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Kernel well-posedness and computation by power series in backstepping output feedback for radially-dependent reaction–diffusion PDEs on multidimensional balls☆

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ABSTRACT

Recently, the problem of boundary stabilization and estimation for unstable linear constant-coefficient reaction–diffusion equation on n-balls (in particular, disks and spheres) has been solved by means of the backstepping method. However, the extension of this result to spatially-varying coefficients is far from trivial. Some early success has been achieved under simplifying conditions, such as radially-varying reaction coefficients under revolution symmetry, on a disk or a sphere. These particular cases notwithstanding, the problem remains open. The main issue is that the equations become singular in the radius; when applying the backstepping method, the same type of singularity appears in the kernel equations. Traditionally, well-posedness of these equations has been proved by transforming them into integral equations and then applying the method of successive approximations. In this case, with the resulting integral equation becoming singular, successive approximations do not easily apply. This paper takes a different route and directly addresses the kernel equations via a power series approach (in the spirit of the method of Frobenius for ordinary differential equations), finding in the process the required conditions for the radially-varying reaction (namely, analyticity and evenness) and showing the existence and convergence of the series solution. This approach provides a direct numerical method that can be readily applied, despite singularities, to both control and observer boundary design problems.

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1. Prologue

It is with admiration that we contribute this article to celebrate the 80th birthday of Professor Art Krener in a special issue in *Systems & Control Letters*—a journal in which he has published some of his most impactful papers, such as his 1983 introduction of nonlinear observers, jointly with Isidori, and his 1984 single-authored introduction of approximate feedback linearization.

Art's contributions are not only deep but also remarkably broad, including nonlinear controllability/observability, nonlinear representation theory, feedback linearization, observer design, control of bifurcations, optimal/bang–bang control, nonlinear H-infinity control, solutions to Hamilton–Jacobi PDEs, and estimation and analysis of stochastic systems.

His career in control theory started with his work on the maximum principle for problems where the first order conditions are inconclusive. In order to generate conclusive results with the help of higher order terms, he turned to Lie algebra and differential geometry tools and employed his extension of Chow's theorem whose answer to the questions of accessibility and reachable states leads to higher order necessary conditions for optimality.

Art then transitioned to representation theory, to answer when a control-affine state–space model can be represented as an input–output model via Volterra series and, somewhat conversely, when a nonlinear input–output map can be represented as a bilinear state–space model. His work was a significant inspiration for our spatial Volterra series approach to control of nonlinear PDEs [1].

Art's most impactful contribution resulted when he turned his attention to controllability in his 1977 IEEE Transactions on Automatic Control paper with mathematical physicist Hermann.

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Krener provided a comprehensive set of results on controllability, observability, and minimality for nonlinear systems, tackling them analogously to the linear case—using rank conditions and leading to decompositions into controllable and uncontrollable subsystems.

The initial idea for the celebrated feedback linearization originated in Krener's necessary and sufficient conditions for the existence of a change of state variables that yields a transformed system that is linear. Brockett, Hunt and Su, and Jacubczyk and Respondek extended this idea by adding feedback to the state transformation, producing the general framework now known as feedback linearization. (One could argue that "PDE backstepping" [2] is a PDE realization of the feedback linearization idea, with a PDE target system extending the finite-dimensional Brunovsky form.) Recognizing himself that feedback linearizability is not satisfied for some important physical systems, Krener then went on to develop a theory of "approximate feedback linearization", published in *Systems & Control Letters* [3].

Pursuing a dual of the feedback linearization approach, for the problem of state estimation, Krener developed an approach to the design of observers for nonlinear systems, in collaboration with Isidori in *Systems & Control Letters* [4], and later with Respondek, which employs a state transformation and the injection of measured output. What we do in Section 6 of the present paper is a PDE analog of Krener's approach.

Art pursued nonlinear control for systems with disturbances in several waves—first in a paper with Isidori, Gori-Giorgi and Monaco in an award-winning paper that gives existence conditions and constructions of decoupling and noninteracting control laws, and then several years later in the context of nonlinear H-infinity control and estimation.

In the early and mid-1990s, inspired by "bifurcation control", wildly popular in the physics community at that time, and linked to interesting applications, Krener and Kang developed control designs for normal forms that include locally quadratic dependencies, which remain the definitive results of the art on this subject.

Art Krener cast an eye on PDE control systems at least as early as the mid-1990s, in the framework of a US Air Force funded project on nonlinear control of jet engine instabilities. While other researchers focused on the first-order Galerkin approximations of the models of rotating stall instabilities in axial flow compressors in jet engines, Art formulated and studied generalized higher-order Moore–Greitzer nonlinear models [5].

It is a delight to see Art Krener take on PDE control as the preoccupation for the present stage of his career [6,7]. Going beyond the conventional development of *operator Riccati equations*, and the limitation to the study of their well posedness, in his trademark fashion, Art is producing computable approximate solutions to *Riccati PDEs* using Al'brekht's approach, which he has already brought to its state-of-the art form for nonlinear and stochastic ODEs.

2. Introduction

In this paper we introduce an explicit boundary output-feedback control law to stabilize an unstable linear *radially-dependent* reaction–diffusion equation on an n-ball (which in 2-D is a disk and in 3-D a sphere).

This paper extends the spherical harmonics [8] approach of [9], which assumed constant coefficients, using some of the ideas of [10]. For a finite number of harmonics, we design boundary feedback laws and output injection gains using the backstepping method [2] (with kernels computed using a power series approach) which allows us to obtain exponential stability of the origin in the L^2 norm. Higher harmonics will be naturally open-loop stable. The required conditions for the radially-varying coefficients are found in the analysis of the numerical method and are non-obvious (evenness of the reaction coefficient). The idea of using a power series to compute backstepping kernels was first seen in [11] (without much analysis of the method itself, but rather numerically optimizing the approximation) and later in [12], where piecewise-smooth kernels require the use of several series. Here, we prove that the method provides a unique converging solution, in the spirit of the method of Frobenius for ordinary differential equations.

Some partial results towards the solution of this problem were obtained in [13] and [14] for the disk and sphere, respectively; however they required symmetry conditions. Older results in this spirit were obtained in [15] and [16]. This paper extends and completes our conference contribution [17] where the ideas where initially presented (without proof).

To the best of our knowledge, this paper presents the first rigorous proof of convergence of a power series solution for the backstepping kernel equations. Thus, this work consolidates the method as a valid alternative to more traditional numerical approaches, which include finite difference approximations of the kernel equations [2,18–20], the use of symbolical successive approximation series [21], or the numerical solution of the integral version of the kernel equations [22,23]. The main advantages of the method are its simplicity (it does not require the sometimes cumbersome conversion to integral equations thus preventing mistakes or any consideration about discrete meshes), speed (modern computing systems can reach high orders of the series in seconds), precision (one reaches a simple polynomial in one variable for the gain at the boundary that does not require interpolation), adaptability (it can be adapted to settings with discontinuous kernels by breaking the domain in pieces, see [12]), and capacity to produce kernels depending on parameters (by symbolically solving the kernel equations). The main drawback is the analyticity requirement of the system coefficients, even though most physical systems and examples seen in backstepping papers indeed possess analytic coefficients, and possibly a slow convergence rate of the series in some cases.

Previous results and applications in multi-dimensional domains include multi-agent deployment in 3-D space [24] (by combining the ideas of [9] and [25]), convection problem on annular domains [21], PDEs with boundary conditions governed by lower-dimensional PDEs [10,26], multi-dimensional cuboid domains [27].

The backstepping method has proved itself to be an ubiquitous method for PDE control, with many other applications including, among others, flow control [28,29], nonlinear PDEs [1], hyperbolic 1-D systems [30–32], or delays [33]. Nevertheless, other design methods are also applicable to the geometry considered in this paper (see for instance [34] or [35]).

The structure of the paper is as follows. In Section 3 we introduce the problem. In Section 4 we state our stability result. We study the well-posedness of the kernels in Section 5, which is the main result of the paper, proving existence of the kernels and providing means for their computation; interestingly, odd and even dimensions require a slightly different approach. We briefly talk next about the observer in Section 6, but skip most details based on its duality with respect to the controller. Then, we give some simulation results in Section 7. We finally conclude the paper with some remarks in Section 8.

3. n-D reaction–diffusion system on an n-ball

Consider the following constant-coefficient reaction–diffusion system in an n-dimensional ball of radius R:

$$\frac{\partial u}{\partial t} = \epsilon \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \right) + \lambda u = \epsilon \, \triangle_n \, u + \lambda(\vec{x}) u, \quad (1)$$

where $u = u(t, \vec{x})$, with $\vec{x} = [x_1, x_2, \ldots, x_n]^T$, is the state variable, evolving for $t > 0$ in the n-ball $B^n(R)$ defined as

$$B^n(R) = \left\{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq R\right\}, \tag{2}$$

with boundary conditions on the boundary of $B^n(R)$, which is the $(n-1)$-sphere $S^{n-1}(R)$ defined as

$$S^{n-1}(R) = \left\{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = R\right\}. \tag{3}$$

The boundary condition is assumed to be of Dirichlet type,

$$u(t, \vec{x})\Big|_{\vec{x}\in S^{n-1}(R)} = U(t, \vec{x}), \tag{4}$$

where $U(t, \vec{x})$ is the actuation. On the other hand the measurement $y(t, \vec{x})$ is defined as

$$y(t, \vec{x}) = \partial_r u(t, \vec{x})\Big|_{\vec{x}\in S^{n-1}(R)}, \tag{5}$$

where ∂_r denotes the derivative in the radial direction (normal to the $(n-1)$-sphere), which would be defined as $\partial_r u(t, \vec{x}) = \vec{\nabla} u \cdot \frac{\vec{x}}{\|\vec{x}\|}$.

Differently from [9], we consider non-constant $\lambda(\vec{x})$ verifying the following assumption.

Assumption 3.1. The coefficient $\lambda(\vec{x})$ is an analytic function of $\vec{x}$ and depends exclusively on the radius $r = \|\vec{x}\|$.

Following [9], both the state and the actuation variable can be written in n-dimensional spherical coordinates, also known as ultraspherical coordinates (see [8], p. 93), which consist of one radial coordinate r and $n-1$ angular coordinates $\vec{\theta}$. Then, using a (complex-valued) Fourier–Laplace series of Spherical Harmonics[1] to handle the angular dependencies, defined as

$$u(t, r, \vec{\theta}) = \sum_{l=0}^{l=\infty} \sum_{m=0}^{m=N(l,n)} u_l^m(r, t) Y_{lm}^n(\vec{\theta}), \tag{6}$$

$$U(t, \vec{\theta}) = \sum_{l=0}^{l=\infty} \sum_{m=0}^{m=N(l,n)} U_l^m(t) Y_{lm}^n(\vec{\theta}), \tag{7}$$

where $N(l, n)$ is the number of (linearly independent) n-dimensional spherical harmonics of degree l, given by $N(0, n) = 1$ (representing the mean value over the n-ball) and, for $l > 0$,

$$N(l, n) = \frac{2l + n - 2}{l} \begin{pmatrix} l + n - 3 \\ l - 1 \end{pmatrix}, \tag{8}$$

with Y_{lm}^n being the mth n-dimensional spherical harmonic of degree l. The coefficients in (6)–(7) are possibly complex-valued.

Following [9] and using (6)–(7) one reaches the following independent complex-valued 1-D reaction–diffusion equation for each spherical harmonic coefficient:

$$\partial_t u_l^m = \frac{\epsilon}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r u_l^m\right) - l(l + n - 2) \frac{\epsilon}{r^2} u_l^m + \lambda(r) u_l^m, \tag{9}$$

evolving in $r \in (0, R]$, $t > 0$, with boundary conditions

$$u_l^m(t, R) = U_l^m(t), \tag{10}$$

In these equations, we have considered Dirichlet boundary conditions. The measurement would be the flux at the boundary, namely $\partial_r u_l^m(t, R)$.

Note that following [36, p. 640], a second boundary condition, reflecting the second-order character of (9) and the need to avoid singular behaviors, can be expressed as:

$$|u_l^m(t, 0)| < \infty. \tag{11}$$

[1] Spherical harmonics were introduced by Laplace to solve the homonymous equation and have been widely used since, particularly in geodesics, electromagnetism and computer graphics. A very complete treatment on the subject can be found in [8].

