Statistical solutions and Kolmogorov entropy for the lattice long-wave-short-wave resonance equations in weighted space *

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Abstract

This article studies the lattice long-wave-short-wave resonance equations in weighted spaces. The authors first prove the global well-posedness of the initial value problem and the existence of the pullback attractor for the process generated by the solution mappings in the weighted space. Then they establish that the process possesses a family of invariant Borel probability measures supported by the pullback attractor. Afterwards, they verify that this family of Borel probability measures satisfies the Liouville theorem and is a statistical solution of the lattice long-wave-short-wave resonance equations. Finally, they prove an upper bound of the Kolmogorov entropy of the statistical solution.

Keywords: Lattice dynamical system; Long-wave-short-wave resonance equations; Weighted space; Statistical solution; Kolmogorov entropy.

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1 Introduction

In this article, we investigate the following initial value problem of the lattice long-wave-short-wave resonance equations in weighted space:

$$i\dot{u}_m - (Au)_m - u_m v_m + i\alpha u_m = f_m(t), \ \ m \in \mathbb{Z}, \ \ t > \tau, \tag{1.1}$$

$$\dot{v}_m + \beta v_m + \gamma(B(|u|^2))_m = g_m(t), \ \ m \in \mathbb{Z}, \ \ t > \tau, \tag{1.2}$$

$$u_m(\tau) = u_{m,\tau}, \quad v_m(\tau) = v_{m,\tau}, \ \ m \in \mathbb{Z}, \ \ t > \tau, \tag{1.3}$$

where $\mathbb{Z}$ is the set of integer numbers, $u_m = u_m(t) \in \mathbb{C}$ and $v_m = v_m(t) \in \mathbb{R}$ are the unknown functions, $\mathbb{C}$ and $\mathbb{R}$ are the set of complex and real numbers respectively, i is the imaginary unit such that $i^2 = -1$, α, β, γ are positive constants, $|u|^2 = (|u_m|^2)_{m\in\mathbb{Z}}$, A and B are the linear operators defined as

$$(Au)_m = 2u_m - u_{m-1} - u_{m+1}, \ \ u = (u_m)_{m\in\mathbb{Z}},$$
$$(Bu)_m = u_{m+1} - u_m, \ \ u = (u_m)_{m\in\mathbb{Z}}.$$

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Equations (1.1)-(1.2) can be regarded as a discrete approximation of the following non-autonomous long-wave-short-wave resonance equations with respect to the space variable $x \in \mathbb{R}$:

$$iu_t + u_{xx} - uv + i\alpha u = f(x,t), \tag{1.4}$$

$$v_t + \beta v + \gamma(|u|^2)_x = g(x,t). \tag{1.5}$$

Equations (1.4)-(1.5) were first derived from the study of surface waves with gravity and capillary mode interactions, as well as in the analysis of the internal waves and Rosby waves (see [2, 12]), where the unknown complex-valued function $u(x,t)$ stands for the envelop of the short wave, and the unknown real-valued function $v(x,t)$ denotes the amplitude of the short wave. The complex-valued function $f(x,t)$ and the real-valued function $g(x,t)$ are both time-dependent external sources. In plasma physics, the long-wave-short-wave resonance equations explains the high-frequency electron plasma resonance and the associated low-frequency ion density perturbations. Equations (1.4)-(1.5) have been extensively studied. For instance, the well-posedness of the Cauchy problem was investigated in [4,17]; the orbital stability of solitary waves was investigated in [13]; the existence of global attractors and random attractors was proved in [18, 19].

The purpose of this article is to investigate the probability of solutions to the lattice long-wave-short-wave resonance equations (1.1)-(1.2) in weighted space. Over the past two decades, various attractors have been the subject of numerous investigations for lattice dynamical systems (LDSs). For example, the global attractors, exponential attractors and their fractal dimension, pullback attractor, uniform attractors for the first-order and second-order LDSs were investigated in [25, 39–41]; [42] researched the random exponential attractor for the stochastic LDSs; the uniform global attractor, pullback attractor and random attractor in weighted space were studied in [1, 14, 15]. For the lattice long-wave-short-wave resonance equations (1.1)-(1.2), the existence of the attractors and kernel sections was verified in [20, 23, 28]; the existence of invariant Borel measures was established in [26]. However, to the best of our knowledge, there is no reference investigating the statistical solution of equations (1.1)-(1.2) in weighted space.

The concept of statistical solutions and invariant measures comes from the study of turbulence in statistical physics (see [10, 11, 24, 27]). In reality, most of the main physical quantities of turbulence (such as velocity field and energy) show strong characteristic after ensemble average with respect to time or space. There are two prevalent definitions of statistical solution, called Foias-Prodi statistical solution [10] and Vishik-Fursikov statistical solution [24], that were originally formulated respectively to describe the probability distribution of solutions in phase space and temporal-spatial space for the incompressible Navier-Stokes equations.

Nowadays, the term "statistical solutions" is usually used as a strict mathematical concept to characterize the probability distribution of solutions of evolutionary equations in the corresponding space. There are several papers studying the invariant measures and statistical solutions of some typical dissipative systems. For instance, reference [8] proved sufficient conditions for the existence of invariant measures for general dissipative semigroups, and this result was extended by [22] to the general dissipative processes; references [5, 6] presented an abstract framework concerning the theory of statistical solutions and trajectory statistical solutions for general evolution equations. Recently, reference [31] established sufficient conditions for the existence of trajectory statistical solutions of the general evolution equations and the result was applied to some typical evolutionary equations (see [16, 30, 33–35]). In addition, references [32, 36] investigated the statistical solutions for the 2D non-autonomous magneto-micropolar fluids and Klein-Gordon-Schrödinger equations, and [37] studied the invariant sample measures for the 2D stochastic Navier-Stokes equations.

The main results of this article are the existence of statistical solutions and upper bound estimation of its Kolmogorov entropy for the lattice long-wave-short-wave resonance equations (1.1)-(1.2) in weighted space $L^2_\rho \times \ell^2_\rho$, where

$$\ell^2_\rho = \Big\{ u = (u_m)_{m\in\mathbb{Z}} : \sum_{m\in\mathbb{Z}} \rho_m u_m^2 < +\infty, \ \ u_m \in \mathbb{R} \Big\}, \tag{1.6}$$

$$L^2_\rho = \Big\{ u = (u_m)_{m\in\mathbb{Z}} : \sum_{m\in\mathbb{Z}} \rho_m |u_m|^2 < +\infty, \ \ u_m \in \mathbb{C} \Big\}, \tag{1.7}$$

here $\rho : \mathbb{Z} \to (0, +\infty), m \to \rho(m) = \rho_m$ are weight functions satisfying some conditions that will be specified in next section. Recall that [26] proved the existence and uniqueness of a family of invariant Borel probability measures for the problem (1.1)-(1.3) in the space $\ell^2 \times L^2$, where

$$\ell^2 = \Big\{ u = (u_m)_{m\in\mathbb{Z}} : \sum_{m\in\mathbb{Z}} u_m^2 < +\infty, \ \ u_m \in \mathbb{R} \Big\}, \tag{1.8}$$

$$L^2 = \Big\{ u = (u_m)_{m\in\mathbb{Z}} : \sum_{m\in\mathbb{Z}} |u_m|^2 < +\infty, \ \ u_m \in \mathbb{C} \Big\}. \tag{1.9}$$

Obviously, the spaces $L^2_\rho \times \ell^2_\rho$ and $L^2 \times \ell^2$ are the same provided that $\rho_m \equiv 1, m = 1, 2, \cdots$. Thus, the result within this article concerning the existence of invariant Borel probability measures is a generalization of that of [26]. However, some new difficulties, together with some new phenomena, will arise when we investigate the existence of statistical solutions and its Kolmogorov ε-entropy in the weighted space $L^2_\rho \times \ell^2_\rho$. Firstly, the weighted functions will produce some additional difficulties when we establish the so-called τ-continuity of the generated process, and this τ-continuity plays the vital role when we construct the statistical solution. Secondly, we discover that the weighted functions influence the Kolmogorov ε-entropy of the statistical solution. In fact, we reveal that the upper bound of the Kolmogorov ε-entropy of the statistical solutions decreases with respect to the weighted functions.

The rest of the article is arranged as follows. In the next section, we prove the global well-posedness of the problem (1.1)-(1.3) in the weighted space. In Section 3, we establish the existence of the bounded pullback absorbing set and the pullback attractor for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ generated by the solution mappings of problem (1.1)-(1.3). In Section 4, we verify the existence of the statistical solutions for equations (1.1)-(1.2). Finally, we give the definition of Kolmogorov ε-entropy for the statistical solution and estimate its upper bound.

2 Global well-posedness

In this section, we will prove the global well-posedness of problem (1.1)-(1.3).

We first introduce some hypotheses of the weighted functions $\rho : \mathbb{Z} \to (0, +\infty), m \to \rho(m) = \rho_m$ that adopted in the weighted spaces ℓ^2_ρ and L^2_ρ.

(H1) Assume that the weighted functions ρ_m are decreasing with respect to $|m|$, and that for all $m \in \mathbb{Z}$ there holds

$$0 < c_* < \rho(m) = \rho_m < c^* < +\infty, \tag{2.1}$$

$$\rho(m \pm 1) = \rho_{m\pm1} \leqslant c_1 \rho(m), \tag{2.2}$$

$$|\rho(m \pm 1) - \rho(m)| \leqslant c_2 \rho(m), \tag{2.3}$$

where c_*, c^*, c_1 and c_2 are positive constants. Moreover, the positive constants α in (1.1), c_1 and c_2 satisfy

$$\varrho := \alpha - c_1 c_2 - c_2 > 0. \tag{2.4}$$

We now illustrate the existence of the weight functions $\rho(m)$ satisfying (H1). Indeed, let us consider the weight function

$$\rho(x) = (1 + \omega^2 x^2)^{-\kappa} + c_*, \ \ x \in \mathbb{R},$$

where $c_* > 0$, $\omega > 0$ and $\kappa > 1/2$. Pick $c^* = c_* + 1$, then the right side of (2.1) is satisfied. It is clearly that $\dfrac{\rho(x \pm 1)}{\rho(x)} \in C(\mathbb{R})$ and

$$\lim_{x \longrightarrow \pm\infty} \frac{\rho(x \pm 1)}{\rho(x)} = \lim_{|m| \longrightarrow \infty} \frac{\rho(m \pm 1)}{\rho(m)} = 1,$$

thus $\sup\limits_{x\in\mathbb{R}} \dfrac{\rho(x \pm 1)}{\rho(x)} < +\infty$. Noticing that

$$\frac{\rho(1-1)}{\rho(1)} = \frac{1 + c_*}{(1 + \omega^2)^{-\kappa} + c_*} = \frac{\rho(-1+1)}{\rho(1)} > 1,$$

we take $c_1 = \sup\limits_{m\in\mathbb{Z}} \dfrac{\rho(m \pm 1)}{\rho(m)}$ and obtain (2.2) with $c_1 > 1$. At the same time, for any $m \in \mathbb{Z} \setminus \{0\}$, by using the mean value theorem we see that there exists $\vartheta_m > 0$ such that

$$|\rho(m \pm 1) - \rho(m)| = |\rho'(m \pm \vartheta_m)| \leqslant \kappa\omega\rho(m \pm \vartheta_m) \leqslant \kappa\omega \max\{\rho(m+1), \rho(m-1)\} \leqslant c_1\kappa\omega\rho(m).$$

For $m = 0$, we have

$$|\rho(\pm 1) - \rho(0)| = |(1 + \omega^2)^{-\kappa} - 1| < 1 < \rho(0).$$

Choose $c_2 = \max\{c_1\kappa\omega, 1\}$, then (2.3) is satisfied.

For the sake of brevity, we use X_ρ to denote ℓ^2_ρ or L^2_ρ and let it be equipped with the inner product and norm as

$$(u, v)_\rho = \sum_{m\in\mathbb{Z}} \rho_m u_m \bar{v}_m, \ \ \|u\|^2_\rho = (u, u)_\rho, \ \ u = (u_m)_{m\in\mathbb{Z}}, \ \ v = (v_m)_{m\in\mathbb{Z}} \in X_\rho,$$

where $\bar{v}_m$ is the conjugate of v_m. Similarly, we denote ℓ^2 or L^2 by X, whose inner product and norm are given by:

$$(u, v) = \sum_{m\in\mathbb{Z}} u_m \bar{v}_m, \ \ \|u\|^2 = (u, u), \ \ u = (u_m)_{m\in\mathbb{Z}}, \ \ v = (v_m)_{m\in\mathbb{Z}} \in X,$$

where $\bar{v}_m$ is the conjugate of v_m. Let

$$E_\rho = L^2_\rho \times \ell^2_\rho,$$

and equip it with the inner product and norm as: for any two elements $\psi^{(k)} = (u^{(k)}, v^{(k)})^T \in E_\rho$ $(k = 1, 2)$,

$$(\psi^{(1)}, \psi^{(2)})_{E_\rho} = (u^{(1)}, u^{(2)})_\rho + (v^{(1)}, v^{(2)})_\rho = \sum_{m\in\mathbb{Z}} \rho_m \big(u^{(1)}_m \bar{u}^{(2)}_m + v^{(1)}_m v^{(2)}_m\big),$$
$$\|\psi\|^2_{E_\rho} = (\psi, \psi)_{E_\rho} = \sum_{m\in\mathbb{Z}} \rho_m |\psi_m|^2 = \sum_{m\in\mathbb{Z}} \rho_m \big(|u_m|^2 + v^2_m\big), \ \ \forall \psi \in E_\rho,$$

where $\bar{u}^{(2)}_m$ stands for the conjugate of $u^{(2)}_m$. Obviously, E_ρ is the weighted Hilbert space.

