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Soluciones Gevrey de sistemas hipergeométricos

por

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A mis padres y a Sebastián.

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Contents

Chapter 1

Introduction

This work is devoted to the study of the irregularity of the GKZ–hypergeometric $\mathcal{D}_{\mathbb{C}^n}$–modules $\mathcal{M}_A(\beta)$ (see Definition 3.1.3). To this end, we explicitly construct Gevrey series solutions of $\mathcal{M}_A(\beta)$ along coordinate subspaces. Let us first recall some general notions and results about the irregularity in $\mathcal{D}$–module Theory.

Let X be a complex manifold and $\mathcal{D}_X$ the sheaf of linear partial differential operators with coefficients in the sheaf of holomorphic functions $\mathcal{O}_X$.

One fundamental problem in the study of the irregularity of any holonomic $\mathcal{D}_X$–module $\mathcal{M}$ is the description of its *analytic slopes* along a smooth hypersurfaces Y in X (see Z. Mebkhout [Meb90]). An analytic slope is a gap $s > 1$ in the Gevrey filtration $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$ of the irregularity complex $\mathrm{Irr}_Y(\mathcal{M})$ (see Definitions 2.4.4 and 2.4.8). The description of the Gevrey series solutions of a holonomic $\mathcal{D}$–module $\mathcal{M}$ along a smooth variety Z is another fundamental problem closely related to its irregularity. In particular, if Z is a smooth hypersurface the index of any non convergent Gevrey solution of $\mathcal{M}$ along Z is an analytic slope of $\mathcal{M}$ along Z (see Definition 2.4.8).

Y. Laurent also defined the *algebraic slopes* of $\mathcal{M}$ along a smooth variety Z (see Definition 2.4.9) as those real numbers $s > 1$ such that the s–micro-characteristic variety of $\mathcal{M}$ with respect to Z is not homogeneous with respect to the filtration by the order of the differential operators. He proved that the set of slopes of $\mathcal{M}$ along Z is a finite set of rational numbers (see [Lau87]).

When $\mathcal{M}$ is a holonomic $\mathcal{D}$–module and Z is a smooth hypersurface, the comparison theorem of the slopes (due to Laurent and Mebkhout [LM99]) states that the algebraic slopes coincide with the analytic ones (see Theorem 2.4.11). Mebkhout has remarked that since 1986 there is a candidate definition for the analytic slopes of a holonomic $\mathcal{D}$-module $\mathcal{M}$ along any smooth subvariety Y using the blowing up $p : \widetilde{X} \longrightarrow X$ of Y in

X and considering the analytic slopes of the holonomic $\mathcal{D}$-module $p^*\mathcal{M}$ along the smooth hypersurface $p^{-1}Y$. However, he also recalled that he doesn't know any significant result related to this definition.

Let us consider the complex manifold $X = \mathbb{C}^n$ and denote $\mathcal{D} := \mathcal{D}_X$. We will also write $\partial_i := \frac{\partial}{\partial x_i}$ for the i–th partial derivative.

Our objects of study are hypergeometric systems, that were introduced by Gel'fand, Graev, Kapranov and Zelevinsky (see [GGZ87] and [GZK89]). They are left $\mathcal{D}$–modules $\mathcal{M}_A(\beta)$ associated with a pair (A, β) where A is a full rank $d \times n$ matrix $A = (a_{ij})$ with integer entries ($d \leq n$) and $\beta \in \mathbb{C}^d$ is a vector of complex parameters (see Definition 3.1.3). A general goal in the study of hypergeometric systems is the description of their invariants in terms of the combinatorics of the pair (A, β).

A good introduction for the theory of hypergeometric systems is [SST00]. These systems are known to be holonomic and their holonomic rank (equivalently, the dimension of the space of holomorphic solutions at nonsingular points) is the normalized volume of the matrix $A = (a_i)_{i=1}^n \in \mathbb{Z}^{d\times n}$ with respect to the lattice $\mathbb{Z}A := \sum_{i=1}^n \mathbb{Z}a_i \subseteq \mathbb{Z}^d$ (see Definition 6.6.1) when either β is generic or I_A is Cohen-Macaulay (see [GZK89], [Ado94]). For results about rank-jumping parameters β see [MMW05], [Ber08] and the references therein. Several authors have studied the holomorphic solutions at nonsingular points of $\mathcal{M}_A(\beta)$ (see for example [GZK89], [SST00] and [OT07]).

Let us explain the structure of this dissertation. In Chapter 2 we recall some general notions and results related to the irregularity in $\mathcal{D}$–module Theory, mainly following [Meb90] and [LM99]. In Chapter 3 we introduce hypergeometric systems following [GGZ87] and [GZK89], recall some well-known results and make some comments and remarks.

Chapter 4 is devoted to the description of the irregularity complex of the hypergeometric systems in two variables by elementary methods. Then in Chapter 5 we compute the cohomology sheaves of the irregularity complex of $\mathcal{M}_A(\beta)$ with respect to its singular locus for any one row integer matrix $A = (a_1 \cdots a_n)$ such that $0 < a_1 < \cdots < a_n$ (see [FC$_2$08]). Our method is to reduce the problem to the case in two variables using deep results in $\mathcal{D}$–module Theory and restrictions theorems.

The structure of Chapter 6 is the following. In Section 6.1 we consider a simplex σ, i.e., a set $\sigma \subseteq \{1, \ldots, n\}$ such that $A_\sigma = (a_i)_{i\in\sigma}$ is an invertible submatrix of A, and we use Γ–series introduced in [GZK89] and slightly generalized in [SST00] to explicitly construct a set of linearly independent Gevrey solutions of $\mathcal{M}_A(\beta)$ along $Y_\sigma = \{x_i = 0 : \; i \notin \sigma\}$. The cardinality of this set of solutions is the normalized volume of A_σ with respect to the lattice

$\mathbb{Z}A$ and we prove that they are Gevrey series of order $s = \max\{|A_\sigma^{-1}a_i| : i \notin \sigma\}$ along the coordinate subspace $Y = \{x_i = 0 : |A_\sigma^{-1}a_i| > 1\} \supseteq Y_\sigma$. Moreover, we also prove that s is their Gevrey index when β is very generic.

In Section 6.2 we construct for any simplex σ and for all β a set of Gevrey series along Y with index s that are solutions of $\mathcal{M}_A(\beta)$ modulo the sheaf of Gevrey series with lower index. This implies for $s > 1$ that s is a slope of $\mathcal{M}_A(\beta)$ along Y when Y is a hyperplane by Lemma 8.0.8, that will be proved in the Appendix.

In Section 6.3 we describe all the slopes along coordinate hyperplanes Y at any point $p \in Y$ (see Theorem 6.3.10). To this end, and using some ideas of [SW08], we prove that the s–micro-characteristic varieties with respect to Y of $\mathcal{M}_A(\beta)$ are homogeneous with respect to the order filtration for all $s \geq 1$ but a finite set of candidates s to algebraic slopes. Then we use the results in Sections 6.1 and 6.2 to prove that all the candidates s to algebraic slopes along hyperplanes occur as the Gevrey index of a Gevrey series solution of $\mathcal{M}_A(\beta)$ modulo convergent series and thus they are analytic slopes. In particular we prove that the set of algebraic slopes of $\mathcal{M}_A(\beta)$ along any coordinate hyperplane is contained in the set of analytic slopes without using the comparison theorem of the slopes [LM99]. We need to use that the set of analytic slopes is contained in the set of algebraic slopes in order to prove that there are no more slopes of $\mathcal{M}_A(\beta)$ along a coordinate hyperplane. Notice that this inclusion of the comparison theorem for the slopes is a consequence of Laurent's index theorem for holomorphic hyperfunctions [Lau99, Corollary 5.3.3] (see also [Meb90, 6.6]).

M. Schulze and U. Walther [SW08] described first the algebraic slopes of $\mathcal{M}_A(\beta)$ along coordinate subspaces assuming that $\mathbb{Z}A = \mathbb{Z}^d$ and that A is *pointed*. A matrix A is said to be pointed if its columns $a_1, \ldots, a_n$ lie in a single open linear half-space of $\mathbb{R}^d$ (equivalently, the associated affine toric variety $\mathcal{V}(I_A)$ passes through the origin). Previous computations of the slopes of $\mathcal{M}_A(\beta)$ along coordinate hyperplanes in the particular cases $d = 1$ and $n = d + 1$ appear in [CT03], [Har04] and [Har03].

In Section 6.5.1 we use the Gevrey series constructed in Section 6.1 and convenient regular triangulations of the matrix A to provide a lower bound for the dimensions of the Gevrey solution spaces. In particular, the lower bound that we obtain for the dimension of the formal solution space of $\mathcal{M}_A(\beta)$ along any coordinate subspace $Y_\tau = \{x_i = 0 : i \notin \tau\}$, $\tau \subseteq \{1, \ldots, n\}$, at generic points of Y_τ is nothing but the normalized volume of the matrix $A_\tau = (a_i)_{i \in \tau}$ with respect to $\mathbb{Z}A$.

In Section 6.5.2 we prove that this lower bound is actually an equality for very generic parameters $\beta \in \mathbb{C}^d$ and then we have the explicit description of the basis of the corresponding Gevrey solution space. Example 6.3.13 shows that this condition on the parameters is necessary in general to obtain a basis. This example also points out a special phenomenon: some algebraic slopes of $\mathcal{M}_A(\beta)$ along coordinate varieties of codimension greater than one do not appear as the Gevrey index of any formal solution modulo convergent series.

Then, in Section 6.6 we assume some conditions ($\mathbb{Z}A = \mathbb{Z}^d$, A is pointed, β is non-rank-jumping and Y is a coordinate hyperplane) in order to use some multiplicity formulas for the s–micro-characteristic cycles of $\mathcal{M}_A(\beta)$ obtained by M. Schulze and U. Walther in [SW08] and general results on the irregularity of holonomic $\mathcal{D}$–modules due to Laurent and Mebkhout [LM99] to compute the dimension of $\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p$ for generic points $p \in Y$. Thus, the set of the classes in $\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p$ of the Gevrey solutions of $\mathcal{M}_A(\beta)$ that we construct along a hyperplane Y is a basis for very generic parameters. Moreover, since $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is a perverse sheaf on Y by a theorem of Z. Mebkhout [Meb90], we know that for $i \geq 1$ the i–th cohomology sheaf of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ has support contained in a subvariety of Y with codimension i. This gives the stalk of the cohomology of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ at generic points of Y. As a consequence, we compute the Newton polygon of $\mathcal{M}_A(\beta)$ along Y with respect to Y at generic points of Y.

Finally, in Chapter 7 (joint work with Uli Walther) we investigate the restriction of $\mathcal{M}_A(\beta)$ with respect to a coordinate subspace, generalizing Corollary 5.1.4.

Chapter 2

Preliminaries I: Irregularity of a holonomic $\mathcal{D}$-module.

In this chapter, we introduce some definitions and well-known results in the general setting of $\mathcal{D}$–module Theory that will be used in the sequel.

2.1 Irregularity of a linear differential operator in one variable.

To set up the problem of computing the Gevrey solutions of a left $\mathcal{D}$–module, we will first recall the situation in the one dimensional case.

Consider an ordinary linear differential operator, of order m,

$$P = a_m \frac{d^m}{dx^m} + \cdots + a_1 \frac{d}{dx} + a_0$$

with $a_i = a_i(x)$ a holomorphic function at the origin in $\mathbb{C}$. Recall that $x = 0$ is a singular point if and only if $a_m(0) = 0$. The *slopes* of P are the slopes of the Newton polygon $N(P)$ of P defined as the convex hull of

$$\bigcup_{i=0}^{m} ((i, i - v(a_i)) + (\mathbb{Z}_{\leq 0})^2)$$

where $v(a_i)$ is the multiplicity of the zero of $a_i(x)$ at $x = 0$. By Fuchs' Theorem, P is regular at $x = 0$ if and only if $N(P)$ is the quadrant

$$(m, m - v(a_m)) + (\mathbb{Z}_{\leq 0})^2.$$

Malgrange-Komatsu's comparison theorem states that P is regular at $x = 0$ if and only if the solution space $\mathrm{Sol}(P, \widehat{\mathcal{O}}/\mathcal{O})$ is zero ([Kom71], [Mal74]). Here P acts naturally on the quotient $\widehat{\mathcal{O}}/\mathcal{O}$ where $\widehat{\mathcal{O}} = \mathbb{C}[[x]]$ (resp. $\mathcal{O} = \mathbb{C}\{x\}$) is the ring of formal (resp. convergent) power series in the variable x. This solution space measures the irregularity of P at $x = 0$ and will be denoted

$$\mathrm{Irr}_0(P) := \mathrm{Sol}(P, \widehat{\mathcal{O}}/\mathcal{O}).$$

Example 2.1.1. *Consider the differential operator $P = x^{k+1}\frac{d}{dx} - k$, $k \geq 1$. The space of holomorphic solutions of P at any point of $\mathbb{C} \setminus \{0\}$ is generated by $f = e^{-1/x^k}$, which has an essential singularity at $x = 0$. The irregularity $\mathrm{Irr}_0(P)$ is generated by the classes in $\widehat{\mathcal{O}}/\mathcal{O}$ of the formal power series:*

$$f_j = \sum_{m \geq 0} \prod_{i=0}^{m-1} \left(m + \frac{j}{k} - i \right) x^{j+k(m+1)}$$

for $j = 0, \ldots, k-1$ since $P(f_j) = -kx^{j+k} \in \mathcal{O}$.

Moreover, the vector space $\mathrm{Irr}_0(P)$ is filtered by the so-called Gevrey solutions of the equation $P(u) = 0$. Let us denote by $\widehat{\mathcal{O}}_s \subset \widehat{\mathcal{O}}$ the subring of Gevrey series of order less than or equal to s (where $s \geq 1$ is a real number). A formal power series

$$f = \sum_i f_i x^i \in \widehat{\mathcal{O}}$$

is in $\widehat{\mathcal{O}}_s$ if and only if the series

$$\rho_s(f) := \sum_i \frac{f_i x^i}{(i!)^{s-1}}$$

is convergent at $x = 0$.

The irregularity $\mathrm{Irr}_0(P)$ is filtered by the solution subspaces

$$\mathrm{Irr}_0^{(s)}(P) := \mathrm{Sol}(P, \widehat{\mathcal{O}}_s/\mathcal{O}).$$

By the comparison theorem of J.P. Ramis [Ram84] the jumps in this filtration are in bijective correspondence with the slopes of $N(P)$ and the dimension of each vector space $\mathrm{Irr}_0^{(s)}(P)$ can also be read on $N(P)$.

In higher dimension, one has analogous results but the situation is much more involved (see Subsection 2.4).

2.2 Gevrey series along a smooth subvariety.

Let X be a complex manifold of dimension $n \geq 1$, $\mathcal{O}_X$ (or simply $\mathcal{O}$) the sheaf of holomorphic functions on X and $\mathcal{D}_X$ (or simply $\mathcal{D}$) the sheaf of linear differential operators with coefficients in $\mathcal{O}_X$. The sheaf $\mathcal{O}_X$ has a natural structure of left $\mathcal{D}_X$–module.

Let Z be a subvariety in X with defining ideal $\mathcal{I}_Z$. We denote by $\mathcal{O}_{X|Z}$ the restriction to Z of the sheaf $\mathcal{O}_X$ (and we will also denote by $\mathcal{O}_{X|Z}$ its extension by 0 on X). Recall that the formal completion of $\mathcal{O}_X$ along Z is defined as

$$\mathcal{O}_{\widehat{X|Z}} := \lim_{\overleftarrow{k}} \mathcal{O}_X/\mathcal{I}_Z^k.$$

By definition $\mathcal{O}_{\widehat{X|Z}}$ is a sheaf on X supported on Z and has a natural structure of left $\mathcal{D}_X$–module. We will also denote by $\mathcal{O}_{\widehat{X|Z}}$ the corresponding sheaf on Z. We denote by $\mathcal{Q}_Z$ the quotient sheaf defined by the following exact sequence

$$0 \to \mathcal{O}_{X|Z} \longrightarrow \mathcal{O}_{\widehat{X|Z}} \longrightarrow \mathcal{Q}_Z \to 0.$$

The sheaf $\mathcal{Q}_Z$ has then a natural structure of left $\mathcal{D}_X$–module.

Remark 2.2.1. *If $X = \mathbb{C}$ and $Z = \{0\}$ then $\mathcal{O}_{\widehat{X|Z},0}$ is nothing but $\mathbb{C}[[x]]$ the ring of formal power series in one variable x, while $\mathcal{O}_{\widehat{X|Z},p} = 0$ for any nonzero $p \in X$. In this case $\mathcal{Q}_{Z,0} = \frac{\mathbb{C}[[x]]}{\mathbb{C}\{x\}}$ and $\mathcal{Q}_{Z,p} = 0$ for $p \neq 0$.*

Definition 2.2.2. *Assume $Z \subset X$ is a smooth subvariety and that around a point $p \in X$ the variety Z is locally defined by $x_{l+1} = \cdots = x_n = 0$ for some system of local coordinates $(x_1, \ldots, x_n)$. A germ*

$$f = \sum_{m \in \mathbb{N}^{n-l}} f_m(x_1, \ldots, x_l) x_{l+1}^{m_{l+1}} \cdots x_n^{m_n} \in \mathcal{O}_{\widehat{X|Z},p}$$

is said to be a Gevrey series of order $s \in \mathbb{R}$ (along Z at the point p) if the power series

$$\rho_s(f) := \sum_{m \in \mathbb{N}^{n-l}} \frac{1}{\prod_{i=l+1}^{n} m_i!^{s-1}} f_m(x_1, \ldots, x_l) x_{l+1}^{m_{l+1}} \cdots x_n^{m_n} \in \mathcal{O}_{\widehat{X|Z},p}$$

is convergent at p.

The sheaf $\mathcal{O}_{\widehat{X|Z}}$ admits a natural filtration by the sub-sheaves $\mathcal{O}_{\widehat{X|Z}}(s)$ of Gevrey series of order s, $1 \leq s \leq \infty$ where $\mathcal{O}_{\widehat{X|Z}}(\infty) = \mathcal{O}_{\widehat{X|Z}}$ by convention. It is clear that $\mathcal{O}_{\widehat{X|Z}}(1) = \mathcal{O}_{X|Z}$. We can also consider the induced filtration on $\mathcal{Q}_Z$, i.e. the filtration by the sub-sheaves $\mathcal{Q}_Z(s)$ defined by the exact sequence:

$$0 \to \mathcal{O}_{X|Z} \longrightarrow \mathcal{O}_{\widehat{X|Z}}(s) \longrightarrow \mathcal{Q}_Z(s) \to 0 \tag{2.1}$$

Definition 2.2.3. *Let Z be a smooth hypersurface in $X = \mathbb{C}^n$ and let p be a point in Z. The Gevrey index of a formal power series $f \in \mathcal{O}_{\widehat{X|Z},p}$ with respect to Z is the smallest $1 \leq s \leq \infty$ such that $f \in \mathcal{O}_{\widehat{X|Z}}(s)_p$.*

2.3 Perverse sheaves

We recall here the definition and some general well-known results about perverse sheaves.

Definition 2.3.1. *A complex* $\mathcal{F}^\bullet \in D^b(\mathbb{C}_X)$ *of sheaves of vector spaces is said to be a* constructible sheaf *if there exists a stratification* (X_λ) *of* X *such that the cohomology sheaves* $\mathcal{H}^i(\mathcal{F}^\bullet)$ *are local systems on each* X_λ.

Definition 2.3.2. *Let* $\mathcal{F} \in D^b_c(\mathbb{C}_X)$ *be a constructible sheaf, then the Euler-Poincaré characteristic of* $\mathcal{F}$ *is the following constructible function on* X*:*

$$\begin{array}{rcl} \chi(\mathcal{F}) : X & \longrightarrow & \mathbb{Z} \\ x & \mapsto & \chi(\mathcal{F})_x = \sum_j (-1)^j \dim_{\mathbb{C}} H^i(\mathcal{F}_x) \end{array}$$

where $H^i(\mathcal{F}_x) = \mathcal{H}^i(\mathcal{F})_x$ *and* $\mathcal{H}^i(\mathcal{F})$ *denotes the* i*–th cohomology sheaf of* $\mathcal{F}$.

Definition 2.3.3. *A constructible sheaf* $\mathcal{F}^\bullet$ *satisfies the* support condition *on* X *if the following conditions hold:*

1. $\mathcal{H}^i(\mathcal{F}^\bullet) = 0$ *for* $i < 0$ *and* $i > n = dim(X)$.
2. *The dimension of the support of* $\mathcal{H}^i(\mathcal{F}^\bullet)$ *is less than or equal to* $n - i$ *for* $0 \leq i \leq n$.

Definition 2.3.4. *A constructible sheaf* $\mathcal{F}^\bullet \in D^b_c(\mathbb{C}_X)$ *is said to be a* perverse sheaf *on* X *if both* $\mathcal{F}^\bullet$ *and its dual* $\mathbb{R}\mathcal{H}om_{\mathbb{C}_X}(\mathcal{F}^\bullet, \mathbb{C}_X)$ *satisfies the support condition.*

Remark 2.3.5. *Let* $Z \subseteq X$ *be a subvariety with codimension* $p \in \mathbb{N}$. *Then a constructible sheaf* $\mathcal{F}^\bullet$ *on* Z *is a perverse sheaf on* Z *if the constructible sheaf on* X *obtained by extending* $\mathcal{F}^\bullet[-p]$ *by* 0 *is a perverse sheaf on* X.

The category $\mathrm{Per}(\mathbb{C}_X)$ of perverse sheaves on X is an abelian category but the category of constructible sheaves $\in D^b_c(\mathbb{C}_X)$ is just additive (see [BBD82]).

By the Riemann-Hilbert correspondence (see [Meb84]) the derived functor

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{O}_X)$$

establishes an equivalence of categories between the category of regular holonomic $\mathcal{D}_X$-modules and the one of perverse sheaves $\mathrm{Per}(\mathbb{C}_X)$.

We will include a proof of the following result for the sake of completeness.

Theorem 2.3.6. *Let $\mathcal{F}^\bullet$ be a perverse sheaf on X, then there exists a Whitney stratification $\{X_\alpha\}_{\alpha\in I}$ of X such that:*

$$\mathrm{Ch}(\mathcal{F}^\bullet) \subseteq \cup_{\alpha\in I} T^*_{X_\alpha} X \tag{2.2}$$

Moreover, for every Whitney stratification of X satisfying (2.2) we have that:

$$\mathcal{H}^i(\mathcal{F}^\bullet)_{|X_\alpha}$$

are locally constants sheaves of finite rank for all $i \in \mathbb{N}$ and $\alpha \in I$.

Proof. Let $\mathcal{F}^\bullet$ be a perverse sheaf on X, then by the Riemann-Hilbert correspondence ([Meb84]), there exists a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}_{\mathcal{F}^\bullet}$ such that

$$\mathcal{F}^\bullet = \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_{\mathcal{F}^\bullet}, \mathcal{O}_X)$$

Thus, by [Kas83, Th. 6.3.1.], we have:

$$\mathrm{Ch}(\mathcal{M}_{\mathcal{F}^\bullet}) = \mathrm{Ch}(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_{\mathcal{F}^\bullet}, \mathcal{O}_X)) = \mathrm{Ch}(\mathcal{F}^\bullet)$$

By some results of M. Kashiwara (see [Kas83, Theorems 5.1.3, 5.1.6] and [Kas83, Lemma 3, page 114]) we have that there exists a Whitney stratification $\{X_\alpha\}_{\alpha\in I}$ of X such that

$$\mathrm{Ch}(\mathcal{M}_{\mathcal{F}^\bullet}) \subseteq \cup_\alpha T^*_{X_\alpha} X \tag{2.3}$$

and that for any Whitney stratification of X satisfying (2.3) the sheaves:

$$\mathcal{H}^i(\mathcal{F}^\bullet)_{|X_\alpha} = \mathcal{E}xt^i(\mathcal{M}_{\mathcal{F}^\bullet}, \mathcal{O}_X)_{|X_\alpha}$$

are locally constants sheaves of finite rank.

□

2.4 Irregularity and slopes of a holonomic $\mathcal{D}$-module

Let X be a complex manifold. We recall here the definition of the irregularity (also called the irregularity complex) of a left coherent $\mathcal{D}_X$–module given by Z. Mebkhout [Meb90, (2.1.2) and page 98].

Recall that if $\mathcal{M}$ is a coherent left $\mathcal{D}_X$–module and $\mathcal{F}$ is any $\mathcal{D}_X$–module, the *solution complex* of $\mathcal{M}$ with values in $\mathcal{F}$ is by definition the complex

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$$

which is an object of $D^b(\mathbb{C}_X)$ the derived category of bounded complexes of sheaves of $\mathbb{C}$–vector spaces on X. The cohomology sheaves of the solution complex are then $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$ (or simply $\mathcal{E}xt^i(\mathcal{M}, \mathcal{F})$) for $i \in \mathbb{N}$.

Definition 2.4.1. [Meb90, (2.1.2) and page 98] *Let Z be a subvariety in X. The irregularity of $\mathcal{M}$ along Z (denoted by $\mathrm{Irr}_Z(\mathcal{M})$) is the solution complex of $\mathcal{M}$ with values in $\mathcal{Q}_Z$, i.e.*

$$\mathrm{Irr}_Z(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{Q}_Z).$$

Definition 2.4.2. *A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is regular with respect to a subvariety $Z \subseteq X$ if $\mathrm{Irr}_Z(\mathcal{M})$ is zero. $\mathcal{M}$ is said to be regular if $\mathrm{Irr}_Z(\mathcal{M})$ is zero for any subvariety Z.*

Proposition 2.4.3. *A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is regular if and only if $\mathrm{Irr}_Z(\mathcal{M})$ is zero for any hypersurface Z.*

If Y is a smooth hypersurface in X we also have the following definition (see [Meb90, Déf. 6.3.7]).

Definition 2.4.4. *For each $1 \leq s \leq \infty$, the irregularity of order s of $\mathcal{M}$ along Y is the complex*

$$\mathrm{Irr}_Y^{(s)}(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{Q}_Y(s)).$$

Remark 2.4.5. *Since $\mathcal{O}_{\widehat{X|Y}}(\infty) = \mathcal{O}_{\widehat{X|Y}}$ we have $\mathrm{Irr}_Y^{(\infty)}(\mathcal{M}) = \mathrm{Irr}_Y(\mathcal{M})$. The support of the irregularity of $\mathcal{M}$ along Z (resp. $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$) is contained in Z (resp. in Y).*

If $X = \mathbb{C}$, $Z = \{0\}$ and $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$ is the $\mathcal{D}_X$–module defined by some nonzero linear differential operator $P(x, \frac{d}{dx})$ with holomorphic coefficients, then $\mathrm{Irr}_Z(\mathcal{M})$ is represented by the complex

$$0 \longrightarrow \frac{\mathbb{C}[[x]]}{\mathbb{C}\{x\}} \xrightarrow{P} \frac{\mathbb{C}[[x]]}{\mathbb{C}\{x\}} \longrightarrow 0$$

where P acts naturally on the quotient $\frac{\mathbb{C}[[x]]}{\mathbb{C}\{x\}}$.

Theorem 2.4.6. [Meb90, Th. 6.3.3] *If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$–module and $Y \subset X$ is a smooth hypersurface, then the complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$ is a perverse sheaf on Y for any $1 \leq s \leq \infty$.*

Remark 2.4.7. *From [Meb90, Cor. 6.3.5] each $\mathrm{Irr}_Y^{(s)}(-)$ for $1 \leq s \leq \infty$, is an exact functor from the category of holonomic $\mathcal{D}_X$-modules to the category of perverse sheaves on Y.*

Moreover, the sheaves $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$, $1 \leq s \leq \infty$, form an increasing filtration of $\mathrm{Irr}_Y^{(\infty)}(\mathcal{M}) = \mathrm{Irr}_Y(\mathcal{M})$. This filtration is called the Gevrey filtration of $\mathrm{Irr}_Y(\mathcal{M})$ (see [Meb90, Sec. 6]).

Let us denote by

$$\mathrm{Gr}_s(\mathrm{Irr}_Y(\mathcal{M})) := \frac{\mathrm{Irr}_Y^{(s)}(\mathcal{M})}{\mathrm{Irr}_Y^{(<s)}(\mathcal{M})}$$

for $1 \leq s \leq \infty$ the graded object associated with the Gevrey filtration of the irregularity $\mathrm{Irr}_Y(\mathcal{M})$ (see [LM99, Sec. 2.4]).

Definition 2.4.8. [LM99, Sec. 2.4] *We say that* $1 \leq s < \infty$ *is an* analytic slope *of* $\mathcal{M}$ *along* Y *at a point* $p \in Y$ *if* p *belongs to the closure of the support of* $\mathrm{Gr}_s(\mathrm{Irr}_Y(\mathcal{M}))$.

Let us denote by $\Lambda = T^*_Y X$ the conormal bundle of Y in X and let $T^*\Lambda$ be the cotangent bundle of Λ. Let F be the filtration on $\mathcal{D}_X$ by the order of the differential operators and let V be the Malgrange-Kashiwara filtration on $\mathcal{D}_X$ with respect to Y. Recall that V is defined by:

$$V_k(\mathcal{D}_X)_p = \{P \in \mathcal{D}_{X,p} : \ \forall j \geq 0, P(\mathcal{I}_{Y,p}^{k+j}) \subseteq \mathcal{I}_{Y,p}^{j}\}$$

for all $p \in X$ and $k \in \mathbb{Z}$, where $\mathcal{I}_Y \subseteq \mathcal{O}_X$ is the defining ideal of Y and $\mathcal{I}_Y^k = \mathcal{O}_X$ for $k < 0$ by convention. For all $s \geq 1$ there exists an intermediate filtration $L_s = F + (s-1)V_Y$ on $\mathcal{D}_X$ (see [Lau87]). The induced filtration on $\mathcal{M}$ by the filtration L_s on $\mathcal{D}_X$ determines a positive cycle of $T^*\Lambda$ denoted by $\mathrm{CCh}^s(\mathcal{M})$. The support $\mathrm{Ch}^s(\mathcal{M})$ of $\mathrm{CCh}^s(\mathcal{M})$ is called the s-micro-characteristic variety of $\mathcal{M}$ with respect to Y. Let $I(s)(\mathcal{M})$ be the analytic closure of the union of the projections on Y of the irreducible non-F-homogeneous components of $\mathrm{Ch}^s(\mathcal{M})$ via $T^*\Lambda \longrightarrow Y$.

Definition 2.4.9. [$\mathrm{Lau}_1$85], [Lau87] $s > 1$ *is an* algebraic slope *of* $\mathcal{M}$ *with respect to* Y *at* $p \in Y$ *if* $p \in I(s)(\mathcal{M})$.

Remark 2.4.10. *The algebraic slopes of a* $\mathcal{D}_X$*-module* $\mathcal{M}$ *with respect to a smooth hypersurface* Y *can be algorithmically computed if* $\mathcal{M}$ *is defined by differential operators with polynomial coefficients [ACG96].*

The following important result generalizes the comparison theorem of J.P. Ramis [Ram84] in one variable.

Theorem 2.4.11. [LM99, Th. 2.5.3] *If* $\mathcal{M}$ *is a holonomic* $\mathcal{D}_X$*–module and* Y *is a smooth hypersurface, then* $s > 1$ *is an analytic slope of* $\mathcal{M}$ *with respect to* Y *at* $p \in Y$ *if and only if* $s > 1$ *is an algebraic slope of* $\mathcal{M}$ *with respect to* Y *at* $p \in Y$.

Chapter 3

Preliminaries II: Hypergeometric systems and Γ–series.

In this chapter we introduce our objects of study: hypergeometric systems (see [GGZ87] and [GZK89]). We also recall some results and make some remarks and comments.

From now on we consider the complex manifold $X = \mathbb{C}^n$ and denote $\mathcal{D} := \mathcal{D}_X$. We will also write $\partial_i := \frac{\partial}{\partial x_i}$ for the i-th partial derivative.

3.1 Hypergeometric systems

Let A be a full rank $d \times n$ matrix $A = (a_{ij})$ with integer entries, $d \leq n$. To this data we can associate an irreducible algebraic variety as follows (see [Stu95] for example).

Definition 3.1.1. *The toric ideal associated with A is the following binomial ideal:*

$$I_A = \langle \Box_u := \partial^{u_+} - \partial^{u_-} : \quad u \in \mathbb{Z}^n,\ Au = 0 \rangle \subseteq \mathbb{C}[\partial_1, \ldots, \partial_n]. \tag{3.1}$$

Here $u = u_+ - u_-$ and $u_+, u_- \in \mathbb{N}^n$ have disjoint supports.

The zeros variety $\mathcal{V}(I_A)$ of I_A in $X = \mathbb{C}^n$ is called the toric variety associated with A.

Remark 3.1.2. *I_A is a prime ideal whose zeros variety $\mathcal{V}(I_A) \subseteq \mathbb{C}^n$ has Krull dimension d (see for example [Stu95]).*

Let $\beta \in \mathbb{C}^d$ be a vector of complex parameters and $1 \leq i \leq d$. Then the linear differential operator:

$$E_i - \beta_i := \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i \tag{3.2}$$

is called the i-th Euler operator associated with (A, β) (or the Euler operator associated with $(a_{i,1} \ \cdots \ a_{i,n})$ and β_i).

Following Gel'fand, Graev, Kapranov and Zelevinsky (see [GGZ87] and [GZK89]), we give the following definition:

Definition 3.1.3. *The hypergeometric system associated with the pair (A, β) is the following left ideal of the Weyl algebra $A_n(\mathbb{C}) = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$:*

$$H_A(\beta) := A_n(\mathbb{C}) I_A + \sum_{i=1}^{d} A_n(\mathbb{C})(E_i - \beta_i)$$

The hypergeometric $\mathcal{D}$–module associated with the pair (A, β) is the quotient sheaf $\mathcal{M}_A(\beta) = \mathcal{D}/\mathcal{D}H_A(\beta)$.

Remark 3.1.4. *Notice that $H_A(\beta) = H_{GA}(G\beta)$ for any invertible matrix G such that GA is an integer matrix.*

The following basic result was proved in [GZK89] when the toric ideal is homogeneous and in [Ado94] for the general case.

Theorem 3.1.5. *$\mathcal{M}_A(\beta)$ is a holonomic $\mathcal{D}$-module for all $\beta \in \mathbb{C}^d$.*

There is a characterization of the regular hypergeometric systems:

Theorem 3.1.6. *For all $\beta \in \mathbb{C}^d$ the following conditions are equivalent:*

i) $H_A(\beta)$ is a regular holonomic ideal.

ii) All the columns of A lie in a hyperplane off the origin.

Proof. The implication $ii) \Longrightarrow i)$ is a theorem of R. Hotta [Hot98, Ch. II, 6.2, Thm.]. The converse result was proved by Saito, Sturmfels and Takayama [SST00, Thm. 2.4.11] when β is generic (see Definition 3.2.4) and by Schulze and Walther [SW08, Corollary 3.16] when A is a pointed matrix such that $\mathbb{Z}A = \mathbb{Z}^d$. On the other hand, when A is non-pointed then $\mathcal{M}_A(\beta)$ is never regular holonomic: the existence of a toric operator $\partial^u - 1 \in I_A$, $u \in \mathbb{N}^n$, implies that the holonomic rank of some initial ideals of $H_A(\beta)$ is zero and this is a contradiction for regular holonomic ideals with positive rank (see [SST00, Thm. 2.5.1.]). □

3.2 Γ–series

In what follows we recall the definition of Γ–series following [GGZ87] and [GZK89, Sec. 1] and in the way these objects are handled in [SST00, Sec. 3.4].

Let the pair (A, β) be as in Section 3.1. Assume $v \in \mathbb{C}^n$. We will consider the Γ–series

$$\varphi_v := x^v \sum_{u \in L_A} \frac{1}{\Gamma(v+u+\mathbf{1})} x^u \in x^v \mathbb{C}[[x_1^{\pm 1}, \ldots, x_n^{\pm n}]] \tag{3.3}$$

where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{N}^n$, $L_A = \mathrm{Ker}(A) \cap \mathbb{Z}^n$ and for a complex vector $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$ one has by definition

$$\Gamma(\gamma) = \prod_{i=1}^{n} \Gamma(\gamma_i)$$

Here Γ is the Euler Gamma function. Notice that the set

$$x^v \mathbb{C}[[x_1^{\pm 1}, \ldots, x_n^{\pm n}]]$$

has a natural structure of left $A_n(\mathbb{C})$-module although it is not a $\mathcal{D}_{X,0}$–module. Nevertheless, if $Av = \beta$ then the expression φ_v formally satisfies the operators defining $\mathcal{M}_A(\beta)$. Let us notice that if $u \in L_A$ then $\varphi_v = \varphi_{v+u}$.

Observe that φ_v is zero if and only if $(v + L_A) \cap (\mathbb{C} \setminus \mathbb{Z}_{<0})^n = \emptyset$. In contrast, the series φ_v does not define a formal power series at any point if $(v + L_A) \cap (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ contains $v + (\mathbb{Z}^n \cap L')$ for some linear subspace $0 \neq L' \subseteq \mathrm{Ker}(A)$.

If $v \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ then the coefficient $\frac{1}{\Gamma(v+u+\mathbf{1})}$ is non-zero for all $u \in L_A$ such that $u_i + v_i \geq 0$ for all i with $v_i \in \mathbb{N}$. In this case, we also have the following equality

$$\frac{\Gamma(v+\mathbf{1})}{\Gamma(v+u+\mathbf{1})} = \frac{[v]_{u_-}}{[v+u]_{u_+}} \tag{3.4}$$

Here

$$[z]_\alpha = \prod_{i:\, \alpha_i > 0} \prod_{j=0}^{\alpha_i - 1} (z_i - j)$$

is the Pochhammer symbol, for any $z \in \mathbb{C}^n$ and any $\alpha \in \mathbb{N}^n$.

Definition 3.2.1. *The* negative support *of a vector $v \in \mathbb{C}^n$ (denoted by* $\mathrm{nsupp}(v)$*) is the set of indices i such that $v_i \in \mathbb{Z}_{<0}$. We say that v has* minimal negative support *if there is no $u \in L_A$ such that* $\mathrm{nsupp}(v+u) \subsetneq \mathrm{nsupp}(v)$.

Assume $v \notin (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ has minimal negative support. The negative support of v is then a non-empty set and $\Gamma(v+\mathbf{1}) = \infty$. Moreover for each $u \in L_A$ at least one coordinate of $v+u$ must be strictly negative (otherwise $\mathrm{nsupp}(v+u) = \emptyset \subsetneq \mathrm{nsupp}(v)$). So $\Gamma(v+u+\mathbf{1}) = \infty$ for all $u \in L_A$ and $\varphi_v = 0$.

If $v \notin (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ does not have minimal negative support then there exists $u \in L_A$ such that $v + u$ has minimal negative support. If $\text{nsupp}(v + u) = \emptyset$ then $\varphi_v = \varphi_{v+u} \neq 0$ while if $\text{nsupp}(v + u) \neq \emptyset$ then $\varphi_v = \varphi_{v+u} = 0$.

Following [SST00, p. 132-133], for any $v \in \mathbb{C}^n$ we will consider the series

$$\phi_v := x^v \sum_{u \in N_v} \frac{[v]_{u_-}}{[v + u]_{u_+}} x^u \tag{3.5}$$

where $N_v = \{u \in L_A \,|\, \text{nsupp}(v + u) = \text{nsupp}(v)\}$.

For $v \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ we have (3.4) and $\Gamma(v + \mathbf{1})\varphi_v = \phi_v$. If $v \notin (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ then the coefficient of x^v in ϕ_v is non-zero (in fact this coefficient is 1) while it is zero in φ_v.

Proposition 3.2.2. [SST00, Prop. 3.4.13] *If $Av = \beta$ then ϕ_v is a solution of the hypergeometric ideal $H_A(\beta)$ (i.e. ϕ_v is formally annihilated by $H_A(\beta)$) if and only if v has minimal negative support.*

In order to simplify notations we will adopt in the sequel the following convention: for $v \in \mathbb{C}^n$ and $u \in L_A$ we will denote

$$\Gamma[v; u] := \frac{[v]_{u_-}}{[v + u]_{u_+}}$$

if $u \in N_v$ and $\Gamma[v; u] := 0$ otherwise. With this convention we have

$$\phi_v = x^v \sum_{u \in L_A} \Gamma[v; u] x^u.$$

We will also use the following definitions.

Definition 3.2.3. *The support of a series $\sum_{v \in \mathbb{C}^n} c_v x^v$ is the set*

$$\{v \in \mathbb{C}^n : \ c_v \neq 0\}.$$

Definition 3.2.4. *$\beta \in \mathbb{C}^d$ is said to be generic if it runs in a Zariski open set. β is said to be very generic if it runs in a countable intersection of Zariski open sets.*

Γ–series φ_v (resp. ϕ_v) are used in [GZK89] (resp. [SST00]) in order to construct a basis of the space of holomorphic solutions of $\mathcal{M}_A(\beta)$ at some nonsingular points in the case when the conditions *i*) and *ii*) of Theorem 3.1.6 are satisfied and β is very generic (resp. generic).

Chapter 4

Irregularity of $\mathcal{M}_A(\beta)$ for $A = (a\ b)$.

4.1 The case of a plane curve

This chapter is devoted to the study of the irregularity of the hypergeometric system associated with an affine plane monomial curve (joint work with F. J. Castro Jiménez [$FC_1$08]). In Section 5.1 (see also [$FC_2$08]) the study of the irregularity of the hypergeometric system associated with any affine monomial curve in $\mathbb{C}^n$ will be reduced to the two dimensional case by using deep results in $\mathcal{D}$-module theory. This justifies our separated treatment for the two variables case in this chapter. In addition, we would like to point out that we will just use elementary methods in this chapter.

We will assume throughout this chapter that $A = (a\ b)$ is an integer row matrix with $0 < a < b$ and $\beta \in \mathbb{C}$. We can assume without loss of generality that a, b are relatively prime (see Remark 3.1.4).

In this case, the hypergeometric $\mathcal{D}$–module $\mathcal{M}_A(\beta)$ is the quotient of $\mathcal{D}$ modulo the sheaf of left ideals generated by the operators

$$P := \Box_{(b,-a)} = \partial_1^b - \partial_2^a$$

and

$$E_A - \beta = ax_1\partial_1 + bx_2\partial_2 - \beta.$$

Sometimes we will write $E = E_A - \beta$ if no confusion is possible.

Although it can be deduced from general results ([GGZ87] and [Ado94, Th. 3.9]) a direct computation shows that the characteristic variety of $\mathcal{M}_A(\beta)$ is

$$\mathrm{Ch}(\mathcal{M}_A(\beta)) = T_X^*X \cup T_Y^*X$$

where $Y = \{x_2 = 0\}$ and then the singular support of $\mathcal{M}_A(\beta)$ is the axis Y.

Our goal here is to explicitly describe the cohomology of $\mathrm{Irr}_Y(\mathcal{M}_A(\beta))$, i.e. we will compute the vector spaces

$$\mathcal{H}^i(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p = \mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$$

for $p \in Y$, $i \in \mathbb{N}^*$ and $1 \leq s \leq \infty$. In the process we will also compute the cohomology of $\mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))$ for all $1 \leq s \leq \infty$.

4.1.1 Holomorphic solutions of $\mathcal{M}_A(\beta)$ at a generic point

By [GZK89, Th. 2] and [Ado94, Cor. 5.21] the dimension of the vector space of holomorphic solutions of $\mathcal{M}_A(\beta)$ at a point $p \in X \setminus Y$ equals b. A basis of such vector space of solutions can be described, using Γ–series (see [GGZ87], [GZK89, Sec. 1] and [SST00, Sec. 3.4], see also Section 3.2) as follows.

