DOI: https://doi.org/10.1093/qjmam/hbac004

Carreau law for non-Newtonian fluid flow through a thin porous media

María ANGUIANO[1], Matthieu BONNIVARD[2], Francisco J. SUÁREZ-GRAU[3]

Abstract

We consider the flow of generalized Newtonian fluid through a thin porous media. The media under consideration is a bounded perforated three dimensional domain confined between two parallel plates, where the distance between the plates is described by a small parameter ε. The perforation consists in an array of solid cylinders, which connect the plates in perpendicular direction, with diameter of size ε and distributed periodically with period ε. The flow is described by the three dimensional incompressible stationary Stokes system with a non-linear viscosity following the *Carreau law*. We study the limit when the thickness tends to zero and prove that the averaged velocity satisfies a non-linear two dimensional homogenized law of *Carreau* type. We illustrate our homogenization result by numerical simulations showing the influence of the Carreau law on the behaviour of the limit system, in the case where the flow is driven by a constant pressure gradient and for different geometries of perforations.

AMS classification numbers: 76A05, 76M50, 76A20, 35B2.

Keywords: Homogenization, non-Newtonian fluid, Carreau law, thin porous media.

1 Introduction

Experimental observations show that the viscosity is constant for some fluids as water or honey, which are called Newtonian fluids. These fluids can be described in a very accurate way by the Navier-Stokes equations. However, many others fluids behave differently, the viscosity of these fluids is no more constant, such as pastes or polymer solutions. These fluids can not be described by the Navier-Stokes equations and they are called non-Newtonian or complex fluids. The simplest idea to describe non-Newtonian fluids is to plot the viscosity measurements versus the imposed shear rate and then, to fit the obtained curve with a simple template viscosity function, adjusting some few parameters. This is the main idea of generalized Newtonian fluids models (also called quasi-Newtonian fluids models), which could be viewed as a first step inside the world of non-Newtonian fluids models (see Saramito [34, Chapter 2] for more details).

In this article we consider the incompressible viscous flow of the generalized Newtonian fluid through a thin porous media, which consists in a domain of small height ε and perforated by periodically distributed solid cylinders with diameter of size ε. The viscosity of the fluid follows the *Carreau law*. This law is commonly used for fluid studied in Chemical Industry and Rheology, for instance in injection moulding of melted polymers, flow of oils, muds, etc.

The incompressible generalized Newtonian fluids are characterized by the viscosity depending on the principal invariants of the symmetric stretching tensor $\mathbb{D}[u]$. If u is the velocity, p the pressure and Du the gradient velocity tensor, $\mathbb{D}[u] = (Du + D^t u)/2$ denotes the symmetric stretching tensor and σ the stress tensor given by $\sigma = -pI + 2\eta_r\mathbb{D}[u]$. The viscosity η_r is constant for a Newtonian fluid but dependent on the shear rate, i.e. $\eta_r = \eta_r(\mathbb{D}[u])$, for viscous non-Newtonian fluids. The deviatoric stress tensor τ, i.e. the part of the total stress

[1]Departamento de Análisis Matemático. Facultad de Matemáticas. Universidad de Sevilla, 41012 Sevilla (Spain) anguiano@us.es

[2]Sorbonne Paris Cité, Laboratoire Jacques-Louis Lions, UMR 7598, UPMC, CNRS. Université Paris Diderot, 75205 Paris (France) bonnivard@ljll.univ-paris-diderot.fr

[3]Departamento de Ecuaciones Diferenciales y Análisis Numérico. Facultad de Matemáticas. Universidad de Sevilla, 41012 Sevilla (Spain) fjsgrau@us.es

tensor that is zero at equilibrium, is then a nonlinear function of the shear rate $\mathbb{D}[u]$, i.e. $\tau = \eta_r(\mathbb{D}[u])\mathbb{D}[u]$ (see Barnes *et al.* [17], Bird *et al.* [18] and Mikelić [32] for more details).

The most widely used laws in engineering practice are the *power law* or Ostwald-de Waele model (Ostwald, 1925; de Waele, 1923) and the *Carreau law* (Carreau, 1972). It is observed that the *power law* correctly describes the behavior of polymers at high shear rates. It offers the advantage of allowing analytical calculations in simple geometries. However, it has the disadvantage of not describing a Newtonian plateau and even predicts an infinite viscosity as the shear rate goes to zero and $r \in (1,2)$ (see Agassant *et al.* [2], p. 49), whereas for real fluids it tends to some constant value η_0 called the zero-shear-rate viscosity. Thus, as a generalization we consider the *Carreau law*, which is an alternative generalized Newtonian model that enables the description of the plateaus in viscosity that are expected when the shear rate is very small or very large. The empiricism for the viscosity η_r used in the *Carreau law* is defined by

$$\eta_r(\mathbb{D}[u]) = (\eta_0 - \eta_\infty)(1 + \lambda|\mathbb{D}[u]|^2)^{\frac{r}{2}-1} + \eta_\infty, \quad 1 < r < 2, \quad \eta_0 > \eta_\infty > 0, \quad \lambda > 0, \tag{1}$$

where η_∞ is the high-shear-rate limit of the viscosity, the parameter λ is a time constant and $r-1$ is a dimensionless constant describing the slope in the power law region. The matrix norm $|\cdot|$ is defined by $|\xi|^2 = Tr(\xi\xi^t)$ with $\xi \in \mathbb{R}^3$. We recall that in this case the viscosity is decreasing with the shear rate and the fluid is said *shear thinning* or *pseudoplastic* (see Saramito [34, Chapter 2]).

Homogenization applied to porous media is a mathematical method that allows to upscale the fundamental equations from continuum physics, being valid at the microscopic level. The homogenization theory of heterogeneous media studies the effects of the micro-structure (i.e. of the pore structure) upon solutions of PDEs of the continuum mechanics, so allow to derive rigorously equations describing filtration of a generalized Newtonian fluid (see Mikelić [32] for more details).

To find the homogenized law describing the generalized Newtonian fluid flow through a porous media $\Omega_\varepsilon \subset \mathbb{R}^3$, which is a domain with fixed height and periodically perforated by obstacles of size ε, Bourgeat and Mikelić in [20] (see also Bourgeat *et al.* [21], Götz and Parhusip [27]) used the homogenization technique called the two-scale convergence. The domain without perforations is the bounded smooth domain $\Omega \subset \mathbb{R}^3$ which is made of two parts, the fluid part Ω_ε and the solid part $\Omega \setminus \Omega_\varepsilon$. Moreover, assuming that the *Reynolds number* is proportional to $\varepsilon^{-\gamma}$ and the flow is sufficiently slow to neglect inertial effects, then the following stationary Stokes system with a non-linear viscosity following the *Carreau law* (1), with $1 < r < 2$, was considered

$$\left\{\begin{array}{rl} -\varepsilon^\gamma \mathrm{div}\,(\eta_r(\mathbb{D}[u_\varepsilon])\mathbb{D}[u_\varepsilon]) + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \mathrm{div}\, u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{array}\right. \tag{2}$$

When ε tends to zero, two types of averaged momentum equations were rigorously derived depending on the value of γ (i.e. the *Reynolds number*) connecting the velocity and the pressure gradient:

- If $\gamma \neq 1$, the homogenized law is the classical 3D Darcy's law for Newtonian fluids

  $$V(x) = \frac{K}{\mu}\left(f(x) - \nabla_x p(x)\right) \text{ in } \Omega, \quad \mathrm{div}_x V(x) = 0 \text{ in } \Omega, \quad V(x)\cdot n = 0 \text{ on } \partial\Omega,$$

  where p is the limit pressure and the *permeability* tensor $K \in \mathbb{R}^{3\times 3}$ is obtained by solving 3D Stokes local problems posed in a reference cell which contains the information of the geometry of the obstacles. The viscosity μ is equal to η_0 if $\gamma < 1$ and equal to η_∞ if $\gamma > 1$.

- If $\gamma = 1$, the mean global filtration velocity as a function of the pressure gradient is given by

  $$V(x) = \mathcal{U}\left(f(x) - \nabla_x p(x)\right) \text{ in } \Omega, \quad \mathrm{div}_x V(x) = 0 \text{ in } \Omega, \quad V(x)\cdot n = 0 \text{ on } \partial\Omega,$$

  where $\mathcal{U} : \mathbb{R}^3 \to \mathbb{R}^3$ is a *permeability* function, not necessary linear, and is defined through the solutions of 3D local Stokes problems with non-linear viscosity following the *Carreau law* and posed in a reference cell.

On the other hand, in [19] Boughanim and Tapiéro considered the polymer flow through a thin slab $\Omega_\varepsilon = \omega \times (0,\varepsilon) \subset \mathbb{R}^3$, where ε is the small thickness of the slab. Starting from the Stokes system (2), with external body force of the form $f = (f', 0)$ such that $f' = (f_1, f_2)$, by using dimension reduction and homogenization techniques, they studied the limit when the thickness tends to zero. According to the value of γ, they proved the following:

- If $\gamma \neq 1$, the homogenization law is the classical linear 2D Reynolds law for Newtonian fluids

$$\begin{cases} V'(x') = \dfrac{1}{6\mu}\left(f'(x') - \nabla_{x'}p(x')\right), \quad V_3(x') = 0 \quad \text{in } \omega, \\ \text{div}_{x'}V(x') = 0 \quad \text{in } \omega, \quad V(x')\cdot n = 0 \quad \text{on } \partial\omega, \end{cases}$$

  where $V' = (V_1, V_2)$, $x' = (x_1, x_2)$. As previously, the viscosity μ is equal to η_0 if $\gamma < 1$ and equal to η_∞ if $\gamma > 1$.

- If $\gamma = 1$, the homogenization law corresponds to a non-linear 2D Reynolds law of Carreau type

$$\begin{cases} V'(x') = 2((f'(x') - \nabla_{x'}p(x')) \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(\frac{1}{2}+\xi)\xi}{\psi(2|f'(x') - \nabla_{x'}p(x')||\xi|)}\, d\xi, \quad V_3(x') = 0 \quad \text{in } \omega, \\ \text{div}_{x'}V'(x') = 0 \quad \text{in } \omega, \quad V'(x')\cdot n = 0 \quad \text{on } \partial\omega, \end{cases}$$

  where the function $\psi = \psi(\tau)$, $\tau \in \mathbb{R}^+$, is the inverse of the equation $\tau = \psi\sqrt{\frac{2}{\lambda}\left(\frac{\psi-\eta_\infty}{\eta_0-\eta_\infty}\right)^{\frac{2}{r-2}} - 1}$.

In this paper, we consider a thin porous media $\Omega_\varepsilon = \omega_\varepsilon \times (0,\varepsilon) \subset \mathbb{R}^3$ of small height ε which is perforated by periodically distributed solid cylinders of diameter of size ε. Here, the bottom of the domain without perforations $\omega \subset \mathbb{R}^2$ is made of two parts, the fluid part ω_ε and the solid part $\omega \setminus \omega_\varepsilon$. This model of thin porous media has been recently introduced by Fabricius *et al.* [25], where the flow of an incompressible viscous fluid described by the stationary Navier-Stokes equations has been studied by the multiscale expansion method, which is a formal tool to analyse homogenization problems. These results have been rigorously proved in [13] using an adaptation, introduced in [10], of the periodic unfolding method from Cioranescu *et al.* [23, 24]. This adaptation consists of a combination of the unfolding method with a rescaling in the height variable, in order to work with a domain of fixed height and to pass to the limit. In particular, the generalized Newtonian fluids obeying the *power law* in the thin porous media Ω_ε have been studied rigorously in Anguiano and Suárez-Grau [10] where we have obtained a 2D Darcy's law when the domain thickness tends to zero (see also [15] for the extension to the case of a thin porous media with an array of cylinders with small diameter). Also, the Bingham plastic behavior in the thin porous media Ω_ε has been studied in [8, 9]. For other studies concerning thin porous media, we refer to Anguiano [3, 4, 5, 6, 7], Anguiano and Suárez-Grau [11, 12, 14], Jouybari and T. S. Lundström [29], Prat and Agaësse [33], Suárez-Grau [35], Yeghiazarian *et al.* [36] and Zhengan and Hongxing [37]. However, as far as we know, in the previous literature there is no study for the homogenization of three-dimensional incompressible stationary Stokes system with a non-linear viscosity following the *Carreau law* in a thin porous media, as we consider in this article.

Therefore, taking into account the previous results, we consider the Stokes system (2) in the thin porous media Ω_ε assuming $\gamma = 1$. This choice is the most challenging one and answers to the question addressed in the paper, namely it preserves the nonlinear character of the flow in the limit. After the homogenization process, we obtain the following lower-dimensional homogenization law

$$\begin{cases} V'(x') = \mathcal{U}\left(f'(x') - \nabla_{x'}p(x')\right), \quad V_3(x') = 0 \text{ in } \omega, \\ \text{div}_{x'}V'(x') = 0 \;\text{ in } \omega, \quad V'(x')\cdot n = 0 \;\text{ on } \partial\omega, \end{cases}$$

where the permeability function $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined through the solutions of 3D local Stokes problems with non-linear viscosity following the *Carreau law* and posed in a reference cell, see Theorem 2.1.