Besides the operators A and B introduced in the Introduction, we define the operator B^* on X_ρ and X as

$$(B^*u)_m = u_{m-1} - u_m, \ \ u = (u_m)_{m\in\mathbb{Z}}.$$

We can check that operators A, B and B^* are all bounded linear operators from X_ρ to X_ρ. Indeed, using (2.1)-(2.3), we obtain

$$\begin{aligned}
\|Bu\|_\rho^2 &= \sum_{m\in\mathbb{Z}} \rho_m |(Bu)_m|^2 = \sum_{m\in\mathbb{Z}} \rho_m |u_{m+1} - u_m|^2 \\
&\leqslant 2\sum_{m\in\mathbb{Z}} \rho_m \big(|u_{m+1}|^2 + |u_m|^2\big) = 2\sum_{m\in\mathbb{Z}} \rho_m |u_{m+1}|^2 + 2\sum_{m\in\mathbb{Z}} \rho_m |u_m|^2 \\
&= 2\sum_{m\in\mathbb{Z}} \rho_{m-1} |u_m|^2 + 2\sum_{m\in\mathbb{Z}} \rho_m |u_m|^2 \leqslant 2\sum_{m\in\mathbb{Z}} c_1\rho_m |u_m|^2 + 2\sum_{m\in\mathbb{Z}} \rho_m |u_m|^2 \\
&= 2(1+c_1)\|u\|_\rho^2, \ \ \forall u = (u_m)_{m\in\mathbb{Z}} \in X_\rho. \qquad (2.5)
\end{aligned}$$

Analogously,

$$\|B^*\|_\rho \leqslant \sqrt{2(1+c_1)}, \quad \|A\|_\rho \leqslant \sqrt{8(1+c_1)}. \qquad (2.6)$$

Now, we write $u = (u_m)_{m\in\mathbb{Z}}$, $|u|^2 = (|u_m|^2)_{m\in\mathbb{Z}}$, $v = (v_m)_{m\in\mathbb{Z}}$, $f(t) = (f_m(t))_{m\in\mathbb{Z}}$, $g(t) = (g_m(t))_{m\in\mathbb{Z}}$, $u_\tau = (u_{m,\tau})_{m\in\mathbb{Z}}$, $v_\tau = (v_{m,\tau})_{m\in\mathbb{Z}}$, and put problem (1.1)-(1.3) as follows:

$$i\dot{u} - Au - uv + i\alpha u = f(t), \ \ t > \tau, \qquad (2.7)$$

$$\dot{v} + \beta v + \gamma B(|u|^2) = g(t), \ \ t > \tau, \qquad (2.8)$$

$$u(\tau) = u_\tau, \ \ v(\tau) = v_\tau, \ \ \tau \in \mathbb{R}. \qquad (2.9)$$

Further, we write $\psi = (u,v)^T$, $F(\psi,t) = \big(-iuv - if(t), g(t) - \gamma B(|u|^2)\big)^T$, and express problem (2.7)-(2.9) as an abstract nonautonomous first-order ODE with respect to time t in E_ρ:

$$\dot{\psi} + \Theta\psi = F(\psi,t), \ \ t > \tau, \qquad (2.10)$$

$$\psi(\tau) = \psi_\tau = (u_\tau, v_\tau)^T \in E_\rho, \ \ \tau \in \mathbb{R}, \qquad (2.11)$$

where I is the identity operator and

$$\Theta = \begin{pmatrix} \alpha I + iA & 0 \\ 0 & \beta I \end{pmatrix}. \qquad (2.12)$$

To ensure the global well-posedness of problem (2.10)-(2.11), we suppose that the functions $f(t) = (f_m(t))_{m\in\mathbb{Z}}$ and $g(t) = (g_m(t))_{m\in\mathbb{Z}}$ satisfy the following conditions.

(H2) Assume $f(t) = (f_m(t))_{m\in\mathbb{Z}} \in C(\mathbb{R}, L_\rho^2)$, $g(t) = (g_m(t))_{m\in\mathbb{Z}} \in C(\mathbb{R}, \ell_\rho^2)$. Moreover, we suppose that there is some continuous function $K(\cdot)$ on the real line, bounded on intervals of the form $(-\infty, t)$, such that

$$\int_{-\infty}^t e^{\sigma s}\|f(s)\|_\rho^2 \mathrm{d}s < e^{(\frac{\sigma}{2}+\delta)t}K(t) < +\infty, \ \ t \in \mathbb{R}, \ \ 0 < \delta < \frac{\sigma}{2} := \min\big\{\frac{\varrho}{4}, \frac{\beta}{4}\big\}, \qquad (2.13)$$

and that

$$\int_{-\infty}^t e^{\sigma s}\|g(s)\|_\rho^2 \mathrm{d}s < +\infty, \ \ t \in \mathbb{R}. \qquad (2.14)$$

For the existence of functions f and g satisfying (H2), one can refer to [26]. In fact, let $\|f(s)\|_\rho^2 \leqslant Me^{\kappa s}$ for all real s, with constants $M > 0$ and $\kappa \geqslant \delta - \frac{\sigma}{2}$. Then

$$\int_{-\infty}^s e^{\sigma s}\|f(s)\|_\rho^2 \mathrm{d}s < e^{(\frac{\sigma}{2}+\delta)s}K(s),$$

with

$$K(s) = \frac{M}{\sigma + \kappa} e^{(\frac{\sigma}{2} + \kappa - \delta)s}.$$

Thus, choosing $\kappa < 0, \kappa = 0$ or $\kappa > 0$, we allow different behavior of the external force f near infinities. The estimation (2.13) will play the essential role when we investigate the existence of a bounded pullback $\mathcal{D}_\sigma$-absorbing set for the process $\{U(t,\tau)\}_{t \geqslant \tau}$ in E_ρ.

Lemma 2.1. *Let assumptions* (H1)-(H2) *hold. Then for any given initial data* $\psi_\tau = (u_\tau, v_\tau)^T \in E_\rho$*, there exists a unique local solution* $\psi(t) = (u(t), v(t))^T \in E_\rho$ *of problem* (2.10)-(2.11) *such that*

$$\psi(\cdot) \in C([\tau, T_0), E_\rho) \cap C^1((\tau, T_0), E_\rho), \tag{2.15}$$

for some $T_0 > \tau$*. Moreover, if* $T_0 < +\infty$*, then* $\lim\limits_{t \to T_0^-} \|\psi(t)\|_{E_\rho} = +\infty$*.*

Proof. It is not difficult to check that Θ is a bounded linear operator that maps E_ρ into itself, and $F(\cdot,\cdot)$ maps $E_\rho \times \mathbb{R}$ into E_ρ. Now let $\mathcal{B}$ be a bounded set in E_ρ and $\psi^{(k)} = (u^{(k)}, v^{(k)})^T \in \mathcal{B}$ $(k = 1, 2)$, denote by $L(\mathcal{B}) = \sup\limits_{\psi \in \mathcal{B}} \|\psi\|^2_{E_\rho}$. Then for any $t \in \mathbb{R}$, we have

$$F(\psi^{(1)}, t) - F(\psi^{(2)}, t) = \big(-i(u^{(1)}v^{(1)} - u^{(2)}v^{(2)}), -\gamma(B(|u^{(1)}|^2) - B(|u^{(2)}|^2))\big)^T \in E_\rho.$$

It is obvious that for every $\psi = (u, v)^T \in E_\rho$ there holds $\|\psi\|^2_{E_\rho} = \|u\|^2_\rho + \|v\|^2_\rho$. Therefore, using (2.5) and some computations, we arrive at

$$\begin{aligned}
&\big\|F(\psi^{(1)}, t) - F(\psi^{(2)}, t)\big\|^2_{E_\rho} \\
=&\big\|u^{(1)}v^{(1)} - u^{(2)}v^{(2)}\|^2_\rho + \big\|\gamma\big(B(|u^{(1)}|^2) - B(|u^{(2)}|^2)\big)\big\|^2_\rho \\
\leqslant&\big\|(u^{(1)}v^{(1)} - u^{(2)}v^{(2)})\big\|^2_\rho + \gamma^2\big\|B\big\|^2_\rho\big\||u^{(1)}|^2 - |u^{(2)}|^2\big\|^2_\rho \\
\leqslant&\big\|u^{(1)}(v^{(1)} - v^{(2)}) + v^{(2)}(u^{(1)} - u^{(2)})\big\|^2_\rho + 2(1 + c_1)\gamma^2\big\||u^{(1)}|^2 - |u^{(2)}|^2\big\|^2_\rho \\
\leqslant&2\big\|u^{(1)}(v^{(1)} - v^{(2)})\big\|^2_\rho + 2\big\|v^{(2)}(u^{(1)} - u^{(2)})\big\|^2_\rho \\
&+ 2(1 + c_1)\gamma^2\big\||u^{(1)}| + |u^{(2)}|\big\|^2_\rho\big\||u^{(1)}| - |u^{(2)}|\big\|^2_\rho \\
\leqslant&2L(\mathcal{B})\big\|\psi^{(1)} - \psi^{(2)}\big\|^2_{E_\rho} + 8(1 + c_1)\gamma^2 L(\mathcal{B})\big\|\psi^{(1)} - \psi^{(2)}\big\|^2_{E_\rho} \\
=&\big(2 + 8(1 + c_1)\gamma^2\big)L(\mathcal{B})\big\|\psi^{(1)} - \psi^{(2)}\big\|^2_{E_\rho},
\end{aligned} \tag{2.16}$$

where $c_1 > 0$ is given by (2.2). By equation (2.16), we see that the map $F(\cdot,\cdot) : E_\rho \times \mathbb{R} \to E_\rho$ is locally Lipschitz with respect to $\psi \in E_\rho$. Using the classical theory of ODEs, we obtain the results of Lemma 2.1. $\square$

We next prove that the local solution obtained in Lemma 2.1 exists globally, by showing that the solution will not blow up at any finite time.

Lemma 2.2. *Let assumptions* (H1)-(H2) *hold, and* $\psi(t) = (u(t), v(t))^T \in E_\rho$ *be the solution of problem* (2.10)-(2.11) *corresponding to initial data* $\psi_\tau = (u_\tau, v_\tau)^T \in E_\rho$ *at initial time* τ*. Then*

$$\|u(t)\|^2_\rho \leqslant e^{-\varrho(t-\tau)}\|u_\tau\|^2_\rho + \frac{e^{-\varrho t}}{\alpha}\int_\tau^t e^{\varrho s}\|f(s)\|^2_\rho \mathrm{d}s, \ \ \forall t > \tau. \tag{2.17}$$

Proof. Taking the imaginary part of the inner product $(L^2_\rho, (\cdot,\cdot)_\rho)$ of (2.7) with u gives

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|u(t)\|^2_\rho + \alpha\|u(t)\|^2_\rho =&\mathbf{Im}(f(t), u(t))_\rho + \mathbf{Im}(Au, u)_\rho \\
\leqslant&\frac{\alpha}{2}\|u(t)\|^2_\rho + \frac{1}{2\alpha}\|f(t)\|^2_\rho + \mathbf{Im}(Au, u)_\rho, \ \ t > \tau.
\end{aligned} \tag{2.18}$$

Write $w_m(t) = \rho_m u_m(t)$, $w(t) = (w_m(t))_{m\in\mathbb{Z}}$. Then by direct computations, we have

$$(Au, u)_\rho = \sum_{m\in\mathbb{Z}} (Bu)_m (B\bar{w})_m, \tag{2.19}$$

$$|(B\bar{w})_m - \rho_m (B\bar{u})_m| = |\bar{u}_{m+1}(\rho_{m+1} - \rho_m)| \leqslant c_2 \rho_m |u_{m+1}|, \tag{2.20}$$

$$\begin{aligned}
\mathbf{Im}(Au, u)_\rho &= \mathbf{Im} \sum_{m\in\mathbb{Z}} (Bu)_m (B\bar{w})_m \\
&= \mathbf{Im} \sum_{m\in\mathbb{Z}} \Big\{ \rho_m (Bu)_m (B\bar{u})_m + (Bu)_m [(B\bar{w})_m - \rho_m (B\bar{u})_m] \Big\} \\
&= \mathbf{Im} \sum_{m\in\mathbb{Z}} (Bu)_m (\rho_{m+1} - \rho_m) \bar{u}_{m+1} = \mathbf{Im} \sum_{m\in\mathbb{Z}} (\rho_{m+1} - \rho_m)(u_{m+1} - u_m)\bar{u}_{m+1} \\
&\leqslant c_2 \sum_{m\in\mathbb{Z}} \rho_m |u_{m+1}||u_m| \leqslant \frac{c_2}{2} \sum_{m\in\mathbb{Z}} (\rho_m |u_{m+1}|^2 + \rho_m |u_m|^2) \\
&\leqslant \frac{c_1 c_2}{2} \|u\|_\rho^2 + \frac{c_2}{2} \|u\|_\rho^2 = \frac{1}{2}(c_1 c_2 + c_2)\|u\|_\rho^2.
\end{aligned} \tag{2.21}$$

Using (2.18) and (2.21), we arrive at

$$\frac{\mathrm{d}}{\mathrm{dt}} \|u(t)\|_\rho^2 + \varrho \|u(t)\|_\rho^2 \leqslant \frac{1}{\alpha} \|f(t)\|_\rho^2, \tag{2.22}$$

where $\varrho = \alpha - c_1 c_2 - c_2 > 0$ is given by (2.4). Applying Gronwall's inequality to (2.22), we deduce (2.17) and complete the proof. $\square$

Lemma 2.3. *Let assumptions* (H1)-(H2) *hold, and* $\psi(t) = (u(t), v(t))^T \in E_\rho$ *be the solution of problem* (2.10)-(2.11) *corresponding to initial data* $\psi_\tau = (u_\tau, v_\tau)^T \in E_\rho$ *at initial time* τ*. Then*

$$\begin{aligned}
\|\psi(t)\|_{E_\rho}^2 \leqslant& \|\psi_\tau\|_{E_\rho}^2 e^{-\sigma(t-\tau)} + c_3 e^{-\sigma t} \int_\tau^t e^{\sigma s} \big(\|f(s)\|_\rho^2 + \|g(s)\|_\rho^2\big) \mathrm{d}s \\
&+ c_3 e^{-\sigma t} \int_\tau^t e^{\sigma s} \|u(s)\|_\rho^4 \mathrm{d}s, \quad \forall t \geqslant \tau,
\end{aligned} \tag{2.23}$$

where σ *is given in* (2.13) *and* $c_3 := \max\big\{\frac{1}{\alpha}, \frac{2}{\beta}, \frac{4\gamma^2(1+c_1)}{\beta}\big\}$.