For $j = 0, \ldots, b-1$ let us consider

$$v^j = (j, \frac{\beta - ja}{b}) \in \mathbb{C}^2$$

and the corresponding Γ–series

$$\phi_{v^j} = x^{v^j} \sum_{m \geq 0} \Gamma[v^j; u(m)] \left(\frac{x_1^b}{x_2^a}\right)^m \in x^{v^j}\mathbb{C}[[x_1, x_2^{-1}]]$$

with $u(m) = (bm, -am) \in L_A = \mathrm{Ker}_{\mathbb{Z}}(A)$, which defines a holomorphic function at any point $p \in X \setminus Y$. This can be easily proven by applying d'Alembert ratio test to the series in $\frac{x_1^b}{x_2^a}$

$$\psi := \sum_{m \geq 0} \Gamma[v^j; u(m)] \left(\frac{x_1^b}{x_2^a}\right)^m.$$

Writing $c_m := \Gamma[v^j; u(m)]$ we have

$$\lim_{m \to \infty} \left| \frac{c_{m+1}}{c_m} \right| = \lim_{m \to \infty} \frac{(am)^a}{(bm)^b} = 0.$$

Thus $\{\phi_{v^j,p} : \ j = 0, \ldots, b-1\}$ is a basis of $\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_X)_p$ at any point $p \in X \setminus Y$. However $\phi_{v^j} \notin \mathcal{O}_X(X \setminus Y)$ if $\frac{\beta - ja}{b} \notin \mathbb{Z}$ because it has monodromy along any loop around the axis Y.

4.2 Free resolution of $\mathcal{M}_A(\beta)$

The aim of this section is to compute a free resolution of $\mathcal{M}_A(\beta)$. This will be useful in Sections 4.3 and 4.4, where we will describe the cohomology of the irregularity complex $\mathrm{Irr}_Y(\mathcal{M}_A(\beta))$.

Lemma 4.2.1. *A free resolution of $\mathcal{M}_A(\beta)$ is given by*

$$0 \longrightarrow \mathcal{D} \xrightarrow{\psi_1} \mathcal{D}^2 \xrightarrow{\psi_0} \mathcal{D} \xrightarrow{\pi} \mathcal{M}_A(\beta) \longrightarrow 0 \tag{4.1}$$

where ψ_0 is defined by the column matrix $(P, E)^t$, ψ_1 is defined by the row matrix $(E + ab, -P)$ and π is the canonical projection.

Proof. It is a well-known result that the stalk $\mathcal{D}_p$ is a flat $A_n(\mathbb{C})$–module, for all $p \in X$. Then, we only need to prove that (4.1) is an exact sequence for $A_n(\mathbb{C})$ instead of $\mathcal{D}$.

We will prove that $\mathrm{Ker}(\psi_0) \subseteq \mathrm{Im}(\psi_1)$ since the rest of inclusions are obvious.

Consider the symbol map

$$\begin{array}{rcl} \sigma : A_n(\mathbb{C}) & \longrightarrow & \mathbb{C}[x_1, x_2, \xi_1, \xi_2] \\ R = \sum_\alpha a_\alpha(x)\partial^\alpha & \mapsto & \sigma(R) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \end{array}$$

where $m = \mathrm{ord}(R)$ is the order of R. Notice that

$$\sigma(R_1R_2) = \sigma(R_1)\sigma(R_2)$$

and that

$$\sigma(R_1 + R_2) = \begin{cases} \sigma(R_1) + \sigma(R_2) & \text{if } \mathrm{ord}(R_1) = \mathrm{ord}(R_2) \\ \sigma(R_1) & \text{if } \mathrm{ord}(R_1) > \mathrm{ord}(R_2) \\ \sigma(R_2) & \text{if } \mathrm{ord}(R_1) < \mathrm{ord}(R_2) \end{cases}$$

Let $(Q, R) \in A_n(\mathbb{C})^2$ be a relation for the pair (P, E), with $Q, R \neq 0$, i.e., $QP + RE = 0$. Let us prove that $(Q, R) \in A_n(\mathbb{C})(E + ab, -P) = \mathrm{Im}(\psi_1)_0$:

Denote $\delta_1 := \mathrm{ord}(Q)$ and $\delta_2 := \mathrm{ord}(R)$. Since $QP + RE = 0$, we have that

$$\delta_1 + b = \mathrm{ord}(QP) = \mathrm{ord}(RE) = \delta_2 + 1$$

and that

$$\sigma(Q)\sigma(P) = -\sigma(R)\sigma(E) \tag{4.2}$$

However, $\sigma(P) = \xi_1^b$ and $\sigma(E) = ax_1\xi_1 + bx_2\xi_2$ are relatively prime so

$$(\sigma(Q), \sigma(R)) = \Lambda_1 \cdot (\sigma(E), \sigma(P))$$

for some polynomial $\Lambda_1 \in \mathbb{C}[x_1, x_2, \xi_1, \xi_2]$ which is homogeneous in the variables ξ_i.

Take an operator $\widetilde{\Lambda}_1 \in A_n(\mathbb{C})$ such that $\sigma(\widetilde{\Lambda}_1) = \Lambda_1$ and consider the relation for (P, E):

$$Q_1P + R_1E = 0$$

where $(Q_1, R_1) = (Q, R) - \widetilde{\Lambda}_1(E + ab, -P)$.

We have that $\mathrm{ord}(Q_1) = \mathrm{ord}(Q - \widetilde{\Lambda}_1(E + ab)) < \delta_1$ and $\mathrm{ord}(R_1) = \mathrm{ord}(R + \widetilde{\Lambda}_1 P) < \delta_2$ by (4.2). Thus, iterating this process leads to a relation:

$$Q_l P + R_l E = 0$$

such that $(Q_l, R_l) = (Q, R) - \sum_{i=1}^{l} \widetilde{\Lambda}_i(E + ab, -P)$, $\widetilde{\Lambda}_i \in A_n(\mathbb{C})$, and either the order of Q_l or the order of R_l is lower than or equal to 0. Since the order of P is $b > 1$ and the order of E is 1, we can assume that the order of Q_l is lower than or equal to 0, i.e., $Q_l = q_l(x) \in \mathbb{C}[x]$. Applying the symbol map we obtain $q_l(x)\sigma(P) = -\sigma(R_l)\sigma(E)$, i.e.,

$$q_l(x)\xi_1^b = -\sigma(R_l)(ax_1\xi_1 + bx_2\xi_2)$$

which is a contradiction (since ξ_1^b and $ax_1\xi_1 + bx_2\xi_2$ are relatively prime) except in the case $R_l = q_l = 0$. Thus $(Q, R) \in A_n(\mathbb{C})(E + ab, -P)$. □

Remark 4.2.2. *For any left $\mathcal{D}$–module $\mathcal{F}$ the solution complex*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{F})$$

is represented by

$$0 \longrightarrow \mathcal{F} \xrightarrow{\psi_0^*} \mathcal{F} \oplus \mathcal{F} \xrightarrow{\psi_1^*} \mathcal{F} \longrightarrow 0$$

where $\psi_0^*(f) = (P(f), E(f))$ and $\psi_1^*(f_1, f_2) = (E+ab)(f_1) - P(f_2)$ for f, f_1, f_2 local sections in $\mathcal{F}$.

4.3 Nullity of $\mathrm{Irr}_Y(\mathcal{M}_A(\beta))_{(0,0)}$

The aim of this section is to prove the following result.

Theorem 4.3.1. *With the previous notations we have $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))_{(0,0)} = 0$ for all $\beta \in \mathbb{C}$ and $1 \leq s \leq \infty$. In other words*

$$\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_{(0,0)} = 0$$

for all $\beta \in \mathbb{C}$, $1 \leq s \leq \infty$ and $i \in \mathbb{N}$. Here the $\mathcal{E}xt$ groups are taken over the sheaf of rings $\mathcal{D}$.

Let us consider the following vector spaces

$$V_A(\beta, s) := \left\{ \sum_{\alpha \in \mathbb{N}^2} a_\alpha x^\alpha \in \mathcal{O}_{X|Y}(s)_{(0,0)} : \ a_\alpha = 0 \text{ if } A\alpha = \beta \right\}$$

and

$$W_A(\beta, s) := \left\{ \sum_{\alpha \in \mathbb{N}^2} a_\alpha x^\alpha \in \mathcal{O}_{X|Y}(s)_{(0,0)} : \ a_\alpha = 0 \text{ if } A\alpha \neq \beta \right\}$$

Remark 4.3.2. *It is clear that* $V_A(\beta,s)\oplus W_A(\beta,s)=\mathcal{O}_{X|Y}(s)_{(0,0)}$. *Moreover,* $W_A(\beta,s)$ *is a vectorial subspace of* $\mathbb{C}[x_1,x_2]$ *with finite dimension. In particular,* $W_A(\beta,s)=0$ *(and so* $V_A(\beta,s)=\mathcal{O}_{X|Y}(s)_{(0,0)}$*) if and only if* $\beta\notin a\mathbb{N}+b\mathbb{N}$.

Lemma 4.3.3. *i) The* $\mathbb{C}$*–linear map*

$$E_A-\beta : V_A(\beta,s)\to V_A(\beta,s)$$

is an automorphism for all $1\le s\le\infty$ *and* $\beta\in\mathbb{C}$. *In particular, if* $\beta\notin a\mathbb{N}+b\mathbb{N}$ *then* $E_A-\beta$ *is an automorphism of* $\mathcal{O}_{X|Y}(s)_{(0,0)}$ *for all* $1\le s\le\infty$.
ii) The $\mathbb{C}$*–linear map*

$$P : W_A(\beta,s)\to W_A(\beta-ab,s)$$

is surjective for all $1\le s\le\infty$ *and* $\beta\in\mathbb{C}$.

Proof. If $\beta\notin ab+a\mathbb{N}+b\mathbb{N}$ then $W_A(\beta-ab,s)=0$ and part *ii)* is obvious in this case. Anyway, for all $\beta\in\mathbb{C}$ and $f\in W_A(\beta-ab,s)$ the solution $h\in W_A(\beta,s)$ to the equation $P(h)=f$ is given by the following recurrence relation for the coefficients h_j and f_j of $x_1^{\frac{\beta-bj}{a}}x_2^j$ and $x_1^{\frac{\beta-bj}{a}-b}x_2^j$ in h and f respectively:

$$h_{k+a(m+1)}=\frac{1}{(k+a(m+1))_a}\left(\left(\frac{\beta-bk}{a}-bm\right)_b h_{k+am}-f_{k+am}\right)\tag{4.3}$$

for $k=0,\dots,a-1$ and $m\in\mathbb{N}$.

Let us prove part *i)*. For $f=\sum_{\alpha\in\mathbb{N}^2}f_\alpha x^\alpha\in\mathbb{C}[[x_1,x_2]]$ we have that

$$(E_A-\beta)(f)=\sum_{\alpha\in\mathbb{N}^2}f_\alpha(A\alpha-\beta)x^\alpha.$$

This implies that $E_A-\beta$ is an automorphism of $V_A(\beta,\infty)$. It is also clear that $E_A-\beta$ is an automorphism of $V_A(\beta,1)$. For any $1<s<\infty$ we have $\rho_s(E_A-\beta)=(E_A-\beta)\rho_s$ and then $E_A-\beta$ is an automorphism of $V_A(\beta,s)$ (see Section 2.2 for the definition of ρ_s). □

Corollary 4.3.4. $E_A-\beta$ *is an automorphism of the vector space* $\mathcal{Q}_Y(s)_{(0,0)}$ *for* $1\le s\le\infty$ *and* $\beta\in\mathbb{C}$.

Proof. [**Theorem 4.3.1**] Let us simply write $E:=E_A-\beta$. The complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))_{(0,0)}$ is represented by the germ at $(0,0)$ of the following complex

$$0\longrightarrow\mathcal{Q}_Y(s)\xrightarrow{\psi_0^*}\mathcal{Q}_Y(s)\oplus\mathcal{Q}_Y(s)\xrightarrow{\psi_1^*}\mathcal{Q}_Y(s)\longrightarrow 0$$

where $\psi_0^*(f)=(P(f),E(f))$ and $\psi_1^*(f_1,f_2)=(E+ab)(f_1)-P(f_2)$ for f,f_1,f_2 germs in $\mathcal{Q}_Y(s)$ (see Remark 4.2.2). In particular, we only need to prove the statement for $i=0,1,2$.

For $i = 0, 2$ the statement follows from Corollary 4.3.4. Let us see the case $i = 1$. Let us consider $(\overline{f}, \overline{g}) \in \operatorname{Ker}(\psi_1^*)_{(0,0)}$ (i.e. $(E + ab)(\overline{f}) = P(\overline{g})$). We want to prove that there exists $\overline{h} \in \mathcal{Q}_Y(s)_{(0,0)}$ such that $P(\overline{h}) = \overline{f}$ and $E(\overline{h}) = \overline{g}$, where $(\overline{\ })$ means modulo $\mathcal{O}_{X|Y,(0,0)} = \mathbb{C}\{x\}$.

From Corollary 4.3.4 we have that there exists a unique $\overline{h} \in \mathcal{Q}_Y(s)_{(0,0)}$ such that $E(\overline{h}) = \overline{g}$. Since $PE = (E + ab)P$ and $(E + ab)(\overline{f}) = P(\overline{g})$ we have:

$$(E + ab)(\overline{f}) = P(\overline{g}) = P(E(\overline{h})) = (E + ab)(P(\overline{h})).$$

Since for all $\beta \in \mathbb{C}$, $E + ab = E_A - (\beta - ab)$ is an automorphism of $\mathcal{Q}_Y(s)_{(0,0)}$ (see Corollary 4.3.4) we also have $\overline{f} = \overline{P(h)}$. So $(\overline{f}, \overline{g}) = (P(\overline{h}), E(\overline{h})) \in \operatorname{Im}(\psi_0^*)_{(0,0)}$. □

Remark 4.3.5. *From Theorem 4.3.1 and the long exact sequence of cohomology associated with the exact sequence (2.1) we have*

$$\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_{(0,0)} \simeq \mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_{(0,0)}$$

for $1 \leq s \leq \infty$, $i \in \mathbb{N}$ *and* $\beta \in \mathbb{C}$. *In fact we have the following two propositions.*

Proposition 4.3.6. *With the previous notations we have*

$$\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_{(0,0)} = 0$$

for all $\beta \notin a\mathbb{N} + b\mathbb{N}$, $1 \leq s \leq \infty$ *and* $i \in \mathbb{N}$.

Proof. Since $\beta \notin a\mathbb{N} + b\mathbb{N}$ then $V_A(\beta) = \mathcal{O}_{X|Y}(s)_{(0,0)}$ and thus $E = E_A - \beta$ is an automorphism of $\mathcal{O}_{X|Y}(s)_{(0,0)}$ by part i) of Lemma 4.3.3. It follows that the proof is analogous to the one of Theorem 4.3.1. □

Proposition 4.3.7. *With the previous notations we have*

$$\dim_{\mathbb{C}}(\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_{(0,0)}) = \begin{cases} 1 & \text{if} \quad i = 0, 1 \\ 0 & \text{if} \quad i \geq 2 \end{cases}$$

for all $\beta \in a\mathbb{N} + b\mathbb{N}$ *and* $1 \leq s \leq \infty$. *Moreover,* $\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_{(0,0)}$ *is generated by a polynomial* ϕ_{v^q} *when* $i = 0$ *and by the class of* $(0, \phi_{v^q})$ *when* $i = 1$ *(see the proof of Lemma 4.4.1 for the definition of* ϕ_{v^q}*).*

Proof. By Remark 4.2.2 it is enough to consider $i = 0, 1, 2$. Let us treat first the case $i = 2$. Let $h \in \mathcal{O}_{X|Y}(s)_{(0,0)}$ and write $h = h_1 + h_2$ with $h_1 \in V_A(\beta - ab, s)$ and $h_2 \in W_A(\beta - ab, s)$. From Lemma 4.3.3 there exist $f \in V_A(\beta - ab, s)$ and $g \in W_A(\beta, s)$ such that $(E + ab)(f) = h_1$ and $P(g) = -h_2$. Then $\psi_1^*(f, g) = (E + ab)(f) - P(g) = h_1 + h_2 = h$ and so the germ of ψ_1^* at $(0, 0)$ is surjective. This implies the statement for $i = 2$.

Let us see now that $\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_{(0,0)}$ has dimension 1. Assume that $h \in \mathcal{O}_{X|Y}(s)_{(0,0)}$ satisfies $P(h) = E(h) = 0$ and write $h = h_1 + h_2$ with $h_1 \in V_A(\beta, s)$ and $h_2 \in W_A(\beta, s)$. We have $E(h_2) = 0$ and then $E(h) = E(h_1) = 0$ implies $h_1 = 0$ because of Lemma 4.3.3. Now, from $P(h) = P(h_2) = 0$ we get $h_2 = \lambda\phi_{v^q}$ for some $\lambda \in \mathbb{C}$ (see Proposition 4.4.4).

Finally, let us prove that $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_{(0,0)}$ has dimension 1. Consider $(f, g) \in \mathcal{O}_{X|Y}(s)_{(0,0)}$ such that $(E + ab)(f) = P(g)$ and write $f = f_1 + f_2, g = g_1 + g_2$ with $f_1 \in V_A(\beta - ab, s)$, $f_2 \in W_A(\beta - ab, s)$, $g_1 \in V_A(\beta, s)$ and $g_2 \in W_A(\beta, s)$. As $(E + ab)(f_2) = 0$ we have $(E + ab)(f) = (E + ab)(f_1) = P(g_1) + P(g_2)$. This implies $P(g_2) = 0$ since $(E + ab)(f_1)$ and $P(g_1)$ belong to $V_A(\beta - ab, s)$. By Lemma 4.3.3 there exists $h_1 \in V_A(\beta, s)$ such that $E(h_1) = g_1$. We also have $(E + ab)(f_1 - P(h_1)) = (E + ab)(f_1) - PE(h_1) = 0$ and again by Lemma 4.3.3 we have $(f_1, g_1) = (P(h_1), E(h_1))$.

By Lemma 4.3.3 there exists $h_2 \in W_A(\beta, a)$ such that $P(h_2) = f_2$. So, $(f_2, g_2) - (P(h_2), E(h_2)) = (0, g_2) = \lambda(0, \phi_{v^q})$ for some $\lambda \in \mathbb{C}$ since $P(g_2) = 0$ (see Proposition 4.4.4). □

4.4 Description of $\mathrm{Irr}_Y(\mathcal{M}_A(\beta))_p$ for $p \in Y$, $p \neq (0, 0)$

We will compute a basis of the vector space $\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ for $1 \leq s \leq \infty$, $i \in \mathbb{N}$, $p \in Y$, $p \neq (0, 0)$. In this section we are writing $p = (\epsilon, 0) \in Y$ with $\epsilon \in \mathbb{C}^*$.

We are going to use Γ–series following ([GGZ87], [GZK89, Section 1]) and in the way they are handled in [SST00, Section 3.4].

We will consider the family $v^k = (\frac{\beta - kb}{a}, k) \in \mathbb{C}^2$ for $k = 0, \ldots, a - 1$. They satisfy $Av^k = \beta$ and the corresponding Γ–series are

$$\phi_{v^k} = x^{v^k} \sum_{m \geq 0} \Gamma[v^k; u(m)] x_1^{-bm} x_2^{am} \in x^{v^k} \mathbb{C}[[x_1^{-1}, x_2]]$$

where $u(m) = (-bm, am)$ for $m \in \mathbb{Z}$.

Although ϕ_{v^k} does not define in general any holomorphic germ at $(0, 0)$ it is clear that it defines a germ $\phi_{v^k,p}$ in $\mathcal{O}_{\widehat{X|Y},p}$ for $k = 0, 1, \ldots, a - 1$. Let us write $x_1 = t_1 + \epsilon$ and remind that $\epsilon \in \mathbb{C}^*$. We have

$$\phi_{v^k,p} = (t_1 + \epsilon)^{\frac{\beta - bk}{a}} x_2^k \sum_{m \geq 0} \Gamma[v^k; u(m)] (t_1 + \epsilon)^{-bm} x_2^{am}.$$

Lemma 4.4.1. *1. If $\beta \in a\mathbb{N} + b\mathbb{N}$ then there exists a unique $0 \leq q \leq a - 1$ such that ϕ_{v^q} is a polynomial. Moreover, the Gevrey index of $\phi_{v^k,p} \in \mathcal{O}_{\widehat{X|Y},p}$ is $\frac{b}{a}$ for $0 \leq k \leq a - 1$ and $k \neq q$.*

2. If $\beta \notin a\mathbb{N}+b\mathbb{N}$ then the Gevrey index of $\phi_{v^k,p} \in \mathcal{O}_{\widehat{X|Y},p}$ is $\frac{b}{a}$ for $0 \le k \le a-1$.

Proof. The notion of Gevrey index is given in Definition 2.2.3. Let us assume first that $\beta \in a\mathbb{N}+b\mathbb{N}$. Then there exists a unique $0 \le q \le a-1$ such that $\beta = qb + a\mathbb{N}$. Then for $m \in \mathbb{N}$ big enough $\frac{\beta - qb}{a} - bm$ is a negative integer and the coefficient $\Gamma[v^q; u(m)]$ is zero.

So ϕ_{v^q} is a polynomial in $\mathbb{C}[x_1, x_2]$ (and then $\phi_{v^q,p}(t_1, x_2)$ is a polynomial in $\mathbb{C}[t_1, x_2]$) since for $\frac{\beta - qb}{a} - bm \ge 0$ the expression

$$x^{v^q} x_1^{-bm} x_2^{am}$$

is a monomial in $\mathbb{C}[x_1, x_2]$.

Let us consider an integer number k with $0 \le k \le a-1$. Assume $\frac{\beta - bk}{a} \notin \mathbb{N}$. Then the formal power series $\phi_{v^k,p}(t_1, x_2)$ is not a polynomial. We will see that its Gevrey index is b/a. It is enough to prove that the Gevrey index of

$$\psi(t_1, x_2) := \sum_{m \ge 0} \Gamma[v^k; u(m)](t_1+\epsilon)^{-bm} x_2^{am} = \sum_{m \ge 0} \Gamma[v^k; u(m)] \left(\frac{x_2^a}{(t_1+\epsilon)^b} \right)^m$$

is b/a, i.e. the series

$$\rho_s(\psi(t_1, x_2)) = \sum_{m \ge 0} \frac{\Gamma[v^k; u(m)]}{(am)!^{s-1}} \left(\frac{x_2^a}{(t_1+\epsilon)^b} \right)^m$$

is convergent for $s \ge b/a$ and divergent for $s < b/a$.

Considering $\rho_s(\psi(t_1, x_2))$ as a power series in $(x_2^a/(t_1+\epsilon)^b)$ and writing

$$c_m := \frac{\Gamma[v^k; u(m)]}{(am)!^{s-1}}$$

we have that

$$\lim_{m \to \infty} \left| \frac{c_{m+1}}{c_m} \right| = \lim_{m \to \infty} \frac{(bm)^b}{(am)^{as}} = \begin{cases} 0 & \text{if } as > b \\ (b/a)^b & \text{if } as = b \\ \infty & \text{if } as < b \end{cases}$$

and then by using the d'Alembert's ratio test it follows that the power series $\rho_s(\psi(t_1, x_2))$ is convergent for $s \ge b/a$ and divergent for $s < b/a$.

□

Proposition 4.4.2. *We have that*

$$\dim_{\mathbb{C}} \left(\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p \right) = a$$

for all $\beta \in \mathbb{C}$, $p \in Y \setminus \{(0,0)\}$.

Proof. Recall that $p = (\epsilon, 0)$ with $\epsilon \in \mathbb{C}^*$. The operators defining $\mathcal{M}_A(\beta)_p$ are (using coordinates (t_1, x_2)) $P = \partial_1^b - \partial_2^a$ and $E_p := at_1\partial_1 + bx_2\partial_2 + a\epsilon\partial_1 - \beta$.

First of all, we will prove the inequality

$$\dim_{\mathbb{C}} \left(\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p \right) \leq a. \tag{4.4}$$

Assume that $f \in \mathbb{C}[[t_1, x_2]]$, $f \neq 0$, satisfies $E_p(f) = P(f) = 0$. Then choosing $\omega \in \mathbb{R}^2_{>0}$ such that $a\omega_2 > b\omega_1$, we have $\text{in}_{(-\omega,\omega)}(E_p) = a\epsilon\partial_1$ and $\text{in}_{(-\omega,\omega)}(P) = \partial_2^a$, where $\text{in}_{(-\omega,\omega)}(-)$ stands for the initial part with respect to the weights $weight(x_i) = -w_i$, $weight(\partial_i) = w_i$.

Then (see [SST00, Th. 2.5.5]) $\partial_1(\text{in}_\omega(f)) = \partial_2^a(\text{in}_\omega(f)) = 0$. So, $\text{in}_\omega(f) = \lambda_l x_2^l$ for some $0 \leq l \leq a-1$ and some $\lambda_l \in \mathbb{C}$. This implies the inequality (4.4).

On the other hand, remind that

$$\phi_{v^k,p} = (t_1 + \epsilon)^{\frac{\beta - bk}{a}} x_2^k \sum_{m \geq 0} \Gamma[v^k; u(m)](t_1 + \epsilon)^{-bm} x_2^{am}$$

is a formal series in $\mathbb{C}[[t_1, x_2]]$ with support contained in $\mathbb{N} \times (k + a\mathbb{N})$ for $k = 0, 1, \ldots, a-1$. Then the family $\{\phi_{v^k,p} \,|\, k = 0, \ldots, a-1\}$ is $\mathbb{C}$-linearly independent and they are solutions of $\mathcal{M}_A(\beta)_p$ in $\mathcal{O}_{\widehat{X|Y},p}$. □

Proposition 4.4.3. *If $\beta \notin a\mathbb{N} + b\mathbb{N}$ then*

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p = \begin{cases} \sum_{k=0}^{a-1} \mathbb{C}\phi_{v^k,p} & \text{if } s \geq \frac{b}{a} \\ 0 & \text{if } s < \frac{b}{a} \end{cases}$$

for all $p = (\epsilon, 0) \in \mathbb{C}^* \times \{0\}$.

Proof. The equality for $s \geq b/a$ follows from Proposition 4.4.2 and Lemma 4.4.1. Moreover, since the series $\phi_{v^k,p}$ have pairwise disjoint supports (see the proof Proposition 4.4.2) we have that any non-trivial linear combination $\sum_{k=0}^{a-1} \lambda_k \phi_{v^k,p}$ with $\lambda_k \in \mathbb{C}$ has Gevrey index equal to b/a if $\beta \notin a\mathbb{N} + b\mathbb{N}$. This implies the equality for $s < b/a$. □

Proposition 4.4.4. *If $\beta \in a\mathbb{N} + b\mathbb{N}$ then*

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p = \begin{cases} \sum_{k=0}^{a-1} \mathbb{C}\phi_{v^k,p} & \text{if } s \geq \frac{b}{a} \\ \mathbb{C}\phi_{v^q} & \text{if } s < \frac{b}{a} \end{cases}$$

for all $p = (\epsilon, 0) \in \mathbb{C}^* \times \{0\}$ where q is the unique $k \in \{0, 1, \ldots, a-1\}$ such that $\beta \in kb + a\mathbb{N}$.

Proof. The proof is analogous to the one of Proposition 4.4.3 and follows from Lemma 4.4.1. □

Lemma 4.4.5. *The germ of* $E := E_A - \beta$ *at any point* $p = (\epsilon, 0) \in \mathbb{C}^* \times \{0\}$ *induces a surjective endomorphism on* $\mathcal{O}_{X|Y}(s)_p$ *for all* $\beta \in \mathbb{C}$, $1 \leq s \leq \infty$.

Proof. We will prove that

$$E_p : \mathcal{O}_{X|Y}(s)_{(0,0)} \longrightarrow \mathcal{O}_{X|Y}(s)_{(0,0)}$$

is surjective (using coordinates (t_1, x_2). It is enough to prove that

$$F := \partial_1 + bx_2u(t_1)\partial_2 - \beta u(t_1)$$

yields a surjective endomorphism on $\mathcal{O}_{X|Y}(s)_{(0,0)}$, where $u(t_1) = (a(t_1 + \epsilon))^{-1} \in \mathbb{C}\{t_1\}$. For $s = 1$, the surjectivity of F follows from Cauchy-Kovalevskaya theorem. To finish the proof it is enough to notice that $\rho_s \circ F = F \circ \rho_s$ for $1 \leq s < \infty$. For $s = \infty$ the result is obvious. □

Corollary 4.4.6. *For all* $p \in Y$, $p \neq (0,0)$, $\beta \in \mathbb{C}$ *and* $1 \leq s \leq \infty$, *we have:*

i) $\mathcal{E}xt^2(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p = 0$.

ii) $\mathcal{E}xt^2(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = 0$.

Proof. i) We first consider the germ at p of the solution complex of $\mathcal{M}_A(\beta)$ as described in Remark 4.2.2 for $\mathcal{F} = \mathcal{O}_{X|Y}(s)$. Then we apply that $E + ab$ is surjective on $\mathcal{O}_{X|Y}(s)_p$ (Lemma 4.4.5).
ii) It follows from *i)* and the long exact sequence in cohomology associated with (2.1). □

4.4.1 Computation of $\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ for $p \in Y$, $p \neq (0,0)$

Lemma 4.4.7. *Assume that* $f \in \mathbb{C}[[t_1, x_2]]$ *satisfies* $E_p(f) = 0$. *Then* $f = \sum_{k=0}^{a-1} f^{(k)}$ *where*

$$f^{(k)} = \sum_{m\geq 0} f_{k+am}(t_1 + \epsilon)^{\frac{\beta - bk}{a} - bm} x_2^{k+am}$$

with $f_{k+am} \in \mathbb{C}$.

Proof. By [SST00, Th. 2.5.5] we have that $\mathrm{in}_{(-\omega,\omega)}(E_p)(\mathrm{in}_\omega(f)) = 0$ for all $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2_{\geq 0}$.

If $\omega_1 > 0$ then $\mathrm{in}_{(-\omega,\omega)}(E_p) = a\epsilon\partial_1$ and so, $\mathrm{in}_\omega(f) \in \mathbb{C}[[x_2]]$ for all ω with $\omega_1 > 0$. On the other hand, if $w_1 = 0$ then $\mathrm{in}_{(-\omega,\omega)}(E_p) = E_p$ and in particular $E_p(\mathrm{in}_{(0,1)}(f)) = 0$. Then $\mathrm{in}_{(-\omega,\omega)}(E_p)(\mathrm{in}_\omega(\mathrm{in}_{(0,1)}(f))) = \epsilon\partial_1(\mathrm{in}_{(0,1)}(f)) = 0$ and so $\mathrm{in}_\omega(\mathrm{in}_{(0,1)}(f)) \in \mathbb{C}[x_2]$, for all $\omega \in \mathbb{R}^2_{>0}$.

This implies the existence of $h(t_1) \in \mathbb{C}[[t_1]]$ with $h(0) \neq 0$ such that $\mathrm{in}_{(0,1)}(f) = x_2^r h(t_1)$ for some $r \in \mathbb{N}$. Take (k, m) the unique pair with $k \in \{0, \dots, a-1\}$ and $m \in \mathbb{N}$ such that $r = k + am$.

There exists $f_{k+am} \in \mathbb{C}^*$ such that t_1 divides

$$\mathrm{in}_{(0,1)}(f) - f_{k+am}(t_1 + \epsilon)^{\frac{\beta - bk}{a} - bm} x_2^{k+am} \in \mathbb{C}[[t_1]] x_2^{k+am}.$$

This and the fact that

$$E_p(\mathrm{in}_{(0,1)}(f) - f_{k+am}(t_1 + \epsilon)^{\frac{\beta - bk}{a} - bm} x_2^{k+am}) = 0$$

imply that $\mathrm{in}_{(0,1)}(f) = f_{k+am}(t_1 + \epsilon)^{\frac{\beta - bk}{a} - bm} x_2^{k+am}$.

We finish by induction by applying the same argument to $f - \mathrm{in}_{(0,1)}(f)$ since $E_p(f - \mathrm{in}_{(0,1)}(f)) = 0$. $\square$

Recall that $Y = \{x_2 = 0\} \subset X = \mathbb{C}^2$ and $v^k = (\frac{\beta - bk}{a}, k)$ for $k = 0, \dots, a-1$.

Remark 4.4.8. *As in the proof of Lemma 4.4.1 if $\beta \in a\mathbb{N} + b\mathbb{N}$ then there exists a unique $0 \leq q \leq a-1$ such that $\beta \in qb + a\mathbb{N}$ and that ϕ_{v^q} is a polynomial.*

Let us write $m_0 = \frac{\beta - qb}{a}$, m' the smallest integer number satisfying $bm' \geq m_0 + 1$ and

$$\widetilde{v^q} := v^q + u(m') = (m_0 - bm', q + am').$$

It is clear that $A\widetilde{v^q} = \beta$ and that $\widetilde{v^q}$ does not have minimal negative support (see Definition 3.2.1, [SST00, p. 132-133]). Then the Γ–series $\phi_{\widetilde{v^q}}$ is not a solution of $H_A(\beta)$. We have

$$\phi_{\widetilde{v^q}} = x^{\widetilde{v^q}} \sum_{m \in \mathbb{N};\, bm \geq m_0 + 1} \Gamma[\widetilde{v^q}; u(m)] x_1^{-bm} x_2^{am}.$$

It is easy to prove that $H_A(\beta)_p(\phi_{\widetilde{v^q},p}) \subset \mathcal{O}_{X,p}$ for all $p = (\epsilon, 0) \in X$ with $\epsilon \neq 0$ (in fact, $(E_A - \beta)(\phi_{\widetilde{v^q}}) = 0$ and $P(\phi_{\widetilde{v^q}})$ is a Laurent monomial term in the variables x_1, x_2 with pole along $x_1 = 0$), and that $\phi_{\widetilde{v^q},p}$ is a Gevrey series of index b/a.

Theorem 4.4.9. *For all $p \in Y \setminus \{(0,0)\}$ and $\beta \in \mathbb{C}$ we have*

$$\dim_{\mathbb{C}}(\mathcal{E}xt^0\,(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p) = \begin{cases} a & \text{if} \quad s \geq b/a \\ 0 & \text{if} \quad s < b/a \end{cases}$$

Moreover, we also have

i) If $\beta \notin a\mathbb{N} + b\mathbb{N}$ then:

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = \sum_{k=0}^{a-1} \mathbb{C}\overline{\phi_{v^k,p}}$$

for all $s \geq b/a$

ii) If $\beta \in a\mathbb{N} + b\mathbb{N}$ then for all $s \geq b/a$ we have :

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = \sum_{k=0, k\neq q}^{a-1} \mathbb{C}\overline{\phi_{v^k,p}} + \mathbb{C}\overline{\phi_{\widetilde{v^q},p}}$$

with $\phi_{\widetilde{v^q}}$ as in Remark 4.4.8.

Here $\overline{\phi}$ stands for the class modulo $\mathcal{O}_{X|Y,p}$ of $\phi \in \mathcal{O}_{X|Y}(s)_p$.

Proof. It follows from Propositions 4.4.3 and 4.4.4, and Theorems 4.4.10 and 4.4.12 below by using the long exact sequence in cohomology. □

4.4.2 Computation of $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ for $p \in Y$, $p \neq (0,0)$

Theorem 4.4.10. *For all $\beta \in \mathbb{C}$ we have*

$$\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p = 0$$

for all $s \geq b/a$ and for all $p \in Y$, $p \neq (0,0)$.

Proof. We will use the germ at p of the solution complex of $\mathcal{M}_A(\beta)$ with values in $\mathcal{F} = \mathcal{O}_{X|Y}(s)$ (see Remark 4.2.2):

$$0 \to \mathcal{O}_{X|Y}(s) \xrightarrow{\psi_0^*} \mathcal{O}_{X|Y}(s) \oplus \mathcal{O}_{X|Y}(s) \xrightarrow{\psi_1^*} \mathcal{O}_{X|Y}(s) \to 0$$

Let us consider $(f,g) \in (\mathcal{O}_{X|Y}(s)_p)^2$ in the germ at p of $\mathrm{Ker}(\psi_1^*)$, i.e.

$$(E_p + ab)(f) = P(g).$$

We want to prove that there exists $h \in \mathcal{O}_{X|Y}(s)_p$ such that $P(h) = f$ and $E_p(h) = g$.

From Lemma 4.4.5, there exists $\widehat{h} \in \mathcal{O}_{X|Y}(s)_p$ such that $E_p(\widehat{h}) = g$. Then:

$$(f,g) = (P(\widehat{h}), E_p(\widehat{h})) + (\widehat{f}, 0)$$

where $\widehat{f} = f - P(\widehat{h}) \in \mathcal{O}_{X|Y}(s)_p$ and $(\widehat{f}, 0) \in \mathrm{Ker}(\psi_1^*)$.

In order to finish the proof it is enough to prove that there exists $h \in \mathcal{O}_{X|Y}(s)_p$ such that $P(h) = \widehat{f}$ and $E_p(h) = 0$.

Since $h, \widehat{f} \in \mathbb{C}[[t_1, x_2]]$, $(E_p + ab)(\widehat{f}) = 0$ and $E_p(h)$ must be 0, it follows from Lemma 4.4.7 that

$$h = \sum_{k=0}^{a-1} \sum_{m \geq 0} h_{k+am} (t_1 + \epsilon)^{\frac{\beta - bk}{a} - bm} x_2^{k+am}$$

and

$$\widehat{f} = \sum_{k=0}^{a-1} \sum_{m \geq 0} f_{k+am} (t_1 + \epsilon)^{\frac{\beta - bk}{a} - b(m+1)} x_2^{k+am}$$

with $h_{k+am}, f_{k+am} \in \mathbb{C}$.

The equation $P(h) = \widehat{f}$ is equivalent to the recurrence relation:

$$h_{k+a(m+1)} = \frac{1}{(k + a(m+1))_a} \left(\left(\frac{\beta - bk}{a} - bm \right)_b h_{k+am} - f_{k+am} \right) \tag{4.5}$$

for $k = 0, \ldots, a-1$ and $m \in \mathbb{N}$. The solution to this recurrence relation proves that there exists $h \in \mathbb{C}[[t_1, x_2]]$ such that $P(h) = \widehat{f}$ and $E_p(h) = 0$.

We need to prove now that $h \in \mathcal{O}_{X|Y}(s)_p$.

Dividing (4.5) by $((k + a(m+1))!)^{s-1}$ we get:

$$\frac{h_{k+a(m+1)}}{(k+a(m+1))!^{s-1}} = \frac{1}{((k+a(m+1))_a)^s} \left(\left(\frac{\beta - bk}{a} - bm \right)_b \frac{h_{k+am}}{(k+am)!^{s-1}} - \frac{f_{k+am}}{(k+am)!^{s-1}} \right)$$

so it is enough to prove that there exists $C, D > 0$ such that

$$\left| \frac{h_{k+am}}{(k + am)!^{s-1}} \right| \leq C D^m \tag{4.6}$$

for all $0 \leq k \leq a-1$ and $m \geq 0$. We will argue by induction on m.

Since $\rho_s(\widehat{f})$ is convergent, there exists $\widetilde{C}, \widetilde{D} > 0$ such that

$$\frac{|f_{k+am}|}{(k + am)!^{s-1}} \leq \widetilde{C} \widetilde{D}^m$$

for all $m \geq 0$ and $k = 0, \ldots, a-1$.

Since $s \geq b/a$, we have

$$\lim_{m \to \infty} \frac{|(\frac{\beta - bk}{a} - bm)_b|}{((k + a(m+1))_a)^s} \leq (b/a)^b$$

and then there exists an upper bound $C_1 > 0$ of the set

$$\left\{ \frac{\left|(\frac{\beta - bk}{a} - bm)_b\right|}{((k + a(m+1))_a)^s} : m \in \mathbb{N} \right\}.$$

Let us consider

$$C = \max\{\widetilde{C}, \frac{|h_k|}{k!^{s-1}}\,; k = 0, \ldots, a-1\}$$

and

$$D = \max\{\widetilde{D}, C_1 + 1\}.$$

So, the case $m = 0$ of (4.6) follows from the definition of C. Assume $|\frac{h_{k+am}}{(k+am)!^{s-1}}| \leq CD^m$. We will prove inequality (4.6) for $m+1$. From the recurrence relation we deduce:

$$\left|\frac{h_{k+a(m+1)}}{(k+a(m+1))!^{s-1}}\right| \leq C_1 \left|\frac{h_{k+am}}{(k+am)!^{s-1}}\right| + \widetilde{C}\widetilde{D}^m$$

and using the induction hypothesis and the definition of C, D we get:

$$\left|\frac{h_{k+a(m+1)}}{(k+a(m+1))!^{s-1}}\right| \leq (C_1+1)CD^m \leq CD^{m+1}.$$

In particular $\rho_s(h)$ converges and $h \in \mathcal{O}_{X|Y}(s)_p$. □

Lemma 4.4.11. *Assume that $h \in \mathcal{O}_{\widehat{X|Y},p}$, $p \in Y$, $p \neq (0,0)$, satisfies $E(h) = 0$ and $P(h) \in \mathcal{O}_{X|Y}(s)_p$ with $s < b/a$. Then:*

i) If $\beta \notin a\mathbb{N} + b\mathbb{N}$ there exists $g \in \mathcal{O}_{X|Y}(s)_p$ with $P(h) = P(g)$ and $E(g) = 0$.

ii) If $\beta \in a\mathbb{N} + b\mathbb{N}$ there exists $g \in \mathcal{O}_{X|Y}(s)_p$ with $P(h) = P(g + \lambda_q \phi_{\widetilde{v^q},p})$ and $E(g) = 0$.

Proof. Since $E(h) = 0$ then $(E + ab)(\widehat{f}) = 0$ for $\widehat{f} := P(h)$. Reasoning as in the proof of Theorem 4.4.10 we have the recurrence relation (4.5) for the coefficients of h and $\widehat{f}$. Let us prove first that for all $k = 0, \ldots, a-1$ such that $\frac{\beta - bk}{a} \notin \mathbb{N}$ there exists $\lambda_k \in \mathbb{C}$ with $h^{(k)} - \lambda_k \phi_{v^k,p} \in \mathcal{O}_{X|Y}(s)_p$.

Since $h_k x_1^{\frac{\beta-bk}{a}} x_2^k$ is holomorphic in a neighborhood of p and $E(x_1^{\frac{\beta-bk}{a}} x_2^k) = 0$ we can assume without loss of generality that $h_k = 0$ obtaining:

$$h_{k+a(m+1)} = -\frac{(\frac{\beta-bk}{a})_{b(m+1)}}{(k+a(m+1))!} \sum_{r=0}^{m} \frac{(k+ar)!}{(\frac{\beta-bk}{a})_{b(r+1)}} f_{k+ar}. \tag{4.7}$$

Recall that the coefficient of $x_1^{\frac{\beta-bk}{a}-bm} x_2^{k+am}$ in $\phi_{v^k,p}$ is

$$\Gamma[v^k; u(m)] = \frac{(\frac{\beta-bk}{a})_{bm} k!}{(k+am)!}.$$

Therefore for all $\lambda_k \in \mathbb{C}$ we get:

$$h_{k+a(m+1)} - \lambda_k \Gamma[v^k; u(m+1)] = \frac{(\frac{\beta-bk}{a})_{b(m+1)}}{(k+a(m+1))!} \left(-k!\lambda_k - \sum_{r=0}^{m} \frac{(k+ar)! f_{k+ar}}{(\frac{\beta-bk}{a})_{b(r+1)}} \right).$$

Since $as < b$ we can choose

$$\lambda_k = -\sum_{r\geq 0} \frac{(k+ar)! f_{k+ar}}{k!(\frac{\beta-bk}{a})_{b(r+1)}} \in \mathbb{C}$$

and, because $f^{(k)} \in \mathcal{O}_{X|Y}(s)_p$, there exist real numbers $C > 0, D > 0$ such that $|f_{k+ar}| \leq CD^r(k+ar)!^{s-1}$ for all $r \geq 0$. Then:

$$h_{k+a(m+1)} - \lambda_k \Gamma[v^k; u(m+1)] = \frac{(\frac{\beta-bk}{a})_{b(m+1)}}{(k+a(m+1))!} \sum_{r\geq m+1} \frac{(k+ar)! f_{k+ar}}{(\frac{\beta-bk}{a})_{b(r+1)}}.$$

Equivalently,

$$h_{k+a(m+1)} - \lambda_k \Gamma[v^k; u(m+1)] = \sum_{r\geq 0} \frac{(k+a(r+m+1))_{ar} f_{k+a(r+m+1)}}{(\frac{\beta-bk}{a} - (m+1)b)_{b(r+1)}}.$$

The series

$$g_m(z) = \sum_{r\geq 0} \frac{(k+a(r+m+1))_{ar}^s}{|(\frac{\beta-bk}{a} - (m+1)b)_{b(r+1)}|} z^r$$

is an entire function in the variable z for all $m \geq 0$. To prove that it is enough to apply the d'Alembert's ratio test using $b > sa$:

$$\lim_{r\to\infty} \frac{(k+a(r+m+1))^s (k+a(r+m+1)-1)^s \cdots (k+a(r+m)+1)^s}{|\frac{\beta-bk}{a} - b(r+m+1)| \cdots |\frac{\beta-bk}{a} - (r+m+2)b+1|} =$$

$$= \lim_{r\to\infty} \frac{(ar)^{as}}{(br)^b} = 0.$$

In particular, $0 < g_m(D) < \infty$ and

$$|h_{k+a(m+1)} - \lambda_k \Gamma[v^k; u(m+1)]| \leq C g_m(D) D^{m+1} (k+am)!^{s-1}.$$

It can be proved (by using elementary properties of the Pochhammer symbol and standard estimates) that there exists $m_2, \widehat{C} \in \mathbb{N}$ such that for all $m \geq m_2$, $g_{m+1}(D) \leq \widehat{C} g_m(D)$. This implies that

$$|g_{m+1}(D)| \leq \widehat{C}^{m+1-m_2} g_{m_2}(D)$$

for all $m \geq m_2$. Then, taking $\widetilde{C} = \widehat{C}^{-m_2} g_{m_2}(D) > 0$, we obtain:

$$|h_{k+a(m+1)} - \lambda_k \Gamma[v^k; u(m+1)]| \leq \widetilde{C} (\widehat{C} D)^{m+1} (k+am)!^{s-1}$$

for all $m \geq 0$. Hence $h^{(k)} - \lambda_k \phi_{v^k,p} \in \mathcal{O}_{X|Y}(s)_p$.

