}\left\|\theta^1\right\|_{H^2},\\ \left|\int_{\partial\mathcal{O}} m|q^\varepsilon|^2\,d\Gamma\right| &\le C\|q^\varepsilon\|_{L^2}\|q^\varepsilon\|_{H^1} \end{aligned}$$

and

(20)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|r^\varepsilon\|^2+\|q^\varepsilon\|^2)+\varepsilon(\|\nabla\times r^\varepsilon\|^2+\|\nabla q^\varepsilon\|^2)\\ &\qquad\le C(\varepsilon+\|\sigma^0\|_{L^\infty}+\|u^0+\varepsilon u^1\|_{L^\infty}+\varepsilon\|\nabla\theta^1\|_{L^\infty})(\|r^\varepsilon\|^2+\|q^\varepsilon\|^2)\\ &\qquad\quad+\frac{\varepsilon}{2}(\|\nabla\times r^\varepsilon\|^2+\|\nabla q^\varepsilon\|^2)\\ &\qquad\quad+C\left[\varepsilon\left(\|u^1\|_{H^2}^2+\|\theta^1\|_{H^2}^2\right)+\|f^\varepsilon\|^2+\|h^\varepsilon\|^2\right], \end{aligned}$$

where the boundary term for r^ε is bounded as in [8, section 2.5].

From Gronwall's inequality and Lemma 3.2, we deduce that

$$\|r^\varepsilon\|_{L^\infty(L^2)}^2+\|q^\varepsilon\|_{L^\infty(L^2)}^2+\varepsilon\left(\|\nabla\times r^\varepsilon\|^2+\|\nabla q^\varepsilon\|^2\right)=O(\varepsilon).$$

Consequently, at time T, since $(u^0,\theta^0)(T,\cdot)=(u^1,\theta^1)(T,\cdot)=(0,0)$, we have

$$\|u^\varepsilon(T,\cdot)\|\le\|\varepsilon r^\varepsilon(T,\cdot)\|\le O(\varepsilon^{3/2}) \quad\text{and}\quad \|\theta^\varepsilon(T,\cdot)\|\le\|\varepsilon^2 q^\varepsilon(T,\cdot)\|\le O(\varepsilon^{5/2}).$$

This concludes the proof of Proposition 3.1 in a special case of the slip boundary condition (9).

3.3. The case of Navier-slip-with-friction boundary conditions. We come back in this section to the general case.

3.3.1. Ansatz with correction term. Let us introduce a smooth function $\varphi:\mathbb{R}^n\mapsto\mathbb{R}$ such that

$$\begin{cases} \varphi=0 \text{ on } \partial\mathcal{O}, \varphi>0 \text{ in } \mathcal{O}, \varphi<0 \text{ in } \mathbb{R}^n\backslash\overline{\mathcal{O}}, \text{ and}\\ |\varphi(x)|=dist(x,\partial\mathcal{O}) \text{ in a small neighborhood of } \partial\mathcal{O}. \end{cases}$$

Then, $\nu=-\nabla\varphi$ near $\partial\mathcal{O}$ and ν can be extended smoothly within the full domain $\mathcal{O}$.

According to the original boundary layer analysis of Navier-slip-with-friction boundary conditions proved in [22] by Iftimie and Sueur, we introduce the following expansions of the variables and the forcing terms:

$$\text{(21)}\quad \begin{cases} u^\varepsilon(t,x) &= u^0(t,x)+\sqrt{\varepsilon}\rho\left(t,x,\varphi(x)/\sqrt{\varepsilon}\right)+\varepsilon u^1(t,x)+\cdots+\varepsilon r^\varepsilon(t,x),\\ p^\varepsilon(t,x) &= p^0(t,x)+\varepsilon p^1(t,x)+\cdots+\varepsilon\pi^\varepsilon(t,x),\\ \theta^\varepsilon(t,x) &= \theta^0(t,x)+\varepsilon^2\theta^1(t,x)+\varepsilon^2 q^\varepsilon(t,x), \end{cases}$$

$$\begin{cases} v^\varepsilon(t,x) &=& v^0(t,x) + \sqrt{\varepsilon} v^\rho\left(t,x,\varphi(x)/\sqrt{\varepsilon}\right) + \varepsilon v^1(t,x), \\ w^\varepsilon(t,x) &=& w^0(t,x) + \varepsilon^2 w^1(t,x), \\ \sigma^\varepsilon(t,x) &=& \sigma^0(t,x). \end{cases}$$

Thus, since u^0 cannot satisfy the Navier-slip-with-friction boundary condition on Λ_T, we introduce in (21) a corrector ρ. This profile is expressed in terms of both the slow spatial variable $x \in \mathcal{O}$ and one fast scalar variable $z = \varphi(x)/\sqrt{\varepsilon}$. In the expansions in (21), the missing terms will be defined below (see section 3.3.3); they will help us to prove that the remainder is small. We use the couples (u^0, θ^0) and (u^1, θ^1) (extended by zero for $t > T$) introduced in the previous sections; see sections 3.2.2 and 3.2.3. The following sections are devoted to determine, analyze, and estimate all the terms.

The boundary layer corrector will be given by the solution to an initial boundary value problem with a boundary condition associated to the extra variable. More precisely, as in [22], we will require that $\rho = \rho(t,x,z)$ satisfies

$$\begin{cases} \partial_t \rho + [(u^0 \cdot \nabla)\rho + (\rho \cdot \nabla) u^0]_{tan} + u^0_\flat z \partial_z \rho - \partial_{zz}\rho = v^\rho & \text{in} \quad \mathbb{R}_+ \times \mathcal{O} \times \mathbb{R}_+, \\ \partial_z \rho(t,x,0) = g^0(t,x) & \text{in} \quad \mathbb{R}_+ \times \mathcal{O}, \\ \rho(t,x,0) \cdot \nu(x) = 0 & \text{in} \quad \mathbb{R}_+ \times \mathcal{O}, \\ \rho(0,x,z) = 0 & \text{in} \quad \mathcal{O} \times \mathbb{R}_+, \end{cases} \tag{22}$$

where we have used the notation

$$u^0_\flat(t,x) := -\frac{u^0(t,x) \cdot \nu(x)}{\varphi(x)} \quad \text{and} \quad g^0(t,x) := 2\chi(x) N(u^0)(t,x) \, \text{in} \mathbb{R}_+ \times \mathcal{O}$$

for a smooth cut-off function χ satisfying $\chi = 1$ in a neighborhood of $\partial\mathcal{O}$.

We can formally obtain (22) by replacing the expression u^ε by $u^0 + \sqrt{\varepsilon}\,\rho(t,x,\varphi(x)/\sqrt{\varepsilon}$ in (7) and keeping the terms of order $\sqrt{\varepsilon}$.

The following points are in order:

- v^ρ must be viewed as a smooth control whose spatial support is located outside of $\overline{\Omega}$. With the help of the transport term, this control will enable us to modify the behavior of ρ inside the physical domain Ω.
- ρ depends on $n+1$ spatial variables (n slow variables x_i and one fast variable z); thus, it is not set in curvilinear coordinates. It is implicitly assumed that ν actually refers to the extension $-\nabla\varphi$ of the normal vector; in turn, this furnishes extensions of the identities in (1).
- We will check that the construction above satisfies $v^\rho \cdot \nu = 0$. Since the equation is linear, it preserves the relation $\rho(0,x,z) \cdot \nu(x) = 0$ at initial time; whence, the boundary profile will be tangential, even inside the domain; see [22, section 2] for more details.
- In (23), the role of the function χ is to ensure that ρ is compactly supported near $\partial\mathcal{O}$.
- Since u^0 is smooth and tangent to the boundary, a Taylor expansion proves that $u^0_\flat$ is smooth in $\overline{\mathcal{O}}$.
- The boundary layer profile ρ does not depend on ε.

3.3.2. The well-prepared dissipation method. Unlike in the previous section, where T is the fixed time control, we will need here virtually long time intervals $[0, T/\varepsilon]$ to dissipate the boundary layer.

The most natural strategy would be to use that u^0 is equal to 0 after time T. Then, (22) would be reduced to a heat equation on the half line $\mathbb{R}^+$ with homogeneous Neumann boundary conditions and the boundary layer would decay. Unfortunately, this is too slow: one can only prove that $\sqrt{\varepsilon}\rho(T/\varepsilon,\cdot,\varphi(\cdot)/\sqrt{\varepsilon})=O(\varepsilon)$ (see [8, section 3.2]); therefore, by dividing by ε, $u(T,\cdot)=O(1)$ and this is not enough to use the local result at the end. This is why we use the source v^ρ to "prepare" the dissipation of the boundary layer.

Let us introduce some weighted Sobolev spaces:

$$H^{m,\ell}(\mathbb{R}):=\left\{f\in H^m(\mathbb{R})\ ;\ \sum_{|\alpha|=0}^{m}\int_{\mathbb{R}}(1+|z|^2)^\ell|\partial_z^\alpha f(z)|^2dz<+\infty\right\},$$

endowed with the corresponding (natural) norms. In [8, Lemma 7], the following result is proved.

Lemma 3.4. *Let us assume that $k\geq 1$, and let $u^0\in C^\infty(\overline{\mathcal{O}}_T;\mathbb{R}^n)$ be a fixed reference flow in* (11). *There exist $v^\rho\in C^\infty(\mathbb{R}_+\times\overline{\mathcal{O}}\times\mathbb{R}_+)$ with $v^\rho\cdot\nu=0$ and support in $(0,T)\times(\overline{\mathcal{O}}\setminus\overline{\Omega})\times\mathbb{R}_+$ such that, for any $j,m\in\mathbb{N}$ and any $\ell=0,1,\dots,k$, the associated boundary layer profile ρ satisfies*

$$\|\rho(t,\cdot,\cdot)\|_{H_x^j(\mathcal{O};H_z^{m,\ell}(\mathbb{R}_+))}\leq C\left|\frac{\log(2+t)}{2+t}\right|^{1/4+(k-\ell)/2},\tag{23}$$

where the positive constant C depends on j, m, ℓ, and u^0 but is independent of t.