In the above equations, the integers m and l stand for the order and degree of the harmonic, respectively. Note that the higher the degree (corresponding to high frequencies), the more "naturally" s Eqs. (9)–(10) is, as seen next. Define the L^2 norm

$$\|f\|_{L^2} = \sqrt{\int_0^R |f(r)|^2 r^{n-1} dr}. \tag{12}$$

and the associated L^2 space as usual, where $|f|^2 = ff^*$, being f^* the complex conjugate of f.

Lemma 3.1. *Given $\lambda(r)$ and R, there exists $L \in \mathbb{N}$ such that, for all $l > L$, the equilibrium $u_l^m \equiv 0$ of system (9)–(10) is open loop exponentially stable, namely, for $U_l^m = 0$ in (10) there exists a positive constant D_1, such that for all t*

$$\|u_l^m(t, \cdot)\|_{L^2} \leq e^{-D_1 t} \|u_l^m(0, \cdot)\|_{L^2}. \tag{13}$$

D_1 is independent of l, and only depends on n, $\lambda(r)$, ϵ, and R, and can be chosen as large as desired just by increasing the values of L.

The proof is skipped as it mimics [10] just by using the L^2 norm as a Lyapunov function and Poincare's inequality.

Thus one only needs to stabilize the unstable mode with $l < L$. Since the different modes are not coupled, it allows us to stabilize them separately and re-assembling them. Moreover since only a finite number of harmonics is stabilized, there is no need to worry about the convergence of the control law as in [9], with its Spherical Harmonics series being just a finite sum.

Our objective can now be stated as follows. Considering only the unstable modes, design an *output-feedback* control law for U_l^m using, for each mode, only the measurement of $\partial_r u_l^m(t, R)$. Our design procedure is established in the next section along with our main stability result.

4. Stability of controlled harmonics

Next, for the unstable modes we design the output-feedback law. The observer and controller are designed separately using the backstepping method, by following [9]; in this reference it is shown that both the feedback and the output injection gains can be found by solving a certain kernel PDE equation, which is essentially the same for both the controller and the observer. Thus, for the sake of brevity and to avoid repetitive material, we only show how to obtain the (full-state) control law, giving the basic observer design and some additional remarks later in Section 6.

4.1. Design of a full-state feedback control law for unstable modes

Based on the backstepping method [2], our idea is utilizing an invertible Volterra integral transformation

$$w_l^m(t, r) = u_l^m(t, r) - \int_0^r K_{lm}^n(r, \rho) u_l^m(t, \rho) d\rho, \tag{14}$$

where the kernel $K_{lm}^n(r, \rho)$ is to be determined, which defined on the domain $\mathcal{T}_k = \{(r, \rho) \in \mathbb{R}^2; 0 \leq \rho \leq r \leq R\}$ to convert the unstable system (9)–(10) into an exponentially target system:

$$\partial_t w_l^m = \epsilon \frac{\partial_r (r^{n-1} \partial_r w_l^m)}{r^{n-1}} - \epsilon l(l + n - 2) \frac{w_l^m}{r^2} - c w_l^m, \tag{15}$$

$$w_l^m(t, R) = 0, \tag{16}$$

where the constant $c > 0$ is an adjustable convergence rate. From (14) and (16), let $r = R$, we obtain the boundary control as the following full-state law

$$U_l^m(t) = \int_0^R K_{lm}^n(R, \rho) u_l^m(t, \rho) d\rho. \tag{17}$$

Following closely the steps of [9] to find conditions for the kernels, and defining $K^n_{lm}(r,\rho) = G^n_{lm}(r,\rho)\rho\left(\frac{\rho}{r}\right)^{l+n-2}$, we finally reach a PDE that the G-kernels need to verify:

$$\frac{\lambda(\rho)+c}{\epsilon}G^n_{lm} = \partial_{rr}G^n_{lm} + (3-n-2l)\frac{\partial_r G^n_{lm}}{r} - \partial_{\rho\rho}G^n_{lm} + (1-n-2l)\frac{\partial_\rho G^n_{lm}}{\rho}, \tag{18}$$

with only one boundary condition:

$$G^n_{lm}(r,r) = -\frac{\int_0^r(\lambda(\sigma)+c)d\sigma}{2r\epsilon}. \tag{19}$$

We assume as usual that these kernel equations are well-posed and the resulting kernel is bounded in $\mathcal{T}$; this will be analyzed later in Section 5, providing also a numerical method for its computation.

4.2. Closed-loop stability analysis of unstable modes

To obtain the stability result of closed-loop system, we need three elements. We begin by stating the stability result for the target system. We follow by obtaining the existence of an inverse transformation that allows us to recover our original variable from the transformed variable. Then we relate the L^2 norm with spherical harmonics. With these elements, we construct the proof of stability mapping the result for the target system to the original system. This is done by showing that the transformation is an invertible map from L^2 into L^2.

We first discuss the stability of the target system, having the following lemma:

Lemma 4.1. *For all $l \in \mathbb{N}$, and for $c \geq 0$, the equilibrium $w^m_l \equiv 0$ of system (15)–(16) is exponentially stable, i.e., there exists a positive constant D_2 such that for all t,*

$$\|w^m_l(t,\cdot)\|_{L^2} \leq e^{-D_2 t}\|w^m_l(0,\cdot)\|_{L^2}, \tag{20}$$

where the constant D_2 is independent of n, l or m, and only depends on c, ϵ, and R; it can be chosen as large as desired just by increasing the value of c.

Proof. Consider the Lyapunov function:

$$V_2(t) = \frac{1}{2}\|w^m_l(t,\cdot)\|^2_{L^2}, \tag{21}$$

then, taking its time derivative, we obtain

$$\dot{V}_2 = \int_0^R \frac{\bar{w}^m_l\partial_t w^m_l + w^m_l\partial_t\bar{w}^m_l}{2}r^{n-1}dr \leq -\left(\frac{\epsilon}{4R^2}+c\right)\|w^m_l\|^2_{L^2} \tag{22}$$

choosing

$$c = D_2 - \frac{\epsilon}{4R^2} \tag{23}$$

we then obtain, independent of the value of n,

$$\dot{V}_2 \leq -2D_2V_2, \tag{24}$$

thus proving the result. □

Lemma 4.2. *For $|l| \leq L$, let c be chosen as in Lemma 4.1, and assume that the kernel $K^n_{lm}(r,\rho)$ is bounded and integrable. The system (9) with boundary control (17) is closed-loop exponentially stable, namely there exists positive constants C and D_2 such that*

$$\|u^m_l(t,\cdot)\|_{L^2} \leq Ce^{-D_2 t}\|u^m_l(0,\cdot)\|_{L^2}, \tag{25}$$

C and D_2 are independent of m or l, and only depend on n, L, $\lambda(r)$, ϵ and R.

Proof. The proof consists of two parts, one is existence of an inverse transformation, and then showing the equivalence of norms of the variables u^n_{lm} and w^n_{lm}; the result then follows from the stability of the target system.

As shown in [9], when $K_n(r,\rho)$ is bounded and integrable, the map (14) is invertible and its inverse transformation is

$$u^m_l(t,r) = w^m_l(t,r) + \int_0^r L^n_{lm}(r,\rho)w^m_l(t,\rho)d\rho, \tag{26}$$

which is also bounded and integrable. Call now $\bar{K}$ and $\bar{L}$ the maximum of the bounds of the function $\check{K}^n_{lm}$ and $\check{L}^n_{lm}$ for a given n and all $l \leq L$ in their respective domains. It is easy to get

$$\|w^m_l(t,\cdot)\|^2_{L^2} \leq M_1\|u^m_l(t,\cdot)\|^2_{L^2}, \tag{27}$$

$$\|u^m_l(t,\cdot)\|^2_{L^2} \leq M_2\|w^m_l(t,\cdot)\|^2_{L^2}. \tag{28}$$

where $M_1 = 2 + R^4\bar{K}/(2n)$ and $M_2 = 2 + R^4\bar{L}/(2n)$. Combining then Lemma 4.1 with the norm equivalence between u^m_l and w^m_l system stated as in (27) and (28), it is easy to obtain

$$\begin{aligned}\|u^m_l(t,\cdot)\|_{L^2} &\leq \sqrt{M_2}\|w^m_l(t,\cdot)\|_{L^2} \leq \sqrt{M_2}e^{-D_2 t}\|w^m_l(0,\cdot)\|_{L^2}\\ &\leq \sqrt{M_1M_2}e^{-D_2 t}\|u^m_l(0,\cdot)\|_{L^2}.\end{aligned} \tag{29}$$

Let $C = \sqrt{M_1M_2}$, the result then follows. □

Note that combining Lemmas 3.1 and 4.2 and taking $D = \min\{D_1, D_2\}$, we get the following stability result for *all* spherical harmonics and thus the full physical system.

Theorem 1. *Under the assumption that the kernel $K^n_{lm}(r,\rho)$ is bounded and integrable, the equilibrium $u^m_l \equiv 0$ of system (9)–(10) under control law (17) is closed-loop exponentially stable, namely, there exists a positive constant D, such that for all t*

$$\|u^m_l(t,\cdot)\|_{L^2} \leq Ce^{-Dt}\|u^m_l(0,\cdot)\|_{L^2}. \tag{30}$$

where D can be chosen as large as desired just by increasing the value of L and c in the control design process.

5. Well-posedness of the kernel equations

Next, we state the main result of the paper, which was in part assumed in Theorem 1, also giving the requirements for $\lambda(r)$. In addition the proof of the result also provides a numerical method to compute the kernels, which is an alternative to successive approximations which do not work in this case (due to the singularities at the origin; see for instance [13] to see the resulting singular integral equation that needs to be solved).

Theorem 2. *Under the assumption that $\lambda(r)$ is an even real analytic function in $[0,R]$, then for a given $n > 1$ and all values of $l \in \mathbb{N}$, there is a unique power series solution $G^n_{lm}(r,\rho)$ for (18)–(19), even in its two variables in the domain $\mathcal{T}$, which is real analytic in the domain. In addition, if $\lambda(r)$ is analytic, but not even, then there is no power series solution to (18)–(19) for most values $l \in \mathbb{N}$.*

The requirement of evenness for $\lambda(r)$ might seem unusual. However, if we carefully consider Assumption 3.1, and since $r = \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$, in physical space $\lambda(\vec{x})$ will be non-analytic, unless it is even. Thus, while solutions to the kernel equations might exist for non-even $\lambda(r)$, we cannot expect them to be analytic. This result notwithstanding, if one is interested in controlling only very low-order harmonics, kernels do exist without this requirement, as shown in [13,14], which only consider the 0-th order harmonic (the mean) respectively for a disk and a sphere, and only require boundedness of $\lambda(r)$.

5.1. Proof of Theorem 2

We start by giving out an algorithmic method to compute the power series for $G^n_{lm}(r,\rho)$, which will allow us to prove Theorem 2 as well as numerically approximating the kernels.

First of all, we show that the evenness of $\lambda(r)$ is a necessary condition to find an analytic solution. Next, it is possible to establish that the series for $G^n_{lm}(r,\rho)$ only has even powers. Exploiting this property to suitably express (18)–(19), we finally show the existence of the power series and thus Theorem 2 follows. Convergence and related issues (radius of convergence) is studied towards the end, finishing the proof.

5.1.1. Computing a power series solution for the kernels

Starting from the most basic assumption of Theorem 2, we consider that $\lambda(r)$ is analytic in $[0,R]$, therefore it can be written as a convergent series (encompassing c and ϵ for notational convenience):

$$\frac{\lambda(r)+c}{\epsilon}=\sum_{i=0}^{\infty}\lambda_i r^i, \tag{31}$$

which, by the evenness of λ, may only contain even powers,[2] this is, $\lambda_i=0$ if i is odd. We then, in the spirit of the method of Frobenius for ordinary differential equations, seek for a solution of (18)–(19) of the form:

$$G^n_{lm}(r,\rho)=\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}C_{ij}r^j\rho^{i-j}\right), \tag{32}$$

where the dependence on n, l and m has been omitted for simplicity (the solution will depend on these values). The series in (32) collects together (in the parenthesis) all the polynomial terms with the same degree.

