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Minimum gradation in greyscales of graphs

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ABSTRACT

In this paper we present the notion of greyscale of a graph as a colouring of its vertices that uses colours from the real interval [0,1]. Any greyscale induces another colouring by assigning to each edge the non-negative difference between the colours of its vertices. These edge colours are ordered in lexicographical decreasing ordering and give rise to a new element of the graph: the gradation vector. We introduce the notion of minimum gradation vector as a new invariant for the graph and give polynomial algorithms to obtain it. These algorithms also output all greyscales that produce the minimum gradation vector. This way we tackle and solve a novel vectorial optimization problem in graphs that may generate more satisfactory solutions than those generated by known scalar optimization approaches.

1. Introduction

Graph colouring problems are among the most important combinatorial optimization problems in graph theory because of their wide applicability in areas such as wiring of printed circuits [1], resource allocation [2,3], frequency assignment problem [4,5], a range of scheduling problems [6,7] or computer register allocation [8,9]. A good collection of samples can be found in [10].

There are some works related to map colouring for which the nature of the colours is essential, whereas the number of them is fixed. The *maximum differential graph colouring problem* [11], or equivalently the *antibandwidth problem*, colours the vertices of the graph in order to maximize the smallest colour difference between adjacent vertices and using exactly as many colours as the number of vertices. On the other hand,

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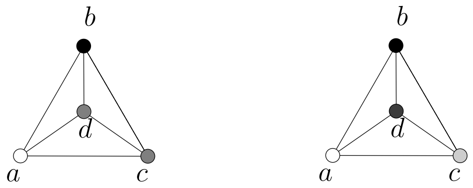


$f([a,b,c,d]) = [0,1,\frac{1}{2},\frac{1}{2}]$ $f'([a,b,c,d]) = [0,1,\frac{1}{3},\frac{2}{3}]$

Fig. 1. Two greyscales f and f' of the graph K_4.

the complementary optimization case, the *bandwidth problem* [12], aims to minimize the maximum colour difference between adjacent vertices. Dillencourt et al. [13] studied a variation of the differential graph colouring problem under the assumption that all colours in the colour spectrum are available. This makes the problem continuous rather than discrete. Recent articles [14] propose to consider a continuous colour spectrum to solve the well-known frequency assignment problem.

The bandwidth and antibandwidth problems attempt to optimize the extreme colours of the edges of the graph, whereas Dillencourt et al. [13] focus on maximizing the sum of the colours of all the edges. Additionally, other previous papers have also explored different sum functions under this last approach (for instance, see [15]). Nonetheless, both cases, extreme values and sum functions, deal with scalar objective functions.

In this line, the present paper tackles mappings taking values within the continuous spectrum $[0,1]$, where 0 and 1 correspond to *white* and *black colours*, respectively, and the rest of the intermediate values are *grey tones*. Graphs in this work are finite, undirected and simple, and are denoted by $G(V,E)$, where V and E are their vertex-set and edge-set, respectively. The number of elements of V and E is denoted by n and m, respectively (for further terminology we follow [16]). We deal with the recent concept of *greyscale* f of G which is a mapping $f : V \longrightarrow [0,1]$ such that $f^{-1}(0) \neq \emptyset$ and $f^{-1}(1) \neq \emptyset$ [17]. For each vertex v of G, we call $f(v)$ the *grey tone* or *colour of* v and notice that two adjacent vertices can have mapped the same grey tone. In particular, values 0 and 1 are called *the extreme tones*. Hence, the notion of *complementary greyscale* arises for each greyscale f such that it maps every vertex v of G to $1 - f(v)$.

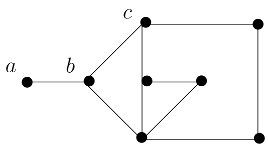
Associated to each greyscale f of the graph $G(V,E)$, the mapping $\widehat{f} : E \to [0,1]$ is defined as $\widehat{f}(e) = |f(u) - f(v)|$ for every edge $e = \{u,v\} \in E$ and represents the gap or increase between the grey tones of the vertices u and v. The value $\widehat{f}(e)$ is also said to be the *grey tone* of the edge e. Thus, we deal with *coloured vertices and edges* by f and $\widehat{f}$, respectively. Note that the same mapping $\widehat{f}$ associated to the greyscale f and its complementary one is obtained. The *gradation vector* associated to the greyscale f of G is the vector $grad(G,f) = (\widehat{f}(e_1), \widehat{f}(e_2), \ldots, \widehat{f}(e_m))$, where the edges of G are indexed such that $\widehat{f}(e_i) \geq \widehat{f}(e_j)$ whether $i < j$, that is, in descending order of their grey tones. For the sake of clarity, and when the graph is fixed, the gradation vector associated to a greyscale f will be denoted by $\mathcal{G}_f$.

Fig. 1 shows two greyscales of the graph K_4, f and f', whose corresponding gradation vectors are $\mathcal{G}_f = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ and $\mathcal{G}_{f'} = (1, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, respectively.

Given two greyscales f and f' of a graph G, we say that f has *better gradation* than f' if the gradation vector $\mathcal{G}_f$ is smaller than $\mathcal{G}_{f'}$ following the lexicographical order, that is, $\mathcal{G}_f < \mathcal{G}_{f'}$. Thus, the descending order of gradation vectors determines the goodness in terms of gradation. Then, f is said to be *smaller or greater by gradation* than f' if $\mathcal{G}_f < \mathcal{G}_{f'}$ or $\mathcal{G}_f > \mathcal{G}_{f'}$, respectively. It can be observed that $\mathcal{G}_f < \mathcal{G}_{f'}$ for the two greyscales in Fig. 1.

A greyscale of a graph G whose gradation vector is minimum is called a *minimum gradation greyscale* of G and the following problem is formulated:

Minimum gradation on graphs (MIGG): given a connected graph $G(V,E)$, finding the minimum gradation vector and all the minimum gradation greyscales.



$$f_1([a, b, \dots, h]) = \left[0, \tfrac{1}{3}, \tfrac{2}{3}, 1, \tfrac{7}{9}, \tfrac{5}{9}, \tfrac{11}{18}, \tfrac{21}{36}\right]$$

$$f_2([a, b, \dots, h]) = \left[0, \tfrac{1}{3}, \tfrac{5}{9}, \tfrac{7}{9}, 1, \tfrac{2}{3}, \tfrac{11}{18}, \tfrac{23}{36}\right]$$

$$\mathcal{G}_{f_1} = \mathcal{G}_{f_2} = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{9}, \tfrac{2}{9}, \tfrac{2}{9}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{36})$$

Fig. 2. The minimum gradation vector can be achieved from different greyscales.