In order to illustrate the influence of the non-linear character of the Carreau law on the behaviour of the two-dimensional limit system, we conclude the paper with a numerical study of a flow of a generalized Newtonian fluid, driven by a constant pressure gradient, in a thin porous medium confined between two parallel plates. Using different geometries for the inclusion T, we show that, depending on the choice of parameters λ and r in the viscosity law (1), the amplitude of the mean filtration velocity associated to a given pressure gradient can be dramatically increased in comparison with the Newtonian case $r = 2$.

The structure of the paper is as follows. In Section 2 we introduce the domain, make the statement of the problem and give the main result (Theorem 2.1). The proof of the main result is provided in Section 3. Section 4 is dedicated to the numerical simulations of the homogenized model, in the case of Carreau fluid driven by a constant pressure gradient. We finish the paper with a list of references.

2 Setting of the problem and main result

Geometrical setting. The periodic porous medium is defined by a domain ω and an associated microstructure, or periodic cell $Y' = (-1/2, 1/2)^2$, which is made of two complementary parts: the fluid part Y'_f, and the solid part T' ($Y'_f \bigcup T' = Y'$ and $Y'_f \bigcap T' = \emptyset$). More precisely, we assume that ω is a smooth, bounded, connected set in $\mathbb{R}^2$, and that T' is an open connected subset of Y' with a smooth boundary $\partial T'$, such that $\bar{T}'$ is strictly included in Y'.

The microscale of a porous medium is a small positive number ε. The domain ω is covered by a regular mesh of square of size ε: for $k' \in \mathbb{Z}^2$, each cell $Y'_{k',\varepsilon} = \varepsilon k' + \varepsilon Y'$ is divided in a fluid part $Y'_{f_{k'},\varepsilon}$ and a solid part $T'_{k',\varepsilon}$, i.e. is similar to the unit cell Y' rescaled to size ε. We define $Y = Y' \times (0,1) \subset \mathbb{R}^3$, which is divided in a fluid part $Y_f = Y'_f \times (0,1)$ and a solid part $T = T' \times (0,1)$, and consequently $Y_{k',\varepsilon} = Y'_{k',\varepsilon} \times (0,1) \subset \mathbb{R}^3$, which is also divided in a fluid part $Y_{f_{k'},\varepsilon}$ and a solid part $T_{k',\varepsilon}$ (see Figures 1 and 2).

We denote by $\tau(\overline{T}'_{k',\varepsilon})$ the set of all translated images of $\overline{T}'_{k',\varepsilon}$. The set $\tau(\overline{T}'_{k',\varepsilon})$ represents the obstacles in $\mathbb{R}^2$.

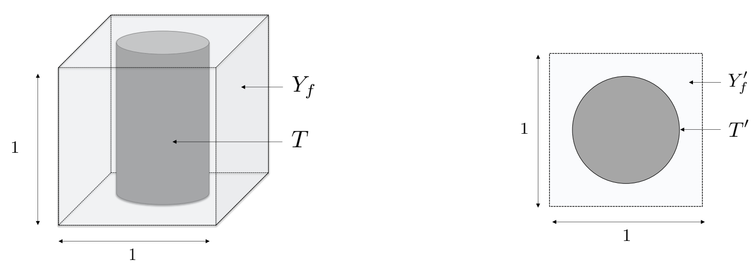

Figure 1: View of the 3D reference cells Y (left) and the 2D reference cell Y' (right).

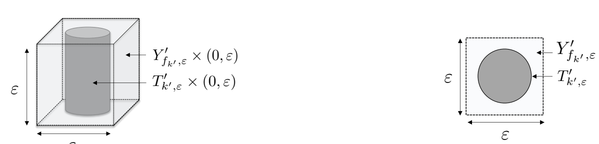

Figure 2: View of the 3D reference cells $Y_{k',\varepsilon}$ (left) and the 2D reference cell $Y'_{k',\varepsilon}$ (right).

The fluid part of the bottom $\omega_\varepsilon \subset \mathbb{R}^2$ of a porous medium is defined by $\omega_\varepsilon = \omega \backslash \bigcup_{k' \in \mathcal{K}_\varepsilon} \bar{T}'_{k',\varepsilon}$, where $\mathcal{K}_\varepsilon = \{k' \in \mathbb{Z}^2 : Y'_{k',\varepsilon} \cap \omega \neq \emptyset\}$. The whole fluid part $\Omega_\varepsilon \subset \mathbb{R}^3$ in the thin porous medium is defined by (see Figure 3)

$$\Omega_\varepsilon = \{(x_1, x_2, x_3) \in \omega_\varepsilon \times \mathbb{R} : 0 < x_3 < \varepsilon\}. \tag{3}$$

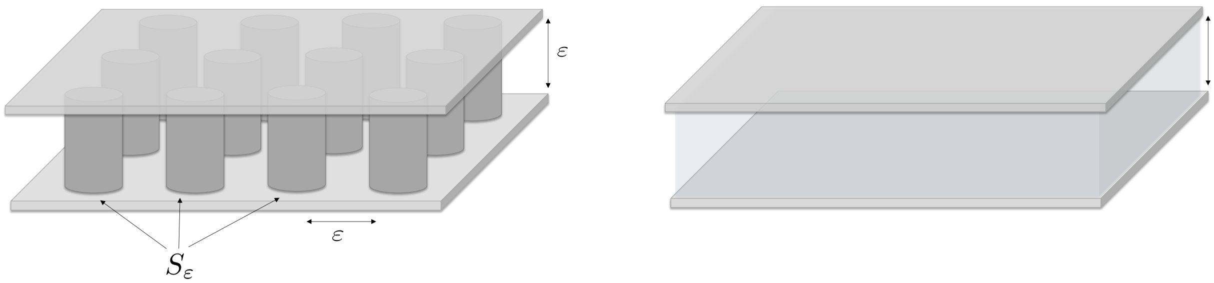

Figure 3: View of the thin porous media Ω_ε (left) and domain without perforations Q_ε (right).

We assume that the obstacles $\tau(\overline{T}'_{k',\varepsilon})$ do not intersect the boundary $\partial\omega$ and we denote by S_ε the set of the solid cylinders contained in Ω_ε, i.e. $S_\varepsilon = \bigcup_{k'\in\mathcal{K}_\varepsilon} T'_{k',\varepsilon} \times (0,\varepsilon)$. We define

$$\widetilde{\Omega}_\varepsilon = \omega_\varepsilon \times (0,1), \quad \Omega = \omega \times (0,1), \quad Q_\varepsilon = \omega \times (0,\varepsilon). \tag{4}$$

We observe that $\widetilde{\Omega}_\varepsilon = \Omega \setminus \bigcup_{k'\in\mathcal{K}_\varepsilon} \bar{T}_{k',\varepsilon}$, and we define $T_\varepsilon = \bigcup_{k'\in\mathcal{K}_\varepsilon} T_{k',\varepsilon}$ as the set of the solid cylinders contained in $\widetilde{\Omega}_\varepsilon$.

To finish, we introduce some notation that will be useful along the paper. The points $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' = (x_1, x_2) \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. We also use the notation x' to denote a generic vector of $\mathbb{R}^2$. Let $C^\infty_\#(Y)$ be the space of infinitely differentiable functions in $\mathbb{R}^3$ that are Y'-periodic. By $L^2_\#(Y)$ (resp. $H^1_\#(Y)$) we denote its completion in the norm $L^2(Y)$ (resp. $H^1(Y)$) and by $L^2_{0,\#}(Y)$ the space of functions in $L^2_\#(Y)$ with zero mean value.

Statement of the problem. Let us consider the following stationary Stokes system with the non-linear viscosity following the *Carreau law* (1) in Ω_ε, with a Dirichlet boundary condition on the exterior boundary ∂Q_ε and the cylinders ∂S_ε,

$$\left\{\begin{array}{rl} -\varepsilon \mathrm{div}\,(\eta_r(\mathbb{D}[u_\varepsilon])\mathbb{D}[u_\varepsilon]) + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \mathrm{div}\, u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial Q_\varepsilon \cup \partial S_\varepsilon, \end{array}\right. \tag{5}$$

where the second member f is of the form

$$f(x) = (f'(x'), 0) \quad \text{with } f' \in L^\infty(\omega)^2. \tag{6}$$

We remark that the assumptions of neglecting the vertical component of the exterior force and the independence of the vertical variable are usual when dealing with fluids in through thin domains (see [19] for more details).

The classical theory from Lions [30] gives the existence of a unique solution $(u_\varepsilon, p_\varepsilon) \in H^1_0(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$, where L^2_0 is the space of functions of L^2 with zero mean value.

Our goal is to study the asymptotic behavior of u_ε and p_ε when ε tends to zero. For this purpose, we use the dilatation in the variable x_3 as follows

$$y_3 = \frac{x_3}{\varepsilon}, \tag{7}$$

in order to have the functions defined in the open set with fixed height $\widetilde{\Omega}_\varepsilon$. Namely, we define $\tilde{u}_\varepsilon$ and $\tilde{p}_\varepsilon$ by

$$\tilde{u}_\varepsilon(x', y_3) = u_\varepsilon(x', \varepsilon y_3), \quad \tilde{p}_\varepsilon(x', y_3) = p_\varepsilon(x', \varepsilon y_3), \quad a.e.\ (x', y_3) \in \widetilde{\Omega}_\varepsilon.$$

Let us introduce some notation which will be useful in the following. For a vectorial function $v = (v', v_3)$ and a scalar function w, we will denote $\mathbb{D}_{x'}[v] = \frac{1}{2}(D_{x'}v + D^t_{x'}v)$ and $\partial_{y_3}[v] = \frac{1}{2}(\partial_{y_3}v + \partial^t_{y_3}v)$, where we denote

$\partial_{y_3} = (0, 0, \frac{\partial}{\partial y_3})^t$. Moreover, associated to the change of variables (7), we introduce the operators: $\mathbb{D}_\varepsilon$, D_ε, div_ε and ∇_ε, by

$$\mathbb{D}_\varepsilon[v] = \frac{1}{2}\left(D_\varepsilon v + D_\varepsilon^t v\right),$$

$$(D_\varepsilon v)_{i,j} = \partial_{x_j} v_i \text{ for } i = 1,2,3,\ j = 1,2, \quad (D_\varepsilon v)_{i,3} = \varepsilon^{-1}\partial_{y_3} v_i \text{ for } i = 1,2,3,$$

$$\mathrm{div}_\varepsilon v = \mathrm{div}_{x'} v' + \varepsilon^{-1}\partial_{y_3} v_3, \quad \nabla_\varepsilon w = (\nabla_{x'} w, \varepsilon^{-1}\partial_{y_3} w)^t.$$

Using the transformation (7), system (5) can be rewritten as

$$\begin{cases} -\varepsilon \mathrm{div}_\varepsilon\left(\eta_r(\mathbb{D}_\varepsilon[\tilde u_\varepsilon])\mathbb{D}_\varepsilon[\tilde u_\varepsilon]\right) + \nabla_\varepsilon \tilde p_\varepsilon = f \text{ in } \widetilde\Omega_\varepsilon, \\ \mathrm{div}_\varepsilon \tilde u_\varepsilon = 0 \text{ in } \widetilde\Omega_\varepsilon, \\ \tilde u_\varepsilon = 0 \text{ on } \partial\Omega \cup \partial T_\varepsilon. \end{cases} \tag{8}$$

Our goal then is to describe the asymptotic behavior of this new sequence $(\tilde u_\varepsilon,\ \tilde p_\varepsilon)$. The sequences of solutions $(\tilde u_\varepsilon,\ \tilde p_\varepsilon) \in H_0^1(\widetilde\Omega_\varepsilon)^3 \times L_0^2(\widetilde\Omega_\varepsilon)$ is not defined in a fixed domain independent of ε but rather in a varying set $\widetilde\Omega_\varepsilon$. In order to pass the limit if ε tends to zero, convergences in fixed Sobolev spaces (defined in Ω) are used which requires first that $(\tilde u_\varepsilon,\ \tilde p_\varepsilon)$ be extended to the whole domain Ω. Then, an extension $(\tilde u_\varepsilon,\ \tilde P_\varepsilon) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ is defined on Ω and coincides with $(\tilde u_\varepsilon,\ \tilde p_\varepsilon)$ on $\widetilde\Omega_\varepsilon$ (we will use the same notation, $\tilde u_\varepsilon$, for the velocity in $\widetilde\Omega_\varepsilon$ and its continuation in Ω).