It is easy to see that the boundary condition (19) implies:

$$\forall i,\quad \sum_{j=0}^{i}C_{ij}=-\frac{\lambda_i}{2(i+1)}, \tag{33}$$

which in particular implies $C_{00}=-\frac{\lambda_0}{2\epsilon}$. On the other hand, the left-hand side of (18) becomes

$$\begin{aligned}\frac{\lambda(\rho)+c}{\epsilon}G^n_{lm}&=\left[\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}C_{ij}r^j\rho^{i-j}\right)\right]\sum_{i=0}^{\infty}\lambda_i\rho^i\\&=\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}B_{ij}r^j\rho^{i-j}\right),\end{aligned} \tag{34}$$

where we have defined

$$B_{ij}=\sum_{k=j}^{i}C_{kj}\lambda_{i-k}. \tag{35}$$

Finally, to express the right-hand side of (18), denote $\gamma=n+2l-2\geq 0$ and define the operators $D_1=\partial_{rr}+(1-\gamma)1/r\partial_r$ and $D_2=-\partial_{\rho\rho}+(-1-\gamma)1/\rho\partial_\rho$. Then

$$D_1G^n_{lm}=\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i}j(j-\gamma)C_{ij}r^{j-2}\rho^{i-j}\right), \tag{36}$$

$$D_2G^n_{lm}=\sum_{i=1}^{\infty}\left(\sum_{j=0}^{i-1}(i-j)(j-i-\gamma)C_{ij}r^j\rho^{i-j-2}\right), \tag{37}$$

and thus, rewriting the sum to be homogeneous with (34),we find

$$(D_1+D_2)G^n_{lm}=\sum_{i=-1}^{\infty}\left(\sum_{j=-1}^{i+1}D_{ij}r^j\rho^{i-j}\right), \tag{38}$$

where, (assuming $C_{ij}=0$ if $i,j<0$ or $j>i$),

$$D_{ij}=(j+2)(j+2-\gamma)C_{(i+2)(j+2)}-(i-j+2)(i-j+2+\gamma)C_{(i+2)j}, \tag{39}$$

Equating (38) and (34), we obtain a system of equations:

$$\forall i\geq -1,\quad D_{i(i+1)}=D_{i(-1)}=0, \tag{40}$$

$$\forall i\geq 0,\ 0\leq j\leq i,\quad (j+2)(j+2-\gamma)C_{(i+2)(j+2)}-(i-j+2)(i-j+2+\gamma)C_{(i+2)j}=B_{ij}. \tag{41}$$

With $\lambda(r)$ and n are fixed, we want to show that the kernel equations are solvable for all values of $l\in\mathbb{N}$. Thus, γ takes increasing values. In addition we can assume $\gamma\neq 1$, since the case $n=3$, $l=0$ was already addressed in [14] showing that it reduces to the usual 1-D kernel equations for parabolic systems [2], which admits a power series solution according to [11].

The first two equalities, if $\gamma\neq 1$, imply

$$\forall i\geq 1,\quad C_{i1}=C_{i(i-1)}=0, \tag{42}$$

and, in particular, $C_{10}=C_{01}=0$, whereas the second equality results in a system of equations that needs to be solved recursively, starting at $i=0$. It can be rewritten as follows to start at $i=2$ (since C_{00}, C_{10} and C_{01} are already determined).

$$\forall i\geq 2,\ 0\leq j\leq i-2,$$

$$(j+2)(j+2-\gamma)C_{i(j+2)}-(i-j)(j-i-\gamma)C_{ij}=\sum_{k=j}^{i-2}C_{kj}\lambda_{i-2-k}. \tag{43}$$

Note that for each $i\geq 2$, there are $i+1$ coefficients in (32) but $i+2$ relations: one from (33), two from (42) and $i-1$ from (43). Thus, it would seem that (33)–(42)–(43) is in general an incompatible system. This is indeed the case if $\lambda(r)$ is not even, i.e., if the series (31) contains odd powers, as shown in the next section.

5.1.2. Evenness requirement of $\lambda(r)$

We start with the following result.

Lemma 5.1. *If $\lambda(r)$ is not even, then there are values of $l\in\mathbb{N}$ for which there is no solution to (18)–(19) in the form of (32).*

Proof. We show that, if there exists i odd such that $\lambda_i\neq 0$, then there is no solution in the form of a power series. First, if $\lambda_1\neq 0$, then from (33) we know that $C_{01}+C_{10}=-\frac{\lambda_1}{4\epsilon}$, however since form (42) one has $C_{01}=C_{10}=0$, this cannot hold. Consider now there is indeed a value $i>1$ for which a coefficient λ_i is distinct from zero and let us show the result by contradiction. Consider the first such i. Now, since the right-hand side of (43) depends on $C_{(i-2)j}$, one gets that for all odd $i'<i$ $C_{i'j}$ must zero from (33)–(42)–(43) all having a zero right-hand side (this can be formalized with an induction argument; we skip the details). Thus, at i, the following system of equations has to be verified:

$$C_{i1}=C_{i(i-1)}=0, \tag{44}$$

$$\sum_{j=0}^{i}C_{ij}=-\frac{\lambda_i}{2\epsilon(i+1)}, \tag{45}$$

[2] It is a known fact of analysis that even functions (respectively, odd functions) contain only even powers (respectively, odd powers) in their Taylor series. This fact has an easy proof by substituting the series in the definition of evenness $f(r)=f(-r)$ (respectively, oddness $f(r)=-f(-r)$) and checking the conditions verified by the coefficients.

and for $0 \le j \le i-2$,

$$(j+2)(j+2-\gamma)C_{i(j+2)} - (i-j)(i-j+\gamma)C_{ij} = 0, \tag{46}$$

Let us consider l sufficiently large such that $\gamma > i$, so that the coefficient $(j+2-\gamma)$ in (46) is distinct from zero in the full range of j, namely $0 \le j \le i-2$. Then none of the coefficients in (46) is zero. Therefore, combining (44) with (46), from C_{i1} we can find C_{i3}, then C_{i5}, and so on. Similarly, from $C_{i(i-1)}$ we can find $C_{i(i-3)}$, $C_{i(i-5)}$ and so on. These two sequences do not overlap because i is odd and therefore, one finds $C_{ij} = 0$ for all $0 \le j \le i$ which is not compatible with (45) unless $\lambda_i = 0$, which contradicts our initial assumption. □

Next we show that evenness of λ implies evenness of the kernels.

Lemma 5.2. *If $\lambda(r)$ is even, then, a solution to (18)–(19) in the form of (32) only has even powers.*

Proof. We need to prove that $C_{ij} = 0$ if either i or j is odd. From the proof of Lemma 5.1, we directly know that for odd i one has $C_{ij} = 0$. Fix, then, i even and consider j odd; for $i = 2$, the result is obvious. Assuming $C_{i'j} = 0$ for all even numbers $i' < i$ and j odd, let us prove the result by induction on the first coefficient. As before, we would need to solve (45)–(43). The right-hand side $B_{ij} = \sum_{k=j}^{i-2} C_{kj}\lambda_{i-2-k}$ of (43) is zero as in (46) by the induction hypothesis (if k even) or directly zero if k odd. Then, following again the proof of Lemma 5.1, we have the same system of Eqs. (45)–(46) for our even i and odd j's. Now:

$$C_{ij} = \frac{(j+2)(j+2-\gamma)}{(i-j)(i-j+\gamma)} C_{i(j+2)},$$

so starting from $C_{i(i-1)} = 0$ we find $C_{i(i-3)} = 0$, then $C_{i(i-5)}$, and so on; however, with i being even, this sequence ends now in C_{i1} (thus, the proof of Lemma 5.1 does not apply because the sequences starting at C_{i1} and $C_{i(i-1)}$ overlap). Thus, one finds $C_{ij} = 0$ for all odd values of j between 1 and $i-1$. □

5.1.3. Well-posedness of the coefficient system

Next, we show that the coefficients of the power series can always be found, which by the previous lemmas only requires studying the even coefficients. For simplification, we redefine (31) and (32) as:

$$\frac{\lambda(r)+c}{\epsilon} = \sum_{i=0}^{\infty} \lambda_i r^{2i}, \quad G^n_{lm}(r,\rho) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} C_{ij} r^{2j} \rho^{2(i-j)} \right), \tag{47}$$

without bothering to redefine the coefficients (note that (35) does not require any change). Defining as well $\gamma' = \frac{\gamma}{2} = \frac{n}{2} + l - 1 \ge 0$, the new system of equations to be solved is

$$\forall i, \quad \sum_{j=0}^{i} C_{ij} = -\frac{\lambda_i}{2(2i+1)}, \tag{48}$$

and

$$\forall i \ge 1, 0 \le j \le i-1, \ (j+1)(j+1-\gamma')C_{i(j+1)} + (i-j)(j-i-\gamma')C_{ij} = \sum_{k=j}^{i-1} C_{kj}\lambda_{i-1-k} = B_{(i-1)j}. \tag{49}$$

Let us outline the solution procedure, and later derive some conclusions. Solving in (49) every C_{ij} as a function of $C_{i(j+1)}$ we get:

$$C_{ij} = \frac{(j+1)(j+1-\gamma')C_{i(j+1)} + B_{(i-1)j}}{(i-j)(i-j+\gamma')}, \tag{50}$$

which can be written more briefly if we define, for $i > 0$ and $0 \le j < i$,

$$a_{ij}(\gamma') = \frac{(j+1)(j+1-\gamma')}{(i-j)(i-j+\gamma')}, \tag{51}$$

as

$$C_{ij} = a_{ij}(\gamma')C_{i(j+1)} + \frac{B_{(i-1)j}}{(i-j)(i-j+\gamma')}. \tag{52}$$

To be able to simplify a bit the equation, redefine

$$\hat{B}_{(i-1)j} = \frac{B_{(i-1)j}}{(i-j)(i-j+\gamma')} \tag{53}$$

then,

$$C_{ij} = a_{ij}(\gamma')C_{i(j+1)} + \hat{B}_{(i-1)j}. \tag{54}$$

and iterating this equality until reaching C_{ii}, we get

$$C_{ij} = \left[\prod_{k=j}^{k=i-1} a_{ik}(\gamma') \right] C_{ii} + \hat{B}_{(i-1)j} + \sum_{r=j+1}^{i-1} \prod_{k=r}^{k=i-1} a_{ik}(\gamma')\hat{B}_{(i-1)r},$$

and inserting this into (48), we reach an equation for C_{ii}, namely

$$C_{ii} = -\frac{1}{\kappa(i,\gamma')} \left[\frac{\lambda_i}{2\epsilon(2i+1)} + H_i \right], \tag{55}$$

where

$$\kappa(i,\gamma') = 1 + \sum_{j=0}^{i-1} \prod_{k=j}^{k=i-1} a_{ik}(\gamma'), \tag{56}$$

and

$$\begin{aligned} H_i &= \sum_{j=0}^{i-1} \hat{B}_{(i-1)j} + \sum_{j=0}^{i-2} \sum_{r=j+1}^{i-1} \prod_{k=r}^{k=i-1} a_{ik}(\gamma')\hat{B}_{(i-1)r} \\ &= \sum_{j=0}^{i-1} \hat{B}_{(i-1)j} + \sum_{r=1}^{i-1} \sum_{j=0}^{r-1} \prod_{k=r}^{k=i-1} a_{ik}(\gamma')\hat{B}_{(i-1)r} \\ &= \sum_{j=0}^{i-1} \hat{B}_{(i-1)j} + \sum_{r=1}^{i-1} r \prod_{k=r}^{k=i-1} a_{ik}(\gamma')\hat{B}_{(i-1)r} \\ &= \sum_{j=0}^{i-1} \left(1 + j \prod_{k=j}^{k=i-1} a_{ik}(\gamma') \right) \hat{B}_{(i-1)j} \end{aligned} \tag{57}$$

It is quite clear that these $\kappa(i,\gamma')$ will play an important role; in particular, if they are non-zero, one can always find a unique solution for the coefficients C_{ij}. Thus one needs to show that $\kappa(i,\gamma') \neq 0$ for any possible i or γ'. The following lemma shows this is indeed the case, by exploiting a connection of the a_{ij} coefficients with Gauss' hypergeometric functions.

Lemma 5.3. *Let i be a positive integer and $\gamma' \ge 0$ a real number. Then, it holds that*

$$\kappa(i,\gamma') = \frac{2i!}{i!} \frac{\Gamma(\gamma'+1)}{\Gamma(i+\gamma'+1)} > 0, \tag{58}$$

where Γ denotes the Gamma function [37, p. 255].