The interest of Lemma 3.4 is twofold:

- The estimates (23) will be used to show that the source terms generated by the boundary layer are integrable in long time and the equation satisfied by the remainder term is well posed.
- Also, they will be used to prove that the boundary layer is sufficiently small at time T/ε.

Remark 3.1. A more ambitious idea would be to design a control strategy to get exactly $\rho(T/\varepsilon,\cdot,\varphi(\cdot)/\sqrt{\varepsilon})\equiv 0$. But, unfortunately, it can be proved that (22) is not null-controllable at time T/ε; see [8, section 3.5].

3.3.3. Technical profiles. For a function $f=f(t,x,z)$, we will use the notation $\{f\}$ to denote its values at points (t,x,z) with $z=\varphi(x)/\sqrt{\varepsilon}$. The full decomposition required for the states and controls will be the following:

$$\left\{\begin{array}{l} u^\varepsilon=u^0+\sqrt{\varepsilon}\{\rho\}+\varepsilon u^1+\varepsilon\nabla\zeta^\varepsilon+\varepsilon\{\beta\}+\varepsilon r^\varepsilon,\quad p^\varepsilon=p^0+\varepsilon\{\psi\}+\varepsilon p^1+\varepsilon\mu^\varepsilon+\varepsilon\pi^\varepsilon,\\ \theta^\varepsilon=\theta^0+\varepsilon^2\theta^1+\varepsilon^2q^\varepsilon,\quad v^\varepsilon=v^0+\sqrt{\varepsilon}\{v^\rho\}+\varepsilon v^1,\quad w^\varepsilon=w^0+\varepsilon^2w^1,\quad \sigma^\varepsilon=\sigma^0.\end{array}\right.\tag{24}$$

The functions β, ζ^ε, and ψ are defined as follows:

$$\begin{array}{l} \beta(t,x,z)=-2e^{-z}N(\rho)(t,x,0)-\nu(x)\displaystyle\int_z^{+\infty}\operatorname{div}\,\rho(t,x,z')\,dz',\\ \left\{\begin{array}{lll}\Delta\zeta^\varepsilon=-\{\operatorname{div}\,\beta\} & \text{in} & \mathcal{O},\\ \partial_\nu\zeta^\varepsilon=-\beta(t,\cdot,0)\cdot\nu & \text{on} & \partial\mathcal{O},\end{array}\right.\\ \left\{\begin{array}{l}\psi=\psi(t,x,z)\text{ satisfies }[(u^0\cdot\nabla)\rho+(\rho\cdot\nabla)u^0]\cdot\nu=\partial_z\psi\\ \text{and }\psi(t,x,z)\to 0\quad\text{as}\quad z\to+\infty.\end{array}\right.\end{array}\tag{25}$$

It is not difficult to check that the definitions in (25) are compatible with (7), and, furthermore, the following estimates hold:

$$
\begin{aligned}
(26)\qquad \|\beta(t,\cdot,\cdot)\|_{H^j_x(\mathcal{O};H^{m,l}_z(\mathbb{R}_+))} &\le C\|\rho(t,\cdot,\cdot)\|_{H^{j+1}_x(\mathcal{O};H^{m+1,l+2}_z(\mathbb{R}_+))},\\
\|\zeta^\varepsilon(t,\cdot)\|_{H^4(\mathcal{O})} &\le C\left(\varepsilon^{-3/4}\|\beta(t,\cdot,\cdot)\|_{H^4_x(\mathcal{O};H^{2,0}_z(\mathbb{R}_+))}\right.\\
&\qquad \left.+\|\rho(t,\cdot,\cdot)\|_{H^3_x(\mathcal{O};H^{0,1}_z(\mathbb{R}_+))}\right),\\
\|\zeta^\varepsilon(t,\cdot)\|_{H^3(\mathcal{O})} &\le C\left(\varepsilon^{-1/4}\|\beta(t,\cdot,\cdot)\|_{H^3_x(\mathcal{O};H^{1,0}_z(\mathbb{R}_+))}\right.\\
&\qquad \left.+\|\rho(t,\cdot,\cdot)\|_{H^2_x(\mathcal{O};H^{0,1}_z(\mathbb{R}_+))}\right),
\end{aligned}
$$

$$
(27)\qquad \|\zeta^\varepsilon(t,\cdot)\|_{H^2(\mathcal{O})} \le C\left(\varepsilon^{1/4}\|\beta(t,\cdot,\cdot)\|_{H^2_x(\mathcal{O};H^{0,0}_z(\mathbb{R}_+))}+\|\rho(t,\cdot,\cdot)\|_{H^1_x(\mathcal{O};H^{0,1}_z(\mathbb{R}_+))}\right),
$$
$$
\|\psi(t,\cdot,\cdot)\|_{H^1_x(\mathcal{O};H^{0,0}_z(\mathbb{R}_+))} \le C\|\rho(t,\cdot,\cdot)\|_{H^2_x(\mathcal{O};H^{0,2}_z(\mathbb{R}_+))}.
$$

3.3.4. Equation and estimates for the remainder. We will now analyze the remainder defined in (24), which is in fact a solution in the domain $\mathbb{R}_+\times\mathcal{O}$ to

(28)
$$
\begin{cases}
\partial_t r^\varepsilon-\varepsilon\Delta r^\varepsilon+(u^\varepsilon\cdot\nabla)r^\varepsilon+\nabla\pi^\varepsilon=\{f^\varepsilon\}-\{A^\varepsilon r^\varepsilon\}+\varepsilon q^\varepsilon e_n+\varepsilon\theta^1 e_n, & \text{in}\quad \mathbb{R}_+\times\mathcal{O},\\
\partial_t q^\varepsilon-\varepsilon\Delta q^\varepsilon+u^\varepsilon\cdot\nabla q^\varepsilon=\{h^\varepsilon\}-B^\varepsilon r^\varepsilon,\quad \operatorname{div} r^\varepsilon=0 & \text{in}\quad \mathbb{R}_+\times\mathcal{O},\\
r^\varepsilon\cdot\nu=0,\quad N(r^\varepsilon)=-N(g^\varepsilon),\quad R(q^\varepsilon)=-R(\theta^1) & \text{on}\quad \mathbb{R}_+\times\partial\mathcal{O},\\
r^\varepsilon(0,\cdot)=0,\quad q^\varepsilon(0,\cdot)=0 & \text{in}\quad \mathcal{O},
\end{cases}
$$

where $g^\varepsilon:=u^1+\nabla\zeta^\varepsilon+\beta|_{z=0}$. We have introduced in (28) the new operators A^ε and B^ε, with

$$
(29)\qquad
\begin{aligned}
&A^\varepsilon r^\varepsilon:=(r^\varepsilon\cdot\nabla)(u^0+\sqrt{\varepsilon}\rho+\varepsilon u^1+\varepsilon\nabla\zeta^\varepsilon+\varepsilon\beta)-(r^\varepsilon\cdot\nu)(\partial_z\rho+\sqrt{\varepsilon}\partial_z\beta),\\
&B^\varepsilon r^\varepsilon:=\varepsilon r^\varepsilon\cdot\nabla\theta^1,
\end{aligned}
$$

and the new forcing terms f^ε and h^ε, with

(30)
$$
\begin{aligned}
f^\varepsilon:=&(\Delta\varphi\partial_z\rho-2(\nu\cdot\nabla)\partial_z\rho+\partial_{zz}\beta)+\sqrt{\varepsilon}(\Delta\rho+\Delta\varphi\partial_z\beta-2(\nu\cdot\nabla)\partial_z\beta)\\
&+\varepsilon(\Delta\beta+\Delta u^1+\Delta\nabla\zeta^\varepsilon)-((\rho+\sqrt{\varepsilon}(\beta+u^1+\nabla\zeta^\varepsilon))\cdot\nabla)(\rho+\sqrt{\varepsilon}(\beta+u^1+\nabla\zeta^\varepsilon))\\
&-(u^0\cdot\nabla)\beta-(\beta\cdot\nabla)u^0-u^0_\flat z\partial_z\beta+(\beta+u^1+\nabla\zeta^\varepsilon)\cdot\nu\partial_z(\rho+\sqrt{\varepsilon}\beta)-\nabla\psi-\partial_t\beta
\end{aligned}
$$

and

$$
(31)\qquad h^\varepsilon:=\varepsilon\Delta\theta^1-(\sqrt{\varepsilon}\rho+\varepsilon(u^1+\nabla\zeta^\varepsilon+\beta))\cdot\nabla\theta^1.
$$

We have to estimate the size of the remainder $(r^\varepsilon,q^\varepsilon)$ at final time and check that it is small. We begin by establishing an energy estimate. Here, we perform computations similar to those in [18, Proposition 1.1] (see also [8, section 4.4]).

Thus, we multiply $(28)_1$ by r^ε and $(28)_2$ by q^ε and we integrate by parts. We proceed as before, term by term, the only difference being the treatment of the terms coming from the boundary.

We recall the following identity, which will be used in what follows:

$$
\int_{\mathcal{O}}(-\Delta u)\cdot v=2\int_{\mathcal{O}}D(u)\cdot D(v)-2\int_{\partial\mathcal{O}}[D(u)\nu]_{tan}\cdot v\,d\Gamma,
$$

where u and v are smooth vector fields such that v is divergence-free and tangential to the boundary.