Our main result referred to the asymptotic behavior of the solution of (8) is given by the following theorem.

Theorem 2.1 (Main Theorem)**.** *There exist $\tilde u \in H_0^1(0,1;L^2(\omega)^3)$ with $\tilde u_3 = 0$ and $\tilde P \in L_0^2(\omega)$, such that the extension $(\tilde u_\varepsilon, \tilde P_\varepsilon)$ of the solution of (8) satisfies the following convergences*

$$\varepsilon^{-1}\tilde u_\varepsilon \rightharpoonup \tilde u \quad \text{weakly in } H^1(0,1;L^2(\omega)^3), \quad \tilde P_\varepsilon \to \tilde P \quad \text{strongly in } L^2(\Omega).$$

Moreover, defining $V(x') = \int_0^1 \tilde u(x', y_3)\,dy_3$, it holds that $(V, \tilde P) \in L^2(\omega)^3 \times (L_0^2(\omega) \cap H^1(\omega))$ is the unique solution of the lower-dimensional homogenization law

$$\begin{cases} V'(x') = \mathcal{U}\left(f'(x') - \nabla_{x'}\tilde P(x')\right), \quad V_3(x') = 0 \quad \text{in } \omega, \\ \mathrm{div}_{x'} V'(x') = 0 \quad \text{in } \omega, \quad V'(x')\cdot n = 0 \quad \text{on } \partial\omega, \end{cases} \tag{9}$$

where the permeability function $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathcal{U}(\xi') = \int_{Y_f} w'_{\xi'}(y)\,dy, \quad \forall\, \xi' \in \mathbb{R}^2, \tag{10}$$

with $w_{\xi'}$, for every $\xi' \in \mathbb{R}^2$, the unique solution of the local Stokes system with the non-linear viscosity following the Carreau law (1),

$$\begin{cases} -\mathrm{div}_y\left(\eta_r\left(\mathbb{D}_y[w_{\xi'}]\right)\mathbb{D}_y[w_{\xi'}]\right) + \nabla_y \pi_{\xi'} = \xi' \quad \text{in } Y_f, \\ \mathrm{div}_y w_{\xi'} = 0 \quad \text{in } Y_f, \\ w_{\xi'} = 0 \quad \text{on } \partial T, \\ (w_{\xi'}, \pi_{\xi'}) \in H^1_\#(Y_f)^3 \times L^2_{0,\#}(Y_f). \end{cases} \tag{11}$$

Remark 2.2. *According to [21, Lemma 2], the permeability function $\mathcal{U}$ is coercive and strictly monotone.*

3 Proof of the main result

In this section we provide the proof of the main result (Theorem 2.1). To to this, first we establish some *a priori* estimates of the solution of (8) and we define the extension of the solution. Second, we introduce the version of the unfolding method depending on ε. Next, a compactness result, which is the main key when we will pass to the limit later, is addressed and finally, the proof of the Theorem 2.1 is given.

3.1 *A priori* estimates.

In this subsection, we establish sharp *a priori* estimates of the dilated solution in $\widetilde{\Omega}_\varepsilon$. To do this, we first need the Poincaré and Korn inequalities in $\widetilde{\Omega}_\varepsilon$, which can be found in [10].

Lemma 3.1 (Remark 4.3-(i) in [10])**.** *There exists a positive constant C, independent of ε, such that*

$$\|\tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} \leq C\varepsilon\|D_\varepsilon\tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}}, \quad \forall\, \tilde{v}\in H^1_0(\widetilde{\Omega}_\varepsilon)^3 \quad (\textit{Poincaré's inequality}), \tag{12}$$

$$\|D_\varepsilon\tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\|\mathbb{D}_\varepsilon[\tilde{v}]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}}, \quad \forall\, \tilde{v}\in H^1_0(\widetilde{\Omega}_\varepsilon)^3 \quad (\textit{Korn's inequality}). \tag{13}$$

We give *a priori* estimates for velocity $\tilde{u}_\varepsilon$ in $\widetilde{\Omega}_\varepsilon$.

Lemma 3.2. *There exists a positive constant C, independent of ε, such that*

$$\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} \leq C\varepsilon, \quad \|D_\varepsilon\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C, \quad \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\,. \tag{14}$$

Proof. Multiplying (8) by $\tilde{u}_\varepsilon$ and integrating over $\widetilde{\Omega}_\varepsilon$, we get

$$\varepsilon(\eta_0-\eta_\infty)\int_{\widetilde{\Omega}_\varepsilon}\left(1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2\right)^{\frac{r}{2}-1}|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2dx'dy_3+\varepsilon\eta_\infty\int_{\widetilde{\Omega}_\varepsilon}|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2dx'dy_3=\int_{\widetilde{\Omega}_\varepsilon}f'\cdot\tilde{u}'_\varepsilon\,dx'dy_3.$$

Taking into account that $\eta_0>\eta_\infty$, and $\lambda>0$, we have

$$\varepsilon(\eta_0-\eta_\infty)\int_{\widetilde{\Omega}_\varepsilon}\left(1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2\right)^{\frac{r}{2}-1}|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2dx'dy_3\geq 0,$$

and then, from Cauchy-Schwarz's inequality and the assumption on f' given in (6), we get

$$\varepsilon\eta_\infty\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}}\leq C\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}.$$

Applying Poincaré's inequality (12) and Korn's inequality (13) in the right-hand side, we have

$$\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}}\leq C\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}},$$

which gives $(14)_3$. Finally, applying again (12) and (13), we obtain $(14)_1$ and $(14)_2$. □

Remark 3.3. *We extend the velocity $\tilde{u}_\varepsilon$ by zero in $\Omega\setminus\widetilde{\Omega}_\varepsilon$ (this is compatible with the homogeneous boundary condition on $\partial\Omega\cup\partial T_\varepsilon$), and denote the extension by same symbol. Obviously, estimates given in Lemma 3.2 remain valid and the extension $\tilde{u}_\varepsilon$ is divergence free too.*

In order to extend the pressure $\tilde{p}_\varepsilon$ to the whole domain Ω and obtain *a priori* estimates, we recall a result in which is concerned with the extension of the pressure p_ε to the whole domain Q_ε. Thus, we first use a restriction operator R^ε from $H^1_0(Q_\varepsilon)^3$ into $H^1_0(\Omega_\varepsilon)^3$, which is introduced in [10] as R^ε_2, next we extend the gradient of the pressure by duality in $H^{-1}(Q_\varepsilon)^3$ and finall, by means of the dilatation, we extend $\tilde{p}_\varepsilon$ to Ω.

Lemma 3.4 (Lemma 4.5-(i) in [10])**.** *There exists a (restriction) operator R^ε acting from $H^1_0(Q_\varepsilon)^3$ into $H^1_0(\Omega_\varepsilon)^3$ such that*

1. *$R^\varepsilon v=v$, if $v\in H^1_0(\Omega_\varepsilon)^3$ (elements of $H^1_0(\Omega_\varepsilon)$ are extended by 0 to Q_ε).*
2. *$\mathrm{div}R^\varepsilon v=0$ in Ω_ε, if $\mathrm{div}\,v=0$ on Q_ε.*
3. *For every $v\in H^1_0(Q_\varepsilon)^3$, there exists a positive constant C, independent of v and ε, such that*

$$\|R^\varepsilon v\|_{L^2(\Omega_\varepsilon)^3}+\varepsilon\|DR^\varepsilon v\|_{L^2(\Omega_\varepsilon)^{3\times 3}}\leq C\left(\|v\|_{L^2(Q_\varepsilon)^3}+\varepsilon\|Dv\|_{L^2(Q_\varepsilon)^{3\times 3}}\right)\,. \tag{15}$$

Using the restriction operator R^ε given in Lemma 3.4, we introduce F_ε in $H^{-1}(Q_\varepsilon)^3$ in the following way

$$\langle F_\varepsilon, v\rangle_{H^{-1}(Q_\varepsilon)^3, H_0^1(Q_\varepsilon)^3} = \langle \nabla p_\varepsilon, R^\varepsilon v\rangle_{H^{-1}(\Omega_\varepsilon)^3, H_0^1(\Omega_\varepsilon)^3}, \quad \text{for any } v \in H_0^1(Q_\varepsilon)^3, \tag{16}$$

and calculate the right hand side of (16) by using the variational formulation of problem (5), which gives

$$\begin{aligned}\langle F_\varepsilon, v\rangle_{H^{-1}(Q_\varepsilon)^3, H_0^1(Q_\varepsilon)^3} = & -\varepsilon(\eta_0-\eta_\infty)\int_{\Omega_\varepsilon}(1+\lambda|\mathbb{D}[u_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}[u_\varepsilon] : DR^\varepsilon v\,dx \\ & -\varepsilon\eta_\infty\int_{\Omega_\varepsilon}\mathbb{D}[u_\varepsilon] : DR^\varepsilon v\,dx + \int_{\Omega_\varepsilon} f'\cdot (R^\varepsilon v)'\,dx\,.\end{aligned} \tag{17}$$

Using Lemma 3.2 for fixed ε, we see that it is a bounded functional on $H_0^1(Q_\varepsilon)$ (see the proof of Lemma 3.5 below), and in fact $F_\varepsilon \in H^{-1}(Q_\varepsilon)^3$. Moreover, $\operatorname{div} v = 0$ implies $\langle F_\varepsilon, v\rangle = 0$, and the DeRham theorem gives the existence of P_ε in $L_0^2(Q_\varepsilon)$ with $F_\varepsilon = \nabla P_\varepsilon$.

Next, we get for every $\tilde v \in H_0^1(\Omega)^3$ where $\tilde v(x', y_3) = v(x', \varepsilon y_3)$, using the change of variables (7), that

$$\langle \nabla_\varepsilon \tilde P_\varepsilon, \tilde v\rangle_{H^{-1}(\Omega)^3, H_0^1(\Omega)^3} = -\int_\Omega \tilde P_\varepsilon \operatorname{div}_\varepsilon \tilde v\,dx'dy_3 = -\varepsilon^{-1}\int_{Q_\varepsilon} P_\varepsilon \operatorname{div} v\,dx = \varepsilon^{-1}\langle \nabla P_\varepsilon, v\rangle_{H^{-1}(Q_\varepsilon)^3, H_0^1(Q_\varepsilon)^3}\,.$$

Using the identification (17) of F_ε, we have

$$\begin{aligned}\langle \nabla_\varepsilon \tilde P_\varepsilon, \tilde v\rangle_{H^{-1}(\Omega)^3, H_0^1(\Omega)^3} = & \;\varepsilon^{-1}\Big(-\varepsilon(\eta_0-\eta_\infty)\int_{\Omega_\varepsilon}(1+\lambda|\mathbb{D}[u_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}[u_\varepsilon] : DR^\varepsilon v\,dx \\ & -\varepsilon\eta_\infty\int_{\Omega_\varepsilon}\mathbb{D}[u_\varepsilon] : DR^\varepsilon v\,dx + \int_{\Omega_\varepsilon} f'\cdot (R^\varepsilon v)'\,dx\Big)\,,\end{aligned}$$

and applying the change of variables (7), we obtain

$$\begin{aligned}\langle \nabla_\varepsilon \tilde P_\varepsilon, \tilde v\rangle_{H^{-1}(\Omega)^3, H_0^1(\Omega)^3} = & -\varepsilon(\eta_0-\eta_\infty)\int_{\widetilde\Omega_\varepsilon}(1+\lambda|\mathbb{D}_\varepsilon[\tilde u_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}_\varepsilon[\tilde u_\varepsilon] : D_\varepsilon \tilde R^\varepsilon \tilde v\,dx'dy_3 \\ & -\varepsilon\eta_\infty\int_{\widetilde\Omega_\varepsilon}\mathbb{D}_\varepsilon[\tilde u_\varepsilon] : D_\varepsilon \tilde R^\varepsilon \tilde v\,dx + \int_{\widetilde\Omega_\varepsilon} f'(x')\cdot (\tilde R^\varepsilon \tilde v)'\,dx'dy_3\,,\end{aligned} \tag{18}$$

where $\tilde R^\varepsilon \tilde v = R^\varepsilon v$ for any $\tilde v \in H_0^1(\Omega)^3$.

Finally, we estimate the right-hand side of (18) and give the estimate of the extended pressure $\tilde P_\varepsilon$.