Proof. Recalling from (56) and (51) the definitions of $\kappa(i,\gamma')$ and a_{ij}, respectively, one has

$$\kappa(i,\gamma') = 1 + \sum_{j=0}^{i-1} \prod_{k=j}^{k=i-1} \frac{(k+1)(k+1-\gamma')}{(i-k)(i-k+\gamma')} \tag{59}$$

which can be rewritten in terms of binomial numbers and rising/falling factorials[3] [38] as

$$\kappa(i,\gamma') = \sum_{j=0}^{i} \binom{i}{j} \frac{(i-\gamma')^{\underline{i-j}}}{(1+\gamma')^{\overline{i-j}}} \tag{60}$$

and reordering the sum and using $\binom{i}{j} = \binom{i}{i-j}$,

$$\kappa(i,\gamma') = \sum_{j=0}^{i} \binom{i}{j} \frac{(i-\gamma')^{\underline{j}}}{(1+\gamma')^{\overline{j}}} = \sum_{j=0}^{i} (-1)^j \binom{i}{j} \frac{(\gamma'-i)^{\overline{j}}}{(1+\gamma')^{\overline{j}}}, \tag{61}$$

where the obvious fact $(x)^{\underline{j}} = (-1)^j(-x)^{\overline{j}}$ has been used. Consider now the finite polynomial $p_i(x,\gamma')$ defined as

$$p_i(x,\gamma) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} \frac{(\gamma'-i)^{\overline{j}}}{(1+\gamma')^{\overline{j}}} x^j \tag{62}$$

From the definition of Gauss' hypergeometric function [37, p. 561], denoted as ${}_2F_1(a,b;c;x)$, in the polynomial case (a or b non-positive integer) and noting $(-1)^j \binom{i}{j} = \frac{(-i)^{\overline{j}}}{j!}$, it is immediate that

$$p_i(x,\gamma) = {}_2F_1(-i,\gamma'-i;1+\gamma';x) \tag{63}$$

and therefore, from Gauss' summation theorem [37, p. 556], which is applicable in this case since $1+2i>0$,

$$\begin{aligned}\kappa(i,\gamma') = p_i(1,\gamma) &= {}_2F_1(-i,\gamma'-i;1+\gamma';1)\\ &= \frac{\Gamma(1+\gamma')}{\Gamma(1+\gamma'+i)}\frac{\Gamma(1+2i)}{\Gamma(1+i)} = \frac{2i!}{i!}\frac{\Gamma(1+\gamma')}{\Gamma(1+\gamma'+i)},\end{aligned} \tag{64}$$

finishing the proof. □

The next result is an immediate conclusion of the positivity of $\kappa(i,\gamma')$:

Lemma 5.4. *If $\lambda(r)$ is even, then, for all values of $l \in \mathbb{R}$, the coefficients in (47) that solve (18)–(19) can be uniquely found up to any order i.*

To conclude the proof of Theorem 2, we need to prove analyticity of the series (47). This step, however, requires splitting the problem in two possible cases: odd dimension (thus, $\gamma' = n/2 + 2l - 1$ is not an integer) and even dimension (γ' integer).

5.1.4. Proof of analyticity for odd dimension

In the odd-dimension case, define the following coefficients:

$$\begin{aligned}&L_{i0} = 1,\\ &L_{ij} = \binom{i}{j} \frac{(i+\gamma')(i-1+\gamma')\dots(i-j+\gamma'+1)}{(1-\gamma')(2-\gamma')\dots(j-\gamma')}, \quad j>0\end{aligned} \tag{65}$$

with L_{i0} defined as 1; these are well-defined given that γ' is non-integer. They can also be expressed as

$$L_{ij} = \binom{i}{j} \frac{\Gamma(1-\gamma')\Gamma(i+1+\gamma')}{\Gamma(j+1-\gamma')\Gamma(i-j+1+\gamma')}$$

Now, in (48)–(49), denote $C_{ij} = L_{ij}\check{C}_{ij}$. Replacing in the recurrence we get

$$\begin{aligned}B_{(i-1)j} &= (j+1)(j+1-\gamma')\binom{i}{j+1}\\ &\quad \frac{\Gamma(1-\gamma')\Gamma(i+1+\gamma')}{\Gamma(j+2-\gamma')\Gamma(i-j+\gamma')}\check{C}_{i(j+1)}\\ &\quad -(i-j)(i-j+\gamma')\binom{i}{j}\\ &\quad \frac{\Gamma(1-\gamma')\Gamma(i+1+\gamma')}{\Gamma(j+1-\gamma')\Gamma(i-j+1+\gamma')}\check{C}_{ij}\\ &= (i-j)(i-j+\gamma')L_{ij}\left(\check{C}_{i(j+1)} - \check{C}_{ij}\right)\end{aligned} \tag{66}$$

Define now

$$\check{B}_{(i-1)j} = \frac{B_{(i-1)j}}{(i-j)(i-j+\gamma')L_{ij}} \tag{67}$$

and the new set of recurrence equations for $\check{C}_{ij}$ becomes rather simple:

$$\forall i, \quad \sum_{j=0}^{i} L_{ij}\check{C}_{ij} = -\frac{\lambda_i}{2(2i+1)}, \tag{68}$$

$$\forall i \geq 1, 0 \leq j \leq i-1, \ \check{C}_{i(j+1)} - \check{C}_{ij} = \check{B}_{(i-1)j}, \tag{69}$$

and the recurrence is easily solvable in terms of one element; for instance, $\check{C}_{ii}$:

$$\check{C}_{ij} = \check{C}_{ii} - \sum_{r=j}^{r=i-1} \check{B}_{(i-1)r} \tag{70}$$

for $j = 1,\dots,i$. Replacing in (68) we reach

$$\sum_{j=0}^{i} L_{ij}\check{C}_{ii} - \sum_{j=0}^{i-1} L_{ij} \sum_{r=j}^{r=i-1} \check{B}_{(i-1)r} = -\frac{\lambda_i}{2(2i+1)}$$

Thus:

$$\begin{aligned}\check{C}_{ii} &= \frac{-\frac{\lambda_i}{2(2i+1)} + \sum_{k=0}^{k=i-1}\sum_{r=k}^{r=i-1} L_{ik}\check{B}_{(i-1)r}}{\sum_{k=0}^{k=i} L_{ik}}\\ &= \frac{-\frac{\lambda_i}{2(2i+1)} + \sum_{r=0}^{r=i-1} \check{B}_{(i-1)r}\left(\sum_{k=0}^{k=r} L_{ik}\right)}{\sum_{k=0}^{k=i} L_{ik}}\end{aligned} \tag{71}$$

Call

$$R_{ij} = \sum_{k=0}^{k=j} L_{ik}, \qquad R_i = \sum_{k=0}^{k=i} L_{ik} = R_{ii}. \tag{72}$$

Then

$$\check{C}_{ii} = \frac{-\frac{\lambda_i}{2(2i+1)} + \sum_{r=0}^{r=i-1} \check{B}_{(i-1)r}R_{ir}}{R_i} \tag{73}$$

Now, solving for the remaining coefficients from (70):

$$\begin{aligned}\check{C}_{ij} &= \frac{-\frac{\lambda_i}{2(2i+1)} + \sum_{r=0}^{r=i-1} \check{B}_{(i-1)r}R_{ir}}{R_i} - \sum_{r=j}^{r=i-1} \check{B}_{(i-1)r}\\ &= \frac{-\frac{\lambda_i}{2(2i+1)} + \sum_{r=0}^{r=i-1} \check{B}_{(i-1)r}R_{ir} - \sum_{r=j}^{r=i-1} \check{B}_{(i-1)r}R_i}{R_i}\\ &= \frac{-\frac{\lambda_i}{2(2i+1)} + \sum_{r=0}^{r=j-1} \check{B}_{(i-1)r}R_{ir} - \sum_{r=j}^{r=i-1} \check{B}_{(i-1)r}(R_i - R_{ir})}{R_i}\end{aligned} \tag{74}$$

Finally, recovering the coefficients C_{ij} from $C_{ij} = L_{ij}\check{C}_{ij}$ and using (67):

$$C_{ij} = \frac{-\frac{\lambda_i}{2(2i+1)}L_{ij} + \sum_{r=0}^{r=j-1} B_{(i-1)r}\frac{L_{ij}R_{ir}}{(i-r)(i-r+\gamma')L_{ir}} - \sum_{r=j}^{r=i-1} B_{(i-1)r}\frac{L_{ij}(R_i-R_{ir})}{(i-r)(i-r+\gamma')L_{ir}}}{R_i}, \tag{75}$$

which is quite explicit.

[3] Rising factorials $(x)^{\overline{n}}$ are sometimes expressed using the Pochhammer's symbol, with a slightly different notation, namely $(x)_n$.

Notice that since $\rho \le r$,

$$|G^n_{lm}(r,\rho)| \le \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}|C_{ij}|r^{2j}\rho^{2(i-j)}\right) \le \sum_{i=0}^{\infty} r^{2i}\left(\sum_{j=0}^{i}|C_{ij}|\right),$$

thus, defining $\alpha_i = \sum_{j=0}^{i}|C_{ij}|$, if we can prove that $\sum_{i=0}^{\infty}\alpha_i r^{2i}$ converges for a certain radius of convergence R, so does $G^n_{lm}(r,\rho)$ for $\rho \le r \le R$, and thus we obtain the required analyticity. Now:

$$\begin{aligned}
\alpha_i &= \sum_{j=0}^{i}|C_{ij}| \\
&\le \frac{|\lambda_i|}{2(2i+1)}\frac{\sum_{j=0}^{i}|L_{ij}|}{|R_i|} + \frac{\sum_{j=0}^{i}\sum_{r=0}^{r=j-1}|B_{(i-1)r}||\frac{L_{ij}R_{ir}}{(i-r)(i-r+\gamma')L_{ir}}|}{|R_i|} \\
&\quad + \frac{\sum_{j=0}^{i}\sum_{r=j}^{r=i-1}|B_{(i-1)r}||\frac{L_{ij}(R_i-R_{ir})}{(i-r)(i-r+\gamma')L_{ir}}|}{|R_i|} \\
&= \frac{|\lambda_i|}{2(2i+1)}\frac{\sum_{j=0}^{i}|L_{ij}|}{|R_i|} + \frac{\sum_{r=0}^{i-1}\sum_{j=r+1}^{i}|B_{(i-1)r}||\frac{L_{ij}R_{ir}}{(i-r)(i-r+\gamma')L_{ir}}|}{|R_i|} \\
&\quad + \frac{\sum_{r=0}^{r=i-1}\sum_{j=0}^{r}|B_{(i-1)r}||\frac{L_{ij}(R_i-R_{ir})}{(i-r)(i-r+\gamma')L_{ir}}|}{|R_i|} \\
&\le \frac{|\lambda_i|}{2(2i+1)}\frac{\sum_{j=0}^{i}|L_{ij}|}{|R_i|} \\
&\quad + \sum_{r=0}^{i-1}|B_{(i-1)r}|\left(\frac{|R_{ir}|\sum_{j=r+1}^{i}|L_{ij}| + |R_i-R_{ir}|\sum_{j=0}^{r}|L_{ij}|}{(i-r)(i-r+\gamma')|L_{ir}||R_i|}\right) \qquad (76)
\end{aligned}$$

To prove the convergence of the power series $\sum_{i=0}^{\infty}\alpha_i r^{2i}$ consider the following lemma, inspired by [39]:

Lemma 5.5. *Consider $g(x) = \sum_{i=0}^{\infty} g_i x^{2i}$ and $h(x) = \sum_{i=0}^{\infty} h_i x^{2i}$ analytic functions, both with radius of convergence R. Let i_0 be a nonnegative integer, let $(a_i)_{i=0}^{\infty}$ be a sequence of real numbers, and define $f(x) = \sum_{i=0}^{\infty} a_i x^{2i}$, where a_i verify, for $i > i_0$*

$$a_i \le b_i|g_i| + c_i\sum_{j=0}^{i-1} a_j|h_{i-1-j}| \qquad (77)$$

where the sequences $b_i, c_i \ge 0$ are decreasing for $i > i_0$, with c_i also verifying $\lim_{i\to\infty} c_i = 0$. Then, $f(x)$ is analytic with radius of convergence at least R.