If $\beta \notin a\mathbb{N} + b\mathbb{N}$ then we have that $g := h - \sum_{k=0}^{a-1} \lambda_k \phi_{v^k,p} \in \mathcal{O}_{X|Y}(s)_p$ satisfies statement *i)*.

ii) If $\beta \in a\mathbb{N} + b\mathbb{N}$ there exists a unique $q \in \{0, 1, \ldots, a-1\}$ verifying $\frac{\beta - bq}{a} = m_0 \in \mathbb{N}$. For $k \neq q$ we have as before that $h^{(k)} - \lambda_k \phi_{v^k,p} \in \mathcal{O}_{X|Y}(s)_p$. For $k = q$ we can assume without loss of generality that $h_{q+am} = 0$ for $m = 0, 1, \ldots, [m_0/b]$ (here $[-]$ denotes the integer part) obtaining an expression for $h_{q+a(m+1)}$ similar to Equality (4.7) for $m \geq [m_0/b]$. Then by using $\phi_{\widetilde{v^q}}$ instead of ϕ_{v^q} we get, alike in *i)*, that $h - \lambda_q \phi_{\widetilde{v^q}} \in \mathcal{O}_{X|Y}(s)_p$. Hence $g := h - \sum_{k=0, k\neq q}^{a-1} \lambda_k \phi_{v^k,p} - \lambda_q \phi_{\widetilde{v^q}} \in \mathcal{O}_{X|Y}(s)_p$ satisfies *ii)*. □

Theorem 4.4.12. *We have*

$$\dim_{\mathbb{C}}(\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p) = \begin{cases} 1 & \text{for} \quad \beta \in a\mathbb{N} + b\mathbb{N} \\ 0 & \text{for} \quad \beta \notin a\mathbb{N} + b\mathbb{N} \end{cases}$$

for all $p \in Y$, $p \neq (0,0)$ *and* $1 \leq s < \frac{b}{a}$. *Moreover, if* $\beta \in a\mathbb{N} + b\mathbb{N}$ *then* $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p$ *is generated by the class of* $(P(\phi_{\widetilde{v^q},p}), 0)$.

Proof. By definition

$$\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p = \frac{\{(f,g) \in (\mathcal{O}_{X|Y}(s)_p)^2 : \ (E + ab)(f) = P(g)\}}{\{(P(h), E(h)) : \ h \in \mathcal{O}_{X|Y}(s)_p\}}$$

As in the proof of Theorem 4.4.10 we can assume $g = 0$ and then $(E + ab)(f) = 0$. This implies that $f = \sum_{k=0}^{a-1} f^{(k)}$ (see Lemma 4.4.7) with

$$f^{(k)} = \sum_{m \geq 0} f_{k+am} x_1^{\frac{\beta - bk}{a} - b(m+1)} x_2^{k+am}$$

We can then consider $h \in \mathcal{O}_{\widehat{X|Y},p}$ such that $P(h) = f$ and $E(h) = 0$ (by using (4.7)) and apply Lemma 4.4.11. Furthermore, it is easy to prove that $P(\phi_{\widetilde{v^q}})$ is a Laurent monomial term with pole along $\{x_1 = 0\}$ and hence holomorphic at any point $p \in Y \setminus \{(0,0)\}$. This finishes the proof. □

Remark 4.4.13. *Notice that the generator* $(P(\phi_{\widetilde{v^q}}), 0)$ *does not define a germ at the origin although the dimension of* $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_{(0,0)}$ *is one (see Proposition 4.3.7). Nevertheless, it can be checked that the class of* $(0, \phi_{v^q})$ *is a generator of* $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p$ *at any point of* $p \in Y$.

Proposition 4.4.14. *For all* $\beta \in \mathbb{C}$ *we have*

$$\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s)) = 0$$

for all $1 \leq s \leq \infty$.

Proof. Since $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = 0$ for $p = (0,0)$ (see Section 4.3) it is enough to prove the equality for all $p \in Y \setminus \{(0,0)\}$.

From Corollary 4.4.6 (for $s = 1$), Theorem 4.4.10 and the long exact sequence in cohomology we get the equality for $s \geq b/a$. Using again Corollary 4.4.6 (for $s = 1$), Theorem 4.4.12, Theorem 4.4.9 (only necessary in the case $\beta \in a\mathbb{N} + b\mathbb{N}$) and the long exact sequence in cohomology we get the equality for $1 \leq s < b/a$.

□

4.5 Remarks and conclusions

Let us summarize the results in this chapter and point out some observations:

1) Michel Granger has remarked that some of the proofs of the results of this chapter can be simplified if one observes that $\mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{F})$ is the mapping cone of the following morphism of Euler-Koszul type complexes induced by P:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{F} & \xrightarrow{P} & \mathcal{F} \\
\downarrow E & & \downarrow (E+ab) \\
\mathcal{F} & \xrightarrow{P} & \mathcal{F} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

For example, since E and $E + ab$ induce automorphisms on $\mathcal{F}_0 = \mathcal{Q}_Y(s)_{(0,0)}$ by Corollary 4.3.4 (i.e. the stalks at the origin of the Euler-Koszul type complexes below are acyclic), we obtain a shorter proof of Theorem 4.3.1.

2) In Sections 4.3 and 4.4 we have proved that the irregularity complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is zero for $1 \leq s < b/a$ and concentrated in degree 0 for $b/a \leq s \leq \infty$. Moreover, we have described a basis of $\mathcal{E}xt^0_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ for $p \in Y$, $p \neq (0,0)$, and $b/a \leq s \leq \infty$ (see Theorem 4.4.9). This description proves in particular that the cohomology of the complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is constructible on Y, with respect to the stratification given by $\{\{(0,0)\}, Y \setminus \{(0,0)\}\}$. This is a particular case of [Meb90, Corollaire 6.2.4] (that uses Laurent's constructibility theorem for holomorphic hyperfunctions [Lau93], Mebkhout's local duality theorem [Meb90, Théorème 6.2.2] and the local biduality theorem for constructible complexes).

3) From the form of the basis described it is also easy to see that the eigenvalues of the monodromy of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ along any loop in Y around the origin are simply

$\exp(\frac{2\pi i(\beta-kb)}{a})$ for $k=0,\ldots,a-1$. Notice that for $\beta \in a\mathbb{N}+b\mathbb{N}$ one eigenvalue (the one for $k=q$) is just 1.

4) We can also give an elementary proof of the fact that the complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is a *perverse sheaf* on Y for any $1 \leq s \leq \infty$ using the previous results. This is a very particular case of a general result of Z. Mebkhout [Meb90, Th. 6.3.3] (see Theorem 2.4.6). To this end, as $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is concentrated in degree 0, it is enough to prove the *co-support* condition, which is equivalent (see [BBD82]) to prove that the hypercohomology $\mathcal{H}^0_{\{p\}}(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))$ with support on $\{p\}$ is zero, for $p \in Y$. This is obvious because $\mathcal{E}xt^0_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))$ has no sections supported on points.

5) We have also proved that the *Gevrey filtration* $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ has a unique gap for $s=b/a$. So the only *analytic slope* of $\mathcal{M}_A(\beta)$ with respect to Y is b/a [Meb90, Déf. 6.3.7]. On the other hand it is also known (see [Har04, Th. 3.3]) that the only *algebraic slope* of $\mathcal{M}_A(\beta)$ is b/a. This fact is a very particular case of the slope comparison theorem of Y. Laurent and Z. Mebkhout [LM99, Th. 2.4.2].

6) Similar methods to the ones presented here can be applied to prove that the irregularity complex $\mathrm{Irr}_Z(\mathcal{M}_{(-a\;b)}(\beta))$ is zero for $Z=\{x_i=0\}$, $i=1,2$. Let us notice that the characteristic variety of $\mathcal{M}_A(\beta)$ is defined in this last case by the ideal $(\xi_1\xi_2, -ax_1\xi_1+bx_2\xi_2)$ and then its singular support is the union of the two coordinate axes in $\mathbb{C}^2$.

Chapter 5

Irregularity of $\mathcal{M}_A(\beta)$ for $A = (a_1 \cdots a_n)$, $0 < a_1 < \cdots < a_n$.

In what follows we will study the irregularity of the hypergeometric $\mathcal{D}$-module $\mathcal{M}_A(\beta)$ associated with an affine monomial curve through the origin in $X = \mathbb{C}^n$ (i.e. when $A = (a_1 \cdots a_n)$ is a row matrix with positive integer entries) and with a parameter $\beta \in \mathbb{C}$. More precisely, we will describe the cohomology sheaves of the irregularity complex of $\mathcal{M}_A(\beta)$ along its singular locus. To this end, we first consider the hypergeometric $\mathcal{D}$-module $\mathcal{M}_A(\beta)$ associated with an affine smooth monomial curve (see Section 5.1). Then we prove that in this case the problem can be reduced to the case of a plane curve treated in Chapter 4 by using restrictions (see Subsection 5.1.1).

The results presented in this chapter are a joint work with F. J. Castro Jiménez [$\mathrm{FC}_2$08].

5.1 The case of a smooth monomial curve

Let $A = (1\, a_2 \cdots a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$ and $\beta \in \mathbb{C}$ throughout this section. The toric ideal I_A is generated in this case by the differential operators:

$$P_{1,i} := \partial_1^{a_i} - \partial_i \;\text{ for } i = 2, \ldots, n.$$

The Euler operator associated with (A, β) is:

$$E - \beta = x_1\partial_1 + a_2x_2\partial_2 + \cdots + a_nx_n\partial_n - \beta$$

Thus, the hypergeometric $\mathcal{D}$–module associated with (A, β) is:

$$\mathcal{M}_A(\beta) = \frac{\mathcal{D}}{\mathcal{D}\langle E - \beta\rangle + \mathcal{D}\langle P_{1,i} :\; i = 2, \ldots, n\rangle}$$

Although it can be deduced from general results (see [GGZ87] and [Ado94, Th. 3.9]), a direct computation shows in this case that the characteristic variety of $\mathcal{M}_A(\beta)$ equals

$$\mathrm{Ch}(\mathcal{M}_A(\beta)) = T^*_X X \cup T^*_Y X$$

where $Y = \{x_n = 0\}$. The module $\mathcal{M}_A(\beta)$ is then holonomic and its singular support is Y. Let us denote by $Z \subset \mathbb{C}^n$ the hyperplane $x_{n-1} = 0$.

One of the main results in this Section is

Theorem 5.1.1. *Let A be an integer row matrix as above and $\beta \in \mathbb{C}$. Then the cohomology sheaves of* $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ *satisfy:*

i) $\mathcal{E}xt^0_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s)) = 0$ *for* $1 \leq s < a_n/a_{n-1}$.

ii) $\mathcal{E}xt^0_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_{|Y\cap Z} = 0$ *for* $1 \leq s \leq \infty$.

iii) $\dim_{\mathbb{C}}\left(\mathcal{E}xt^0_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p\right) = a_{n-1}$ *for* $s \geq a_n/a_{n-1}$ *and* $p \in Y \setminus Z$.

iv) $\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s)) = 0$ *for* $i \geq 1$ *and* $1 \leq s \leq \infty$.

The main ingredients in the proof of Theorem 5.1.1 are:

1. The corresponding results for the case of affine monomial plane curves (see Chapter 4).
2. Corollary 5.1.4.
3. Cauchy-Kovalevskaya Theorem for Gevrey series (see [LM02, Cor. 2.2.4]).
4. The perversity of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ [Meb90, Th. 6.3.3].
5. Theorem 2.3.6 (that is a consequence of the Riemann-Hilbert correspondence [Meb84] and Kashiwara's constructibility theorem [Kas75]).

We will also describe a basis of the solution vector space in part *iii)* of Theorem 5.1.1 (see Theorem 5.1.23).

5.1.1 Reduction of the number of variables by restriction

In the sequel we will use some results concerning restriction of hypergeometric systems.

Theorem 5.1.2. [CT03, Th. 4.4] *Let $A = (1\ a_2\ \cdots\ a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$ and $\beta \in \mathbb{C}$. Then for $i = 2, \ldots, n$, the restriction of $\mathcal{M}_A(\beta)$ to $\{x_i = 0\}$ is isomorphic to the $\mathcal{D}'$–module*

$$\mathcal{M}_A(\beta)_{|\{x_i=0\}} := \frac{\mathcal{D}}{\mathcal{D}H_A(\beta) + x_i\mathcal{D}} \cong \frac{\mathcal{D}'}{\mathcal{D}'H_{A'}(\beta)}$$

where $A' = (1\ a_2\ \cdots\ a_{i-1}\ a_{i+1}\ \cdots\ a_n)$ and $\mathcal{D}'$ is the sheaf of linear differential operators with holomorphic coefficients on $\mathbb{C}^{n-1}$ (with coordinates $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$).

Theorem 5.1.3. *Let $A = (1\ ka\ kb)$ be an integer row matrix with $1 \leq a < b$, $1 < ka < kb$ and a, b relatively prime. Then for all $\beta \in \mathbb{C}$ there exist $\beta_0, \ldots, \beta_{k-1} \in \mathbb{C}$ such that the restriction of $\mathcal{M}_A(\beta)$ to $\{x_1 = 0\}$ is isomorphic to the $\mathcal{D}'$–module*

$$\mathcal{M}_A(\beta)_{|\{x_1=0\}} := \frac{\mathcal{D}}{\mathcal{D}H_A(\beta) + x_1\mathcal{D}} \simeq \bigoplus_{i=0}^{k-1} \mathcal{M}_{A'}(\beta_i)$$

where $\mathcal{D}'$ is the sheaf of linear differential operators on the plane $\{x_1 = 0\}$ and $A' = (a\ b)$. Moreover, for all but finitely many $\beta \in \mathbb{C}$ we can take $\beta_i = \frac{\beta - i}{k}$, $i = 0, 1, \ldots, k-1$.

An ingredient in the proof of Theorem 5.1.1 is the following

Corollary 5.1.4. *Let $A = (1\ a_2\ \cdots\ a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$ and $\beta \in \mathbb{C}$. Then there exist $\beta_i \in \mathbb{C}$, $i = 0, \ldots, k-1$ such that the restriction of $\mathcal{M}_A(\beta)$ to $X' = (x_1 = x_2 = \cdots = x_{n-2} = 0)$ is isomorphic to the $\mathcal{D}'$–module*

$$\mathcal{M}_A(\beta)_{|X'} := \frac{\mathcal{D}}{\mathcal{D}H_A(\beta) + (x_1, x_2, \cdots, x_{n-2})\mathcal{D}} \simeq \bigoplus_{i=0}^{k-1} \mathcal{M}_{A'}(\beta_i)$$

where $\mathcal{D}'$ is the sheaf of linear differential operators on X', $A' = \frac{1}{k}(a_{n-1}\ \ a_n)$ and $k = \gcd(a_{n-1}, a_n)$. Moreover, for all but finitely many $\beta \in \mathbb{C}$ we can take $\beta_i = \frac{\beta - i}{k}$, $i = 0, 1, \ldots, k-1$.

Let us fix some notations and state some preliminary results in order to prove Theorem 5.1.3.

Notation 5.1.5. *Let A be an integer $d \times n$–matrix of rank d and $\beta \in \mathbb{C}^n$. For any weight vector $\omega \in \mathbb{R}^n$ and any ideal $J \subset \mathbb{C}[\partial] = \mathbb{C}[\partial_1, \ldots, \partial_n]$ we denote by $\mathrm{in}_\omega(J)$ the initial ideal of J with respect to the graduation on $\mathbb{C}[\partial]$ induced by w. According to [SST00, p. 106] the* fake initial ideal *of $H_A(\beta)$ is the ideal*

$$\mathrm{fin}_\omega(H_A(\beta)) = A_n \mathrm{in}_\omega(I_A) + A_n(A\theta - \beta)$$

where $\theta = (\theta_1, \ldots, \theta_n)$ and $\theta_i = x_i\partial_i$.

Assume that $A = (1\ ka\ kb)$ is an integer row matrix with $1 \leq a < b$, $1 < ka < kb$ and a, b relatively prime.

Let us write $P_1 = \partial_2^b - \partial_3^a$, $P_2 = \partial_1^{ka} - \partial_2$, $P_3 = \partial_1^{kb} - \partial_3$ and $E = \theta_1 + ka\theta_2 + kb\theta_3 - \beta$. It is clear that $P_1 \in H_A(\beta) = \langle P_2, P_3, E \rangle \subset A_3$.

Let us consider $\prec$ a monomial order on the monomials in A_3 satisfying:

$$\left.\begin{array}{l} \gamma_1 + a\gamma_2 + b\gamma_3 < \gamma_1' + a\gamma_2' + b\gamma_3' \\ \text{or} \\ \gamma_1 + a\gamma_2 + b\gamma_3 = \gamma_1' + a\gamma_2' + b\gamma_3' \text{ and } 3a\gamma_2 + 2b\gamma_3 < 3a\gamma_2' + 2b\gamma_3' \end{array}\right\} \Rightarrow$$

$$\Rightarrow x^\alpha \partial^\gamma \prec x^{\alpha'} \partial^{\gamma'}$$

Write $\omega = (1, 0, 0)$ and let us denote by $\prec_\omega$ the monomial order on the monomials in A_3 defined as

$$x^\alpha \partial^\gamma \prec_\omega x^{\alpha'} \partial^{\gamma'} \overset{\text{Def.}}{\Longleftrightarrow} \begin{cases} \gamma_1 - \alpha_1 < \gamma_1' - \alpha_1' \\ \text{or} \\ \gamma_1 - \alpha_1 = \gamma_1' - \alpha_1' \text{ and } x^\alpha \partial^\gamma \prec x^{\alpha'} \partial^{\gamma'} \end{cases}$$

Since $\mathrm{g.c.d.}(a, b) = 1$, there exist $\alpha, \gamma \in \mathbb{N}$ such that $\alpha a - \gamma b = 1$. If $a = 1$ then we can take $\alpha = 1$ and $\gamma = 0$. Otherwise, we can assume that $0 < \gamma < a$ and $0 < \alpha < b$. Then we have the following result.

Lemma 5.1.6. *Let $A = (1\ ka\ kb)$ be a row matrix with integer entries and $1 \leq a < b$ relatively prime integers, $1 < ka < kb$. If $a > 1$, then we have the following Groebner basis of $I_A \subseteq A_3(\mathbb{C})$ with respect to $\prec_\omega$:*

$$\mathfrak{G}_A = \{P_1, P_2, Q_j, \widetilde{Q}_j : j = 1, \ldots, a-1\}$$

where $Q_j = \partial_1^{jk} \partial_3^{j\gamma - l_j a} - \partial_2^{j\alpha - l_j b}$ and $\widetilde{Q}_j = \partial_1^{jk} \partial_2^{-j\alpha + (l_j+1)b} - \partial_3^{-j\gamma + (l_j+1)a}$. Here $l_j \in \mathbb{N}$ is the unique non-negative integer such that $0 < j\gamma - l_j a < a$, $j = 1, \ldots, a-1$. If $a = 1$, the reduced Groebner basis of $I_A \subseteq A_3(\mathbb{C})$ with respect to $\prec_\omega$ is simply $\mathfrak{G}_A = \{P_1, P_2\}$.

Proof. This is a technical result that follows easily from the application of Buchberger's algorithm in the Weyl algebra A_3 to the given set of differential operators. □

Lemma 5.1.7. *Let $A = (1\ ka\ kb)$ be an integer row matrix with $1 \leq a < b$, $1 < ka < kb$ and a, b relatively prime. Then*

$$\mathrm{fin}_\omega(H_A(\beta)) = A_3 \mathrm{in}_\omega(I_A) + A_3 E = A_3(P_1, E, \partial_1^k)$$

for $\beta \notin \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and for all $\beta \in \mathbb{N}^*$ big enough.

Proof. By Lemma 5.1.6 we have that

$$\mathrm{in}_\omega(I_A) = \langle P_1, \partial_1^{ka}, \partial_1^{jk}\partial_3^{j\gamma - l_j a}, \partial_1^{jk}\partial_2^{-j\alpha+(l_j+1)b} : j = 1, 2, \ldots, a-1\rangle$$

where $l_j \in \mathbb{N}$ is the unique non-negative integer such that $0 < j\gamma - l_j a < a$, $j = 1, \ldots, a-1$. Thus,

$$\mathrm{fin}_\omega(H_A(\beta)) = \langle E, P_1, \partial_1^{ka}, \partial_1^{jk}\partial_3^{j\gamma - l_j a}, \partial_1^{jk}\partial_2^{-j\alpha+\widetilde{l}_j a} : j = 1, 2, \ldots, a-1\rangle$$

and it is enough to prove that $\partial_1^k \in \mathrm{fin}_\omega(H_A(\beta))$.

We have in particular that $\partial_1^a, \partial_1^k\partial_2^{b-\alpha}, \partial_1^k\partial_3^\gamma \in \mathrm{in}_\omega(I_A) \subseteq \mathrm{fin}_\omega(H_A(\beta))$. Let us see that this implies that $\partial_1^k \in \mathrm{fin}_\omega(H_A(\beta))$ for β as in the statement.

Let $r_1 \in \mathbb{N}$ be the smallest non-negative integer such that $\partial_1^{(r_1+1)k} \in \mathrm{fin}_\omega(H_A(\beta))$ and let us assume to the contrary that $r_1 \geq 1$.

Let $r_2 \in \mathbb{N}$ be the smallest non-negative integer such that

$$\partial_1^{k(r_1+1)-1}\partial_2^{r_2+1} \in \mathrm{fin}_\omega(H_A(\beta))$$

and $r_3 \in \mathbb{N}$ be the smallest non-negative integer such that

$$\partial_1^{k(r_1+1)-1}\partial_2^{r_2}\partial_3^{r_3+1} \in \mathrm{fin}_\omega(H_A(\beta)).$$

Since $E \in \mathrm{fin}_\omega(H_A(\beta))$ then $\partial_1^{k(r_1+1)-1}\partial_2^{r_2}\partial_3^{r_3}E \in \mathrm{fin}_\omega(H_A(\beta))$. Moreover,

$$\partial_1^{k(r_1+1)-1}\partial_2^{r_2}\partial_3^{r_3}E = (E + k(r_1+1) - 1 + kar_2 + kbr_3)\partial_1^{k(r_1+1)-1}\partial_2^{r_2}\partial_3^{r_3}$$

and using that $\partial_1^{(r_1+1)k}, \partial_1^{(r_1+1)k-1}\partial_2^{r_2+1}, \partial_1^{k(r_1+1)-1}\partial_2^{r_2}\partial_3^{r_3+1} \in \mathrm{fin}_\omega(H_A(\beta))$ we deduce that $(-\beta + k(r_1+1) - 1 + kar_2 + kbr_3)\partial_1^{k(r_1+1)-1}\partial_2^{r_2}\partial_3^{r_3} \in \mathrm{fin}_\omega(H_A(\beta))$. Then for $\beta \notin \mathbb{N}^*$ or $\beta \in \mathbb{N}^*$ big enough, we have $\partial_1^{k(r_1+1)-1}\partial_2^{r_2}\partial_3^{r_3} \in \mathrm{fin}_\omega(H_A(\beta))$. From the definition of r_3 it is clear that $\partial_1^{k(r_1+1)-1}\partial_2^{r_2} \in \mathrm{fin}_\omega(H_A(\beta))$ and then $\partial_1^{k(r_1+1)-1} \in \mathrm{fin}_\omega(H_A(\beta))$ (by the definition of r_2). Anlogously, from the fact that $\partial_1^{k(r_1+1)-j} \in \mathrm{fin}_\omega(H_A(\beta))$ with $1 \geq j \geq k-1$ and using that $k(r_1+1) - j \geq k+1$ (because $r_1 \geq 1$ and $j \leq k-1$), it follows that $\partial_1^{k(r_1+1)-(j+1)} \in \mathrm{fin}_\omega(H_A(\beta))$. Finally, $\partial_1^{kr_1} \in \mathrm{fin}_\omega(H_A(\beta))$, which is a contradiction with the definition of r_1 unless $r_1 = 0$, i.e. $\partial_1^k \in \mathrm{fin}_\omega(H_A(\beta))$. □

Definition 5.1.8. [SST00, Def. 5.1.1] *Let $I \subseteq A_n$ be a holonomic ideal (i.e. A_n/I is holonomic) and $\widetilde{\omega} \in \mathbb{R}^n \setminus \{0\}$. The b-function I with respect to $\widetilde{\omega}$ is the monic generator of the ideal*

$$\mathrm{in}_{(-\widetilde{\omega},\widetilde{\omega})}(I) \cap \mathbb{C}[\tau]$$

where $\tau = \widetilde{\omega}_1\theta_1 + \cdots + \widetilde{\omega}_n\theta_n$ and $\mathrm{in}_{(-\widetilde{\omega},\widetilde{\omega})}(I)$ is the initial ideal of I with respect to the weight vector $(-\widetilde{\omega}, \widetilde{\omega})$.

Corollary 5.1.9. *Let $A = (1\ ka\ kb)$ be an integer row matrix with $1 \le a < b$, $1 < ka < kb$ and a, b relatively prime. Then the b–function of $H_A(\beta)$ with respect to $\omega = (1,0,0)$ is*

$$b(\tau) = \tau(\tau - 1)\cdots(\tau - (k-1))$$

for all but finitely many $\beta \in \mathbb{C}$.

Proof. From [SST00, Th. 3.1.3] for all but finitely many $\beta \in \mathbb{C}$ we have

$$\mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) = \mathrm{fin}_\omega(H_A(\beta)).$$

Then by using Lemma 5.1.7 we get

$$\mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) = A_3(P_1, E, \partial_1^k)$$

for all but finitely many $\beta \in \mathbb{C}$. An easy computation shows that $\{P_1, E, \partial_1^k\}$ is a Groebner basis of the ideal $\mathrm{in}_{(-\omega,\omega)}(H_A(\beta))$ with respect to any monomial order $>$ satisfying $\theta_3 > \theta_1, \theta_2$ and $\partial_2^b > \partial_3^a$. In particular we can consider the lexicographic order

$$x_3 > x_2 > \partial_2 > \partial_3 > x_1 > \partial_1$$

which is an elimination order for x_1 and ∂_1. So we get

$$\mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) \cap \mathbb{C}[x_1]\langle\partial_1\rangle = \langle\partial_1^k\rangle$$

and since $x_1^k\partial_1^k = \theta_1(\theta_1 - 1)\cdots(\theta_1 - (k-1))$, we have

$$\mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) \cap \mathbb{C}[\theta_1] = \langle\theta_1(\theta_1 - 1)\cdots(\theta_1 - (k-1))\rangle$$

This proves Corollary 5.1.4. □

Remark 5.1.10. *Corollary 5.1.9 can be related to [CT03, Th. 4.3] which proves that for $A = (1\ a_2\ \cdots\ a_n)$ with $1 < a_2 < \cdots < a_n$, the b–function of $H_A(\beta)$ with respect to e_i is $b(\tau) = \tau$, for $i = 2, \ldots, n$. Here $e_i \in \mathbb{R}^n$ is the vector with a 1 in the i–th coordinate and 0 elsewhere.*

Recall (see e.g. [SST00, Def. 1.1.3]) that a Groebner basis of a left ideal $I \subset A_n$ with respect to $(-\omega, \omega) \in \mathbb{R}^{2n}$ (or simply with respect to $\omega \in \mathbb{R}^n$) is a finite subset $G \subset I$ such that $I = A_n G$ and $\mathrm{in}_{(-\omega,\omega)}(I) = A_n \mathrm{in}_{(-\omega,\omega)}(G)$ where $\mathrm{in}_{(-\omega,\omega)}(G) = \{\mathrm{in}_{(-\omega,\omega)}(P) \mid P \in G\}$. Then the proof of Corollary 5.1.9 also proves the following

Lemma 5.1.11. *Let $A = (1\ ka\ kb)$ be an integer row matrix with $1 \le a < b$, $1 < ka < kb$ and a, b relatively prime. For all but finitely many $\beta \in \mathbb{C}$, a Groebner basis of $H_A(\beta) \subset A_3$ with respect to $\omega = (1,0,0)$ is*

$$\{P_1, P_2, P_3, E, R\}$$

for some $R \in A_3$ satisfying $\mathrm{in}_{(-\omega,\omega)}(R) = \partial_1^k$.

The following Proposition is a particular case of [Ber08, Th. 6.5.] (see also [Sai01, Th. 2.1]) and it will be used later.

Proposition 5.1.12. *Assume $A = (a_1\ a_2 \cdots a_n)$ is an integer row matrix with $0 < a_1 < a_2 < \cdots < a_n$. For $\beta, \beta' \in \mathbb{C}^d$ we have that $\mathcal{M}_A(\beta) \simeq \mathcal{M}_A(\beta')$ if and only if one of the following conditions holds:*

i) $\beta, \beta' \in \mathbb{N}A$.

ii) $\beta, \beta' \in \mathbb{Z} \setminus \mathbb{N}A$.

iii) $\beta, \beta' \notin \mathbb{Z}$ *but* $\beta - \beta' \in \mathbb{Z}$.

Proof. (**Theorem 5.1.3**) We have $A = (1\ ka\ kb)$ with $1 \leq a < b$, $1 < ka < kb$ and a, b relatively prime. From Proposition 5.1.12 it is enough to compute the restriction for all but finitely many $\beta \in \mathbb{C}$. We will compute the restriction of $\mathcal{M}_A(\beta)$ to $\{x_1 = 0\}$ by using an algorithm by T. Oaku and N. Takayama [SST00, Algorithm 5.2.8].

Let $r = k - 1$ be the biggest integer root of the Bernstein polynomial $b(\tau)$ of $H_A(\beta)$ with respect to $\omega = (1, 0, 0)$ (see Corollary 5.1.9). Recall that $\mathcal{D}' = \mathcal{D}_{\mathbb{C}^2}$ and that we are using (x_2, x_3) as coordinates in $\mathbb{C}^2$. We consider the free $\mathcal{D}'$–module with basis $\mathcal{B}_{k-1} := \{\partial_1^i :\ i = 0, 1, \ldots, k-1\}$:

$$(\mathcal{D}')^{r+1} = (\mathcal{D}')^k \simeq \bigoplus_{i=0}^{k-1} \mathcal{D}' \partial_1^i.$$

To apply [SST00, Algorithm 5.2.8] to our case we will use the elements with ω–order less than or equal to $k - 1$, in the Groebner basis $\mathcal{G} := \{P_1, P_2, P_3, E, R\}$ of $H_A(\beta)$. Recall that $\mathcal{G}$ is given by Lemma 5.1.11 for all but finitely many $\beta \in \mathbb{C}$. Each operator $\partial_1^i P_1$, $\partial_1^i E$, $i = 0, \ldots, k-1$, must be written as a $\mathbb{C}$–linear combination of monomials $x^u \partial^v$ and then substitute $x_1 = 0$ into this expression. The result is an element of $(\mathcal{D}')^k = \mathcal{D}'\mathcal{B}_k$. In this case we get:

$$(\partial_1^i P_1)_{|x_1=0} = P_1 \partial_1^i,\ (\partial_1^i E)_{|x_1=0} = (kax_2\partial_2 + kbx_3\partial_3 - \beta + i)\partial_1^i,\ i = 0, \ldots, k-1$$

and this proves the theorem. □

Remark 5.1.13. *Let us consider $A = (1\ a_2\ \cdots\ a_n)$, $1 < a_2 < \cdots < a_n$, $k = \gcd(a_{n-1}, a_n)$ and $A' = \frac{1}{k}(a_{n-1}, a_n)$. We can apply Cauchy-Kovalevskaya Theorem for Gevrey series (see [LM02, Cor. 2.2.4]), Corollary 5.1.4 and [CT03, Prop. 4.2] to the hypergeometric system $\mathcal{M}_A(\beta)$ to prove that for each $\beta \in \mathbb{C}$ there exist $\beta_i \in \mathbb{C}$, $i = 0, \ldots, k-1$, and a quasi-isomorphism*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_{|X'} \xrightarrow{\simeq} \bigoplus_{i=0}^{k-1} \mathbb{R}\mathcal{H}om_{\mathcal{D}_{X'}}(\mathcal{M}_{A'}(\beta_i), \mathcal{O}_{\widehat{X'|Y'}}(s))$$

for all $1 \leq s \leq \infty$ *where* $Y = \{x_n = 0\}$, $X' = \{x_1 = x_2 = \cdots = x_{n-2} = 0\}$ *and* $Y' = Y \cap X'$. *Moreover, for all but finitely many* β *we can take* $\beta_i = \frac{\beta - i}{k}$. *Notice that coordinates in* X, Y, X', Y' *are* $x = (x_1, \ldots, x_n)$, $y = (x_1, \ldots, x_{n-1})$, $x' = (x_{n-1}, x_n)$ *and* $y' = (x_{n-1})$ *respectively.*

The last quasi-isomorphism induces a $\mathbb{C}$*–linear isomorphism*

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_{(0,\ldots,0,\epsilon_{n-1},0)} \xrightarrow{\simeq} \bigoplus_{i=0}^{k-1} \mathcal{E}xt^j_{\mathcal{D}_{X'}}(\mathcal{M}_{A'}(\beta_i), \mathcal{O}_{\widehat{X'|Y'}}(s)))_{(\epsilon_{n-1},0)}$$

for all $\epsilon_{n-1} \in \mathbb{C}$, $s \geq 1$ *and* $j \in \mathbb{N}$ *and we also have equivalent results for* $\mathcal{Q}_Y(s)$ *and* $\mathcal{Q}_{Y'}(s)$ *instead of* $\mathcal{O}_{X|Y}(s)$ *and* $\mathcal{O}_{\widehat{X'|Y'}}(s)$.

In particular, using [FC$_1$08, Proposition 5.9], we have:

Proposition 5.1.14. *Let* $A = (1\ a_2\ \cdots\ a_n)$ *be an integer row matrix with* $1 < a_2 < \cdots < a_n$. *Then for all* $\beta \in \mathbb{C}$

$$\dim_{\mathbb{C}}(\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_{(0,\ldots,0,\epsilon_{n-1},0)}) = \begin{cases} a_{n-1} & \text{if } s \geq a_n/a_{n-1},\ j = 0 \\ & \text{and } \epsilon_{n-1} \neq 0. \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 5.1.15. *Let* $A = (1\ a_2\ \cdots\ a_n)$ *be an integer row matrix with* $1 < a_2 < \cdots < a_n$. *Then for all* $\beta \in \mathbb{C}$

$$\mathrm{Ch}(\mathrm{Irr}^{(s)}_Y(\mathcal{M}_A(\beta))) \subseteq T^*_Y Y \bigcup T^*_{Z \cap Y} Y$$

for $s \geq \frac{a_n}{a_{n-1}}$.

Proof. Here $\mathrm{Ch}(\mathrm{Irr}^{(s)}_Y(\mathcal{M}_A(\beta)))$ is the characteristic variety of the perverse sheaf $\mathrm{Irr}^{(s)}_Y(\mathcal{M}_A(\beta))$ (see e.g. [LM99, Sec. 2.4]). The Corollary follows from the inclusion

$$\mathrm{Ch}^{(s)}(\mathcal{M}_A(\beta)) \subset T^*_X X \cup T^*_Y X \cup T^*_Z X$$

for $s \geq a_n/a_{n-1}$ and then by applying [LM99, Prop. 2.4.1]. □

Proof. (**Theorem 5.1.1**) Let us consider the Whitney stratification $Y = Y_1 \cup Y_2$ of $Y = \{x_n = 0\} \subset \mathbb{C}^n$ defined as

$$Y_1 := Y \setminus (Y \cap Z) \equiv \mathbb{C}^{n-2} \times \mathbb{C}^*.$$

$$Y_2 := Y \cap Z \equiv \mathbb{C}^{n-2} \times \{0\}.$$

As $\mathrm{Irr}^{(s)}_Y(\mathcal{M}_A(\beta))$ is a perverse sheaf for all $\beta \in \mathbb{C}$ and $1 \leq s \leq \infty$ [Meb90, Th. 6.3.3] we can apply the Riemann-Hilbert correspondence (see [Meb84] and [KK79], [Kas84]), Kashiwara's constructibility theorem [Kas75] and Corollary 5.1.15, to prove that

$$\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_{|Y_j}$$

is a locally constant sheaf of finite rank for all $i \in \mathbb{N}$, $j = 1, 2$. To finish the proof it is enough to apply Proposition 5.1.14. □

5.1.2 Gevrey solutions of $\mathcal{M}_A(\beta)$

We will compute a basis of the vector spaces $\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ for all $p \in Y$, $1 \leq s \leq \infty$, $\beta \in \mathbb{N}$, $i \in \mathbb{N}$ and $A = (1 \; a_2 \; \cdots \; a_n)$ is an integer row matrix with $1 < a_2 < \cdots < a_n$. In fact it is enough to achieve the computation for $i = 0$ and $p \in Y \setminus Z$, otherwise the corresponding germ is zero by Theorem 5.1.1.

Lemma 5.1.16. *Let $A = (1 \; a_2 \; \cdots \; a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$ and $\omega \in \mathbb{R}^n_{>0}$ satisfying*

a) $w_i > a_i\omega_1$ *for* $2 \leq i \leq n-2$ *or* $i = n$

b) $a_{n-1}\omega_1 > \omega_{n-1}$

c) $\omega_{n-1} > \omega_1, \ldots, \omega_{n-2}$

Then $H_A(\beta)$ has a_{n-1} exponents with respect to ω and they have the form

$$v^j = (j, 0, \ldots, 0, \frac{\beta - j}{a_{n-1}}, 0) \in \mathbb{C}^n$$

$j = 0, 1, \ldots, a_{n-1} - 1.$

Proof. The notion of *exponent* is given in [SST00, page 92]. The toric ideal I_A is generated by $P_{1,i} = \partial_1^{a_i} - \partial_i \in \mathbb{C}[\partial]$, $i = 2, \ldots, n$.

Let $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n_{>0}$ be a weight vector satisfying the statement of the lemma. We have:

$$\mathrm{in}_{(-\omega,\omega)} P_{1,i} = \begin{cases} \partial_i & \text{if } i = 2, \ldots, n-2, n \\ \partial_1^{a_{n-1}} & \text{if } i = n-1 \end{cases}$$

In particular $\{P_{1,i} : \; i = 2, \ldots, n\}$ is a Groebner basis of I_A with respect to $(-\omega, \omega)$ and then

$$\mathrm{in}_\omega I_A = \langle \partial_2, \ldots, \partial_{n-2}, \partial_1^{a_{n-1}}, \partial_n \rangle.$$

The standard pairs of $\mathrm{in}_\omega(I_A)$ are ([SST00, Sec. 3.2]):

$$\mathcal{S}(\mathrm{in}_\omega(I_A)) = \{(\partial_1^j, \{n-1\}) : \; j = 0, 1, \ldots, a_{n-1} - 1\}$$

To the standard pair $(\partial_1^j, \{n-1\})$ we associate, following [SST00, page 108], the *fake exponent*

$$v^j = (j, 0, \ldots, 0, \frac{\beta - j}{a_{n-1}}, 0)$$

of the module $\mathcal{M}_A(\beta)$ with respect to ω. It is easy to prove that these fake exponents are in fact exponents since they have minimal negative support [SST00, Th. 3.4.13]. □

Remark 5.1.17. *With the above notation, the* Γ*–series* ϕ_{v^j} *associated with* v^j *for* $j = 0, \ldots, a_{n-1} - 1$*, is defined as*

$$\phi_{v^j} = x^{v^j} \sum_{u \in L_A} \Gamma[v^j; u] x^u$$

where $L_A = \mathrm{Ker}_{\mathbb{Z}}(A)$ *is the lattice generated by the vectors* $\{u^2, \ldots, u^n\}$ *and* u^i *is the* $(i-1)$*–th row of the matrix*

$$\begin{pmatrix} -a_2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{n-2} & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ a_{n-1} & 0 & 0 & \cdots & 0 & 0 & -1 & 0 \\ -a_n & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For any $\mathbf{m} = (m_2, \ldots, m_n) \in \mathbb{Z}^{n-1}$ *let us denote* $u(\mathbf{m}) := \sum_{i=2}^n m_i u^i \in L_A$*. We can write*

$$\phi_{v^j} = x^{v^j} \sum_{\substack{m_2, \ldots, m_{n-1}, m_n \geq 0 \\ \sum_{i \neq n-1} a_i m_i \leq j + a_{n-1} m_{n-1}}} \Gamma[v^j; u(\mathbf{m})] x^{u(\mathbf{m})}$$

for $j = 0, 1, \ldots, a_{n-1} - 1$*. We have for* $\mathbf{m} = (m_2, \ldots, m_n) \in \mathbb{N}^{n-1}$ *such that* $j - \sum_{i \neq n-1} a_i m_i + a_{n-1} m_{n-1} \geq 0$

$$\Gamma[v^j; u(\mathbf{m})] = \frac{(\frac{\beta - j}{a_{n-1}})_{m_{n-1}} j!}{m_2! \cdots m_{n-2}! m_n! (j - \sum_{i \neq n-1} a_i m_i + a_{n-1} m_{n-1})!}$$

and

$$x^{u(\mathbf{m})} = x_1^{-\sum_{i \neq n-1} a_i m_i + a_{n-1} m_{n-1}} x_2^{m_2} \cdots x_{n-2}^{m_{n-2}} x_{n-1}^{-m_{n-1}} x_n^{m_n}$$

Proposition 5.1.18. *Let* $A = (1 \; a_2 \; \cdots \; a_n)$ *be an integer row matrix with* $1 < a_2 < \cdots < a_n$*,* $Y = \{x_n = 0\} \subset X$ *and* $Z = \{x_{n-1} = 0\} \subset X$*. Then we have:*

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p = \sum_{j=0}^{a_{n-1}-1} \mathbb{C}\phi_{v^j, p}$$

for all $\beta \in \mathbb{C}$*,* $p \in Y \setminus Z$*.*

Proof. Step 1.- Using [GZK89] and [SST00] we will describe a_{n-1} linearly independent solutions of $\mathcal{M}_A(\beta)_p$, living in some Nilsson series ring. Then, using initial ideals, we will prove that an upper bound of the dimension of $\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p$ is a_{n-1} for p in $Y \setminus Z$.