Lemma 3.5. *There exists a positive constant C independent of ε, such that*

$$\|\tilde P_\varepsilon\|_{L^2(\Omega)} \le C\,, \quad \|\nabla_\varepsilon \tilde P_\varepsilon\|_{H^{-1}(\Omega)^3} \le C. \tag{19}$$

Proof. Applying the dilatation in (15), we have that $\tilde R^\varepsilon \tilde v$ satisfies the following estimate

$$\|\tilde R^\varepsilon \tilde v\|_{L^2(\widetilde\Omega_\varepsilon)^3} + \varepsilon\|D_\varepsilon \tilde R^\varepsilon \tilde v\|_{L^2(\widetilde\Omega_\varepsilon)^{3\times 3}} \le C\left(\|\tilde v\|_{L^2(\Omega)^3} + \varepsilon\|D_\varepsilon \tilde v\|_{L^2(\Omega)^{3\times 3}}\right), \tag{20}$$

and since $\varepsilon \ll 1$, we have

$$\|\tilde R^\varepsilon \tilde v\|_{L^2(\widetilde\Omega_\varepsilon)^3} \le C\|\tilde v\|_{H_0^1(\Omega)^3}, \quad \|D_\varepsilon \tilde R^\varepsilon \tilde v\|_{L^2(\widetilde\Omega_\varepsilon)^{3\times 3}} \le \frac{C}{\varepsilon}\|\tilde v\|_{H_0^1(\Omega)^3}. \tag{21}$$

Taking into account that $1 < r < 2$, we have the continuous embedding $L^2(\widetilde{\Omega}_\varepsilon) \subset L^{2r-2}(\widetilde{\Omega}_\varepsilon)$. Then, from Cauchy-Schwarz's inequality, we can deduce

$$\begin{aligned}
&\int_{\widetilde{\Omega}_\varepsilon} (1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}^\varepsilon \tilde{v}\, dx' dy_3 \\
&\leq C\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \|D_\varepsilon \tilde{R}^\varepsilon \tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} + C\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|^{r-1}_{L^{2r-2}(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \|D_\varepsilon \tilde{R}^\varepsilon \tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \\
&\leq C\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \|D_\varepsilon \tilde{R}^\varepsilon \tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} + C\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|^{r-1}_{L^{2}(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \|D_\varepsilon \tilde{R}^\varepsilon \tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}},
\end{aligned}$$

and using the last estimate in (14) and the last estimate of the dilated restricted operator given in (21), we obtain

$$\left|\varepsilon(\eta_0-\eta_\infty)\int_{\widetilde{\Omega}_\varepsilon} (1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}^\varepsilon \tilde{v}\, dx' dy_3\right| \leq C\|\tilde{v}\|_{H^1_0(\Omega)^3}. \tag{22}$$

Moreover, from Cauchy-Schwarz's inequality, the last estimate in (14), the assumption of f' given in (6) and estimates of the dilated restricted operator given in (21), we obtain

$$\left|\varepsilon\eta_\infty\int_{\widetilde{\Omega}_\varepsilon} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}^\varepsilon \tilde{v}\, dx\right| \leq C\varepsilon\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \|D_\varepsilon \tilde{R}^\varepsilon \tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\|\tilde{v}\|_{H^1_0(\Omega)^3}, \tag{23}$$

$$\left|\int_{\widetilde{\Omega}_\varepsilon} f'\cdot(\tilde{R}^\varepsilon\tilde{v})'\, dx' dy_3\right| \leq C\|\tilde{R}^\varepsilon \tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} \leq C\|\tilde{v}\|_{H^1_0(\Omega)^3}. \tag{24}$$

Then, taking into account (22)-(24) in (18), we get

$$\left|\langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v}\rangle_{H^{-1}(\Omega)^3, H^1_0(\Omega)^3}\right| \leq C\|\tilde{v}\|_{H^1_0(\Omega)^3}.$$

This implies the second estimate in (19) and then, using the Nečas inequality, there exists a representative $\tilde{P}_\varepsilon \in L^2_0(\Omega)$ such that

$$\|\tilde{P}_\varepsilon\|_{L^2(\Omega)} \leq C\|\nabla \tilde{P}_\varepsilon\|_{H^{-1}(\Omega)^3} \leq C\|\nabla_\varepsilon \tilde{P}_\varepsilon\|_{H^{-1}(\Omega)^3},$$

which implies the first estimate in (19). □

3.2 Adaptation of the unfolding method.

The change of variables (7) does not provide the information we need about the behavior of $\tilde{u}_\varepsilon$ in the microstructure associated to $\widetilde{\Omega}_\varepsilon$. To solve this difficulty, we use an adaptation introduced in [10] of the unfolding method from [23].

Let us recall that this adaptation of the unfolding method divides the domain $\widetilde{\Omega}_\varepsilon$ in cubes of lateral length ε and vertical length 1. Thus, given $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon) \in H^1_0(\Omega)^3 \times L^2_0(\Omega)$, we define $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ by

$$\hat{u}_\varepsilon(x',y) = \tilde{u}_\varepsilon\left(\varepsilon\kappa\left(\frac{x'}{\varepsilon}\right)+\varepsilon y', y_3\right), \quad \hat{P}_\varepsilon(x',y) = \tilde{P}_\varepsilon\left(\varepsilon\kappa\left(\frac{x'}{\varepsilon}\right)+\varepsilon y', y_3\right), \quad \text{a.e. } (x',y)\in\omega\times Y, \tag{25}$$

assuming $\tilde{u}_\varepsilon$ and $\tilde{P}_\varepsilon$ are extended by zero outside ω, where the function $\kappa : \mathbb{R}^2 \to \mathbb{Z}^2$ is defined by

$$\kappa(x') = k' \Longleftrightarrow x' \in Y'_{k',1}, \quad \forall\, k' \in \mathbb{Z}^2.$$

Remark 3.6. *We make the following comments:*

- *The function κ is well defined up to a set of zero measure in $\mathbb{R}^2$ (the set $\cup_{k'\in\mathbb{Z}^2}\partial Y'_{k',1}$). Moreover, for every $\varepsilon > 0$, we have*

$$\kappa\left(\frac{x'}{\varepsilon}\right) = k' \Longleftrightarrow x' \in Y'_{k',\varepsilon}.$$

- *For $k' \in \mathcal{K}_\varepsilon$, the restrictions of $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ to $Y'_{k',\varepsilon} \times Y$ does not depend on x', whereas as a function of y it is obtained from $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ by using the change of variables $y' = \frac{x'-\varepsilon k'}{\varepsilon}$, which transforms $Y_{k',\varepsilon}$ into Y.*

Following the proof of [10, Lemma 4.9], we have the following estimates relating $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ and $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$.

Lemma 3.7. *The sequence $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ defined by (25) satisfies the following estimates*

$$\begin{gathered}\|\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^3} \le \|\tilde{u}_\varepsilon\|_{L^2(\Omega)^3},\\ \|D_{y'}\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^{3\times 2}} \le \varepsilon\|D_{x'}\tilde{u}_\varepsilon\|_{L^2(\Omega)^{3\times 2}}, \quad \|\partial_{y_3}\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^3} \le \|\partial_{y_3}\tilde{u}_\varepsilon\|_{L^2(\Omega)^3},\\ \|\mathbb{D}_{y'}[\hat{u}_\varepsilon]\|_{L^2(\omega\times Y)^{3\times 2}} \le \varepsilon\|\mathbb{D}_{x'}[\tilde{u}_\varepsilon]\|_{L^2(\Omega)^{3\times 2}}, \quad \|\partial_{y_3}[\hat{u}_\varepsilon]\|_{L^2(\omega\times Y)^3} \le \|\partial_{y_3}[\tilde{u}_\varepsilon]\|_{L^2(\Omega)^3},\\ \|\hat{P}_\varepsilon\|_{L^2(\omega\times Y)} \le \|\tilde{P}_\varepsilon\|_{L^2(\Omega)}.\end{gathered} \tag{26}$$

Now, from estimates of the extended velocity (14) and pressure (19) together with Lemma 3.7, we have the following estimates for $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$.

Lemma 3.8. *There exists a constant $C > 0$ independent of ε, such that $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ defined by (25) satisfies*

$$\|\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^3} \le C\varepsilon, \quad \|D_y\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^{3\times 3}} \le C\varepsilon, \quad \|\mathbb{D}_y[\hat{u}_\varepsilon]\|_{L^2(\omega\times Y)^{3\times 3}} \le C\varepsilon, \tag{27}$$

$$\|\hat{P}_\varepsilon\|_{L^2(\omega\times Y)} \le C. \tag{28}$$

3.3 Compactness results.

We analyze the asymptotic behavior of sequences of the extension of $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ and $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$, when ε tends to zero.

Lemma 3.9. *There exist $\tilde{u} \in H^1_0(0,1;L^2(\omega)^3)$ where $\tilde{u}_3 = 0$, $\hat{u} \in L^2(\omega; H^1_\#(Y)^3)$, with $\hat{u} = 0$ on $\omega \times T$ such that $\int_Y \hat{u}(x', y)\,dy = \int_0^1 \tilde{u}(x', y_3)\,dy_3$ with $\int_Y \hat{u}_3(x', y)\,dy = 0$, such that*

$$\varepsilon^{-1}\tilde{u}_\varepsilon \rightharpoonup (\tilde{u}', 0) \text{ weakly in } H^1(0,1;L^2(\omega)^3), \tag{29}$$

$$\varepsilon^{-1}\hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ weakly in } L^2(\omega; H^1(Y)^3). \tag{30}$$

Moreover, $\tilde{u}$ and $\hat{u}$ satisfy the following divergence conditions

$$\mathrm{div}_{x'}\left(\int_0^1 \tilde{u}'(x', y_3)\,dy_3\right) = 0 \text{ in } \omega, \quad \left(\int_0^1 \tilde{u}'(x', y_3)\,dy_3\right)\cdot n = 0 \text{ in } \partial\omega, \tag{31}$$

$$\mathrm{div}_y\,\hat{u}(x', y) = 0 \text{ in } \omega\times Y_f, \quad \mathrm{div}_{x'}\left(\int_{Y_f} \hat{u}'(x', y)\,dy\right) = 0 \text{ in } \omega, \quad \left(\int_{Y_f} \hat{u}'(x', y)\,dy\right)\cdot n = 0 \text{ on } \partial\omega. \tag{32}$$

Proof. Arguing as in [10, Lemma 5.2.-(i)], we obtain convergence (29) and divergence condition (31). Moreover, proceeding similarly as in [10, Lemma 5.4.-(i)] we deduce convergence (30) and divergence conditions (32). □

Lemma 3.10. *For a subsequence of ε still denoted by ε, there exists $\tilde{P} \in L^2_0(\omega)$ such that*

$$\tilde{P}_\varepsilon \to \tilde{P} \text{ strongly in } L^2(\Omega), \tag{33}$$

$$\hat{P}_\varepsilon \to \tilde{P} \text{ strongly in } L^2(\omega\times Y). \tag{34}$$

Proof. The first estimate in (19) implies, up to a subsequence, the existence of $\tilde{P} \in L^2_0(\Omega)$ such that

$$\tilde{P}_\varepsilon \rightharpoonup \tilde{P} \text{ weakly in } L^2(\Omega). \tag{35}$$

Also, from the second estimate in (19), by noting that $\partial_{y_3}\tilde{P}_\varepsilon/\varepsilon$ also converges weakly in $H^{-1}(\Omega)$, we obtain $\partial_{y_3}\tilde{P} = 0$ and so $\tilde{P}$ is independent of y_3. Moreover, if we argue as in [20, Lemma 4.4], we have that the convergence (35) of the pressure $\tilde{P}_\varepsilon$ is in fact strong. Since $\tilde{P}_\varepsilon$ has null mean value in Ω, then $\tilde{P}$ has null mean value in ω, which concludes the proof of (33). Finally, the strong convergence of $\hat{P}_\varepsilon$ given in (34) follows from [24, Proposition 1.9-(ii)] and the strong convergence of $\tilde{P}_\varepsilon$ given in (33). □

3.4 Proof of Theorem 2.1.

The proof will be divided in two steps. In the first step, we obtain the homogenized behavior given by a coupled system, with a Carreau like macroviscosity, and in the second step we decouple it to obtain the macroscopic law.