Proof. Since g and h analytic with radius of convergence R we can write $|g_i|, |h_i| \le MR^{-2i}$, where the definition as power series of squares has been taken into account. Thus:

$$a_i \le b_i MR^{-2i} + c_i\sum_{j=0}^{i-1} a_j MR^{-2i+1+2j} \qquad (78)$$

Define $\breve{a}_i = a_i$ for $i \le i_0$ and, for $i > i_0$, $\breve{a}_i = b_i MR^{-2i} + c_i\sum_{j=0}^{i-1}\breve{a}_j MR^{-2i+1+2j}$. Obviously $a_i \le \breve{a}_i$ and therefore the radius of convergence of $f(x)$ would be at least the radius of convergence of $\breve{f}(x) = \sum_{i=0}^{\infty}\breve{a}_i x^{2i}$. Now:

$$\begin{aligned}
\breve{a}_{i+1} &= b_{i+1}MR^{-2i-2} + c_{i+1}\sum_{j=0}^{i}\breve{a}_j MR^{-2i+2j-1} \\
&= b_{i+1}MR^{-2i-2} + c_{i+1}M\breve{a}_i + c_{i+1}R^{-2}\sum_{j=0}^{i-1}\breve{a}_j MR^{-2i2+j+1} \qquad (79)
\end{aligned}$$

It is sufficient to compute

$$\begin{aligned}
&\lim_{i\to\infty}\frac{\breve{a}_{i+1}}{\breve{a}_i} \\
&= \lim_{i\to\infty}\frac{b_{i+1}MR^{-2i-2} + c_{i+1}M\breve{a}_i + c_{i+1}R^{-2}\sum_{j=0}^{i-1}\breve{a}_j MR^{-2i+2j+1}}{\breve{a}_i} \\
&= \lim_{i\to\infty} Mc_{i+1} \\
&\quad + \lim_{i\to\infty}\frac{b_{i+1}MR^{-2i-2} + c_{i+1}R^{-2}\sum_{j=0}^{i-1}\breve{a}_j MR^{-2i+2j+1}}{b_i MR^{-2i} + c_i\sum_{j=0}^{i-1}\breve{a}_j MR^{-2i+1+2j}} \\
&= R^{-2}\lim_{i\to\infty}\frac{b_{i+1} + c_{i+1}\sum_{j=0}^{i-1}\breve{a}_j R^{2j+1}}{b_i + c_i\sum_{j=0}^{i-1}\breve{a}_j R^{1+2j}} \\
&\le R^{-2}\lim_{i\to\infty}\frac{b_i + c_i\sum_{j=0}^{i-1}\breve{a}_j R^{2j+1}}{b_i + c_i\sum_{j=0}^{i-1}\breve{a}_j R^{1+2j}} \\
&= R^{-2}, \qquad (80)
\end{aligned}$$

where the inequality holds for sufficiently large $i > i_0$ and thus in the limit, therefore proving the lemma. □

To apply Lemma 5.5 to (76) we need to bound some of the terms. In particular, if we are able to find b_i and c_i such that

$$\begin{aligned}
&\frac{\sum_{j=0}^{i}|L_{ij}|}{2(2i+1)|R_i|} \le b_i, \\
&\max_{r\in\{0,\dots,i-1\}}\left(\frac{|R_{ir}|\sum_{j=r+1}^{i}|L_{ij}| + |R_i-R_{ir}|\sum_{j=0}^{r}|L_{ij}|}{(i-r)(i-r+\gamma')|L_{ir}||R_i|}\right) \le c_i, \qquad (81)
\end{aligned}$$

we get

$$\begin{aligned}
\alpha_i &\le b_i|\lambda_i| + c_i\sum_{r=0}^{i-1}|B_{(i-1)r}| \\
&= b_i|\lambda_i| + c_i\sum_{r=0}^{i-1}\sum_{k=r}^{i-1}|C_{kr}||\lambda_{i-1-k}| \\
&= b_i|\lambda_i| + c_i\sum_{k=0}^{i-1}\sum_{r=0}^{k}|C_{kr}||\lambda_{i-1-k}| \\
&= b_i|\lambda_i| + c_i\sum_{k=0}^{i-1}\alpha_k|\lambda_{i-1-k}|. \qquad (82)
\end{aligned}$$

Thus, assuming that b_i and c_i verify the conditions given in Lemma 5.5, and given that $\lambda(x)$ has a radius of convergence of at least R, we see that $G^n_{lm}(r,\rho)$ converges and defines an analytic function for $\rho \le r \le R$, thus proving Theorem 2 for the odd-dimension case.

It remains to find such b_i and c_i. Proceeding exactly as in Lemma 5.3 with a slight modification, we directly find

$$\begin{aligned}
R_i &= {}_2F_1(-i,-(\gamma'+i);1-\gamma';1) = \frac{\Gamma(1-\gamma')\Gamma(2i+1)}{\Gamma(-\gamma'+1+i)\Gamma(i+1)} \\
&= \frac{2i!}{i!}\frac{\Gamma(1-\gamma')}{\Gamma(1-\gamma'+i)} \qquad (83)
\end{aligned}$$

Now, let $N = \gamma' - 1/2$ and $i > 2N$. One can see that for $1 \le j \le N$,

$$\begin{aligned}
|L_{ij}| &= \binom{i}{j}\frac{(i+\gamma')(i-1+\gamma')\dots(i-j+\gamma'+1)}{(\gamma'-1)(\gamma'-2)\dots(\gamma'-j)} \\
&\le |L_{iN}| \\
&= \binom{i}{N}\frac{(i+\gamma')(i-1+\gamma')\dots(i+3/2)}{(\gamma'-1)(\gamma'-2)\dots(1/2)} \\
&\le \frac{i!}{N!(i-N)!}\frac{(i+N+1)!}{(i+1)!}\frac{1}{(N-1)!} \qquad (84)
\end{aligned}$$

and

$$|R_i| = \frac{2i!}{i!}\left|\frac{\Gamma(\gamma'-1)}{\Gamma(i+1-\gamma')}\right| \ge \frac{2i!}{i!(i-N)!N!}. \qquad (85)$$

Thus, for $i > 2N$, calling d_i the following sequence

$$d_i = \frac{\sum_{j=0}^{N} |L_{ij}|}{|R_i|} \leq \frac{i!(i+N+1)!}{(i+1)(N-1)!2i!}, \tag{86}$$

it is clear that d_i is a decreasing sequence, since from the ratio test $r = \lim_{i\to\infty} \frac{d_{i+1}}{d_i} = 1/4$.

Now, set $i_0 = 2N$. For $i > i_0$,

$$\frac{\sum_{j=0}^{i} |L_{ij}|}{2\epsilon(2i+1)|R_i|} = \frac{\sum_{j=0}^{N} |L_{ij}| + \sum_{j=N+1}^{i} |L_{ij}|}{2(2i+1)|R_i|} \leq \frac{d_i+1}{2(2i+1)} = b_i \tag{87}$$

It is obvious that b_i is decreasing, since d_i is decreasing.

Now we need to find a sequence c_i for the second term in (81). First of all,

$$\frac{|R_{ir}| \sum_{j=r+1}^{i} |L_{ij}| + |R_i - R_{ir}| \sum_{j=0}^{r} |L_{ij}|}{(i-r)(i-r+\gamma')|L_{ir}||R_i|} \leq \frac{2\sum_{j=r+1}^{i} |L_{ij}| \sum_{j=0}^{r} |L_{ij}|}{(i-r)(i-r+\gamma')|L_{ir}||R_i|} \tag{88}$$

The following lemmas are needed to find a bound to (88).

Lemma 5.6. *Let $N = \gamma' - 1/2$ and $i > 2N+1$. Then, define $j^* = \lfloor \frac{i-1+N}{2} \rfloor$. It holds that $|L_{ij}| \leq |L_{ij^*}|$.*

Proof. Consider the ratio $\frac{|L_{ij+1}|}{|L_{ij}|}$. It is easy to see that

$$\frac{|L_{ij+1}|}{|L_{ij}|} = \frac{(i-j+\gamma')(i-j)}{|j+1-\gamma'|(j+1)} \tag{89}$$

Now, if $j \leq N < i/2 - 1/2$, then $|j+1-\gamma'| = \gamma' - j - 1 > \gamma' - 1 = N + 1/2$. Then, we have

$$\frac{|L_{ij+1}|}{|L_{ij}|} = \frac{(i-j+\gamma')(i-j)}{(\gamma'-j-1)(j+1)} > \frac{i/2(i/2+\gamma')}{(N+1/2)(N+1)} \geq 1. \tag{90}$$

Thus, the sequence always increases as long as $j \leq N$, and we can look for a maximum $j^* > N$. Then, for $j > N$, denote the ratio of (89) by f:

$$f = \frac{|L_{ij+1}|}{|L_{ij}|} = \frac{(i-j+\gamma')(i-j)}{(j+1-\gamma')(j+1)} \tag{91}$$

Now, $f \leq 1$ implies $(i-j)^2 + \gamma'(i-j) \leq (j+1)^2 - \gamma'(j+1)$. Thus, $(i-j)^2 - (j+1)^2 + \gamma'(i+1) \leq 0$. Manipulating the expression, we find $(i^2-1) - 2j(i+1) + \gamma'(i+1) \leq 0$ and canceling the term $(i+1)$ the following inequality for j is reached:

$$j \leq \frac{i-1+\gamma'}{2} \tag{92}$$

Therefore, if (and only if) the bound given by (92) is verified, $\frac{|L_{ij+1}|}{|L_{ij}|} \leq 1$. Therefore, we conclude that the maximum of the sequence $|L_{ij}|$ is reached at

$$j = j^* = \lfloor \frac{i-1+\gamma'}{2} \rfloor = \lfloor \frac{i-1+N}{2} + 1/4 \rfloor = \lfloor \frac{i-1+N}{2} \rfloor \tag{93}$$

thus finishing the proof. □

Lemma 5.7. *Let $N = \gamma' - 1/2$ and $i > 2N+1$. Then we have $\frac{\sum_{j=r+1}^{i} |L_{ij}| \sum_{j=0}^{r} |L_{ij}|}{(i-r)(i-r+\gamma')|L_{ir}} \leq 2L_{ij^*}$, where j^* is defined in (93).*

Proof. Now, to bound the term $\sum_{j=r+1}^{i} |L_{ij}| \sum_{j=0}^{r} |L_{ij}|$, consider two possibilities and use Lemma 5.6. If $r < j^*$, then $\sum_{j=r+1}^{i} |L_{ij}| \sum_{j=0}^{r} |L_{ij}| \leq (i-r)(r+1)|L_{ij^*}||L_{ir}|$. On the other hand, if $r \geq j^*$, then $\sum_{j=r+1}^{i} |L_{ij}| \sum_{j=0}^{r} |L_{ij}| \leq (i-r)(r+1)|L_{ir+1}||L_{ij^*}|$. Therefore, if $r < j^*$:

$$\frac{\sum_{j=r+1}^{i} |L_{ij}| \sum_{j=0}^{r} |L_{ij}|}{(i-r)(i-r+\gamma')|L_{ir}|} \leq \frac{(r+1)|L_{ij^*}|}{i-r+\gamma'} \leq \frac{(j^*+1)|L_{ij^*}|}{i-j^*+\gamma'}$$

else, if $r \geq j^*$,

$$\frac{\sum_{j=r+1}^{i} |L_{ij}| \sum_{j=0}^{r} |L_{ij}|}{(i-r)(i-r+\gamma')|L_{ir}|} \leq \frac{(r+1)|L_{ir+1}||L_{ij^*}|}{(i-r+\gamma')|L_{ij}|}$$

and using (91),

$$\frac{\sum_{j=r+1}^{i} |L_{ij}| \sum_{j=0}^{r} |L_{ij}|}{(i-r)(i-r+\gamma')|L_{ir}|} \leq \frac{(i-r)|L_{ij^*}|}{(r+1-\gamma')} \leq \frac{(i-j^*)|L_{ij^*}|}{(j^*+1-\gamma')}$$

Now, since $\frac{i-1+N}{2} \leq j^* \leq \frac{i+N}{2}$, one has that

$$\frac{j^*+1}{i-j^*+\gamma'} \leq \frac{\frac{i+N}{2}+1}{i-\frac{i+N}{2}+N+1/2} = \frac{i+N+2}{i+N+1} < 2,$$

and similarly,

$$\frac{i-j^*}{j^*+1-\gamma'} \leq \frac{i-\frac{i-1+N}{2}}{\frac{i-1+N}{2}+1/2-N} = \frac{i+1-N}{i-N} < 2,$$

thus concluding the proof. □

Thus, we are left with showing that $\frac{|L_{ij^*}|}{|R_i|}$ is decreasing, which is expected, since L_{ij^*} is one of the elements that appear in the sum R_i. Using the expression (83) and the formula for L_{ij} that involves the Gamma function, we obtain the following:

$$\frac{|L_{ij^*}|}{|R_i|} = \frac{i!^2\Gamma(1-\gamma')\Gamma(i+1+\gamma')\Gamma(i+1-\gamma')}{2i!j^*!(i-j^*)!\Gamma(\gamma'-1)\Gamma(j^*+1-\gamma')\Gamma(i-j^*+1+\gamma')}. \tag{94}$$