Proof. Taking the real part of the inner product of (2.10) with $\psi(t) = (u(t), v(t))^T$ in E_ρ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \|\psi\|_{E_\rho}^2 + \mathbf{Re}(\Theta\psi, \psi)_{E_\rho} = \mathbf{Re}(F(\psi, t), \psi)_{E_\rho}, \quad \forall t \geqslant \tau. \tag{2.24}$$

We first estimate the term $\mathbf{Re}(\Theta\psi, \psi)_{E_\rho}$. Direct computations gives

$$\mathbf{Re}(\Theta\psi, \psi)_{E_\rho} = \mathbf{Re}((\alpha u + iAu, \beta v)^T, (u, v)^T)_{E_\rho} = \alpha \|u\|_\rho^2 + \beta \|v\|_\rho^2 - \mathbf{Im}(Au, u)_\rho. \tag{2.25}$$

Note that we have denoted by $w_m(t) = \rho_m u_m(t)$ and $w(t) = (w_m(t))_{m\in\mathbb{Z}}$. Using (2.2) and the similar derivations as those as (2.21), we have

$$
\begin{aligned}
&-\mathbf{Im}(Au,u)_\rho = -\mathbf{Im}\sum_{m\in\mathbb{Z}}(Bu)_m(B\bar{w})_m\\
=&-\mathbf{Im}\sum_{m\in\mathbb{Z}}\Big\{\rho_m(Bu)_m(B\bar{u})_m+(Bu)_m[(B\bar{w})_m-\rho_m(B\bar{u})_m]\Big\}\\
=&-\mathbf{Im}\sum_{m\in\mathbb{Z}}\Big\{(Bu)_m[(B\bar{w})_m-\rho_m(B\bar{u})_m]\Big\} = -\mathbf{Im}\sum_{m\in\mathbb{Z}}(Bu)_m(\bar{w}_{m+1}-\bar{w}_m-\rho_m(B\bar{u})_m)\\
=&-\mathbf{Im}\sum_{m\in\mathbb{Z}}(Bu)_m(\rho_{m+1}-\rho_m)\bar{u}_{m+1} = -\mathbf{Im}\sum_{m\in\mathbb{Z}}(\rho_{m+1}-\rho_m)(u_{m+1}-u_m)\bar{u}_{m+1}\\
\geqslant&-c_2\mathbf{Im}\sum_{m\in\mathbb{Z}}\rho_m(u_{m+1}-u_m)\bar{u}_{m+1}\\
\geqslant&-c_2\sum_{m\in\mathbb{Z}}\rho_m|u_{m+1}||u_m| \geqslant -\frac{c_2}{2}\sum_{m\in\mathbb{Z}}(\rho_m|u_{m+1}|^2+\rho_m|u_m|^2)\\
=&-\frac{c_2}{2}\sum_{m\in\mathbb{Z}}\rho_{m-1}|u_m|^2-\frac{c_2}{2}\sum_{m\in\mathbb{Z}}\rho_m|u_m|^2 = -\frac{c_2}{2}\sum_{m\in\mathbb{Z}}\rho_{m-1}|u_m|^2-\frac{c_2}{2}\|u\|_\rho^2\\
\geqslant&-\frac{c_2}{2}\sum_{m\in\mathbb{Z}}c_1\rho_m|u_m|^2-\frac{c_2}{2}\|u\|_\rho^2\\
=&-\frac{c_1c_2}{2}\|u\|_\rho^2-\frac{c_2}{2}\|u\|_\rho^2 = -\frac{1}{2}(c_1c_2+c_2)\|u\|_\rho^2.
\end{aligned}
\tag{2.26}
$$

Inserting (2.26) into (2.25) yields

$$
\mathbf{Re}(\Theta\psi,\psi)_{E_\rho} \geqslant \alpha\|u\|_\rho^2+\beta\|v\|_\rho^2-\frac{1}{2}(c_1c_2+c_2)\|u\|_\rho^2. \tag{2.27}
$$

For the term $\mathbf{Re}(F(\psi,t),\psi)_{E_\rho}$, we use Cauchy's inequality to derive

$$
\begin{aligned}
\mathbf{Re}\big(F(\psi,t),\psi\big)_{E_\rho} =&\mathbf{Re}\big((-iuv-if(t),g(t)-\gamma B(|u|^2))^T,(u,v)^T\big)_{E_\rho}\\
=&\mathbf{Im}(f(t),u)_\rho+(g(t),v)_\rho-(\gamma B(|u|^2),v)_\rho\\
\leqslant&\frac{\alpha}{2}\|u(t)\|_\rho^2+\frac{1}{2\alpha}\|f(t)\|_\rho^2+\frac{\beta}{4}\|v(t)\|_\rho^2+\frac{1}{\beta}\|g(t)\|_\rho^2\\
&+\frac{\beta}{4}\|v(t)\|_\rho^2+\frac{\gamma^2}{\beta}\|B(|u|^2)\|_\rho^2\\
\leqslant&\frac{\alpha}{2}\|u(t)\|_\rho^2+\frac{1}{2\alpha}\|f(t)\|_\rho^2+\frac{\beta}{2}\|v(t)\|_\rho^2+\frac{1}{\beta}\|g(t)\|_\rho^2+\frac{\gamma^2\|B\|_\rho^2}{\beta}\|u\|_\rho^4.
\end{aligned}
\tag{2.28}
$$

From (2.24) and (2.27)-(2.28), it follows that

$$
\frac{\mathrm{d}}{\mathrm{dt}}\|\psi(t)\|_{E_\rho}^2+\sigma\|\psi(t)\|_{E_\rho}^2 \leqslant c_3(\|f(t)\|_\rho^2+\|g(t)\|_\rho^2+\|u\|_\rho^4). \tag{2.29}
$$

Applying Gronwall's inequality to (2.29), we obtain (2.23). The proof of Lemma 2.3 is complete. □

By definition, a continuous process $\{U(t,\tau)\}_{t\geqslant\tau}$ in the phase space E_ρ means $\{U(t,\tau)\}_{t\geqslant\tau}$ is a two-parameter family of mappings in E_ρ satisfying:

(a) $U(t,s)U(s,\tau)=U(t,\tau)$, $\forall\, t\geqslant s\geqslant\tau,\tau\in\mathbb{R}$;

(b) $U(\tau,\tau)=\mathrm{Id}$ (identity operator), $\tau\in\mathbb{R}$;

(c) For given t and τ with $t\geqslant\tau$, the mapping $U(t,\tau)$ is continuous from E_ρ to E_ρ.

Combining Lemma 2.1 and Lemma 2.3, it follows that, to every given initial datum $\psi_\tau = (u(\tau), v(\tau))^T \in E_\rho$ at every initial time τ, problem (2.10)-(2.11) corresponds uniquely a global solution $\psi(\cdot,\tau;\psi_\tau) = (u(\cdot), v(\cdot))^T \in E_\rho$ on $[\tau, +\infty)$. Hence the solution mappings $U(t,\tau)$ from initial datum ψ_τ to solution $\psi(t,\tau;\psi_\tau)$ defined via

$$U(t,\tau): E_\rho \ni \psi_\tau = (u_\tau, v_\tau)^T \longmapsto \psi(t,\tau;\psi_\tau) = U(t,\tau)\psi_\tau \in E_\rho, \ \ \forall t \geqslant \tau \in \mathbb{R}, \tag{2.30}$$

generate a process $\{U(t,\tau)\}_{t\geqslant\tau}$ on E_ρ. Next we prove the continuity of $\{U(t,\tau)\}_{t\geqslant\tau}$.

Lemma 2.4. *Let assumptions* (H1)-(H2) *hold. Then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *defined by* (2.30) *is continuous on* E_ρ*, that is, for any given* $t,\tau \in \mathbb{R}$ *with* $\tau \leqslant t$*, the mapping* $U(t,\tau): E_\rho \longmapsto E_\rho$ *is continuous.*

Proof. Let $\mathcal{B}$ be a bounded subset of E_ρ, and $t,\tau \in \mathbb{R}$ given with $\tau \leqslant t$. Let $\psi^{(k)}(\tau) = \psi_\tau^{(k)} = (u_\tau^{(k)}, v_\tau^{(k)})^T \in \mathcal{B}$ $(k = 1,2)$ be the initial data at initial time τ, and $\psi^{(k)}(t) = U(t,\tau)\psi_\tau^{(k)}$ the corresponding solutions of problem (2.10)-(2.11). Write $\tilde{u}(\cdot) = u^{(1)}(\cdot) - u^{(2)}(\cdot)$, $\tilde{v}(\cdot) = v^{(1)}(\cdot) - v^{(2)}(\cdot)$, $\tilde{\psi}(\cdot) = \psi^{(1)}(\cdot) - \psi^{(2)}(\cdot)$. Then we have

$$\frac{\mathrm{d}\tilde{\psi}(t)}{\mathrm{d}t} + \Theta\tilde{\psi}(t) = F(\psi^{(1)}, t) - F(\psi^{(2)}, t), \ \ t > \tau, \tag{2.31}$$

$$\tilde{\psi}(\tau) = \psi_\tau^{(1)} - \psi_\tau^{(2)}. \tag{2.32}$$

Taking the real part of the inner product of (2.31) with $\tilde{\psi}$ in E_ρ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\tilde{\psi}\|_{E_\rho}^2 + \mathbf{Re}\big(\Theta\tilde{\psi}, \tilde{\psi}\big)_{E_\rho} = \mathbf{Re}\big(F(\psi^{(1)}, t) - F(\psi^{(2)}, t), \tilde{\psi}\big)_{E_\rho}, \ \ t > \tau. \tag{2.33}$$

From (2.27), we find

$$\begin{aligned}\mathbf{Re}(\Theta\tilde{\psi}, \tilde{\psi})_{E_\rho} &\geqslant \alpha\|\tilde{u}\|_\rho^2 + \beta\|\tilde{v}\|_\rho^2 - \frac{1}{2}(c_1c_2 + c_2)\|\tilde{u}\|_\rho^2 = \varrho\|\tilde{u}\|_\rho^2 + \beta\|\tilde{v}\|_\rho^2 + \frac{1}{2}(c_1c_2 + c_2)\|\tilde{u}\|_\rho^2 \\ &\geqslant \varrho\|\tilde{u}\|_\rho^2 + \beta\|\tilde{v}\|_\rho^2.\end{aligned} \tag{2.34}$$

At the same time, from (2.16), we see that there exists a constant $c_4 = c_4(t,\tau,\mathcal{B}) > 0$, such that

$$\mathbf{Re}\big(F(\psi^{(1)}, t) - F(\psi^{(2)}, t), \tilde{\psi}\big)_{E_\rho} \leqslant c_4\|\tilde{\psi}(t)\|_{E_\rho}^2. \tag{2.35}$$

Combining (2.33)-(2.35), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\psi}(t)\|_{E_\rho}^2 + 2(\sigma - c_4)\|\tilde{\psi}(t)\|_{E_\rho}^2 \leqslant 0. \tag{2.36}$$

Applying Gronwall's inequality to (2.36), we derive

$$\|\tilde{\psi}(t)\|_{E_\rho}^2 \leqslant \|\tilde{\psi}(\tau)\|_{E_\rho}^2 e^{-2(\sigma - c_4)(t-\tau)}.$$

This completes the proof of Lemma 2.4. □

3 Existence of the pullback attractor

In this section, we will first prove that the process $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses a bounded pullback absorbing set and pullback asymptotic nullness in E_ρ. Then we verify that $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses a pullback attractor. For the definitions concerning the bounded pullback absorbing set, pullback asymptotic nullness and pullback attractor, one can refer to reference [7].

Henceforth, we denote by $\mathcal{O}(E_\rho)$ the family of all nonempty subsets of E_ρ and consider the families of nonempty sets $\widehat{D}_0 = \{D_0(s)|s \in \mathbb{R}\} \subseteq \mathcal{O}(E_\rho)$ parameterized by time t. Let $\mathcal{D}_\sigma$ be the class of families of nonempty subsets $\widehat{D} = \{D(s)|s \in \mathbb{R}\} \subseteq \mathcal{O}(E_\rho)$ which satisfies

$$\mathcal{D}_\sigma = \Big\{\widehat{D} = \{D(s)|s \in \mathbb{R}\}\big| \lim_{s\to-\infty} e^{\frac{\sigma s}{2}} \sup_{\psi\in D(s)} \|\psi\|^2_{E_\rho} = 0\Big\}. \tag{3.1}$$

We next prove the existence of the pullback $\mathcal{D}_\sigma$-absorbing set for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in E_ρ.

Lemma 3.1. *Let assumptions* (H1)-(H2) *hold. Then the process $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses a bounded pullback $\mathcal{D}_\sigma$-absorbing set $\widehat{\mathcal{B}_0} = \{\mathcal{B}_0(s)|s \in \mathbb{R}\}$, that is, for each $t \in \mathbb{R}$ and any $\widehat{D} = \{D(s)|s \in \mathbb{R}\} \in \mathcal{D}_\sigma$, $\exists \tau_0 = \tau_0(t, \widehat{D}) \leqslant t$ such that $U(t,\tau)D(\tau) \subset \mathcal{B}_0(t)$, $\forall \tau \leqslant \tau_0$, where $\mathcal{B}_0(s) = \mathcal{B}_0(0; R_\sigma(s))$ is a closed ball in E_ρ with center zero and radius $R_\sigma(s)$.*

Proof. Let t and $\widehat{D} = \{D(s)|s \in \mathbb{R}\} \in \mathcal{D}_\sigma$ be given. Then Lemma 2.2 and Lemma 2.3 show that for any $\psi_\tau \in D(\tau)$ there holds

$$\begin{aligned}\|U(t,\tau)\psi_\tau\|^2_{E_\rho} \leqslant& \|\psi_\tau\|^2_{E_\rho} e^{-\sigma(t-\tau)} + c_3 e^{-\sigma t} \int_\tau^t e^{\sigma s}\big(\|f(s)\|^2_\rho + \|g(s)\|^2_\rho\big)\mathrm{d}s \\ &+ c_3 e^{-\sigma t} \int_\tau^t e^{\sigma s}\Big(e^{-\varrho(s-\tau)}\|u_\tau\|^2_\rho + \frac{e^{-\varrho s}}{\alpha}\int_\tau^s e^{\varrho\theta}\|f(\theta)\|^2_\rho \mathrm{d}\theta\Big)^2 \mathrm{d}s, \ \ \forall t \geqslant \tau. \end{aligned} \tag{3.2}$$

According to assumption (H2) and (3.1), it is clear that

$$\lim_{\tau\to-\infty} \|\psi_\tau\|^2_{E_\rho} e^{-\sigma(t-\tau)} = 0, \tag{3.3}$$

$$\int_{-\infty}^t e^{\sigma s}\big(\|f(s)\|^2_\rho + \|g(s)\|^2_\rho\big)\mathrm{d}s < +\infty. \tag{3.4}$$

For the third term on the right-hand side of (3.2), we express it as

$$\int_\tau^t e^{\sigma s}\Big(e^{-\varrho(s-\tau)}\|u_\tau\|^2_\rho + \frac{e^{-\varrho s}}{\alpha}\int_\tau^s e^{\varrho\theta}\|f(\theta)\|^2_\rho \mathrm{d}\theta\Big)^2 \mathrm{d}s = I_1(\tau) + I_2(\tau) + I_3(\tau), \tag{3.5}$$

where

$$\begin{aligned} I_1(\tau) &= \int_\tau^t e^{\sigma s}\|u_\tau\|^4_\rho e^{-2\varrho(s-\tau)}\mathrm{d}s, \\ I_2(\tau) &= \frac{2}{\alpha}\int_\tau^t e^{\sigma s}\|u_\tau\|^2_\rho e^{-2\varrho s+\varrho\tau}\int_\tau^s e^{\varrho\theta}\|f(\theta)\|^2_\rho \mathrm{d}\theta\mathrm{d}s, \\ I_3(\tau) &= \frac{1}{\alpha^2}\int_\tau^t e^{(\sigma-2\varrho)s}\Big(\int_\tau^s e^{\varrho\theta}\|f(\theta)\|^2_\rho\mathrm{d}\theta\Big)^2\mathrm{d}s. \end{aligned}$$

From (3.1), it is easy to see that

$$\begin{aligned} I_1(\tau) =& \big(\|u_\tau\|^2_\rho e^{\frac{\sigma}{2}\tau}\big)^2 \int_\tau^t e^{\sigma s} e^{-2\varrho(s-\tau)} e^{-\sigma\tau}\mathrm{d}s \\ =& \frac{1}{2\varrho-\sigma}\big(\|u_\tau\|^2_\rho e^{\frac{\sigma}{2}\tau}\big)^2\big(1 - e^{-(2\varrho-\sigma)(t-\tau)}\big) \\ \leqslant& \frac{\big(\|\psi_\tau\|^2_{E_\rho} e^{\frac{\sigma}{2}\tau}\big)^2}{2\varrho-\sigma} \longrightarrow 0, \ \text{ as } \ \tau\to-\infty. \end{aligned} \tag{3.6}$$

By (H2) and (2.13), we have

$$\int_{-\infty}^t e^{\varrho s}\|f(s)\|^2_\rho\mathrm{d}s = \int_{-\infty}^t e^{(\varrho-\sigma)s}e^{\sigma s}\|f(s)\|^2_\rho\mathrm{d}s \leqslant e^{(\varrho-\frac{\sigma}{2}+\delta)t}K(t) < +\infty. \tag{3.7}$$