First of all, the series

$$\{\phi_{v^j} \mid j = 0, \ldots, a_{n-1} - 1\} \subset x^{v^j} \mathbb{C}[[x_1^{\pm 1}, x_2, \ldots, x_{n-2}, x_{n-1}^{-1}, x_n]]$$

described in Remark 5.1.17, are linearly independent since $\mathrm{in}_\omega(\phi_{v^j}) = x^{v^j}$ for $0 \leq j \leq a_{n-1} - 1$. They are solutions of the system $\mathcal{M}_A(\beta)$ (see [GGZ87], [GZK89, Sec. 1],[SST00, Sec. 3.4]).

On the other hand

$$\dim_{\mathbb{C}} \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p \leq a_{n-1} \tag{5.1}$$

for $p = (\epsilon_1, \ldots, \epsilon_{n-1}, 0)$, $\epsilon_{n-1} \neq 0$. This follows from the following facts:
a) The initial ideal $\mathrm{in}_\omega(I_A)$ equals $\langle \partial_2, \ldots, \partial_{n-2}, \partial_1^{a_{n-1}}, \partial_n \rangle$ for ω as in Lemma 5.1.16.
b) The germ of E at p is nothing but $E_p := E + \sum_{i=1}^{n-1} a_i \epsilon_i \partial_i$ (here $a_1 = 1$) and satisfies

$$\mathrm{in}_{(-\omega,\omega)}(E_p) = a_{n-1}\epsilon_{n-1}\partial_{n-1}.$$

c) By [SST00, Th. 2.5.5] if $f \in \mathcal{O}_{\widehat{X|Y},p}$ is a solution of the ideal $H_A(\beta)$ then $\mathrm{in}_\omega(f)$ must the annihilated by $\mathrm{in}_{(-\omega,\omega)}(H_A(\beta))$. That proves inequality (5.1).

Step 2.- It is easy to prove, using standard estimates, that the series $\phi_{v^j,p}$ are in fact in $\mathcal{O}_{\widehat{X|Y},p}$ for all $p \in Y \setminus Z$. Then by step 1, they form a basis of the the vector space $\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p$ for $p \in Y \setminus Z$. □

Remark 5.1.19. *We will prove (see Theorem 5.1.21) that the Gevrey index of $\phi_{v^j,p}$ is a_n/a_{n-1} for $\beta \in \mathbb{C}$ and $p \in Y \setminus Z$ except for $\beta \in \mathbb{N}$ and $j = q$ the unique integer $0 \leq q \leq a_{n-1} - 1$ such that $\beta - q \in a_{n-1}\mathbb{N}$. In that case ϕ_{v^q} is a polynomial (see details in Remark 5.1.22).*

Proposition 5.1.20. *Let $A = (1\ a_2\ \cdots\ a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$, $Y = \{x_n = 0\} \subset X$. Then we have:*

i) For each $\beta \in \mathbb{C} \setminus \mathbb{N}$ we have $\mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}) = 0$ for all $j \in \mathbb{N}$.

ii) For each $\beta \in \mathbb{N}$ the sheaf $\mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})$ is locally constant on Y of rank 1 for $j = 0, 1$ and it is zero for $j \geq 2$.

Proof. Recall that the characteristic variety of $\mathcal{M}_A(\beta)$ is

$$\mathrm{Ch}(\mathcal{M}_A(\beta)) = T^*_X X \cup T^*_Y X.$$

Then from Kashiwara's constructibility theorem [Kas75] we have that, for all $j \in \mathbb{N}$, the sheaf

$$\mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{O}_X)_{|Y} = \mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_{|Y} \tag{5.2}$$

is locally constant.

Assume $\beta \notin \mathbb{N}$. From Corollary 5.1.4 and Proposition 5.1.12 we have that there exists $m \in \mathbb{N}$ such that $\mathcal{M}_A(\beta) \simeq \mathcal{M}_A(\beta - m)$ and

$$\mathcal{M}_A(\beta)_{|X'} \simeq \mathcal{M}_A(\beta - m)_{|X'} \simeq \bigoplus_{i=0}^{k-1} \mathcal{M}_{(a'_{n-1}\ a'_n)}\left(\frac{\beta - m - i}{k}\right)$$

with $X' = \{x_1 = \cdots = x_{n-2} = 0\} \subset X$, $k = \gcd(a_{n-1}, a_n)$ and $a'_\ell = \frac{a_\ell}{k}$ for $\ell = n-1, n$. Then by applying Cauchy-Kovalevskaya Theorem (see Remark 5.1.13) we get:

$$\mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_{|X'} \simeq \bigoplus_{i=0}^{k-1} \mathcal{E}xt^j(\mathcal{M}_{(a'_{n-1}\ a'_n)}\left(\frac{\beta - m - i}{k}\right), \mathcal{O}_{X'|Y'})$$

with $Y' = X' \cap Y$.

As $\beta \notin \mathbb{N}$ then $\frac{\beta-m-i}{k} \notin a'_{n-1}\mathbb{N} + a'_n\mathbb{N}$ for $i = 0, \ldots, k-1$. Then part *i)* follows from Proposition 4.3.6.

Assume now $\beta \in \mathbb{N}$. From Corollary 5.1.4 and Proposition 5.1.12 we have that there exists $m \in \mathbb{N}$ such that $\mathcal{M}_A(\beta) \simeq \mathcal{M}_A(\beta + m)$ and

$$\mathcal{M}_A(\beta)_{|X'} \simeq \mathcal{M}_A(\beta + m)_{|X'} \simeq \bigoplus_{i=0}^{k-1} \mathcal{M}_{(a'_{n-1}\ a'_n)}\left(\frac{\beta + m - i}{k}\right).$$

Then by applying Cauchy-Kovalevskaya Theorem (see Remark 5.1.13) we get:

$$\mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_{|X'} \simeq \bigoplus_{i=0}^{k-1} \mathcal{E}xt^j(\mathcal{M}_{(a'_{n-1}\ a'_n)}\left(\frac{\beta + m - i}{k}\right), \mathcal{O}_{X'|Y'}).$$

By [$\mathrm{FC}_1$08] this last module is in fact equal to

$$\mathcal{E}xt^j(\mathcal{M}_{(a'_{n-1}\ a'_n)}\left(\frac{\beta + m - i_0}{k}\right), \mathcal{O}_{X'|Y'})$$

where i_0 is the unique integer number such that $0 \leq i_0 \leq k-1$ and $\beta + m - i_0 \in k\mathbb{N}$. Then part *ii)* follows from Proposition 4.3.7. □

Theorem 5.1.21. *Let $A = (1\ a_2\ \cdots\ a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$, $Y = \{x_n = 0\} \subset X$ and $Z = \{x_{n-1} = 0\} \subset X$. Then we have:*

i)

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p = \bigoplus_{j=0}^{a_{n-1}-1} \mathbb{C}\phi_{v^j,p}$$

for all $\beta \in \mathbb{C}$, $p \in Y \setminus Z$ and $s \geq a_n/a_{n-1}$.

ii)

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p = \begin{cases} 0 & \text{if } \beta \notin \mathbb{N} \\ \mathbb{C}\phi_{v^q} & \text{if } \beta \in \mathbb{N} \end{cases}$$

for all $p \in Y \setminus Z$ *and* $1 \leq s < a_n/a_{n-1}$*, where* q *is the unique element in* $\{0, 1, \dots, a_{n-1} - 1\}$ *satisfying* $\frac{\beta - q}{a_{n-1}} \in \mathbb{N}$ *and* ϕ_{v^q} *is a polynomial.*

Proof. i) Let us consider $a_n/a_{n-1} \leq s \leq \infty$ and $p \in Y \setminus Z$. Assume first that $\beta \notin \mathbb{N}$. By Proposition 5.1.20 and the long exact sequence of cohomology associated with the short exact sequence (2.1) we have that

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p \simeq \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$$

and by Theorem 5.1.1 this last vector space has dimension a_{n-1}. As

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p \subset \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p$$

part *i)* follows from Proposition 5.1.18 if $\beta \notin \mathbb{N}$.

Assume now $\beta \in \mathbb{N}$. Applying the long exact sequence of cohomology associated with the short exact sequence (2.1), Theorem 5.1.1 and Proposition 5.1.20 we get an exact sequence of vector spaces

$$0 \to \mathbb{C} \to \mathcal{L}_1 \to \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p \to \mathbb{C} \to \mathcal{L}_2 \to 0$$

where $\mathcal{L}_1 = \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p$ and $\mathcal{L}_2 = \mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p$. Let us write $\nu_i = \dim_{\mathbb{C}}(\mathcal{L}_i)$. By Theorem 5.1.1 we also have $\nu_1 = a_{n-1} + \nu_2$. On the other hand, by Proposition 5.1.18, we know that $\nu_1 \leq a_{n-1}$. This implies $\nu_1 = a_{n-1}$ and $\mathcal{L}_2 = \{0\}$. In particular we have the equality

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p \subset \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p$$

part *i)* also follows from Proposition 5.1.18 if $\beta \in \mathbb{N}$.
Let us prove part *ii)*. First of all, by Theorem 5.1.1, $\mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = 0$ for all $j \in \mathbb{N}$. Then the result follows from the long exact sequence of cohomology associated with the short exact sequence (2.1) and Proposition 5.1.20. □

Remark 5.1.22. *Let us recall here the notations introduced in Lemma 5.1.16. For* $A = (1\ a_2\ \cdots\ a_n)$ *an integer row matrix with* $1 < a_2 < \cdots < a_n$ *and* $\omega \in \mathbb{R}^n_{>0}$ *generic and satisfying*

1. $w_i > a_i\omega_1$*, for* $2 \leq i \leq n-2$ *or* $i = n$
2. $a_{n-1}\omega_1 > \omega_{n-1}$

3. $\omega_{n-1} > \omega_1, \ldots, \omega_{n-2}$

we have proved (see Lemma 5.1.16) that $H_A(\beta)$ *has* a_{n-1} *exponents with respect to* ω *and that they have the form:*

$$v^j = (j, 0, \ldots, 0, \frac{\beta - j}{a_{n-1}}, 0) \in \mathbb{C}^n$$

$j = 0, 1, \ldots, a_{n-1} - 1.$

The corresponding Γ*–series* ϕ_{v^j} *is defined as:*

$$\phi_{v^j} = x^{v^j} \sum_{\substack{m_2,\ldots,m_{n-1},m_n \geq 0 \\ \sum_{i\neq n-1} a_i m_i \leq j + a_{n-1}m_{n-1}}} \Gamma[v^j; u(\mathbf{m})] x^{u(\mathbf{m})}$$

for $j = 0, 1, \ldots, a_{n-1} - 1$*, where for any* $\mathbf{m} = (m_2, \ldots, m_n) \in \mathbb{Z}^{n-1}$ *we denote* $u(\mathbf{m}) := \sum_{i=2}^n m_i u^i \in L_A$*.*

For $\mathbf{m} = (m_2, \ldots, m_n) \in \mathbb{N}^{n-1}$ *such that* $j - \sum_{i\neq n-1} a_i m_i + a_{n-1}m_{n-1} \geq 0$*, we have*

$$\Gamma[v^j; u(\mathbf{m})] = \frac{(\frac{\beta-j}{a_{n-1}})_{m_{n-1}} j!}{m_2! \cdots m_{n-2}! m_n! (j - \sum_{i\neq n-1} a_i m_i + a_{n-1}m_{n-1})!}$$

and

$$x^{u(\mathbf{m})} = x_1^{-\sum_{i\neq n-1} a_i m_i + a_{n-1}m_{n-1}} x_2^{m_2} \cdots x_{n-2}^{m_{n-2}} x_{n-1}^{-m_{n-1}} x_n^{m_n}.$$

If $\beta \in \mathbb{N}$ *then there exists a unique* $0 \leq q \leq a_{n-1} - 1$ *such that* $\beta - q \in a_{n-1}\mathbb{N}$*. Let us write* $m_0 = \frac{\beta - q}{a_{n-1}}$*.*

Then for $m \in \mathbb{N}$ *big enough* $m_0 - a_n m$ *is a negative integer and the coefficient* $\Gamma[v^q; u(\mathbf{m})]$ *is zero and then* ϕ_{v^q} *is a polynomial in* $\mathbb{C}[x]$*.*

Recall that $u^{n-1} = (a_{n-1}, 0, \ldots, -1, 0) \in L_A$ *and let us write*

$$\widetilde{v^q} = v^q + (m_0 + 1)u^{n-1} = (q + (m_0 + 1)a_{n-1}, 0, \ldots, 0, -1, 0) = (\beta + a_{n-1}, 0, \ldots, 0, -1, 0).$$

We have $A\widetilde{v^q} = \beta$ *an the corresponding* Γ*–series is*

$$\phi_{\widetilde{v^q}} = x^{\widetilde{v^q}} \sum_{\mathbf{m} \in M(q)} \Gamma[\widetilde{v^q}; u(\mathbf{m})] x^{u(\mathbf{m})}$$

where for $\mathbf{m} = (m_2, \ldots, m_n) \in \mathbb{Z}^n$ *one has* $u(\mathbf{m}) = \sum_{i=2}^n m_i u^i$ *and*

$$M(q) := \{(m_2, \ldots, m_n) \in \mathbb{N}^{n-1} \,|\, q + (m_0 + m_{n-1} + 1)a_{n-1} - \sum_{i\neq n-1} a_i m_i \geq 0\}.$$

Let us notice that $\widetilde{v^q}$ *does not have minimal negative support (see [SST00, p. 132-133]) and then the* Γ*–series* $\phi_{\widetilde{v^q}}$ *is not a solution of* $H_A(\beta)$*. We will prove in Theorem 5.1.23 that* $H_A(\beta)_p(\phi_{\widetilde{v^q},p}) \subset \mathcal{O}_{X,p}$ *for all* $p \in Y \setminus Z$ *and that* $\phi_{\widetilde{v^q},p}$ *is a Gevrey series of index* a_n/a_{n-1}*.*

The second main result of this Section is the following

Theorem 5.1.23. *Let $A = (1\ a_2\ \cdots\ a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$, $Y = \{x_n = 0\} \subset X$ and $Z = \{x_{n-1} = 0\} \subset X$. Then for all $p \in Y \setminus Z$ and $s \geq a_n/a_{n-1}$ we have:*

i) If $\beta \notin \mathbb{N}$, then:

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = \bigoplus_{j=0}^{a_{n-1}-1} \mathbb{C}\overline{\phi_{v^j,p}}.$$

ii) If $\beta \in \mathbb{N}$, then there exists a unique $q \in \{0, \ldots, a_{n-1} - 1\}$ such that $\frac{\beta - q}{a_{n-1}} \in \mathbb{N}$ and we have:

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = \bigoplus_{q \neq j=0}^{a_{n-1}-1} \mathbb{C}\overline{\phi_{v^j,p}} \oplus \mathbb{C}\overline{\phi_{\widetilde{v^q},p}}.$$

Here $\overline{\phi}$ stands for the class modulo $\mathcal{O}_{X|Y,p}$ of $\phi \in \mathcal{O}_{\widehat{X|Y},p}(s)$.

Proof. Part *i)* follows from Theorem 5.1.21 and Proposition 5.1.20 using the long exact sequence of cohomology.

Let us prove *ii)*. Since $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s)) = 0$ (see Theorem 5.1.1) and applying Theorem 5.1.21, Proposition 5.1.20 and the long exact sequence in cohomology we get that

$$\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_{|Y \setminus Z}$$

is zero for $s \geq a_n/a_{n-1}$ and locally constant of rank 1 for $1 \leq s < a_n/a_{n-1}$. We also have that

$$\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_{|Y \cap Z}$$

is locally constant of rank 1 for all $s \geq 1$.

Assume $s \geq a_n/a_{n-1}$. We consider the following long exact sequence associated with the short exact sequence (2.1) (with $p \in Y \setminus Z$ and $\mathcal{M} = \mathcal{M}_A(\beta)$)

$$0 \to \mathcal{E}xt^0(\mathcal{M}, \mathcal{O}_{X|Y})_p \to \mathcal{E}xt^0(\mathcal{M}, \mathcal{O}_{X|Y}(s))_p \xrightarrow{\rho} \mathcal{E}xt^0(\mathcal{M}, \mathcal{Q}_Y(s))_p \to \mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_{X|Y})_p \to 0 \tag{5.3}$$

We also have

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_p \simeq \mathbb{C}$$

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s))_p \simeq \mathbb{C}^{a_{n-1}}$$

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p \simeq \mathbb{C}^{a_{n-1}}$$

$$\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_p \simeq \mathbb{C}$$

Since $\beta \in \mathbb{N}$ there exists a unique $q = 0, 1, \ldots, a_{n-1} - 1$ such that $\frac{\beta - q}{a_{n-1}} \in \mathbb{N}$ and then $\phi_{v^q} \in \mathbb{C}[x]$ generates $\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_p = \mathrm{Ker}(\rho)$.

Using the above exact sequence and the first isomorphism theorem we get that the family

$$\{\overline{\phi_{v^j,p}} : \ 0 \leq j \leq a_{n-1} - 1, j \neq q\}$$

is linearly independent in $\mathcal{Q}_Y(s)_p$ for all $p \in Y \setminus Z$.

In a similar way to the proof of Theorem 5.1.1 it can be proved that $\phi_{\widetilde{v^q},p} \in \mathcal{O}_{X|Y}(s)_p$ for all $p \in Y \setminus Z$ and $s \geq a_n/a_{n-1}$.

Writing $t_{n-1} = x_{n-1}^{-1}$ and defining:

$$\psi_{\widetilde{v^q}}(x_1, \ldots, x_{n-2}, t_{n-1}, x_n) := x^{-\widetilde{v^q}} \phi_{\widetilde{v^q}}(x_1, \ldots, x_{n-2}, \frac{1}{t_{n-1}}, x_n)$$

we have that

$$\psi_{\widetilde{v^q}} \in \mathbb{C}[[x_1, \ldots, x_{n-2}, t_{n-1}, x_n]]$$

Taking the subsum of $\psi_{\widetilde{v^q}}$ for $m_2 = \cdots = m_{n-2} = 0,\ m_n = a_{n-1}m,\ m_{n-1} = a_n m, m \in \mathbb{N}$, we get the power series

$$\sum_{m \geq 0} c_m \left(t_{n-1}^{a_n} x_n^{a_{n-1}}\right)^m$$

where

$$c_m = \frac{(-1)^{a_n m}(a_n m)!}{(a_{n-1}m)!}$$

This power series has Gevrey index a_n/a_{n-1} with respect to $x_n = 0$. Then $\phi_{\widetilde{v^q}}$ has Gevrey index a_n/a_{n-1}.

We have $E(\phi_{\widetilde{v^q}}) = P_i(\phi_{\widetilde{v^q}}) = 0$, for all $i = 1, 2, \ldots, n-2, n$ and $P_{n-1}(\phi_{\widetilde{v^q}})$ is a meromorphic function with poles along Z (and holomorphic on $X \setminus Z$):

$$P_{n-1}(\phi_{\widetilde{v^q}}) = \sum_{\underline{m} \in \widetilde{M}(q)} \frac{(\beta + a_{n-1})! x_1^{q - \sum_{i \neq n-1} a_i m_i + a_{n-1}(m_0+1)} x_2^{m_2} \cdots x_{n-2}^{m_{n-2}} x_{n-1}^{-1} x_n^{m_n}}{m_2! \cdots m_{n-2}! m_n! (q - \sum_{i \neq n-1} a_i m_i + a_{n-1}(m_0 + 1))!}$$

where

$$\widetilde{M}(q) = \{(m_2, \ldots, m_{n-2}, m_n) \in \mathbb{N}^{n-2} | \sum a_i m_i \leq q + a_{n-1}(m_0 + 1) = \beta + a_{n-1}\}$$

is a finite set (recall that $m_0 = \frac{\beta - q}{a_{n-1}} \in \mathbb{N}$).

In particular, $H_A(\beta)(\phi_{\widetilde{v^q}}) \subseteq \mathcal{O}_X(X \setminus Z)$.

So,

$$\overline{\phi_{\widetilde{v^q},p}} \in \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$$

for all $p \in Y \setminus Z$ and $s \geq a_n/a_{n-1}$.

In order to finish the proof we will see that for all $\lambda_j \in \mathbb{C}$ ($j = 0, \ldots, a_{n-1}-1$; $j \neq q$) and for all $p \in Y \setminus Z$ we have

$$\phi_{\widetilde{v^q},p} - \sum_{j\neq q} \lambda_j \phi_{v^j,p} \notin \mathcal{O}_{X|Y,p}.$$

Let us write

$$\psi_{v^j}(x_1, \ldots, x_{n-2}, t_{n-1}, x_n) := \phi_{v^j}(x_1, \ldots, x_{n-2}, \frac{1}{t_{n-1}}, x_n)$$

Assume to the contrary that there exist $p \in Y \setminus Z$ and $\lambda_j \in \mathbb{C}$ such that:

$$\phi_{\widetilde{v^q},p} - \sum_{j\neq q} \lambda_j \phi_{v^j,p} \in \mathcal{O}_{X|Y,p}$$

Let us consider the holomorphic function at p defined as

$$f := x^{\widetilde{v^q}} \psi_{\widetilde{v^q},p} - \sum_{j\neq q} \lambda_j \psi_{v^j,p}$$

We have the following equality of holomorphic functions at p:

$$\rho_s(f + \sum_{j\neq q} \lambda_j \psi_{v^j}) = \rho_s(x^{\widetilde{v^q}} \psi_{\widetilde{v^q}})$$

for $s > a_n$.

The function $\rho_s(x^{\widetilde{v^q}} \psi_{\widetilde{v^q}})$ is holomorphic in $\mathbb{C}^n$ while each $\rho_s(\psi_{v^j})$ has the form $t_{n-1}^{-\frac{\beta-j}{a_{n-1}}} \psi_j$ with ψ_j holomorphic in $\mathbb{C}^n$.

Making a loop around the t_{n-1} axis ($\log t_{n-1} \mapsto \log t_{n-1} + 2\pi i$) we get the equality:

$$\rho_s(\widehat{f} + \sum_{j\neq q} c_j \lambda_j \psi_{v^j}) = \rho_s(x^{\widetilde{v^q}} \psi_{\widetilde{v^q}})$$

where $c_j = e^{-\frac{\beta-j}{a_{n-1}} 2\pi i} \neq 1$ (since $\frac{\beta-j}{a_{n-1}} \notin \mathbb{Z}$ for all $j \neq q$) and $\widehat{f}$ is obtained from f after the loop. Since f is holomorphic at p then $\widehat{f}$ also is. Subtracting both equalities we get:

$$\rho_s(\widehat{f} - f + \sum_{j\neq q} (c_j - 1) \lambda_j \psi_{v^j}) = 0$$

and then

$$\sum_{j\neq q}(c_j-1)\lambda_j\psi_{v^j}=f-\widehat{f}$$

in the neighborhood of p. This contradicts the fact that the power series $\{\phi_{v^j}:\ j\neq q, 0\leq j\leq a_{n-1}-1\}$ are linearly independent modulo $\mathcal{O}_{X|Y,p}$ (here we have $c_j-1\neq 0$). This proves the theorem. □

Corollary 5.1.24. *If $\beta\in\mathbb{N}$ then for all $p\in Y\setminus Z$ the vector space*

$$\mathcal{E}xt^1(\mathcal{M}_A(\beta),\mathcal{O}_{X|Y})_p$$

is generated by the class of:

$$(P_2(\phi_{\widetilde{v^q}}),\ldots,P_{n-1}(\phi_{\widetilde{v^q}}),P_n(\phi_{\widetilde{v^q}}),E(\phi_{\widetilde{v^q}}))=$$

$$(0,\ldots,0,\sum_{\underline{m}\in\widetilde{M}(q)}\frac{(\beta+a_{n-1})!\,x_1^{q-\sum_{i\neq n-1}a_im_i+a_{n-1}(m_0+1)}x_2^{m_2}\cdots x_{n-2}^{m_{n-2}}x_{n-1}^{-1}x_n^{m_n}}{m_2!\cdots m_{n-2}!m_n!(q-\sum_{i\neq n-1}a_im_i+a_{n-1}(m_0+1))!},0,0)$$

in

$$\frac{(\mathcal{O}_{X|Y})_p^n}{\mathrm{Im}(\psi_0^*,\mathcal{O}_{X|Y})_p}$$

where

$$\widetilde{M}(q)=\{(m_2,\ldots,m_{n-2},m_n)\in\mathbb{N}^{n-2}\mid\sum a_im_i\leq q+a_{n-1}(m_0+1)=\beta+a_{n-1}\}$$

is a finite set (with $m_0=\frac{\beta-q}{a_{n-1}}\in\mathbb{N}$) and ψ_0^* being the dual map of

$$\begin{array}{rcl}\psi_0:\mathcal{D}^n & \longrightarrow & \mathcal{D}\\ (Q_1,\ldots,Q_n) & \mapsto & \sum_{j=2}^n Q_jP_j+Q_nE\end{array}$$

Proof. It follows from the proof of Theorem 5.1.23 since $\mathcal{E}xt^1(\mathcal{M}_A(\beta),\mathcal{O}_{X|Y})_p\simeq\mathbb{C}$ for all $p\in Y\setminus Z$ and moreover

$$(P_2(\phi_{v^j}),\ldots,P_n(\phi_{v^j}),E(\phi_{v^j}))=\underline{0}$$

for $0\leq j\leq a_{n-1}-1$, $j\neq q$. □

Remark 5.1.25. *We can also compute the holomorphic solutions of $\mathcal{M}_A(\beta)$ at any point in $X\setminus Y$ for $A=(1\ a_2\ \ldots\ a_n)$ with $1<a_2<\ldots<a_n$ and for any $\beta\in\mathbb{C}$, where $Y=\{x_n=0\}\subset X=\mathbb{C}^n$. We consider the vectors $w^j=(j,0,\ldots,0,\frac{\beta-j}{a_n})\in\mathbb{C}^n$, $j=0,1,\ldots,a_n-1$ then the germs at $p\in X\setminus Y$ of the series solutions $\{\phi_{w^j}:\ j=0,1,\ldots,a_n-1\}$ is a basis of $\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}_A(\beta),\mathcal{O}_X)_p$.*

We have summarized the main results of this Section in Figure 5.1. Here $A=(1\ a_2\ \cdots\ a_n)$, $s\geq\frac{a_n}{a_{n-1}}$, $p\in Y\setminus Z$, $z\in Y\cap Z$, $\beta_{\rm esp}\in\mathbb{N}$ and $\beta_{\rm gen}\notin\mathbb{N}$.

(z,β_{esp}) (p,β_{esp}) / (z,β_{gen}) (p,β_{gen})	$\mathcal{E}xt^0(\mathcal{M}_A(\beta),\mathcal{F})$		$\mathcal{E}xt^1(\mathcal{M}_A(\beta),\mathcal{F})$	
$\mathcal{F}=\mathcal{O}_{X\vert Y}$	1	1	1	1
	0	0	0	0
$\mathcal{F}=\mathcal{O}_{X\vert Y}(s)$	1	a_{n-1}	1	0
	0	a_{n-1}	0	0
$\mathcal{F}=\mathcal{Q}_Y(s)$	0	a_{n-1}	0	0
	0	a_{n-1}	0	0

Figure 5.1: Dimension of the germs of $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}_A(\beta),\mathcal{F})$

5.2 The case of a monomial curve

Let $A=(a_1\ a_2\ \cdots\ a_n)$ be an integer row matrix with $1<a_1<a_2<\cdots<a_n$ and assume without loss of generality $\gcd(a_1,\ldots,a_n)=1$.

In this Section we will compute the dimension of the germs of the cohomology of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ at any point in $Y=\{x_n=0\}\subseteq X=\mathbb{C}^n$ for all $\beta\in\mathbb{C}$ and $1\leq s\leq\infty$.

We will consider the auxiliary matrix $A'=(1\ a_1\ \cdots\ a_n)$ and the corresponding hypergeometric ideal $H_{A'}(\beta)\subset A_{n+1}$ where A_{n+1} is the Weyl algebra of linear differential operators with coefficients in the polynomial ring $\mathbb{C}[x_0,x_1,\ldots,x_n]$. We denote ∂_0 the partial derivative with respect to x_0.

In this Section we denote $X'=\mathbb{C}^{n+1}$ and we identify $X=\mathbb{C}^n$ with the hyperplane $\{x_0=0\}$ in X'. If $\mathcal{D}_{X'}$ is the sheaf of linear differential operators with holomorphic coefficients in X' then the analytic hypergeometric system associated with (A',β), denoted by $\mathcal{M}_{A'}(\beta)$, is by definition the quotient of $\mathcal{D}_{X'}$ by the sheaf of ideals generated by the hypergeometric ideal $H_{A'}(\beta)\subset A_{n+1}$ (see Section 3.1).

One of the main results in this Section is the following

Theorem 5.2.1. *Let $A'=(1\ a_1\ \cdots\ a_n)$ an integer row matrix with $1<a_1<\cdots<a_n$ and $\gcd(a_1,\ldots,a_n)=1$. For each $\beta\in\mathbb{C}$ there exists $\beta'\in\mathbb{C}$ such that the restriction of $\mathcal{M}_{A'}(\beta)$ to $X=\{x_0=0\}\subset X'$ is the $\mathcal{D}_X$–module*

$$\mathcal{M}_{A'}(\beta)_{|X}:=\frac{\mathcal{D}_{X'}}{\mathcal{D}_{X'}H_{A'}(\beta)+x_0\mathcal{D}_{X'}}\simeq\mathcal{M}_A(\beta')$$

where $A=(a_1\ a_2\ \cdots\ a_n)$. Moreover, for all but finitely many β we have $\beta'=\beta$.

Proof. Following [SST00], we will use the notations defined in Notation 5.1.5 and Definition

5.1.8. For $i = 1, 2, \ldots, n$ let us consider $\delta_i \in \mathbb{N}$ the smallest integer satisfying $1 + \delta_i a_i \in \sum_{j \neq i} a_j \mathbb{N}$. Such a δ_i exists because $\gcd(a_1, \ldots, a_n) = 1$.

Let us consider $\rho_{ij} \in \mathbb{N}$ such that

$$1 + \delta_i a_i = \sum_{j \neq i} \rho_{ij} a_j.$$

Then the operator $Q_i := \partial_0 \partial_i^{\delta_i} - \partial^{\rho_i}$ belongs to $I_{A'}$ where $\partial^{\rho_i} = \prod_{j \neq 0, i} \partial_j^{\rho_{ij}}$. Moreover, for $\omega = (1, 0, \ldots, 0) \in \mathbb{N}^{n+1}$ we have $\mathrm{in}_{(-\omega,\omega)}(Q_i) = \partial_0 \partial_i^{\delta_i} \in \mathrm{in}_\omega I_{A'}$ for $i = 1, \ldots, n$.

We also have that $P_1 = \partial_0^{a_1} - \partial_1 \in I_{A'}$ and $\mathrm{in}_{(-\omega,\omega)} P_1 = \partial_0^{a_1} \in \mathrm{in}_\omega I_{A'}$. Then

$$\mathrm{in}_\omega I_{A'} \supseteq \langle \partial_0^{a_1}, \partial_0 \partial_1^{\delta_1} \ldots, \partial_0 \partial_n^{\delta_n}, T_1, \ldots, T_r \rangle \tag{5.4}$$

for any binomial generating system $\{T_1, \ldots, T_r\} \subseteq \mathbb{C}[\partial_1, \ldots, \partial_n]$ of the ideal $I_A = I_{A'} \cap \mathbb{C}[\partial_1, \ldots, \partial_n]$ (notice that $u \in L_A = \mathrm{Ker}_{\mathbb{Z}}(A) \subset \mathbb{Z}^n \Longleftrightarrow (0, u) \in L_{A'} = \mathrm{Ker}_{\mathbb{Z}}(A') \subset \mathbb{Z}^{n+1}$).

Using (5.4) we can prove (similarly to the proof of Lemma 5.1.7 for $k = 1$) that for $\beta \notin \mathbb{N}^*$ or $\beta \in \mathbb{N}^*$ big enough, we have

$$\partial_0 \in \mathrm{fin}_\omega(H_{A'}(\beta)) = \mathrm{in}_\omega I_{A'} + \langle E' \rangle \tag{5.5}$$

where $E' = E + x_0 \partial_0$ and $E := E(\beta) = \sum_{i=1}^n a_i x_i \partial_i - \beta$. So there exists $R \in H_{A'}(\beta)$ such that $\partial_0 = \mathrm{in}_{(-\omega,\omega)}(R)$. In particular we have

$$\langle H_A(\beta), \partial_0 \rangle \subseteq \mathrm{fin}_\omega(H_{A'}(\beta)) \subseteq \mathrm{in}_{(-\omega,\omega)}(H_{A'}(\beta))$$

and the b–function of $H_{A'}(\beta)$ with respect to ω is $b(\tau) = \tau$. So the restriction of $\mathcal{M}_{A'}(\beta)$ to $\{x_0 = 0\}$ is a cyclic $\mathcal{D}_X$-module (see [SST00, Algorithm 5.2.8]).

In order to compute $\mathcal{M}_{A'}(\beta)_{|\{x_0=0\}}$ we will follow [SST00, Algorithm 5.2.8]. First of all we need to describe the form of a Groebner basis of $H_{A'}(\beta)$ with respect to ω. Let $\{T_1, \ldots, T_r, R_1, \ldots, R_\ell\}$ be a Groebner basis of $I_{A'}$ with respect to ω. So we have

$$I_{A'} = \langle T_1, \ldots, T_r, R_1, \ldots, R_\ell \rangle$$

and

$$\mathrm{in}_\omega I_{A'} = \langle T_1, \ldots, T_r, \mathrm{in}_{(-\omega,\omega)} R_1, \ldots, \mathrm{in}_{(-\omega,\omega)} R_\ell \rangle.$$

If, for some $i = 0, \ldots, \ell$, the ω-order of $\mathrm{in}_{(-\omega,\omega)} R_i$ is 0, then $\mathrm{in}_{(-\omega,\omega)} R_i = R_i \in I_{A'} \cap \mathbb{C}[\partial_1, \ldots, \partial_n] = I_A$ and then $\mathrm{in}_{(-\omega,\omega)} R_i = R_i \in \langle T_1, \ldots, T_r \rangle$.

If the ω-order of $\mathrm{in}_{(-\omega,\omega)} R_i$ is greater than or equal to 1, then ∂_0 divide $\mathrm{in}_{(-\omega,\omega)} R_i$. Then, according (5.5), for $\beta \notin \mathbb{N}^*$ or $\beta \in \mathbb{N}^*$ big enough, we have

$$\mathrm{fin}_\omega(H_{A'}(\beta)) = \langle \partial_0, E, T_1, \ldots, T_r \rangle = \langle \partial_0 \rangle + A_{n+1} H_A(\beta) \subseteq \mathrm{in}_{(-\omega,\omega)}(H_{A'}(\beta)) \tag{5.6}$$

From [SST00, Th. 3.1.3], for all but finitely many $\beta \in \mathbb{C}$, we have

$$\mathrm{in}_{(-\omega,\omega)}(H_{A'}(\beta)) = \langle \partial_0, E, T_1, \ldots, T_r \rangle = \langle \partial_0 \rangle + A_{n+1} H_A(\beta). \tag{5.7}$$

So, for all but finitely many $\beta \in \mathbb{C}$, the set

$$\mathcal{G} = \{R, R_1, \ldots, R_\ell, E', T_1, \ldots, T_r\}$$

is a Groebner basis of $H_{A'}(\beta)$ with respect to ω, since first of all $\mathcal{G}$ is a generating system of $H_{A'}(\beta)$ and on the other hand $\mathrm{in}_{(-\omega,\omega)}(H_{A'}(\beta)) = A_{n+1}\mathrm{in}_{(-\omega,\omega)}(\mathcal{G})$.

We can now follow [SST00, Algorithm 5.2.8], as in the proof of Theorem 5.1.3, to prove the result for all but finitely many $\beta \in \mathbb{C}$. Then, to finish the proof it is enough to apply Proposition 5.1.12 for A'. $\square$

Remark 5.2.2. *Recall that $Y = \{x_n = 0\} \subset X = \mathbb{C}^n$ and $Z = \{x_{n-1} = 0\} \subset X$. Let us denote $Y' = \{x_n = 0\} \subset X'$, $Z' = \{x_{n-1} = 0\} \subset X'$. Notice that $Y = Y' \cap X$ and $Z = Z' \cap X$.*

By using Cauchy-Kovalevskaya Theorem for Gevrey series (see [LM02, Cor. 2.2.4]), [CT03, Proposition 4.2] and Theorem 5.2.1, we get, for all but finitely many $\beta \in \mathbb{C}$ and for all $1 \leq s \leq \infty$, the following isomorphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_{X'}}(\mathcal{M}_{A'}(\beta), \mathcal{O}_{\widehat{X'|Y'}}(s))_{|X} \xrightarrow{\simeq} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}(s)).$$

We also have the following

Theorem 5.2.3. *Let $A = (a_1\ a_2\ \cdots\ a_n)$ be an integer row matrix with $1 < a_1 < a_2 < \cdots < a_n$ and $\gcd(a_1, \ldots, a_n) = 1$. Then for all $\beta \in \mathbb{C}$ we have*

i) $\mathcal{E}xt^0_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s)) = 0$ *for* $1 \leq s < a_n/a_{n-1}$.

ii) $\mathcal{E}xt^0_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_{|Y\cap Z} = 0$ *for* $1 \leq s \leq \infty$.

iii) $\dim_{\mathbb{C}}(\mathcal{E}xt^0_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p) = a_{n-1}$ *for* $a_n/a_{n-1} \leq s \leq \infty$ *and* $p \in Y \setminus Z$.

iv) $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s)) = 0$, *for* $i \geq 1$ *and* $1 \leq s \leq \infty$.

Here $Y = \{x_n = 0\} \subset \mathbb{C}^n$ and $Z = \{x_{n-1} = 0\} \subset \mathbb{C}^n$.

Proof. It follows from Remark 5.2.2, Theorem 5.1.1 and Proposition 5.1.12. $\square$

Remark 5.2.4. *With the notations of Theorem 5.2.3, a basis of the $\mathbb{C}$–vector space $\mathcal{E}xt^0_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ for any $\frac{a_n}{a_{n-1}} \leq s \leq \infty$, $p \in Y \setminus Z$ and $\beta \in \mathbb{C}$ is given* *by the "substitution" (in a sense to be made precise) of $x_0 = 0$ in the basis of $\mathcal{E}xt^0_{\mathcal{D}'}(\mathcal{M}_{A'}(\beta), \mathcal{Q}_{Y'}(s))_{(0,p)}$ described in Theorem 5.1.23.*

Remind that for $A' = (1\ a_1\ \dots\ a_n)$ and $\beta \in \mathbb{C}$ the Γ–series described in Section 5.1 are

$$\phi_{v^j} = (x')^{v^j} \sum_{\substack{m_1,\dots,m_{n-1},m_n \geq 0 \\ \sum_{i\neq n-1} a_i m_i \leq j + a_{n-1}m_{n-1}}} \Gamma[v^j; u(\mathbf{m})](x')^{u(\mathbf{m})}$$

where $x' = (x_0, x_1, \dots, x_n)$, $\quad v^j = (j, 0, \dots, 0, \frac{\beta - j}{a_{n-1}}, 0) \in \mathbb{C}^{n+1}$ for $j = 0, 1, \dots, a_{n-1} - 1$ and for $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ we have

$$(x')^{u(\mathbf{m})} = x_0^{-\sum_{i\neq n-1} a_i m_i + a_{n-1}m_{n-1}} x_1^{m_1} \cdots x_{n-2}^{m_{n-2}} x_{n-1}^{-m_{n-1}} x_n^{m_n}.$$

For $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$ such that $j - \sum_{i\neq n-1} a_i m_i + a_{n-1}m_{n-1} \geq 0$ we have

$$\Gamma[v^j; u(\mathbf{m})] = \frac{(\frac{\beta - j}{a_{n-1}})_{m_{n-1}}\, j!}{m_1! \cdots m_{n-2}! m_n! (j - \sum_{i\neq n-1} a_i m_i + a_{n-1} m_{n-1})!}.$$

Since $I_A = I_{A'} \cap \mathbb{C}[\partial_1, \dots, \partial_n]$, if $I_{A'}(f) = 0$ then $I_A(f_{|x_0=0}) = 0$ for every formal power series $f \in \mathcal{O}_{\widehat{X'|Y'},p'}$ where $p' = (0, p) \in Y' \cap \{x_0 = 0\} = Y$.

Furthermore, a Laurent monomial $(x')^{w'}$ is annihilated by the Euler operator associated with (A', β) if and only if $A'w' = \beta$ and after the substitution $x_0 = 0$ this monomial becomes zero or x^w (in the case $w' = (0, w)$) which are both annihilated by the Euler operator associated with (A, β), since $Aw = A'w' = \beta$ in the case $w' = (0, w)$.

Hence, for $p \in Y$, every formal series solution $f \in \mathcal{O}_{\widehat{X'|Y'},(0,p)}$ of $\mathcal{M}_{A'}(\beta)$ becomes, after the substitution $x_0 = 0$, a formal series solution $f_{|x_0=0} \in \mathcal{O}_{\widehat{X|Y},p}$ of $\mathcal{M}_A(\beta)$. The analogous result is also true for convergent series solutions at a point of $x_0 = 0$.

After the substitution $x_0 = 0$ in the series ϕ_{v^j} we get

$$\phi_{v^j|x_0=0} = \sum_{\substack{m_1,\dots,m_{n-1},m_n \geq 0 \\ \sum a_i m_i = j + a_{n-1}m_{n-1}}} \frac{(\frac{\beta - j}{a_{n-1}})_{m_{n-1}} j! x_1^{m_1} \cdots x_{n-2}^{m_{n-2}} x_{n-1}^{\frac{\beta - j}{a_{n-1}} - m_{n-1}} x_n^{m_n}}{m_1! \cdots m_{n-2}! m_n!}$$

for $j = 0, 1, \dots, a_{n-1} - 1$.

The summation before is taken over the set

$$\Delta_j := \{(m_1, \dots, m_n) \in \mathbb{N}^n : \sum_{i\neq n-1} a_i m_i = j + a_{n-1} m_{n-1}\}.$$

It is clear that $(0, \dots, 0) \in \Delta_0$ and for $j \geq 1$, Δ_j is a non empty set since $\gcd(a_1, \dots, a_n) = 1$. *Moreover Δ_j is a countably infinite set for $j \geq 0$. To this end take some $\underline{\lambda} := (\lambda_1, \dots, \lambda_n) \in \Delta_j$. Then $\underline{\lambda} + \mu(0, \dots, 0, a_n, a_{n-1})$ is also in Δ_j for all $\mu \in \mathbb{N}$.*

The series $\phi_{v^j|x_0=0}$ is a Gevrey series of order $s=\frac{a_n}{a_{n-1}}$ since ϕ_{v^j} also is. We will see that in fact the Gevrey index of $\phi_{v^j|x_0=0}$ is $\frac{a_n}{a_{n-1}}$ for $j=0,\ldots,a_{n-1}-1$ such that $\frac{\beta-j}{a_{n-1}}\notin\mathbb{N}$. To this end let us consider the subsum of $\phi_{v^j|x_0=0}$ over the set of $(m_1,\ldots,m_n)\in\mathbb{N}^n$ of the form $\underline{\lambda}^{(j)}+\mathbb{N}(0,\ldots,0,a_n,a_{n-1})$ for some fixed $\underline{\lambda}^{(j)}\in\Delta_j$. Then we get the series:

$$\frac{j!(\frac{\beta-j}{a_{n-1}})_{\lambda_{n-1}^{(j)}}x_1^{\lambda_1^{(j)}}\cdots x_{n-2}^{\lambda_{n-2}^{(j)}}x_{n-1}^{\frac{\beta-j}{a_{n-1}}-\lambda_{n-1}^{(j)}}}{\lambda_1^{(j)}!\cdots\lambda_{n-2}^{(j)}!}\sum_{m\geq 0}\frac{(\frac{\beta-j}{a_{n-1}}-\lambda_{n-1}^{(j)})_{a_n m}x_{n-1}^{-a_n m}x_n^{\lambda_n^{(j)}+a_{n-1}m}}{(\lambda_n^{(j)}+a_{n-1}m)!}$$

and it can be proven, by using d'Alembert ratio test, that its Gevrey index equals $\frac{a_n}{a_{n-1}}$ at any point in $Y\setminus Z$, for any $j=0,\ldots,a_{n-1}-1$ such that $\frac{\beta-j}{a_{n-1}}\notin\mathbb{N}$.