Notice that, given a graph, the minimum gradation vector is unique but different minimum gradation greyscales which give rise to it can exist (see an example in Fig. 2).

We define the restricted version of the minimum gradation problem when the grey tones of some vertices are known a priori, and the aim is to obtain the minimum gradation vector preserving the fixed grey tones. This situation leads to the concept of incomplete greyscale. Given a graph $G(V,E)$ and a nonempty proper subset V_c of V, *an incomplete* V_c*-greyscale* of G is a mapping on V_c to the interval $[0,1]$. A greyscale f is *compatible* with an incomplete V_c-greyscale g if $f(u) = g(u)$ for all $u \in V_c$.

Restricted minimum gradation on graphs (RMIGG): given a connected graph $G(V,E)$ and an incomplete V_c-greyscale g of G, finding the gradation vector that is minimum among all the gradation vectors of greyscales compatible with g, as well as determining all these greyscales.

Solving each of these problems means finding the appropriate minimum gradation vector and all their minimum gradation greyscales except the complementary ones. Note that the MIGG and RMIGG problems are posed for connected graphs but general graphs can be also considered, and in this case each connected component has to be studied separately.

In an analogous way, the notion of *contrast in greyscales of graphs* is widely studied in work [17] by the same authors of this paper. Particularly, the maximum contrast problem is formulated by using the vector $(\widehat{f}(e_m), \widehat{f}(e_{m-1}), \dots, \widehat{f}(e_1))$ associated to the greyscale f; observe that the lexicographical ascending ordering is considered. This new problem belongs to the NP-hard class since it is related to the chromatic number problem as it is proved in [17].

Now let us focus on the contribution of this new notion we have just introduced. At present, although the statement of these problems seems to be quite simple, we have not found minimum gradation problems studied in these terms in the literature. The gradation vector leads us to a vectorial objective function which allocates grey tones in a manner which is both local and global: it is local due to the fact that the colour of every particular edge belongs to the gradation vector; and it is global because all edges of the graph participate in the vectorial objective function. Recall that the classical minimax criterium minimizes the maximal component of the vector, while the minisum criterium minimizes the sum of all components of the vector. Fig. 3 visually displays an example of the goodness of minimum gradation vectors versus scalar optimization. Every vertex has been associated to a *big pixel* which is coloured with its grey tone, and these *big pixels* are next to each other according to the adjacencies between vertices. The idea of gradation is clearly better shown in Fig. 3(a) in the sense that the changes of colours are smooth.

Minimum gradation in greyscales could contribute to numerous problems concerning real networks. For instance, the community detection problem has become extremely useful in a wide variety of different areas such as biological, social, technological, and information networks. This problem aims to identify special groups of vertices in a graph with high concentrations of edges within such vertices and low concentrations between these groups. This peculiarity of real networks is called community structure [18], or clustering in graphs. For an extensive report on this topic see [19].

The graph may have a hierarchical structure, that is, it may arrange several levels of grouping of vertices, with small clusters included within large clusters, which are in turn included in larger clusters, etcetera. In such cases, hierarchical clustering algorithms may be used [20], that is, clustering techniques that disclose the multilevel structure of the graph. The starting point of any hierarchical clustering method is the definition

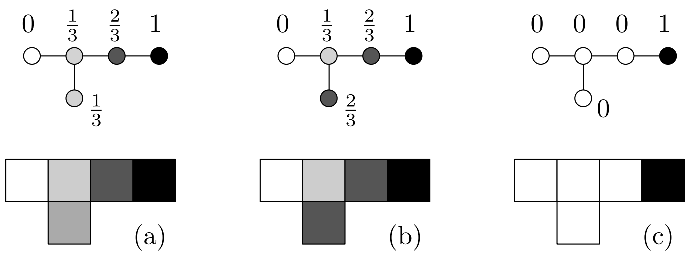


Fig. 3. Comparison of minimum greyscales according to different criteria about the colours of edges: (a) minimum gradation vector, (b) minimax scalar criterium; and (c) minisum scalar criterium.

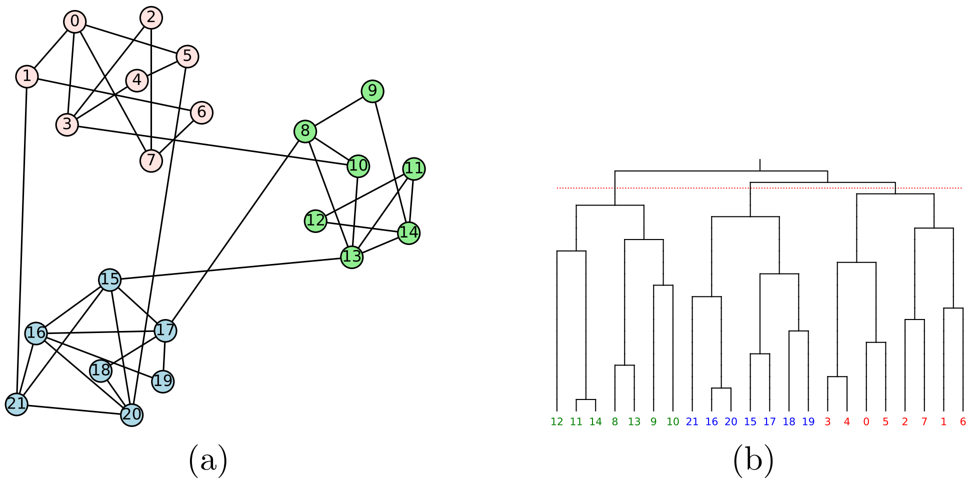


Fig. 4. (a) A simple graph with three communities. (b) Hierarchical clustering from a minimum gradation greyscale for that graph.

of a similarity or weight measure between groups of vertices or clusters. Then, these clusters can iteratively be merged by agglomerative algorithms if their similarity is sufficiently high, and starting with single vertex clusters.

In the literature, many different weights have been proposed to be used in hierarchical clustering algorithms. In this line, the concept of gradation vector allows us to define a new weight measure which quantifies similarities between vertex clusters, such that a pair of vertices will have high similarity value if their distance in the graph is small, where edges are weighted by the values of $\hat{f}$. Then, it is possible to manage similarities between two clusters working with the mean among all the weighted distances of pairs of vertices, one belonging to each cluster. Hence, similarities are expressed in terms of edge colours, and high similarity values are obtained for small gradation vectors. Furthermore, the image $[0, 1]$ of any greyscale leads to the needed variation range for weights in hierarchical clustering algorithms. In particular, the values 0 and 1 correspond to maximum and minimum similarities, respectively.