It follows that

$$-\varepsilon\int_{\mathcal{O}}\Delta r^\varepsilon\cdot r^\varepsilon = 2\varepsilon\|D(r^\varepsilon)\|^2 + 2\varepsilon\int_{\partial\mathcal{O}}([Mr^\varepsilon]_{tan}+N(g^\varepsilon))\cdot r^\varepsilon\,d\Gamma,$$

and, consequently, for any $\lambda>0$,

$$\begin{aligned}2\left|\int_{\partial\mathcal{O}}([Mr^\varepsilon]_{tan}+N(g^\varepsilon))\cdot r^\varepsilon\right| &\le 2\int_{\partial\mathcal{O}}|Mr^\varepsilon\cdot r^\varepsilon|\,d\Gamma+\int_{\partial\mathcal{O}}|N(g^\varepsilon)\cdot r^\varepsilon|\,d\Gamma\\ &\le \lambda\|\nabla r^\varepsilon\|^2 + C_\lambda(\|r^\varepsilon\|^2+\|N(g^\varepsilon)\|^2_{L^2(\partial\mathcal{O})})\\ &\le \lambda\|\nabla r^\varepsilon\|^2 + C_\lambda(\|r^\varepsilon\|^2+\|g^\varepsilon\|^2_{H^2(\mathcal{O})}).\end{aligned}\tag{32}$$

Let us absorb the term $\|\nabla r^\varepsilon\|^2$ in the right-hand side of (32). Thanks to the classical Korn's inequality, since $\operatorname{div} r^\varepsilon=0$ in $\mathcal{O}$ and $r^\varepsilon\cdot\nu=0$ on $\partial\mathcal{O}$, we have

$$\|r^\varepsilon\|^2_{H^1}\le C_K\|r^\varepsilon\|^2+C_K\|D(r^\varepsilon)\|^2$$

for some $C_K>0$. Choosing $\lambda=1/(2C_K)$, we get:

$$\begin{aligned}\frac{d}{dt}\|r^\varepsilon\|^2+\varepsilon\|D(r^\varepsilon)\|^2 &\le \Big(\|\sigma^0\|_\infty + C\varepsilon+\|\{f^\varepsilon\}\|+2\|\{A^\varepsilon\}\|_\infty\Big)\|r^\varepsilon\|^2\\ &\quad+\Big(C\varepsilon\|g^\varepsilon\|^2_{H^2}+\|\{f^\varepsilon\}\|+\varepsilon\|\theta^1\|^2\Big)+\varepsilon\|q^\varepsilon\|^2\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\|q^\varepsilon\|^2+\varepsilon\|\nabla q^\varepsilon\|^2 &\le \Big(\|\sigma^0\|_\infty+\|\{h^\varepsilon\}\|+\|B^\varepsilon\|_\infty+C\varepsilon\Big)\|q^\varepsilon\|^2\\ &\quad+\Big(\|\{h^\varepsilon\}\|+C\varepsilon\|\theta^1\|^2_{H^2}\Big)+\|B^\varepsilon\|_\infty\|r^\varepsilon\|^2.\end{aligned}$$

Adding these two estimates, we see that

$$\begin{aligned}&\frac{d}{dt}(\|r^\varepsilon\|^2+\|q^\varepsilon\|^2)+\varepsilon(\|D(r^\varepsilon)\|^2+\|\nabla q^\varepsilon\|^2)\\ &\le\Big(\|\sigma^0\|_\infty+C\varepsilon+\|\{f^\varepsilon\}\|+2\|\{A^\varepsilon\}\|_\infty+\|\{h^\varepsilon\}\|+\|B^\varepsilon\|_\infty\Big)\Big(\|r^\varepsilon\|^2+\|q^\varepsilon\|^2\Big)\\ &\qquad+\Big(C\varepsilon\|g^\varepsilon\|^2_{H^2}+\|\{f^\varepsilon\}\|+\|\{h^\varepsilon\}\|+C\varepsilon\|\theta^1\|^2_{H^2}\Big).\end{aligned}$$

Applying Gronwall's inequality in the interval $(0,T/\varepsilon)$ and using the fact that the initial state vanishes and

$$\|\{A^\varepsilon\}\|_{L^1(L^\infty)}+\|B^\varepsilon\|_{L^1(L^\infty)}=O(1),\tag{33}$$

$$\varepsilon\|\theta^1\|^2_{L^2(H^2)}+\varepsilon\|g^\varepsilon\|^2_{L^2(H^2)}=O(\varepsilon^{1/4}),\tag{34}$$

$$\|\{f^\varepsilon\}\|_{L^1(L^2)}+\|\{h^\varepsilon\}\|_{L^1(L^2)}=O(\varepsilon^{1/4}),\tag{35}$$

we obtain

$$\|r^\varepsilon\|^2_{L^\infty(L^2)}+\|q^\varepsilon\|^2_{L^\infty(L^2)}+\varepsilon\Big(\|D(r^\varepsilon)\|^2_{L^2(L^2)}+\|\nabla q^\varepsilon\|^2_{L^2(L^2)}\Big)=O(\varepsilon^{1/4}).\tag{36}$$

The estimates (33)–(35) hold in the whole interval $[0,+\infty)$. The estimates for $\{A^\varepsilon\}$, g^ε, and $\{f^\varepsilon\}$ can be found in [8, section 4.4]. Here, we give some details to obtain the estimates for B^ε, θ^1, and $\{h^\varepsilon\}$, which are new.

First, B^ε and θ^1 can be easily bounded using (29) and Lemma 3.2. This yields $\|B^\varepsilon\|_{L^1(L^\infty)} = O(1)$ and $\varepsilon\|\theta^1\|^2_{L^2(H^2)} = O(\varepsilon)$.

Now, let us justify the estimate of $\{h^\varepsilon\}$. The first term of $\{h^\varepsilon\}$ is $O(\varepsilon)$, thanks to the regularity of θ^1.

The second term of $\{h^\varepsilon\}$, one can be treated as follows:

$$\begin{aligned}
&\|\sqrt{\varepsilon}\{\rho\}(t,\cdot)\cdot\nabla\theta^1(t,\cdot)\| \\
&\qquad \le C\sqrt{\varepsilon}\|\{\rho\}(t,\cdot)\|_{H^1}\|\nabla\theta^1(t,\cdot)\|_{H^1} \\
&\qquad \le C\left(\sqrt{\varepsilon}\|\rho(t,\cdot,\cdot)\|_{H^1_x(H^{0,0}_z)} + \|\{\partial_z\rho\}(t,\cdot)\|\right)\|\nabla\theta^1(t,\cdot)\|_{H^1} \\
&\qquad \le C\left(\sqrt{\varepsilon}\|\rho(t,\cdot,\cdot)\|_{H^1_x(H^{0,0}_z)} + \varepsilon^{1/4}\|\rho(t,\cdot,\cdot)\|_{H^1_x(H^{1,0}_z)}\right)\|\nabla\theta^1(t,\cdot)\|_{H^1} \\
&\qquad \le C\varepsilon^{1/4}\|\rho(t,\cdot,\cdot)\|_{H^1_x(H^{1,0}_z)}\|\nabla\theta^1(t,\cdot)\|_{H^1},
\end{aligned}$$

where we have used that the fast scaling variable enables us to "win" a factor $\varepsilon^{1/4}$; see [22, Lemma 3] and the Sobolev embedding $H^1(\mathcal{O}) \hookrightarrow L^4(\mathcal{O})$ which is valid in two and three dimensions. Then, integrating this last inequality with respect to time over $(0, T/\varepsilon)$, using the fact that θ^1 is bounded in $L^\infty(0,T;H^3(\mathcal{O}))$ and Lemma 3.4 for $k=4$ and noting that there exists a positive constant $C>0$ such that $(\log s)/s \le Cs^{-1/2}$, for every $s \ge 1$, we see that

$$\|\sqrt{\varepsilon}\{\rho\}\cdot\nabla\theta^1\|_{L^1(L^2)} = O(\varepsilon^{1/4}).$$

The third term of $\{h^\varepsilon\}$ is $O(\varepsilon)$, thanks to the regularity of u^1 and θ^1.

For the fourth term of $\{h^\varepsilon\}$, using (26) and (27), we have

$$\begin{aligned}
&\|\varepsilon\nabla\zeta^\varepsilon(t,\cdot)\cdot\nabla\theta^1(t,\cdot)\| \\
&\qquad \le C\varepsilon\|\nabla_x\zeta^\varepsilon(t,\cdot)\|_{H^1}\|\nabla\theta^1(t,\cdot)\|_{H^1} \\
&\qquad \le C\varepsilon\|\zeta^\varepsilon(t,\cdot)\|_{H^2}\|\nabla\theta^1(t,\cdot)\|_{H^1} \\
&\qquad \le C\varepsilon\left(\varepsilon^{1/4}\|\beta(t,\cdot,\cdot)\|_{H^2_x(H^{0,0}_z)} + \|\rho(t,\cdot,\cdot)\|_{H^1_x(H^{0,1}_z)}\right)\|\nabla\theta^1(t,\cdot)\|_{H^1} \\
&\qquad \le C\varepsilon\|\rho(t,\cdot,\cdot)\|_{H^3_x(H^{1,2}_z)}\|\nabla\theta^1(t,\cdot)\|_{H^1}.
\end{aligned}$$

Integrating this last inequality with respect to time, and using Lemma 3.4 for $k=3$ and, again, the fact that θ^1 is bounded in $L^\infty(0,T;H^3(\mathcal{O}))$, we find that

$$\|\varepsilon\nabla\zeta^\varepsilon\cdot\nabla\theta^1\|_{L^1(L^2)} = O(\varepsilon^{1/4}).$$

The last term of $\{h^\varepsilon\}$ can be estimated in a similar way, using (26).