Step 1. From Lemmas 3.9 and 3.10, we prove that the sequence $(\hat{u}_\varepsilon, \tilde{P}_\varepsilon)$ converges to $(\hat{u}, \tilde{P}) \in L^2(\omega; H^1_\#(Y_f)^3) \times (L^2_0(\omega) \cap H^1(\omega))$, which are the unique solutions of the following two-pressures generalized Newtonian Stokes problem with the non-linear viscosity following the *Carreau law* (1),

$$\left\{\begin{array}{rl} -\mathrm{div}_y\left(\eta_r\left(\mathbb{D}_y[\hat{u}]\right)\mathbb{D}_y[\hat{u}]\right) + \nabla_y\hat{\pi} = f' - \nabla_{x'}\tilde{P} & \text{in} \quad \omega\times Y_f, \\ \mathrm{div}_y\hat{u} = 0 & \text{in} \quad \omega\times Y_f, \\ \mathrm{div}_{x'}\left(\displaystyle\int_{Y_f}\hat{u}'\,dy\right) = 0 & \text{in } \omega, \\ \left(\displaystyle\int_{Y_f}\hat{u}'\,dy\right)\cdot n = 0 & \text{on } \partial\omega, \\ \hat{u} = 0 & \text{in } \omega\times T, \\ \hat{\pi}\in L^2(\omega; L^2_{0,\#}(Y_f)). & \end{array}\right. \tag{36}$$

Divergence conditions $(36)_{2,3,4}$ and condition $(36)_5$ follow from Lemma 3.9. To prove that $(\hat{u}, \tilde{P})$ satisfies the momentum equation given in (36), we choose a test function $v(x', y) \in \mathcal{D}(\omega; C^\infty_\#(Y)^3)$ with $v(x', y) = 0$ in $\omega\times T$ (thus, $v(x', x'/\varepsilon, y_3)$ belongs to $H^1_0(\widetilde{\Omega}_\varepsilon)^3$). Multiplying (8) by $v(x', x'/\varepsilon, y_3)$, integrating by parts, and taking into account the extension of $\tilde{u}_\varepsilon$ and $\tilde{P}_\varepsilon$, we have

$$\begin{aligned} &\varepsilon(\eta_0-\eta_\infty)\int_\Omega (1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : \left(\mathbb{D}_{x'}[v] + \varepsilon^{-1}\mathbb{D}_y[v]\right)\,dx'dy_3 \\ &+\varepsilon\eta_\infty\int_\Omega \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : \left(\mathbb{D}_{x'}[v] + \varepsilon^{-1}\mathbb{D}_y[v]\right)\,dx'dy_3 \\ &-\int_\Omega \tilde{P}_\varepsilon\left(\mathrm{div}_{x'}v' + \varepsilon^{-1}\mathrm{div}_y v\right)\,dx'dy_3 = \int_\Omega f'\cdot v'\,dx'dy + O_\varepsilon, \end{aligned}$$

where O_ε is a generic real sequence which tends to zero with ε and can change from line to line.

By the change of variables given in Remark 3.6, we obtain

$$\begin{aligned} &(\eta_0-\eta_\infty)\int_{\omega\times Y}(1+\lambda|\varepsilon^{-1}\mathbb{D}_y[\hat{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\left(\varepsilon^{-1}\mathbb{D}_y[\hat{u}_\varepsilon]\right) : \mathbb{D}_y[v]\,dx'dy \\ &+\eta_\infty\int_{\omega\times Y}\varepsilon^{-1}\mathbb{D}_y[\hat{u}_\varepsilon] : \mathbb{D}_y[v]\,dx'dy \\ &-\int_{\omega\times Y}\hat{P}_\varepsilon\,\mathrm{div}_{x'}v'\,dx'dy - \varepsilon^{-1}\int_{\omega\times Y}\hat{P}_\varepsilon\,\mathrm{div}_y v\,dx'dy = \int_{\omega\times Y} f'\cdot v'\,dx'dy + O_\varepsilon. \end{aligned} \tag{37}$$

Now, let us define the functional J_r by

$$J_r(v) = \frac{\eta_0-\eta_\infty}{r\lambda}\int_{\omega\times Y}(1+\lambda|\mathbb{D}_y[v]|^2)^{\frac{r}{2}}dx'dy + \frac{\eta_\infty}{2}\int_{\omega\times Y}|\mathbb{D}_y[v]|^2dx'dy.$$

Observe that J_r is convex and Gateaux differentiable on $L^2(\omega; H^1_\#(Y)^3)$ (see [16, Proposition 2.1 and Section 3] for more details) and $A_r = J'_r$ is given by

$$(A_r(w), v) = (\eta_0-\eta_\infty)\int_{\omega\times Y}(1+\lambda|\mathbb{D}_y[w]|^2)^{\frac{r}{2}-1}\mathbb{D}_y[w] : \mathbb{D}_y[v]dx'dy + \eta_\infty\int_{\omega\times Y}\mathbb{D}_y[w] : \mathbb{D}_y[v]dx'dy.$$

Applying [30, Proposition 1.1., p.158], in particular, we have that A_r is monotone, i.e.

$$(A_r(w) - A_r(v), w - v) \geq 0, \quad \forall w, v \in L^2(\omega; H^1_\#(Y)^3). \tag{38}$$

On the other hand, for all $\varphi \in \mathcal{D}(\omega; C^\infty_\#(Y)^3)$ satisfying the divergence conditions $\mathrm{div}_{x'} \int_Y \varphi'\, dy = 0$ in ω and $\mathrm{div}_y \varphi = 0$ in $\omega \times Y$, we choose v_ε defined by

$$v_\varepsilon = \varphi - \varepsilon^{-1}\hat{u}_\varepsilon,$$

as a test function in (37). Taking into account that $\mathrm{div}_\varepsilon \tilde{u}_\varepsilon = 0$, we get that $\varepsilon^{-1}\mathrm{div}_y \hat{u}_\varepsilon = 0$, and then we obtain

$$(A_r(\varepsilon^{-1}\hat{u}_\varepsilon), v_\varepsilon) - \int_{\omega\times Y} \hat{P}_\varepsilon\, \mathrm{div}_{x'} v'_\varepsilon\, dx'dy = \int_{\omega\times Y} f' \cdot v'_\varepsilon\, dx'dy + O_\varepsilon,$$

which is equivalent to

$$(A_r(\varphi) - A_r(\varepsilon^{-1}\hat{u}_\varepsilon), v_\varepsilon) - (A_r(\varphi), v_\varepsilon) + \int_{\omega\times Y} \hat{P}_\varepsilon\, \mathrm{div}_{x'} v'_\varepsilon\, dx'dy = -\int_{\omega\times Y} f' \cdot v'_\varepsilon\, dx'dy + O_\varepsilon.$$

Due to (38), we can deduce

$$(A_r(\varphi), v_\varepsilon) - \int_{\omega\times Y} \hat{P}_\varepsilon\, \mathrm{div}_{x'} v'_\varepsilon\, dx'dy \geq \int_{\omega\times Y} f' \cdot v'_\varepsilon\, dx'dy + O_\varepsilon,$$

i.e.

$$\begin{aligned}&(\eta_0 - \eta_\infty) \int_{\omega\times Y} (1 + \lambda|\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[v_\varepsilon] dx'dy + \eta_\infty \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[v_\varepsilon] dx'dy \\ &- \int_{\omega\times Y} \hat{P}_\varepsilon\, \mathrm{div}_{x'} v'_\varepsilon\, dx'dy \geq \int_{\omega\times Y} f' \cdot v'_\varepsilon\, dx'dy + O_\varepsilon.\end{aligned} \tag{39}$$

Now, we pass to the limit in every terms.

From convergence (30), passing to the limit when ε tends to zero, we have that the first and second terms converge to

$$(\eta_0 - \eta_\infty) \int_{\omega\times Y} (1 + \lambda|\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}]\, dx'dy + \eta_\infty \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}]\, dx'dy.$$

From convergences (30) and (34), the third term converges to

$$\int_{\omega\times Y} \tilde{P}\, \mathrm{div}_{x'}(\varphi' - \hat{u}')\, dx'dy.$$

Since $\tilde{P}$ does not depend on y, by using the divergences conditions $\mathrm{div}_{x'} \int_Y \varphi\, dy = 0$ and $(32)_2$, we have

$$\int_{\omega\times Y} \tilde{P}\, \mathrm{div}_{x'}(\varphi' - \hat{u}')\, dx'dy = \int_\omega \tilde{P}\, \mathrm{div}_{x'} \left(\int_Y (\varphi' - \hat{u}') dy \right) dx' = 0.$$

Thus, we deduce that the variational inequality (39) converges to the following one

$$\begin{aligned}&(\eta_0 - \eta_\infty) \int_{\omega\times Y} (1 + \lambda|\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}]\, dx'dy \\ &+\eta_\infty \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}]\, dx'dy \geq \int_{\omega\times Y} f' \cdot (\varphi' - \hat{u}')\, dx'dy,\end{aligned}$$

which by Minty's lemma, see [30, Chapter 3, Lemma 1.2], is equivalent to

$$-\mathrm{div}_y\left(\eta_r(\mathbb{D}_y[\hat{u}])\mathbb{D}_y[\hat{u}]\right) = f' \quad \text{in} \quad \omega \times Y.$$

By density

$$\begin{aligned}&(\eta_0-\eta_\infty)\int_{\omega\times Y}(1+\lambda|\mathbb{D}_y[\hat{u}]|^2)^{\frac{r}{2}-1}\mathbb{D}_y[\hat{u}]:\mathbb{D}_y[v]\,dx'dy\\&+\eta_\infty\int_{\omega\times Y}\mathbb{D}_y[\hat{u}]:\mathbb{D}_y[v]\,dx'dy=\int_{\omega\times Y}f'\cdot v'\,dx'dy,\end{aligned}\tag{40}$$

holds for every v in the Hilbert space $\mathcal{V}$ defined by

$$\mathcal{V}=\left\{\begin{array}{l}v(x',y)\in L^2(\omega;H^1_\#(Y)^3)\text{ such that}\\ \mathrm{div}_{x'}\left(\displaystyle\int_{Y_f}v(x',y)\,dy\right)=0\text{ in }\omega,\quad\left(\displaystyle\int_{Y_f}v(x',y)\,dy\right)\cdot n=0\text{ on }\partial\omega\\ \mathrm{div}_y v(x',y)=0\text{ in }\omega\times Y_f,\quad v(x',y)=0\text{ in }\omega\times T\end{array}\right\}.$$

Reasoning as in [1, Lemma 1.5], the orthogonal of $\mathcal{V}$, a subset of $L^2(\omega;H^{-1}_\#(Y)^3)$, is made of gradients of the form $\nabla_{x'}\tilde{\pi}(x')+\nabla_y\hat{\pi}(x',y)$, with $\tilde{\pi}(x')\in H^1(\omega)/\mathbb{R}$ and $\hat{\pi}(x',y)\in L^2(\omega;L^2_\#(Y_f)/\mathbb{R})$. Thus, integrating by parts, the variational formulation (40) is equivalent to the two-pressures generalized Newtonian Stokes problem (36). It remains to prove that $\tilde{\pi}$ coincides with pressure $\tilde{P}$. This can be easily done passing to the limit similarly as above by considering the test function φ, which is divergence-free only in y, and by identifying limits. It holds then that $\tilde{P}\in L^2_0(\omega)\cap H^1(\omega)$. From [20, Theorem 2], problem (36) admits a unique solution $(\hat{u},\hat{\pi},\tilde{P})\in L^2(\omega;H^1_\#(Y_f)^3)\times L^2(\omega;L^2_{0,\#}(Y_f))\times(L^2_0(\omega)\cap H^1(\omega))$ and then, the entire sequence $(\hat{u}_\varepsilon,\hat{P}_\varepsilon)$ converges to $(\hat{u},\tilde{P})$.

Step 2. In this step we give an approximation of the model (36), where the macroscopic scale is totally decoupled from the microscopic one. To do this, we seek a global filtration velocity of the form given in (10), i.e.

$$V(x')=\mathcal{U}(f'(x')-\nabla_{x'}\tilde{P}(x'))\quad\text{in }\omega,\tag{41}$$

where $\mathcal{U}:\mathbb{R}^2\to\mathbb{R}^3$ is a permeability function, not necessary linear, and $V(x')=\int_0^1\tilde{u}(x',y_3)\,dy_3=\int_Y\hat{u}(x',y)\,dy$ with $\mathrm{div}_{x'}V'=0$ in ω and $V'\cdot n=0$ on $\partial\omega$.

Using the idea from [21] to decouple the homogenized problems of *Carreau* type, for every $\xi'\in\mathbb{R}^2$ we consider the function $\mathcal{U}:\mathbb{R}^2\to\mathbb{R}^3$ given by

$$\mathcal{U}(\xi')=\int_{Y_f}w_{\xi'}(y)\,dy,$$

where $w_{\xi'}$ denotes the unique solution of the local Stokes problem given by (11), see [20, Theorem 2]. Thus, $(\hat{u},\hat{\pi})$ takes the form

$$\hat{u}(x',y)=w_{f'(x')-\nabla_{x'}\tilde{P}(x')}(y),\quad\hat{\pi}(x',y)=\pi_{f'(x')-\nabla_{x'}\tilde{P}(x')}(y)\quad\text{in }\omega\times Y,$$

and then, from the relation $V(x')=\int_Y\hat{u}(x',y)\,dy$ with $\int_Y\hat{u}_3(x',y)\,dy=0$ given in Lemma 3.9, we deduce the filtration velocity (41), where $V_3=0$. Moreover, from second and third conditions given in (32) together with (41), we deduce

$$\mathrm{div}_{x'}V'=0\quad\text{in }\omega,\quad V'\cdot n=0\quad\text{on }\partial\omega.$$

Since $V_3=0$, to simplify the notation, we redefine the definition of $\mathcal{U}$ by the expression given in (10) and then, we get $\mathcal{U}:\mathbb{R}^2\to\mathbb{R}^2$, which concludes the proof of (9). Finally, from [21, Theorem 1], the macroscopic problem (9) has a unique solution $(V,\tilde{P})\in L^2(\omega)^3\times(L^2_0(\omega)\cap H^1(\omega))$ and Theorem 2.1 is proved.