Now, the decreasing character of the sequence is established as follows (consider the case where $i+N$ is odd, so that $j^* = \frac{i-1+N}{2}$; the even case is analogous). Consider Stirling's approximation to the factorial, namely $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Then:

$$\frac{i!^2}{2i!j^*!(i-j^*)!} = \frac{i!^2}{2i!\left(\frac{i-1+N}{2}\right)!\left(\frac{i+1-N}{2}\right)!} \approx \sqrt{\frac{i}{\pi(i^2-(N-1)^2)}}\frac{1}{2^{2i}}\frac{e^i}{\left(\frac{i-1+N}{2}\right)^{\frac{i-1+N}{2}}\left(\frac{i+1-N}{2}\right)^{\frac{i+1-N}{2}}}. \tag{95}$$

On the other hand, Stirling's approximation to the Gamma function [37, p. 257] reads $\Gamma(z) \approx \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z$, thus

$$\begin{aligned} &\frac{\Gamma(i+1-\gamma')\Gamma(i+1+\gamma')}{\Gamma(j^*+1-\gamma')\Gamma(i-j^*+1+\gamma')} \\ &= \frac{\Gamma(i+1/2-N)\Gamma(i+3/2+N)}{\Gamma(\frac{i-N}{2})\Gamma(\frac{i+N}{2}+2)} \\ &\approx \frac{1}{e^i}\sqrt{\frac{(\frac{i-N}{2})(\frac{i+N}{2}+2)}{(i+1/2-N)(i+3/2+N)}} \\ &\quad\times\frac{(i+1/2-N)^{i+1/2-N}(i+3/2+N)^{i+3/2+N}}{(\frac{i-N}{2})^{\frac{i-N}{2}}(\frac{i+N}{2}+2)^{\frac{i+N}{2}+2}}. \end{aligned} \tag{96}$$

Putting together (95)–(96), we obtain the following.

$$\frac{|L_{ij^*}|}{|R_i|} \approx \frac{i\Gamma(1-\gamma')}{\sqrt{\pi}\Gamma(\gamma'-1)}\sqrt{f_1(i)}f_2(i)^i f_3(i), \tag{97}$$

where we have broken the approximation into three functions:

$$f_1(i) = \frac{i}{\pi(i^2-(N-1)^2)}\frac{(\frac{i-N}{2})(\frac{i+N}{2}+2)}{(i+1/2-N)(i+3/2+N)}, \tag{98}$$

$$f_2(i) = \frac{(i+1/2-N)(i+3/2+N)}{\sqrt{(i-1+N)(i+1-N)(i-N)(i+N+4)}}, \tag{99}$$

$$f_3(i) = \frac{(i+1/2-N)^{1/2-N}(i+3/2+N)^{3/2+N}}{\left(\frac{i-1+N}{2}\right)^{\frac{-1+N}{2}}\left(\frac{i+1-N}{2}\right)^{\frac{1-N}{2}}(\frac{i-N}{2})^{\frac{-N}{2}}(\frac{i+N}{2}+2)^{\frac{N}{2}+2}}. \tag{100}$$

Notice that clearly $\lim_{i\to\infty} f_1(i) = 0$ (since $f_1(i)$ behaves like $\mathcal{O}(1/i)$ for large i), $\lim_{i\to\infty} f_2(i) = 1$, and $\lim_{i\to\infty} f_3(i) = 16$, thus it only remains to compute $\lim_{i\to\infty} f_2(i)^i$, which is an indeterminate of the kind 1^∞. Resolving it (the details are omitted for brevity) one obtains that the limit is indeed 1. Thus, it is possible to find the decreasing sequence c_i in (81), concluding the proof of convergence and analyticity in odd dimension.

5.1.5. Proof of analyticity for even dimension

The fact that γ' is an integer makes the odd approach a priori impossible, since (65) would not be well defined (it would contain divisions by zero). However, to overcome that difficulty, we employ a partial solution for the kernel equations, to the order $\gamma'-1$, which helps to regularize the problem.

For this, consider $F(r,\rho) = \sum_{i=0}^{\gamma'-1} r^{2i}\phi_i(\rho^2)$. Replacing this function in (18)–(19) results in

$$\sum_{i=0}^{\gamma'-1} r^{2i}\frac{\lambda(\rho^2)+c}{\epsilon}\phi_i(\rho^2) =$$
$$\sum_{i=0}^{\gamma'-1}\left[4i(i-\gamma')r^{2i-2}\phi_i(\rho^2) - r^{2i}\left(4\rho^2\phi_i''(\rho^2) + 2(2+\gamma')\phi'(\rho^2)\right)\right]$$

and one gets the following recursive set of ODEs:

$$\frac{\lambda(\rho^2)+c}{\epsilon}\phi_i(\rho^2) = 4(i+1)(i+1-\gamma')\phi_{i+1}(\rho^2)$$
$$-4\rho^2\phi_i''(\rho^2) - 2(2+\gamma')\phi'_{\gamma'-1}(\rho^2)$$

which is solved starting at $i=\gamma'-1$:

$$\frac{\lambda(\rho^2)+c}{\epsilon}\phi_{\gamma'-1}(\rho^2) = -4\rho^2\phi_{\gamma'-1''}(\rho^2) - 2(2+\gamma')\phi'_{\gamma'-1}(\rho^2)$$

This can be written as

$$4x\phi_{\gamma'-1''} + 2(2+\gamma')\phi'_{\gamma'-1} + \frac{\lambda(x)+c}{\epsilon}\phi_{\gamma'-1} = 0$$

which is an ODE with a regular singular point at $x=0$. By applying the Frobenius method [40, Chapter 36] one can rewrite this equation as

$$4x^2\phi_{\gamma'-1''} + 2x(2+\gamma')\phi'_{\gamma'-1} + \frac{\lambda(x)+c}{\epsilon}x\phi_{\gamma'-1} = 0$$

and its indicial equation is $r(r-1)+(1+\gamma'/2)r = 0$, thus $r_1 = 0$ and $r_2 = \gamma'/2$ (non-integer). We are interested in the solution of the form $\phi_{\gamma'/2} = \sum_{i=0}^{\infty} a_i\rho^{2i}$ and discard the other solution. By Fuchs' theorem [41, p. 146] this solution is analytic where $\lambda(x)$ is analytic, thus the radius of convergence of the resulting $\phi_{\gamma'-1}(\rho^2)$ is greater than one. Next, for $i=\gamma'-2$ up to $i=0$:

$$4x\phi_i'' + 2(2+\gamma')\phi_i' + \frac{\lambda(x)+c}{\epsilon}\phi_i = 4(i+1)(i+1-\gamma')\phi_{i+1}(x)$$

which, has the same indicial equation and again, also admits a solution in the required form. Applying once more Fuchs' theorem, this solution is analytic in intervals where both $\lambda(x)$ and ϕ_{i+1} are analytic. Thus, by induction, we find a family of solutions such that the radius of convergence of all ϕ_i is greater than R.

The solutions just found have a degree of freedom (the first coefficient a_i of their power series, which is $\phi_i(0)$). The idea is to construct the solution such that the boundary condition $G^n_{lm}(r,r) = H(r)$ is satisfied up to order $2\gamma'-2$. Thus: $F(r,r) = \sum_{i=0}^{\gamma'-1} r^{2i}\phi_i(r^2)$ and expanding in power series $\phi_i(r^2)$:

$$F(r,r) = \sum_{i=0}^{\gamma'-1} r^{2i}\left(\phi_i(0) + \frac{r^2}{1!}\phi_i'(0) + \frac{r^4}{2!}\phi_i''(0) + \dots\right)$$

Thus:

$$\phi_0(0) = H(0), \tag{101}$$
$$\phi_1(0) + \frac{1}{1!}\phi_0'(0) = \frac{1}{1!}H'(0), \tag{102}$$
$$\phi_2(0) + \frac{1}{1!}\phi_1'(0) + \frac{1}{2!}\phi_0''(0) = \frac{1}{2!}H''(0), \tag{103}$$
$$\dots \tag{104}$$
$$\phi_{\gamma-1}(0) + \dots + \frac{1}{(\gamma-1)!}\phi_0^{(\gamma-1)}(0) = \frac{1}{(\gamma-1)!}H^{(\gamma-1)}(0). \tag{105}$$

It can be shown that this scheme produces valid initial values for the ϕ_i's. However, an easier approach is to follow the general series approach of Section 5.1.1 up to order $i=\gamma'-1$. By the uniqueness of the series development and identifying coefficients, it can easily be shown that $\phi_i(0) = C_{ii}$.

Next, calling $G^n_{lm}(r,\rho) = \check{G}^n_{lm}(r,\rho) + F(r,\rho)$ the new boundary condition for the PDE becomes: $\check{G}^n_{lm}(r,r) = H(r) - F(r,r)$ which starts at order $2\gamma'$. Thus, we can propose $\check{G}^n_{lm}(r,\rho) = r^\gamma F_2(r,\rho)$. One can see that the PDE for F_2 is

$$\frac{\lambda(\rho)+c}{\epsilon}F_2(r,\rho) = \partial_{rr}F_2(r,\rho) + (1+\gamma)\frac{\partial_r F_2(r,\rho)}{r}$$
$$-\partial_{\rho\rho}F_2(r,\rho) - (1+\gamma)\frac{\partial_\rho F_2(r,\rho)}{\rho}, \tag{106}$$

and following previous sections, calling $\psi(r^2) = \frac{H(r)-F(r,r)}{r^{2\gamma'}}$ and abusing the notation by keeping the same name for the coefficients C_{ij}, one can find a power series development $F_2(r,\rho) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} C_{ij}r^{2j}\rho^{2(i-j)}\right)$ as

$$\forall i, \quad \sum_{j=0}^{i} C_{ij} = -\frac{\psi_i}{2(2i+1)}, \tag{107}$$

$$\forall i \geq 1, 0 \leq j \leq i-1, \quad (j+1)(j+1+\gamma')C_{i(j+1)}$$
$$-(i-j)(i-j+\gamma')C_{ij} = B_{(i-1)j}, \tag{108}$$

Now the approach of Section 5.1.4 becomes applicable and even easier, since all coefficients are positive. Indeed, define

$$L_{i0} = 1, \qquad L_{ij} = \binom{i}{j}$$
$$\frac{(i+\gamma')(i-1+\gamma')\dots(i-j+\gamma'+1)}{(1+\gamma')(2+\gamma')\dots(j+\gamma')} > 0, \quad j>0 \tag{109}$$

Mimicking Section 5.1.4 we reach

$$\begin{aligned}\alpha_n &= \sum_{j=0}^{i}|C_{ij}| \\ &\leq \frac{|\psi_i|}{2(2i+1)}\frac{\sum_{j=0}^{i}|L_{ij}|}{|R_i|} \\ &\quad + \sum_{r=0}^{i-1}|B_{(i-1)r}|\left(\frac{|R_{ir}|\sum_{j=r+1}^{i}|L_{ij}| + |R_i - R_{ir}|\sum_{j=0}^{r}|L_{ij}|}{(i-r)(i-r+\gamma')|L_{ir}||R_i|}\right) \\ &= \frac{|\psi_i|}{2(2i+1)} + \sum_{r=0}^{i-1}|B_{(i-1)r}|\frac{2R_{ir}(R_i-R_{ir})}{(i-r)(i-r+\gamma')L_{ir}R_i},\end{aligned} \tag{110}$$

where the last step can be performed due to the positivity of the redefined coefficients L_{ij} compared to Section 5.1.4. Again, we

apply Lemma 5.5 to (110). In this case, we define

$$b_i = \frac{1}{2\epsilon(2i+1)} \geq 0$$

which is already a decreasing sequence. Then we need to find c_i such that

$$\max_{r\in\{0,\dots,i-1\}} \frac{2R_{ir}(R_i - R_{ir})}{(i-r)(i-r+\gamma')L_{ir}R_i} \leq c_i \tag{111}$$

and c_i should be shown to be decreasing (for sufficiently large i) and convergent to zero. Consider the following lemma.