Given any connected graph, we have implemented the above procedure to disclose the possible cluster structure of the graph. Fig. 4 shows a graph (left) whose three communities are perfectly identified by the dendrogram (right) obtained from a minimum gradation greyscale.

It is well known that graph theory is used to modelize many kinds of networks services. For water supply networks, problems such as minimizing the amount of dissipated power in the water network and establishing pressure control techniques, among others, are studied. In [21], the method of graph partitioning is proposed to solve them, which consists of creating subsystems to simplify water balance and identify water losses. In

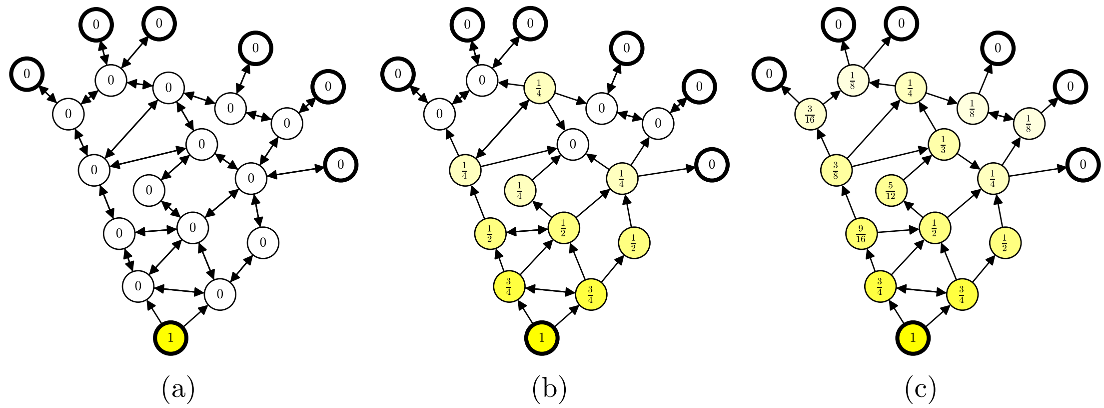


Fig. 5. Pressure assignment to the nodes of a water distribution network. There is a source vertex and six sink vertices, highlighted in thick black line. The direction of flow ranges from high to low pressure. In (a) the minisum criterium is applied. In (b) a solution under the minimax criterium is shown. In (c) a better pressure assignment is shown, obtained from a solution of the RMIGG problem.

this line, the above posed technique to divide the graph into clusters or subsystems based on minimum gradation may be applied to this water network partitioning problem.

On the other hand, a minimum gradation greyscale could be useful to model a possible almost uniform distribution of a service through a network, from sources to sinks. Given a water supply network and the graph that models it, any minimum gradation greyscale of the graph would help to design an appropriate water distribution through the network, in such a manner that water pressure losses between contiguous pipes would be minimized. Thus, sources and sinks could be represented by vertices coloured with extreme tones and the aim of obtaining the minimum gradation vector preserving these fixed grey tones is set out. That is, the restricted minimum gradation problem on graphs could provide a model to tackle the different problems that arise in water supply networks.

Fig. 5 shows three possible pressure assignments to the nodes of a simplified network model of fluid distribution. The extreme tone 1 has been preassigned to a single source vertex while the value 0 is preassigned to the six sinks of the network. The solutions provided by optimal values of scalar functions following minisum criterium (a) and minimax criterium (b) are worse than that obtained from the result of the RMIGG problem (c).

Therefore, in this paper we present the notions of greyscale of a graph and minimum gradation vector as a novel vectorial optimization problem in graphs. The outline of the paper is as follows: Section 2 formally introduces theoretical results concerning the key of minimum colour assignment on graphs. In Section 3, the polynomial nature of both problems, *minimum gradation* and *restricted minimum gradation on graphs* is proved by designing algorithms that provide minimum gradation vectors and all greyscales that give rise to them. Finally, in Section 4 we conclude with some remarks and highlight some open problems.

2. Uniform colour assignment in gradation

In this section results about the nature of gradation problems are established, which let us prove the correctness of our polynomial procedures in Section 3.

The notion of distance in graphs and, in particular, the geodesics play an essential role in the MIGG problem. Given a connected graph G, the *distance* $d(u,v)$ between two vertices u and v in G is the length of a shortest path joining them; a $u-v$ *path* is a path joining the u and v vertices of G and a $u-v$ *geodesic* is a shortest $u-v$ path. The *diameter* $d(G)$ is the length of any longest geodesic and two vertices u and v are *antipodal* if $d(u,v)=d(G)$.

Given a greyscale f of a connected graph $G(V,E)$, the *edge-colour-increase mapping* F is defined as the mapping $F : V \times V \longrightarrow [0,1]$ such that

$$F(u,v) = \begin{cases} \frac{|f(u)-f(v)|}{d(u,v)} & \text{if } u \neq v, \\ 0 & \text{if } u = v. \end{cases}$$

The value of $F(u,v)$ can be viewed as 'the amount of colour' that every edge of any $u-v$ geodesic would be given whether the colour increase between u and v were fairly distributed along the $u-v$ geodesic. Throughout this section, the relationship between $\hat{f}$ and $F(u,v)$ is studied.

Our next results are established on the set of geodesics of the given graph and they state several links between the values of f, $\hat{f}$ and F, according to the position of the vertices into the geodesics.

Lemma 2.1. *Let f be a greyscale of a graph $G(V,E)$ and let u and v be a pair of vertices of G. For each vertex w of each $u-v$ geodesic and different from u and v, it holds that,*

$$F(u,v) < max\{F(u,w), F(w,v)\} \quad or \quad F(u,v) = F(u,w) = F(w,v).$$

Moreover, the above equalities only hold whenever $f(w)$ belongs to the interval of extremes $f(u)$ and $f(v)$.

Proof. For the sake of clarity and without loss of generality we may suppose that $f(u) \leq f(v)$. It is clear that $d(u,v) = d(u,w) + d(w,v)$, and since w is different from u and v, then $d(u,w) < d(u,v)$ and $d(w,v) < d(u,v)$. We proceed according to the relative size of $f(w)$ with respect to $f(u)$ and $f(v)$.