4 Numerical simulations

The objective of this section is to investigate numerically the behaviour of a flow of a Carreau fluid between two parallel plates, separated by a thin porous medium of width ε, as described in Section 2.

The viscosity (1) used in the Carreau law depends on four rheological parameters: η_0, η_∞, r and λ. Let us first specify our choice of parameters. In many applications (see for instance [18]), η_∞ is very small compared with η_0. For this reason, we arbitrarily fix $\eta_0 = 1$ and $\eta_\infty = 10^{-3}$ in the following numerical tests. As regards r and λ, we take $r \in \{1.3, 1.5, 1.7, 2\}$ and $\lambda \in \{1, 10, 100\}$. The value $r = 2$ corresponds to the Newtonian case of a fluid of constant viscosity $\eta_0 = 1$, that we consider as a reference case. Reducing the value of r and multiplying the value of λ by a factor 10 from one simulation to the other will give access to a large panel of nonlinear behaviours for the simulated fluid.

In order to avoid boundary effects on the lateral parts of the boundary of the domain Ω_ε, we assume that ω is a square $\omega = (-L, L)^2$ and impose periodic boundary conditions on $\partial Q_\varepsilon = \partial\omega \times (0, \varepsilon)$ in system (5). Hence, we consider the system

$$\left\{\begin{array}{rl} -\varepsilon \mathrm{div}\,(\eta_r(\mathbb{D}[u_\varepsilon])\mathbb{D}[u_\varepsilon]) + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \mathrm{div}\, u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial S_\varepsilon, \\ u_\varepsilon(x_1, -L) = u_\varepsilon(x_1, L), & x_1 \in (-L, L), \\ u_\varepsilon(-L, x_2) = u_\varepsilon(L, x_2), & x_2 \in (-L, L), \end{array}\right. \tag{42}$$

In this case, Lemma 3.9 needs to be slightly modified to take into account the periodicity on $\partial\omega$. More precisely, conditions (31) and (32) are replaced by

$$\mathrm{div}_{x'}\left(\int_0^1 \tilde{u}'(x', y_3)\, dy_3\right) = 0 \ \text{ in } \omega, \tag{43}$$

$$\mathrm{div}_y\, \hat{u}(x', y) = 0 \ \text{ in } \omega \times Y_f, \quad \mathrm{div}_{x'}\left(\int_{Y_f} \hat{u}'(x', y)\, dy\right) = 0 \ \text{ in } \omega. \tag{44}$$

The free divergence boundary conditions express the fact that for any function φ in $H^1(\omega)$, satisfying periodic boundary conditions on $\partial\omega$,

$$\int_\omega \left(\int_0^1 \tilde{u}'(x', y_3)\, dy_3\right) \varphi(x')\, dx' = 0 \quad \text{and} \quad \int_\omega \left(\int_{Y_f} \hat{u}'(x', y)\, dy\right) \varphi(x')\, dx' = 0. \tag{45}$$

Indeed, multiplying the condition $\mathrm{div}_\varepsilon \tilde{u}_\varepsilon = 0$ by such function φ, and integrating by parts over $\omega \times (0, 1)$, one obtains

$$-\int_\omega \Big(\int_0^1 \tilde{u}_\varepsilon'(x', y_3) dy_3\Big) \cdot \nabla_{x'}\varphi(x') dx' + \int_{\partial\omega} \Big(\int_0^1 \tilde{u}_\varepsilon'(x', y_3) dy_3\Big) \cdot n(x')\varphi(x')\, d\mathcal{H}^1(x') = 0$$

where $\mathcal{H}^1$ stands for the Hausdorff measure of dimension 1. By periodicity of the normal vector n, assuming that both $\tilde{u}_\varepsilon'$ and φ are periodic with respect to x', we see that the boundary term in the previous relation vanishes. Passing to the limit by (29), we deduce the first equality in (45). The second equality is proved analogously.

In the case of periodic lateral boundary conditions, the limit system (9) is replaced accordingly by

$$\left\{\begin{array}{l} V'(x') = \mathcal{U}\left(f'(x') - \nabla_{x'}\tilde{P}(x')\right), \quad V_3(x') = 0 \quad \text{in } \omega, \\ \mathrm{div}_{x'} V'(x') = 0 \ \text{ in } \omega, \end{array}\right. \tag{46}$$

where the permeability function $\mathcal{U}$ and the corresponding cell problem are respectively given by (10) and (11).

Throughout this numerical section, we assume that the flow is driven by a constant pressure gradient in the horizontal direction, which amounts to considering a constant external force $f = (f', 0)$ with $f' \in \mathbb{R}^2$. Such assumption is realistic in applications like enhanced oil recovery, where the flow, frequently described by the Carreau law in the engineering literature, is mainly driven by the pressure difference between the injection point and the well [18, Chapter 4]. Since f' is constant, one gets that $\tilde{P} \equiv 0$ and V' is also constant, given by $V' = \int_{Y_f} w'_{f'}(y)\,dy$ where $w'_{f'}$ is the solution to system (11) with $\xi' = f'$.

In order to compute the solution to system (11), we rely on a mixed formulation of the problem and solve it by a finite element method, using FreeFem++ software [28]. Since the system (11) is nonlinear (for $1 < r < 2$), we solve the corresponding mixed formulation by a fixed point algorithm (see for instance [34, Section 2.8]). We consider the Taylor-Hood approximation for the velocity-pressure pair, *i.e.* P_2 elements for the velocity field and P_1 elements for the pressure. It is well known that this choice is compatible with the Babuška-Brezzi condition [22, 26].

The three-dimensional mesh of the cell Y_f is obtained by constrained Delaunay tetrahedralization. The results that we present in the next subsections are obtained using approximately 8000 tetrahedra in the mesh of Y_f.

We propose to explore numerically the influence of the amplitude of the constant pressure gradient f', and of its orientation, on the mean filtration velocity V'. In our simulations, we will use different shapes for the inclusion T, that are presented in the next paragraph.

4.1 Geometry of the inclusion T

In the numerical simulations of system (11), we have chosen to consider two types of shapes for the horizontal projection T' of the solid inclusion $T = T' \times (0,1)$: ellipses and rectangles.

- Ellipses are built using two possible values for each semi-axis: 0.1 and 0.3. This leads to 4 different geometries: two disks of respective radius 0.1 and 0.3, and the ellipse of semi-major axis 0.3 and semi-minor axis 0.1, parallel to the x or the y axis (see Fig. 4). These shapes will be numbered E_1, E_2, E_3 and E_4 in the rest of this section.
- Similarly, we construct 4 rectangular shapes by selecting the length of opposite sides in $\{0.15, 0.35\}$ (see Fig. 5). These shapes are numbered R_1, R_2, R_3 and R_4.

These shapes are commonly used in the literature on flows in porous media, and offer a variety of geometric features, such as size, regularity and isotropy, that favor comparisons.

4.2 Influence of the amplitude of the imposed pressure gradient f'

In order to test the impact of the variations of $|f'|$, we have computed the mean filtration velocity V' associated to a pressure gradient f' directed by e_1, in the form $f' = (f_1, 0)$ with $f_1 \in (0,1)$. In that case, V' is also directed by e_1, and reads $V' = (V_1, 0)$. The results that we have obtained are plotted in Fig. 6 (in the case of elliptic inclusions E_1 to E_4) and in Fig. 7 (in the case of rectangular inclusions R_1 to R_4).

We can observe that the amplitude of the mean filtration velocity, which is simply equal to $|V'| = V_1$ in this setting, increases as r is diminished and λ is augmented. For $\lambda = 1$, V_1 remains very close to the reference value (Newtonian case $r = 2$). On the opposite, increasing λ up to $\lambda = 100$ affects strongly the behaviour of the fluid as a function of r. For instance, fixing $\lambda = 100$ and $f_1 = 1$, and taking $r = 1.3$ drastically increases the velocity V_1 with respect to the computed value for $r = 2$.

Comparing E_1 to E_4 and R_1 to R_4, we observe that, as expected, the filtration velocity is smaller for large volume inclusions than for small volume ones. Our results also confirm the intuition that aligning the obstacle in the sense of the flow (as E_2 and R_2), or perpendicularly to it (as E_3 and R_3), affects the filtration velocity: V_1 is much smaller for E_3 (resp. R_3) than for E_2 (resp. R_2). Let us notice that this effect is much more pronounced for rectangles that for ellipses, which is probably linked with the difference of regularity of the boundaries of the corresponding shapes.

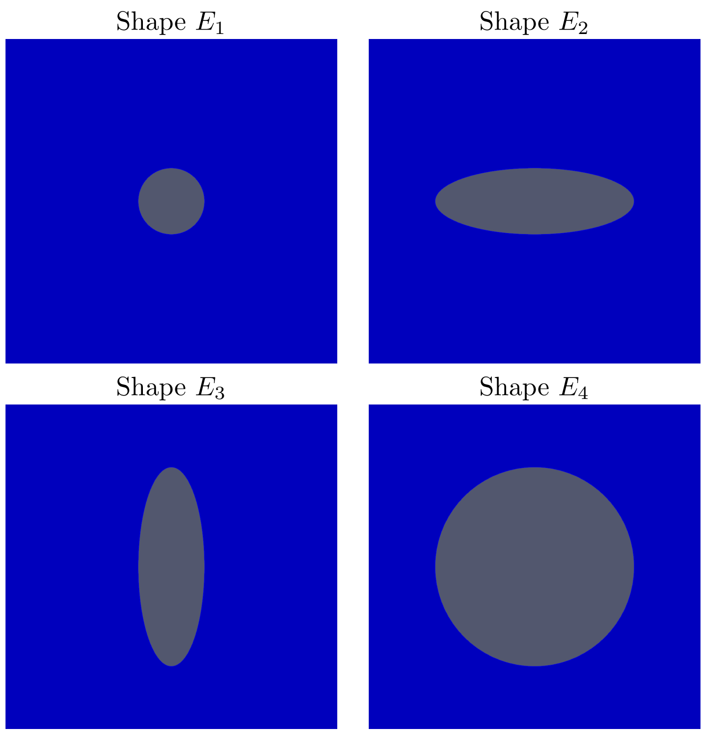

Figure 4: Representation of $2D$ reference cells Y', in the case of elliptic shapes for T' inclusions (in grey), surrounded by Y'_f (in blue). From left to right and top to bottom: disk or radius 0.1, ellipse of semi-major axis 0.3 and semi-minor axis 0.1, oriented along the x direction and the y direction resp., and disk of radius 0.3. These shapes are numbered E_1, E_2, E_3 and E_4.

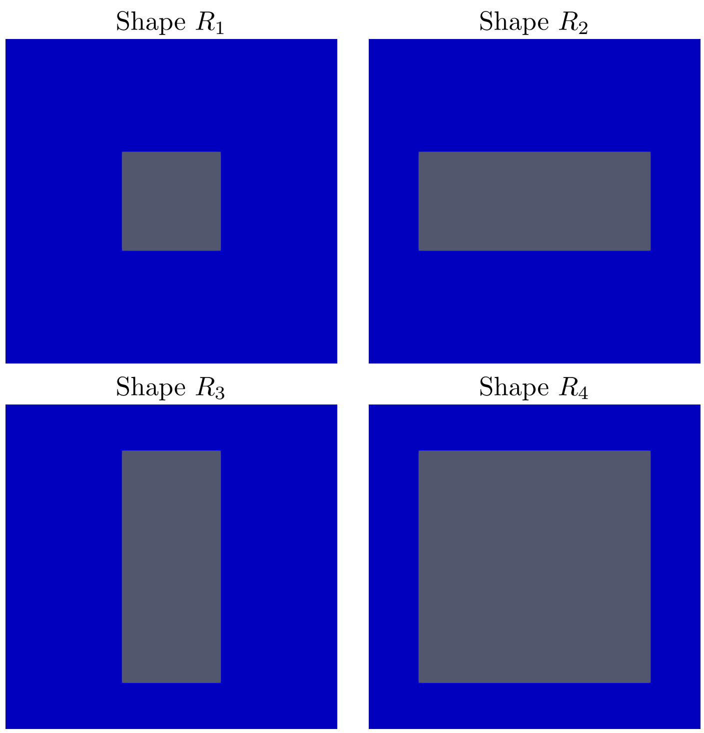

Figure 5: Representation of $2D$ reference cells Y', in the case of rectangular shapes for T' inclusions (in grey), surrounded by Y'_f (in blue). From left to right and top to bottom: square or side 0.15, rectangle of length 0.35 and width 0.15, oriented along the x direction and the y direction resp., and square of side 0.35. These shapes are numbered R_1, R_2, R_3 and R_4.