For all $j=0,\ldots,a_{n-1}-1$ we have

$$\phi_{v^j|x_0=0}\in x_{n-1}^{\frac{\beta-j}{a_{n-1}}}\mathbb{C}[[x_1,\ldots,x_{n-2},x_{n-1}^{-1},x_n]]$$

and in particular these a_{n-1} series are linearly independent and hence a basis of $\mathcal{E}xt^0(\mathcal{M}_A(\beta),\mathcal{O}_{X|Y}(s))_p$ for $s\geq a_n/a_{n-1}$ and $p\in Y\setminus Z$) if all of them are nonzero (see Theorem 5.1.21).

If there exists $0\leq j\leq a_{n-1}-1$ such that $\phi_{v^j|x_0=0}=0$ then we have that ϕ_{v^j} is a polynomial divisible by x_0 and this happens if and only if $\beta\in\mathbb{N}\setminus\mathbb{N}A$.

In this last case we do not get a basis of $\mathcal{E}xt^0(\mathcal{M}_A(\beta),\mathcal{O}_{X|Y}(s))_p$ by the previous procedure. We will proceed as follows. Let us consider $w'=(0,\omega)\in\mathbb{N}^{n+1}$ such that $\beta':=\beta-A'w'\in\mathbb{Z}_{<0}$. Then taking the basis $\{\phi_{v^j}\,|\,j=0,\ldots,a_{n-1}-1\}$ of $\mathcal{E}xt^0(\mathcal{M}_{A'}(\beta'),\mathcal{O}_{\widehat{X'|Y'}}(s))_{(0,p)}$ given by Theorem 5.1.21 and after the substitution $x_0=0$, we get a basis of $\mathcal{E}xt^0(\mathcal{M}_A(\beta'),\mathcal{O}_{X|Y}(s))_p$ for $s\geq a_n/a_{n-1}$ and $p\in Y\setminus Z$.

Since $\beta,\beta'\in\mathbb{Z}\setminus\mathbb{N}A$ then

$$\partial^w:\mathcal{M}_A(\beta')\to\mathcal{M}_A(\beta)$$

is an isomorphism (see [SW07, Remark 3.6] and [Ber08, Lemma 6.2]) and we can use this isomorphism to obtain a basis of $\mathcal{E}xt^0(\mathcal{M}_A(\beta),\mathcal{O}_{X|Y}(s))_p$.

Using previous discussion and similar ideas to the ones of Section 5.1 (we will use the notations therein) we can prove the following Theorem.

Theorem 5.2.5. *Let $A=(a_1\ a_2\ \cdots\ a_n)$ be an integer row matrix with $0<a_1<a_2<\cdots<a_n$, $Y=\{x_n=0\}\subset X$ and $Z=\{x_{n-1}=0\}\subset X$. Then for all $p\in Y\setminus Z$, $\beta\in\mathbb{C}$ and $s\geq a_n/a_{n-1}$ we have:*

i) If $\beta \notin \mathbb{N}$, then:

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = \bigoplus_{j=0}^{a_{n-1}-1} \mathbb{C}\overline{(\phi_{v^j|x_0=0})_p}.$$

ii) If $\beta \in \mathbb{N}$, then there exists a unique $q \in \{0, \ldots, a_{n-1}-1\}$ such that $\frac{\beta-q}{a_{n-1}} \in \mathbb{N}$ and we have:

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = \bigoplus_{q\neq j=0}^{a_{n-1}-1} \mathbb{C}\overline{(\phi_{v^j|x_0=0})_p} \oplus \mathbb{C}\overline{(\phi_{\widetilde{v^q}|x_0=0})_p}.$$

Here $\overline{\phi}$ stands for the class modulo $\mathcal{O}_{X|Y,p}$ of $\phi \in \mathcal{O}_{\widehat{X|Y},p}(s)$.

Remark 5.2.6. *We can also compute the holomorphic solutions of $\mathcal{M}_A(\beta)$ at any point in $X \setminus Y$ for $A = (a_1\ a_2\ \ldots\ a_n)$ with $0 < a_1 < a_2 < \ldots < a_n$ and for any $\beta \in \mathbb{C}$, where $Y = \{x_n = 0\} \subset X = \mathbb{C}^n$ (see [FC$_1$08, Sec. 2.1] and Remark 5.1.25). As in the beginning of Section 5.2 let us consider the auxiliary matrix $A' = (1\ a_1\ a_2\ \ldots\ a_n)$ and the notation therein.*

Let us consider the vectors $w^j = (j, 0, \ldots, 0, \frac{\beta-j}{a_n}) \in \mathbb{C}^{n+1}$, $j = 0, 1, \ldots, a_n - 1$. Then the germs at $p' = (0, p) \in X' \setminus Y'$ (with $p \in X \setminus Y$) of the series solutions $\{\phi_{w^j} : j = 0, 1, \ldots, a_n - 1\}$ is a basis of $\mathcal{E}xt^0_{\mathcal{D}'}(\mathcal{M}_{A'}(\beta), \mathcal{O}_{X'})_{p'}$. Taking

$$\{\phi_{w^j|x_0=0} : j = 0, 1, \ldots, a_n - 1\}$$

we get a basis of $\mathcal{E}xt^0_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_X)_p$ for $\beta \in \mathbb{C}$ such that $\beta \notin \mathbb{N} \setminus \mathbb{N}A$ at any point $p \in X \setminus Y$. When $\beta \in \mathbb{N} \setminus \mathbb{N}A$ we can proceed as in Remark 5.2.4.

5.3 Remarks and conclusions

1) In Sections 5.1 and 5.2 we have proved that the irregularity complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is zero for $1 \leq s < a_n/a_{n-1}$ and concentrated in degree 0 for $a_n/a_{n-1} \leq s \leq \infty$ (see Theorems 5.1.1 and 5.2.3). Here A is a row integer matrix $(a_1\, a_2 \cdots a_n)$ with $0 < a_1 < a_2 < \cdots < a_n$ and β is a parameter in $\mathbb{C}$. We have reduced the case $a_1 > 1$ to the one where $a_1 = 1$ and then to the two dimensional case treated in Chapter 4.

2) We have described a basis of $\mathcal{E}xt^0_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ for $p \in Y \setminus Z$ and $a_n/a_{n-1} \leq s \leq \infty$ (see Theorems 5.1.23 and 5.2.5). Here $Y = \{x_n = 0\} \subset X = \mathbb{C}^n$ and $Z = \{x_{n-1} = 0\} \subset X$. From the form of the basis it is easy to see that the eigenvalues of the corresponding monodromy, with respect to Z, are simply $\exp(\frac{2\pi i(\beta-k)}{a_{n-1}})$ for $k = 0, \ldots, a_{n-1} - 1$. Notice that for $\beta \in \mathbb{Z}$ one eigenvalue (the one corresponding to the unique $k = 0, \ldots, a_{n-1} - 1$ such that $\frac{\beta-k}{a_{n-1}} \in \mathbb{Z}$) is just 1. See Remark 5.2.4 for notations.

Chapter 6

Some results regarding the irregularity of $\mathcal{M}_A(\beta)$

Let $A = (a_1 \cdots a_n)$ be a full rank matrix with columns $a_j \in \mathbb{Z}^d$ and $\beta \in \mathbb{C}^d$. This Chapter is devoted to the study of the Gevrey solutions of $\mathcal{M}_A(\beta)$ along coordinate subspaces and their relation with the irregularity of $\mathcal{M}_A(\beta)$ with respect to such subspaces.

Let us first introduce some notation. For any set $\tau \subseteq \{1, \ldots, n\}$ we denote by $\operatorname{conv}(\tau)$ the convex hull of $\{a_i : \ i \in \tau\} \subseteq \mathbb{R}^d$ and by Δ_τ the convex hull of $\{a_i : \ i \in \tau\} \cup \{\mathbf{0}\} \subseteq \mathbb{R}^d$. We shall identify τ with the set $\{a_i : \ i \in \tau\}$ and with $\operatorname{conv}(\tau)$. We also denote by A_τ the matrix given by the columns of A indexed by τ. A coordinate subspace is denoted by $Y_\tau := \{x_i = 0 : \ i \notin \tau\}$ for some $\tau \subseteq \{1, \ldots, n\}$. We will also write $\overline{\tau} = \{1, \ldots, n\} \setminus \tau$ and $x_\tau = (x_i)_{i \in \tau}$. We shall use the notation $|\cdot|$ for the sum of the coordinates of a vector as well as for the modulus of a complex number.

The following definition will be used in Sections 6.1 and 6.2.

Definition 6.0.1. *A formal series*

$$f = \sum_{m \in \mathbb{N}^{n-r}} f_m(x_\tau) x_{\overline{\tau}}^m \in \mathbb{C}\{x_\tau - p_\tau\}[[x_{\overline{\tau}}]]$$

is said to be Gevrey of multi-order $\mathbf{s} = (s_i)_{i \notin \tau} \in \mathbb{R}^{n-r}$ along Y_τ at $p \in Y_\tau$ if the series

$$\rho_{\mathbf{s}}^\tau(f) := \sum_{m \in \mathbb{N}^{n-r}} \frac{f_m(x_\tau)}{m!^{\mathbf{s}-\mathbf{1}}} x_{\overline{\tau}}^m$$

is convergent at p. Here we denote $m!^{\mathbf{s}-\mathbf{1}} = \prod_{i \notin \tau} (m_i!)^{s_i - 1}$.

Remark 6.0.2. *Notice that any Gevrey series of multi-order $\mathbf{s} = (s_i)_{i \notin \tau}$ along Y_τ at $p \in Y_\tau$ is also a Gevrey series of order $s = \max\{s_i : \ i \notin \tau\}$ along Y_τ at p (see Definition 2.2.2).*

6.1 Gevrey solutions of $\mathcal{M}_A(\beta)$ associated with a $(d-1)$–simplex.

Fix a set $\sigma \subseteq \{1,\ldots,n\}$ with cardinality d and $\det(A_\sigma) \neq 0$ throughout this section. Then Δ_σ is a d-simplex and σ is a $(d-1)$-simplex. The normalized volume of Δ_σ with respect to $\mathbb{Z}A$ is

$$\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) = \frac{d!\,\mathrm{vol}(\Delta_\sigma)}{[\mathbb{Z}^d : \mathbb{Z}A]} = \frac{|\det(A_\sigma)|}{[\mathbb{Z}^d : \mathbb{Z}A]}$$

where $\mathrm{vol}(\Delta_\sigma)$ denotes the Euclidean volume of Δ_σ. The aims of this section are: 1) to explicitly construct $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ linearly independent formal solutions of $\mathcal{M}_A(\beta)$ along the subspace $Y_\sigma = \{x_i = 0 :\ i \notin \sigma\}$ at any point of $Y_\sigma \cap \{x_j \neq 0 :\ j \in \sigma\}$ and 2) to prove that these series are Gevrey series along Y_σ of multi-order $(s_i)_{i\notin\sigma}$ with $s_i = |A_\sigma^{-1} a_i|$.

We reorder the variables in order to have $\sigma = \{1,\ldots,d\}$ for simplicity. Then a basis of $\mathrm{Ker}(A) = \{u \in \mathbb{Q}^n :\ Au = 0\}$ is given by the columns of the matrix:

$$B_\sigma = \begin{pmatrix} -A_\sigma^{-1} A_{\overline{\sigma}} \\ I_{n-d} \end{pmatrix} = \begin{pmatrix} -A_\sigma^{-1} a_{d+1} & -A_\sigma^{-1} a_{d+2} & \cdots & -A_\sigma^{-1} a_n \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & 1 \end{pmatrix}$$

Set

$$v^{\mathbf{k}} = (A_\sigma^{-1}(\beta - \sum_{i\notin\sigma} k_i a_i), \mathbf{k})$$

and observe that $Av^{\mathbf{k}} = \beta$ for all $\mathbf{k} = (k_i)_{i\notin\sigma} \in \mathbb{N}^{n-d}$. Hence, according to Lemma 1 in Section 1.1 of [GZK89], we have that $\varphi_{v^{\mathbf{k}}}$ (see (3.3)) is annihilated by the operators (3.1) and (3.2).

Set $\Lambda_{\mathbf{k}} := \{\mathbf{k}+\mathbf{m} = (k_i+m_i)_{i\in\overline{\sigma}} \in \mathbb{N}^{n-d} :\ \sum_{i\in\overline{\sigma}} a_i m_i \in \mathbb{Z}A_\sigma\}$. It is clear that

$$\varphi_{v^{\mathbf{k}}} = x_\sigma^{A_\sigma^{-1}\beta} \sum_{\mathbf{k}+\mathbf{m}\in\Lambda_{\mathbf{k}}} \frac{x_\sigma^{-A_\sigma^{-1}(\sum_{i\notin\sigma}(k_i+m_i)a_i)} x_{\overline{\sigma}}^{\mathbf{k}+\mathbf{m}}}{\Gamma(A_\sigma^{-1}(\beta - \sum_{i\notin\sigma}(k_i+m_i)a_i) + \mathbf{1})(\mathbf{k}+\mathbf{m})!}$$

is a formal series along $Y_\sigma = \{x_i = 0 :\ i \notin \sigma\}$ at any point of $Y_\sigma \cap \{x_j \neq 0 :\ j \in \sigma\}$ because the coefficient of $x_{\overline{\sigma}}^{\mathbf{k}+\mathbf{m}}$ defines a germ of holomorphic function at any point of $Y_\sigma \cap \{x_j \neq 0 :\ j \in \sigma\}$. Notice that $\varphi_{v^{\mathbf{k}}}$ is zero if and only if for all $\mathbf{m} \in \Lambda_{\mathbf{k}}$, $A_\sigma^{-1}(\beta - \sum_{i\notin\sigma}(k_i+m_i)a_i)$ has at least one negative integer coordinate.

Let us consider the lattice $\mathbb{Z}\sigma = \mathbb{Z}A_\sigma = \sum_{i\in\sigma} \mathbb{Z}a_i$ contained in $\mathbb{Z}A$.

Lemma 6.1.1. *The following statements are equivalent for all* $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}^{n-d}$*:*

1) $v^{\mathbf{k}} - v^{\mathbf{k}'} \in \mathbb{Z}^n$

2) $[A_{\overline{\sigma}}\mathbf{k}] = [A_{\overline{\sigma}}\mathbf{k}']$ *in* $\mathbb{Z}A/\mathbb{Z}\sigma$.

3) $\Lambda_{\mathbf{k}} = \Lambda_{\mathbf{k}'}$.

Lemma 6.1.2. *We have the equality:*

$$\{\Lambda_{\mathbf{k}} :\ \mathbf{k} \in \mathbb{Z}^{n-d}\} = \{\Lambda_{\mathbf{k}} :\ \mathbf{k} \in \mathbb{N}^{n-d}\}$$

and the cardinality of this set is $[\mathbb{Z}A : \mathbb{Z}\sigma] = \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$.

Proof. The equality is clear because $A_{\overline{\sigma}}\mathbf{c} \in \mathbb{Z}\sigma$ for $\mathbf{c} = |\det(A_\sigma)| \cdot (1, \dots, 1) \in (\mathbb{N}^*)^{n-d}$ and then for any $\mathbf{k} \in \mathbb{Z}^{n-d}$ there exists $\alpha \in \mathbb{N}$ such that $\mathbf{k} + \alpha\mathbf{c} \in \mathbb{N}^{n-d}$ and $\Lambda_{\mathbf{k}} = \Lambda_{\mathbf{k}+\alpha\mathbf{c}}$.

$\forall\overline{\lambda} \in \mathbb{Z}A/\mathbb{Z}\sigma$ there exists $\mathbf{k} \in \mathbb{Z}^{n-d}$ with $\overline{A_{\overline{\sigma}}\mathbf{k}} = \overline{\lambda} \in \mathbb{Z}A/\mathbb{Z}\sigma$. Then by the equivalence of 2) and 3) in Lemma 6.1.1 we have that $\{\Lambda_{\mathbf{k}} :\ \mathbf{k} \in \mathbb{Z}^{n-d}\}$ has the same cardinality as the finite group $\mathbb{Z}A/\mathbb{Z}\sigma$. □

Remark 6.1.3. *Notice that, for all* $\mathbf{k}, \mathbf{k}' \in \mathbb{N}^{n-d}$ *such that* $v^{\mathbf{k}} - v^{\mathbf{k}'} \in \mathbb{Z}^n$ *we have that* $\varphi_{v^{\mathbf{k}}} = \varphi_{v^{\mathbf{k}'}}$ *and in the other case we have that* $\varphi_{v^{\mathbf{k}}}$, $\varphi_{v^{\mathbf{k}'}}$ *have disjoint supports.*

Remark 6.1.4. *One may consider* $\mathbf{k}(1), \dots, \mathbf{k}(r) \in \mathbb{N}^{n-d}$ *such that*

$$\mathbb{Z}A/\mathbb{Z}\sigma = \{[A_{\overline{\sigma}}\mathbf{k}(i)] :\ i = 1, \dots, r\}$$

with $r = [\mathbb{Z}A : \mathbb{Z}\sigma]$. *Then the set in Lemma 6.1.2 is equal to* $\{\ \Lambda_{\mathbf{k}(i)} :\ i = 1, \dots, r\}$ *and it determines a partition of* $\mathbb{N}^{n-d}$, *i.e.,* $\Lambda_{\mathbf{k}(i)} \cap \Lambda_{\mathbf{k}(j)} = \emptyset$ *if* $i \neq j$ *and* $\cup_{i=1}^{r}\Lambda_{\mathbf{k}(i)} = \mathbb{N}^{n-d}$.

We have described $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ formal solutions of $\mathcal{M}_A(\beta)$ along Y_σ associated with a simplex σ having pairwise disjoint supports. Thus, they are linearly independent if none of them is zero.

Remark 6.1.5. *For very generic* β *(see Definition 3.2.4) we have that* $v^{\mathbf{k}}$ *does not have any negative integer coordinate for all* $\mathbf{k} \in \mathbb{N}^{n-d}$. *In particular, if* β *is very generic we have that* $\varphi_{v^{\mathbf{k}}} \neq 0$, $\forall\mathbf{k}$. *More precisely, very generic parameter vectors* β *lie in the complement of a locally finite arrangement of countable many hyperplanes that depend on* A.

From now on, we consider the Γ–series ϕ_v (3.5) defined in [SST00, p. 132-133] because they are not zero for any $v \in \mathbb{C}^n$ and they will be especially useful in Section 6.2.

Remark 6.1.6. *Observe that any $u \in L_A$ has the form $(-\sum_{j\notin\sigma} r_j A_\sigma^{-1} a_j, \mathbf{r})$ with $\mathbf{r} = (r_j)_{j\notin\sigma} \in \mathbb{Z}^{n-d}$ such that $A_{\overline{\sigma}}\mathbf{r} = \sum_{j\notin\sigma} r_j a_j \in \mathbb{Z}\sigma$. Then we can choose $\mathbf{k} \in \mathbb{N}^{n-d}$ such that $v^{\mathbf{k}}$ has minimal negative support because we do not change the class of $\sum_{j\notin\sigma} k_j a_j$ modulo $\mathbb{Z}A_\sigma$ when replacing $\mathbf{k}$ by $\mathbf{k}+\mathbf{r} \in \mathbb{N}^{n-d}$. Then the new series $\phi_\sigma^{\mathbf{k}} := \phi_{v^{\mathbf{k}}} \neq 0$ has the form:*

$$\phi_\sigma^{\mathbf{k}} = \sum_{\mathbf{k}+\mathbf{m}\in S_{\mathbf{k}}} \frac{[v^{\mathbf{k}}]_{u(\mathbf{m})_-}}{[v^{\mathbf{k}} + u(\mathbf{m})]_{u(\mathbf{m})_+}} x^{v^{\mathbf{k}}+u(\mathbf{m})}$$

where

$$S_{\mathbf{k}} := \{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k}} : \operatorname{nsupp}(v^{\mathbf{k}+\mathbf{m}}) = \operatorname{nsupp}(v^{\mathbf{k}})\} \subseteq \Lambda_{\mathbf{k}}$$

and $u(\mathbf{m}) = (-\sum_{i\notin\sigma} m_i A_\sigma^{-1} a_i, \mathbf{m})$ for $\mathbf{m} = (m_i)_{i\notin\sigma} \in \mathbb{Z}^{n-d}$. It is clear that $\mathbf{k}+\mathbf{m} \in S_{\mathbf{k}}$ if and only if $v^{\mathbf{k}+\mathbf{m}} \in v^{\mathbf{k}} + N_{v^{\mathbf{k}}}$.

Remark 6.1.7. *Using that $S_{\mathbf{k}} \subseteq \Lambda_{\mathbf{k}}$, $\forall \mathbf{k} \in \mathbb{N}^{n-d}$, and Remark 6.1.4 we have that two series in $\{\phi_\sigma^{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^{n-d}\}$ are either equal up to multiplication by a nonzero scalar or they have disjoint supports. Thus, the set $\{\phi_\sigma^{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^{n-d}\}$ has $\operatorname{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ linearly independent formal series solutions of $\mathcal{M}_A(\beta)$ along Y_σ at any point of $Y_\sigma \cap \{x_j \neq 0 : j \in \sigma\}$ for all $\beta \in \mathbb{C}^d$.*

Example 6.1.8. *Let $A = (a_1\ a_2\ a_3) \in \mathbb{Z}^{2\times 3}$ be the matrix with columns:*

$$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad a_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The kernel of A is generated by $u = (6, 1, -2)$ and so $L_A = \mathbb{Z}u$. Then the hypergeometric system associated with A and $\beta \in \mathbb{C}^2$ is generated by the differential operators:

$$\Box_u = \partial_1^6 \partial_2 - \partial_3^2,\ E_1 - \beta_1 = x_1\partial_1 + 3x_3\partial_3 - \beta_1,\ E_2 - \beta_2 = 2x_2\partial_2 + x_3\partial_3 - \beta_2.$$

In this example $\mathbb{Z}A = \mathbb{Z}^2$, A is pointed and $\sigma = \{1, 2\}$ is a simplex with normalized volume $\operatorname{vol}_{\mathbb{Z}A}(\Delta_\sigma) = |\det(A_\sigma)| = 2$ (see Figure 6.1).

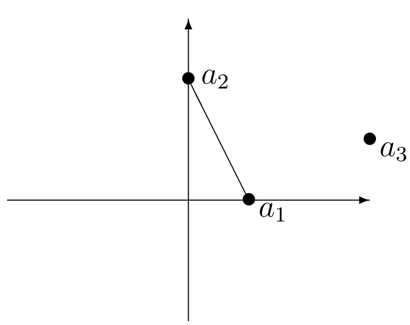

Figure 6.1

Two convenient vectors associated with σ are

$$v^0 = (\beta_1, \beta_2/2, 0) \text{ and } v^1 = (\beta_1 - 3, (\beta_2 - 1)/2, 1).$$

The associated series:

$$\phi_{v^0} = \sum_{m \geq 0} \frac{[\beta_1]_{6m} [\beta_2/2]_m}{(2m)!} x_1^{\beta_1 - 6m} x_2^{\beta_2/2 - m} x_3^{2m}$$

and

$$\phi_{v^1} = \sum_{m \geq 0} \frac{[\beta_1 - 3]_{6m} [(\beta_2 - 1)/2]_m}{(2m+1)!} x_1^{\beta_1 - 3 - 6m} x_2^{(\beta_2 - 1)/2 - m} x_3^{1+2m}$$

are formal series along $Y_\sigma = \{x_3 = 0\}$ at any point of $Y_\sigma \cap \{x_1 x_2 \neq 0\}$ that are annihilated by the Euler operators $E_1 - \beta_1$, $E_2 - \beta_2$ because $Av^k = \beta$ and by the toric operator $\Box_u$ since v^k has minimal negative support for all $\beta \in \mathbb{C}^2$ for $k = 0, 1$.

The following Lemma is very related to [OT07, Lemma 1], [GZK89, Proposition 1, Section 1.1.] and [PST05, Proposition 5], that deal with the convergence of Γ-series that are solutions of regular hypergeometric systems.

Lemma 6.1.9. *Assume that $\{b_i\}_{i=d+1}^n$ is a set of vectors in $\mathbb{Q}^d \times \mathbb{N}^{n-d}$, $\mathbf{k} \in \mathbb{Z}^{n-d}$. Let us denote $u(\mathbf{m}) = \sum_{i=d+1}^n m_i b_i$ and consider a set $D_\mathbf{k} \subseteq \{\mathbf{k} + \mathbf{m} \in \mathbb{N}^{n-d} : u(\mathbf{m}) \in \mathbb{Z}^n\}$ and a vector $v \in \mathbb{C}^n$ such that* $\operatorname{nsupp}(v + u(\mathbf{m})) = \operatorname{nsupp}(v)$ *for any $\mathbf{m} \in D_\mathbf{k} - \mathbf{k}$. Then for all $\mathbf{s} \in \mathbb{R}^{n-d}$ the following statements are equivalent:*

1) $\displaystyle\sum_{\mathbf{k}+\mathbf{m} \in D_\mathbf{k}} \frac{[v]_{u(\mathbf{m})_-}}{[v + u(\mathbf{m})]_{u(\mathbf{m})_+}} y^{\mathbf{k}+\mathbf{m}}$ *is Gevrey of multi-order* $\mathbf{s}$ *along $y = 0$.*

2) $\displaystyle\sum_{\mathbf{k}+\mathbf{m} \in D_\mathbf{k}} \frac{u(\mathbf{m})_-!}{u(\mathbf{m})_+!} y^{\mathbf{k}+\mathbf{m}}$ *is Gevrey of multi-order* $\mathbf{s}$ *along $y = 0$.*

3) $\displaystyle\sum_{\mathbf{k}+\mathbf{m} \in D_\mathbf{k}} \prod_{j=d+1}^{n} (k_j + m_j)!^{-|b_j|} y^{\mathbf{k}+\mathbf{m}}$ *is Gevrey of multi-order* $\mathbf{s}$ *along $y = 0$.*

In particular, for $\mathbf{s} = (s_{d+1}, \ldots, s_n)$ with $s_i = 1 - |b_i|$, $i = d+1 \ldots, n$, 1),2) and 3) are satisfied. Moreover, 1), 2) and 3) are also equivalent if we write order *s instead of* multi-order $\mathbf{s}$ *and all these series are Gevrey of order $s = \max_i \{1 - |b_i|\}$.*

Proof. $\forall \alpha \in \mathbb{C}$, $\forall m \in \mathbb{N}$ with $[\alpha]_m \neq 0$ there exists $C, D > 0$ such that:

$$C^m |[\alpha]_m| \leq m! \leq |[\alpha]_m| D^m \tag{6.1}$$

For the proof of (6.1) it is enough to consider $c_m := [\alpha]_m/m!$ and see that $\lim_{m\to\infty} |c_{m+1}/c_m| = 1$. The proof for (6.1) with $\alpha + m$ instead of α is analogous. It follows that 1) and 2) are equivalent.

We can use Stirling's formula $m! \sim \sqrt{2\pi m}(m/e)^m$ in order to prove that $\forall m \in \mathbb{N}$, $\forall q \in \mathbb{Q}_+$ with $qm \in \mathbb{N}$, there exist $C', D' > 0$ verifying:

$$(C')^m m!^q \leq (qm)! \leq (D')^m m!^q \tag{6.2}$$

Take $\lambda \in \mathbb{N}^*$ such that $\lambda b_i \in \mathbb{Z}^n$ for all $i = d+1, \ldots, n$. Then by (6.2) we have 2) if and only if the series $\sum_{\mathbf{k}+\mathbf{m}\in D_{\mathbf{k}}} \frac{((\lambda u(\mathbf{m})_-)!)^{1/\lambda}}{((\lambda u(\mathbf{m})_+)!)^{1/\lambda}} y^{\mathbf{k}+\mathbf{m}}$ is Gevrey of multi-order $\mathbf{s}$ along $y = 0$.

For the rest of the proof we assume for simplicity that $D_{\mathbf{k}} \subseteq (\mathbf{k} + \mathbb{N}^{n-d}) \cap \mathbb{N}^{n-d}$. The equivalence of 2) and 3) can be proven without this assumption but it is necessary to distinguish more cases.

Observe that $\lambda u(m)_+ - \lambda u(m)_- = \sum_{i=d+1}^n \lambda(b_i)_+ m_i - \sum_{i=d+1}^n \lambda(b_i)_- m_i$ and $u(m)_+, u(m)_-$, $\sum_{i=d+1}^n \lambda(b_i)_+ m_i$, $\sum_{i=d+1}^n \lambda(b_i)_- m_i \in \mathbb{N}^n$. However, $u(m)_+$, $u(m)_-$ have disjoint supports while $\sum_{i=d+1}^n \lambda(b_i)_+ m_i$ and $\sum_{i=d+1}^n \lambda(b_i)_- m_i$ do not have disjoint supports in general.

On the other hand, for all $m, n \in \mathbb{N}$ with $n \leq m$ we have that:

$$(m-n)! \leq \frac{m!}{n!} \leq 2^m (m-n)! \tag{6.3}$$

Then by (6.3) we have 2) if and only if

$$\sum_{\mathbf{k}+\mathbf{m}\in S} \left(\frac{(\sum_{i=d+1}^n \lambda(b_i)_- m_i)!}{(\sum_{i=d+1}^n \lambda(b_i)_+ m_i)!} \right)^{1/\lambda} y^{\mathbf{k}+\mathbf{m}} \tag{6.4}$$

is Gevrey of multi-order $\mathbf{s}$ along $y = 0$.

If we replace m by $m + n$ in (6.3) and multiply by $n!$ we obtain a formula that can be generalized by induction. We obtain that $\forall m_{d+1}, \ldots, m_n \in \mathbb{N}$ there exist $C'', D'' > 0$ such that:

$$(C'')^{\sum_j m_j} \prod_i m_i! \leq (\sum m_i)! \leq (D'')^{\sum_j m_j} \prod_i m_i! \tag{6.5}$$

A combination of (6.5) and (6.2) proves that 3) holds if and only if (6.4) is Gevrey of multi-order $\mathbf{s}$ along $y = 0$.

Finally, it is clear that 3) is true for $s_i = 1 - |b_i|$, $i = d+1, \ldots, n$. □

Remark 6.1.10. *From the proof of the equivalence of 1) and 2) in Lemma 6.1.9 it can be deduced that after applying $\rho_{\mathbf{s}}$ to the series in 1) there exists an open set W such that this series converges in W and $0 \in W$ does not depend on v but on $D_k - k$.*

Consider $\mathbf{s} = (s_j)_{j \notin \sigma}$ with

$$s_j := |A_\sigma^{-1} a_j|, \; j \notin \sigma$$

and $s = \max_i \{s_i\}$ throughout this section.

Lemma 6.1.11. *For all $\mathbf{k} \in \mathbb{N}^{n-d}$ the series*

$$\psi_\sigma^{\mathbf{k}} := \sum_{\mathbf{k}+\mathbf{m} \in S_k} \frac{[v^k]_{u(m)_-}}{[v^{\mathbf{k}} + u(\mathbf{m})]_{u(\mathbf{m})_+}} y^{\mathbf{k}+\mathbf{m}}$$

is Gevrey of multi-order $\mathbf{s}$ along $y = \mathbf{0} \in \mathbb{C}^{n-d}$. Moreover, if β is very generic then it has Gevrey index s along $y = \mathbf{0} \in \mathbb{C}^{n-d}$.

Proof. It follows from Lemma 6.1.9 (if we take b_{d+i} equal to the i-th column of B_σ, $D_{\mathbf{k}} = S_{\mathbf{k}}$ and $v = v^{\mathbf{k}}$) that $\psi_\sigma^{\mathbf{k}}$ is Gevrey of multi-order $\mathbf{s}$ along $y = \mathbf{0}$.

If β is very generic we have that $S_{\mathbf{k}} = \Lambda_{\mathbf{k}}$ and it is obvious that the series in 3) of Lemma 6.1.9 has Gevrey index s in this case. □

Corollary 6.1.12. *The series $\phi_\sigma^{\mathbf{k}}$ is Gevrey of multi-order $\mathbf{s} = (s_j)_{j \notin \sigma}$ along Y_σ at any point of $Y_\sigma \cap \{x_i \neq 0 : \; i \in \sigma\}$. If β is very generic then it is Gevrey with index s along Y_σ.*

Proof. If we take $y = (y_j)_{j \notin \sigma}$ with $y_j := x_\sigma^{-A_\sigma^{-1} a_j} x_j$, $j \notin \sigma$, then $\phi_\sigma^{\mathbf{k}}(x) = x_\sigma^{A_\sigma^{-1}\beta} \psi_\sigma^{\mathbf{k}}(y)$ and the result follows from Lemma 6.1.11. □

Example 6.1.13. ***(Continuation of Example 6.1.8)*** *We have that*

$$\rho_s(\phi_{v^0}) = x_1^{\beta_1} x_2^{\beta_2/2} \sum_{m \geq 0} \frac{[\beta_1]_{6m} [\beta_2/2]_m}{(2m)!^s} \left(\frac{x_3^2}{x_1^6 x_2} \right)^m$$

It is easy to see that $\rho_s(\phi_{v^0})$ has a nonempty domain of convergence if and only if $s \geq 7/2$ when $\beta_1, \beta_2/2 \notin \mathbb{N}$ (use D'Alembert criterion for the series in one variable $y = x_3^2/(x_1^6 x_2)$). Then ϕ_{v^0} is a Gevrey series solution of $\mathcal{M}_A(\beta)$ with index $s = 7/2$ along $Y_\sigma = \{x_3 = 0\}$ at any point of $Y_\sigma \cap \{x_1 x_2 \neq 0\}$. Nevertheless, ϕ_{v^0} is a finite sum if either $\beta_1 \in \mathbb{N}$ or $\beta_2/2 \in \mathbb{N}$ and so it has the same convergence domain as the (multi-valued) function $x_1^{\beta_1} x_2^{\beta_2/2}$. If both $\beta_1, \beta_2/2 \in \mathbb{N}$, then ϕ_{v^0} is a polynomial.

Analogously, ϕ_{v^1} is a Gevrey series solution of order $s = 7/2$ along Y_σ at any point of $Y_\sigma \cap \{x_1 x_2 \neq 0\}$. It has Gevrey index $s = 7/2$ if $\beta_1 - 3, (\beta_2 - 1)/2 \notin \mathbb{N}$ and it is convergent otherwise.

Notice that $s = 7/2$ is the unique algebraic slope of $\mathcal{M}_A(\beta)$ along $Y_\sigma = \{x_3 = 0\}$ at $\mathbf{0} \in \mathbb{C}^3$ (see [SW08] or [Har03]).

The convergence domain of $\rho_{\mathbf{s}}^{\emptyset}(\psi_{\sigma}^{\mathbf{k}})$ contains $\{y \in \mathbb{C}^{n-d} : \ |y_j| < R, \ j \notin \sigma\}$ for certain $R > 0$. In particular, $\rho_{\mathbf{s}}^{\sigma}(\phi_{v^{\mathbf{k}}})$ converges in

$$\{x \in \mathbb{C}^n : \prod_{i \in \sigma} x_i \neq 0, \ |x_j| < R|x_{\sigma}^{A_{\sigma}^{-1} a_j}|, \ \forall j \notin \sigma\}.$$

The unique hyperplane that contains σ is

$$H_{\sigma} = \{\mathbf{y} \in \mathbb{R}^d : \ |A_{\sigma}^{-1}\mathbf{y}| = 1\}$$

and we denote by $H_{\sigma}^{-} := \{\mathbf{y} \in \mathbb{R}^d : \ |A_{\sigma}^{-1}\mathbf{y}| < 1\}$ (resp. by $H_{\sigma}^{+} := \{\mathbf{y} \in \mathbb{R}^d : \ |A_{\sigma}^{-1}\mathbf{y}| > 1\}$) the open affine half-space that contains (resp. does not contain) the origin $\mathbf{0} \in \mathbb{R}^d$.

Recall that $\mathbf{s} = (s_i)_{i \notin \sigma}$ where $s_i = |A_{\sigma}^{-1} a_i|$ is the unique rational number such that $a_i / s_i \in H_{\sigma}$. Moreover, $s_i > 1$ (resp. $s_i < 1$) if and only if $a_i \in H_{\sigma}^{+}$ (resp. $a_i \in H_{\sigma}^{-}$). Taking the set

$$\tau = \{i : \ a_i \notin H_{\sigma}^{+}\}$$

and $\mathbf{s}' = (s_i)_{i \notin \tau}$ we have that $\rho_{\mathbf{s}'}^{\tau}(\phi_{v^{\mathbf{k}}})$ converges in the open set

$$U_{\sigma}' := \{x \in \mathbb{C}^n : \prod_{i \in \sigma} x_i \neq 0, \ |x_j| < R|x_{\sigma}^{A_{\sigma}^{-1} a_j}|, \ \forall a_j \in (H_{\sigma} \setminus \sigma) \cup H_{\sigma}^{+}\}.$$

This implies that $\phi_{v^{\mathbf{k}}}$ is Gevrey of multi-order $\mathbf{s}'$ along Y_{τ} at any point of $U_{\sigma}' \cap Y_{\tau}$. Then, if we consider

$$U_{\sigma} := \{x \in \mathbb{C}^n : \prod_{i \in \sigma} x_i \neq 0, \ |x_j| < R|x_{\sigma}^{A_{\sigma}^{-1} a_j}|, \ \forall a_j \in H_{\sigma} \setminus \sigma\} \tag{6.6}$$

the following result is obtained.

Theorem 6.1.14. *For any set ς with $\sigma \subseteq \varsigma \subseteq \tau$ the series*

$$\phi_{\sigma}^{\mathbf{k}} = \sum_{\mathbf{k}+\mathbf{m} \in S_{\mathbf{k}}} \frac{[v^{\mathbf{k}}]_{u(\mathbf{m})_{-}}}{[v^{\mathbf{k}} + u(\mathbf{m})]_{u(\mathbf{m})_{+}}} \, x^{v^{\mathbf{k}} + u(\mathbf{m})}$$

is a Gevrey series solution of $\mathcal{M}_A(\beta)$ of order $s = \max\{s_i = |A_{\sigma}^{-1} a_i| : \ i \notin \sigma\}$ along Y_{ς} at any point of $Y_{\varsigma} \cap U_{\sigma}$. If β is very generic then s is its Gevrey index.

Remark 6.1.15. *If $H_{\sigma} \cap \{a_i : \ i = 1, \ldots, n\} = \sigma$ then $U_{\sigma} = \{\prod_{i \in \sigma} x_i \neq 0\}$.*

Remark 6.1.16. *Recall that in Theorem 6.1.14 the vector $v^{\mathbf{k}} = (A_{\sigma}^{-1}(\beta - A_{\bar{\sigma}}\mathbf{k}), \mathbf{k})$ has minimal negative support because we have chosen $\mathbf{k} \in \Lambda_{\mathbf{k}}$ this way (see Remark 6.1.6). This guarantees that $\phi_{v^{\mathbf{k}}}$ is annihilated by I_A by [SST00, Section 3.4] (see Proposition 3.2.2). However, this series is also Gevrey of order s when $\mathbf{k}$ does not satisfy this condition.*

6.2 Slopes of $\mathcal{M}_A(\beta)$ associated with a $(d-1)$–simplex.

In the context of Section 6.1 we fix a simplex $\sigma \subseteq A$ with $\det(A_\sigma) \neq 0$ and consider $\mathbf{s} = (s_i)_{i \notin \sigma}$ where $s_i = |A_\sigma^{-1} a_i|$. We consider $\tau = \{j : \; a_j \notin H_\sigma^+\} \supseteq \sigma$ and the coordinate subspace $Y_\tau = \{x_j = 0 : \; j \notin \tau\}$ in this section.

Our purpose here is to construct one nonzero Gevrey series solution of $\mathcal{M}_A(\beta)$ in $(\mathcal{O}_{X|Y_\tau}(s)/\mathcal{O}_{X|Y_\tau}(< s))_p$ for $p \in Y_\tau \cap U_\sigma$ with support contained in the set $\Lambda_{\mathbf{k}} \subseteq \mathbb{N}^{n-d}$ in the partition of $\mathbb{N}^{n-d}$ (see Remark 6.1.4) for all $\beta \in \mathbb{C}^d$. In particular we will prove the following result:

Proposition 6.2.1. *For $s = \max\{s_i = |A_\sigma^{-1} a_i| : \; i \notin \sigma\}$, for all $p \in Y_\tau \cap U_\sigma$ and for all $\beta \in \mathbb{C}^d$:*

$$\dim(\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y_\tau}(s)/\mathcal{O}_{X|Y_\tau}(< s)))_p \geq \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma).$$

As a consequence of Proposition 6.2.1 and Lemma 8.0.8, we obtain the following result that justifies the name of this section:

Corollary 6.2.2. *If Y_τ is a coordinate hyperplane (equivalently, the cardinality of τ is $n-1$) and $s = |A_\sigma^{-1} a_{\overline{\tau}}| > 1$ then s is an analytic slope of $\mathcal{M}_A(\beta)$ along Y_τ at any point in the closure of $Y_\tau \cap U_\sigma$.*

Remark 6.2.3. *By Theorem 6.1.14 we only need Lemma 8.0.8 for the proof of Corollary 6.2.2 if β is not very generic.*

Remark 6.2.4. *Observe that $\mathbf{0}$ is in the closure of $Y_\tau \cap U_\sigma$. However, by Remark 8.0.9 we have that s is a slope along Y_τ at any point of Y_τ.*

Let us proceed with the construction of the announced series and the proof of Proposition 6.2.1.

We identify $\mathbf{k}+\mathbf{m} \in \mathbb{N}^{n-d}$ with $v^{\mathbf{k}+\mathbf{m}} = (A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m})), \mathbf{k}+\mathbf{m}) \in \mathbb{C}^d \times \mathbb{N}^{n-d}$ and establish a partition of $\Lambda_{\mathbf{k}}$ in terms of the negative support of the vector $v^{\mathbf{k}+\mathbf{m}} \in \mathbb{C}^d \times \mathbb{N}^{n-d}$ as follows. For any subset $\eta \subseteq \sigma$ set:

$$\Lambda_{\mathbf{k},\eta} := \{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k}} : \; \mathrm{nsupp}(A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m}))) = \eta\}.$$

Consider the set

$$\Omega_{\mathbf{k}} := \{\eta \subseteq \sigma : \; \Lambda_{\mathbf{k},\eta} \neq \emptyset\}.$$

Then it is clear that $\{\Lambda_{\mathbf{k},\eta} : \eta \in \Omega_{\mathbf{k}}\}$ is a partition of $\Lambda_{\mathbf{k}}$. Moreover $\Lambda_{\mathbf{k},\eta}$ is the intersection of a polytope with $\Lambda_{\mathbf{k}}$ because the conditions

$$\mathrm{nsupp}(A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m}))) = \eta$$

are equivalent to inequalities of type:

$$(A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m})))_i < 0$$

for $i \in \eta$ and

$$(A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m})))_j \geq 0$$

for $j \notin \eta$ such that $(A_\sigma^{-1}(\beta - A_{\overline{\sigma}}\mathbf{k}))_j \in \mathbb{Z}$.