Therefore, we can conclude that there exists a positive constant $\tilde{K_1}(t)$, which depends only on t and function $K(\cdot)$, such that

$$
\begin{aligned}
I_2(\tau) =&\frac{2}{\alpha}\|u_\tau\|_\rho^2 e^{\frac{\sigma}{2}\tau}e^{(\varrho-\frac{\sigma}{2})\tau}\int_\tau^t\Big[e^{-(2\varrho-\sigma)s}\int_\tau^s e^{\varrho\theta}\|f(\theta)\|_\rho^2\mathrm{d}\theta\Big]\mathrm{d}s\\
\leqslant&\frac{2}{\alpha}\|\psi_\tau\|_{E_\rho}^2 e^{\frac{\sigma}{2}\tau}e^{(\varrho-\frac{\sigma}{2})\tau}\int_\tau^t e^{(\sigma-2\varrho)s}e^{(\varrho-\frac{\sigma}{2}+\delta)s}K(s)\mathrm{d}s\\
=&\frac{2}{\alpha}\|\psi_\tau\|_{E_\rho}^2 e^{\frac{\sigma}{2}\tau}e^{\delta\tau}e^{(\varrho-\frac{\sigma}{2}-\delta)\tau}\int_\tau^t e^{(\frac{\sigma}{2}-\varrho+\delta)s}K(s)\mathrm{d}s\\
\leqslant&\frac{4}{\alpha(2\varrho-\sigma-2\delta)}\|\psi_\tau\|_{E_\rho}^2 e^{\frac{\sigma}{2}\tau}e^{\delta\tau}\tilde{K_1}(t)\big(1-e^{-(\varrho-\frac{\sigma}{2}-\delta)(t-\tau)}\big)\\
\leqslant&\frac{4}{\alpha(2\varrho-\sigma-2\delta)}\|\psi_\tau\|_{E_\rho}^2 e^{\frac{\sigma}{2}\tau}e^{\delta\tau}\tilde{K_1}(t)\longrightarrow 0,\ \text{ as }\ \tau\to-\infty.
\end{aligned}\tag{3.8}
$$

Also by (H2), (2.13) and (3.7), we have

$$
\begin{aligned}
I_3(\tau) =& \frac{1}{\alpha^2}\int_\tau^t e^{(\sigma-2\varrho)s}e^{(\sigma-2\varrho-2\delta)s}\left(\int_\tau^s e^{\varrho\theta}\|f(\theta)\|_\rho^2\mathrm{d}\theta\right)^2 e^{(-\sigma+2\varrho+2\delta)s}\mathrm{d}s\\
\leqslant& \frac{1}{\alpha^2}\int_{-\infty}^t e^{(\sigma-2\varrho)s}\left(e^{(\frac{\sigma}{2}-\varrho-\delta)s}\int_{-\infty}^s e^{\varrho\theta}\|f(\theta)\|_\rho^2\mathrm{d}\theta\right)^2 e^{(-\sigma+2\varrho+2\delta)s}\mathrm{d}s\\
\leqslant& \frac{1}{\alpha^2}\int_{-\infty}^t e^{2\delta s}K^2(s)\mathrm{d}s\leqslant\frac{1}{2\delta\alpha^2}e^{2\delta t}\tilde{K}_2(t)<+\infty,\ \ \forall\tau<t,
\end{aligned}\tag{3.9}
$$

where $\tilde{K}_2(t)$ is a bounded quantity relying only on the function $K(\cdot)$ and t. Now, we choose $R_\sigma(t)>0$ such that

$$
R_\sigma^2(t)=1+c_3e^{-\sigma t}\int_{-\infty}^t e^{\sigma s}\big(\|f(s)\|_\rho^2+\|g(s)\|_\rho^2\big)\mathrm{d}s+\frac{c_3e^{-\sigma t}}{\alpha^2}\int_{-\infty}^t e^{(\sigma-2\varrho)s}\Big(\int_{-\infty}^s e^{\varrho\theta}\|f(\theta)\|_\rho^2\mathrm{d}\theta\Big)^2\mathrm{d}s.
$$

Then, from (2.23) we conclude that the family of closed balls $\widehat{\mathcal{B}_0}=\{\mathcal{B}_0(0;R_\sigma(t))|t\in\mathbb{R}\}$ is the desired bounded pullback $\mathcal{D}_\sigma$-absorbing set for $\{U(t,\tau)\}_{t\geqslant\tau}$ in E_ρ. The proof is complete. $\square$

The following lemma shows the pullback $\mathcal{D}_\sigma$-asymptotic nullness of the process $U(t,\tau)_{t\geqslant\tau}$ in E_ρ.

Lemma 3.2. *Let assumptions* (H1)-(H2) *hold. Then* $\forall\, t\in\mathbb{R}$, $\forall\widehat{D}=\{D(s)|s\in\mathbb{R}\}\in\mathcal{D}_\sigma$ *and* $\forall\varepsilon>0$, $\exists M_0=M_0(t,\varepsilon,\widehat{D})\in\mathbb{N}$ *and* $\tau_0=\tau_0(t,\varepsilon,\widehat{D})\leqslant t$ *such that*

$$
\sup_{\psi_\tau\in D(\tau)}\sum_{|m|\geqslant M_0}\rho_m|(U(t,\tau)\psi_\tau)_m|^2\leqslant\varepsilon^2,\ \ \forall\tau\leqslant\tau_0,\tag{3.10}
$$

where $|\psi_m|^2=|u_m|^2+v_m^2$, $\psi=(\psi_m)_{m\in\mathbb{Z}}=(u_m,v_m)^T\in E_\rho$.

Proof. According to Urysohn's Lemma, we choose a smooth function $\chi(\cdot)\in C^1(\mathbb{R}_+,\mathbb{R}_+)$ such that

$$
\begin{cases}
\chi(x)=0, & 0\leqslant x\leqslant 1,\\
0\leqslant\chi(x)\leqslant 1, & 1\leqslant x\leqslant 2,\\
\chi(x)=1, & x\geqslant 2,\\
|\chi'(x)|\leqslant\chi_0, & x\geqslant 0,
\end{cases}\tag{3.11}
$$

where $\chi_0>0$ is a positive constant. Consider any given $\widehat{D}=\{D(s)|s\in\mathbb{R}\}\in\mathcal{D}_\sigma$ and t, we denote by $\psi(t)=\psi(t;\tau,\psi_\tau)=U(t,\tau)\psi_\tau=(u(t),v(t))^T$ the solution of problem (2.10)-(2.11) with initial value $\psi_\tau\in D(\tau)$ at the initial time $\tau\in\mathbb{R}$. Let $M>0$ be a natural number and set

$$
\begin{aligned}
\xi_m=\chi(\frac{|m|}{M})u_m,\ \ \zeta_m=\chi(\frac{|m|}{M})v_m,\ \ \phi_m=(\xi_m,\zeta_m)^T,\ \ w_m=\rho_m u_m,\ \ z_m=\rho_m\xi_m,\ \ m\in\mathbb{Z},\\
\xi=(\xi_m)_{m\in\mathbb{Z}},\ \ \zeta=(\zeta_m)_{m\in\mathbb{Z}},\ \ \phi=(\phi_m)_{m\in\mathbb{Z}},\ \ w=(w_m)_{m\in\mathbb{Z}},\ \ z=(z_m)_{m\in\mathbb{Z}},\ \ m\in\mathbb{Z}.
\end{aligned}
$$

Taking the real part of the inner product of (2.10) with $\phi = (\phi_m)_{m\in\mathbb{Z}}$ in E_ρ yields

$$\mathbf{Re}\big(\dot{\psi}, \phi\big)_{E_\rho} + \mathbf{Re}\big(\Theta\psi, \phi\big)_{E_\rho} = \mathbf{Re}\big(F(\psi, t), \phi\big)_{E_\rho}. \tag{3.12}$$

We next compute the three terms in (3.12) one by one.

Firstly, direct computations gives

$$\begin{aligned}\mathbf{Re}\big(\dot{\psi}, \phi\big)_{E_\rho} =&(\dot{u}, \xi)_\rho + (\dot{v}, \zeta)_\rho = \sum_{m\in\mathbb{Z}} \rho_m \dot{u}_m \xi_m + \sum_{m\in\mathbb{Z}} \rho_m \dot{v}_m \zeta_m \\ =&\sum_{m\in\mathbb{Z}} \rho_m \dot{u}_m \chi(\frac{|m|}{M})\bar{u}_m + \sum_{m\in\mathbb{Z}} \rho_m \dot{v}_m \chi(\frac{|m|}{M}) v_m \\ =&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|\psi_m|^2. \end{aligned} \tag{3.13}$$

Secondly,

$$\begin{aligned}\mathbf{Re}\big(\Theta\psi, \phi\big)_{E_\rho} =&\mathbf{Re}((\alpha u + iAu, \beta v)^T, (\xi, \zeta)^T)_{E_\rho} = \alpha(u, \xi)_\rho - \mathbf{Im}(Au, \xi)_\rho + \beta(v, \zeta)_\rho \\ =&\alpha \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|u_m|^2 + \beta \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|v_m|^2 - \mathbf{Im}(Au, \xi)_\rho. \end{aligned} \tag{3.14}$$

For the term $-\mathbf{Im}(Au, \xi)_\rho$, we have

$$\begin{aligned}-\mathbf{Im}(Au, \xi)_\rho =& - \mathbf{Im}(Au, z) = -\mathbf{Im} \sum_{m\in\mathbb{Z}} (Bu)_m (B\bar{z})_m \\ =& - \mathbf{Im}\Big(\sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})(Bu)_m (B\bar{w})_m + \sum_{m\in\mathbb{Z}} (Bu)_m \big[(B\bar{z})_m - \chi(\frac{|m|}{M})(B\bar{w})_m\big]\Big). \end{aligned} \tag{3.15}$$

From (2.2) and (2.26), we deduce

$$\begin{aligned}& - \mathbf{Im} \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})(Bu)_m (B\bar{w})_m \\ \geqslant& - \frac{c_2}{2} \sum_{m\in\mathbb{Z}} \big(\rho_m \chi(\frac{|m|}{M})|u_{m+1}|^2 + \rho_m \chi(\frac{|m|}{M})|u_m|^2\big) \\ \geqslant& - \frac{c_1 c_2}{2} \sum_{m\in\mathbb{Z}} \rho_{m+1} \chi(\frac{|m|}{M})|u_{m+1}|^2 - \frac{c_2}{2} \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|u_m|^2 \\ =& - \frac{c_1 c_2}{2} \sum_{m\in\mathbb{Z}} \rho_{m+1} \Big[\chi(\frac{|m|}{M}) - \chi(\frac{|m+1|}{M}) + \chi(\frac{|m+1|}{M})\Big]|u_{m+1}|^2 - \frac{c_2}{2} \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|u_m|^2. \end{aligned} \tag{3.16}$$

Using the differential mean value theorem and (3.11), we obtain

$$\sum_{m\in\mathbb{Z}} \rho_{m+1} \Big[\chi(\frac{|m|}{M}) - \chi(\frac{|m+1|}{M})\Big]|u_{m+1}|^2 \leqslant \frac{\chi_0}{M} \sum_{m\in\mathbb{Z}} \rho_{m+1}|u_{m+1}|^2 = \frac{\chi_0}{M}\|u\|_\rho^2. \tag{3.17}$$

Inserting (3.17) into (3.16) yields

$$\begin{aligned}& - \mathbf{Im} \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})(Bu)_m (B\bar{w})_m \\ \geqslant& - \frac{c_1 c_2}{2} \Big[\frac{\chi_0}{M}\|u\|_\rho^2 + \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|u_m|^2\Big] - \frac{c_2}{2} \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|u_m|^2. \end{aligned} \tag{3.18}$$

At the same time, performing direct computations and estimation, we arrive at

$$
\begin{aligned}
&-\mathbf{Im}\sum_{m\in\mathbb{Z}}(Bu)_m\Big[(B\bar{z})_m-\chi(\frac{|m|}{M})(B\bar{w})_m\Big]\\
=&-\mathbf{Im}\sum_{m\in\mathbb{Z}}(Bu)_m\Big(\chi(\frac{|m+1|}{M})-\chi(\frac{|m|}{M})\Big)\rho_{m+1}\bar{u}_{m+1}\\
\geqslant&-\frac{\chi_0}{M}\sum_{m\in\mathbb{Z}}\rho_{m+1}|(Bu)_m||u_{m+1}|\\
\geqslant&-\frac{\chi_0}{2M}\sum_{m\in\mathbb{Z}}\rho_{m+1}\big(|(Bu)_m|^2+|u_{m+1}|^2\big).
\end{aligned}\tag{3.19}
$$

From (2.2) and (2.5), we see that

$$
\sum_{m\in\mathbb{Z}}\rho_{m+1}((Bu)_m)^2\leqslant c_1\sum_{m\in\mathbb{Z}}\rho_m((Bu)_m)^2=c_1\|Bu\|_\rho^2\leqslant 2c_1(c_1+1)\|u\|_\rho^2.\tag{3.20}
$$

It then follows from (3.19) and (3.20) that

$$
-\mathbf{Im}\sum_{m\in\mathbb{Z}}(Bu)_m\Big[(B\bar{z})_m-\chi(\frac{|m|}{M})(B\bar{w})_m\Big]\geqslant-\frac{\chi_0}{2M}(\|u\|_\rho^2+2c_1(c_1+1)\|u\|_\rho^2).\tag{3.21}
$$

Combining (3.15), (3.18) and (3.21), we obtain

$$
\begin{aligned}
-\mathbf{Im}(Au,\xi)_\rho\geqslant&-\frac{c_1c_2}{2}\Big[\frac{\chi_0}{M}\|u\|_\rho^2+\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})|u_m|^2\Big]-\frac{c_2}{2}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})|u_m|^2\\
&-\frac{\chi_0}{2M}(\|u\|_\rho^2+2c_1(c_1+1)\|u\|_\rho^2)\\
=&-\frac{c_1c_2+c_2}{2}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})|u_m|^2-\frac{(c_1c_2+1+2c_1(c_1+1))\chi_0}{2M}\|u\|_\rho^2.
\end{aligned}\tag{3.22}
$$

Thus, taking (3.14) and (3.22) into account, we have

$$
\begin{aligned}
\mathbf{Re}(\Theta\psi,\phi)_{E_\rho}\geqslant&\alpha\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})|u_m|^2+\beta\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})|v_m|^2\\
&-\frac{c_1c_2+c_2}{2}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})|u_m|^2-\frac{(c_1c_2+1+2c_1(c_1+1))\chi_0}{2M}\|u\|_\rho^2.
\end{aligned}\tag{3.23}
$$