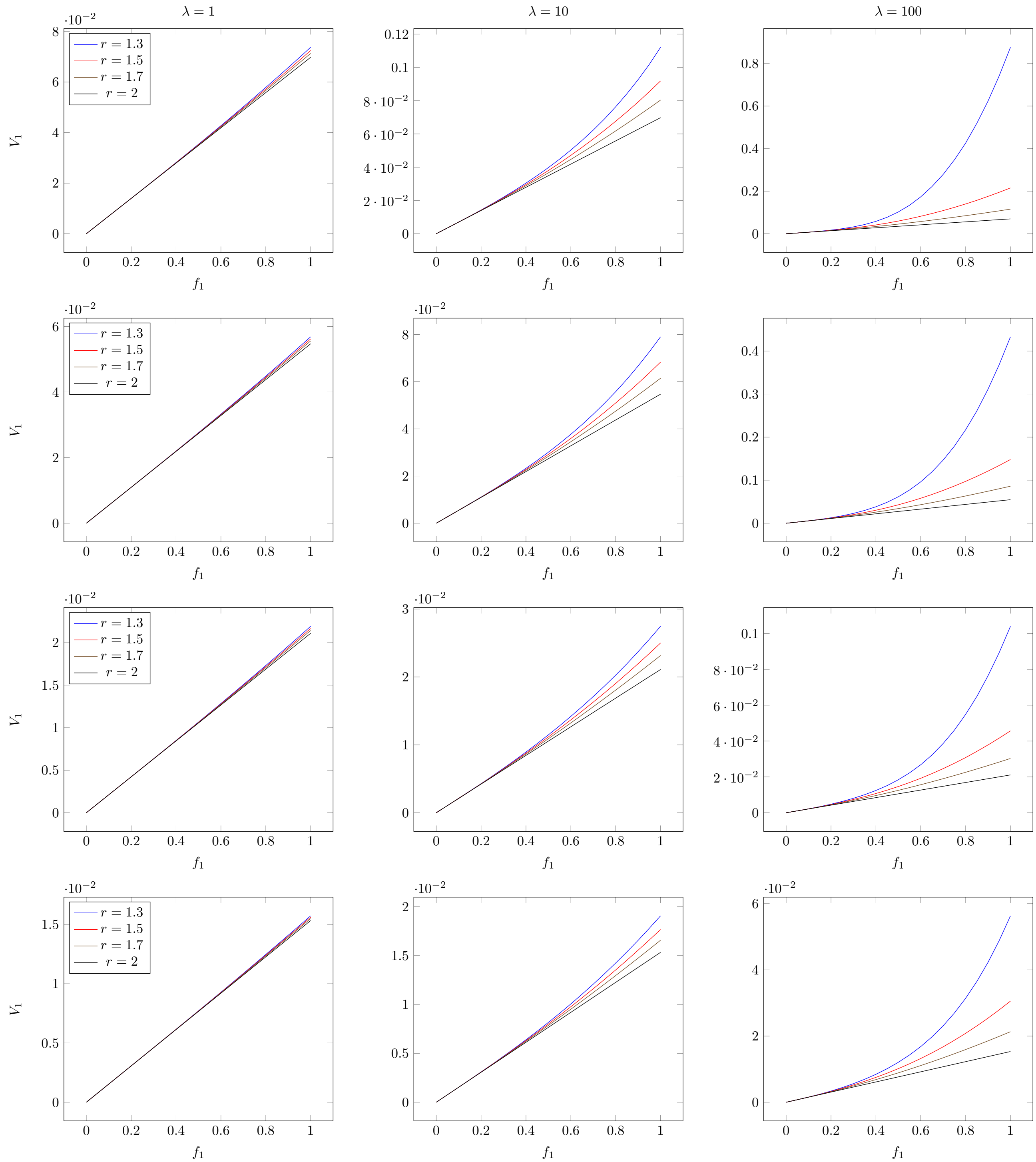

Figure 6: Component V_1 of the mean filtration velocity V' V' plotted against f_1, with $f' = (f_1, 0)$, for $r \in \{1.3, 1.5, 1.7, 2\}$, in the case of elliptic inclusions E_1 (first line), E_2 (second line), E_3 (third line) and E_4 (fourth line). From left to right: $\lambda = 1, \lambda = 10, \lambda = 100$.

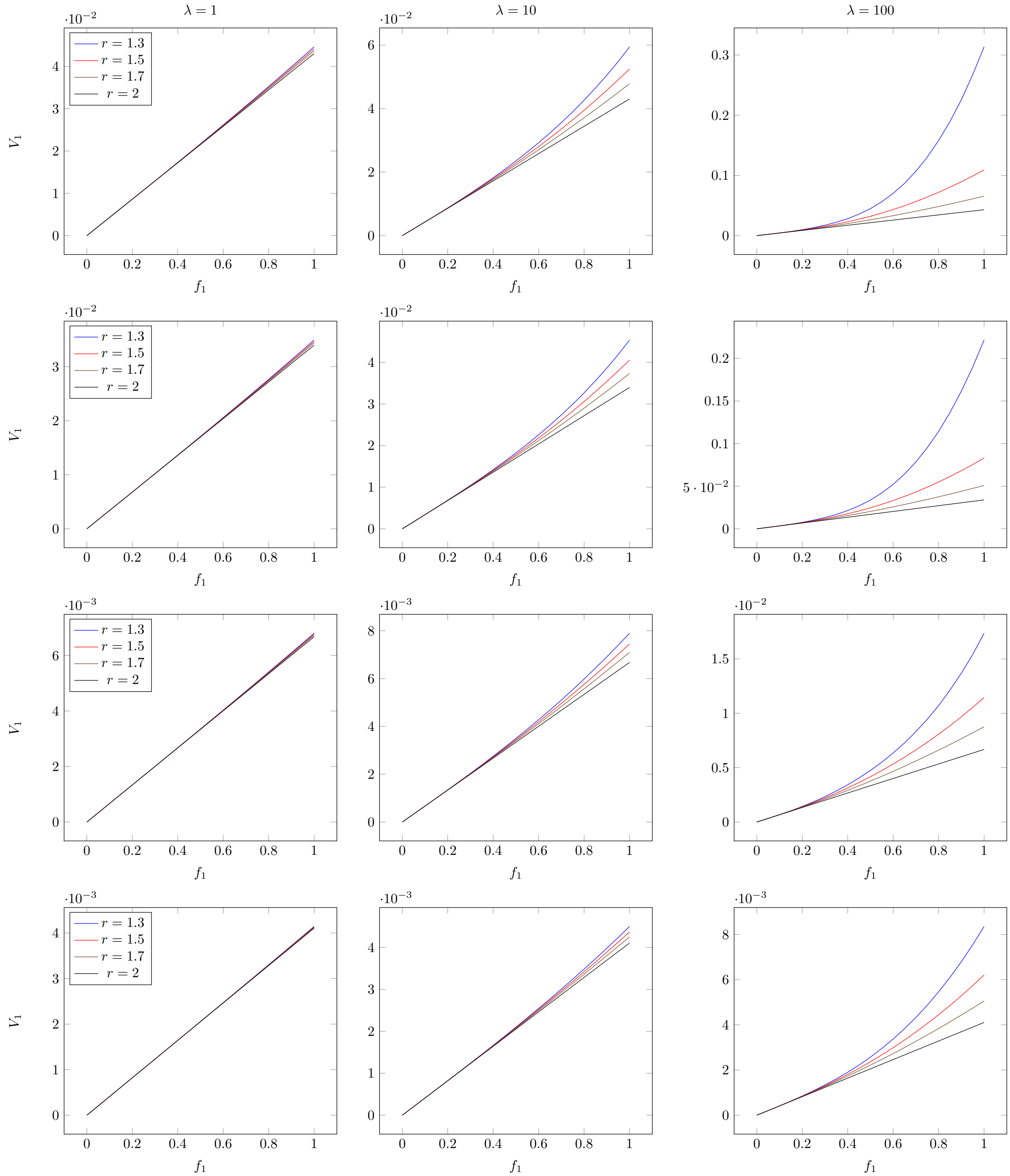

Figure 7: Component V_1 of the mean filtration velocity V' plotted against f_1, with $f' = (f_1, 0)$, for $r \in \{1.3, 1.5, 1.7, 2\}$, in the case of rectangular inclusions R_1 (first line), R_2 (second line), R_3 (third line) and R_4 (fourth line). From left to right: $\lambda = 1, \lambda = 10, \lambda = 100$.

4.3 Dependency on the orientation of the pressure gradient

We complete our previous results on the amplitude of f' by testing the influence of its orientation, on both the orientation and amplitude of V'. To this aim, we consider a family of pressure gradients $f' = (\cos\theta, \sin\theta)$ and the anisotropic shapes of inclusions E_2 and R_2. By symmetry, we can restrict the angular parameter θ to $\theta \in [0, \pi/2]$. The results that we obtain are represented in Fig. 8 (for ellipse E_2) and in Fig. 9 (for rectangle R_2).

As observed in the previous paragraph, for $\lambda = 1$, the value of r value does not seem to have a strong impact on the behaviour of the model. For both shapes E_2 and R_2, the value and orientation of V' remain essentially the same as in the reference Newtonian case $r = 2$. On the other hand, for $\lambda = 100$, the influence of r becomes clear. For example, in case of inclusion R_2, the maximal amplitude of the velocity (achieved for $\theta = 0$) is close to 0.2 with $r = 1.3$, while it was merely equal to 0.03 for $r = 2$. A comparable change of order of magnitude can be observed for E_2.

As regards the orientation of the vector V', we may observe a slight difference of orientation in case $\lambda = 100$, between the reference case $r = 2$ and the other cases $r \in \{1.3, 1.5, 1.7\}$, concerning mainly the values of angle θ close to $\pi/2$. However, from a global perspective, it appears that modifiying λ and r does not have a definite influence on the response of the system to a rotation of f'.

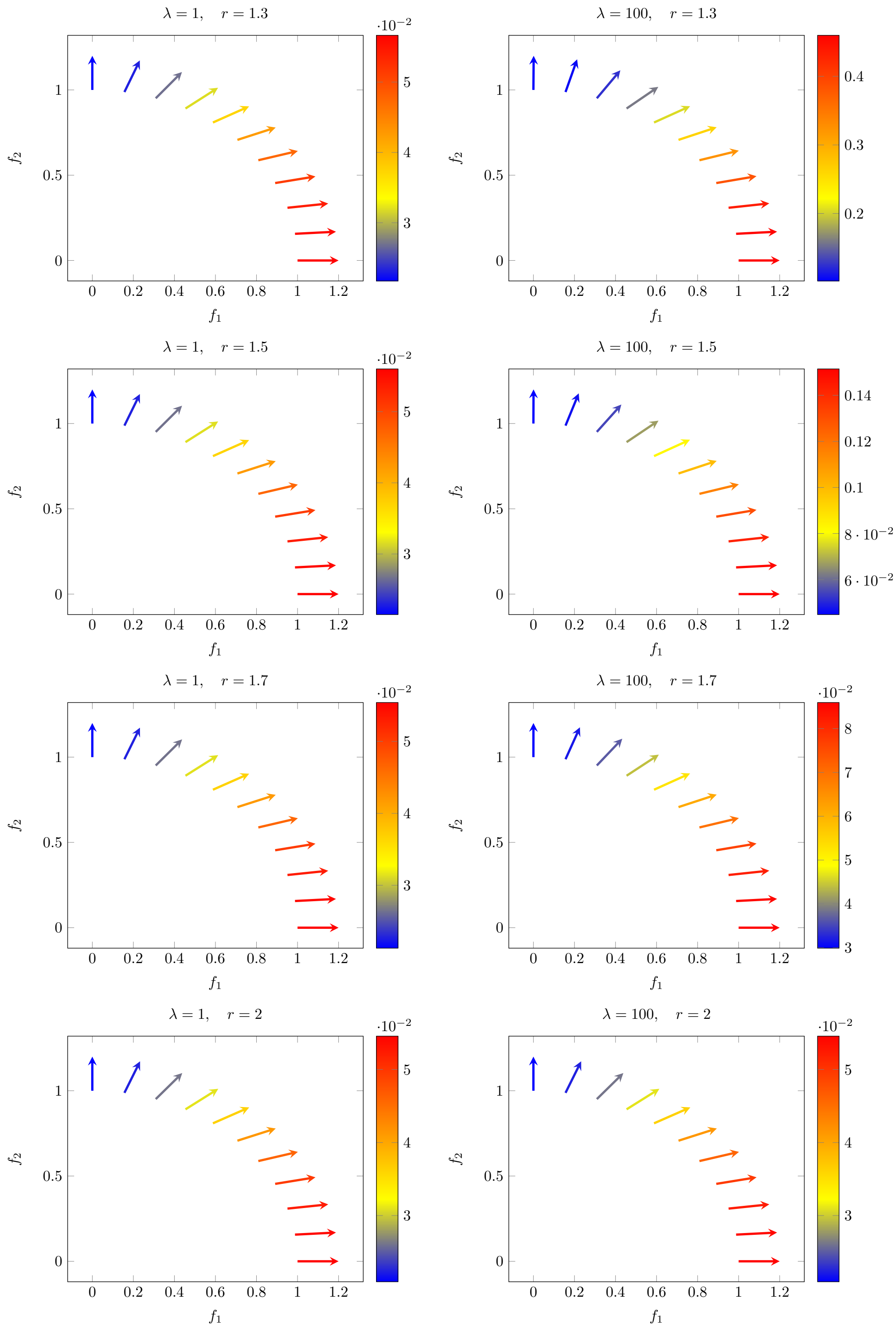

Figure 8: Representation of V' when f' is a unit vector of the form $(f_1, f_2) = (\cos\theta, \sin\theta)$ with $\theta \in [0, \pi/2]$, in the case of an elliptic inclusion E_2. Each vector V' is represented by a vector of length 0.2, localized at point (f_1, f_2) and colored according to $|V'|$. The left column corresponds to $\lambda = 1$ and the right one to $\lambda = 100$. Each line from top to bottom corresponds respectively to $r = 1.3$, $r = 1.5$, $r = 1.7$ and $r = 2$.