3.4. Towards the trajectory. In this section, we deduce a small-time global approximate controllability result to the smooth trajectories by arguing as in [8, section 5]. For this purpose, we will use once more Lemma 3.4 and the estimates (36) on the remainder.

Let $(u^\varepsilon, p^\varepsilon, \theta^\varepsilon)$ be the solution to (7). First, during the interval $[0,T]$, we put

(37)
$$\left\{\begin{array}{l} u^\varepsilon = u^0 + \sqrt{\varepsilon}\{\rho\} + \varepsilon u^{1,\varepsilon} + \varepsilon\nabla\zeta^\varepsilon + \varepsilon\{\beta\} + \varepsilon r^\varepsilon, \quad p^\varepsilon = p^0 + \varepsilon\{\psi\} + \varepsilon p^{1,\varepsilon} + \varepsilon\mu^\varepsilon + \varepsilon\pi^\varepsilon, \\ \theta^\varepsilon = \theta^0 + \varepsilon^2\theta^1 + \varepsilon^2 q^\varepsilon, \quad v^\varepsilon = v^0 + \sqrt{\varepsilon}\{v^\rho\} + \varepsilon v^{1,\varepsilon}, \quad w^\varepsilon = w^0 + \varepsilon^2 w^{1,\varepsilon}, \quad \sigma^\varepsilon = \sigma^0, \end{array}\right.$$

where $u^{1,\varepsilon}(0,\cdot) = u_*$, $\theta^{1,\varepsilon}(0,\cdot) = \theta_*$, $u^{1,\varepsilon}(T,\cdot) = \overline{u}(\varepsilon T,\cdot)$, and $\theta^{1,\varepsilon}(T,\cdot) = \overline{\theta}(\varepsilon T,\cdot)$. The couple $(u^{1,\varepsilon}, \theta^{1,\varepsilon})$ solves, together with some $p^{1,\varepsilon}$, the first-order system (15), and obviously $u^{1,\varepsilon}$ and $\theta^{1,\varepsilon}$ depend on ε. However, since the reference trajectory is of class C^∞, all the required estimates can be made independent of ε. In a second step, for large times $t \geq T$, we modify the expansions and set

$$\left\{ \begin{array}{l} u^\varepsilon = \sqrt{\varepsilon}\{\rho\} + \varepsilon\overline{u}(\varepsilon t, \cdot) + \varepsilon\nabla\zeta^\varepsilon + \varepsilon\{\beta\} + \varepsilon r^\varepsilon, \quad p^\varepsilon = \varepsilon^2\overline{p}(\varepsilon t, \cdot) + \varepsilon\mu^\varepsilon + \varepsilon\pi^\varepsilon, \\ \theta^\varepsilon = \varepsilon^2\overline{\theta}(\varepsilon t, \cdot) + \varepsilon^2 q^\varepsilon, \quad v^\varepsilon = \sqrt{\varepsilon}v^\rho + \varepsilon^2\overline{v}, \quad w^\varepsilon = \varepsilon^3\overline{w}. \end{array} \right. \tag{38}$$

Note that, for $t \geq T$, we have $u^0 = 0$ and (u^1, θ^1) is the "main" trajectory. Changing (37) by (38) allows us to get rid of some terms in the equations satisfied by the remainder. Indeed, terms such as $\varepsilon\Delta u^1$, $\varepsilon(u^1\cdot\nabla)u^1$, $\varepsilon u^1\cdot\nabla\theta^1$, and $\varepsilon\Delta\theta^1$ will no longer appear in (30) and (31) because they are already taken into account by $(\overline{u}, \overline{\theta})$. Actually, despite the presence of the profile (u^1, θ^1) in both steps, the estimates obtained for the remainder profile are as in section 3.3.4.

Let us introduce

$$u^{(\varepsilon)}(t,x) := \frac{1}{\varepsilon}u^\varepsilon\left(\frac{t}{\varepsilon}, x\right) \quad \text{and} \quad \theta^{(\varepsilon)}(t,x) := \frac{1}{\varepsilon^2}\theta^\varepsilon\left(\frac{t}{\varepsilon}, x\right).$$

Then, thanks to (26), (27), and (36), we see that

$$\begin{aligned} &\left\| u^{(\varepsilon)}(T,\cdot) - \overline{u}(T,\cdot) \right\| \\ &\quad = \left\| \varepsilon^{-1/2}\{\rho\}(T/\varepsilon,\cdot) + \nabla\zeta^\varepsilon(T/\varepsilon,\cdot) + \{\beta\}(T/\varepsilon,\cdot) + r^\varepsilon(T/\varepsilon,\cdot) \right\| \\ &\quad \leq \varepsilon^{-1/2}\|\{\rho\}(T/\varepsilon,\cdot)\| + \varepsilon^{1/4}\|\beta(T/\varepsilon,\cdot,\cdot)\|_{H^2_x(\mathcal{O};H^{0,0}_z(\mathbb{R}_+))} \\ &\qquad + \|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^1_x(\mathcal{O};H^{0,1}_z(\mathbb{R}_+))} + \|\{\beta\}(T/\varepsilon,\cdot)\| + \|r^\varepsilon(T/\varepsilon,\cdot)\| \\ &\quad \leq \varepsilon^{-1/2}\|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^0_x(\mathcal{O};H^{0,0}_z(\mathbb{R}_+))} + \varepsilon^{1/4}\|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^3_x(\mathcal{O};H^{1,2}_z(\mathbb{R}_+))} \\ &\qquad + \|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^1_x(\mathcal{O};H^{0,1}_z(\mathbb{R}_+))} + \|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^1_x(\mathcal{O};H^{1,2}_z(\mathbb{R}_+))} + O(\varepsilon^{1/8}). \end{aligned}$$

We can use (23) to estimate the terms containing ρ in the estimates above. First, recall that there exists a positive constant $C > 0$ such that $(\log s)/s \leq Cs^{-1/2}$ for every $s \geq 1$. Then, by taking ε sufficiently small, the following is found for $k \geq 2$:

$$\begin{aligned} \varepsilon^{-\frac{1}{2}}\|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^0_x(\mathcal{O};H^{0,0}_z(\mathbb{R}_+))} &\leq C\varepsilon^{-1/2}\left|\frac{\log(2+T/\varepsilon)}{2+T/\varepsilon}\right|^{1/4+k/2} \leq C\varepsilon^{-3/8+k/4}, \\ \varepsilon^{\frac{1}{4}}\|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^3_x(\mathcal{O};H^{1,2}_z(\mathbb{R}_+))} &\leq C\varepsilon^{1/4}\left|\frac{\log(2+T/\varepsilon)}{2+T/\varepsilon}\right|^{-3/4+k/2} \leq C\varepsilon^{-1/8+k/4}, \\ \|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^1_x(\mathcal{O};H^{0,1}_z(\mathbb{R}_+))} &\leq C\left|\frac{\log(2+T/\varepsilon)}{2+T/\varepsilon}\right|^{-1/4+k/2} \leq C\varepsilon^{-1/8+k/4}, \\ |\rho(T/\varepsilon,\cdot,\cdot)|_{H^1_x(\mathcal{O};H^{1,2}_z(\mathbb{R}_+))} &\leq C\left|\frac{\log(2+T/\varepsilon)}{2+T/\varepsilon}\right|^{-3/4+k/2} \leq C\varepsilon^{-3/8+k/4}. \end{aligned}$$

Finally, we choose k large enough, we conclude that $\|u^{(\varepsilon)}(T,\cdot) - \overline{u}(T,\cdot)\| = O(\varepsilon^{1/8})$, and, from (36), we have $\|\theta^{(\varepsilon)}(T,\cdot) - \overline{\theta}(T,\cdot)\| = \|q^\varepsilon(T/\varepsilon,\cdot)\| = O(\varepsilon^{1/8})$.

This concludes the proof of Proposition 3.1.

4. Local controllability of the Boussinesq system. The results in this section are relatively well known. For clarity, they will be specified and their proof will be sketched to some extent.

Let ω_c and ω be two nonempty open sets such that $\omega_c \subset\subset \omega \subset\subset \mathcal{O} \setminus \overline{\Omega}$, and let χ_ω be a cut-off function such that $\chi_\omega = 0$ outside ω and $\chi_\omega = 1$ in ω_c.