Lemma 5.8. *Consider L_{ij} as defined in (109) and R_i, R_{ij} as defined in (109). Then:*

1. $L_{i(i-j)} = L_{ij}$
2. $R_i - R_{ir} = R_{i(i-r-1)}$
3. *Let* $F(i,r) = \frac{R_{ir}(R_i-R_{ir})}{(i-r)(i-r+\gamma')L_{ir}}$ *for i, r nonnegative integers with $r < i$. Then we have $F(i,r) = F(i, i-r-1)$.*
4. *If $r \leq i/2$, $R(i,r) \leq (r+1)L_{ir}$.*
5. *It holds that* $\max_{r\in\{0,\dots,i-1\}} \frac{2R_{ir}(R_i-R_{ir})}{(i-r)(i-r+\gamma')L_{ir}} \leq 2\frac{i+2}{i(i+2\gamma')}R_i$.

Proof. When writing $L_{ij} = \binom{i}{j} \frac{\Gamma(1+\gamma')\Gamma(i+1+\gamma')}{\Gamma(j+1+\gamma')\Gamma(i-j+1+\gamma')}$ the first property becomes evident, whereas the second property is immediate from the first since $R_i - R_{ij} = \sum_{k=j+1}^{k=i} L_{ik} = \sum_{k=0}^{i-j-1} L_{i(i-k)} = \sum_{k=0}^{i-j-1} L_{ik} = R_{i(i-j-1)}$.

For the third property, note that

$$\begin{aligned} F(i,r) &= \frac{(r+1)(r+1+\gamma)}{(i-r)(i-r+\gamma')}\frac{L_{i(i-r-1)}}{L_{ir}}F(i,i-r-1) \\ &= \frac{(r+1)(r+1+\gamma)}{(i-r)(i-r+\gamma')}\frac{L_{i(r+1)}}{L_{ir}}F(i,i-r-1) \\ &= F(i,i-r-1) \end{aligned} \tag{112}$$

The fourth property is obvious, noting that $L_{ij} \leq L_{i(j+1)}$ for $j < i/2$. Finally, for the last property, first note that it is only required to study $0 \leq r \leq i/2$ given the third property. Now:

$$\begin{aligned} F(i,r) &= \frac{R_{ir}(R_i - R_{ir})}{(i-r)(i-r+\gamma')L_{ir}} \\ &\leq \frac{(r+1)R_{i(i-r-1)}}{(i-r)(i-r+\gamma')} \\ &\leq \frac{r+1}{(i-r)(i-r+\gamma')}R_i \end{aligned} \tag{113}$$

and since this is an increasing function of r for $0 \leq r < i$, we can bound it by its value at $r = i/2$, thus proving the final property. □

Therefore, setting $c_i = 4\frac{i+2}{i(i+2\gamma')}$, a sequence that decreases to zero, we can apply Lemma 5.5 to (110) and follow the same steps as in Section 5.1.4 to obtain the result of Theorem 2 for the case of even dimensions.

6. Observer design

This section designs an observer for (9)–(10) from the measured output $\partial_r u_l^m(t,R)$ as follows:

$$\begin{aligned} \partial_t \hat{u}_l^m &= \epsilon\frac{\partial_r\left(r^{n-1}\partial_r\hat{u}_l^m\right)}{r^{n-1}} - l(l+n-2)\frac{\epsilon}{r^2}\hat{u}_l^m + \lambda(r)\hat{u}_l^m \\ &+ p_{lm}^n(r)(\partial_r u_l^m(t,R) - \partial_r\hat{u}_l^m(t,R)), \end{aligned} \tag{114}$$

with boundary condition

$$\hat{u}_l^m(t,R) = U_l^m(t). \tag{115}$$

We need to design the output injection gain $p_{lm}^n(r)$. Following closely [9], define the observer error as $\tilde{u} = u - \hat{u}$. The observer error dynamics is given by

$$\begin{aligned} \frac{\partial\tilde{u}_{lmt}}{\partial t} &= \frac{\epsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\tilde{u}_{lm}\right) - l(l+n-2)\frac{\epsilon}{r^2}\tilde{u}_{lm} \\ &+ \lambda(r)\tilde{u}_{lm} - p_{lm}^n(r)\partial_r\tilde{u}_{lm}(t,R), \end{aligned} \tag{116}$$

with boundary conditions

$$\tilde{u}_{lm}(t,R) = 0. \tag{117}$$

Next, we use the backstepping method to find a value of $p_{lm}^n(r)$ that guarantees the convergence of $\tilde{u}$ to zero. This ensures that the observer estimates tend to the true state values. Our approach to designing $p(r)$ is to find a mapping that transforms (116) into the following target system

$$\frac{\partial\tilde{w}_{lmt}}{\partial t} = \frac{\epsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\tilde{w}_{lm}\right) - c\tilde{w}_{lm} - l(l+n-2)\frac{\epsilon}{r^2}\tilde{w}_{lm}, \tag{118}$$

with boundary conditions

$$\tilde{w}_{lm}(t,R) = 0. \tag{119}$$

The transformation is defined as follows:

$$\tilde{u}_{lm}(t,r) = \tilde{w}_{lm}(t,r) - \int_r^R P_{lm}^n(r,\rho)\tilde{w}_{lm}(t,\rho)d\rho, \tag{120}$$

and then $p_{lm}^n(r)$ will be found from the transformation kernel as an additional condition.

From [9], one obtains the following PDE that the kernel must verify:

$$\begin{aligned} &\frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r P_{lm}^n\right) - \partial_\rho\left(\rho^{n-1}\partial_\rho\left(\frac{P_{lm}^n}{\rho^{n-1}}\right)\right) \\ &- l(l+n-2)\left(\frac{1}{r^2} - \frac{1}{\rho^2}\right)P_{lm}^n = -\frac{\lambda(r)}{\epsilon}P_{lm}^n \end{aligned} \tag{121}$$

In addition, we find a value for the output injection gain kernel

$$p_{lm}^n(r) = \epsilon P_{lm}^n(r,R) \tag{122}$$

In addition, the following boundary condition must be verified.

$$\begin{aligned} 0 &= \lambda(r) + \epsilon\left(\partial_r P_{lm}^n(r,\rho)\right)\Big|_{\rho=r} + \frac{\epsilon}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}P_{lm}^n(r,r)\right) \\ &+ \epsilon\partial_\rho\left(\frac{P_{lm}^n(r,\rho)}{\rho^{n-1}}\right)\Big|_{\rho=r} r^{n-1}, \end{aligned} \tag{123}$$

which can be written as

$$\begin{aligned} 0 &= \lambda(r) + \epsilon\partial_r P_{lm}^n(r,r) + \epsilon\frac{d}{dr}\left(P_{lm}^n(r,r)\right) + (n-1)\frac{\epsilon P_{lm}^n(r,r)}{r} \\ &+ \epsilon\partial_\rho P_{lm}^n(r,r) - (n-1)\frac{\epsilon P_{lm}^n(r,r)}{r}. \end{aligned} \tag{124}$$

Following [9], and after some computations, we reach the boundary conditions for the kernel equations as follows:

$$P_{lm}^n(0,\rho) = 0, \quad \forall l \neq 0 \tag{125}$$

$$P_{lmr}^n(0,\rho) = 0, \quad \forall l \neq 1 \tag{126}$$

$$P_{lm}^n(r,r) = -\frac{\int_0^r \lambda(\sigma)d\sigma}{2\epsilon}. \tag{127}$$

It turns out that the observer kernel equation can be transformed into the control kernel equation, therefore obtaining a similar explicit result. For this, define

$$\check{P}_{lm}^n(r,\rho) = \frac{\rho^{n-1}}{r^{n-1}}P_{lm}^n(\rho,r), \tag{128}$$

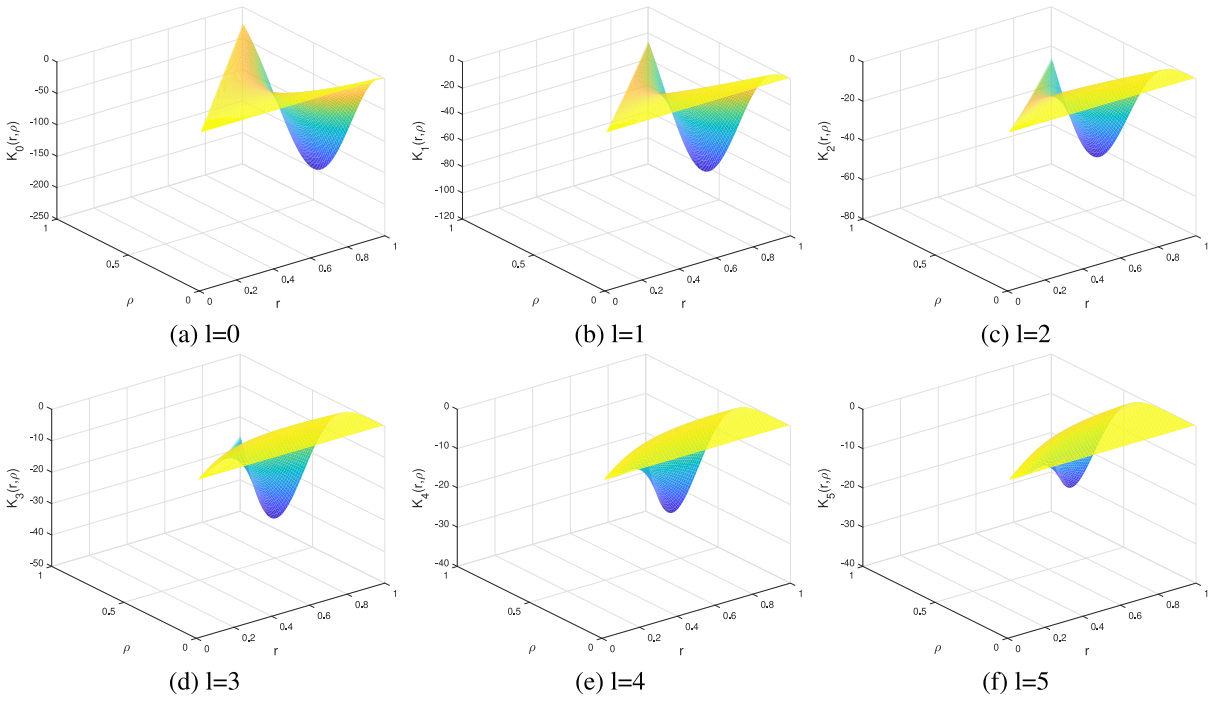


Fig. 1. Polynomial approximation of the first six control gains $K_l(r,\rho)$, $l=0,1,\ldots,5$.

and it can be verified that the equation now governing $\check{P}^n_{lm}(r,\rho)$ is exactly the equation satisfied by $K^n_{lm}(r,\rho)$. Thus $\check{P}^n_{lm}(r,\rho) = K^n_{lm}(r,\rho)$ and we can apply our previous result of Section 5.

The observer error dynamics has the same stability properties derived in Section 4 for the closed-loop system under full state control. As in the controller case, only a limited number of modes need to be estimated; namely, those that are not naturally stable by Lemma 4.1, this being the main difference from the result given in [9].

Finally, the controller–observer augmented system can be proved closed-loop stable as in [9], using the separation principle given the linearity of the system, with desired convergence rate, and without much modification; we skip the details, which requires going up to H^1 stability, as in [9].

7. Implementation and simulation study

In this section, the simulation experiment on a three-dimensional unity ball ($n=3, R=1$) is taken as an example to illustrate the effectiveness of the proposed control, and some implementation remarks.

The system with the output feedback control law is simulated on $0 \le t \le 2$ s with the following parameters: $\epsilon = 1$, $\lambda(r) = 10r^4+50r^2+50$, $c=3$. We consider that the system initially has the random quantity $u_0 \in [0, 10]$, and the observer's initial condition is set as the actual state plus an error of normal distribution with zero mean and $\sigma^2 = 0.5$.

Fig. 1 shows the plots of the polynomial approximation of kernels K^3_{lm}, which is obtained by first expanding $\frac{\lambda(r)+c}{\epsilon}$ by using (31), and then finding the coefficients of (32) up to a cutoff in the pth powers by solving recursively (39)–(41) for each i up to p; in each step one needs to compute the coefficients B_{ij} given by (35) from the previously-found coefficients C_{ij}. The value of K does not depend on m, so we omit this subindex, and l is varied from zero to the value given by Lemma 3.1. The value of p is chosen as $p=15$. Applying Lemma 3.1, one can obtain l to be 11; however, here, to save space, we only show the first six approximate numerical solutions of the control gains. As shown in Fig. 1, we find that K_l becomes increasingly smaller as l increases.