1. If $f(w) \notin [f(u), f(v)]$, it is easy to check that $F(u,v) < F(w,v)$ or $F(u,v) < F(u,w)$ and so $F(u,v) < \max\{F(u,w), F(w,v)\}$ trivially follows.
2. If $f(u) \leq f(w) \leq f(v)$, we prove that it is not possible $F(u,v) > \max\{F(u,w), F(w,v)\}$ and if $F(u,v) = \max\{F(u,w), F(w,v)\}$, then the three values of F are equal.
   Assume to the contrary that $F(u,v) > \max\{F(u,w), F(w,v)\}$. Hence, $f(w) - f(u) < d(u,w)F(u,v)$ and $f(v) - f(w) < d(v,w)F(u,v)$, and then $f(v) - f(u) < (d(u,w) + d(w,v))F(u,v) = d(u,v)F(u,v) \Rightarrow F(u,v) < F(u,v)$, which is a contradiction.
   Next, if $F(u,v) = \max\{F(u,w), F(w,v)\}$ we assume without loss of generality that $F(u,v) = F(u,w)$. Then, by replacing $f(w) - f(u) = (f(v) - f(u))\frac{d(u,w)}{d(u,v)}$ in $F(w,v) = \frac{(f(v)-f(u))-(f(w)-f(u))}{d(w,v)}$, we obtain

$$F(w,v) = \frac{(f(v)-f(u))[d(u,v)-d(u,w)]}{d(u,v)d(w,v)} = \frac{(f(v)-f(u))d(v,w)}{d(u,v)d(w,v)} = F(u,v). \quad \square$$

The following result establishes connections between the mappings $\widehat{f}$ and F on geodesics.

Lemma 2.2. *Let f be a greyscale of a graph $G(V,E)$, let u and v be a pair of vertices of G and let P_{u-v} be a $u-v$ geodesic. The following statements hold:*

1. $F(u,v) \leq \max_{e \in P_{u-v}} \widehat{f}(e)$.
2. *If* $F(u,v) = \max_{e \in P_{u-v}} \widehat{f}(e)$ *then* $F(u,v) = \widehat{f}(e)$ *for every edge* $e \in P_{u-v}$.

Proof. For the sake of simplicity and without loss of generality, let the alternating sequence of vertices and edges $\{u = w_0, e_1, w_1, e_2, w_2, \ldots, w_{l-1}, e_l, v = w_l\}$ be the $u-v$ geodesic P_{u-v} where $f(u) \leq f(v)$, and hence $l = d(u,v)$.

1. A stronger assertion will be stated, that is

$$F(u,v) \leq \max\{\widehat{f}(e_1), \ldots, \widehat{f}(e_i), F(w_i, v)\}$$

for $i = 1, \ldots, l-1$. For $i = 1$, Lemma 2.1 applied to w_1 of P_{u-v} and the fact that $\widehat{f}(e_1) = F(u, w_1)$ lead trivially to $F(u, v) \leq \max\{\widehat{f}(e_1), F(w_1, v)\}$.
Inductively, let us suppose that $F(u, v) \leq \max\{\widehat{f}(e_1), \ldots, \widehat{f}(e_i), F(w_i, v)\}$. Lemma 2.1 is again applied, in this case to w_{i+1} as vertex of the path $\{w_i, e_{i+1}, w_{i+1}, \ldots, v\}$, obtaining that

$$F(w_i, v) \leq \max\{F(w_i, w_{i+1}), F(w_{i+1}, v)\} = \max\{\widehat{f}(e_{i+1}), F(w_{i+1}, v)\}.$$

Thus, the induction hypothesis and this inequality about $F(w_i, v)$ give rise to the result for $i+1$.

2. Assume to the contrary that there exists e_j an edge of P_{u-v} such that $\widehat{f}(e_j) < F(u, v)$. The following intervals are considered for $1 \leq i \leq l$:

$$I_i = \begin{cases} [f(w_{i-1}), f(w_i)] & \text{if} \quad f(w_{i-1}) < f(w_i) \\ [f(w_i), f(w_{i-1})] & \text{if} \quad f(w_i) < f(w_{i-1}) \\ \emptyset & \text{if} \quad f(w_{i-1}) = f(w_i) \end{cases}$$

Thus, the union of these intervals is a cover of $[f(u), f(v)]$ and therefore the following contradiction is achieved:

$$\begin{aligned} f(v) - f(u) &\leq \sum_{i=1}^{l} |f(w_{i-1}) - f(w_i)| = \sum_{i=1}^{l} \widehat{f}(e_i) = \widehat{f}(e_j) + \sum_{i=1, i\neq j}^{l} \widehat{f}(e_i) \leq \\ &\leq \widehat{f}(e_j) + (l-1) \max_{1\leq i\leq l} \widehat{f}(e_i) = \widehat{f}(e_j) + (l-1)F(u, v) < \\ &< F(u, v) + (l-1)F(u, v) = lF(u, v) = f(v) - f(u). \quad \square \end{aligned}$$

The following result highlights the key role that the maximum value of the edge-colour-increase mapping F plays in the MIGG problem.

Corollary 2.3. *Let f be a greyscale of a connected graph $G(V, E)$ and let u and v be a pair of vertices of G such that $F(u, v) = \max_{a,b\in V} F(a, b)$ and $f(u) \leq f(v)$. Then,*

1. *$f(w) = f(u) + d(u, w)F(u, v)$ for each vertex w of a $u - v$ geodesic.*
2. *$\widehat{f}(e) = F(u, v)$ for each edge e of a $u - v$ geodesic.*

Proof.

1. If w is u or v, then the result holds trivially. Otherwise, since $F(u, v) = \max_{a,b\in V} F(a, b)$, in particular, it holds that

$$F(u, v) \geq \max\{F(u, w), F(w, v)\}.$$

Then, by Lemma 2.1, $F(u, v) = F(u, w) = F(w, v)$ and $f(w)$ belongs to the interval $[f(u), f(v)]$. Now, the result follows immediately:

$$F(u, v) = F(u, w) = \frac{f(w) - f(u)}{d(u, w)} \Rightarrow f(w) = f(u) + d(u, w)F(u, v).$$

2. Let a and b be the vertices of the edge e. Statement *1* is applied to a and b, and since $d(u, b) = d(u, a) \pm 1$, it holds that $f(a) = f(u) + d(u, a)F(u, v)$ and $f(b) = f(u) + (d(u, a) \pm 1)F(u, v)$.
Then, $\widehat{f}(e) = |f(b) - f(a)| = |\pm F(u, v)| = F(u, v)$. $\square$

Next, a property for minimum gradation greyscales related to antipodal vertices which are white-and-black coloured is established. Furthermore, the first components of the minimum gradation vector are characterized.