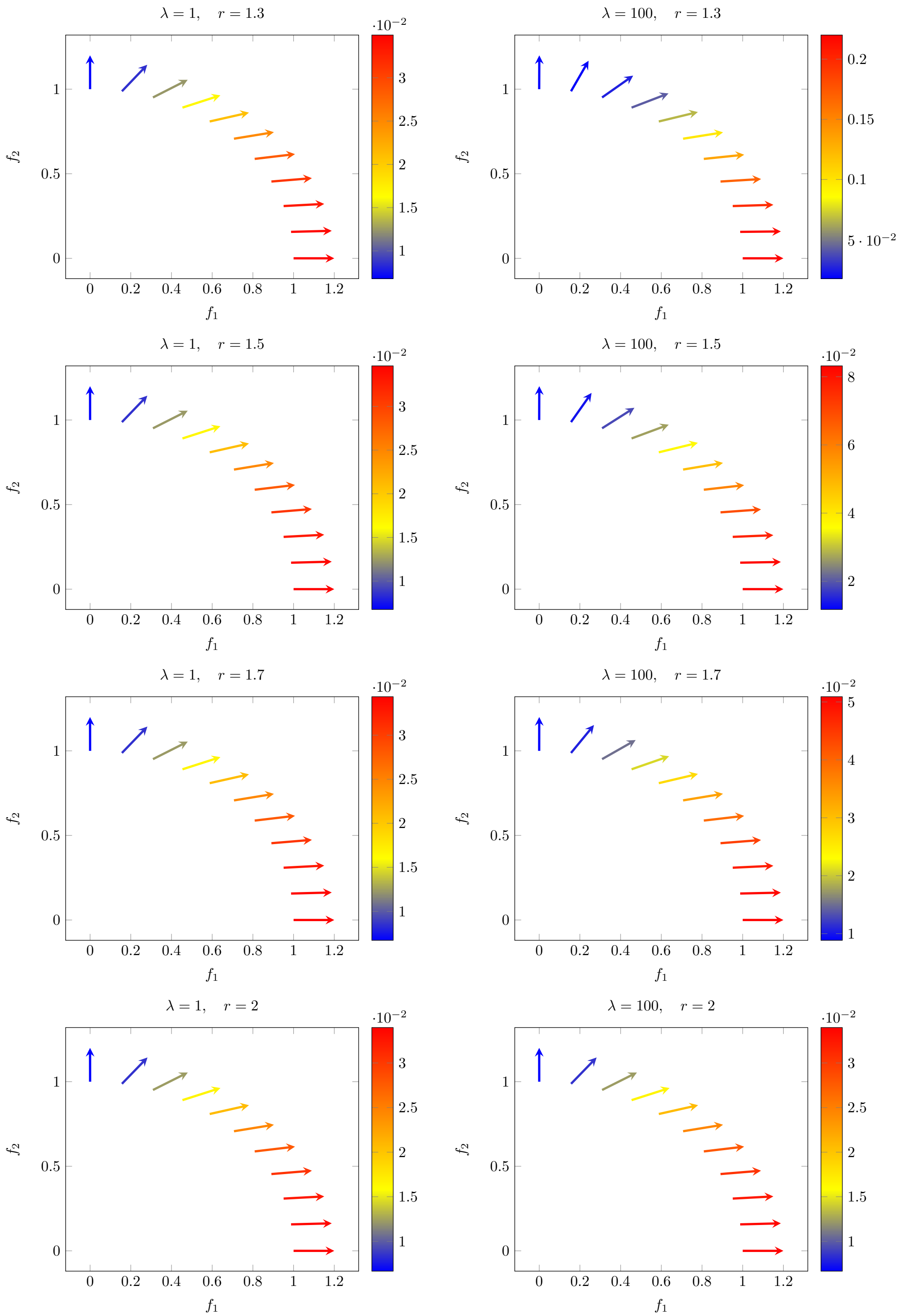

Figure 9: Representation of V' when f' is a unit vector of the form $(f_1, f_2) = (\cos\theta, \sin\theta)$ with $\theta \in [0, \pi/2]$, in the case of a rectangular inclusion R_2. Each vector V' is represented by a vector of length 0.2, localized at point (f_1, f_2) and colored according to $|V'|$. The left column corresponds to $\lambda = 1$ and the right one to $\lambda = 100$. Each line from top to bottom corresponds respectively to $r = 1.3$, $r = 1.5$, $r = 1.7$ and $r = 2$.

References

[1] G. Allaire, One-phase Newtonian flow, in: *Homogenization and Porous Media, Interdisciplinary Applied Mathematics Series*, **6**, Springer-Verlag, New York, (1997), 77-94.

[2] J.F. Agassant, P. Avenas, M. Vincent, B. Vergnes, P.J. Carreau, Polymer Processing, Principles and Modelling, Carl Hanser Verlag, Munich, 2017.

[3] M. Anguiano, Darcy's laws for non-stationary viscous fluid flow in a thin porous medium, *Math. Meth. Appl. Sci.*, **40**, No. 8 (2017), 2878–2895.

[4] M. Anguiano, On the non-stationary non-Newtonian flow through a thin porous medium, *ZAMM-Z. Angew. Math. Mech.*, **97**, (2017), 895–915.

[5] M. Anguiano, Derivation of a quasi-stationary coupled Darcy-Reynolds equation for incompressible viscous fluid flow through a thin porous medium with a fissure, *Math. Meth. Appl. Sci.*, **40**, (2017), 4738-4757.

[6] M. Anguiano, Homogenization of a non-stationary non-Newtonian flow in a porous medium containing a thin fissure, *Eur. J. Appl. Math.*, **30** (2), (2019), 248–277.

[7] M. Anguiano, Reaction-Diffusion Equation on Thin Porous Media, *Bull. Malays. Math. Sci. Soc.*, **44**, (2021), 3089–3110.

[8] M. Anguiano and R. Bunoiu, On the flow of a viscoplastic fluid in a thin periodic domain, in *Integral Methods in Science and Engineering* (eds. C. Constanda and P. Harris), Birkhäuser, Cham, (2019), 15–24.

[9] M. Anguiano and R. Bunoiu, Homogenization of Bingham flow in thin porous media, *Netw. Heterog. Media*, **15**, No. 1 (2020), 87–110.

[10] M. Anguiano and F.J. Suárez-Grau, Homogenization of an incompressible non-Newtonian flow through a thin porous medium, *Z. Angew. Math. Phys.*, (2017), 68:45.

[11] M. Anguiano and F.J. Suárez-Grau, Derivation of a coupled Darcy-Reynolds equation for a fluid flow in a thin porous medium including a fissure *Z. Angew. Math. Phys.*, (2017), 68: 52.

[12] M. Anguiano and F.J. Suárez-Grau, Analysis of the effects of a fissure for a non-Newtonian fluid flow in a porous medium, *Communications in Mathematical Sciences*, **16**, (2018), 273 – 292.

[13] M. Anguiano and F.J. Suárez-Grau, The transition between the Navier-Stokes equations to the Darcy equation in a thin porous medium, *Mediterr. J. Math.*, (2018), 15:45.

[14] M. Anguiano and F.J. Suárez-Grau, Newtonian fluid flow in a thin porous medium with non-homogeneous slip boundary conditions *Networks and Heterogeneous Media*, **14**, (2019) 289-316.

[15] M. Anguiano and F.J. Suárez-Grau, Lower-dimensional nonlinear Brinkman's law for non-Newtonian flows in a thin porous medium, *Mediterr. J. Math.*, (2021), 18:175.

[16] J. Baranger and K. Najib, Analyse numerique des ecoulements quasi-Newtoniens dont la viscosite obeit a la loi puissance ou la loi de carreau, *Numer. Math.*, **58** (1990), 35-49.

[17] H.A. Barnes, J.F. Hutton and K. Walters, *An introduction to rheology*, Amsterdam, Elsevier, 1989.

[18] R.B. Bird, R.C. Armstrong and O. Hassager, *Dynamics of Polymeric Liquids, Volume 1: Fluid Mechanics*, Wiley, 1977.

[19] F. Boughanim and R. Tapiéro, Derivation of the two-dimensional Carreau law for a quasi-newtonian fluid flow through a thin slab, *Appl. Anal.*, **57**, (1995), 243–269.

[20] A. Bourgeat and A. Mikelić, Homogenization of a polymer flow through a porous medium, *Nonlinear Anal.*, **26**, (1996), 1221–1253.

[21] A. Bourgeat, O. Gipouloux and E. Marušić-Paloka, Filtration law for polymer flow through porous media, *Multiscale Model. Sim.*, **1**, (2003), 432–457.

[22] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag New York, Inc. (1991).

[23] D. Cioranescu, A. Damlamian and G. Griso, The periodic unfolding method in homogenization, *SIAM J. Math. Anal.*, **40**, No. 4 (2008), 1585–1620.

[24] D. Cioranescu, A. Damlamian and G. Griso, *The periodic unfolding method: theory and applications to partial differential problems*, Series in Contemporary Mathematics, **3**, Springer, Singapore, 2018.

[25] J. Fabricius, J.G. I. Hellström, T.S. Lundström, E. Miroshnikova and P. Wall, Darcy's law for flow in a periodic thin porous medium confined between two parallel plates, *Transp. Porous Med.*, **115**, (2016), 473–493.

[26] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, Springer Publishing Company, Incorporated (2011).

[27] T. Götz and H.A. Parhusip, On an asymptotic expansion for Carreau fluids in porous media , *Journal of Engineering Mathematics,* **51** (2005) 351–365.

[28] F. Hecht, New Development in FreeFem++, *J. Numer. Math.* **20** (2012) 251–265.

[29] N. F. Jouybari and T. S. Lundström, Investigation of Post-Darcy Flow in Thin Porous Media, *Transport in Porous Media*, **138**, (2021), 157–184.

vertical beams,

[30] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[31] A. Mikelić, Non-Newtonian Flow, in: *Homogenization and Porous Media, Interdisciplinary Applied Mathematics Series*, **6**, Springer-Verlag, New York, (1997), 45-68.

[32] A. Mikelić, An introduction to the homogenization modeling of non-Newtonian and electrokinetic flows in porous media, in: *Non-Newtonian Fluid Mechanics and Complex Flows. Lecture Notes in Mathematics*, **2212**, Springer, 2018.

[33] M. Prat and T. Agaësse, Thin Porous Media, in *Handbook of Porous Media* (ed. Kambiz Vafai), CRC PressEditors, (2015), 89-112.

[34] P. Saramito, *Complex fluids: Modeling and Algorithms, Mathématiques et Applications*, Springer, 2016.

[35] F.J. Suárez-Grau, Mathematical modeling of micropolar fluid flows through a thin porous medium, J. Eng. Math., 126: 7 (2021).

[36] L. Yeghiazarian, K. Pillai and R. Rosati, Thin Porous Media, *Transp. Porous Med.*, **115**, (2016), 407-410.

[37] Y. Zhengan and Z. Hongxing, Homogenization of a stationary Navier-Stokes flow in porous medium with thin film, *Acta Math. Sci.*, **28**, (2008), 963–974.