The goal of this section is to prove the local exact controllability to the trajectories of the following Boussinesq system with distributed controls:

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = \theta e_n + v\chi_\omega, \quad \operatorname{div} u = 0 & \text{in} \quad \mathcal{O}_T, \\ \partial_t \theta - \Delta\theta + u\cdot\nabla\theta = w\chi_\omega & \text{in} \quad \mathcal{O}_T, \\ u\cdot\nu = 0, \quad N(u) = 0, \quad R(\theta) = 0 & \text{on} \quad \Lambda_T, \\ u(0,\cdot) = u_*, \quad \theta(0,\cdot) = \theta_* & \text{in} \quad \mathcal{O}. \end{cases} \tag{39}$$

Since (39) is nonlinear, we first begin by proving a (global) null controllability result for the following system:

$$\begin{cases} \partial_t z - \Delta z + ((a+b)\cdot\nabla)z + (z\cdot\nabla)b + \nabla q = he_n + v\chi_\omega, \quad \operatorname{div} z = 0 & \text{in} \quad \mathcal{O}_T, \\ \partial_t h - \Delta h + (a+b)\cdot\nabla h + z\cdot\nabla c = w\chi_\omega & \text{in} \quad \mathcal{O}_T, \\ z\cdot\nu = 0, \quad N(z) = 0, \quad R(h) = 0 & \text{on} \quad \Lambda_T, \\ z(0,\cdot) = z_*, \quad h(0,\cdot) = h_* & \text{in} \quad \mathcal{O}, \end{cases} \tag{40}$$

where the vector fields a, b, and M and the scalar functions c and m satisfy the following assumptions:

$$\begin{aligned} &(a,b,c) \in L^\infty(0,T;H\times H\times L^2(\mathcal{O})) \cap L^\infty(\mathcal{O}_T)^{2n+1}, \ (a_t,b_t,c_t) \in L^2(0,T;L^r(\mathcal{O})^{2n+1}), \\ &M \in E := H^{1-\ell}(0,T;W^{\vartheta_1,\vartheta_1+1}(\partial\mathcal{O})^{n\times n}) \cap H^{(3-\ell)/2}(0,T;H^{\vartheta_2}(\partial\mathcal{O})^{n\times n}), \\ &m \in F := H^{1-\ell}(0,T;W^{\vartheta_1,\vartheta_1+1}(\partial\mathcal{O})) \cap H^{(3-\ell)/2}(0,T;H^{\vartheta_2}(\partial\mathcal{O})), \end{aligned} \tag{41}$$

where $\ell \in (0,1/2)$ is arbitrarily close to $1/2$, $r = 2n$, $\vartheta_2 = (1/2)(3-n) + (1-\ell)(n-2)$, and $\vartheta_1 > 1$ (arbitrarily small) if $n = 3$ and $\vartheta_1 = 1$ if $n = 2$. From well-known Sobolev embeddings, we deduce at once that $E \hookrightarrow L^\infty((0,T)\times\partial\mathcal{O})^{n\times n}$ and $F \hookrightarrow L^\infty((0,T)\times\partial\mathcal{O})$.

It is well known that the null controllability of (40) is equivalent to the observability of the adjoint system

$$\begin{cases} -\partial_t\phi - \Delta\phi - (a\cdot\nabla)\phi - D(\phi)b + \nabla\pi = c\nabla\psi, \quad \operatorname{div}\phi = 0 & \text{in} \quad \mathcal{O}_T, \\ -\partial_t\psi - \Delta\psi - (a+b)\cdot\nabla\psi = \phi\cdot e_n & \text{in} \quad \mathcal{O}_T, \\ \phi\cdot\nu = 0, \quad N(\phi) = 0, \quad R(\psi) = 0 & \text{on} \quad \Lambda_T, \\ \phi(T,\cdot) = \phi_*, \quad \psi(T,\cdot) = \psi_* & \text{in} \quad \mathcal{O}. \end{cases} \tag{42}$$

The desired observability inequality will be a consequence of a global Carleman inequality for (42); see Proposition 4.1 below.

4.1. Carleman estimates. Before stating the required inequalities, let us introduce several classical weights in the study of Carleman estimates for parabolic equations; see [13]. The basic weight will be a function $\eta^0 \in C^2(\overline{\mathcal{O}})$ verifying

$$\eta^0 > 0 \quad \text{in} \quad \mathcal{O}, \quad \eta^0 \equiv 0 \quad \text{on} \quad \partial\mathcal{O}, \quad |\nabla\eta^0| > 0 \quad \text{in} \quad \overline{\mathcal{O}\setminus\omega'},$$

where $\omega' \subset\subset \omega_c$ is a nonempty open set. The existence of η^0 is proved in [13].

Thus, for any $\lambda > 0$ we set

$$\begin{aligned} \alpha(x,t) &= \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0(x)}}{t^4(T-t)^4}, \quad \alpha^*(t) = \max_{x\in\overline{\mathcal{O}}}\alpha(x,t), \quad \widehat{\alpha}(t) = \min_{x\in\overline{\mathcal{O}}}\alpha(x,t), \\ \xi(x,t) &= \frac{e^{\lambda\eta^0(x)}}{t^4(T-t)^4}, \qquad \xi^*(t) = \min_{x\in\overline{\mathcal{O}}}\xi(x,t), \quad \widehat{\xi}(t) = \max_{x\in\overline{\mathcal{O}}}\xi(x,t). \end{aligned}$$

We also introduce the following notation:

$$I(s,\lambda;\phi) \;=\; \iint_{\mathcal{O}_T} e^{-2s\alpha}\left[s^3\lambda^4\xi^3|\phi|^2 + s\lambda^2\xi|\nabla\phi|^2 + s^{-1}\xi^{-1}(|\phi_t|^2+|\Delta\phi|^2)\right]dx\,dt,$$

where s and λ are positive real numbers and $\phi=\phi(t,x)$.

The following Carleman inequality holds:

Proposition 4.1. *Assume that the assumptions* (41) *are fulfilled. There exist positive constants $\widetilde{\lambda}$, $\widetilde{s}$, and $C=C(\mathcal{O},\omega_c)$ such that, for any $(\phi_*,\psi_*)\in H\times L^2(\mathcal{O})$, the corresponding solution to* (42) *verifies*

$$I(s,\lambda;\phi)+I(s,\lambda;\psi)\le C(1+T^2)s^{15/2}\lambda^8\iint_{(0,T)\times\omega_c} e^{-4s\hat{\alpha}+2s\alpha^*}\hat{\xi}^{15/2}(|\phi|^2+|\psi|^2)\,dx\,dt \tag{43}$$

for all $\lambda\ge\widetilde{\lambda}$ and $s\ge\widetilde{s}$. Furthermore, $\widetilde{\lambda}$ and $\widetilde{s}$ have the form $\widetilde{\lambda}=\widetilde{\lambda}_0e^{\widetilde{\lambda}_1T}$ and $\widetilde{s}=\widetilde{s}_0e^{\lambda\widetilde{s}_1}(T^4+T^8)$, where $\widetilde{\lambda}_0$, $\widetilde{\lambda}_1$, and $\widetilde{s}_0$ only depend on $\|a\|_\infty$, $\|b\|_\infty$, $\|c\|_\infty$, $\|a_t\|_{L^2(L^r)}$, $\|b_t\|_{L^2(L^r)}$, $\|c_t\|_{L^2(L^r)}$, and $\|M\|_E$, and $\|m\|_F$ and $\widetilde{s}_1$ only depend on $\mathcal{O}$ and ω_c.

The proof of Proposition 4.1 consists of three steps: (i) global Carleman estimates for ϕ and ψ (see [18, Proposition 2.1] and [27, Appendix D]); (ii) estimates of the pressure by a local term using elliptic Carleman inequalities (see [24]); (iii) estimates of local integrals of $\Delta\phi$ and ϕ_t by using global energy estimates. For more details, we refer the reader to [27, Appendix E] and [4, Appendix C].

4.2. Null controllability of the linearized system. In what follows, we take $s=\widetilde{s}$ and $\lambda=\widetilde{\lambda}$.

In this section, we prove the null controllability of the linear system (40) as a consequence of the inequality (43). To this end, let us introduce the space where the controls are searched for:

$$\mathcal{H}:=[H^1(0,T;L^2(\mathcal{O})^n)\cap C^0([0,T];H^1(\mathcal{O})^n)]\times[H^1(0,T;L^2(\mathcal{O}))\cap C^0([0,T];H^1(\mathcal{O}))].$$

Proposition 4.2. *Let $(z_*,h_*)\in H\times L^2(\mathcal{O})$ be given, and suppose that* (41) *holds. Then, there exist controls $(v,w)\in\mathcal{H}$ such that the corresponding solution to* (40) *satisfies*

$$z(T,\cdot)=0 \quad \textit{and} \quad h(T,\cdot)=0.$$

Moreover, the following estimate holds:

$$\begin{aligned}\|\kappa^{1/2}v\chi_\omega\|+\|\kappa^{1/2}w\chi_\omega\|+\|v\|_{H^1(L^2)}+\|v\|_{L^\infty(H^1)}+\|w\|_{H^1(L^2)}+\|w\|_{L^\infty(H^1)}&\\ \le C(\|z_*\|+\|h_*\|),&\end{aligned}$$

where the positive constant C depends only on $\mathcal{O}$, ω, T, $\|a\|_\infty$, $\|b\|_\infty$, $\|c\|_\infty$, $\|a_t\|_{L^2(L^r)}$, $\|b_t\|_{L^2(L^r)}$, $\|c_t\|_{L^2(L^r)}$, and $\|M\|_E$ and $\|m\|_F$, $\kappa(t)=e^{4s\hat{\alpha}-2s\alpha^*}\hat{\xi}^{-15/2}$, $\hat{\alpha}$, α^*, and $\hat{\xi}$ are defined in section 4.1.

The proof of Proposition 4.2 is based on a penalized *Hilbert uniqueness method;* it follows the ideas of [18, section 3.1]. The details can be found in [27, Proposition 3.17].

4.3. Local exact controllability to the trajectories of the Boussinesq system. We now prove the local exact controllability to the trajectories of (39).