Theorem 2.4. *Let f be a minimum gradation greyscale of a graph G. Then,*

1. *If u and v are vertices of G such that $f(u) = 0$ and $f(v) = 1$, then u and v are antipodal vertices.*
2. *At least the first $d(G)$ components of the minimum gradation vector of G are $\frac{1}{d(G)}$.*

Proof. Given any pair of antipodal vertices u and v of G, let $f\langle u,v\rangle$ be the mapping on $V(G)$ defined as $f\langle u,v\rangle(w) = \frac{d(w,u)-d(w,v)+d(G)}{2d(G)}$. It is straightforward that $f\langle u,v\rangle$ is a greyscale of G such that $f\langle u,v\rangle(u) = 0$ and $f\langle u,v\rangle(v) = 1$.

Moreover, for any edge $\{w_1, w_2\}$ and for any vertex t, it is easy to check that $d(w_1,t)-d(w_2,t) \in \{-1, 0, 1\}$. Hence, we get the value

$$\widehat{f}\langle u,v\rangle(\{w_1,w_2\}) = \left|\frac{d(w_2,u) - d(w_2,v) + d(G) - d(w_1,u) + d(w_1,v) - d(G)}{2d(G)}\right|.$$

Therefore, the range of $\widehat{f}\langle u,v\rangle$ is $\{\frac{1}{d(G)}, \frac{1}{2d(G)}, 0\}$.

Hence, as f is a minimum gradation greyscale, $\widehat{f}(e) \leq \frac{1}{d(G)}$ for all $e \in E$ and so $\max_{e\in E} \widehat{f}(e) \leq \frac{1}{d(G)}$.

1. Since $f(u) = 0$ and $f(v) = 1$, $F(u,v) = \frac{1}{d(u,v)}$ and by Lemma 2.2 (part 1), $F(u,v) = \frac{1}{d(u,v)} \leq \max_{e\in P_{u-v}} \widehat{f}(e) \leq \max_{e\in E} \widehat{f}(e) \leq \frac{1}{d(G)}$. Then $d(u,v) \geq d(G)$, that is, $d(u,v) = d(G)$.
2. For a pair of vertices u and v given by Statement *1* and according to Lemma 2.2 (part 1), it holds that $F(u,v) = \frac{1}{d(G)} = \max_{e\in P_{u-v}} \widehat{f}(e)$, where P_{u-v} is any $u-v$ geodesic, obviously, of length $d(G)$. Last, by Lemma 2.2 (part 2), $F(u,v) = \frac{1}{d(G)} = \widehat{f}(e)$ for every edge $e \in P_{u-v}$. □

The characterization of the minimum gradation vector for trees can be directly obtained from the above corollary.

Corollary 2.5. *The minimum gradation vector of a tree T of diameter $d(T)$ is the vector whose $d(T)$ first components are equal to $\frac{1}{d(T)}$ and the remaining components are null.*

3. Solving the minimum gradation problems

Our next aim is to design algorithms which provide all the minimum gradation greyscales of a connected graph, for both MIGG and RMIGG problems. These greyscales are obtained in a stepwise manner by incomplete greyscales such that each of these is compatible with the previous one. Thus, an iterative procedure based on the operation of deleting coloured edges and isolated coloured vertices is carried out.

Firstly, we devise the V_c-COMPATIBLE-COMPLETE-MAPPING common subroutine which is applied to solve both the different RMIGG problems according to the possible existence of the extreme tones as prefixed colours, and the MIGG problem.

PROCEDURE: V_c-COMPATIBLE-COMPLETE-MAPPING
Input: An incomplete V_c-greyscale g of a connected graph $G(V,E)$.
Output: A mapping f on V compatible with g.

1. Initialize $G_1(V_1,E_1) \leftarrow G(V,E)$
2. Initialize $i \leftarrow 1$
3. Initialize $V_1^1 \leftarrow V_1$ and the number of connected components $l(i) \leftarrow 1$
4. **For** $u \in V_c$ **do** $f(u) = g(u)$
5. **While** $|V_c| < |V|$ **do**

(a) Compute the distance matrix D_i of graph G_i
(b) Compute the finite value $M_i = \max_{1\leq j\leq l(i)}\{F(a,b) : \{a,b\} \subseteq V_i^j \cap V_c\}$ and the set $S_i = \{\{u,v\} \subseteq V_i^j \cap V_c : F(u,v) = M_i, 1 \leq j \leq l(i)\}$, where distances are taken from D_i
(c) Initialize $A \leftarrow \emptyset$
(d) **For** each $\{u,v\} \in S_i$ and considering distances from D_i **do**

   i. Compute $A_{u,v} = \{w \in V_i : w \in$u-vgeodesic of $G_i\}$
   ii. **For** each $w \in A_{u,v}$ **do**

$$f(w) = \begin{cases} f(u) + d(w,u)\ M_i & \text{if} \quad f(u) \leq f(v) \\ f(v) + d(w,v)\ M_i & \text{if} \quad f(u) > f(v) \end{cases}$$

   iii. Actualize $A \leftarrow A \cup A_{u,v}$

(e) Actualize $V_c \leftarrow V_c \cup A$
(f) Let $G_{i+1}(V_{i+1}, E_{i+1})$ be the subgraph of $G_i(V_i, E_i)$ obtained by deleting all the edges w_1w_2 with w_1, $w_2 \in A$ and removing the resulting isolated vertices. Let $l(i+1)$ be the number of connected components of G_{i+1}. Let V_{i+1}^j be the vertex-sets of the connected components of G_{i+1} for $j = 1 \ldots l(i+1)$
(g) **If** $S_i = \emptyset$, each set V_i^j contains exactly one vertex w^j in V_c **then**

   i. **For** $j := 1$ to $l(i)$ **do** $f(u) = f(w^j)$ with $u \in V_i^j$
   ii. Actualize $V_c \leftarrow V_c \cup V_i^1 \cup \cdots \cup V_i^{l(i)}$

(h) Actualize $i \leftarrow i + 1$

In general terms, the V_c-Compatible-Complete-Mapping iterative procedure extends the input incomplete V_c-greyscale to a mapping on V by assigning colours based on maximum values of edge-colour-increase mappings. Afterwards, this mapping will lead us to the solution of the migg problem and the different rmigg problems according to the possible existence of the extreme tones as prefixed colours.

The main step of the algorithm is the while-loop of Step 5, which works while vertices without colour exist. Thus, in every execution of this step, the maximum value of the edge-colour-increase mapping and all the pairs of vertices that reach this value are computed, and Step 5(d)ii assigns colours to the vertices lying on the geodesics joining such pairs of vertices. Next, edges with coloured extreme vertices and isolated coloured vertices are removed.