For any $\eta \in \Omega_{\mathbf{k}}$ the series $\phi_{v^{\mathbf{k}+\mathbf{m}}}$ for $\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}$ depends on $\Lambda_{\mathbf{k},\eta}$ but not on $\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}$ up to multiplication by nonzero scalars. Let us fix any $\widetilde{\mathbf{k}} \in \Lambda_{\mathbf{k},\eta}$ and set:

$$\phi_{\mathbf{k},\eta} := \phi_{v^{\widetilde{\mathbf{k}}}}.$$

Observe that the support of the series $\phi_{\mathbf{k},\eta}$ is:

$$\operatorname{supp}(\phi_{\mathbf{k},\eta}) = \{v^{\mathbf{k}+\mathbf{m}} : \ \mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}\}.$$

If the set $\Omega_{\mathbf{k}}$ has only one element η then $\Lambda_{\mathbf{k}} = \Lambda_{\mathbf{k},\eta}$ and the series $\phi_\sigma^{\mathbf{k}} = \phi_{\mathbf{k},\eta}$ is a nonzero Gevrey series solution of $\mathcal{M}_A(\beta)$ in $\mathcal{O}_{X|Y_\tau}(s) \setminus \mathcal{O}_{X|Y_\tau}(<s)$ at any point of $Y_\tau \cap U_\sigma$ (see Theorem 6.1.14).

All the series in the finite set $\{\phi_{\mathbf{k},\eta} : \mathbf{k} \in \mathbb{N}^{n-d}, \eta \in \Omega_{\mathbf{k}}\}$ are Gevrey series along $Y_\sigma = \{x_i = 0 : \ i \notin \sigma\}$ with multi-order $\mathbf{s}$ at points of $Y_\sigma \cap \{x_j \neq 0 : \ j \in \sigma\}$. In fact, these series are Gevrey of order s along Y_τ at any point of $Y_\tau \cap U_\sigma$ and they are all annihilated by the Euler operators.

For all $\eta \in \Omega_{\mathbf{k}}$, the support of the series $\phi_{\mathbf{k},\eta}$ is $\operatorname{supp}(\phi_{\mathbf{k},\eta}) = \{v^{\mathbf{k}+\mathbf{m}} : \ \mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}\}$ and $\cup_{\eta\in\Omega_{\mathbf{k}}} \Lambda_{\mathbf{k},\eta} = \Lambda_{\mathbf{k}}$. Then there exists $\eta \in \Omega_{\mathbf{k}}$ such that $\phi_{\mathbf{k},\eta} \in \mathcal{O}_{X|Y_\tau}(s)$ has Gevrey index s. But a series ϕ_v is annihilated by I_A if and only if v has minimal negative support (see [SST00], Section 3.4.) so if we take $\eta' \in \Omega_{\mathbf{k}}$ with minimal cardinality then $\phi_{\mathbf{k},\eta'} \in \mathcal{O}_{X|Y_\tau}(s)$ is a solution of $\mathcal{M}_A(\beta)$. In general, we cannot take $\eta = \eta'$.

The following Lemma is the key of the proof of Proposition 6.2.1.

Lemma 6.2.5. *Consider an element η of the set*

$$\{\eta' \in \Omega_{\mathbf{k}} : \ \phi_{\mathbf{k},\eta'} \textit{ has Gevrey index } s\}$$

with minimal cardinality. Then $\square_u(\phi_{\mathbf{k},\eta}) \in \mathcal{O}_{X|Y_\tau}(<s)$ for all $u \in L_A$.

Proof. Consider $\Lambda_{\mathbf{k},\eta}$ with η as above and $u \in L_A$. Then there exists $\widetilde{\mathbf{m}} \in \mathbb{Z}^{n-d}$ such that $u = (-A_\sigma^{-1}A_{\overline{\sigma}}\widetilde{\mathbf{m}}, \widetilde{\mathbf{m}})$ and then

$$\square_u = \partial_\sigma^{(A_\sigma^{-1}A_{\overline{\sigma}}\widetilde{\mathbf{m}})_-}\partial_{\overline{\sigma}}^{\widetilde{\mathbf{m}}_+} - \partial_\sigma^{(A_\sigma^{-1}A_{\overline{\sigma}}\widetilde{\mathbf{m}})_+}\partial_{\overline{\sigma}}^{\widetilde{\mathbf{m}}_-}.$$

On the other hand, the series $\phi_{\mathbf{k},\eta}$ has the form:

$$\phi_{\mathbf{k},\eta} = \sum_{\mathbf{k}+\mathbf{m}\in\Lambda_{\mathbf{k},\eta}} c_{\mathbf{k}+\mathbf{m}} x_{\sigma}^{A_{\sigma}^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}+\mathbf{m}}$$

where $c_{\mathbf{k}+\mathbf{m}} \in \mathbb{C}$ verifies that $c_{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}/c_{\mathbf{k}+\mathbf{m}}$ is a rational function on $\mathbf{m}$ (recall that there exists $\widetilde{\mathbf{k}} \in \Lambda_{k,\eta}$ such that $c_{\mathbf{k}+\mathbf{m}} = \frac{[v^{\widetilde{\mathbf{k}}}]_{u(\mathbf{k}-\widetilde{\mathbf{k}}+\mathbf{m})_-}}{[v^{\mathbf{k}+\mathbf{m}}]_{u(\mathbf{k}-\widetilde{\mathbf{k}}+\mathbf{m})_+}}$ by definition of $\phi_{\mathbf{k},\eta}$).

A monomial $x^{v^{\mathbf{k}+\mathbf{m}}-u_-} = x^{v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}-u_+}$ appearing in $\Box_u(\phi_{\mathbf{k},\eta})$ comes from the monomials $x^{v^{\mathbf{k}+\mathbf{m}}}$ and $x^{v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}}$ after one applies ∂^{u_-} and ∂^{u_+} respectively.

If $\mathbf{k}+\mathbf{m}, \mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\eta}$ then the monomial $x^{v^{\mathbf{k}+\mathbf{m}}-u_-}$ appears in $\partial^{u_-}(\phi_{\mathbf{k},\eta})$ and $\partial^{u_+}(\phi_{\mathbf{k},\eta})$ with the same coefficients so it doesn't appear in the difference.

If $\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}$ but $\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \notin \Lambda_{\mathbf{k},\eta}$ (the case $\mathbf{k}+\mathbf{m} \notin \Lambda_{\mathbf{k},\eta}$ but $\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\eta}$ is analogous), we can distinguish two cases:

1) There exists i such that $v_i^{\mathbf{k}+\mathbf{m}} \in \mathbb{N}$ but $v_i^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}} < 0$ so $u_i = v_i^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}} - v_i^{\mathbf{k}+\mathbf{m}} < 0$. Then $\partial^{u_-}(x^{v^{\mathbf{k}+\mathbf{m}}}) = 0$ and $x^{v^{\mathbf{k}+\mathbf{m}}-u_-}$ does not appear in $\Box_u(\phi_{\mathbf{k},\eta})$.

2) We have $\operatorname{nsupp}(v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}) = \varsigma \subsetneq \operatorname{nsupp}(v^{\mathbf{k}+\mathbf{m}}) = \eta$. Then $[v^{\mathbf{k}+\mathbf{m}}]_{u_-} \neq 0$ and the coefficient of $x^{v^{\mathbf{k}+\mathbf{m}}-u_-}$ in $\Box_u(\phi_{\mathbf{k},\eta})$ is $c_{\mathbf{k}+\mathbf{m}}[v^{\mathbf{k}+\mathbf{m}}]_{u_-} \neq 0$. Furthermore, $\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\varsigma}$ with $\varsigma \in \Omega_{\mathbf{k}}$ such that $\phi_{\mathbf{k},\varsigma}$ is Gevrey of index $s' < s$ because we chose η that way.

By 1), 2) and the analogous cases when $\mathbf{k}+\mathbf{m} \notin \Lambda_{\mathbf{k},\eta}$ but $\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\eta}$, we have:

$$\Box_u(\phi_{\mathbf{k},\eta}) = \sum_{\varsigma'} \sum_{\substack{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}\in\Lambda_{\mathbf{k},\eta} \\ \mathbf{k}+\mathbf{m}\in\Lambda_{\mathbf{k},\varsigma'}}} c_{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}[v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}]_{u_+} x^{v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}-u_+} -$$

$$-\sum_{\varsigma} \sum_{\substack{\mathbf{k}+\mathbf{m}\in\Lambda_{\mathbf{k},\eta} \\ \mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}\in\Lambda_{\mathbf{k},\varsigma}}} c_{\mathbf{k}+\mathbf{m}}[v^{\mathbf{k}+\mathbf{m}}]_{u_-} x^{v^{\mathbf{k}+\mathbf{m}}-u_-} \tag{6.7}$$

Here, $\varsigma, \varsigma' \subseteq \eta$ varies in a subset of the finite set $\Omega_{\mathbf{k}}$ whose elements ς'' verify that the series $\phi_{\mathbf{k},\varsigma''}$ has Gevrey index $s'' < s$. Let us denote by $\widetilde{s} < s$ the maximum of these s''.

Since $c_{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}/c_{\mathbf{k}+\mathbf{m}}$, $[v^{\mathbf{k}+\mathbf{m}}]_{u_-}$ and $[v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}]_{u_+}$ are rational functions on $\mathbf{m}$ the series $\Box_u(\phi_{\mathbf{k},\eta})$ has Gevrey index at most the maximum of the Gevrey index of the series

$$\sum_{\varsigma'} \sum_{\substack{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}\in\Lambda_{\mathbf{k},\eta} \\ \mathbf{k}+\mathbf{m}\in\Lambda_{\mathbf{k},\varsigma'}}} c_{\mathbf{k}+\mathbf{m}} x^{v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}-u_+}$$

$$\sum_{\varsigma} \sum_{\substack{\mathbf{k}+\mathbf{m}\in\Lambda_{\mathbf{k},\eta} \\ \mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}\in\Lambda_{\mathbf{k},\varsigma}}} c_{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}} x^{v^{\mathbf{k}+\mathbf{m}}-u_-}$$

which is at most $\widetilde{s} < s$.

It follows that $\Box_u(\phi_{\mathbf{k},\eta}) \in \mathcal{O}_{X|Y_\tau}(< s)$ for all $u \in L_A$ while $\phi_{\mathbf{k},\eta}$ has Gevrey index s. $\Box$

Moreover the classes of the series $\{\phi_{\mathbf{k},\eta_{\mathbf{k}}} : \mathbf{k} \in \mathbb{N}^{n-d}\}$ (with $\eta_{\mathbf{k}} \in \Omega_{\mathbf{k}}$ chosen as η in Lemma 6.2.5) in $(\mathcal{O}_{X|Y_\tau}(s)/\mathcal{O}_{X|Y_\tau}(< s))_p$, $p \in Y_\tau \cap U_\sigma$, are linearly independent since the support of $\phi_{\mathbf{k},\eta}$ restricted to the variables x_i with $i \notin \sigma$ is $\Lambda_{\mathbf{k},\eta_{\mathbf{k}}} \subseteq \Lambda_{\mathbf{k}}$ and $\{\Lambda_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^{n-d}\}$ is a partition of $\mathbb{N}^{n-d}$. This finishes the proof of Proposition 6.2.1.

6.3 Slopes of $\mathcal{M}_A(\beta)$ along coordinate hyperplanes

In this section we will describe all the slopes of $\mathcal{M}_A(\beta)$ along coordinate hyperplanes. First, we recall here the definition of (A, L)-umbrella [SW08], but we will slightly modify the notation in [SW08] for technical reasons. Consider any full rank matrix $A = (a_1 \ \cdots \ a_n) \in \mathbb{Z}^{d\times n}$ and $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n_{>0}$.

Definition 6.3.1. *Set $a_j^{\mathbf{s}} := a_j/s_j$, $j = 1, \ldots, n$, and let*

$$\Delta_A^{\mathbf{s}} := \operatorname{conv}(\{a_i^{\mathbf{s}} : \ i = 1, \ldots, n\} \cup \{\mathbf{0}\})$$

be the so-called $(A, \mathbf{s})$-polyhedron.

The $(A, \mathbf{s})$-umbrella is the set $\Phi_A^{\mathbf{s}}$ of faces of $\Delta_A^{\mathbf{s}}$ which do not contain the origin. $\Phi_A^{\mathbf{s},q} \subseteq \Phi_A^{\mathbf{s}}$ denotes the subset of faces of dimension q for $q = 0, \ldots, d-1$.

The following statement is very similar to [SW08, Lemma 2.13]. The difference here is that we do not assume that A is pointed and that we just consider $\mathbf{s} \in \mathbb{R}^n$ such that $s_i > 0$ for all $i = 1, \ldots, n$. For this reason we slightly modify a part of the proof of [SW08, Lemma 2.13].

Lemma 6.3.2. *Let $\widetilde{I}_A^{\mathbf{s}}$ be the ideal of $\mathbb{C}[\xi_1, \ldots, \xi_n]$ generated by the following elements:*

i) $\xi_{i_1} \cdots \xi_{i_r}$ where $a_{i_1}/s_{i_1}, \ldots, a_{i_r}/s_{i_r}$ do not lie in a common facet of $\Phi_A^{\mathbf{s}}$.

ii) $\xi^{u_+} - \xi^{u_-}$ where $u \in L_A$ and $\operatorname{supp}(u)$ is contained in a facet of $\Phi_A^{\mathbf{s}}$.

Then $\widetilde{I}_A^{\mathbf{s}} = \sqrt{\operatorname{in}_{\mathbf{s}}(I_A)}$.

Proof. The proofs of [SW08, Lemma 2.12] and [SW08, Lemma 2.13] use [SW08, Lemma 2.8] and [SW08, Lemma 2.10], but they do no use that A is pointed elsewhere. We rewrite here the proofs of [SW08, Lemma 2.8] and [SW08, Lemma 2.10] without the pointed assumption on A but for $\mathbf{s} \in \mathbb{R}^n_{>0}$. Let us prove in particular that $\mathrm{in}_{\mathbf{s}}(I_A) \subseteq \widetilde{I}^{\mathbf{s}}_A \subseteq \sqrt{\mathrm{in}_{\mathbf{s}}(I_A)}$.

For the proof of the inclusion $\mathrm{in}_{\mathbf{s}}(I_A) \subseteq \widetilde{I}^{\mathbf{s}}_A$ we only need to prove that $\forall u \in \mathbb{Z}^n$ with $Au = 0$ then $\mathrm{in}_{\mathbf{s}}(\Box_u) \in \widetilde{I}^{\mathbf{s}}_A$.

If $\mathrm{supp}(u) \subseteq \tau \in \Phi^{\mathbf{s}}_A$ then $\exists h_\tau \in \mathbb{Q}^d$ such that $\langle h_\tau, a_i/s_i \rangle = 1$, $\forall i \in \tau$, i.e., $\langle h_\tau, a_i \rangle = s_i$, $\forall i \in \tau$. Since $Au = 0$ and $\mathrm{supp}(u) \subseteq \tau$ we have

$$0 = \langle h_\tau, Au \rangle = \langle h_\tau A, u \rangle = \sum_{i \in \tau} s_i u_i = \sum_{i=1}^{n} s_i u_i \tag{6.8}$$

so $\mathrm{in}_{\mathbf{s}}(\Box_u) = \xi^{u_+} - \xi^{u_-}$ which lies in $\widetilde{I}^{\mathbf{s}}_A$ by definition.

Assume there exists $\tau \in \Phi^{\mathbf{s}}_A$ such that $\mathrm{supp}(u_+) \subseteq \tau$ and $\mathrm{supp}(u_-) \subsetneq \tau'$ for any $\tau' \in \Phi^{\mathbf{s}}_A$. $h_\tau(a_i) = s_i \ \forall i \in \tau$ but $h_\tau(a_j) < s_i \ \forall i \notin \tau$. Since $Au = 0$ then $Au_+ = Au_-$ and by the assumption

$$\sum_{i=1}^{n} s_i (u_+)_i = \langle h_\tau A, u_+ \rangle = \langle h_\tau, Au_+ \rangle = \langle h_\tau, Au_- \rangle < \sum_{i=1}^{n} s_i (u_-)_i$$

so $\mathrm{in}_{\mathbf{s}}(\Box_u) = \xi^{u_-}$ is a multiple of $\displaystyle\prod_{j \in \mathrm{supp}(u_-)} \xi_j$ which is an element of the type of $i)$ by assumption.

The case $\mathrm{supp}(u_-) \subseteq \tau$ and $\mathrm{supp}(u_-) \subsetneq \tau'$ for any $\tau' \in \Phi^{\mathbf{s}}_A$ is analogous to the previous case. If there is no face containing $\mathrm{supp}(u_+)$ nor $\mathrm{supp}(u_-)$ it is trivial that $\xi^{u_+}, \xi^{u_+} \in \widetilde{I}^{\mathbf{s}}_A$ and so $\xi^{u_+} - \xi^{u_+} \in \widetilde{I}^{\mathbf{s}}_A$. Finally, since $Au_+ = Au_-$ it is not possible that $\varsigma = \mathrm{supp}(u_+) \subseteq \tau$ and $\varsigma' = \mathrm{supp}(u_-) \subseteq \tau'$ for any $\tau, \tau' \in \Phi^{\mathbf{s}, d-1}_A$ such that $\tau \neq \tau'$ (because this implies that $\mathrm{pos}(\varsigma) \cap \mathrm{pos}(\varsigma') = \{\mathbf{0}\}$).

Let us prove the inclusion $\widetilde{I}^{\mathbf{s}}_A \subseteq \sqrt{\mathrm{in}_{\mathbf{s}}(I_A)}$. It is clear that the elements of type $ii)$ lies in $\mathrm{in}_{\mathbf{s}}(I_A) \subseteq \sqrt{\mathrm{in}_{\mathbf{s}}(I_A)}$ so we only need to prove that the elements of type $i)$ belong to $\sqrt{\mathrm{in}_{\mathbf{s}}(I_A)}$:

If $a_{i_1}/s_{i_1}, \ldots, a_{i_r}/s_{i_r}$ do not lie in a common facet of $\Phi^{\mathbf{s}}_A$ then $\exists \mathbf{a}$ such that:

(1) $\mathbf{a} \in \mathrm{conv}(a_{i_1}/s_{i_1}, \ldots, a_{i_r}/s_{i_r})$.

(2) $\mathbf{a}$ lies in the interior of $\Delta^{\mathbf{s}}_A$.

By (1) we can write:

$$\mathbf{a} = \sum_{j=1}^{r} \epsilon_j a_{i_j}/s_{i_j} \tag{6.9}$$

with $\sum_j \epsilon_j = 1$ and $\epsilon_j \geq 0$, $\forall j$.

And by (2) there exists $t > 1$ such that ta still belongs to $\Delta_A^{\mathbf{s}}$ and we can write:

$$t\mathbf{a} = \sum_{i=1}^{n} \eta_i a_i/s_i \tag{6.10}$$

with $\sum_j \eta_j = 1$ and $\eta_j \geq 0$, $\forall j$.

Finally, by (6.9) and (6.10) we have:

$$\sum_{i=1}^{r}(t\epsilon_j)a_{i_j}/s_{i_j} = \sum_{i=1}^{n} \eta_i a_i/s_i.$$

Then there exists $\lambda \in \mathbb{N}^*$ such that

$$P = \prod_{j=1}^{r} \partial_{i_j}^{\lambda t \epsilon_j/s_j} - \prod_{j=1}^{n} \partial_j^{\lambda \eta_j/s_j} \in I_A$$

and the $\mathbf{s}$-degree of the first monomial is λt while the $\mathbf{s}$-degree of the second monomial is λ so $\mathrm{in}_{\mathbf{s}}(P) = \prod_{j=1}^{r} \partial_{i_j}^{\lambda t\epsilon_j/s_j} \in \mathrm{in}_{\mathbf{s}}(I_A)$. This implies that $\xi_{i_1} \cdots \xi_{i_r} \in \sqrt{\mathrm{in}_{\mathbf{s}}(I_A)}$. □

Let $\tau \subseteq \{1, \ldots, n\}$ be a set with cardinality $l \geq 0$ and consider the coordinate subspace $Y_\tau = \{x_i = 0 : \ i \notin \tau\}$ with dimension l.

The special filtration

$$L_s := F + (s-1)V_\tau$$

with $s \geq 1$ is an intermediate filtration between the filtration F by the order of the differential operators and the Malgrange-Kashiwara filtration with respect to Y_τ that we denote by V_τ. Recall that V_τ is associated with the weights -1 for the variables $x_{\overline{\tau}}$, 1 for $\partial_{\overline{\tau}}$ and 0 for the rest of the variables.

We shall identify $s \in \mathbb{R}_{>0}$ with $(s_1, \ldots, s_n)$ throughout this section, where $s_i = 1$ if $i \in \tau$ and $s_i = s$ if $i \notin \tau$. Then $(L_s)_{n+j} = s_j$ for all $j = 1, \ldots, n$.

Lemma 6.3.3. *Assume $s > 1$ is such that $\Phi_A^s = \Phi_A^{s+\epsilon} = \Phi_A^{s-\epsilon}$ for sufficiently small $\epsilon > 0$. Then the ideal $\widetilde{I}_A^s$ is homogeneous with respect to F and V_τ. In particular $\mathcal{V}(\widetilde{I}_A^s + \langle A\mathbf{x}\underline{\xi}\rangle)$ is a bi-homogeneous variety in $\mathbb{C}^{2n}$.*

Proof. We only need to prove that the elements in Lemma 6.3.2, *ii*), are bi-homogeneous. From the proof of Lemma 6.3.2 we deduce that they are L_s-homogeneous. Since $\Phi_A^s = \Phi_A^{s+\epsilon} = \Phi_A^{s-\epsilon}$ we have that they are also $L_{s\pm\epsilon}$-homogeneous for all $\epsilon > 0$ small enough. Since $L_{s\pm\epsilon} = F + (s \pm \epsilon - 1)V_\tau$ we obtain that they are F-homogeneous and V_τ-homogeneous. □

Lemma 6.3.4. $\dim_{\mathbb{C}}(\mathcal{V}(\text{in}_{L_s}(I_A)) \cap \mathcal{V}(A\mathbf{x}\xi)) \leq n$.

Proof. Let $\omega \in \mathbb{R}^n_{>0}$ be a generic weight vector such that $\text{in}_\omega(\text{in}_{L_s}(I_A))$ is a monomial ideal. For $\epsilon > 0$ small enough $\text{in}_\omega(\text{in}_{L_s}(I_A)) = \text{in}_{\widetilde{\omega}}(I_A)$ for $\widetilde{\omega} = s + \epsilon\omega \in \mathbb{R}^n_{>0}$.

Choose any monomial order $<$ in $\mathbb{C}[x,\xi]$ that refines the partial order given by $(u,v) := (1-\epsilon\omega_1, \ldots, 1-\epsilon\omega_n; \epsilon\omega_1, \ldots, \epsilon\omega_n) \in \mathbb{R}^{2n}_{>0}$. It is clear that $\text{in}_{(u,v)}(Ax\xi)_i = (Ax\xi)_i$ for all $i = 1, \ldots, d$ and that $\text{in}_{(u,v)}(\text{in}_{L_s}(I_A)) = \text{in}_{\widetilde{\omega}}(I_A)$. Then

$$\text{in}_{\widetilde{\omega}}(I_A) + \langle Ax\xi \rangle \subseteq \text{in}_{(u,v)}(\text{in}_{L_s}(I_A) + \langle Ax\xi \rangle)$$

and so we have that:

$$E_<(\text{in}_{L_s}(I_A) + \langle A\mathbf{x}\xi \rangle) = E_<(\text{in}_{(u,v)}(\text{in}_{L_s}(I_A) + \langle Ax\xi \rangle)) \supset E_<(\text{in}_{\widetilde{\omega}}(I_A) + \langle A\mathbf{x}\xi \rangle) \quad (6.11)$$

where $E_<(I) := \{(\alpha,\gamma) \in \mathbb{N}^{2n} : \text{in}_<(P) = c_{\alpha,\gamma}x^\alpha\xi^\gamma, \; P \in I \setminus \{0\}\}$ for any ideal $I \subseteq \mathbb{C}[x,\xi]$. The inclusion (6.11) implies that the Krull dimension of the residue ring $\mathbb{C}[x,\xi]/(\text{in}_{L_s}(I_A) + \langle A\mathbf{x}\xi \rangle)$ is at most the one of $\mathbb{C}[x,\xi]/(\text{in}_{\widetilde{\omega}}(I_A) + \langle A\mathbf{x}\xi \rangle)$.

Then it is enough to prove that $\mathbb{C}[x,\xi]/(\text{in}_{\widetilde{\omega}}(I_A) + \langle A\mathbf{x}\xi \rangle)$ has Krull dimension n. Since $M = \text{in}_{\widetilde{\omega}}(I_A)$ is a monomial ideal then:

$$\text{in}_{\widetilde{\omega}}(I_A) = \cap_{(\partial^b,\sigma)\in\mathcal{S}(M)} \langle \xi_j^{b_j+1} : \; j \notin \sigma \rangle$$

where $S(M)$ denotes the set of standard pairs of M (see [SST00, Section 3.2]). This implies that

$$\mathcal{V}(\text{in}_{\widetilde{\omega}}(I_A) + \langle A\mathbf{x}\xi \rangle) = \cup_{(\partial^b,\sigma)\in\mathcal{S}(M)} \mathcal{V}(\langle \xi_j : \; j \notin \sigma \rangle + \langle Ax\xi \rangle).$$

By [SST00, Corollary 3.2.9.], the columns of A indexed by σ are linearly independent when $(\partial^b, \sigma) \in \mathcal{S}(M)$, so the dimension of each component

$$\mathcal{V}(\langle \xi_j : \; j \notin \sigma \rangle + \langle Ax\xi \rangle) = \mathcal{V}(\langle \xi_j : \; j \notin \sigma \rangle + \langle x_j\xi_j : \; j \in \sigma \rangle)$$

is n. □

Lemma 6.3.5. *Under the assumptions of lemma 6.3.3 we have that s is not an algebraic slope of $\mathcal{M}_A(\beta)$ along Y_τ at any point of Y_τ.*

Proof. We know that:

$$\mathrm{Ch}^s(\mathcal{M}_A(\beta)) = \mathcal{V}(\sqrt{\mathrm{in}_{L_s}(H_A(\beta))}) \subseteq \mathcal{V}(\sqrt{\mathrm{in}_{L_s}(I_A)}) \cap \mathcal{V}(A\mathbf{x}\xi) = \mathcal{V}(\widetilde{I}_A^s + \langle A\mathbf{x}\xi\rangle).$$

Hence the s-characteristic variety of $\mathcal{M}_A(\beta)$ is contained in a bi-homogeneous variety of dimension at most n when the assumptions in Lemma 6.3.3 are satisfied. Since $\mathrm{Ch}^s(\mathcal{M}_A(\beta))$ is known to be purely n-dimensional, each irreducible component is an irreducible component of $\mathcal{V}(\widetilde{I}_A^s + \langle A\mathbf{x}\xi\rangle)$ and so it is also bi-homogeneous. Moreover, this is true not only at the origin $x = 0 \in \mathbb{R}^n$ but also at any point of Y_τ because $(L_s)_i = 0$ for $i \in \tau$ and $Y_\tau = \{x_i = 0 : \ i \notin \tau\}$. Then s is not an algebraic slope of $\mathcal{M}_A(\beta)$ along Y_τ at any point of Y_τ. □

Remark 6.3.6. *Observe that after the proof of Lemma 6.3.5 we have the equality in Lemma 6.3.4.*

Remark 6.3.7. *A consequence of Lemma 6.3.5 is that $\mathcal{M}_A(\beta)$ has no algebraic slopes along $\mathbf{0} \in \mathbb{C}^n$ at $\mathbf{0}$.*

Example 6.3.8. *Let $A = (a_1\ a_2\ a_3\ a_4)$ be the non-pointed matrix with columns*

$$a_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ a_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ a_3 = \begin{pmatrix} -3 \\ -2 \end{pmatrix},\ a_4 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

and consider the associated hypergeometric system:

$$H_A(\beta) = I_A + \langle x_1\partial_1 - 3x_3\partial_3 + 2x_4\partial_4 - \beta_1, -x_1\partial_1 + x_2\partial_2 - 2x_3\partial_3 + 2x_4\partial_4 - \beta_2\rangle$$

where $I_A = \langle \partial_1\partial_2\partial_3\partial_4 - 1, \partial_1\partial_2^3 - \partial_3\partial_4^2, \partial_3^2\partial_4^3 - \partial_2^2\rangle$ and $\beta_1, \beta_2 \in \mathbb{C}$.

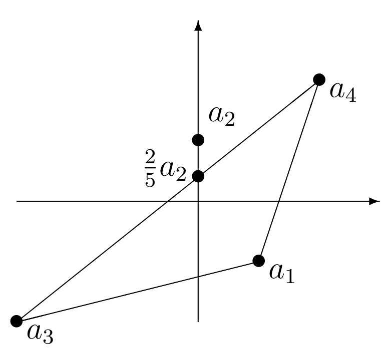

Figure 6.2

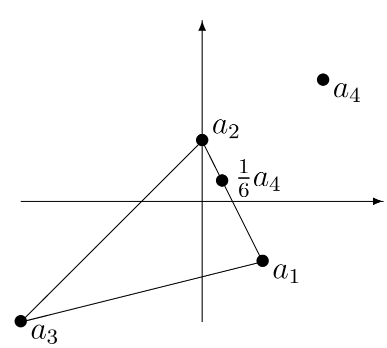

Figure 6.3

From Lemma 6.3.5 we deduce that there is not any algebraic slope along a coordinate subspace different from $Y = \{x_2 = 0\}$ and $Z = \{x_4 = 0\}$. By Corollary 6.2.2 and using again Lemma 6.3.5 we know that the unique slope of $\mathcal{M}_A(\beta)$ along Y is $|A_\sigma^{-1}a_2| = 5/2$ with

$\sigma = \{3,4\}$ *and that the unique slope of* $\mathcal{M}_A(\beta)$ *along* Z *is* $|A_{\overline{\sigma}}^{-1}a_4| = 6$ *with* $\overline{\sigma} = \{1,2\}$. *Notice that* $2a_2/5$ *lies in the affine line passing through* a_3 *and* a_4 *(see Figure 6.2) and that* $a_4/6$ *lies in the affine line passing through* a_1 *and* a_2 *(see Figure 6.3). We also can construct* $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) = 2$ *Gevrey solutions of* $\mathcal{M}_A(\beta)$ *along* Y *(it is analogous for* Z*) as follows.*

The matrix B_σ *is*

$$B_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \\ 5/2 & -3/2 \end{pmatrix}$$

and we consider the vectors $v^1 = (0,0,A_\sigma^{-1}\beta) = (0,0,-\beta_1+\beta_2,-\beta_1+\frac{3}{2}\beta_2)$ *and* $v^2 = (0,1,A_\sigma^{-1}(\beta-a_2)) = (0,1,-\beta_1+\beta_2-1,-\beta_1+\frac{3}{2}(\beta_2-1))$.

If none of $\beta_1-\beta_2, -\beta_1+\frac{3}{2}\beta_2$ *and* $-\beta_1+\frac{3}{2}(\beta_2-1)$ *are integers then the series* ϕ_{v^1} *and* ϕ_{v^2} *are Gevrey series solutions along* Y *of* $\mathcal{M}_A(\beta)$ *with index* $5/2$ *at any point of* $Y \cap \{x_1x_2 \neq 0\}$. *In other case, we can replace the vectors* v^i *by* $v^{i,k} := v^i + k(0,1,-1,-3/2)$ *with* $k \in 2\mathbb{N}$ *big enough in order to obtain Gevrey solutions* $\phi_{v^{i,k}}$ *of* $\mathcal{M}_A(\beta)$ *modulo convergent series at any point of* $Y \cap \{x_1x_2 \neq 0\}$ *with index* $5/2$.

Denote for $s > 1$:

$$\Omega_{Y_\tau}^{(s)} = \{\sigma \subseteq \tau : \det(A_\sigma) \neq 0, \max\{|A_\sigma^{-1}a_i| : i \notin \tau\} = s, |A_\sigma^{-1}a_j| \leq 1, \forall j \in \tau\}.$$

Then we have the following result.

Lemma 6.3.9. *If* $\sigma \in \Omega_{Y_\tau}^{(s_0)} \neq \emptyset$ *then for all* $p \in Y_\tau \cap U_\sigma$*:*

1) s_0 *is the Gevrey index of a solution of* $\mathcal{M}_A(\beta)$ *in* $\mathcal{O}_{\widehat{X|Y_\tau},p}$ *for very generic parameters* $\beta \in \mathbb{C}^d$.

2) s_0 *is the Gevrey index of a solution of* $\mathcal{M}_A(\beta)$ *in* $\mathcal{O}_{\widehat{X|Y_\tau},p}/\mathcal{O}_{X|Y_\tau(<s_0),p}$ *for all* $\beta \in \mathbb{C}^d$.

3) *If* Y_τ *is a hyperplane, then* s_0 *is the Gevrey index of a solution of* $\mathcal{M}_A(\beta)$ *in* $\mathcal{O}_{\widehat{X|Y_\tau},p}/\mathcal{O}_{X|Y_\tau,p}$ *for all* $\beta \in \mathbb{C}^d$.

Proof. We consider any $\sigma \in \Omega_{Y_\tau}^{(s_0)}$. If β is very generic the Gevrey series solutions of $\mathcal{M}_A(\beta)$ along Y_τ associated with σ, $\{\phi_\sigma^{\mathbf{k}}\}_{\mathbf{k}}$ (see Section 6.1), have Gevrey index $s_0 = \max\{|A_\sigma^{-1}a_i| : i \in \tau\}$ along Y_τ at $p \in Y_\tau \cap U_\sigma$. If β is not very generic we can proceed as in Section 6.2 in order to construct a Gevrey series associated with σ with index s_0 which is a solution of $\mathcal{M}_A(\beta)$ in $(\mathcal{O}_{X|Y}(s_0)/\mathcal{O}_{X|Y}(<s_0))_p$ for all $p \in Y \cap U_\sigma$. By a similar argument to the one in the proof of Lemma 8.0.8 the result is obtained. □

Assume that Y is a coordinate hyperplane for the remainder of this section. We can reorder the variables so that $Y = \{x_n = 0\}$.

Theorem 6.3.10. *For all $p \in Y$ the following statements are equivalent:*

1) Φ_A^s jumps at $s = s_0$.

2) $\Omega_Y^{(s_0)} \neq \emptyset$.

3) s_0 is an analytic slope of $\mathcal{M}_A(\beta)$ along Y at p.

4) s_0 is an algebraic slope of $\mathcal{M}_A(\beta)$ along Y at p.

Proof. We will prove first the equivalence of 1) and 2). Assume there exists $\sigma \in \Omega_Y^{(s_0)} \neq \emptyset$, then $H_\sigma = \{y \in \mathbb{R}^d : \; |A_\sigma^{-1} y| = 1\}$ is the only hyperplane containing a_i for all $i \in \sigma$ and $|A_\sigma^{-1}(a_n/(s_0+\epsilon))| = s_0/(s_0+\epsilon) < 1$, $\forall \epsilon > 0$. Hence $a_n/s_0 \in H_\sigma$ but $a_n/(s_0+\epsilon) \notin H_\sigma$, $\forall \epsilon > 0$.

Consider $\eta = \{i : \; a_i \in H_\sigma\}$, then $\eta \in \Phi_A^{s_0+\epsilon, d-1}$, $\forall \epsilon > 0$ and $n \notin \eta$ while $\eta \cup \{n\} \in \Phi_A^{s_0, d-1}$, so Φ_A^s jumps at $s = s_0$.

Conversely if $\Omega_Y^{(s_0)} = \emptyset$ then $\forall \sigma \subseteq \{1, 2, \ldots, n-1\}$ such that $|A_\sigma^{-1} a_i| \leq 1$ for all $i = 1, \ldots, n-1$ we have $|A_\sigma^{-1} a_n| < s_0$ or $|A_\sigma^{-1} a_n| > s_0$.

Consider $\epsilon > 0$ small enough such that $|A_\sigma^{-1} a_n| < s_0 \pm \epsilon$ if $|A_\sigma^{-1} a_n| < s_0$ and $|A_\sigma^{-1} a_n| > s_0 \pm \epsilon$ if $|A_\sigma^{-1} a_n| > s_0$ for all simplices σ such that $|A_\sigma^{-1} a_i| \leq 1$ for all $i = 1, \ldots, n-1$.

Let us prove that $\Phi_A^{s_0, d-1} = \Phi_A^{s_0 \pm \epsilon, d-1}$.

Assume first that $n \notin \eta \subseteq \{1, \ldots, n\}$. Then:

$\eta \in \Phi_A^{s_0, d-1} \iff \exists \sigma \subseteq \eta$ such that $|A_\sigma^{-1} a_i| = 1$ for $i \in \eta$, $|A_\sigma^{-1} a_i| < 1$ for $i \notin \eta \cup \{n\}$ and $|A_\sigma^{-1} a_n| < s_0 \iff \exists \sigma \subseteq \eta$ such that $|A_\sigma^{-1} a_i| = 1$ for $i \in \eta$, $|A_\sigma^{-1} a_i| < 1$ for $i \notin \eta \cup \{n\}$ and $|A_\sigma^{-1} a_n| < s_0 \pm \epsilon \iff \eta \in \Phi_A^{s_0 \pm \epsilon, d-1}$.

If $n \in \eta \subseteq \{1, \ldots, n\}$ and $\dim(\mathrm{conv}(\eta \setminus \{n\})) = d-1$ then there exists a simplex $\sigma \subseteq \eta \setminus \{n\}$ such that $\det(A_\sigma) \neq 0$. Then $\eta \notin \Phi_A^{s_0, d-1}$ because in such a case $|A_\sigma^{-1} a_i| \leq 1$ for all $i \neq n$, $|A_\sigma^{-1} a_n| = s_0$ and so $\sigma \in \Omega_Y^{(s_0)}$, a contradiction. Moreover $\eta \notin \Phi_A^{s_0 \pm \epsilon, d-1}$ for $\epsilon > 0$ small enough because $|A_\sigma^{-1} a_n|$ is a fixed value while $s_0 \pm \epsilon$ varies with ϵ.

Finally, if $n \in \eta \subseteq \{1, \ldots, n\}$ and $\dim(\mathrm{conv}(\eta \setminus \{n\})) < d-1$ then there exists a hyperplane $H' = \{\mathbf{y} \in \mathbb{R}^d : \; h'(\mathbf{y}) = 0\}$ that contains $\mathbf{0} \in \mathbb{R}^d$ and a_i for all $i \in \eta \setminus \{n\}$. We also can choose the linear function h' in the definition of H' such that $h'(a_n) = 1$. In this case:

$\eta \in \Phi_A^{s_0,d-1} \Longleftrightarrow \eta \backslash \{n\} \in \Phi_A^{s_0,d-2}$ and $\exists H'' = \{\mathbf{y} \in \mathbb{R}^d : \ h''(\mathbf{y}) = 1\}$ such that $h''(a_i) = 1$ for $i \in \eta \setminus \{n\}$, $h''(a_n) = s_0$ and $h''(a_j) < 1$ for $j \notin \eta$. This imply for $h := h'' \pm \epsilon h'$ that $h(a_i) = 1$ for all $i \in \eta \setminus \{n\}$, $h(a_n) = s_0 \pm \epsilon$ and $h(a_j) = h''(a_j) \pm \epsilon h'(a_j) < 1$ for $j \notin \eta$ and $\epsilon > 0$ small enough because $h''(a_j) < 1$ for $j \notin \eta$. Hence $\eta \in \Phi_A^{s_0 \pm \epsilon, d-1}$.

We have proved that $\Phi_A^{s_0,d-1} \subseteq \Phi_A^{s_0 \pm \epsilon, d-1}$. This implies equality since they are (A,s)-umbrellas of the same matrix A and $s = s_0 \pm \epsilon > 0$ (in particular $\cup_{\eta \in \Phi_A^{s,d-1}} \operatorname{pos}(\eta) = \operatorname{pos}(A)$ for all $s > 0$). Moreover, the (A,s)-umbrellas are determined by their facets, so $\Phi_A^{s_0} = \Phi_A^{s_0 \pm \epsilon}$.

The implication 2) $\Longrightarrow$ 3) is a direct consequence of Lemma 6.3.9 if p belongs to the closure of $Y \cap U_\sigma$ for some $\sigma \in \Omega_Y^{(s_0)}$ (for example, if $p = 0$). Nevertheless, since the analytic slopes are found in relatively open subsets of the hyperplane Y we can use the constructibility of the slopes in order to prove the result at any point of Y (see Remark 8.0.9). The implication 3) $\Longrightarrow$ 4) follows from Laurent's index theorem for holomorphic hyperfunctions [Lau99] (see also [Meb90, 6.6]). Finally, the implication 4) $\Longrightarrow$ 1) is nothing but Lemma 6.3.5. □

Remark 6.3.11. *In Theorem 6.3.10, the equivalence of 3) and 4) is a particular case of the comparison theorem of the slopes [LM99]. Notice that we don't need to use this theorem for the implication* 4) $\Longrightarrow$ 3)*.*

Remark 6.3.12. *Notice that if Y is a coordinate hyperplane then every algebraic slope s_0 of $\mathcal{M}_A(\beta)$ along Y is the Gevrey index of certain Gevrey solutions of $\mathcal{M}_A(\beta)$ along Y modulo convergent series. Example 6.3.13 shows that this is not true for coordinate subspaces of codimension greater than one.*

Example 6.3.13. *Let $\mathcal{M}_A(\beta)$ be the hypergeometric $\mathcal{D}$-module associated with the matrix*

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

and the parameter vector $\beta \in \mathbb{C}^2$. In this case $n = 3 = d + 1$ and so the toric ideal is principal $I_A = \langle \partial_1^3 - \partial_2 \partial_3 \rangle$.

If we take $Y = \{x_2 = x_3 = 0\}$ then the only algebraic slope of $\mathcal{M}_A(\beta)$ along Y at $p \in Y$ is $s_0 = 3/2$ (observe Figure 6.4 and see [SW08] since A is pointed). Nevertheless, we will prove that if $\beta_2 \notin \mathbb{Z}$ then for all $s \geq 1$, $\mathcal{H}^0(\operatorname{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)) = 0$:

For any formal series $f = \sum_{m \in \mathbb{N}^2} f_m(x_1) x_2^{m_2} x_3^{m_3}$ along Y at $p = (p_1, 0, 0) \in Y$ then

$$(E_2 - \beta_2)(f) = \sum_{m \in \mathbb{N}^2} (m_2 - m_3 - \beta_2) f_m(x_1) x_2^{m_2} x_3^{m_3}$$

and hence $(E_2 - \beta_2)(f) \in \mathcal{O}_{X,p}$ *(resp.* $(E_2 - \beta_2)(f) = 0$*) if and only if* $f \in \mathcal{O}_{X,p}$ *(resp.* $f = 0$*) because* $(m_2 - m_3 - \beta_2) \neq 0, \forall m_2, m_3 \in \mathbb{N}$.

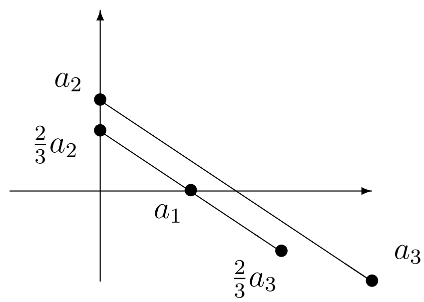

Figure 6.4

On the other hand, if $\beta_2 \in \mathbb{Z}$ *we can take* $k \in \mathbb{N}$ *the minimum natural number such that* $v = (\beta_1 - 3k, \beta_2 + k, k) \in \mathbb{C} \times \mathbb{N}^2$ *has minimal negative support. Since* $Av = \beta$ *then*

$$\phi_v = \sum_{m \geq 0} \frac{k![\beta_1 - 3k]_{3m}}{(k+m)![\beta_2 + k + m]_m} x_1^{\beta_1 - 3(k+m)} x_2^{\beta_2 + k + m} x_3^{k+m}$$

is a formal solution of $\mathcal{M}_A(\beta)$ *along* Y *at any point* $p \in Y$ *with* $p_1 \neq 0$*. In fact* ϕ_v *has Gevrey index* $s_0 = 3/2$ *if* $\beta_1 - 3k \notin \mathbb{N}$ *and it is a polynomial when* $\beta_1 - 3k \in \mathbb{N}$*. In this last case, if we consider* $v' = v + k'u$ *with* $u = (-3, 1, 1) \in L_A$ *and* $k' \in \mathbb{N}$ *such that* $v_1' < 0$ *then* $\phi_{v'}$ *is a Gevrey series of index* s_0 *and* $P(\phi_{v'})$ *is convergent along* Y *at any point* $p \in Y \setminus \{0\}$*.*

Thus, the algebraic slope $s_0 = 3/2$ *is the index of a Gevrey solution of* $\mathcal{M}_A(\beta)$ *along* Y *if and only if* $\beta_2 \in \mathbb{Z}$*. Observe that "the special parameters" are not contained in a Zariski closed set but in a countable union of them. Note also that* I_A *is Cohen-Macaulay and then it is known that the set of rank-jumping parameters is empty.*

6.4 Regular triangulations and $(A, \mathbf{s})$-umbrellas.

The aims of this section are to compare the notion of $(A, \mathbf{s})$-umbrella in [SW08] with the one of regular triangulation of the matrix A (see for example [Stu95]), to show that the common domain of definition of the constructed Gevrey series solutions $\phi_\sigma^{\mathbf{k}}$ is nonempty when σ varies in a regular triangulation and to prove the existence of convenient regular triangulations.