Let $(\overline{u}, \overline{p}, \overline{\theta})$ be an uncontrolled solution to (39), that is, a triplet satisfying

$$\begin{cases} \partial_t \overline{u} - \Delta \overline{u} + (\overline{u} \cdot \nabla)\overline{u} + \nabla \overline{p} = \overline{\theta} e_n, \quad \text{div}\, \overline{u} = 0 & \text{in} \quad \mathcal{O}_T, \\ \partial_t \overline{\theta} - \Delta \overline{\theta} + \overline{u} \cdot \nabla \overline{\theta} = 0 & \text{in} \quad \mathcal{O}_T, \\ \overline{u} \cdot \nu = 0, \quad N(\overline{u}) = 0, \quad R(\overline{\theta}) = 0 & \text{on} \quad \Lambda_T, \\ \overline{u}(0,\cdot) = \overline{u}_*, \quad \overline{\theta}(0,\cdot) = \overline{\theta}_* & \text{in} \quad \mathcal{O}. \end{cases}$$

Let us assume that the following holds:

$$\begin{aligned} &\overline{u} \in X := H^{(3-\ell)/2}(0,T; H^{\vartheta_2+1/2}(\mathcal{O})^n \cap H) \cap H^{1-\ell}(0,T; W^{\vartheta_1+1/2,\vartheta_1+1}(\mathcal{O})^n), \\ &\overline{u}_* \in H^3(\mathcal{O})^n \cap H, \quad N(\overline{u}_*) = 0 \quad \text{on} \quad \partial\mathcal{O}, \\ &\overline{\theta} \in Y := H^{(3-\ell)/2}(0,T; H^{\vartheta_2+1/2}(\mathcal{O})) \cap H^{1-\ell}(0,T; W^{\vartheta_1+1/2,\vartheta_1+1}(\mathcal{O})), \\ &\overline{\theta}_* \in H^3(\mathcal{O}), \quad R(\overline{\theta}_*) = 0 \quad \text{on} \quad \partial\mathcal{O}, \end{aligned} \tag{44}$$

with ℓ, r, ϑ_1, and ϑ_2 as in the beginning of section 4.

Proposition 4.3. *Assume that $T > 0$ and $(\overline{u}, \overline{u}_*, \overline{\theta}, \overline{\theta}_*)$ satisfies* (44). *Then, there exists $\delta_T > 0$ such that, for every $(u_*, \theta_*) \in [H^3(\mathcal{O})^n \cap H] \times H^3(\mathcal{O})$ satisfying $\|u_* - \overline{u}_*\|_{H^3} \leq \delta_T$, $\|\theta_* - \overline{\theta}_*\|_{H^3} \leq \delta_T$ and the compatibility conditions*

$$N(u_*) = 0, \qquad R(\theta_*) = 0 \quad \textit{on} \quad \partial\mathcal{O},$$

one can find controls $(v, w) \in \mathcal{H}$ and associated solutions (u, p, θ) to (39) *with*

$$u(T,\cdot) = \overline{u}(T,\cdot) \quad \textit{and} \quad \theta(T,\cdot) = \overline{\theta}(T,\cdot) \quad \textit{in } \mathcal{O}.$$

The proof is based on a Kakutani's fixed-point theorem. It is a straightforward adaptation of the argument in [18, section 3.2]. The details can be found in [27, Proposition 3.18] and [4, Proposition 4.3]. See also [29] for a similar result.

5. Global controllability to the trajectories. Let us explain how the previous arguments can be chained in order to prove the main result, that is, Theorem 1.1.

First, we reduce the controllability to weak trajectories to the controllability to smooth trajectories as follows.

Despite $(\overline{u}, \overline{p}, \overline{\theta})$ only being a weak solution in $[0, T]$, there exists a time interval $[\tau_1, \tau_2] \subset (0, T)$ such that $(\overline{u}, \overline{p}, \overline{\theta})$ is smooth in $[\tau_1, \tau_2]$. This statement follows from classical results; indeed, one can easily adapt [21, Theorems 2, 3, and 9] or [30, Remark 3.2] (written for the Navier–Stokes equations with Dirichlet boundary conditions and source terms) to our context.

Then, we can start our control strategy by doing nothing in $[0, \tau_1]$, that is, taking $v = w = \sigma = 0$ in (5). The weak trajectory will move from (u_*, θ_*) to some $(u, \theta)(\tau_1, \cdot)$, which must be viewed as the new initial data. Hence, without loss of generality, we can work with a smooth reference trajectory.

We split the control strategy into four steps.

Step 1 - Regularization of the data: We begin by extending Ω to a new domain $\mathcal{O}$, as explained in section 2.1. We also use Proposition 2.1 to guarantee the existence of $(u_*, \theta_*) \in H \times L^2(\mathcal{O})$ and $\sigma_* \in C_c^\infty(\omega_0)$ satisfying (4). We set $\sigma(t, x) := \varsigma(t/T)\sigma_*(x)$ with ς a smooth nonnegative decreasing function such that $\varsigma \equiv 1$ near 0 and $\varsigma \equiv 0$ near 1/8. The function σ must satisfy the compatibility condition $\text{div}\, u_* = \sigma(0, \cdot)$. Then, we let the system (5) evolve with $v = w = 0$ in the time interval $(0, T/8)$ in order to reach some data $(u, \theta)(T/8, \cdot) \in H \times L^2(\mathcal{O})$. Next, by using the smoothing effect of the uncontrolled Boussinesq system starting from divergence-free data (see Lemma 2.1), we deduce that there exists $T_1 \in (T/8, T/4)$ such that $(u,\theta)(T_1,\cdot) \in [H^3(\mathcal{O})^n \cap H] \times H^3(\mathcal{O})$. Accordingly, we can apply Lemma 3.3.

Step 2 - Global approximate controllability result in $L^2(\mathcal{O})^{n+1}$: Let us set $T_2 := T/2$. Starting from the new initial data $(u,\theta)(T_1,\cdot)$, we use the global approximate controllability result stated in Proposition 3.1 in a time interval of size $T_2 - T_1 \geq T/4$. Thus, for any $\delta > 0$, we can build a trajectory starting from $(u,\theta)(T_1,\cdot)$ such that

$$\|(u,\theta)(T_2,\cdot) - (\overline{u},\overline{\theta})(T_2,\cdot)\| \leq \delta.$$

Step 3 - Regularizing argument: Now, we use again Lemma 2.1 to deduce the existence of a time $T_3 \in (T_2, 3T/4)$ such that

$$\|(u,\theta)(T_3,\cdot) - (\overline{u},\overline{\theta})(T_3,\cdot)\|_{H^3 \times H^3} \leq \Psi_{T/4}(\delta).$$

In particular, we can take δ small enough such that

$$\Psi_{T/4}(\delta) \leq \delta_{T/4},$$

where $\delta_{T/4}$ is the radius of local controllability result given in Proposition 4.3 and the function $\Psi_{T/4}$ appears in the regularity result for the free Boussinesq system; see Lemma 2.1.

Step 4 - Local controllability in $H^3(\mathcal{O})^{n+1}$: Finally, we use the local controllability result in $[T_3, T_3 + T/4]$ and get

$$(u,\theta)(T_3 + T/4,\cdot) = (\overline{u},\overline{\theta})(T_3 + T/4,\cdot).$$

Then, extending the control by zero for $t \in [T_3 + T/4, T]$, we get (3) and the proof is complete.

Remark 5.1. A detailed analysis of the proofs of the results in sections 3 to 4 shows that the (intermediate) global approximate controllability result holds as soon as the components of $\bar{u}$ and $\bar{\theta}$ belong to $L^\infty(0,T;H^3(\mathcal{O})) \cap C^0([0,T];H^2(\mathcal{O}))$ and the local exact controllability result holds as soon as $(\overline{u},\overline{\theta})$ satisfies (44).

6. Additional comments and open questions.

6.1. Controlling with fewer controls. A natural extension of the main result would be the global exact controllability with a reduced number of controls acting on a small part of the boundary. Unfortunately, in order to solve this problem, we cannot use the extension domain technique.

However, in the spirit of [10, 29] one could try to establish a small-time global null controllability for the internal control system (5) in two dimensions by acting only on the temperature. The intuition behind a result of this kind is the following: the temperature θ is directly controlled by w; then, θ acts through the coupling term θe_2 to control the component u_2 and then u_2 acts as bilinear control through the term $u_2 \partial_{x_2} u_1$ to control the component u_1.

One can also try get a global control result acting only on the motion, that is, with $w = 0$ in (5); for some local results in this direction, see [3, 10].

Note that, in the case of Neumann conditions on θ, i.e., with $m \equiv 0$, we have an obstruction: Indeed, the total thermal energy associated with θ is conserved and we have

$$\int_\Omega \theta(T,x)\,dx = \int_\Omega \theta_0(x)\,dx.$$

However, one could try to control to zero any initial data for the temperature $\theta_0 \in L_0$, where L_0 is a closed linear subspace of $L^2(\Omega)$ given by

$$L_0 = \left\{ \theta_* \in L^2(\Omega) : \int_\Omega \theta_*(x)\, dx = 0 \right\},$$

which is invariant for the equation of temperature.

Results of these kinds will be analyzed in the near future

6.2. Other boundary conditions for the velocity field. Another natural question is whether Theorem 1.1 holds with u subject to other boundary conditions.

By imposing Dirichlet boundary (no-slip) conditions on the velocity, we face a very well known and challenging open problem related to a conjecture by Jacques-Louis Lions. As pointed out in [8], the boundary layer found in the presence of Dirichlet conditions has a behavior which is not as "good" as in the case of Navier boundary conditions. This implies many difficulties to estimate the boundary layer profiles and the remainder terms. As an attempt to deal with this problem, we refer the reader to [9], where the authors prove that a kind of global boundary null controllability result holds if we allow a distributed force, which can be chosen arbitrarily small in any Sobolev norm in space; see also [20] for related results.

6.3. Other boundary conditions for the temperature. Let us see that Theorem 1.1 holds with Dirichlet boundary conditions on the temperature.

To prove this, we can adapt the strategy of the proof of Theorem 1.1 (see section 5). After the extension and regularization steps, the initial temperature θ_* vanishes on the whole boundary $\partial\mathcal{O}$. Then, the temperature θ^1, which solves $(15)_2$, preserves this property in $[0, T/\varepsilon]$.