Remark 3.1. For every edge e of G either its vertices belong to the set A at only one execution of Step 5d since it may be removed in Step 5f, or its vertices has been assigned the same colour in Step 5g. In the second case, e contributes with $\widehat{f}(e) = 0$ in $grad(G, f)$.

In the first case, by applying Corollary 2.3.*2.* to each pair of vertices in S_i and every connected component of G_i at which the maximum value M_i is reached, Step 5d will produce $\widehat{f}(e) = M_i$.

Remark 3.2. Note that the mapping generated by V_c-Compatible-Complete-Mapping procedure and the input incomplete V_c-greyscale have the same range of grey tones. Therefore, that mapping is not necessarily a greyscale due to the possible nonexistence of the extreme tones as values reached by the mapping. Additionally, the grey tone assigned by Step 5(d)ii can also be obtained as follows:

$$f(w) = \begin{cases} f(v) - d(w,v)\ M_i & \text{if} \quad f(u) \leq f(v) \\ f(u) - d(w,u)\ M_i & \text{if} \quad f(u) > f(v) \end{cases}$$

Next, the rmigg problems are resolved according to the nature of the prefixed colours, that is, distinguishing whether or not the extreme tones are prefixed values.

Lemma 3.3. *The procedure V_c-Compatible-Complete-Mapping is finite and has time computational complexity of $O(n^4)$ for any connected graph of order n.*

Proof. At least one vertex is coloured at every iteration of the while-loop in Step 5, either by Substep 5(d)ii or 5g, and hence it ends after at most $|V| - |V_c|$ iterations. The time complexity of computing distance matrices (Step 5a) dominates the time complexity of the rest of the steps and that can be done in $O(n^3)$ time applying the Floyd–Warshall algorithm [22,23]. So, the while-loop and the Step 5a determine the polynomial time of the V_c-Compatible-Complete-Mapping algorithm, that is, $O(n^4)$ time. □

Theorem 3.4. *Let g be an incomplete V_c-greyscale of a connected graph $G(V, E)$ such that the extreme tones 0 and 1 are reached by g. Then it is possible to solve the associated RMIGG problem in $O(n^4)$ time.*

Moreover, the solution to RMIGG problem is unique and the associated greyscale is provided by the V_c-Compatible-Complete-Mapping algorithm.

Proof. The proof splits into three claims.

Claim 1. *The algorithm outputs a well-defined greyscale f compatible with g.*

The values of f are either the values of g (Step 4) or are assigned by Step 5(d)ii to vertices belonging to $u - v$ geodesics such that $M_i = F(u, v)$. Owing to Step 4, f is compatible with g and since both extreme tones are prefixed colours, its range is the interval $[0, 1]$.

Let us check that the colour assignment by Step 5(d)ii is consistent, that is, both when a vertex in V_c is again coloured by Step 5(d)ii in the *ith*-iteration; and in the case of a vertex belonging to different such geodesics.

First, let w be a vertex with colour $f(w)$ belonging to a $u-v$ geodesic such that $M_i = F(u, v)$ (maximum value of Step 5b); in particular, $F(u, v)$ is greater than $F(u, w)$ and $F(w, v)$. Hence, by applying Lemma 2.1 for f, u, v and the subgraph induced by $V_{i-1} \cap V_c$, we can affirm $F(u, v) = F(u, w) = F(w, v)$ and, moreover $f(w)$ belongs to the interval of extremes $f(u)$ and $f(v)$. Whether $f(u) \leq f(v)$, it holds that

$$F(u, v) = F(u, w) \Rightarrow M_i = \frac{f(w) - f(u)}{d(w, u)} \Rightarrow f(w) = f(u) + d(w, u)\, M_i.$$

In other words, the value assigned to w by Step 5(d)ii coincides with the previous colour of w. When $f(u) > f(v)$ the reasoning is similar taking into account that $F(u, v) = F(w, v)$.

On the other hand, let $u_1 - v_1$ and $u_2 - v_2$ be two geodesics such that $M_i = F(u_1, v_1) = F(u_2, v_2)$ (we assume, without loss of generality, $f(u_1) \leq f(v_1)$ and $f(u_2) \leq f(v_2)$) and let w be a vertex in $V - V_c$ belonging to both geodesics. In order to prove $f(w) = f(u_1) + d(u_1, w)\, M_i = f(u_2) + d(u_2, w)\, M_i$, let us suppose on the contrary that $f(u_1) + d(u_1, w)\, M_i > f(u_2) + d(u_2, w)\, M_i$ (the arguments are similar if the inequality "<" is assumed), and therefore

$$f(u_1) > f(u_2) + (d(u_2, w)) - d((u_1, w))\, M_i, \tag{1}$$

From $M_i = F(u_1, v_1)$ it holds that

$$f(v_1) = f(u_1) + (d(u_1, w) + d(w, v_1))\, M_i. \tag{2}$$

Taking into account (1) and (2),

$$f(v_1) > f(u_2) + (d(u_2, w) + d(w, v_1))\, M_i \geq f(u_2) + d(u_2, v_1)\, M_i \Rightarrow$$
$$\Rightarrow \frac{f(v_1) - f(u_2)}{d(u_2, v_1)} \geq M_i.$$

which is a contradiction.

This finishes the proof of Claim 1.

Claim 2. *$grad(G,f) = (M_1,\ldots,M_1,M_2,\ldots,M_2,\ldots,M_r,\ldots,M_r,0,\ldots,0)$, where r is the number of executions of the while-loop, possibly with no zeros.*

By Remark 3.1, let us consider the edges that contribute with $M_i \neq 0$ to $grad(G,f)$.

It is necessary to guarantee that the sequence of maximum values computed in Step 5b is strictly decreasing in i. Let M_i and M_{i+1} be the maximum values of F on G_i and G_{i+1} and computed by the iterations i and $i+1$ of Step 5b, respectively. Henceforth we will check $M_i > M_{i+1}$, for $i = 1,2,\ldots,r-1$.