For any subset $\sigma \subseteq \{1, \ldots, n\}$ we will write $\mathrm{pos}(\sigma) = \sum_{i \in \sigma} \mathbb{R}_{\geq 0} a_i \subseteq \mathbb{R}^d$. Recall that we identify σ with $\{a_i : i \in \sigma\}$.

Definition 6.4.1. *A triangulation of A is a set* T *whose elements are subsets of columns of A verifying:*

1) $\{\mathrm{pos}(\sigma) :\ \sigma \in \mathrm{T}\}$ *is a simplicial fan.*

2) $\mathrm{pos}(A) = \cup_{\sigma\in\mathrm{T}} \mathrm{pos}(\sigma)$.

A vector $\omega \in \mathbb{R}^n$ defines a collection T_ω of subsets of columns of A as follows:

$\sigma \subseteq \{a_1, \ldots, a_n\}$ is a face of T_ω, $\sigma \in \mathrm{T}_\omega$, if there exists a vector $\mathbf{c} \in \mathbb{R}^d$ such that

$$\langle \mathbf{c}, a_j\rangle = \omega_j \text{ for all } j \in \sigma$$

and

$$\langle \mathbf{c}, a_j\rangle < \omega_j \text{ for all } j \notin \sigma.$$

Remark 6.4.2. *We will say that $\omega \in \mathbb{R}^n$ is generic when the collection T_ω is a simplicial complex and a triangulation of A.*

Definition 6.4.3. *A triangulation* T *is said to be regular if there exists a generic $\omega \in \mathbb{R}^n$ such that* $\mathrm{T} = \mathrm{T}_\omega$.

Observe that the collection $\{\mathrm{pos}(\sigma) :\ \sigma \in \Phi_A^{\mathbf{s}}\}$ is a polyhedral fan. When $\mathbf{s} \in \mathbb{R}^n_{>0}$ is generic it is a simplicial fan and so $\Phi_A^{\mathbf{s}}$ is a triangulation of A. In fact it is a regular triangulation because for any $\mathbf{s} \in \mathbb{R}^n_{>0}$ we have that $\Phi_A^{\mathbf{s}} = \mathrm{T}_{\mathbf{s}}$:

$\sigma \in \Phi_A^{\mathbf{s}} \iff \exists \mathbf{c} \in \mathbb{R}^d |\ \langle \mathbf{c}, a_i/s_i\rangle = 1,\ \forall i \in \sigma$, and $\langle \mathbf{c}, a_i/s_i\rangle < 1,\ \forall i \notin \sigma \iff \exists \mathbf{c} |\ \langle \mathbf{c}, a_i\rangle = s_i,\ \forall i \in \sigma$ and $\langle \mathbf{c}, a_i\rangle < s_i,\ \forall i \notin \sigma \iff \sigma \in \mathrm{T}_{\mathbf{s}}$.

Given a $(d-1)$-simplex $\sigma \in \mathrm{T}_\omega$ there exists $\mathbf{c} \in \mathbb{R}^d$ such that $\mathbf{c}A_\sigma = \omega_\sigma$ and $\mathbf{c}A_{\overline{\sigma}} < \omega_{\overline{\sigma}}$. This is equivalent to:

$$\mathbf{c} = \omega_\sigma A_\sigma^{-1},\ \omega_\sigma A_\sigma^{-1} A_{\overline{\sigma}} < \omega_{\overline{\sigma}}.$$

But this happens if and only if $\omega \in C(\sigma) := \{\omega \in \mathbb{R}^n :\ \omega B_\sigma > 0\}$ which is an open convex polyhedral rational cone of dimension n. Then we can write

$$C(\sigma) = \{\omega \in \mathbb{R}^n :\ \sigma \in \mathrm{T}_\omega\}$$

and for any regular triangulation $\mathrm{T} = \mathrm{T}_{\omega_0}$ we have

$$\omega_0 \in C(\mathrm{T}) := \bigcap_{\sigma\in\mathrm{T}} C(\sigma).$$

Hence $C(\mathrm{T}) = \{\omega \in \mathbb{R}^n :\ \mathrm{T}_\omega = \mathrm{T}\}$ is a nonempty open rational convex polyhedral cone. It is clear that $\cup_{\mathrm{T}} \overline{C(\mathrm{T})} = \mathbb{R}^n$ where T runs over all regular triangulations of A and $\overline{C(\mathrm{T})}$

denotes the Euclidean closure of $C(\mathrm{T})$. More precisely, there exists a polyhedral fan with support $\mathbb{R}^n$ such that T_ω is constant for $\omega \in \mathbb{R}^n$ running in any relatively open cone of this polyhedral fan. We also can restrict this fan to $\mathbb{R}^n_{>0}$ and obtain that the $(A, \mathbf{s})$-umbrella is constant for $\mathbf{s} \in \mathbb{R}^n_{>0}$ running in any relatively open cone.

Remark 6.4.4. *Recall from (6.6) that*

$$U_\sigma = \{x \in \mathbb{C}^n : \prod_{i\in\sigma} x_i \neq 0, \ (-\log|x_1|, \ldots, -\log|x_n|)B_{\sigma,j} > -\log R, \ \forall a_j \in H_\sigma \setminus \sigma\}$$

where $B_{\sigma,j}$ is the j-th column of B_σ, i.e. the vector with σ-coordinates $-A_\sigma^{-1}a_j$ and $\overline{\sigma}$-coordinates equal to the j-th column of the identity matrix of order $n-d$. Then U_σ contains those points $x \in \mathbb{C}^n \cap \{\prod_{i\in\sigma} x_i \neq 0\}$ for which

$$(-\log|x_1|, \ldots, -\log|x_n|)$$

lies in a sufficiently far translation of the cone $C(\sigma)$ inside itself. Then for any regular triangulation T of A we have that $\cap_{\sigma\in\mathrm{T}} U_\sigma$ is a nonempty open set since it contains those points $x \in \mathbb{C}^n \cap \{\prod_{i\in\sigma} x_i \neq 0 : \ \sigma \in \mathrm{T}\}$ for which $(-\log|x_1|, \ldots, -\log|x_n|) \in \mathbb{R}^n$ lies in a sufficiently far translation of the nonempty open cone $C(\mathrm{T})$ inside itself.

Lemma 6.4.5. *Given a full rank matrix $A \in \mathbb{Z}^{d\times n}$ with $d \leq n$ and a lattice Λ with $A \subseteq \Lambda \subseteq \mathbb{Z}^d$ there exists a regular triangulation T of A such that*

$$\mathrm{vol}_\Lambda(\Delta_A) = \sum_{\sigma\in\mathrm{T}, \dim\sigma = d-1} \mathrm{vol}_\Lambda(\Delta_\sigma) \tag{6.12}$$

Proof. The volume function vol_Λ with respect to a lattice Λ is nothing but the Euclidean volume function normalized so that the unit simplex in Λ has volume one. Hence, it is enough to prove the result for the Euclidean volume.

Take $\omega = (\omega_1, \ldots, \omega_n)$ with $\omega_i = 1$ for all $i = 1, \ldots, n$. If all the facets τ of Δ_A (with $\mathbf{0} \notin \tau$) contain exactly d columns of A then T_ω is a regular triangulation of A that verifies (6.12).

Let us denote by $H_{\omega,\tau}$ the unique hyperplane that contains $\{a_i/\omega_i : \ i \in \tau\}$ for a facet $\tau \in T_\omega$. Assume now that there exists a facet $\tau \in T_\omega$ with at least cardinality $d+1$. Then we can take $i \in \tau$ such that $H_{\omega,\tau}$ is the unique hyperplane that contains $\{a_j/\omega_j : \ j \in \tau \setminus \{i\}\}$. Consider all the hyperplanes $H_{\omega,\tau'} \neq H_{\omega,\tau}$ determined by facets $\tau' \in T_\omega$ such that $a_i \notin \tau'$. Then a_i/ω_i lies in the open set $\cap_{\tau'} H^-_{\omega,\tau'}$ and so $a_i/(\omega_i - \epsilon)$ does too for $\epsilon > 0$ small enough. This means that we do not modify the sets in T_ω that not contain a_i via replacing ω_i by $\omega_i - \epsilon > 0$. If $a_i \in \tau' \in T_\omega$ and $\tau' \setminus \{a_i\}$ does not determine $H_{\omega,\tau'}$ then τ' is not modified neither with this replacing. We just modify the facets $\tau' \in T_\omega$ such that $a_i \in \tau'$ and $\tau' \setminus \{a_i\}$ determine $H_{\omega,\tau'}$. Such a kind of facet is replaced by more than one facet with vertices contained in τ' and hence each of the new facets contain less columns of A than the original one. This process finishes in a finite number of steps and yields to a vector $\omega \in \mathbb{R}^n_{>0}$ such that T_ω is a regular triangulation (i.e. there is not any $\tau \in T_\omega$ with cardinality greater than d). Moreover, T_ω satisfies (6.12) because (6.12) is satisfied at any step of the process to construct T_ω. $\square$

6.5 Gevrey solutions of $\mathcal{M}_A(\beta)$ along coordinate subspaces.

6.5.1 Lower bound for the dimension.

In this section we provide an optimal lower bound in terms of volumes of polytopes of the dimension of $\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y_\tau}(s))_p$, $s \in \mathbb{R}$, for generic points $p \in Y_\tau = \{x_i = 0 : \ i \notin \tau\}$ and for all $\beta \in \mathbb{C}^d$. To this end we will use regular triangulations $\mathrm{T}(\tau)$ of the submatrix $A_\tau = (a_i)_{i\in\tau}$ of A and Theorem 6.1.14.

Consider the submatrix $A_\tau = (a_i)_{i\in\tau}$ of A. If the rank of A_τ is d then there exists a regular triangulation $\mathrm{T}(\tau)$ of A_τ such that

$$\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau) = \sum_{\sigma\in\mathrm{T}(\tau), \dim\sigma=d-1} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) \tag{6.13}$$

because of Lemma 6.4.5. If the rank of A_τ is lower than d then this equality holds for any regular triangulation of the matrix A_τ since all the volumes in (6.13) are zero.

For all $s \in \mathbb{R}$ we consider the following subset of $\mathrm{T}(\tau)$:

$$\mathrm{T}(\tau, s) := \{\sigma \in \mathrm{T}(\tau) : \ \dim(\sigma) = d-1, \ a_j/s \notin H_\sigma^+ \ \forall j \notin \tau\}.$$

The following theorem is the main result in this section.

Theorem 6.5.1. *For all $\tau \subseteq \{1, \ldots, n\}$,*

$$\dim_{\mathbb{C}} \mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau}})_p \geq \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau) \tag{6.14}$$

for p in the nonempty relatively open set $W_{\mathrm{T}(\tau)} := Y_\tau \cap (\bigcap_{\sigma\in\mathrm{T}(\tau)} U_\sigma)$. More precisely,

$$\dim_{\mathbb{C}} \mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y_\tau}(s))_p \geq \sum_{\sigma\in\mathrm{T}(\tau,s)} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) \tag{6.15}$$

for all $s \in \mathbb{R}$ and p in the nonempty relatively open set $W_{\mathrm{T}(\tau,s)} := Y_\tau \cap (\bigcap_{\sigma\in\mathrm{T}(\tau,s)} U_\sigma)$.

Proof. $W_{\mathrm{T}(\tau)} \subseteq W_{\mathrm{T}(\tau,s)}$ are nonempty relatively open subsets of Y_τ because $\mathrm{T}(\tau)$ is a regular triangulation of A_τ (use Remark 6.4.4 for A_τ instead of A).

For each fixed $(d-1)$-simplex $\sigma \in \mathrm{T}(\tau, s)$, we have that $|A_\sigma^{-1} a_j| \leq 1$ for all $j \in \tau$ and $|A_\sigma^{-1} a_j| \leq s$ for all $j \notin \tau$ and we can construct $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ Gevrey solutions of $\mathcal{M}_A(\beta)$ of order s along Y_τ at any point of $Y_\tau \cap U_\sigma$ by Theorem 6.1.14. These $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ series $\{\phi_\sigma^{\mathbf{k}}\}_{\mathbf{k}}$ are linearly independent because they have pairwise disjoint supports. The linear independency of the set of all $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ series $\phi_\sigma^{\mathbf{k}}$ when σ varies in $\mathrm{T}(\tau)$ is also clear if we assume that β is very generic (because this implies that they have pairwise disjoint supports).

If β is not very generic some of the series could be equal up to multiplication by a nonzero scalar. In such a case one can proceed similarly to the proof of Theorem 3.5.1. in [SST00]:

We introduce a perturbation $\beta \mapsto \beta + \epsilon\beta'$ with $\beta' \in \mathbb{C}^d$ such that $\beta + \epsilon\beta'$ is very generic for $\epsilon \in \mathbb{C}$ with $|\epsilon| > 0$ small enough (it is enough to consider $\beta' \in \mathbb{C}^d$ such that $(A_\sigma^{-1}\beta')_i \neq 0$ for all $i = 1, \ldots, d$ and $\sigma \in \mathrm{T}(\tau)$).

Consider the set $\{\phi_\sigma^{\mathbf{k}} : \ \sigma \in \mathrm{T}(\tau), \mathbf{k} \in \mathbb{N}^{n-d}\}$ with $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ Gevrey series solutions of $\mathcal{M}_A(\beta + \epsilon\beta')$ with disjoint supports. We will denote these series by $\phi_\sigma^{\mathbf{k}}(\beta + \epsilon\beta')$ in this proof. It is clear that $\phi_\sigma^{\mathbf{k}}(\beta + \epsilon\beta') = \phi_{v_\sigma^{\mathbf{k}}(\beta+\epsilon\beta')}$ for

$$v_\sigma^{\mathbf{k}}(\beta + \epsilon\beta') = v_\sigma^{\mathbf{k}}(\beta) + \epsilon v_\sigma^{\mathbf{0}}(\beta').$$

Here $v_\sigma^{\mathbf{k}}(\beta)$ has σ-coordinates $A_\sigma^{-1}(\beta - A_{\overline{\sigma}}\mathbf{k})$ and $\overline{\sigma}$-coordinates $\mathbf{k}$. Similarly, $v_\sigma^{\mathbf{0}}(\beta')$ has σ-coordinates $A_\sigma^{-1}\beta'$ and $\overline{\sigma}$-coordinates $\mathbf{0}$. Let T be a regular triangulation of A such that $\mathrm{T}(\tau) \subseteq \mathrm{T}$. For any $\phi_\sigma^{\mathbf{k}}(\beta)$ we can assume without loss of generality that $v_\sigma^{\mathbf{k}}(\beta)$ has minimal negative support, $\phi_\sigma^{\mathbf{k}}(\beta) = \phi_{v_\sigma^{\mathbf{k}}(\beta)}$ and $\mathrm{in}_\omega(\phi_\sigma^{\mathbf{k}}(\beta)) = x^{v_\sigma^{\mathbf{k}}(\beta)}$ for some fixed generic $\omega \in \mathcal{C}(\mathrm{T})$. Then for two simplices $\sigma, \sigma' \in \mathrm{T}(\tau)$ we have that $\phi_{v_\sigma^{\mathbf{k}}(\beta)} = c\phi_{v_{\sigma'}^{\mathbf{k}'}(\beta)}$ for some $c \in \mathbb{C}$ if and only if $v_\sigma^{\mathbf{k}}(\beta) = v_{\sigma'}^{\mathbf{k}'}(\beta)$.

Let us denote $\nu = \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$. Since $\beta + \epsilon\beta'$ is very generic, there exist ν $\mathbb{C}(\epsilon)$-linearly independent Gevrey series solutions of $\mathcal{M}_A(\beta)$ along Y_τ of the form

$$\phi_\sigma^{\mathbf{k}}(\beta + \epsilon\beta') = \sum_{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k}}} q_{k+m}(\epsilon) x^{v_\sigma^{\mathbf{k}}(\beta+\epsilon\beta') + u(\mathbf{m})}$$

where

$$q_{k+m}(\epsilon) = \frac{[v_\sigma^{\mathbf{k}}(\beta) + \epsilon v_\sigma^{\mathbf{0}}(\beta')]_{u(\mathbf{m})_-}}{[v_\sigma^{\mathbf{k}}(\beta) + \epsilon v_\sigma^{\mathbf{0}}(\beta') + u(m)]_{u(\mathbf{m})_-}}$$

for $\sigma \in \mathrm{T}(\tau)$ and $\mathbf{k} \in \mathbb{N}^{n-d}$ verifying that $\phi_\sigma^{\mathbf{k}}(\beta) = \phi_{v_\sigma^{\mathbf{k}}}(\beta)$. Observe that for all $\mathbf{k} + \mathbf{m} \in \Lambda_{\mathbf{k}}$ we can write

$$x^{v_\sigma^{\mathbf{k}}(\beta) + \epsilon v_\sigma^{\mathbf{0}}(\beta') + u(m)} = e^{\epsilon \log x_\sigma^{A_\sigma^{-1}\beta'}} x^{v_\sigma^{\mathbf{k}}(\beta)}.$$

Then we have:

$$\phi_\sigma^{\mathbf{k}}(\beta+\epsilon\beta') = e^{\epsilon \log x_\sigma^{A_\sigma^{-1}\beta'}} \sum_{\mathbf{k}+\mathbf{m}\in\Lambda_{\mathbf{k}}} q_{k+m}(\epsilon) x^{v_\sigma^{\mathbf{k}}(\beta)+u(\mathbf{m})}.$$

It is clear that $q_{\mathbf{k}+\mathbf{m}}(\epsilon)$ is a rational function on ϵ and it has a pole of order $\mu_{\mathbf{k}+\mathbf{m}}$ with $0 \leq \mu_{\mathbf{k}+\mathbf{m}} \leq d$. On the other hand

$$e^{\epsilon \log x_\sigma^{A_\sigma^{-1}\beta'}} = \sum_{l\geq 0} \frac{(\log(x_\sigma^{A_\sigma^{-1}(\beta')}))^l}{l!}\epsilon^l$$

so we can expand the series $\epsilon^\mu \phi_\sigma^{\mathbf{k}}(\beta+\epsilon\beta')$ (with $\mu = \max\{\mu_{\mathbf{k}+\mathbf{m}}\} \leq d$) and write it in the form $\sum_{j\geq 0}\phi_j(x)\epsilon^j$ where $\phi_0(x)\neq 0$ and $\phi_j(x)$ are Gevrey solutions of $\mathcal{M}_A(\beta)$ along Y_τ that converge in a common relatively open subset of Y_τ for all j.

After a reiterative process making convenient linear combinations of the series and dividing by convenient powers of ϵ, one obtain ν Gevrey solutions of $\mathcal{M}_A(\beta+\epsilon\beta')$ of the form $\sum_{j\geq 0}\psi_{i,j}(x)\epsilon^j$ where $\psi_{i,0}(x)\neq 0$, $i=1,\ldots,\nu$, are linearly independent. Then we can substitute $\epsilon = 0$ and obtain the desired ν linearly independent Gevrey series solutions of $\mathcal{M}_A(\beta)$. The logarithms $\log(x_i)$ just appear for $i\in\sigma$ with σ varying in $\mathrm{T}(\tau)$ at any step of the process. Thus the $\nu = \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ final series just have logarithms $\log(x_i)$ with $i\in\tau$ and they are Gevrey series solutions of $\mathcal{M}_A(\beta)$ along Y_τ at points of $W_{\mathrm{T}(\tau)}$. This proves (6.14). Moreover, it is clear that the Gevrey index cannot increase with this process and so (6.15) can be proved with the same argument. □

Remark 6.5.2. *The proof of Proposition 5.2. in [Sai02] guarantees that all the series solutions obtained after the process that we mention in the proof of Theorem 6.5.1 have the form*

$$\sum_v g_v(log(x_i):\ i\in\tau)x^v$$

with $g_v(y_\tau)$ *a polynomial in* $\mathbb{C}[y_\tau^u:\ u\in L_{A_\tau}]$.

Remark 6.5.3. *Theorem 6.5.1 generalizes [SST00, Theorem 3.5.1] and [Tak07, Corollary 1] (taking* $\tau=\{1,\ldots,n\}$ *and* $s=1$ *in (6.15)), that establish that the holonomic rank of a hypergeometric system (i.e. the dimension of the space of holomorphic solutions at nonsingular points) is greater than or equal to* $\mathrm{vol}_{\mathbb{Z}A}(\Delta_A)$. *A more precise statement than [Tak07, Corollary 1] is given in [MMW05]: the holonomic rank of* $\mathcal{M}_A(\beta)$ *is upper semi-continuous in* $\beta\in\mathbb{C}^d$ *with the Zariski topology.*

Remark 6.5.4. *Different regular triangulations* $\mathrm{T}(\tau)$ *of* A_τ *verifying the condition (6.13) will produce different sets with* $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ *linearly independent solutions of* $\mathcal{M}_A(\beta)$ *in* $\mathcal{O}_{\widehat{X|Y_\tau},p}$ *for* p *in pairwise disjoint open subsets* $W_{\mathrm{T}(\tau)}$ *of* Y_τ. *It is natural to ask whether*

$\cup_{\mathrm{T}(\tau)}\overline{W_{\mathrm{T}(\tau)}} = Y_\tau$ *for* $\mathrm{T}(\tau)$ *running over all possible regular triangulations* $\mathrm{T}(\tau)$ *of* A_τ *verifying (6.12). We have that* $\cup_{\mathrm{T}(\tau)}\overline{C(\mathrm{T}(\tau))} = \mathbb{R}^l$ *and that there exists* $w, w' \in C(\mathrm{T}(\tau))$ *verifying two-sided Abel lemma:*

$$w + C(\mathrm{T}(\tau)) \subseteq -\operatorname{Log} W_{\mathrm{T}(\tau)} \subseteq w' + C(\mathrm{T}(\tau))$$

where $\operatorname{Log} : \mathbb{C}^l \longrightarrow \mathbb{R}^l$ *is the map* $\operatorname{Log}(x_1, \ldots, x_l) = (\log|x_1|, \ldots, \log|x_l|)$*. This should be contrasted with [PST05, Lemma 11]. However, an argument similar to that of Remark 8.0.9 proves that (6.14) and (6.15)) hold at generic points of* Y_τ*.*

Remark 6.5.5. *If there are no more than* d *columns of* A_τ *in the same facet of* Δ_τ *then by Theorem 6.1.14 all the series above are Gevrey of the corresponding order along* Y_τ *at any point of* $Y_\tau \cap (\cap_{\sigma\in \mathrm{T}(\tau)}\{\prod_{i\in\sigma} x_i \neq 0\})$*.*

6.5.2 Dimension for very generic parameters.

In Subsection 6.5.1 we proved the lower bound (6.14) by explicitly constructing $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ Gevrey series solutions of $\mathcal{M}_A(\beta)$ along Y_τ in certain relatively open subsets of Y_τ. The aim of this section is to prove that equality holds if β is very generic.

Let $\tau \subseteq \{1, \ldots, n\}$ be a subset with cardinality l, $1 \leq l \leq n-1$, and recall that we denote $Y_\tau = \{x_i = 0 : \ i \notin \tau\}$.

Theorem 6.5.6. *For generic* $p \in Y_\tau$ *and very generic* β*,*

$$\dim_{\mathbb{C}} \mathcal{H}om(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau}})_p = \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau).$$

Remark 6.5.7. *Theorem 6.5.6 implies that equality holds in (6.15) for very generic parameters* $\beta \in \mathbb{C}^d$ *because the* $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ *Gevrey series* $\phi_\sigma^{\mathbf{k}}$ *with* $\sigma \in \mathrm{T}(\tau)$ *have pairwise disjoint supports and their index along* Y_τ *is* $\max\{|A_\sigma^{-1}a_j| : \ j \notin \tau\}$*.*

Corollary 6.5.8. *If* $\beta \in \mathbb{C}^d$ *is very generic then*

$$\dim_{\mathbb{C}} \mathcal{H}^0(\mathrm{Irr}_{Y_\tau}^{(s)}(\mathcal{M}_A(\beta))_p \geq \sum_{\sigma\in \mathrm{T}(\tau,s)\setminus \mathrm{T}(\tau,1)} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) \tag{6.16}$$

for generic $p \in Y_\tau$*.*

Proof. It follows from Theorem 6.5.6, Remark 6.5.7 and the exact sequence $0 \to \mathcal{H}om(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y_\tau}) \longrightarrow \mathcal{H}om(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y_\tau}(s)) \longrightarrow \mathcal{H}^0(\mathrm{Irr}_{Y_\tau}^{(s)}(\mathcal{M}_A(\beta))$. □

Lemma 6.5.9. *If* $f = \sum_{m\in\mathbb{N}^{n-l}} f_m(x_\tau)x_{\overline{\tau}}^m \in \mathcal{O}_{\widehat{X|Y_\tau,p}}$ *is a formal solution of* $\mathcal{M}_A(\beta)$*, then* $f_m(x_\tau) \in \mathcal{O}_{Y_\tau,p}$ *is a holomorphic solution of* $\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}}m)$ *for all* $m \in \mathbb{N}^{n-l}$*.*

Proof. It is clear that $I_A \cap \mathbb{C}[\partial_\tau] = I_{A_\tau}$. Then for any differential operators $P \in I_{A_\tau} \subseteq \mathbb{C}[\partial_\tau]$ we have that

$$0 = P(f) = \sum_{m \in \mathbb{N}^{n-l}} P(f_m(x_\tau)) x_{\overline{\tau}}^m$$

and this implies that $P(f_m(x_\tau)) = 0$ for all $m \in \mathbb{N}^{n-l}$.

Let Θ denote the vector with coordinates $\Theta_i = x_i \partial_i$ for $i = 1, \ldots, n$. Then $A\Theta - \beta = A_\tau \Theta_\tau + A_{\overline{\tau}} \Theta_{\overline{\tau}} - \beta$ and

$$\mathbf{0} = (A\Theta - \beta)(f) = \sum_{m \in \mathbb{N}^{n-l}} (A_\tau \Theta_\tau + A_{\overline{\tau}} m - \beta)(f_m(x_\tau)) x_{\overline{\tau}}^m$$

so $f_m(x_\tau)$ must be annihilated by the Euler operators $A_\tau \Theta_\tau - (\beta - A_{\overline{\tau}} m)$. □

Corollary 6.5.10. *If* $\operatorname{rank}(A_\tau) < d$ *and* $\beta \in \mathbb{C}^d$ *is very generic then*

$$dim_{\mathbb{C}} \mathcal{H}om(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau}}) = 0.$$

Proof. If $\operatorname{rank}(A_\tau) < d$, then there exists a nonzero vector $\gamma \in \mathbb{Q}^d$ such that the vector γA_τ is zero. If β is very generic $(\gamma A_\tau \Theta_\tau - \gamma(\beta - A_{\overline{\tau}} m) = -\gamma(\beta - A_{\overline{\tau}} m) \neq 0$ is a nonzero constant that is a linear combination of the Euler operators in the definition of $\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}} m)$ and so $\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}} m) = 0$. By Lemma 6.5.9, the coefficients in $\mathcal{O}_{Y_\tau, p}$ of any formal solution f of $\mathcal{M}_A(\beta)$ in $\mathcal{O}_{\widehat{X|Y_\tau}, p}$ must be solutions of $\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}} m) = 0$. This implies that the coefficients of f are zero and so $f = 0$. □

Remark 6.5.11. *By Corollary 6.5.10 we have the equality in Theorem 6.5.6 holds when* $\operatorname{rank}(A_\tau) < d$*. For the remainder of this section we shall assume that* $\operatorname{rank}(A_\tau) = d$ *and then* $l \geq d$*.*

The following Lemma is a direct consequence of results from [Ado94] and [GZK89].

Lemma 6.5.12. *If* β *is very generic and* $p \in Y_\tau$*, then for all* $m \in \mathbb{N}^{n-l}$*:*

$$\dim_{\mathbb{C}} \mathcal{H}om(\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}} m), \mathcal{O}_{Y_\tau})_p \leq \operatorname{vol}_{\mathbb{Z}\tau}(\Delta_\tau).$$

Equality holds if p *does not lie in the singular locus of* $\mathcal{M}_{A_\tau}(\beta)$ *(which does not depend on* β*).*

Let us consider $\mathrm{T}(\tau)$ a regular triangulation of A_τ verifying (6.13).

Lemma 6.5.13. *If* $\beta \in \mathbb{C}^d$ *is very generic, then any formal solution* $f = \sum_{m \in \mathbb{N}^{n-l}} f_m(x_\tau) x_{\overline{\tau}}^m \in \mathcal{O}_{\widehat{X|Y_\tau}, p}$ *of* $\mathcal{M}_A(\beta)$*,* $p \in W_{\mathrm{T}(\tau)} \subseteq Y_\tau$*, can be written as follows:*

$$f = \sum_{\sigma \in \mathrm{T}(\tau)} \sum_{\mathbf{m} \in \mathbb{N}^{n-d}} c_{\sigma, \mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}} \mathbf{m})} x_{\overline{\sigma}}^{\mathbf{m}}.$$

Proof. By Lemma 6.5.12 a basis of $\mathcal{H}om(\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}}\mathbf{m}_{\overline{\tau}}, \mathcal{O}_{Y_\tau, p})$ for $p \in W_{\mathrm{T}(\tau)} \subseteq Y_\tau$ is given by the $\mathrm{vol}_{\mathbb{Z}\tau}(\Delta_\tau)$ series $\phi_\sigma^{\mathbf{k}}$ with σ running in the $(d-1)$-simplices of $\mathrm{T}(\tau)$ and $\Lambda_{\mathbf{k}}$ running in the partition of $\mathbb{N}^{l-d}$ (see Remark 6.1.4 and apply it to the matrix A_τ with l columns and $\sigma \subseteq \tau$). In particular we obtain that:

$$f_{m_{\overline{\tau}}}(x_\tau) = \sum_{\sigma \in \mathrm{T}(\tau)} \sum_{m_{\overline{\sigma}\cap\tau} \in \mathbb{N}^{l-d}} c_{\sigma, \mathbf{m}_{\overline{\sigma}}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\tau}}\mathbf{m}_{\overline{\tau}} - A_{\overline{\sigma}\cap\tau}\mathbf{m}_{\overline{\sigma}\cap\tau})} x_{\overline{\sigma}\cap\tau}^{\mathbf{m}_{\overline{\sigma}\cap\tau}}$$

and this implies the result. □

Using the partition $\{\Lambda_{\mathbf{k}(i)} : \ i = 1, \ldots, r\}$ of $\mathbb{N}^{n-d}$ (see Remark 6.1.4) with $r = [\mathbb{Z}A : \mathbb{Z}\sigma]$ we can write the formal solution in Lemma 6.5.13 as:

$$f = \sum_{\sigma \in \mathrm{T}(\tau)} \sum_{i=1}^{r} \sum_{\mathbf{k}(i)+\mathbf{m} \in \Lambda_{\mathbf{k}(i)}} c_{\sigma, \mathbf{k}(i)+\mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}(i)+\mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}(i)+\mathbf{m}}.$$

Let us denote by $v_\sigma^{\mathbf{k}(i)+\mathbf{m}}$ the exponent of $x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}(i)+\mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}(i)+\mathbf{m}}$.

Since Euler operators $E_i - \beta_i$ annihilate every monomial $x^{v_\sigma^{\mathbf{k}(i)+\mathbf{m}}}$ appearing in f we just need to use toric operators $\Box_u = \partial^{u_+} - \partial^{u_-}$ with $u \in L_A = \mathrm{Ker}(A) \cap \mathbb{Z}^n$ in order prove that f is annihilated by $H_A(\beta)$ if and only if the formal series

$$\sum_{\mathbf{k}(i)+\mathbf{m} \in \Lambda_{\mathbf{k}(i)}} c_{\sigma, \mathbf{k}(i)+\mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}(i)+\mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}(i)+\mathbf{m}}$$

is annihilated by $H_A(\beta)$ for all $\sigma \in \mathrm{T}(\tau)$ and $i = 1, \ldots, r$.

This is clear because $v_\sigma^{\mathbf{k}(i)+\mathbf{m}} - v_{\sigma'}^{\mathbf{k}(j)+\mathbf{m}'} \in \mathbb{Z}^n$ if and only if $\sigma = \sigma'$ and $i = j$ (because β is very generic and for fixed σ we have Lemma 6.1.1). Recall here that for $u \in L_A$ any pair of monomials x^v, $x^{v'}$ verify that $\partial^{u_-}(x^v) = [v]_{u_-} x^{v-u_-}$ and $\partial^{u_+}(x^{v'}) = [v']_{u_+} x^{v'-u_+}$ and $x^{v-u_-} = x^{v'-u_+}$ if and only if $v - v' = u$.

Moreover, a series $\sum_{\mathbf{k}(i)+\mathbf{m} \in \Lambda_{\mathbf{k}(i)}} c_{\sigma, \mathbf{k}(i)+\mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}(i)+\mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}(i)+\mathbf{m}}$ is annihilated by I_A if and only if it is $c\phi_\sigma^{\mathbf{k}(i)}$ for certain $c \in \mathbb{C}$.

Thus we obtain that any formal solution of $\mathcal{M}_A(\beta)$ along Y_τ at $p \in W_{\mathrm{T}(\tau)} \subseteq Y_\tau$ is a linear combination of the linearly independent formal solutions $\phi_\sigma^{\mathbf{k}}$ with $\sigma \in T(\tau)$ and $\Lambda_k \in \{\Lambda_{k(i)} : \ 1 \leq i \leq \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) = [\mathbb{Z}A : \mathbb{Z}\sigma]\}$ the partition of $\mathbb{N}^{n-d}$ associated with σ (see Remark 6.1.4). That is, we have a basis with cardinality $\sum_{\sigma \in T(\tau)} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) \overset{(6.13)}{=} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$. This finishes the proof of Theorem 6.5.6.

6.6 Irregularity of $\mathcal{M}_A(\beta)$ along coordinate hyperplanes under some conditions on (A, β).

Assume throughout this section that A is a pointed matrix such that $\mathbb{Z}A = \mathbb{Z}^d$ and that Y is a coordinate hyperplane. Then we have that the irregularity complex of order s, $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$, is a perverse sheaf on Y (see [Meb04]). This implies in particular the existence of an analytic subvariety $S \subseteq Y$ with codimension $q > 0$ in Y such that for all $p \in Y \setminus S$:

$$\chi(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))))_p = \dim(\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p) \tag{6.17}$$

Here $\chi(\mathcal{F}) = \sum_{i\geq 0}(-1)^i \dim(\mathcal{H}^i(\mathcal{F}))$ denotes the Euler-Poincaré characteristic of a bounded constructible complex of sheaves $\mathcal{F} \in \mathrm{D}_c^b(\mathbb{C}_Y)$. The characteristic cycle of $\mathcal{F} \in \mathrm{D}_c^b(\mathbb{C}_Y)$ is the unique lagrangian cycle

$$\mathrm{CCh}(\mathcal{F}) = m_Y T_Y^* Y + \sum_{\alpha\colon \dim Y_\alpha < \dim Y} m_\alpha T_{Y_\alpha}^* Y \subseteq T^*Y$$

that satisfies the index formula:

$$\chi(\mathcal{F}) = \mathrm{Eu}(m_Y Y + \sum_{\alpha\colon \dim Y_\alpha < \dim Y} (-1)^{\mathrm{codim}_Y(Y_\alpha)} m_\alpha \overline{Y_\alpha})$$

where Eu denotes the Euler's isomorphism between the group of cycles on Y and the group of constructible functions on Y with integer values. Thus by (6.17) we have that for all $p \in Y \setminus S$:

$$\dim(\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p = \mathrm{Eu}(\mathrm{CCh}(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))))_p = m_Y \tag{6.18}$$

where m_Y is the multiplicity of $T_Y^* Y$ in $\mathrm{CCh}(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))$.

Y. Laurent and Z. Mebkhout provided a formula in [LM99] to obtain the cycle $\mathrm{CCh}(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))$ in terms of the $(1+\epsilon)$-characteristic cycle and the $(s+\epsilon)$-characteristic cycle of $\mathcal{M}_A(\beta)$ for $\epsilon > 0$ small enough. By [LM99] in order to compute the multiplicity m_Y of $T_Y^* Y$ in $\mathrm{CCh}(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))$ we only need to know the multiplicity of $T_X^* X$ and $T_Y^* X$ in the $(1+\epsilon)$-characteristic cycle of $\mathcal{M}_A(\beta)$ and the $(s+\epsilon)$-characteristic cycle of $\mathcal{M}_A(\beta)$ with respect to Y for $\epsilon > 0$ small enough.

We are going to use the multiplicities formula for the s-characteristic cycle of $\mathcal{M}_A(\beta)$ obtained by M. Schulze and U. Walther in [SW08] in the case when A is pointed and β is non-rank-jumping. First of all we need to recall some definitions given in [SW08].

Let us consider $\Phi_A^s \ni \tau \subseteq \tau' \in \Phi_A^{s,d-1}$ and the natural projection

$$\pi_{\tau,\tau'} : \mathbb{Z}\tau' \to \mathbb{Z}\tau' / (\mathbb{Z}\tau' \cap \mathbb{Q}\tau).$$

Definition 6.6.1. *In a lattice Λ, the volume function vol_Λ is normalized so that the unit simplex of Λ has volume 1. We abbreviate $\mathrm{vol}_{\tau,\tau'} := \mathrm{vol}_{\pi_{\tau,\tau'}(\mathbb{Z}\tau')}$.*

Definition 6.6.2. *For $\Phi_A^s \ni \tau \subseteq \tau' \in \Phi_A^{s,d-1}$, define the polyhedra*

$$P_{\tau,\tau'} := \mathrm{conv}(\pi_{\tau,\tau'}(\tau' \cup \{0\})), \quad Q_{\tau,\tau'} := \mathrm{conv}(\pi_{\tau,\tau'}(\tau' \setminus \tau))$$

where conv *means to take the convex hull.*

The following theorem was proven by M. Schulze and U. Walther (see [SW08, Th. 4.21] and [SW08, Cor. 4.12]).

Theorem 6.6.3. *For generic $\beta \in \mathbb{C}^d$ (more precisely, non-rank-jumping) and $\tau \in \Phi_A^s$, the multiplicity of $\overline{C}_A^\tau$ in the s-characteristic cycle of $\mathcal{M}_A(\beta)$ is:*

$$\mu_A^{s,\tau} = \sum_{\tau \subseteq \tau' \in \Phi_A^s} [\mathbb{Z}^d : \mathbb{Z}\tau] \cdot [(\mathbb{Z}\tau' \cap \mathbb{Q}\tau) : \mathbb{Z}\tau] \cdot \mathrm{vol}_{\tau,\tau'}(P_{\tau,\tau'} \setminus Q_{\tau,\tau'}).$$

Here $\overline{C}_A^\tau$ is the closure in T^*X of the conormal space to the orbit $O_A^\tau \subseteq T_0^*X$, where O_A^τ is the orbit of $1_\tau \in \{0,1\}^n$ ($(1_\tau)_i = 1$ if $a_i \in \tau$, $(1_\tau)_i = 0$ if $a_i \notin \tau$) by the d-torus action:

$$\begin{array}{ccc} (\mathbb{C}^*)^d \times T_0^*X & \longrightarrow & T_0^*X \\ (t,\xi) & \mapsto & t\cdot\xi := (t^{a_1}\xi_1, \ldots, t^{a_n}\xi_n) \end{array}$$

Assume that $Y = \{x_n = 0\}$ by reordering the variables. We are interested in the multiplicities of $\overline{C}_A^\emptyset = T_X^*X$ and $\overline{C}_A^{\{n\}} = T_Y^*X$ in the r-characteristic cycles of $\mathcal{M}_A(\beta)$ for $r = s+\epsilon$ and $r = 1+\epsilon$ with $\epsilon > 0$ small enough. In particular, we need to compute $\mu_A^{s+\epsilon,\emptyset}$, $\mu_A^{s+\epsilon,\{n\}}$, $\mu_A^{1+\epsilon,\emptyset}$ and $\mu_A^{1+\epsilon,\{n\}}$.

It is a classical result that $\mu_A^{1,\emptyset} = \mathrm{rank}(\mathcal{M}_A(\beta)) = \mathrm{vol}_{\mathbb{Z}^d}(\Delta_A)$ for generic β (see [GZK89], [Ado94]).

From [SW08, Corollay 4.22] if $\tau = \emptyset$ then

$$\mu_A^{s,\emptyset} = \mathrm{vol}_{\mathbb{Z}^d}(\cup_{\tau' \in \Phi_A^{s,d-1}}(\Delta_{\tau'}^1 \setminus \mathrm{conv}(\tau'))).$$

Since $\Phi_A^{s+\epsilon}$ is constant for $\epsilon > 0$ small enough we have that all its faces τ are F-homogeneous and then $\mathrm{vol}_{\mathbb{Z}^d}(\mathrm{conv}(\tau)) = 0$. As a consequence,

$$\mu_A^{s+\epsilon,\emptyset} = \mathrm{vol}_{\mathbb{Z}^d}(\cup_{\tau' \in \Phi_A^{s+\epsilon,d-1}}(\Delta_{\tau'}^1)) \tag{6.19}$$

for all $\epsilon > 0$ small enough. Let us compute $\mu_A^{s+\epsilon,\{n\}}$ for $s \geq 1$ and $\epsilon > 0$ small enough.