Indeed, since the flow u^0 is parallel to the boundary, particles on the boundary cannot enter in the domain $\mathcal{O}$, i.e., $\Phi^0(s; t, x) \in \partial\mathcal{O}$ for all $s, t \in [0, T/\varepsilon]$ and $x \in \partial\mathcal{O}$ (see [1, Theorem 5.1]). Thus, in the estimates of the reminder q^ε (see section 3.2.4), we get a zero boundary integral

$$\int_{\partial\mathcal{O}} q^\varepsilon \frac{\partial q^\varepsilon}{\partial \nu}\, d\Gamma = \int_{\partial\mathcal{O}} \theta^1 \frac{\partial q^\varepsilon}{\partial \nu}\, d\Gamma = 0.$$

Finally, a local control result for the Boussinesq system with Navier-slip-with-friction boundary conditions for the velocity field and Dirichlet boundary conditions on the temperature can also be deduced, and this completes the argument.

6.4. Some possible extensions. Theorem 1.1 can be easily extended to cover global control properties of a few systems of the Navier–Stokes and Boussinesq kinds. For example, it can be applied to some pollution models, where the motion and temperature PDEs are coupled to one or several additional transport-diffusion-reaction equations. Some results will be given in the near future.

Nevertheless, there are other situations where the extension of the result seems more (or much more) delicate. One of them concerns "complete" or "full" Boussinesq systems. By this we mean the equations

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = \theta e_n, \quad \operatorname{div} u = 0 & \text{in} \quad (0,T) \times \Omega, \\ \partial_t \theta - \Delta \theta + u \cdot \nabla \theta = (\nabla u + \nabla u^t) \cdot \nabla u & \text{in} \quad (0,T) \times \Omega, \end{cases}$$

completed with initial conditions and boundary control requirements as before. Another one is the variable density Navier–Stokes system

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{in} \quad (0,T) \times \Omega, \\ \rho(\partial_t u + (u \cdot \nabla) u) - \Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 & \text{in} \quad (0,T) \times \Omega, \end{cases}$$

this time completed with initial conditions for u and ρ and, again, boundary controls acting on u.

Appendix A. Regularity of the uncontrolled Boussinesq system. Let us present the proof of Lemma 2.1. In the followinglowing, let us assume that M and m are regular enough. In what follows, we will use Korn's inequality recurrently.

LEMMA A.1 (second Korn inequality). *There exist two positive constants $C_1, C_2 > 0$ such that, for every $u \in H^1(\mathcal{O})^n$, one has*

$$C_1 (\|u\| + \|D(u)\|) \leq \|u\|_{H^1} \leq C_2 (\|u\| + \|D(u)\|).$$

We will also need the following results.

LEMMA A.2. *There exist positive constants $C_l, C_r, K > 0$ such that, for every $u \in H^1(\mathcal{O})^n$, we have*

$$C_l \|u\|_{K,M} \leq \|u\|_{H^1} \leq C_r \|u\|_{K,M},$$

where $\|u\|_{K,M} := (K\|u\|^2 + \int_{\partial\mathcal{O}} Mu \cdot u + \|D(u)\|^2)^{1/2}$.

LEMMA A.3. *There exist positive constants $C_l, C_r, \gamma > 0$ such that, for every $\theta \in H^1(\mathcal{O})$, we have*

$$C_l \|\theta\|_{\gamma,m} \leq \|\theta\|_{H^1} \leq C_r \|\theta\|_{\gamma,m},$$

where $\|\theta\|_{\gamma,m} := (\gamma\|\theta\|^2 + \int_{\partial\mathcal{O}} m|\theta|^2 + \|\nabla\theta\|^2)^{1/2}$.

The proofs of these two lemmas rely on the interpolation inequality [2, Theorem III.2.36]. In particular, it is used that there exists a positive constant C such that

$$\|u\|_{L^2(\partial\mathcal{O})} \leq C\|u\|^{1/2}\|u\|_{H^1}^{1/2} \quad \forall u \in H^1(\mathcal{O}).$$

LEMMA A.4 (Proposition III.2.35 in [2]). *Let $p \in [1, +\infty]$ and $q \in [p, p^*]$, where p^* is the critical exponent associated with p. Then, there exists $C > 0$ such that*

$$\|u\|_{L^q} \leq C\|u\|_{L^p}^{1+n/q-n/p}\|u\|_{W^{1,p}}^{n/p-n/q} \quad \forall u \in W^{1,p}(\mathcal{O}).$$

LEMMA A.5 (pages 490–494 in [18]). *Let $f \in L^2(\mathcal{O})^n$ and $g \in H^{1/2}(\partial\mathcal{O})^n$. Then, there exists a unique strong solution $(u, p) \in H^2(\mathcal{O})^n \times H^1(\mathcal{O})$ to the Stokes problem*

$$\begin{cases} -\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 & \text{in} \quad \mathcal{O}, \\ u \cdot \nu = 0, \quad N(u) = g & \text{on} \quad \partial\mathcal{O}, \end{cases}$$

and there exists a positive constant $C > 0$ such that

$$\|u\|_{H^2} + \|p\|_{H^1} \leq C(\|f\| + \|g\|_{H^{1/2}}).$$

Moreover, if $f \in H^k(\mathcal{O})^n$ and $g \in H^{k+1/2}(\partial\mathcal{O})^n$ for some $k \geq 0$, then $(u, p) \in H^{k+2}(\mathcal{O})^n \times H^{k+1}(\mathcal{O})$ and we have

$$\|u\|_{H^{k+2}} + \|p\|_{H^{k+1}} \leq C(\|f\|_{H^k} + \|g\|_{H^{k+1/2}}).$$

LEMMA A.6. *Let $S : D(S) \to L^2_{\mathrm{div}}(\mathcal{O})^n$ be the Stokes operator, where $D(S) = \{v \in H^2(\mathcal{O})^n \cap L^2_{\mathrm{div}}(\mathcal{O})^n : N(v) = 0\}$ and $S := -\mathbb{P}\Delta$. There exists a positive constant $C > 0$ such that, for every $u \in D(S)$, we have*

$$\|u\|_{H^2} \le C\left(\|Su\| + \|u\|_{H^1}\right).$$

Moreover, if $Su \in H^k(\mathcal{O})^n$ for some $k \ge 0$, then $u \in H^{k+2}(\mathcal{O})^n$ and we have

$$\|u\|_{H^{k+2}} \le C(\|Su\|_{H^k} + \|u\|_{H^{k+1}}).$$

LEMMA A.7. *Let $u \in H^1(\mathcal{O})$ satisfying $\Delta u \in L^2(\mathcal{O})$ and $\frac{\partial u}{\partial \nu} + mu = 0 \quad$ on $\quad \partial\mathcal{O}$. Then, there exists a constant $C > 0$, only depending on $\mathcal{O}$, such that*

$$\|u\|_{H^2} \le C(\|\Delta u\| + \|mu\|_{H^{1/2}(\partial\mathcal{O})}).$$

Moreover, if $\Delta u \in H^k(\mathcal{O})$ for some $k \ge 0$, then $u \in H^{k+2}(\mathcal{O})$ and we have

$$\|u\|_{H^{k+2}} \le C(\|\Delta u\|_{H^k} + \|mu\|_{H^{k+1/2}(\partial\mathcal{O})}).$$

This last result is a consequence of [2, Theorem III.4.3].

Throughout the proof of Lemma 2.1, we will accept that the constants C can increase from line to line and depend on T and the trajectory $(\overline{u}, \overline{\theta})$. For simplicity, we will only consider the 3D case. The proof is split in several steps.

STEP 1 - WEAK ESTIMATES IN $(0, T/3)$. Let us first multiply $(6)_1$ by r and $(6)_2$ by q, integrate by parts, and sum. We get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|r\|^2 + \|q\|^2\right) + 2\|Dr\|^2 + \|\nabla q\|^2 + 2\int_{\partial\mathcal{O}} Mr\cdot r + \int_{\partial\mathcal{O}} m|q|^2 \\ = (qe_n, r) - \int_{\mathcal{O}} (r\cdot\nabla)\overline{u}\cdot r - \int_{\mathcal{O}} r\cdot\nabla\overline{\theta}\, q.\end{aligned}$$

From the Cauchy–Schwarz and Young inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|r\|^2 + \|q\|^2\right) + 2\|Dr\|^2 + \|\nabla q\|^2 + 2\int_{\partial\mathcal{O}} Mr\cdot r + \int_{\partial\mathcal{O}} m|q|^2 \le C(\|r\|^2 + \|q\|^2).$$

Using Lemmas A.2 and A.3, we deduce that

$$\frac{1}{2}\frac{d}{dt}\left(\|r\|^2 + \|q\|^2\right) + \frac{2}{C_l^2}(\|r\|^2_{H^1} + \|q\|^2_{H^1}) \le (C + 2K)\|r\|^2 + (C + 2\gamma)\|q\|^2. \tag{45}$$

By applying Gronwall's lemma, we have for a.e. $t \in [0, T]$ that

$$\|r(t,\cdot)\|^2 + \|q(t,\cdot)\|^2 + \int_0^t \left(\|r(s,\cdot)\|^2_{H^1} + \|q(s,\cdot)\|^2_{H^1}\right) ds \le e^{Ct}\left(\|r_*\|^2 + \|q_*\|^2\right). \tag{46}$$

Therefore, from the mean value theorem, we deduce by contradiction that there exists $0 \le t_1 \le T/3$ such that

$$\|r(t_1,\cdot)\|^2_{H^1} + \|q(t_1,\cdot)\|^2_{H^1} \le C_1\left(\|r_*\|^2 + \|q_*\|^2\right) \tag{47}$$

for a positive constant C_1 independent of t_1.