Here, let us d_i denote the distance measured in G_i and is listed in the matrix D_i; it is clear that $d_i(u,v) \leq d_{i+1}(u,v)$. Let also u_{i+1} and v_{i+1} be two vertices such that $M_{i+1} = F(u_{i+1},v_{i+1})$ on G_{i+1}. The executions at which the vertices u_{i+1} and v_{i+1} are coloured determine three cases:

1. Both vertices u_{i+1} and v_{i+1} have been coloured before the ith-iteration takes place. Then, if $\{u_{i+1},v_{i+1}\} \notin S_i$, it follows that
$$M_i \geq \frac{|f(u_{i+1})-f(v_{i+1})|}{d_i(u_{i+1},v_{i+1})} \geq \frac{|f(u_{i+1})-f(v_{i+1})|}{d_{i+1}(u_{i+1},v_{i+1})} = M_{i+1}.$$
Otherwise, the fact that $\{u_{i+1},v_{i+1}\} \in S_i$ leads to
$$d_i(u_{i+1},v_{i+1}) < d_{i+1}(u_{i+1},v_{i+1})$$
and then
$$M_i = \frac{|f(u_{i+1})-f(v_{i+1})|}{d_i(u_{i+1},v_{i+1})} > \frac{|f(u_{i+1})-f(v_{i+1})|}{d_{i+1}(u_{i+1},v_{i+1})} = M_{i+1}.$$
2. Precisely one of the vertices u_{i+1} and v_{i+1} is coloured at the ith-iteration and the other one has been coloured in a previous iteration. Without loss of generality, $f(u_{i+1}) \leq f(v_{i+1})$ can be assumed and then we distinguish two possibilities depending on whether either u_{i+1} receives its grey tone at the ith-iteration or v_{i+1} does.
   (a) If the vertex u_{i+1} is coloured at the ith-iteration, it belongs to some $u_i - v_i$ geodesic such that $M_i = F(u_i,v_i)$ on G_i ($f(u_i) \leq f(v_i)$ can be assumed) and so it holds that
$$f(u_i) = f(u_{i+1}) - d_i(u_i,u_{i+1})\,M_i. \tag{3}$$
On the other hand, v_{i+1} is coloured before the ith-iteration and the inequality between M_i and M_{i+1} is achieved by distinguishing if $\{u_i,v_{i+1}\}$ belongs or not to S_i.
In case that $\{u_i,v_{i+1}\} \notin S_i$, the equality in (3) leads to:
$$\begin{aligned} M_i > \frac{f(v_{i+1})-f(u_i)}{d_i(u_i,v_{i+1})} &\geq \frac{f(v_{i+1})-f(u_i)}{d_i(u_i,u_{i+1})+d_i(u_{i+1},v_{i+1})} \geq \\ &\geq \frac{f(v_{i+1})-f(u_{i+1})+d_i(u_i,u_{i+1})\,M_i}{d_i(u_i,u_{i+1})+d_{i+1}(u_{i+1},v_{i+1})} = \\ &= \frac{d_{i+1}(u_{i+1},v_{i+1})\,M_{i+1}+d_i(u_i,u_{i+1})\,M_i}{d_i(u_i,u_{i+1})+d_{i+1}(u_{i+1},v_{i+1})}. \end{aligned}$$
Observe that if $M_i \leq M_{i+1}$, then
$$M_i > \frac{[d_{i+1}(u_{i+1},v_{i+1})+d_i(u_i,u_{i+1})]\,M_i}{d_i(u_i,u_{i+1})+d_{i+1}(u_{i+1},v_{i+1})} = M_i,$$
which is impossible, and therefore $M_i > M_{i+1}$.

In case that $\{u_i, v_{i+1}\} \in S_i$, as u_{i+1} and v_{i+1} belong to the connected component G_{i+1}, and since $d_i(u_{i+1}, v_{i+1}) < d_{i+1}(u_{i+1}, v_{i+1})$ it follows that $d_i(u_i, v_{i+1}) < d_i(u_i, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1})$ and

$$M_i = \frac{f(v_{i+1}) - f(u_i)}{d_i(u_i, v_{i+1})} > \frac{f(v_{i+1}) - f(u_i)}{d_i(u_i, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1})},$$

and the reasoning goes on as in the lines above.

(b) If the vertex v_{i+1} is coloured at the ith-iteration, similar arguments lead to the result $M_i > M_{i+1}$ taking into account two facts: v_{i+1} belongs to some $u_i - v_i$ geodesic such that $M_i = F(u_i, v_i)$ on G_i and $f(u_i) \le f(v_i)$, which implies that $f(v_{i+1}) = f(v_i) - d_i(v_i, v_{i+1})\, M_i$, and the membership or not of $\{u_{i+1}, v_i\}$ in S_i.

3. Both vertices u_{i+1} and v_{i+1} are coloured by the ith-iteration. Therefore, two pairs of vertices exist $\{u_i^1, v_i^1\}$ and $\{u_i^2, v_i^2\}$ of G_i such that $M_i = F(u_i^1, v_i^1) = F(u_i^2, v_i^2)$ and the vertices u_{i+1} and v_{i+1} belong to some $u_i^1 - v_i^1$ and $u_i^2 - v_i^2$ geodesic, respectively (without loss of generality we may suppose that $f(u_{i+1}) \le f(v_{i+1})$, $f(u_i^1) \le f(v_i^1)$ and $f(u_i^2) \le f(v_i^2)$). Then,

$$f(u_{i+1}) = f(u_i^1) + d_i(u_i^1, u_{i+1})\, M_i \Rightarrow f(u_i^1) = f(u_{i+1}) - d_i(u_i^1, u_{i+1})\, M_i \tag{4}$$

$$f(v_{i+1}) = f(v_i^2) - d_i(v_i^2, v_{i+1})\, M_i \Rightarrow f(v_i^2) = f(v_{i+1}) + d_i(v_i^2, v_{i+1})\, M_i \tag{5}$$

If $\{u_i^1, v_i^2\} \notin S_i$ and from (4) and (5), it holds that

$$\begin{aligned} M_i &> \frac{f(v_i^2) - f(u_i^1)}{d_i(u_i^1, v_i^2)} \ge \frac{f(v_{i+1}) + d_i(v_i^2, v_{i+1})M_i - f(u_{i+1}) + d_i(u_i^1, u_{i+1})M_i}{d_i(u_i^1, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1}) + d_i(v_{i+1}, v_i^2)} = \\ &= \frac{d_{i+1}(u_{i+1}, v_{i+1})\, M_{i+1} + d_i(v_i^2, v_{i+1})\, M_i + d_i(u_i^1, u_{i+1})\, M_i}{d_i(u_i^1, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1}) + d_i(v_{i+1}, v_i^2)} \end{aligned}$$

Observe that if we assume $M_i \le M_{i+1}$ implies

$$M_i > \frac{[d_i(u_i^1, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1}) + d_i(v_{i+1}, v_i^2)]\, M_i}{d_i(u_i^1, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1}) + d_i(v_{i+1}, v_i^2)} = M_i,$$

which is impossible, and hence $M_i > M_{i+1}$.

It remains to consider the case $\{u_i^1, v_i^2\} \in S_i$. The inequality $M_i > M_{i+1}$ is achieved by applying that $d_i(u_i^1, v_i^2) < d_i(u_i^1, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1}) + d_i(v_{i+1}, v_i^2)$, which follows from the connection of u_{i+1} and v_{i+1} in G_{i+1}.