Finally, let us estimate the term $\mathbf{Re}(F(\psi,t),\phi)_{E_\rho}$. In fact,

$$
\begin{aligned}
\mathbf{Re}\big(F(\psi,t),\phi\big)_{E_\rho}=&\mathbf{Re}\big((-iuv-if(t),g(t)-\gamma B(|u|^2))^T,(\xi,\zeta)^T\big)_{E_\rho}\\
=&\mathbf{Im}(uv,\xi)_\rho+\mathbf{Im}(f(t),\xi)_\rho+(g(t),\zeta)_\rho-(\gamma B(|u|^2),\zeta)_\rho\\
=&\mathbf{Im}(f(t),\xi)_\rho+(g(t),\zeta)_\rho-(\gamma B(|u|^2),\zeta)_\rho,
\end{aligned}\tag{3.24}
$$

since that $\mathbf{Im}(uv,\xi)_\rho=\mathbf{Im}\sum\limits_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})|u_m|^2v_m=0$. For the first two terms on the right-hand side of (3.24), we use Cauchy's inequality to obtain

$$
\mathbf{Im}(f(t),\xi)_\rho\leqslant\frac{\alpha}{2}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})|u_m(t)|^2+\frac{1}{2\alpha}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})|f_m(t)|^2,\tag{3.25}
$$

$$
(g(t),\zeta)_\rho\leqslant\frac{\beta}{4}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})|v_m(t)|^2+\frac{1}{\beta}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})|g_m(t)|^2.\tag{3.26}
$$

We next estimate the last term on the right-hand side of (3.24). Using (2.2) and Cauchy's inequality, we derive

$$
\begin{aligned}
-(\gamma B(|u|^2),\zeta)_\rho =& -\gamma\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})(|u_{m+1}|^2-|u_m|^2)v_m\\
\leqslant&\frac{\beta}{4}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})v_m^2+\frac{\gamma^2}{\beta}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})(|u_{m+1}|^2-|u_m|^2)^2\\
\leqslant&\frac{\beta}{4}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})v_m^2+\frac{2\gamma^2}{\beta}\sum_{m\in\mathbb{Z}}\rho_m|u_{m+1}|^2\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})|u_{m+1}|^2\\
&+\frac{2\gamma^2}{\beta}\sum_{m\in\mathbb{Z}}\rho_m|u_m|^2\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})|u_m|^2\\
\leqslant&\frac{\beta}{4}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})v_m^2+\frac{2\gamma^2c_1\|u\|_\rho^2}{\beta}\frac{1}{c_*}\underbrace{\sum_{m\in\mathbb{Z}}\rho_{m+1}\chi(\frac{|m|}{M})|u_{m+1}|^2}_{I_4}\\
&+\frac{2\gamma^2\|u\|_\rho^2}{\beta}\frac{1}{c_*}\underbrace{\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m|}{M})|u_m|^2}_{I_5}.
\end{aligned}
\tag{3.27}
$$

To estimate the terms I_4 and I_5 in (3.27), we write

$$
p_m=\chi(\frac{|m-1|}{M})u_m,\ \ q_m=\rho_m p_m,\ \ p=(p_m)_{m\in\mathbb{Z}},\ \ q=(q_m)_{m\in\mathbb{Z}}.
$$

Taking the imaginary part of the inner product $(L^2_\rho,(\cdot,\cdot)_\rho)$ of equation (2.7) with p gives

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m-1|}{M})|u_m|^2+\alpha\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m-1|}{M})|u_m|^2\\
=&\mathbf{Im}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m-1|}{M})f_m\bar{u}_m+\mathbf{Im}(Au,p)_\rho\\
\leqslant&\frac{\alpha}{2}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m-1|}{M})|u_m(t)|^2+\frac{1}{2\alpha}\sum_{m\in\mathbb{Z}}\rho_m\chi(\frac{|m-1|}{M})|f_m(t)|^2+\mathbf{Im}(Au,p)_\rho.
\end{aligned}
\tag{3.28}
$$

By direct computations, we have

$$
\begin{aligned}
\mathbf{Im}(Au,p)_\rho=&\mathbf{Im}\sum_{m\in\mathbb{Z}}(Bu)_m(B\bar{q})_m\\
=&\mathbf{Im}\Big(\sum_{m\in\mathbb{Z}}\chi(\frac{|m-1|}{M})(Bu)_m(B\bar{w})_m+\sum_{m\in\mathbb{Z}}(Bu)_m\big[(B\bar{q})_m-\chi(\frac{|m-1|}{M})(B\bar{w})_m\big]\Big).
\end{aligned}
\tag{3.29}
$$

Using (2.2) and (2.21), together with differential mean value theorem and (3.11), we deduce

$$
\begin{aligned}
&\mathbf{Im}\sum_{m\in\mathbb{Z}}\chi(\frac{|m-1|}{M})(Bu)_m(B\bar{w})_m\\
=&\mathbf{Im}\sum_{m\in\mathbb{Z}}\Big\{\rho_m\chi(\frac{|m-1|}{M})(Bu)_m(B\bar{u})_m+\chi(\frac{|m-1|}{M})(Bu)_m[(B\bar{w})_m-\rho_m(B\bar{u})_m]\Big\}\\
=&\mathbf{Im}\sum_{m\in\mathbb{Z}}\chi(\frac{|m-1|}{M})(Bu)_m(\rho_{m+1}-\rho_m)\bar{u}_{m+1}\\
\leqslant& c_2\sum_{m\in\mathbb{Z}}\chi(\frac{|m-1|}{M})\rho_m|u_{m+1}||u_m|\leqslant\frac{c_2}{2}\sum_{m\in\mathbb{Z}}\chi(\frac{|m-1|}{M})\big(\rho_m|u_{m+1}|^2+\rho_m|u_m|^2\big)\\
\leqslant&\frac{c_1c_2}{2}\sum_{m\in\mathbb{Z}}\Big[\chi(\frac{|m-1|}{M})-\chi(\frac{|m|}{M})+\chi(\frac{|m|}{M})\Big]\rho_{m+1}|u_{m+1}|^2+\frac{c_2}{2}\sum_{m\in\mathbb{Z}}\chi(\frac{|m-1|}{M})\rho_m|u_m|^2\\
\leqslant&\frac{c_1c_2\chi_0}{2M}\|u\|_\rho^2+\frac{c_1c_2+c_2}{2}\sum_{m\in\mathbb{Z}}\chi(\frac{|m-1|}{M})\rho_m|u_m|^2. \qquad (3.30)
\end{aligned}
$$

Also, we obtain by using (2.2) and (2.5) that

$$
\begin{aligned}
&\mathbf{Im}\sum_{m\in\mathbb{Z}}(Bu)_m\Big[(B\bar{q})_m-\chi(\frac{|m-1|}{M})(B\bar{w})_m\Big]\\
=&\mathbf{Im}\sum_{m\in\mathbb{Z}}(Bu)_m\big(\chi(\frac{|m|}{M})-\chi(\frac{|m-1|}{M})\big)\rho_{m+1}\bar{u}_{m+1}\\
\leqslant&\frac{\chi_0}{M}\sum_{m\in\mathbb{Z}}\rho_{m+1}|(Bu)_m||u_{m+1}|\leqslant\frac{\chi_0}{2M}\sum_{m\in\mathbb{Z}}\rho_{m+1}\big(|(Bu)_m|^2+|u_{m+1}|^2\big)\\
\leqslant&\frac{\chi_0(1+2c_1(c_1+1))}{2M}\|u\|_\rho^2. \qquad (3.31)
\end{aligned}
$$

Inserting (3.30) and (3.31) into (3.29) gives

$$
\mathbf{Im}(Au,p)_\rho\leqslant\frac{c_1c_2+c_2}{2}\sum_{m\in\mathbb{Z}}\chi(\frac{|m-1|}{M})\rho_m|u_m|^2+\frac{\big(c_1c_2+1+2c_1(c_1+1)\big)\chi_0}{2M}\|u\|_\rho^2. \qquad (3.32)
$$

It then follows from (3.28) and (3.32) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}}\rho_{m+1}\chi(\frac{|m|}{M})|u_{m+1}|^2+\varrho\sum_{m\in\mathbb{Z}}\rho_{m+1}\chi(\frac{|m|}{M})|u_{m+1}|^2\\
\leqslant&\frac{1}{\alpha}\sum_{m\in\mathbb{Z}}\rho_{m+1}\chi(\frac{|m|}{M})|f_{m+1}(t)|^2+\frac{\big(c_1c_2+1+2c_1(c_1+1)\big)\chi_0}{M}\|u\|_\rho^2. \qquad (3.33)
\end{aligned}
$$

Applying Gronwall's inequality to (3.33),

$$
\begin{aligned}
I_4=&\sum_{m\in\mathbb{Z}}\rho_{m+1}\chi(\frac{|m|}{M})|u_{m+1}(t)|^2\\
\leqslant&\sum_{m\in\mathbb{Z}}\rho_{m+1}\chi(\frac{|m|}{M})|u_{m+1}(\tau)|^2e^{-\varrho(t-\tau)}+\frac{1}{\alpha}\int_\tau^t\Big(\sum_{|m|\geqslant M}\rho_{m+1}|f_{m+1}(s)|^2\Big)e^{-\varrho(t-s)}\mathrm{d}s\\
&+\frac{\big(c_1c_2+1+2c_1(c_1+1)\big)\chi_0}{\varrho M}\|u\|_\rho^2\\
\leqslant&e^{-\varrho(t-\tau)}\|u(\tau)\|_\rho^2+\frac{e^{-\varrho t}}{\alpha}\int_\tau^t\Big(\sum_{|m|\geqslant M}\rho_{m+1}|f_{m+1}(s)|^2\Big)e^{\varrho s}\mathrm{d}s\\
&+\frac{\big(c_1c_2+1+2c_1(c_1+1)\big)\chi_0}{\varrho M}\|u\|_\rho^2. \qquad (3.34)
\end{aligned}
$$

Parallel to I_1, we can obtain

$$
\begin{aligned}
I_5 =& \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|u_m(t)|^2 \\
\leqslant& \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|u_m(\tau)|^2 e^{-\varrho(t-\tau)} + \frac{1}{\alpha}\int_\tau^t \Big(\sum_{|m|\geqslant M} \rho_m |f_m(s)|^2\Big) e^{-\varrho(t-s)}\mathrm{d}s \\
&+ \frac{\big(c_1c_2+1+2c_1(c_1+1)\big)\chi_0}{\varrho M}\|u\|_\rho^2 \\
\leqslant& e^{-\varrho(t-\tau)}\|u(\tau)\|_\rho^2 + \frac{e^{-\varrho t}}{\alpha}\int_\tau^t \Big(\sum_{|m|\geqslant M} \rho_m |f_m(s)|^2\Big) e^{\varrho s}\mathrm{d}s \\
&+ \frac{\big(c_1c_2+1+2c_1(c_1+1)\big)\chi_0}{\varrho M}\|u\|_\rho^2. \qquad (3.35)
\end{aligned}
$$

Combining (3.24)-(3.27) and (3.34)-(3.35), we have

$$
\begin{aligned}
&\mathbf{Re}(F(\psi,t),\phi)_{E_\rho} \\
\leqslant& \frac{\alpha}{2}\sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|u_m(t)|^2 + \frac{1}{2\alpha}\sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|f_m(t)|^2 \\
&+ \frac{\beta}{2}\sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|v_m(t)|^2 + \frac{1}{\beta}\sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|g_m(t)|^2 \\
&+ \frac{2(c_1+1)\gamma^2\|u\|_\rho^2}{\beta c_*}e^{-\varrho(t-\tau)}\|u(\tau)\|_\rho^2 + \frac{2(c_1+1)\gamma^2\|u\|_\rho^2 e^{-\varrho t}}{\alpha\beta c_*}\int_\tau^t \Big(\sum_{|m|\geqslant M} \rho_m |f_m(s)|^2\Big) e^{\varrho s}\mathrm{d}s \\
&+ \frac{2(c_1+1)\gamma^2\|u\|_\rho^2}{\beta c_*}\cdot\frac{\big(c_1c_2+1+2c_1(c_1+1)\big)\chi_0}{\varrho M}\|u\|_\rho^2. \qquad (3.36)
\end{aligned}
$$

We next will use the notation $a \lesssim b$ to stand for $a \leqslant cb$ for a general constant $c > 0$ that just depends on the parameters from our problem and will not produce confusion. At this stage, we can derive from (3.13), (3.23) and (3.36) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|\psi_m|^2 + \varrho\sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|u_m|^2 + \beta\sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|v_m|^2 \\
\lesssim& \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|f_m(t)|^2 + \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|g_m(t)|^2 + \|u_\tau\|_\rho^2 e^{-\varrho(t-\tau)}\|u(t)\|_\rho^2 \\
&+ \|u(t)\|_\rho^2 e^{-\varrho t}\int_\tau^t \Big(\sum_{|m|\geqslant M} \rho_m |f_m(s)|^2\Big) e^{\varrho s}\mathrm{d}s + \frac{\|u(t)\|_\rho^4}{M} + \frac{\|u(t)\|_\rho^2}{M}. \qquad (3.37)
\end{aligned}
$$

Now Lemma 3.1 shows that there exists a time $\tau_1 = \tau_1(t,\widehat{D}) \leqslant t$ such that

$$
\|u(t)\|_\rho^2 \leqslant R_\sigma^2(t), \ \ \forall \tau \leqslant \tau_1.
$$

Hence, (3.37) implies that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|\psi_m|^2 + \sigma\sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|\psi_m|^2 \\
\lesssim& \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|f_m(t)|^2 + \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M})|g_m(t)|^2 + e^{-\varrho(t-\tau)}\|\psi_\tau\|_{E_\rho}^2 R_\sigma^2(t) \\
&+ R_\sigma^2(t)e^{-\varrho t}\int_\tau^t \Big(\sum_{|m|\geqslant M} \rho_m |f_m(s)|^2\Big) e^{\varrho s}\mathrm{d}s + \frac{R_\sigma^4(t)}{M} + \frac{R_\sigma^2(t)}{M}, \ \ \forall \tau \leqslant \tau_1. \qquad (3.38)
\end{aligned}
$$

Note that $R_\sigma(t) < +\infty$ for given t. Thus, for any $\varepsilon > 0$ there exist $M_1 = M_1(t,\varepsilon) \in \mathbb{N}$ and $\tau_2 = \tau_2(t,\varepsilon)$ such that

$$\frac{R_\sigma^2(t) + R_\sigma^4(t)}{M} \leqslant \frac{\sigma\varepsilon^2}{9}, \ \ \forall M \geqslant M_1, \tag{3.39}$$

$$\|\psi_\tau\|_{E_\rho}^2 e^{-\varrho(t-\tau)} R_\sigma^2(t) \leqslant \frac{\sigma\varepsilon^2}{9}, \ \ \forall \tau \leqslant \tau_2. \tag{3.40}$$

At the same time, from (3.7) we see that $\displaystyle\int_{-\infty}^{t} e^{\varrho s}\|f(s)\|_\rho^2 \mathrm{d}s < +\infty$ for given t. Therefore, for above given t and $\varepsilon > 0$, there exists $M_2 = M_2(t,\varepsilon) \in \mathbb{N}$ such that

$$R_\sigma^2(t) e^{-\varrho t} \int_\tau^t \Big(\sum_{|m|\geqslant M} \rho_m |f_m(s)|^2\Big) e^{\varrho s} \mathrm{d}s \leqslant \frac{\sigma\varepsilon^2}{9}, \ \ \forall M \geqslant M_2. \tag{3.41}$$

Inserting (3.39)-(3.41) into (3.38) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M}) |\psi_m|^2 + \sigma \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M}) |\psi_m|^2 \\
&\lesssim \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M}) |f_m(t)|^2 + \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M}) |g_m(t)|^2 + \frac{\sigma\varepsilon^2}{3}.
\end{aligned} \tag{3.42}$$

Applying Gronwall's inequality to (3.42) implies

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M}) |\psi_m(t)|^2 &\lesssim \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M}) |\psi_m(\tau)|^2 e^{-\sigma(t-\tau)} + e^{-\sigma t} \int_\tau^t e^{\sigma s} \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M}) |f_m(s)|^2 \mathrm{d}s \\
&\quad + e^{-\sigma t} \int_\tau^t e^{\sigma s} \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M}) |g_m(s)|^2 \mathrm{d}s + \frac{\varepsilon^2}{3}.
\end{aligned} \tag{3.43}$$

Now, from assumption (H2) we see that, for above given t and $\varepsilon > 0$, there exists some $M_3 = M_3(t,\varepsilon) \in \mathbb{N}$ such that

$$e^{-\sigma t}\Big(\int_\tau^t e^{\sigma s} \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M}) |f_m(s)|^2 \mathrm{d}s + \int_\tau^t e^{\sigma s} \sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M}) |g_m(s)|^2 \mathrm{d}s\Big) \leqslant \frac{\varepsilon^2}{3}, \ \ \forall M \geqslant M_3. \tag{3.44}$$

Finally, from (3.1) we can conclude that for above given t and $\varepsilon > 0$, there is a time $\tau_3 = \tau_3(t,\varepsilon,\widehat{D})$ such that

$$\sum_{m\in\mathbb{Z}} \rho_m \chi(\frac{|m|}{M}) |\psi_m(\tau)|^2 e^{-\sigma(t-\tau)} \leqslant e^{-\sigma(t-\tau)} \sup_{\psi_\tau \in D(\tau)} \|\psi_\tau\|_{E_\rho}^2 \leqslant \frac{\varepsilon^2}{3}, \ \ \forall \tau \leqslant \tau_3. \tag{3.45}$$

Let

$$M_0 = \max\{M_1, M_2, M_3\}, \ \ \tau_0 = \min\{\tau_1, \tau_2, \tau_3\}.$$

Then, it follows from (3.43)-(3.45) that (3.10) holds. The proof of Lemma 3.2 is complete. □

In terms of Lemma 3.1, Lemma 3.2 and [29, Lemma 2.1], we obtain the main result of this section as follows.