STEP 2 - STRONG ESTIMATES IN $(t_1, 2T/3)$. Let $\mathbb{P}$ be the classical Leray projector. We multiply $(6)_1$ and $(6)_2$ by $-Sr$ and $-\Delta q$, respectively, and then integrate by parts. Since M is symmetric, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|Dr\|^2 + \int_{\partial\mathcal{O}} Mr\cdot r\right) + \|Sr\|^2 \\
&\quad = \int_{\partial\mathcal{O}} (M_t)r\cdot r + \int_{\mathcal{O}}\left((r\cdot\nabla)r\cdot Sr + (\overline{u}\cdot\nabla)r\cdot Sr + (r\cdot\nabla)\overline{u}\cdot Sr - (qe_n, Sr)\right) \\
&\quad \le C\|r\|_{H^1}^2 + \tfrac{1}{2}\|Sr\|^2 + C\|q\|^2 + \|r\|_{L^6}^2\|\nabla r\|_{L^3}^2.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|\nabla q\|^2 + \int_{\partial\mathcal{O}} m|q|^2\right) + \|\Delta q\|^2 &= \frac{1}{2}\int_{\partial\mathcal{O}} (m_t)q\cdot q + (r\cdot\nabla q, \Delta q) \\
&\quad + (\overline{u}\cdot\nabla q, \Delta q) + (r\cdot\nabla\overline{\theta}, \Delta q) \\
&\le C\|q\|_{H^1}^2 + \frac{1}{2}\|\Delta q\|^2 + C\|r\|^2 + \|r\|_{L^6}^2\|\nabla q\|_{L^3}^2.
\end{aligned}$$

Multiplying (45) by $\varsigma = \max\{K, \gamma\}$, adding the above inequalities, and using Lemmas A.2–A.7, we deduce the following:

$$\begin{aligned}
&\frac{d}{dt}\left(\|r\|_{\varsigma,M}^2 + \|q\|_{\varsigma,m}^2\right) + \|r\|_{H^2}^2 + \|q\|_{H^2}^2 \\
&\quad \le C(\|r\|_{\varsigma,M}^2 + \|q\|_{\varsigma,m}^2 + \|r\|_{L^6}^2\|\nabla r\|_{L^3}^2 + \|r\|_{L^6}^2\|\nabla q\|_{L^3}^2) \\
&\quad \le C\left[(\|r\|_{\varsigma,M}^2 + \|q\|_{\varsigma,m}^2) + (\|r\|_{\varsigma,M}^2 + \|q\|_{\varsigma,m}^2)^3\right].
\end{aligned} \tag{48}$$

Introducing $Y(t) := \|r(t,\cdot)\|_{\varsigma,M}^2 + \|q(t,\cdot)\|_{\varsigma,m}^2$, we see that Y is a.e. differentiable and, from (48), we have that

$$Y' \le C(Y^3 + Y). \tag{49}$$

In view of (49), we obtain

$$Y(t)^2 \le \frac{e^{C(t-t_1)}Y(t_1)^2}{Y(t_1)^2 + 1 - e^{C(t-t_1)}Y(t_1)^2}.$$

Let us take $t - t_1 \le \tau_1$ small enough such that $e^{C(t-t_1)} \le 1 + \frac{1}{2Y(t_1)^2}$. Then, $Y(t)^2 \le 2e^{C(t-t_1)}Y(t_1)^2$ and, from (47), we deduce that $Y(t) \le CY_*$, where $Y_* := \|r_*\|^2 + \|q_*\|^2$. Therefore,

$$\|r(t,\cdot)\|_{\varsigma,M}^2 + \|q(t,\cdot)\|_{\varsigma,m}^2 + \int_{t_1}^{t}(\|r(s,\cdot)\|_{H^2}^2 + \|q(s,\cdot)\|_{H^2}^2)ds \le CY_* + C(Y_* + Y_*^3)\tau_1.$$

Taking τ_1 small enough such that $\tau_1 \le (1 + Y_*^2)^{-1}$, we have that $CY_* + C(Y_* + Y_*^3)\tau_1 \le C_2Y_*$. Therefore, one has

$$\|r(t,\cdot)\|_{\varsigma,M}^2 + \|q(t,\cdot)\|_{\varsigma,m}^2 + \int_{t_1}^{t}(\|r(s,\cdot)\|_{H^2}^2 + \|q(s,\cdot)\|_{H^2}^2)ds \le C_2\left(\|r_*\|^2 + \|q_*\|^2\right) \tag{50}$$

for $t_1 \le t \le t_1 + \tau_1$. This ensures the existence of $t_1 \le t_2 < \min\{2T/3, t_1 + \tau_1\}$ such that

$$\|r(t_2,\cdot)\|_{H^2}^2 + \|q(t_2,\cdot)\|_{H^2}^2 \le \frac{C_2}{\tau_1}\left(\|r_*\|^2 + \|q_*\|^2\right).$$

Step 3 - Third energy estimate in (t_2, T). At this point, we differentiate (6) with respect to time and multiply by $\partial_t r$ and $\partial_t q$. Then, we integrate by parts to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|r_t\|^2 + 2\|Dr_t\|^2 + 2\int_{\partial\mathcal{O}} Mr_t\cdot r_t \\ = -2\int_{\partial\mathcal{O}} M_t r\cdot r_t + (q_t e_n, r_t) - (r_t\cdot\nabla)r\cdot r_t - (\overline{u}_t\cdot\nabla)r\cdot r_t \\ -(r_t\cdot\nabla)\overline{u}\cdot r_t - (r\cdot\nabla)\overline{u}_t\cdot r_t \\ \le C\left(\|r\|_{H^1}\|r_t\|_{H^1} + \|q_t\|^2 + \|r_t\|^2 + \|r_t\|_3\|\nabla r\|\|r_t\|_6 + \|r\|_{H^1}^2\right)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|q_t\|^2 + \|\nabla q_t\|^2 + \int_{\partial\mathcal{O}} m|q_t|^2 \\ = -\int_{\partial\mathcal{O}} m_t q q_t - ((r_t+\overline{u})\cdot\nabla q, q_t) - (r_t\cdot\nabla\overline{\theta}, q_t) - (r\cdot\nabla\overline{\theta}_t, q_t) \\ \le C\left(\|q\|_{H^1}\|q_t\|_{H^1} + \|q\|_{H^1}^2 + \|q_t\|^2 + \|r_t\|^2 + \|r_t\|_3\|\nabla q\|\|r_t\|_6\right).\end{aligned}$$

Consequently, using Lemmas A.2–A.4 and adding the two above inequalities, we have

$$\begin{aligned}\frac{d}{dt}\left(\|r_t\|^2 + \|q_t\|^2\right) + \|r_t\|_{H^1}^2 + \|q_t\|_{H^1}^2 \\ \le C\left(\left(\|r\|_{H^1}^4 + \|q\|_{H^1}^4 + 1\right)\|r_t\|^2 + \|q_t\|^2 + \|r\|_{H^1}^2 + \|q\|_{H^1}^2\right).\end{aligned}$$

Now, introducing $Z(t) := \|r_t(t,\cdot)\|^2 + \|q_t(t,\cdot)\|^2$, we find from (50) that

$$Z' \le C[(1+Y_*^2)Z + Y_*]$$

for $t_2 \le t \le t_1 + \tau_1$. By applying Gronwall's lemma, we have for a.e. $t \in [t_2, t_1+\tau_1]$

$$Z(t) \le e^{C(1+Y_*^2)(t-t_2)}\left(Z(t_2) + CY_*(t-t_2)\right).$$

Since we have $Z(t_2) \le \Psi_1(Y_*)$ for some nonnegative regular Ψ_1 with $\Psi_1(0) = 0$, we find that $Z(t) \le \Psi_2(Y_*)$, with

$$\Psi_2(s) := e^{C(1+s^2)}(\Psi_1(s) + Cs) \quad \forall s \ge 0.$$

Therefore,

(51)

$$\|r_t(t,\cdot)\|^2 + \|q_t(t\cdot)\|^2 + \int_{t_2}^t \left(\|r_t(s,\cdot)\|_{H^1}^2 + \|q_t(s,\cdot)\|_{H^1}^2\right) ds \le \Psi_3(Y_*) \quad \forall t \in [t_2, t_1+\tau_1],$$

where $\Psi_3(s) := C[(1+s^2)\Psi_2(s) + s]$. In particular, this yields the existence of $t_3 \in (t_2, t_1+\tau_1)$ such that

$$\|r_t(t_3,\cdot)\|_{H^1}^2 + \|q_t(t_3,\cdot)\|_{H^1}^2 \le \frac{\Psi_3(Y_*)}{(t_1 - t_2 + \tau_1)}. \tag{52}$$

Actually, it is not difficult to check that the set of times $t_3 \in (t_2, t_1+\tau_1)$ satisfying (52) has a positive measure.

Step 4 - Conclusion. Using (50) and (51), we deduce an estimate of r in $L^\infty(H^2)$. It suffices to view $(6)_1$ as a family of Stokes problems (see Lemma A.5 and the arguments presented in [30, Theorem 3.8]). Then, looking at $(6)_2$ as a family of elliptic problems, we also find $L^\infty(H^2)$ estimates for q; see Lemma A.7. Both estimates depend on Y_* continuously. Therefore, repeating the procedure, we see that $(r(t_3), q(t_3)) \in H^3 \times H^3$ with an estimate of the form $\Psi(Y_*)$.

Acknowledgments. The authors would like to thank Sergio Guerrero and Frédéric Marbach for several fruitful discussions about this work. The authors also would like to thank the anonymous referees for their remarks that helped to improve a first version of the present work a lot.

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