Therefore, Claim 2 is shown.

Claim 3. *Unicity of f. f is the only greyscale compatible with g such that its gradation vector*

$$grad(G, f) = (M_1, \dots, M_1, M_2, \dots, M_2, \dots, M_r, \dots, M_r, 0, \dots, 0)$$

is minimum among all gradation vectors of such greyscales.

Let C_k be the vertex-set of G containing the vertices that have been coloured at any of the first k executions of the while-loop in Step 5, for $k = 1 \dots r$, being r the total number of executions of Step 5. Given a minimum gradation greyscale h compatible with g we prove by induction on k that $h(w) = f(w)$ for all $w \in V$.

For $k = 1$, every vertex of C_1 belongs to some $u - v$ geodesic $P_{u-v} = \{u = w_0, e_1, w_1, e_2, w_2, \dots, w_{l-1}, e_l, v = w_l\}$ (alternating sequence of vertices and edges) such that $g(u) = f(u) = h(u)$ and $g(v) = f(v) = h(v)$. There does not exist an edge e_i such that $\widehat{h}(e_i) > M_1$ due to the minimality of $\mathcal{G}_h$ and the existence of $\mathcal{G}_f$. The next argument guarantees the non-existence of an edge e_i such that $\widehat{h}(e_i) < M_1$, therefore $\widehat{h}(e_i) = \widehat{f}(e_i) = M_1$ for all edges of P_{u-v}. Both values of any greyscale on consecutive vertices of P_{u-v} define intervals whose union is a cover of the interval $[g(u), g(v)]$ or $[g(v), g(u)]$. The lengths of these intervals, that is, the grey tones of the corresponding edges, are all M_1 for f; and so, also for the greyscale h. Thus, if there exists one of them less than M_1 there must exists another one greater than M_1,

However, this fact is not possible owing to the minimality of $\mathcal{G}_h$ and the existence of $\mathcal{G}_f$.

Then, $f(w_i) = h(w_i)$ for all vertices of P_{u-v} since $\widehat{f}(e_i) = \widehat{h}(e_i) = M_1$ for all edges of P_{u-v} and $f(u) = h(u) = g(u)$ and $f(v) = h(v) = g(v)$.

For the induction step, the same previous reasoning is applied to the elements of the geodesics taking part in the execution $k+1$ of Step 5, since the extreme vertices of such geodesics belong to C_k and therefore their grey tones assigned by f and h are equal.

Therefore, proof of Claim 3 ends.

Finally, the time complexity follows from Lemma 3.3. □

The next result solves the RMIGG problem in the case of only one type of extreme colour is present, either white or black, among the prefixed values.

Theorem 3.5. *Let g be an incomplete V_c-greyscale of a connected graph $G(V, E)$ such that only one extreme tone, either 0 or 1, is reached by g. Then it is possible to solve the associated RMIGG problem in $O(n^5)$ time.*

Proof. Without loss of generality, we may suppose that g reaches the white colour 0 but not the black colour 1. There exists a vertex $w \in V - V_c$ such that $f(w) = 1$ for any greyscale f compatible with g. Then, for every vertex $w \in V - V_c$ a new incomplete greyscale g_w is defined such that $g_w(u) = g(u)$ whether $u \in V_c$ and $g_w(w) = 1$. In accordance with Theorem 3.4 for g_w, only one greyscale f_w exists, that is compatible with g_w whose gradation vector is minimum among all gradation vectors of greyscales compatible with g_w.

Among these $|V - V_c|$ greyscales f_w, those whose gradation vector is minimum are the solutions of the RMIGG problem, and they have been obtained by running the Compatible-Complete-Mapping polynomial procedure $|V - V_c|$ times. Hence $O(n^5)$ is achieved for this problem. □

Now, the following result solves the proposed problems in case of neither black nor white appearing among the prefixed values.

Theorem 3.6. *It is possible to resolve both RMIGG and MIGG problems in at most $O(n^6)$ time.*

Proof. This proof is similar to the proof of Theorem 3.5, but in this case a new incomplete $V_c \cup \{w_1, w_2\}$-greyscale is defined for every pair of vertices w_1 and w_2 of $V - V_c$ such that $g_{\{w_1,w_2\}}(u) = g(u)$ whether $u \in V_c$, $g_{\{w_1,w_2\}}(w_1) = 0$ and $g_{\{w_1,w_2\}}(w_2) = 1$. Since there are $\binom{|V - V_c|}{2}$ of these incomplete greyscales and the Compatible-Complete-Mapping polynomial procedure provides only one greyscale for each one of them, this problem can be solved in $O(n^6)$ time. Finally, we can consider MIGG problem as a particular case of RMIGG problem for $V_c = \emptyset$. □

In accordance with Theorem 2.4 the actual computational cost of the V_c-Compatible-Complete-Mapping procedure applied to the MIGG problem ($V_c = \emptyset$) can be reduced. By only taking into account the pairs of antipodal vertices to be coloured with the extreme tones, instead of all the pairs of vertices of the graph. Such pairs of antipodal vertices can be obtained form distance matrix D_1 in Step 5a of first while-loop without extra computational cost. On the other hand, the V_c-Compatible-Complete-Mapping procedure has to be applied a quadratic number of times in the worst case, once for each pair of vertices coloured with black and white. These executions can be performed in parallel computation and since we are dealing with minimax problems, after each iteration of the while-loop, it suffices to continue with the executions that lead to the minimum value for M_i. The rest of these executions of the V_c-Compatible-Complete-Mapping procedure can be discarded and aborted.

4. Conclusions and future works

This article introduces the new concept of gradation of a graph which is related to vertex and edge colourings. This way, polynomial algorithms have been designed to solve minimum gradation problems, taking into account whether or not prefixed colours exist. Nevertheless, the algorithms developed in this paper have high computational complexity, thus it is essential further efforts are made to improve the computational time required to solve these gradation problems. Since the time complexities are determined by the computation of the distance matrix of the graph, it is suggested the research of different resolution techniques so as to reduce the computational times. Additionally, it would also be interesting to pose gradation in digraphs, studying a possible more suitable way of assigning colours to the directed edges.

Finally, some practical applications cited in the introduction deserve a more in-depth study on their utility and possible improvement compared to already known results. Such is the case of hierarchical clustering algorithms for which we have indicated a similarity function based on a minimum gradation greyscale. Other possible applications in the context of distribution networks on graphs also be beneficial and lead to future work.

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