Theorem 3.1. *Let assumptions* (H1)-(H2) *hold. Then in E_ρ, the process $\{U(t,\tau)\}_{t\geqslant\tau}$ corresponding to the problem* (2.10)-(2.11) *has a pullback $\mathcal{D}_\sigma$-attractor (denoted by) $\hat{\mathcal{A}}_{\mathcal{D}_\sigma} = \{\mathcal{A}_{\mathcal{D}_\sigma}(t) : t \in \mathbb{R}\}$ satisfying*

(a) *Compactness: $\forall t \in \mathbb{R}$, $\mathcal{A}_{\mathcal{D}_\sigma}(t)$ is a nonempty compact subset of E_ρ;*

(b) *Invariance:* $U(t,\tau)\mathcal{A}_{\mathcal{D}_\sigma}(\tau)=\mathcal{A}_{\mathcal{D}_\sigma}(t),\ \ \forall t\geqslant\tau$;

(c) *Pullback attraction:* $\hat{\mathcal{A}}_{\mathcal{D}_\sigma}$ *is pullback* $\mathcal{D}_\sigma$*-attracting in the following sense*

$$\lim_{\tau\to-\infty}\mathrm{dist}_{E_\rho}\big(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}_\sigma}(t)\big)=0,\ \ \forall\widehat{D}=\{D(s)|s\in\mathbb{R}\}\in\mathcal{D}_\sigma,t\in\mathbb{R},$$

where $\mathrm{dist}_{E_\rho}(\cdot,\cdot)$ *denotes the Hausdorff semidistance in* E_ρ.

4 Construction of the statistical solutions

In this section, we will use the generalized Banach limit and the pullback attractor $\hat{\mathcal{A}}_{\mathcal{D}_\sigma}$ obtained in Theorem 3.1 to construct the statistical solutions for equation (2.10).

Lemma 4.1. *Let assumptions* (H1)-(H2) *hold. Then for every given* $t\in\mathbb{R}$ *and* $\psi_*\in E_\rho$, *the* E_ρ*-valued mapping* $\tau\longmapsto U(t,\tau)\psi_*$ *is continuous and bounded on* $(-\infty,t]$.

Proof. For every given $t\in\mathbb{R}$ and $\psi_*=(u_*,v_*)^T\in E_\rho$, we derive from (2.17) and (2.23) that

$$\begin{aligned}\|U(t,\tau)\psi_*\|^2_{E_\rho}\leqslant&\|\psi_*\|^2_{E_\rho}e^{-\sigma(t-\tau)}+c_3e^{-\sigma t}\int_{-\infty}^t e^{\sigma s}\big(\|f(s)\|^2_\rho+\|g(s)\|^2_\rho\big)\mathrm{d}s\\&+c_3e^{-\sigma t}\int_{-\infty}^t e^{\sigma s}\Big(e^{-\varrho(s-\tau)}\|u_*\|^2_\rho+\frac{e^{-\varrho s}}{\alpha}\int_{-\infty}^s e^{\varrho\theta}\|f(\theta)\|^2_\rho\mathrm{d}\theta\Big)^2\mathrm{d}s,\ \ \forall t\geqslant\tau.\end{aligned}\tag{4.1}$$

From the proof of Lemma 3.1 and (H2) we are not difficult to see that the right-hand side of (4.1) is bounded by a constant independent of τ. Therefore, $\|U(t,\cdot)\psi_*\|_{E_\rho}$ is bounded on $(-\infty,t]$.

Next we prove the E_ρ-valued mapping $\tau\longmapsto U(t,\tau)\psi_*$ is continuous. Let $s_*\in(-\infty,t]$ be given. We just need to prove that to any $\varepsilon>0$, and the above given t and ψ_*, there corresponds $\delta=\delta(\varepsilon,s_*,t,\psi_*)>0$, such that

$$\text{for every }\ r\in(s_*-\delta,s_*+\delta)\quad\Longrightarrow\quad\|U(t,r)\psi_*-U(t,s_*)\psi_*\|_{E_\rho}<\varepsilon.\tag{4.2}$$

Without loss of generality, we assume $r<s_*$. Write

$$\psi^{(1)}(\cdot)=U(\cdot,s_*)U(s_*,r)\psi_*,\ \ \psi^{(2)}(\cdot)=U(\cdot,s_*)\psi_*,\ \ \psi^{(1)}(\cdot)-\psi^{(2)}(\cdot)=\tilde{\psi}(\cdot),$$

then $\tilde{\psi}(\cdot)$ satisfies

$$\frac{\mathrm{d}\tilde{\psi}(t)}{\mathrm{d}t}+\Theta\tilde{\psi}(t)=F(\psi^{(1)},t)-F(\psi^{(2)},t),\ \ t>s_*,\tag{4.3}$$

$$\tilde{\psi}(s_*)=U(s_*,r)\psi_*-\psi_*.\tag{4.4}$$

Taking the real part of the inner product of (4.3) with $\tilde{\psi}$ in E_ρ gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\psi}(t)\|^2_{E_\rho}+\mathbf{Re}\big(\Theta\tilde{\psi}(t),\tilde{\psi}(t)\big)_{E_\rho}=\mathbf{Re}\big(F(\psi^{(1)},t)-F(\psi^{(2)},t),\tilde{\psi}(t)\big)_{E_\rho}.\tag{4.5}$$

For the term $\mathbf{Re}\big(\Theta\tilde{\psi}(t),\tilde{\psi}(t)\big)_{E_\rho}$ in (4.5), we derive from (2.27) that

$$\begin{aligned}\mathbf{Re}(\Theta\tilde{\psi}(t),\tilde{\psi}(t))_{E_\rho}\geqslant&\alpha\|\tilde{u}(t)\|^2_\rho+\beta\|\tilde{v}(t)\|^2_\rho-\frac{1}{2}(c_1c_2+c_2)\|\tilde{u}\|^2_\rho=\varrho\|\tilde{u}\|^2_\rho+\beta\|\tilde{v}\|^2_\rho+\frac{1}{2}(c_1c_2+c_2)\|\tilde{u}\|^2_\rho\\\geqslant&\varrho\|\tilde{u}\|^2_\rho+\beta\|\tilde{v}\|^2_\rho,\ \ \forall t>s_*.\end{aligned}\tag{4.6}$$

From (4.1) and (H2), we see that there exists a constant $c_5=c_5(t,s_*,\psi_*)>0$ such that

$$\|\psi^{(1)}(t)\|^2_{E_\rho}+\|\psi^{(2)}(t)\|^2_{E_\rho}\leqslant c_5.\tag{4.7}$$

Now (2.16) and (4.7) imply that there exists a constant $c_6 = c_6(c_5) > 0$ such that

$$\mathbf{Re}(F(\psi^{(1)},t) - F(\psi^{(2)},t), \tilde{\psi}(t))_{E_\rho} \leqslant c_6 \|\tilde{\psi}(t)\|^2_{E_\rho}. \tag{4.8}$$

Combining (4.5)-(4.6) and (4.8), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\tilde{\psi}(t)\|^2_{E_\rho} + \sigma\|\tilde{\psi}(t)\|^2_{E_\rho} \leqslant c_6\|\tilde{\psi}(t)\|^2_{E_\rho}. \tag{4.9}$$

Using Gronwall's inequality to (4.9) gives

$$\begin{aligned}\|\tilde{\psi}(t)\|^2_{E_\rho} &\leqslant \|\tilde{\psi}(s_*)\|^2_{E_\rho} \exp\Big\{ \int_{s_*}^t 2(c_6 - \sigma)\mathrm{d}\theta \Big\} \\ &= \|U(s_*, r)\psi_* - \psi_*\|^2_{E_\rho} \exp\Big\{ \int_{s_*}^t 2(c_6 - \sigma)\mathrm{d}\theta \Big\},\end{aligned} \tag{4.10}$$

which indicates that $\|U(s_*, r)\psi_* - \psi_*\|^2_{E_\rho}$ is as small as needed provided that $|r - s_*|$ is small enough. Therefore, the E_ρ-valued function $\tau \longmapsto U(t,\tau)\psi_*$ is continuous on $(-\infty, t]$. □

To state the existence of the Borel probability measure, we recall the definition of generalized Banach limit.

Definition 4.1. ([22]) *A generalized Banach limit is any linear functional, which we denote by* $\mathrm{LIM}_{t\to-\infty}$*, defined on the space of all bounded real-valued functions on* $\mathbb{R}$ *and satisfying*

(1) $\mathrm{LIM}_{t\to-\infty}h(t) \geqslant 0$ *for nonnegative functions* $h(\cdot)$ *on* $(-\infty,+\infty)$*;*

(2) $\mathrm{LIM}_{t\to-\infty}h(t) = \lim\limits_{t\to-\infty} h(t)$ *if the usual limit* $\lim\limits_{t\to-\infty} h(t)$ *exists.*

Let $\mathcal{B}(\mathbb{R})$ be the collection of all real-valued bounded functions on $(-\infty,+\infty)$. For any generalized Banach limit $\mathrm{LIM}_{t\to-\infty}$, we have the following useful property ([11, (1.38)] or [8, (2.3)])

$$|\mathrm{LIM}_{t\to-\infty}h(t)| \leqslant \limsup_{t\to-\infty} |h(t)|, \ \ \forall\, h(\cdot) \in \mathcal{B}(\mathbb{R}). \tag{4.11}$$

In terms of Theorem 3.1, Lemma 4.1 and the abstract result [22, Theorem 3.1, Theorem 4.1][1] we have the following result.

Theorem 4.1. *Let assumptions* (H1)-(H2) *hold. Then for given generalized Banach limit* $\mathrm{LIM}_{\tau\to-\infty}$ *and every* $\zeta_* = (u_*, v_*)^T \in E_\rho$*, there exists a unique family of Borel probability measures* $\{\mu_t\}_{t\in\mathbb{R}}$ *on* E_ρ *such that the support of the measure* μ_t *is contained in* $\mathcal{A}_{\mathcal{D}_\sigma}(t)$*, and*

$$\begin{aligned}\mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t \Phi\big(U(t,s)\zeta_*\big)\mathrm{d}s &= \int_{\mathcal{A}_{\mathcal{D}_\sigma}(t)} \Phi(\psi)\mathrm{d}\mu_t(\psi) = \int_{E_\rho} \Phi(\psi)\mathrm{d}\mu_t(\psi) \\ &= \mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t \int_{E_\rho} \Phi\big(U(t,s)\psi\big)\mathrm{d}\mu_s(\psi)\mathrm{d}s,\end{aligned} \tag{4.12}$$

where $\Phi \in C(E_\rho)$ *and* $C(E_\rho)$ *is the set of all real-valued continuous functionals on* E_ρ*. Moreover,* μ_t *is invariant in the sense that*

$$\int_{\mathcal{A}_{\mathcal{D}_\sigma}(t)} \Phi(\psi)\mathrm{d}\mu_t(\psi) = \int_{\mathcal{A}_{\mathcal{D}_\sigma}(\tau)} \Phi\big(U(t,\tau)\psi\big)\mathrm{d}\mu_\tau(\psi), \ \ t \geqslant \tau. \tag{4.13}$$

[1]In the following Theorem 4.1 we pick a fixed point ζ_* in E_ρ. In this case, the abstract result [22, Theorem 3.1, Theorem 4.1] is completely correct.

In the following we will establish that the family of Borel probability measures $\{\mu_t\}_{t\in\mathbb{R}}$ obtained in Theorem 4.1 is a statistical solution of equation (2.10). Write equation (2.10) as

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = H(\psi,t) := F(\psi(t),t) - \Theta\psi, \quad t \in \mathbb{R}. \tag{4.14}$$

Then for each $t \in \mathbb{R}$, we see from the proof of Lemma 2.1 that $H(\psi,t) : E_\rho \times \mathbb{R} \longmapsto E_\rho$ is continuous. To specify the definition of statistical solution, we need introduce the class $\mathcal{T}$ of test functions. We expect that the test function $\Psi \in \mathcal{T}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(\psi(t)) = \big(\Psi'(\psi), H(\psi,t)\big)_{E_\rho}, \quad t \in \mathbb{R}, \tag{4.15}$$

for every solution $\psi(t)$ of equation (4.14).