Consider any $\tau \in \Phi_A^{s+\epsilon,d-1}$ such that $n \in \tau$. Since $\epsilon > 0$ is generic ($\Phi_A^{t,d-1}$ is locally constant at $t = s+\epsilon$) we have that $a_n \notin \mathbb{Q}(\tau \setminus \{a_n\})$ and hence there exists certain $(d-1)$-simplices $\sigma_1, \ldots, \sigma_r$ such that $n \in \sigma_i \subseteq \tau$, $\tau = \cup_i \sigma_i$, $\sigma_i \cap \sigma_j$ is a k-simplex with $k \leq d-2$

($\{\sigma_1, \ldots, \sigma_r\}$ defines a triangulation of τ). Then $\mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau) = \sum_{i=1}^r \mathrm{vol}_{\mathbb{Z}^d}(\Delta_{\sigma_i})$ and we want to prove that

$$\mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau) = [\mathbb{Z}^d : \mathbb{Z}\tau] \cdot [\mathbb{Z}\tau \cap \mathbb{Q}a_n : \mathbb{Z}a_n] \cdot \mathrm{vol}_{\{n\},\tau}(P_{\{n\},\tau} \setminus Q_{\{n\},\tau}) \tag{6.20}$$

Since $\mathbb{Z}\sigma_i \subseteq \mathbb{Z}\tau \subseteq \mathbb{Z}^d$ then $\mathrm{vol}_{\mathbb{Z}^d}(\Delta_{\sigma_i}) = [\mathbb{Z}^d : \mathbb{Z}\sigma_i] = [\mathbb{Z}^d : \mathbb{Z}\tau] \cdot [\mathbb{Z}\tau : \mathbb{Z}\sigma_i]$ so we only need to prove:

$$\sum_{i=1}^r [\mathbb{Z}\tau : \mathbb{Z}\sigma_i] = [\mathbb{Z}\tau \cap \mathbb{Q}a_n : \mathbb{Z}a_n] \cdot \mathrm{vol}_{\{n\},\tau}(P_{\{n\},\tau} \setminus Q_{\{n\},\tau}).$$

But $a_n \notin \mathbb{Q}(\tau \setminus \{a_n\})$ implies that $[\mathbb{Z}\tau \cap \mathbb{Q}a_n : \mathbb{Z}a_n] = 1$ and τ is F-homogeneous so we have to prove that:

$$\mathrm{vol}_{\{n\},\tau}(P_{\{n\},\tau}) = \sum_{i=1}^r [\mathbb{Z}\tau : \mathbb{Z}\sigma_i].$$

We observe that $\pi_{\{n\},\tau}(\tau \cup \{0\}) = (\tau \setminus \{a_n\}) \cup \{0\}$ in $\mathbb{Z}\tau/(\mathbb{Z}\tau \cap \mathbb{Q}a_n) = \mathbb{Z}(\tau \setminus \{a_n\})$. Consider a $(d-2)$-simplex $\widetilde{\sigma}$ such that $\mathbb{Z}\widetilde{\sigma} = \mathbb{Z}(\tau \setminus \{a_n\})$. Since $a_n \notin \sum_{i \in \tau \setminus \{n\}} \mathbb{Q}a_i$ there exists a hyperplane H such that $a_i \in H$ for all $i \in \tau \setminus \{n\}\}$, $0 \in H$ and $\widetilde{\sigma} \subseteq H$. Recall that the Euclidean volume of the convex hull of a bounded polytope Δ contained in a hyperplane $H \subseteq \mathbb{R}^d$ and a point $c \notin H$ is the product of the relative volume $\mathrm{vol}_{rel}(\Delta)$ of Δ and the distance $\mathrm{d}(c, H)$ from c to H divided by $d!$. Hence, we have the following equalities:

$$\mathrm{vol}_{\{n\},\tau}(P_{\{n\},\tau}) = \frac{\mathrm{vol}_{rel}(\Delta_{\tau \setminus \{n\}})}{\mathrm{vol}_{rel}(\Delta_{\widetilde{\sigma}})} = \frac{\mathrm{vol}(\Delta_\tau)}{\mathrm{vol}(\Delta_{\widetilde{\sigma} \cup \{n\}})} = \sum_{i=1}^r \frac{\mathrm{vol}(\Delta_{\sigma_i})}{\mathrm{vol}(\Delta_{\widetilde{\sigma} \cup \{n\}})} =$$
$$= \sum_{i=1}^r \frac{[\mathbb{Z}^d : \mathbb{Z}\sigma_i]}{[\mathbb{Z}^d : \mathbb{Z}\tau]} = \sum_{i=1}^r [\mathbb{Z}\tau : \mathbb{Z}\sigma_i].$$

We have proved (6.20) and as a consequence the following Lemma.

Lemma 6.6.4. *Consider $s \geq 1$ and β non-rank-jumping. Then for all $\epsilon > 0$ small enough:*

$$\mu_A^{s+\epsilon,\{n\}} = \sum_{n \in \tau \in \Phi_A^{s+\epsilon}} \mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau).$$

We close this section with the following result about the irregularity along any coordinate hyperplane Y of the hypergeometric system $\mathcal{M}_A(\beta)$ associated with a full rank pointed matrix A with $\mathbb{Z}A = \mathbb{Z}^d$.

Theorem 6.6.5. *If $\beta \in \mathbb{C}^d$ is generic (more precisely, non-rank-jumping) then*

$$\dim_{\mathbb{C}}(\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p) = \sum_{n \notin \tau \in \Phi_A^s \setminus \Phi_A^1} \mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau)$$

for all $p \in Y \setminus S$, where S is a subvariety of Y with $\dim S < \dim Y$. Then, for very generic β the nonzero classes in $\mathcal{Q}_Y(s)$ of the constructed series $\phi_\sigma^{\mathbf{k}}$ with $\sigma \in \mathrm{T}'$ form a basis in their common domain of definition $U \subseteq Y$.

Proof. Using the results of [LM99] we have that

$$\dim_{\mathbb{C}}(\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p) = \mu_A^{s+\epsilon,\emptyset} - \mu_A^{1+\epsilon,\emptyset} + \mu_A^{1+\epsilon,\{n\}} - \mu_A^{s+\epsilon,\{n\}}$$

Then by Lemma 6.6.4 and (6.19) we obtain

$$\dim_{\mathbb{C}}(\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p) = \sum_{n \notin \tau \in \Phi_A^{s+\epsilon}} \mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau) - \sum_{n \notin \tau \in \Phi_A^{1+\epsilon}} \mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau)$$

But the set of facets $\tau \in \Phi_A^{s+\epsilon}$ such that $n \notin \tau$ coincide with the set of facets $\tau \in \Phi_A^s$ such that $n \notin \tau$ when $\epsilon > 0$ is small enough because, in that case, for any $(d-1)$-simplex $\sigma \subseteq \tau$ $|A_\sigma^{-1} a_n| < s$ if and only if $|A_\sigma^{-1} a_n| < s + \epsilon$ and the value of $|A_\sigma^{-1} a_j| \leq 1$ does not depend on s. On the other hand, $\{\tau : \ n \notin \tau \in \Phi_A^1\} \subseteq \{\tau : \ n \notin \tau \in \Phi_A^s\}$. Thus, we have the result. □

Remark 6.6.6. *Notice that Theorem 6.6.5 implies that under the assumptions of this section equality holds in (6.16).*

6.7 Remarks and conclusions

1) An anonymous referee of the paper [FC$_2$08] asked us the following question. Is there some understanding how Gevrey solutions of $\mathcal{M}_A(\beta)$ relate to solutions of $\mathcal{M}_{A^h}(\beta^h)$ with A^h the matrix obtained from A by adding a row of 1's and then a column equal to the first unit vector? The idea is to consider a regular triangulation T for the matrix A^h containing a regular triangulation $\mathrm{T}(\tau)$ of A_τ^h such that the added column $a_0 = \binom{1}{\mathbf{0}}$ is a vertex of any d-simplex in $\mathrm{T}(\tau)$. For any d-simplex $\{a_0\} \cup \sigma \in T(\tau)$, the dehomogenization (in the sense of [OT07, Definition 2]) of the $\mathrm{vol}_{\mathbb{Z}A^h}(\Delta_{\{a_0\}\cup\sigma}) = \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ holomorphic solutions $\phi_\sigma^{\mathbf{k}}$ of $\mathcal{M}_{A^h}(\beta^h)$ associated with $\{a_0\} \cup \sigma$ are Gevrey solutions of $\mathcal{M}_A(\beta)$ with respect to Y_τ associated with σ.

2) Given a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ and a smooth subvariety $Y \subseteq X$, Z. Mebkhout defined the Newton polygon $N(\mathcal{M}, Y)_\alpha$ of $\mathcal{M}$ with respect to Y along each irreducible component $Y_\alpha \subseteq Y$ of the characteristic variety of $\mathrm{Irr}_Y(\mathcal{M})$ (see [Meb96]). More precisely, if we consider the slopes $1 < s_r < \ldots < s_1 < 1$ of $\mathcal{M}$ with respect to Y and denote by m_{α,s_i} the multiplicity of $T_{Y_\alpha}^* Y$ in the characteristic cycle of $\mathrm{Gr}_{s_i}(\mathrm{Irr}_Y(\mathcal{M}))$, then $N(\mathcal{M}, Y)_\alpha$ is the convex hull of $(0,0) - \mathbb{N}^2$ and the points

$$(\sum_{j=1}^{i}(s_j - 1)m_{\alpha,s_j}, -\sum_{j=1}^{i} m_{\alpha,s_j})$$

for $i = 1, \ldots, r$. Thus, under the assumptions of Section 6.6 we can compute the Newton polygon $N(\mathcal{M}_A(\beta), Y)_Y$ of $\mathcal{M}_A(\beta)$ with respect to $Y = \{x_n = 0\}$ along Y. To this end, recall that for $Y_\alpha = Y$ we have that $m_{Y,s} = m_{\alpha,s} = \mu_A^{s+\epsilon,\emptyset} - \mu_A^{s+\epsilon,\{n\}} + \mu_A^{s-\epsilon,\{n\}} - \mu_A^{s-\epsilon,\emptyset}$ for $\epsilon > 0$ small enough (applying the results of [LM99] to the case $\mathcal{M} = \mathcal{M}_A(\beta)$ and $Y = \{x_n = 0\}$). Now we can use Lemma 6.6.4 and (6.19) to conclude that $m_{Y,s} = \sum_{\eta: a_n/s \in H_\eta} \operatorname{vol}_{\mathbb{Z}^d}(\Delta_\eta)$, where η denotes a facet of Δ_τ with $0 \notin \eta$ and H_η is the unique hyperplane that contains η. There are a finite set of rational numbers $1 < s_r < \ldots < s_1 < \infty$ such that a_n/s_i lie at least in one hyperplane H_η supported in one facet η of Δ_τ with $0 \notin \eta$. We have that $s_i = |A_\sigma^{-1} a_n|$ for any $(d-1)$-simplex $\sigma \subseteq H_\eta \cap \{a_j : \ j = 1, \ldots, n-1\}$ with $a_n/s_i \in H_\eta$. This implies that $N(\mathcal{M}_A(\beta), Y)_Y$ is the convex hull of $(0,0) - \mathbb{N}^2$ and the points

$$(\sum_{j=1}^{i}(s_j - 1) \sum_{\eta:\ a_n/s_j \in H_\eta} \operatorname{vol}_{\mathbb{Z}A}(\Delta_\eta), -\sum_{j=1}^{i} \sum_{\eta:\ a_n/s_j \in H_\eta} \operatorname{vol}_{\mathbb{Z}A}(\Delta_\eta))$$

for $i = 1, \ldots, r$.

Let us illustrate this comment with an example. Let A be the matrix with columns $a_1 = \binom{1}{0}$, $a_2 = \binom{1}{1}$, $a_3 = \binom{0}{2}$, $a_4 = \binom{3}{1}$ and consider $\tau = \{1, 2, 3\}$. Then $Y_\tau = \{x_4 = 0\}$, Δ_τ is the convex hull of $\{a_1, a_2, a_3, 0\} \subseteq \mathbb{R}^2$ and $a_4 \notin \Delta_\tau$ (see Figure 6.5).

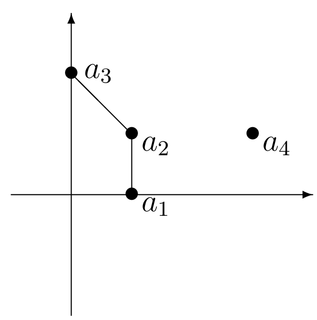

Figure 6.5

The unique regular triangulation $\mathrm{T}(\tau)$ of A_τ that satisfies (6.13) is determined by $\sigma_1 = \{a_1, a_2\}$ and $\sigma_2 = \{a_2, a_3\}$. Notice that $\mathbb{Z}A = \mathbb{Z}^2$, $\operatorname{vol}_{\mathbb{Z}^2}(\Delta_{\sigma_1}) = 1$, $\operatorname{vol}_{\mathbb{Z}^2}(\Delta_{\sigma_2}) = 2$ and $1 < s_2 = |A_{\sigma_1}^{-1} a_4| = 2 < s_1 = |A_{\sigma_1}^{-1} a_4| = 3 < +\infty$. Thus, the Newton polygon of $\mathcal{M}_A(\beta)$ with respect to Y_τ at a generic point $p \in Y_\tau$ is the convex hull of $((0,0) - \mathbb{N}^2)$ and the points $(2, -1)$ and $(4, -3)$ (see Figure 6.6).

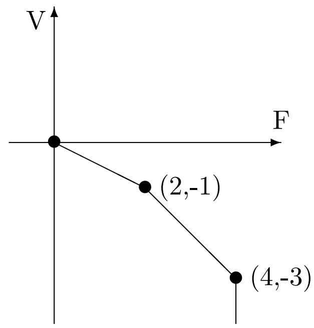

Figure 6.6 $N(\mathcal{M}_A(\beta), Y_\tau)_p$

3) Corollary 6.2.2 uses that Y_τ has codimension one because otherwise the irregularity complex of a holonomic $\mathcal{D}$-module with respect to Y_τ is not a perverse sheaf and the irregularity complexes of order s along Y_τ do not determine a Gevrey filtration.

Chapter 7

Restriction of $\mathcal{M}_A(\beta)$ with respect to a coordinate subspace

This Chapter is a joint work with Uli Walther (Purdue University). Our aim here is the computation of the restriction of $\mathcal{M}_A(\beta)$ with respect to a coordinate subspace under some conditions to be precise in the sequel. In particular, we generalize [CT03, Th. 4.4] and Theorem 5.1.3 (that imply Corollary 5.1.4).

Let $A = (a_1 \ \cdots \ a_n) \in \mathbb{Z}^{d\times n}$ be a full rank matrix with integer entries. Consider a subset $\tau \subseteq \{1,\ldots,n\}$ and denote $Y_\tau = \{x_i = 0 : \ i \notin \tau\}$. Let $i_\tau : Y_\tau \hookrightarrow X = \mathbb{C}^n$ denote the natural inclusion. Then we will prove the following result:

Theorem 7.0.1. *If one of the following conditions holds:*

i) $\beta \in \mathbb{C}^d$ *is generic and* $\mathbb{Q}_{\geq 0}A = \mathbb{Q}_{\geq 0}A_\tau$.

ii) $\beta \in \mathbb{C}^d$ *is very generic and* $\operatorname{rank}(A_\tau) = d$.

then the (derived) inverse image of $\mathcal{M}_A(\beta)$ *is given by*

$$\mathbb{L}^{-k} i_\tau^* \mathcal{M}_A(\beta) \simeq \begin{cases} \oplus_{\lambda\in\Omega}\mathcal{M}_A(\beta-\lambda) & \text{if } k=0 \\ 0 & \text{if } k>0 \end{cases}$$

for some subset Ω *of* $\{A_{\overline{\tau}}\mathbf{k} : \ \mathbf{k}\in\mathbb{Z}^{n-d}\}$ *with cardinality* $[\mathbb{Z}A : \mathbb{Z}\tau]$ *generating* $\mathbb{Z}A/\mathbb{Z}\tau$.

Notation 7.0.2. *For any subset* $\tau \subseteq \{1,\ldots,n\}$ *we shall write* x_τ *and* ∂_τ *for* $(x_i)_{i\in\tau}$ *and* $(\partial_i)_{i\in\tau}$ *respectively. We will consider the polynomial ring* $R_\tau = \mathbb{C}[\partial_\tau]$ *and the Weyl Algebra* $D_\tau = \mathbb{C}[x_\tau]\langle\partial_\tau\rangle$. *We also denote* $S_\tau = \mathbb{C}[t^{a_i} : i\in\tau] \simeq R_\tau/I_\tau$, *where* I_τ *is the toric ideal associated with the submatrix* A_τ.

We can prove Theorem 7.0.1 using D_A (resp. D_τ) instead of the sheaf of linear differential operators $\mathcal{D}_A$ (resp. $\mathcal{D}_\tau$) because of the flatness of the fibers of $\mathcal{D}_A$ (resp. $\mathcal{D}_\tau$) with respect to D_A (resp. D_τ).

Set the $\mathbb{Z}^d$-grading given by $\deg(\partial_i) = -a_i = -\deg(x_i)$ for $i = 1, \ldots, n$ in D_A. This is compatible with a $\mathbb{Z}^d$-grading in S_A, $\deg(\partial_i) = -a_i$.

We know that $\mathbb{L}i_\tau^* \mathcal{M}_A(\beta)$ is quasi-isomorphic to the Koszul complex $K_\bullet(x_i \cdot : \ i \notin \tau; \mathcal{M}_A(\beta))$ as a complex of left $\mathcal{D}_\tau$-modules (see for example [MT04, Proposition 3.1]).

Recall that the actions $x_i\cdot$, $i \notin \tau$, are endomorphisms of left $\mathcal{D}_\tau$-modules but they are not endomorphisms of left $\mathcal{D}_A$-modules.

On the other hand we have by [MMW05, Theorem 6.6] that the Koszul complex $\mathcal{K}_\bullet(E_A - \beta, S_A)$ (see Definition 4.2 in [MMW05]) is a resolution of $\mathcal{M}_A(\beta)$ whenever β is not rank-jumping for A (see [MMW05, Theorem 6.6]). Thus, we substitute $\mathcal{M}_A(\beta)$ by $\mathcal{K}_\bullet(E_A - \beta, S_A)$ in $K_\bullet(x_i \cdot : \ i \notin \tau; \mathcal{M}_A(\beta))$ obtaining a double complex $C_{\bullet,\bullet} = K_\bullet(x_i \cdot : \ i \notin \tau; \mathcal{K}_\bullet(E_A - \beta, S_A))$.

Recall that $\mathcal{K}_\bullet(E_A - \beta, S_A) = \oplus_{\alpha \in \mathbb{Z}^d} K_\bullet(E_A - \beta - \alpha, (D_A \otimes_{R_A} S_A)_\alpha)$ by Lemma 4.3 in [MMW05]). Notice that $x_i\cdot$ sends an element in $(D_A \otimes_{\mathbb{C}[\partial_A]} S_A)_\alpha$ to an element in $(D_A \otimes_{R_A} S_A)_{\alpha + a_i}$ and that $x_i(E_A - \beta - \alpha) = (E_A - \beta - \alpha - a_i)x_i$. Thus, all the diagrams in $C_{\bullet,\bullet}$ are commutative.

On the other hand, we have the following natural isomorphism of D_A-modules, which is also observed in [MMW05],

$$\begin{array}{ccc} (D_A \otimes_{R_A} S_A) & \longrightarrow & \mathbb{C}[x_{\overline{\tau}}] \otimes_{\mathbb{C}} (D_\tau \otimes_{R_\tau} S_A) \\ x_\tau^{\mu_\tau} x_{\overline{\tau}}^{\mu_{\overline{\tau}}} \partial_\tau^{\nu_\tau} \partial_{\overline{\tau}}^{\nu_{\overline{\tau}}} \otimes m & \mapsto & x_{\overline{\tau}}^{\mu_{\overline{\tau}}} \otimes (x_\tau^{\mu_\tau} \partial_\tau^{\nu_\tau}) \otimes \partial_{\overline{\tau}}^{\nu_{\overline{\tau}}} m \end{array} \tag{7.1}$$

Remark 7.0.3. *The i-th row of the double complex $C_{\bullet,\bullet}$ is a direct sum of $\binom{d}{i}$ copies of the complex $K_\bullet(x_i \cdot : \ i \notin \tau, ; D_A \otimes_{R_A} S_A)$. Thus, using the natural isomorphism (7.1), we have that this row is quasi-isomorphic to a direct sum of $\binom{d}{i}$ copies of the complex:*

$$K_\bullet(x_i \cdot : \ i \notin \tau; \mathbb{C}[x_{\overline{\tau}}]) \otimes_{\mathbb{C}} (D_\tau \otimes_{R_\tau} S_A) \tag{7.2}$$

Let π be the natural projection of $C_{\bullet,\bullet}$ to $K_\bullet(x_i \cdot : i \notin \tau; \mathcal{M}_A(\beta))$ and let η be the natural projection of $C_{\bullet,\bullet}$ to $\mathbb{C} \otimes_{\mathbb{C}} \mathcal{K}_\bullet(E_\tau - \beta, S_A)$. Consider the induced morphisms of the total complexes:

$$\operatorname{Tot}(C_{\bullet,\bullet}) \xrightarrow{\pi} \operatorname{Tot}(K_\bullet(x_i \cdot : i \notin \tau; \mathcal{M}_A(\beta))) = K_\bullet(x_i \cdot : i \notin \tau; \mathcal{M}_A(\beta)) \tag{7.3}$$

and

$$\mathrm{Tot}(C_{\bullet,\bullet}) \xrightarrow{\eta} \mathrm{Tot}(\mathbb{C} \otimes_{\mathbb{C}} \mathcal{K}_{\bullet}(E_\tau - \beta, S_A)) = \mathcal{K}_{\bullet}(E_\tau - \beta, S_A). \tag{7.4}$$

Since $\mathcal{K}_{\bullet}(E_A - \beta, S_A)$ is a resolution of $\mathcal{M}_A(\beta)$ if β is not rank-jumping (see [MMW05, Theorem 6.6]), we have that both complexes in (7.3) compute $\mathrm{Tor}_{\bullet}^{D_A}(\frac{D_A}{\sum_{i\in\tau} x_i D_A}, \mathcal{M}_A(\beta))$ and π is a quasi-isomorphism. Moreover, η is a quasi-isomorphism too because of Remark 7.0.3. Thus, we have the following result.

Lemma 7.0.4. *If β is not rank-jumping for A then*

$$\mathbb{L}i_\tau^* \mathcal{M}_A(\beta) \simeq \mathcal{K}_{\bullet}(E_\tau - \beta, S_A)$$

as complexes of left $\mathcal{D}_\tau$-modules.

Remark 7.0.5. *Let us denote $\mathcal{H}_i(E_\tau - \beta, S_A) = \mathcal{H}_i(\mathcal{K}_{\bullet}(E_\tau - \beta, S_A))$. For all $\beta \in \mathbb{C}^d$, we also have that $i_\tau^* \mathcal{M}_A(\beta) \simeq \mathcal{H}_0(E_\tau - \beta, S_A)$ because $\mathcal{M}_A(\beta) = \mathcal{H}_0(E_A - \beta, S_A)$, $i_\tau^*(\mathcal{K}_{\bullet}(E_A - \beta, S_A)) = \frac{D_A}{\sum_{i\in\tau} x_i D_A} \otimes_{D_A} \mathcal{K}_{\bullet}(E_A - \beta, S_A)$ is quasi-isomorphic to $\mathcal{K}_{\bullet}(E_\tau - \beta, S_A)$ and i_τ^* is right exact.*

It is clear that $\mathbb{Q}_{\geq 0} A = \mathbb{Q}_{\geq 0} A_\tau$ implies that $\mathrm{rank}(A_\tau) = \mathrm{rank}(A) = d$ and this last condition is equivalent to $[\mathbb{Z}A : \mathbb{Z}\tau] < +\infty$.

Consider $S_A = \mathbb{C}[\mathbb{N}A]$ and $S_\tau = \mathbb{C}[\mathbb{N}A_\tau]$. Then the assumption $\mathbb{Q}_{\geq 0} A = \mathbb{Q}_{\geq 0} A_\tau$ guarantees that S_A is a finitely generated $\mathbb{Z}^d$-graded S_τ-module and so it is a toric R_τ-module (see Definition 4.5. and Example 4.7 in [MMW05]). If we don't assume $\mathbb{Q}_{\geq 0} A = \mathbb{Q}_{\geq 0} A_\tau$ but only $\mathrm{rank}(A_\tau) = \mathrm{rank}(A) = d$ then we have that S_A is a weakly toric R_τ-module (see [SW07]).

Notice that there exists a subset $\Omega \subseteq \mathbb{Z}A_{\overline{\tau}}$ with cardinality $[\mathbb{Z}A : \mathbb{Z}\tau]$ such that

$$\oplus_{\lambda\in\Omega} S_\tau(\lambda) \subseteq S_A$$

where $S_\tau(\lambda) = t^\lambda S_\tau$, and there are not more (shifted) copies of S_τ in $S_A \setminus \oplus_{\lambda\in\Omega} S_\tau(\lambda)$.

If we assume $i)$ we have that the quotient $Q = S_A/(\oplus_{\lambda\in\Omega} S_\tau(\lambda))$ verifies that $\dim(Q) < d$. We can consider the long exact sequence of Euler-Koszul homology associated to the short exact sequence

$$0 \to \oplus_{\lambda\in\Omega} S_\tau(\lambda) \longrightarrow S_A \longrightarrow Q \to 0$$

By [MMW05, Proposition 5.3] vanishing of $\mathcal{H}_i(E_\tau - \beta, Q)$ for all $i \geq 0$ is equivalent to $-\beta \notin \mathrm{qdeg}(Q)$. This last condition is satisfied by generic parameters vectors β when $\mathbb{Q}_{\geq 0} A = \mathbb{Q}_{\geq 0} A_\tau$ because $\dim(Q) < d$ in that case. For all $\beta \notin -\mathrm{qdeg}(Q)$, we have

$$\mathcal{H}_i(E_\tau - \beta, S_A) \simeq \mathcal{H}_i(E_\tau - \beta, \oplus_{\lambda\in\Omega} S_\tau(\lambda)) =$$
$$\oplus_{\lambda\in\Omega} \mathcal{H}_i(E_\tau - \beta, S_\tau(\lambda)) \simeq \oplus_{\lambda\in\Omega} \mathcal{H}_i(E_\tau - \beta + \lambda, S_\tau)(\lambda)$$

for all $i \geq 0$. Then, we use that $\mathcal{H}_0(E_\tau - \beta + \lambda, S_\tau) = \mathcal{M}_{A_\tau}(\beta - \lambda)$ to obtain Theorem 7.0.1. Furthermore, notice that $\mathcal{H}_k(E_\tau - \beta + \lambda, S_\tau) = 0$ for all $k > 1$ if $\beta - \lambda$ is not rank-jumping for A_τ for all $\lambda \in \Omega$ and this implies that $\mathbb{L}^{-k} i_\tau^* \mathcal{M}_A(\beta) = 0$ for all $k > 0$. If we assume *ii)* then the argument is similar if we use [SW07, Theorem 5.4] instead of [MMW05, Proposition 5.3]. This finishes the proof of Theorem 7.0.1.

Remark 7.0.6. *If $\mathcal{M}$ is a holonomic $\mathcal{D}$-module and Y_τ is non-characteristic for $\mathcal{M}$ it is known that $\mathbb{L}^{-k} i_\tau^* \mathcal{M} = 0$ for all $k > 1$ and that the holonomic rank of $\mathcal{M}$ coincide with the holonomic rank of $i_\tau^* \mathcal{M}$ (see for example [MT04]). We notice that Y_τ is non-characteristic for $\mathcal{M}_A(\beta)$ if and only if τ contains all the columns of A that are vertices of Δ_A.*

Example 7.0.7. *Let us consider the matrix $A = (a_0\ a_1\ a_2\ a_3\ a_4)$ with $a_i = \binom{1}{i}$ for $i = 0, 1, 2, 3, 4$. In this case we have that $\mathbb{Q}_{\geq 0} A \cap \mathbb{Z} A = \mathbb{N} A$. Thus, I_A is normal and hence Cohen-Macaulay. This implies that the holonomic rank of $\mathcal{M}_A(\beta)$ equals $\mathrm{vol}_{\mathbb{Z}A}(\Delta_A) = 4$ for all $\beta \in \mathbb{C}^2$ (see [MMW05, Corollary 9.2]). Consider $\tau = \{0, 1, 3, 4\}$ and $Y_\tau = \{x_2 = 0\}$. In particular, by Remark 7.0.6 we have that the holonomic rank of $i_\tau^* \mathcal{M}_A(\beta)$ is 4 for all $\beta \in \mathbb{C}^2$.*

On the other hand, the toric ideal I_τ associated with A_τ is not Cohen-Macauly and $\beta = \binom{1}{2}$ is a rank-jumping parameter for $\mathcal{M}_{A_\tau}(\beta)$. More precisely, the holonomic rank of $\mathcal{M}_{A_\tau}(\beta)$ is $\mathrm{vol}_{\mathbb{Z}\tau}(\Delta_\tau) = 4$ for all $\beta \in \mathbb{C}^2 \setminus \{\binom{1}{2}\}$ while the holonomic rank of $\mathcal{M}_{A_\tau}(\beta)$ is 5 for $\beta = \binom{1}{2}$ (see [ST98]). This implies that for $\beta = \binom{1}{2}$, $\mathcal{M}_{A_\tau}(\beta)$ cannot be isomorphic to $i_\tau^* \mathcal{M}_A(\beta')$ for any $\beta' \in \mathbb{C}^2$.

Chapter 8

Appendix

Let us denote $\mathcal{O}_{X|Y}(< s) := \cup_{s'<s}\mathcal{O}_{X|Y}(s')$ for $s \in \mathbb{R}$. For the sake of completeness we include a proof of the following result.

Lemma 8.0.8. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module such that there exists a series $f \in \mathcal{O}_{X|Y}(s)_p$ with Gevrey index $s > 1$ whose class in*

$$(\mathcal{O}_{X|Y}(s)/\mathcal{O}_{X|Y}(< s))_p$$

is a solution of $\mathcal{M}$, for all p in a relatively open set $U \subseteq Y$. Then s is a slope of $\mathcal{M}$ along Y at any point in the closure of U.

Proof. Any holonomic $\mathcal{D}$-module is cyclic (see [Bjö93, Proposition 3.1.5]). Thus, we can assume without loss of generality that $\mathcal{M} = \mathcal{D}/\mathcal{I}$ with $\mathcal{I}$ a sheaf of ideals generated by some differential operators $P_1, \ldots, P_m \in \mathcal{D}(U)$. Then, by the assumption, there exists $s_i < s$ such that $P_i(f_p) \in \mathcal{O}_{X|Y}(s_i)_p$, $i = 1, \ldots, m$. For $s' = \max\{s_i\} < s$ we have that $(\overline{P_i(f)})_{i=1}^m \in (\mathcal{Q}_Y(s'))^m$ verifies all the left $\mathcal{D}$-relations verified by $(P_i)_{i=1}^m$. Thus, we can consider its class in $\mathcal{H}^1(\mathrm{Irr}_Y^{(s')}(\mathcal{M}))$.

Since Y is a smooth hypersurface, $\mathrm{Irr}_Y^{(s')}(M)$ is a perverse sheaf on Y [Meb90]. In particular, the support S of the sheaf $\mathcal{H}^1(\mathrm{Irr}_Y^{(s')}(\mathcal{M}))$ has at most dimension equal to $\dim Y - 1$, so the relatively open set $U' = U \setminus \overline{S} \subseteq Y$ verifies that $\mathcal{H}^1(\mathrm{Irr}_Y^{(s')}(\mathcal{M}))_{|U'} = 0$ and its closure is equal to the one of U.

In particular the class of $(\overline{P_i(f)})_{i=1}^m \in (\mathcal{Q}_Y(s'))^m$ in $\mathcal{H}^1(\mathrm{Irr}_Y^{(s')}(\mathcal{M}))_{|U'}$ is zero. This implies the existence of $\overline{h} \in \mathcal{Q}_Y(s')$ such that $(\overline{P_i(h)})_{i=1}^m = (\overline{P_i(f)})_{i=1}^m$ in $(Q_Y(s')_p)^m$. Equivalently, $P_i(f-h)$ is convergent at any point of U' for all $i = 1, \ldots, m$, and we also have that $f - h$ has Gevrey index s because f has Gevrey index s and h has Gevrey index $s' < s$.

The last assertion means that

$$\overline{f-h} \in \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{Q}_Y(s))_{|U'} \setminus \bigcup_{s'<s} \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{Q}_Y(s'))_{|U'}$$

and therefore s is a slope of $\mathcal{M}$ along Y at any point in the closure of U. □

Remark 8.0.9. *By the results of [Meb90] there exists a Whitney stratification $\{Y_\alpha\}_\alpha$ of Y such that $\mathcal{H}^i(\mathrm{Irr}_Y^{(s)}(\mathcal{M}))_{|Y_\alpha}$ are locally constant sheaves for all $s \geq 1$ and $i \geq 0$. If Y is an irreducible algebraic hypersurface and Y_α are algebraic subvarieties then the set $Y_\gamma = Y \setminus \cup_{\dim Y_\alpha < n-1} Y_\alpha$ is a connected stratum (see [Gro62, Théorème 2.1.]). Thus, if $U \cap Y_\gamma$ is a relatively open set in Y_γ and s is a slope of $\mathcal{M}$ along Y at any point of U, we have that s is a slope of $\mathcal{M}$ along Y at any point of Y_γ. This implies that s is a slope of $\mathcal{M}$ along Y at any point of Y by Definition 2.4.8 because Y is the analytic closure of Y_γ.*

Resumen en español

Esta memoria versa sobre la irregularidad de los $\mathcal{D}$-módulos hipergeométricos $\mathcal{M}_A(\beta)$ (cf. Definición 3.1.3), introducidos por Gelfand, Graev, Kapranov y Zelevinsky [GGZ87], [GZK89]. Nuestro estudio se basa principalmente en la construcción explícita de soluciones Gevrey de estos $\mathcal{D}$-módulos a lo largo de variedades de coordenadas y en el uso de resultados generales de la Teoría de $\mathcal{D}$-módulos.

La memoria consta de una introducción (Capítulo 1), dos capítulos de resultados preliminares (Capítulos 2 y 3), cuatro capítulos con resultados originales (Capítulos 4,5,6 y 7) y un apéndice (Capítulo 8).

Comencemos introduciendo algunas nociones y resultados generales de la teoría de irregularidad de los $\mathcal{D}$-módulos. Sean X una variedad analítica compleja y $\mathcal{D}_X$ el haz de operadores diferenciales lineales con coeficientes en el haz de funciones holomorfas $\mathcal{O}_X$.

Un problema fundamental en el estudio de la irregularidad de cualquier $\mathcal{D}_X$–módulo holónomo $\mathcal{M}$ es la descripción de sus pendientes analíticas a lo largo de una hipersuperficie Y de X (cf. Z. Mebkhout [Meb90]). Una pendiente analítica $s > 1$ es un salto en la filtración Gevrey $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$ del complejo de irregularidad $\mathrm{Irr}_Y(\mathcal{M})$ (cf. Definiciones 2.4.4 y 2.4.8). La descripción de las soluciones Gevrey de un $\mathcal{D}$-módulo holónomo a lo largo de una variedad lisa Z es otro problema fundamental relacionado estrechamente con la irregularidad. En particular, si Z es una hipersuperficie lisa, el índice de cualquier solución Gevrey de $\mathcal{M}$ a lo largo de Z no convergente es una pendiente analítica de $\mathcal{M}$ a lo largo de Z.

Por otro lado, Y. Laurent definió las pendientes algebraicas de un D_X-módulo coherente $\mathcal{M}$ a lo largo de una variedad lisa Z (cf. Definición 2.4.9) como aquéllos números reales $s > 1$ tales que la variedad s–micro-característica de $\mathcal{M}$ respecto de Z no sea homogénea respecto de la filtración por el orden de los operadores diferenciales. Además, probó que las pendientes de $\mathcal{M}$ a lo largo de Z son racionales y que el conjunto de ellas es finito (cf. [Lau87]).

Si $\mathcal{M}$ es un $\mathcal{D}$-módulo holónomo y Z es una hipersuperficie lisa, el teorema de comparación de las pendientes [LM99] afirma que las pendientes analíticas y las algebraicas coinciden (cf. Teorema 2.4.11). Mebkhout ha comentado que desde 1986 se ha propuesto una definición de pendiente analítica de un $\mathcal{D}$-módulo holónomo $\mathcal{M}$ a lo largo de una subvariedad lisa Y usando una explosión $p : \widetilde{X} \longrightarrow X$ de Y en X y considerando las pendientes analíticas del $\mathcal{D}$-módulo holónomo $p^*\mathcal{M}$ a lo largo de la hipersuperficie lisa $p^{-1}Y$. Sin embargo, también hizo destacar que no conoce resultados significativos relativos a esta definición.

Consideremos la variedad analítica compleja $X = \mathbb{C}^n$ y denotemos $\mathcal{D} := \mathcal{D}_X$. También denotamos $\partial_i := \frac{\partial}{\partial x_i}$ la derivada parcial i–ésima.

Nuestros objetos de estudio son los $\mathcal{D}$–módulos hipergeométricos $\mathcal{M}_A(\beta)$, que están asociados a un par (A, β), donde $A = (a_{ij})$ es una matriz entera $d \times n$ de rango máximo $d \leq n$ y $\beta \in \mathbb{C}^d$ es un vector de parámetros complejos (cf. Definición 3.1.3). Un objetivo general en el estudio de estos sistemas es la descripción de sus invariantes en términos de la combinatoria del par (A, β).

El libro [SST00] es una buena introducción a la teoría de sistemas hipergeométricos. Estos sistemas son holónomos y la dimensión del espacio de sus soluciones holomorfas en un punto no singular es igual al volumen normalizado de la envolvente convexa de las columnas de $A = (a_i)_{i=1}^n \in \mathbb{Z}^{d\times n}$ y el origen respecto del retículo $\mathbb{Z}A := \sum_{i=1}^n \mathbb{Z}a_i \subseteq \mathbb{Z}^d$ (cf. Definición 6.6.1) cuando β es genérico o I_A es Cohen-Macaulay (cf. [GZK89], [Ado94]). En [MMW05], [Ber08] y las referencias que se encuentran en ellos, pueden encontrarse resultados precisos sobre los saltos de rango de $\mathcal{M}_A(\beta)$ al variar β. Varios autores han estudiado las soluciones holomorfas de $\mathcal{M}_A(\beta)$ en puntos no singulares (cf. [GZK89], [SST00] y [OT07]).

Expliquemos la estructura de esta memoria. En el Capítulo 2 recordamos conceptos y resultados generarles sobre la irregularidad en teoría de $\mathcal{D}$-módulos, principalmente siguiendo [Meb90] y [LM99]. En el Capítulo 3 introducimos los sistemas hipergeométricos [GGZ87], [GZK89], y exponemos algunos resultados conocidos.

En el Capítulo 4 describimos completamente el complejo de irregularidad de los sistemas hipergeométricos en dos variables por métodos elementales. En el Capítulo 5 calculamos los haces de cohomología del complejo de irregularidad de $\mathcal{M}_A(\beta)$ a lo largo de su lugar singular para toda matriz fila de enteros $A = (a_1 \cdots a_n)$ tal que $0 < a_1 < \cdots < a_n$ (cf. [$FC_2$08]). Nuestro método es reducir el problema al caso de dos variables y usar algunos teoremas de restricción y resultados profundos de Teoría de $\mathcal{D}$–módulos.

La estructura del Capítulo 6 es la siguiente. En la Sección 6.1 consideramos un símplice σ, i.e., un conjunto $\sigma \subseteq \{1, \ldots, n\}$ tal que la submatriz $A_\sigma = (a_i)_{i\in\sigma}$ de A es invertible, y usamos ciertas Γ–series introducidas en [GZK89] y generalizadas en [SST00] para construir explícitamente un conjunto linealmente independiente de soluciones Gevrey de $\mathcal{M}_A(\beta)$ a lo largo de $Y_\sigma = \{x_i = 0 : \ i \notin \sigma\}$. El cardinal de este conjunto de soluciones es el volumen normalizado de A_σ respecto al retículo $\mathbb{Z}A$. También probamos que tienen orden Gevrey $s = \max\{|A_\sigma^{-1} a_i| : \ i \notin \sigma\}$ a lo largo del subespacio $Y = \{x_i = 0 : \ |A_\sigma^{-1} a_i| > 1\} \supseteq Y_\sigma$ y que s es su índice Gevrey cuando β es muy genérico.

En la Sección 6.2 construímos para cada símplice σ y cada vector de parámetros β un conjunto de series Gevrey a lo largo de Y con índice s que son soluciones de $\mathcal{M}_A(\beta)$ módulo el haz de series Gevrey con índice más pequeño. Esto implica que si Y es un hiperplano y $s > 1$ se tiene que s es una pendiente analítica de $\mathcal{M}_A(\beta)$ a lo largo de Y debido al Lema 8.0.8, probado en el Apéndice.

En la Sección 6.3 describimos todas las pendientes a lo largo de hiperplanos de coordenada Y en cualquier punto $p \in Y$ (cf. Teorema 6.3.10). Con este propósito, y usando algunas ideas de [SW08], probamos que las variedades s–micro-características a lo largo de Y de $\mathcal{M}_A(\beta)$ son homogéneas respecto a la filtración por el orden para todo $s \geq 1$ excepto un conjunto finito de valores de s que son posibles pendientes algebraicas. A continuación, usamos los resultados de las Secciones 6.1 y 6.2 para probar que toda posible pendiente algebraica s a lo largo de un hiperplano es el índice de Gevrey de una solución formal de $\mathcal{M}_A(\beta)$ módulo series convergentes y, por tanto, son pendientes analíticas. En particular, probamos que el conjunto de pendientes algebraicas de $\mathcal{M}_A(\beta)$ a lo largo de un hiperplano de coordenadas está contenido en el conjunto de pendientes analíticas sin usar el teorema de comparación de las pendientes [LM99]. Necesitamos usar entonces que el conjunto de las pendientes analíticas está contenido en el conjunto de las pendientes algebraicas para probar que $\mathcal{M}_A(\beta)$ no tiene más pendientes a lo largo de hiperplanos de coordenadas. Esta inclusión del teorema de comparación para las pendientes es consecuencia del teorema del índice de Laurent para las hiperfunciones holomorfas [Lau99, Corollary 5.3.3] (cf. [Meb90, 6.6]).

M. Schulze y U. Walther [SW08] describieron anteriormente las pendientes algebraicas de $\mathcal{M}_A(\beta)$ a lo largo de variedades de coordenadas bajo la hipótesis de que $\mathbb{Z}A = \mathbb{Z}^d$ y que exista un semiespacio abierto que contenga todas las columnas $a_1, \ldots, a_n$ de A. En [CT03], [Har04] y [Har03] se pueden encontrar cálculos previos de las pendientes de $\mathcal{M}_A(\beta)$ a lo largo de hiperplanos de coordenadas en el caso particular $d = 1$ y $n = d + 1$.

En la Sección 6.5.1 usamos las series Gevrey construídas en la Sección 6.1 y ciertas triangulaciones convenientes de la matriz A para calcular una cota inferior de las dimensiones de los espacios de soluciones Gevrey de $\mathcal{M}_A(\beta)$ a lo largo de cada variedad de coordenadas $Y_\tau = \{x_i = 0 : \ i \notin \tau\}$, $\tau \subseteq \{1, \ldots, n\}$, en puntos genéricos de Y_τ. En particular, la cota inferior obtenida es igual al volumen normalizado de la matriz $A_\tau = (a_i)_{i\in\tau}$ respecto de $\mathbb{Z}A$.

En la Sección 6.5.2 probamos que la mencionada cota inferior es una igualdad para parámetros $\beta \in \mathbb{C}^d$ muy genéricos y, por tanto, tenemos una descripción explícita de la base del correspondiente espacio de soluciones Gevrey. El ejemplo 6.3.13 prueba que la condición impuesta sobre los parámetros es necesaria en general para obtener una base. Además, este ejemplo muestra un fenómeno especial: algunas pendientes algebraicas de $\mathcal{M}_A(\beta)$ a lo largo de variedades de coordenadas de codimensión mayor que uno no son el índice de Gevrey de ninguna solución formal de $\mathcal{M}_A(\beta)$ módulo el haz de funciones holomorfas restringido a la variedad.

Más adelante, en la Sección 6.6 imponemos algunas condiciones ($\mathbb{Z}A = \mathbb{Z}^d$, las columnas de A se encuentran en un semiespacio abierto, β es genérico e Y es un hiperplano de coordenadas) con el objeto de usar algunas fórmulas de multiplicidades de los ciclos s–micro-característicos de $\mathcal{M}_A(\beta)$ obtenidos por M. Schulze y U. Walther en [SW08] y resultados generales sobre la irregularidad de los $\mathcal{D}$–módulos holónomos debidos a Laurent y Mebkhout [LM99] para calcular la dimensión de $\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p$ en puntos genéricos $p \in Y$. Por tanto, el conjundo de las clases en $\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p$ de las soluciones Gevrey de $\mathcal{M}_A(\beta)$ que construímos a lo largo de un hiperplano Y es una base para parámetros muy genéricos. Además, puesto que $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ es un haz perverso sobre Y por un teorema de Z. Mebkhout [Meb90], sabemos que para $i \geq 0$ el soporte del i–ésimo haz de cohomología de $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ está contenido en una subvariedad de Y de codimensión i. Tenemos entonces la fibra de los haces de cohomología de $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ en puntos genéricos de Y. Como consecuencia, se calcula de forma explícita el polígono de Newton de $\mathcal{M}_A(\beta)$ a lo largo de de Y respecto de Y en puntos genéricos de Y.

Por último, el Capítulo 7 (trabajo conjunto con Uli Walther) está dedicado al cálculo de la restricción de $\mathcal{M}_A(\beta)$ a lo largo de una variedad de coordenadas. En particular, se generaliza el Corolario 5.1.4.

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