Definition 4.2. *We define the class $\mathcal{T}$ of test functions to be the set of real-valued functionals $\Psi = \Psi(\cdot)$ on E_ρ that are bounded on bounded subset of E_ρ and satisfy:*

(a) *for each $\psi \in E_\rho$, the Frechét derivative $\Psi'(\psi)$ exists: for any $\psi \in E_\rho$ there exists an element $\Psi'(\psi) \in E_\rho$ such that*

$$\lim_{\|\phi\|_{E_\rho}\to 0} \frac{|\Psi(\psi+\phi) - \Psi(\psi) - (\Psi'(\psi),\phi)_{E_\rho}|}{\|\phi\|_{E_\rho}} = 0, \qquad \phi \in E_\rho;$$

(b) *the mapping $\psi \longmapsto \Psi'(\psi)$ is continuous and bounded as a functional from E_ρ to E_ρ;*

(c) *for every solution $\psi(t)$ of equation* (4.14), (4.15) *holds true.*

Definition 4.3. *A family of Borel probability measures $\{\nu_t\}_{t\in\mathbb{R}}$ on E_ρ is called a statistical solution of equation* (4.14) *if the following conditions are satisfied:*

(a) *for every $\Phi \in C(E_\rho)$, the function $t \mapsto \int_{E_\rho} \Phi(\psi)\mathrm{d}\nu_t(\psi)$ is continuous;*

(b) *for almost $t \in \mathbb{R}$, the function $\psi \mapsto \big(H(\psi,t),\phi\big)_{E_\rho}$ is ν_t-integrable for every $\phi \in E_\rho$. Moreover, the map*

$$t \mapsto \int_{E_\rho} \big(H(\psi,t),\phi\big)_{E_\rho} \mathrm{d}\nu_t(\psi)$$

belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ for every $\phi \in E_\rho$;

(c) *for every test function $\Psi \in \mathcal{T}$, there holds that*

$$\int_{E_\rho} \Psi(\psi)\mathrm{d}\nu_t(\psi) - \int_{E_\rho} \Psi(\psi)\mathrm{d}\nu_\tau(\psi) = \int_\tau^t \int_{E_\rho} \big(\Psi'(\psi), H(\psi,s)\big)_{E_\rho} \mathrm{d}\nu_s(\psi)\mathrm{d}s, \ \ \forall t,\tau \in \mathbb{R}.$$

The main result of this section reads as follows.

Theorem 4.2. *Let assumptions* (H1)-(H2) *hold. Then the family of Borel probability measures $\{\mu_t\}_{t\in\mathbb{R}}$ obtained in Theorem 4.1 is a statistical solution of equation* (4.14).

Proof. We prove that the family of invariant Borel probability measures $\{\mu_t\}_{t\in\mathbb{R}}$ obtained in Theorem 4.1 satisfies the defining properties (a)-(c) of Definition 4.3.

Firstly, for any given $t_* \in \mathbb{R}$, we derive from (4.12) and (4.13) that

$$\int_{E_\rho} \Phi(\psi)\mathrm{d}\mu_t(\psi) - \int_{E_\rho} \Phi(\psi)\mathrm{d}\mu_{t_*}(\psi) = \int_{\mathcal{A}_{\mathcal{D}_\sigma}(t_*)} \big(\Phi(U(t,t_*)\psi) - \Phi(\psi)\big)\mathrm{d}\mu_{t_*}(\psi), \ \ \forall t > t_*. \tag{4.16}$$

Because that $\|U(t,t_*)\psi-\psi\|_{E_\rho}\longrightarrow 0$ in E_ρ as $t\to t_*^+$, $\Phi\in C(E_\rho)$ and that $\mathcal{A}_{\mathcal{D}_\sigma}(t_*)$ is compact in E_ρ, (4.16) implies

$$\lim_{t\to t_*^+}\int_{E_\rho}\Phi(\psi)\mathrm{d}\mu_t(\psi)=\int_{E_\rho}\Phi(\psi)\mathrm{d}\mu_{t_*}(\psi),\ \ \forall\Phi\in C(E_\rho).$$

Similarly,

$$\lim_{t\to t_*^-}\int_{E_\rho}\Phi(\psi)\mathrm{d}\mu_t(\psi)=\int_{E_\rho}\Phi(\psi)\mathrm{d}\mu_{t_*}(\psi),\ \ \forall\Phi\in C(E_\rho).$$

Hence,

$$\lim_{t\to t_*}\int_{E_\rho}\Phi(\psi)\mathrm{d}\mu_t(\psi)=\int_{E_\rho}\Phi(\psi)\mathrm{d}\mu_{t_*}(\psi),\ \ \forall\Phi\in C(E_\rho),$$

and property (a) is proved.

Secondly, we verify property (b). Note that we have proved that μ_t is carried by $\mathcal{A}_{\mathcal{D}_\sigma}(t)\subset E_\rho$ for every $t\in\mathbb{R}$. Now, for every $\phi\in E_\rho$, we define $\Upsilon_\phi(\cdot):E_\rho\longmapsto\mathbb{R}$ by

$$\Upsilon_\phi(\psi)=(H(\psi,t),\phi)_{E_\rho},\ \ \psi=(u,v)^T\in E_\rho. \tag{4.17}$$

We next establish $\Upsilon_\phi(\cdot)\in C(E_\rho)$. Let $\psi_*=(u_*,v_*)^T\in E_\rho$ be fixed and consider $\psi=(u,v)^T\in E_\rho$ with $\|\psi_*-\psi\|_{E_\rho}\leqslant 1$. Then

$$\begin{aligned}|\Upsilon_\phi(\psi_*)-\Upsilon_\phi(\psi)|&=|\big(H(\psi_*,\cdot)-H(\psi,\cdot),\phi\big)_{E_\rho}|\\&\leqslant|\big(F(\psi_*,t)-F(\psi,t),\phi\big)_{E_\rho}|+|\big(\Theta\psi_*-\Theta\psi,\phi\big)_{E_\rho}|.\end{aligned} \tag{4.18}$$

On the one side, (2.12) shows that Θ: $E_\rho\longmapsto E_\rho$ is a bounded linear operator. Indeed, there exists a constant $c_7=c_7(\alpha,\beta)>0$ such that

$$|\big(\Theta\psi_*-\Theta\psi,\phi\big)_{E_\rho}|\leqslant c_7\|\psi_*-\psi\|_{E_\rho}\|\phi\|_{E_\rho}. \tag{4.19}$$

On the other hand, (2.16) shows that there exists a constant $c_8=c_8(c_1,\gamma,\psi_*)>0$ such that

$$|\big(F(\psi_*,t)-F(\psi,t),\phi\big)_{E_\rho}|\leqslant c_8\|\psi_*-\psi\|_{E_\rho}\|\phi\|_{E_\rho}. \tag{4.20}$$

Inserting (4.19) and (4.20) into (4.18) yields

$$|\Upsilon_\phi(\psi_*)-\Upsilon_\phi(\psi)|\leqslant(c_7+c_8)\|\psi_*-\psi\|_{E_\rho}\|\phi\|_{E_\rho}, \tag{4.21}$$

which implies that the real-valued function $\Upsilon_\phi(\cdot)$ defined by (4.17) belongs to $C(E_\rho)$. Then we conclude from (4.12) that for every $\phi\in E_\rho$, the function $\psi\mapsto(H(\psi,\cdot),\phi)_{E_\rho}=\Upsilon_\phi(\psi)$ is μ_t-integrable. Meanwhile, we have proved in property (a) that the function

$$t\longmapsto\int_{E_\rho}\big(H(\psi,t),\phi\big)_{E_\rho}\mathrm{d}\mu_t(\psi)=\int_{E_\rho}\Upsilon_\phi(\psi)\mathrm{d}\mu_t(\psi)$$

is continuous on $\mathbb{R}$. It obviously belongs to $L^1_{\text{loc}}(\mathbb{R})$.

Lastly, for any solution $\psi(\cdot)$ of (4.14) and any $\Psi\in\mathcal{T}$, we deduce from (4.15) that

$$\Psi(\psi(t))-\Psi(\psi(\tau))=\int_\tau^t\big(\Psi'(\psi(\theta)),H(\psi(\theta),\theta)\big)_{E_\rho}\mathrm{d}\theta. \tag{4.22}$$

Now for any $s<\tau$, let $\psi_*\in E_\rho$ and $\psi(\theta)=U(\theta,s)\psi_*$, $\theta\geqslant s$. Then we use (4.22) to get

$$\Psi(U(t,s)\psi_*)-\Psi(U(\tau,s)\psi_*)=\int_\tau^t\big(\Psi'(U(\theta,s)\psi_*),H(U(\theta,s)\psi_*,\theta)\big)_{E_\rho}\mathrm{d}\theta. \tag{4.23}$$

Using (4.12), (4.23) and Fubini's Theorem, we arrive at

$$
\begin{aligned}
&\int_{E_\rho} \Psi(\psi)\mathrm{d}\mu_t(\psi) - \int_{E_\rho} \Psi(\psi)\mathrm{d}\mu_\tau(\psi) \\
=&\int_{\mathcal{A}_{\mathcal{D}_\sigma}(t)} \Psi(\psi)\mathrm{d}\mu_t(\psi) - \int_{\mathcal{A}_{\mathcal{D}_\sigma}(\tau)} \Psi(\psi)\mathrm{d}\mu_\tau(\psi) \\
=&\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau \int_{E_\rho} \big(\Psi(U(t,s)\psi_*) - \Psi(U(\tau,s)\psi_*)\big)\mathrm{d}\mu_s(\psi_*)\mathrm{d}s \\
=&\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau \int_{E_\rho}\int_\tau^t \big(\Psi'(U(\theta,s)\psi_*), H(U(\theta,s)\psi_*,\theta)\big)_{E_\rho}\mathrm{d}\theta\mathrm{d}\mu_s(\psi_*)\mathrm{d}s \\
=&\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau \int_\tau^t\int_{E_\rho} \big(\Psi'(U(\theta,s)\psi_*), H(U(\theta,s)\psi_*,\theta)\big)_{E_\rho}\mathrm{d}\mu_s(\psi_*)\mathrm{d}\theta\mathrm{d}s.
\end{aligned}
$$

Taking into account the invariance property of the process $U(\theta,s)=U(\theta,\tau)U(\tau,s)$ and the measure μ_t (see (4.13)) we have

$$
\begin{aligned}
&\int_{E_\rho} \big(\Psi'(U(\theta,s)\psi_*), H(U(\theta,s)\psi_*,\theta)\big)_{E_\rho}\mathrm{d}\mu_s(\psi_*) \\
=&\int_{E_\rho} \big(\Psi'(U(\theta,\tau)U(\tau,s)\psi_*), H(U(\theta,\tau)U(\tau,s)\psi_*,\theta)\big)_{E_\rho}\mathrm{d}\mu_s(\psi_*) \\
=&\int_{E_\rho} \big(\Psi'(U(\theta,\tau)\psi_*), H(U(\theta,\tau)\psi_*,\theta)\big)_{E_\rho}\mathrm{d}\mu_\tau(\psi_*).
\end{aligned}
\tag{4.24}
$$

The right-hand side of (4.24) is independent of s. Therefore,

$$
\begin{aligned}
&\int_{\mathcal{A}_{\mathcal{D}_\sigma}(t)} \Psi(\psi)\mathrm{d}\mu_t(\psi) - \int_{\mathcal{A}_{\mathcal{D}_\sigma}(\tau)} \Psi(\psi)\mathrm{d}\mu_\tau(\psi) \\
=&\int_\tau^t \int_{E_\rho} \big(\Psi'(U(\theta,\tau)\psi_*), H(U(\theta,\tau)\psi_*,\theta)\big)_{E_\rho}\mathrm{d}\mu_\tau(\psi_*)\mathrm{d}\theta \\
=&\int_\tau^t \int_{E_\rho} \big(\Psi'(\psi), H(\psi(s),s)\big)_{E_\rho}\mathrm{d}\mu_s(\psi)\mathrm{d}s.
\end{aligned}
\tag{4.25}
$$

The proof of Theorem 4.2 is complete. □

Remark 4.1. *Since μ_t is carried by $\mathcal{A}_{\mathcal{D}_\sigma}(t)\subset E_\rho$ for every $t\in\mathbb{R}$, equation* (4.25) *shows that $\{\mu_t\}_{t\in\mathbb{R}}$ satisfies property (c) of Definition 4.3. We want to point out that if statistical equilibrium has been reached by the lattice long-wave-short-wave resonance equations, then its statistical informations do not change with time, that is $\Psi'(\psi(t))=0$. In this situation,* (4.25) *implies*

$$
\int_{\mathcal{A}_{\mathcal{D}_\sigma}(t)} \Psi(\psi)\mathrm{d}\mu_t(\psi) = \int_{\mathcal{A}_{\mathcal{D}_\sigma}(\tau)} \Psi(\psi)\mathrm{d}\mu_\tau(\psi),\ \ t\geqslant\tau,
\tag{4.26}
$$

which indicates that the shape of the pullback attractor $\mathcal{A}_{\mathcal{D}_\sigma}(\cdot)$ could change with the evolution of time from τ to t, but the "total measures" of $\mathcal{A}_{\mathcal{D}_\sigma}(\tau)$ and $\mathcal{A}_{\mathcal{D}_\sigma}(t)$ coincide with each other. This is the result of Liouville Theorem from Statistical Mechanics.

5 Kolmogorov entropy of the statistical solutions

The goal of this section is to formulate the concept of the Kolmogorov ε-entropy for the statistical solutions $\{\mu_t\}_{t\in\mathbb{R}}$ in the weighted space E_ρ, and then estimate its upper bound.

From now on, we use supp μ_t to denote the support of μ_t. Notice that supp $\mu_t\subset\mathcal{A}_{\mathcal{D}_\sigma}(t)$ for every $t\in\mathbb{R}$. Hence, the open balls in E_ρ with radii ε which is necessary to cover $\mu_t\subset\mathcal{A}_{\mathcal{D}_\sigma}(t)$ shall also cover supp μ_t. Since $\mathcal{A}_{\mathcal{D}_\sigma}(t)$ is compact in E_ρ for every $t\in\mathbb{R}$, the minimal number $\mathbb{N}_\varepsilon(\mathcal{A}_{\mathcal{D}_\sigma}(t))$ of open balls in E_ρ with radii ε which is necessary to cover $\mathcal{A}_{\mathcal{D}_\sigma}(t)$ is finite for any $\varepsilon>0$.

Definition 5.1. *Let $\{\mu_t\}_{t\in\mathbb{R}}$ be the statistical solution obtained in Theorem 4.2. For each $\varepsilon>0$, let $\mathbb{N}_\varepsilon(\mu_t,E_\rho)=\mathbb{N}_\varepsilon(\mu_t)$ be the minimal number of open balls in E_ρ with radii ε which is necessary to cover supp μ_t. The number $\mathbf{K}_\varepsilon(\mu_t)=\mathbf{K}_\varepsilon(\mu_t,E_\rho)=\ln\mathbb{N}_\varepsilon(\mu_t)$ is called the Kolmogorov ε-entropy of the statistical solution $\{\mu_t\}_{t\in\mathbb{R}}$ in E_ρ.*

From Definition 5.1 we see that the Kolmogorov ε-entropy of the statistical solution μ_t is given via that of supp $\mu_t\subset\mathcal{A}_{\mathcal{D}_\sigma}(t)$. In fact, the Kolmogorov ε-entropy of $\mathcal{A}_{\mathcal{D}_\sigma}(t)$ is defined in the same way as Definition 5.1. We know that the number

$$d_F(\mathcal{A}_{\mathcal{D}_\sigma}(t))=\limsup_{\varepsilon\to0^+}\frac{\log_2(\mathbb{N}_\varepsilon(\mathcal{A}_{\mathcal{D}_\sigma}(t)))}{\log_2(1/\varepsilon)}$$

is called the fractal dimension of $\mathcal{A}_{\mathcal{D}_\sigma}(t)$. This number is known also as the box dimension, metric dimension, or entropy dimension. Note that $\mathcal{A}_{\mathcal{D}_\sigma}(t)$ is a compact set in the Hilbert space E_ρ. If $d_F(\mathcal{A}_{\mathcal{D}_\sigma}(t))\leqslant m$ (m is a nonnegative integer), then $\mathcal{A}_{\mathcal{D}_\sigma}(t)$ has topological dimension less than or equal to $2m+1$ and is homeomorphic to a subset of $\mathbb{R}^N$, where $N\leqslant 2m+1$ (cf. [9]).

Lemma 5.1. ([21]) *Let $n\in\mathbb{N}$, $\Lambda_1=\{x=(x_m)_{|m|\leqslant n}:x_m\in\mathbb{R},|x_m|\leqslant r\}\subset\mathbb{R}^{2n+1}$ be a regular polyhedron. Then Λ_1 can be covered by $\mathbb{N}_\varepsilon(\Lambda_1)=\big([r\cdot\frac{2}{\varepsilon}\cdot\sqrt{2n+1}]+1\big)^{2n+1}$ balls in $\mathbb{R}^{2n+1}$ with radii $\frac{\varepsilon}{2}$, where $[x]$ denote maximum integer $\leqslant x$. Also the regular polyhedron $\Lambda_2=\{x=(x_m)_{|m|\leqslant n}:x_m\in\mathbb{C},|x_m|\leqslant r\}\subset\mathbb{C}^{2n+1}$ can be covered by $\mathbb{N}_\varepsilon(\Lambda_2)=\big([r\cdot\frac{2}{\varepsilon}\cdot\sqrt{2n+1}]+1\big)^{2(2n+1)}$ balls in $\mathbb{C}^{2n+1}$ with radii $\frac{\varepsilon}{2}$.*

Next we prove the upper bound estimation for the Kolmogorov ε-entropy of the statistical solutions $\{\mu_t\}_{t\in\mathbb{R}}$.

Theorem 5.1. *Let assumptions* (H1)-(H2) *hold. Then the Kolmogorov ε-entropy of the statistical solutions $\{\mu_t\}_{t\in\mathbb{R}}$ obtained in Theorem 4.2 satisfies*

$$\mathbf{K}_\varepsilon(\mu_t)\leqslant 3(2M_*+1)\ln\Big(\big[\frac{R_\sigma(t)}{\sqrt{\rho_{M_*}}}\cdot\frac{2\sqrt{2}}{\varepsilon}\cdot\sqrt{2M_*+1}\big]+1\Big),\ \ t\in\mathbb{R},\tag{5.1}$$

where $M_*=M_*(\frac{\varepsilon}{2\sqrt{2}},t,R_\sigma(t))$ is the minimal positive integer such that

$$\sup_{(\psi_m)_{m\in\mathbb{Z}}=\psi\in\mathcal{B}_0(t)}\Big(\sum_{|m|>M_*}\rho_m|\psi_m|^2\Big)^{\frac12}\leqslant\frac{\varepsilon}{2}.$$

Proof. For given $t\in\mathbb{R}$ and $\varepsilon>0$, Lemma 3.2 shows that there exists some $M_*=M_*(\frac{\varepsilon}{2\sqrt{2}},t,R_\sigma(t))\in\mathbb{N}$ such that

$$\sup_{(\psi_m)_{m\in\mathbb{Z}}=\psi\in\mathcal{B}_0(t)}\sum_{|m|>M_*}\rho_m|\psi_m|^2\leqslant\frac{\varepsilon^2}{4}.\tag{5.2}$$

Now for any $\psi=(\psi_m)_{m\in\mathbb{Z}}=(u_m,v_m)^T_{m\in\mathbb{Z}}\in\mathcal{A}_{\mathcal{D}_\sigma}(t)\subset\mathcal{B}_0(t)$, we decompose ψ into two parts as

$$\psi=z+y=(z_m)_{\in\mathbb{Z}}+(y_m)_{m\in\mathbb{Z}},\tag{5.3}$$

where

$$z_m=(\varphi_m,\eta_m)^T=\begin{cases}\psi_m, & |m|\leqslant M_*,\\ 0, & |m|>M_*,\end{cases}\qquad y_m=\begin{cases}0, & |m|\leqslant M_*,\\ \psi_m, & |m|>M_*.\end{cases}\tag{5.4}$$

Obviously,

$$\|y\|_{E_\rho} =\Big(\sum_{|m|>M_*}\rho_m|\psi_m|^2\Big)^{\frac{1}{2}} \leqslant \frac{\varepsilon}{2}, \tag{5.5}$$

$$\|z\|_{E_\rho}^2 = \sum_{m\in\mathbb{Z}}\rho_m|z_m|^2 = \sum_{|m|\leqslant M_*}\rho_m|z_m|^2 = \sum_{|m|\leqslant M_*}\rho_m|\psi_m|^2 \leqslant \|\psi\|_{E_\rho}^2 \leqslant R_\sigma^2(t), \tag{5.6}$$

where $|z_m|^2 = |\varphi_m|^2 + \eta_m^2$. From (5.6) we see that

$$\sum_{|m|\leqslant M_*(\varepsilon)}\rho_m(|\varphi_m|^2+\eta_m^2) \leqslant \|\psi\|_{E_\rho}^2 \leqslant R_\sigma^2(t),$$

which implies that

$$|\varphi_m| \leqslant \frac{R_\sigma(t)}{\sqrt{\rho_m}} \leqslant \frac{R_\sigma(t)}{\sqrt{\rho_{M_*}}}, \quad |\eta_m| \leqslant \frac{R_\sigma(t)}{\sqrt{\rho_m}} \leqslant \frac{R_\sigma(t)}{\sqrt{\rho_{M_*}}}, \quad \forall\, |m| \leqslant M_*,$$

because that ρ_m is decreasing with respect to $|m|$. For the regular polyhedron

$$\Gamma_1 = \Big\{\eta = (\eta_m)_{|m|\leqslant M_*} : \eta_m \in \mathbb{R}, |\eta_m| \leqslant \frac{R_\sigma(t)}{\sqrt{\rho_{M_*}}}\Big\} \subset \mathbb{R}^{2M_*+1},$$

Lemma 5.1 indicates that Γ_1 can be covered, under the usual norm of $\mathbb{R}^{2M_*+1}$, by

$$N_\varepsilon^{(1)}(\Gamma_1) = \Big(\Big[\frac{R_\sigma(t)}{\sqrt{\rho_{M_*}}}\cdot\frac{2\sqrt{2}}{\varepsilon}\cdot\sqrt{2M_*+1}\Big]+1\Big)^{2M_*+1}$$

balls in $\mathbb{R}^{2M_*+1}$ with radii $\frac{\varepsilon}{2\sqrt{2}}$. Similarly, the regular polyhedron

$$\Gamma_2 = \Big\{\varphi = (\varphi_m)_{|m|\leqslant M_*} : \varphi_m \in \mathbb{C}, |\varphi_m| \leqslant \frac{R_\sigma(t)}{\sqrt{\rho_{M_*}}}\Big\} \subset \mathbb{C}^{2M_*+1}$$

can be covered, under the usual norm of $\mathbb{C}^{2M_*+1}$, by

$$N_\varepsilon^{(2)}(\Gamma_2) = \Big(\Big[\frac{R_\sigma(t)}{\sqrt{\rho_{M_*}}}\cdot\frac{2\sqrt{2}}{\varepsilon}\cdot\sqrt{2M_*+1}\Big]+1\Big)^{2(2M_*+1)}$$

balls in $\mathbb{C}^{2M_*+1}$ with radii $\frac{\varepsilon}{2\sqrt{2}}$. Hence, the polyhedron

$$\begin{aligned}\Gamma =&\Gamma_1\times\Gamma_2 = \Big\{z = (\varphi_m,\eta_l)^T_{|m|,|l|\leqslant M_*} : |\varphi_m| \leqslant \frac{R_\sigma(t)}{\sqrt{\rho_{M_*}}},\ \ |\eta_l| \leqslant \frac{R_\sigma(t)}{\sqrt{\rho_{M_*}}}\Big\}\\ &\subset\mathbb{C}^{2M_*+1}\times\mathbb{R}^{2M_*+1}\end{aligned}$$

can be covered, under the usual norm of $\mathbb{C}^{2M_*+1}\times\mathbb{R}^{2M_*+1}$, by

$$\begin{aligned}N_\varepsilon(\Gamma) =&N_\varepsilon^{(1)}(\Gamma_1)\times N_\varepsilon^{(2)}(\Gamma_2)\\ =&\Big(\Big[\frac{R_\sigma(t)}{\sqrt{\rho_{M_*}}}\cdot\frac{2\sqrt{2}}{\varepsilon}\cdot\sqrt{2M_*+1}\Big]+1\Big)^{3(2M_*+1)}\end{aligned}$$

balls in $\mathbb{C}^{2M_*+1}\times\mathbb{R}^{2M_*+1}$ with radii $\frac{\varepsilon}{2}$. Let the centers of these $\frac{\varepsilon}{2}$ balls be

$$z_k^* = (\varphi_{km}^*,\eta_{kl}^*)_{|m|,|l|\leqslant M_*} \subset \mathbb{C}^{2M_*+1}\times\mathbb{R}^{2M_*+1},\ \ k = 1,2,\cdots,N_\varepsilon(\Gamma),$$

and write

$$\begin{aligned}\hat{z}_k =&\begin{cases} z_k^*, & \max\{|m|,|l|\} \leqslant M_*,\\ 0, & \max\{|m|,|l|\} > M_*,\end{cases} \qquad k = 1,2,\cdots,N_\varepsilon(\Gamma),\\ \tilde{z} =&(z_m)_{|m|\leqslant M_*},\end{aligned}$$

then $\hat{z}_k \in E_\rho$, $k = 1, 2, \cdots, N_\varepsilon(\Gamma)$, $\tilde{z} \in \mathbb{C}^{2M_*+1} \times \mathbb{R}^{2M_*+1}$. Above analyses show that for any $z = (z_m)_{m\in\mathbb{Z}}$ defined by (5.3) and (5.4), there exists $\hat{z}_k (1 \leqslant k \leqslant N_\varepsilon(\Gamma))$ such that

$$\|z - \hat{z}_k\|_{E_\rho} = \|\tilde{z} - z_k^*\|_{\mathbb{C}^{2M_*+1}\times\mathbb{R}^{2M_*+1}} \leqslant \frac{\varepsilon}{2}.$$

Therefore, for each $\psi = (\psi_m)_{m\in\mathbb{Z}} \in \mathcal{A}_{\mathcal{D}_\sigma}(t) \subset \mathcal{B}_0(t)$,

$$\|\psi - \hat{z}_k\|_{E_\rho} = \|z + y - \hat{z}_k\|_{E_\rho} \leqslant \|z - \hat{z}_k\|_{E_\rho} + \|y\|_{E_\rho} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which shows that $\mathcal{A}_{\mathcal{D}_\sigma}(t)$ can be covered by $N_\varepsilon(\Gamma)$ balls in E_ρ with centers $\hat{z}_k$ and radii ε. The proof is complete. □

6 Conclusions and remarks

In this article, we first prove the existence of statistical solutions for the lattice long-wave-short-wave resonance equations in weighted space, and then present an upper bound estimation of the Kolmogorov ε-entropy for the obtained statistical solutions. The result concerning the existence of invariant Borel probability measures is a generalization of that of [26], from usual $L^2 \times \ell^2$ space to weighted space $L^2_\rho \times \ell^2_\rho$. There are some new difficulties, together with some new phenomena, arise during our investigation. Firstly, the weighted functions produce some additional difficulties when we establish the so-called τ-continuity of the generated process, and this τ-continuity plays the essential role in the construction of the statistical solution. Secondly, we observe that the upper bound of the Kolmogorov ε-entropy of the statistical solutions decreases with respect to the weighted functions.

It is worth remarking that the method used in this article is also valid for some other model lattice system, such as the lattice reaction-diffusion equations (parabolic problem) and lattice nonlinear wave equations (hyperbolic problem) [39], and the coupled lattice Klein-Gordon-Schrödinger equations ([29]), etc. Particularly, reference [38] investigated the statistical solutions and Liouville theorem for the following second order lattice systems

$$\epsilon(t)\ddot{u}_m + \dot{u}_m + (2u_m - u_{m-1} - u_{m+1}) + \lambda_m u_m + f_m(u_m) = g_m(t), \;\; t > \tau, \;\; m \in \mathbb{Z}, \tag{6.1}$$

with varying coefficients on the time-dependent phase spaces. We predict that the results of [38] can be extended to the time-dependent weighted spaces. This is an interesting issue and we will investigate it in another paper.

Finally, we want point out that the fact that system (1.1)-(1.2) is of infinite ODEs produces considerable difficulties in numerical simulation. It seems more reasonable to present numerical examples for the finite-dimensional truncated approximate ODEs of system (1.1)-(1.2) to verify the theory of this article. For each positive integer $n \in \mathbb{N}$, we consider the following finite-dimensional truncated approximate ODEs of system (1.1)-(1.2):

$$\begin{cases} i\dot{u}_{-n} - 2u_{-n} + u_{-n-1} + u_{-n+1} - u_{-n}v_{-n} + i\alpha u_{-n} = f_{-n}(t), \\ \dot{v}_{-n} + \beta v_{-n} + \gamma(|u_{-n}|^2 - |u_{-n-1}|^2) = g_{-n}(t), \\ i\dot{u}_{-n+1} - 2u_{-n+1} + u_{-n} + u_{-n+2} - u_{-n+1}v_{-n+1} + i\alpha u_{-n+1} = f_{-n+1}(t), \\ \dot{v}_{-n+1} + \beta v_{-n+1} + \gamma(|u_{-n+1}|^2 - |u_{-n}|^2) = g_{-n+1}(t), \\ \vdots \\ i\dot{u}_n - 2u_n + u_{n-1} + u_{n+1} - u_n v_n + i\alpha u_n = f_n(t), \\ \dot{v}_n + \beta v_n + \gamma(|u_n|^2 - |u_{n-1}|^2) = g_n(t), \end{cases} \tag{6.2}$$

with the initial conditions

$$u_m(\tau) = u_{m,\tau}, \quad \dot{u}_m(\tau) = u_{1m,\tau}, \quad z_m(\tau) = z_{m,\tau}, \quad m = -n, -n+1, \cdots, n, \;\; \tau \in \mathbb{R}, \tag{6.3}$$

where functions $f_m(t)$ and $g_m(t)$, $|m| \leqslant n$, are exactly as the same ones as in (1.1) and (1.2). Obviously, we can use the same procedure as that in this article to obtain the existence and an upper bound of the Kolmogorov ε-entropy of statistical solution (denoted by $\mu_t^{(n)}$) for system (6.2) in finite-dimensional weighted space

$$\begin{aligned} E_\rho^{2n+1} =& \Big\{ u = (u_m)_{|m|\leqslant n} : \sum_{m=-n}^{n} \rho_m u_m^2 < +\infty, \ \ u_m \in \mathbb{C} \Big\} \\ & \times \Big\{ u = (u_m)_{|m|\leqslant n} : \sum_{m=-n}^{n} \rho_m u_m^2 < +\infty, \ \ u_m \in \mathbb{R} \Big\}. \end{aligned}$$

Then we can use the method of [3] to proceed analysis and simulation, verifying the theory of this article. In addition, the convergence of the statistical solution $\mu_t^{(n)}$ of the finite-dimensional truncated approximate ODEs (6.2) to the statistical solution μ_t of the original system (1.1)-(1.2) as $n \to \infty$ will rationalize above numerical simulation. We will prove this convergence and present some numerical examples in another paper.

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