In order to avoid a dramatic increase in the complexity of simulation caused by the high dimension, in our simulations, we employ a method also based on spherical harmonic expansions, which greatly reduces the error. Thus, we only calculate the harmonics u^m_l that only need discretization in the radial direction, and then we sum up a finite number S of harmonics to recover u. When $S>0$ is a large enough integer, the error caused by the use of a finite number of harmonics is much smaller than the angular discretization error. Thus, the simulation is carried out using the formula

$$u(t,r,\theta_1,\theta_2) = \sum_{l=0}^{l=S}\sum_{m=-l}^{m=l} u^m_l(t,r)Y^3_{lm}(\theta_1,\theta_2) \tag{129}$$

which is a truncated variant of (6), where the spherical harmonics are defined as

$$Y^3_{lm}(\theta_1,\theta_2) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P^3_{lm}(\cos(\theta_1))e^{im\theta_2} \tag{130}$$

with P^3_{lm} the associated Legendre polynomial defined as

$$P^3_{lm}(s) = \frac{1}{2^l l!}(1-s^2)^{m/2}\frac{d^{l+m}}{ds^{l+m}}(s^2-1)^l \tag{131}$$

Figs. 2 and 3 illustrate the transients of open-loop and closed-loop responses at different times, respectively, where the color denotes the value of the position at this time. The evolutions of the L^2 norm of u are plotted in Fig. 2(d) and Fig. 3(e), respectively. Note that in these figures, the ranges of color bars are different and thus avoid too uniform colors in Fig. 3.

When the open-loop and closed-loop evolution is compared directly, the validity of the proposed method is illustrated more intuitively. Fig. 3(f) shows the L^2 norm of the observation error, from which it can be found that the system begins to converge to its zero equilibrium after the observation error has already settled to zero as well. The evolutions at different layers, namely $r=0.002$, $r=0.3$, $r=0.5$, and $r=0.8$, are shown in Fig. 4(a)–(c), and the observer errors are presented in Fig. 4(b), (d). For clarity, only the first 0.4 s of the response are shown here. Fig. 5 shows the control effort at the boundary. It can be seen that

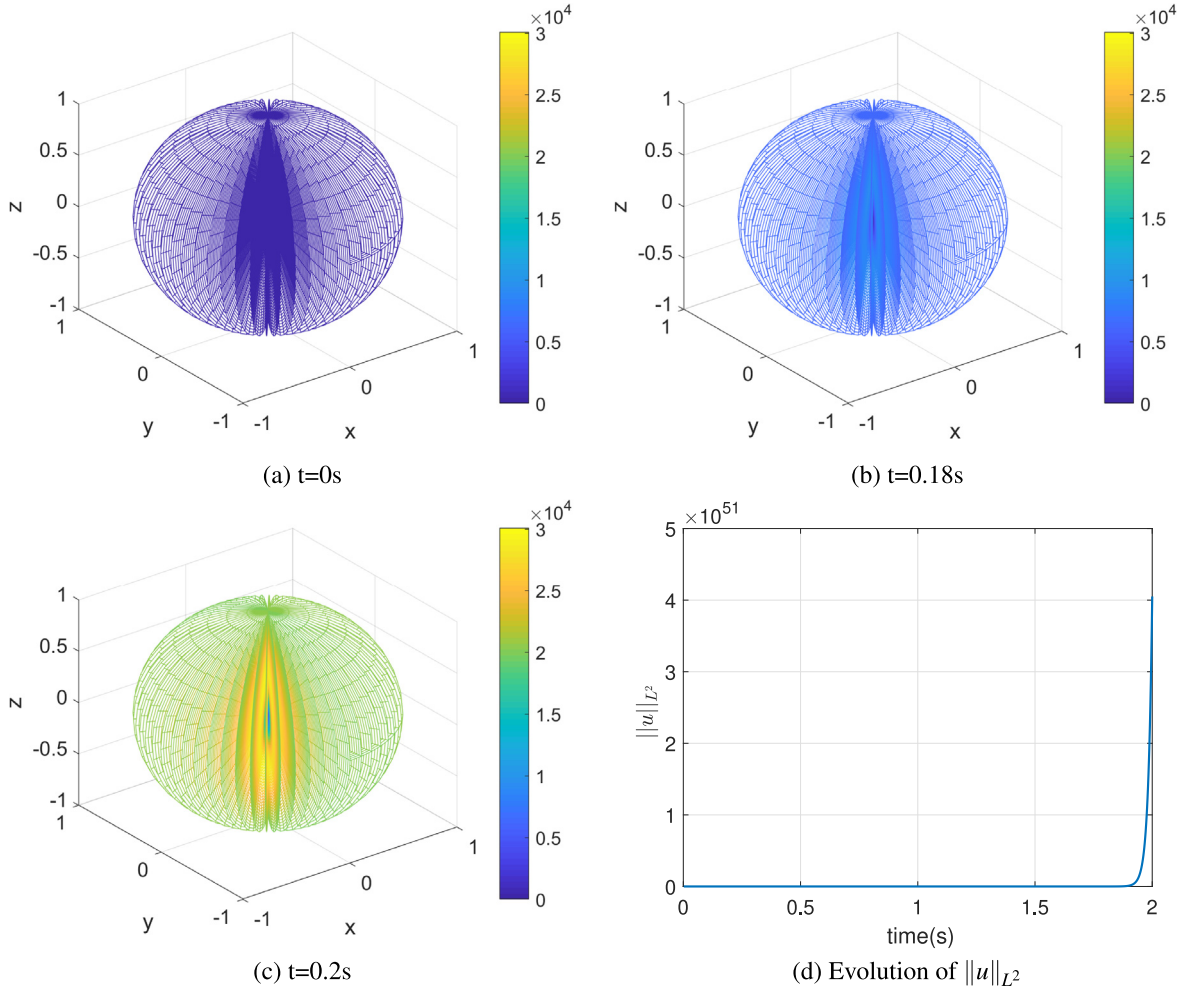


Fig. 2. Open-loop evolution. (a)–(c) Transient states. (d) L^2 norm of state u.

the system driven by the proposed boundary control eventually converges after a short-term fluctuation.

8. Conclusion

We have shown a design to stabilize a radially varying reaction–diffusion equation on an n ball, by using an output-feedback boundary control law (with boundary measurements as well) designed through a backstepping method. The radially varying case proves to be a challenge, as the kernel equations become singular in radius; when applying the backstepping method, the same type of singularity appears in the kernel equations, and successive approximations become difficult to use. Using a power series approach, a solution is found, thus providing a numerical method that can be readily applied to both control and observer boundary design. In addition, the required conditions for the radially varying coefficients (analyticity and evenness) are revealed.

This result can be extended in several ways. If one has Neumann boundary conditions at the controlled boundary (which implies then that one is measuring the state at the boundary instead of its normal derivative), or even Robin boundary conditions, the method can be extended straightforwardly, since the transformation itself does not change and, therefore, the backstepping kernels remain the same. Only the particular control/observer gains, deduced from the backstepping kernels, would change; as well as a small modification on Lemma 4.1 to account for the change in the boundary conditions.

In practice, this result can be of interest for the deployment of multi-agent systems, following the spirit of [24]; thus, the radial domain mirrors a radial topology of interconnected agents that follow the reaction–diffusion dynamics to converge to equilibria that represent different deployment profiles. Since the plant can be chosen as desired (thereby setting the behavior of the agents), the use of analytic reaction coefficients is not actually a restriction, but opens the door to richer families of deployment profiles compared to the constant coefficient case of [24].

However, the theoretical side of the result needs to be further investigated; an avenue of research that can be explored is the relaxation of the analyticity hypothesis by using reaction coefficients belonging to the Gevrey family; the kernels can then be analyzed to verify if they are still analytic, or rather Gevrey-type kernels, or simply do not converge. Also, the rate of convergence of the obtained power series is of interest and shall be explored. We have experimentally observed that the rate of convergence of the series representation of $\lambda(r)$ has a considerable influence on it. In addition, one could also explore how fast the series converges in the case of constant λ, since an explicit solution is known from [9]. In particular, the worst case in a domain with radius R would be given by the convergence rate of the Maclaurin series of $\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\epsilon}}R\right]$ where I_1 is a modified Bessel function of order 1. Since this function behaves quite closely to an exponential if its argument is large (which would be the case with slowest convergence), the number of required terms would be given by

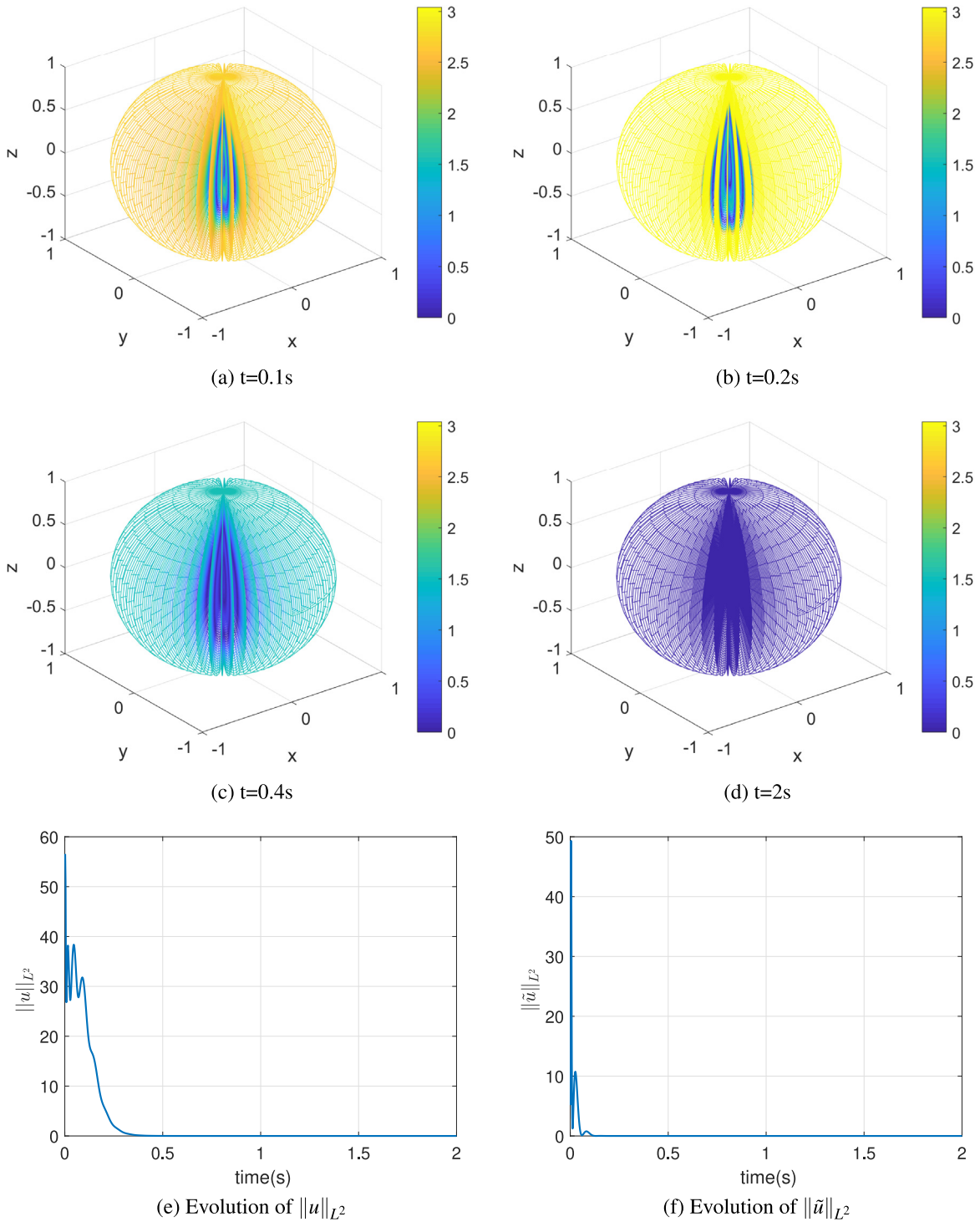


Fig. 3. Closed-loop evolution using output feedback control. (Note the different upper ranges of the color bars in Figs. 2 and 3.) (a)–(d) Transient states. (e) L^2 norm of state u. (f) L^2 norm of observation error $\tilde{u}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

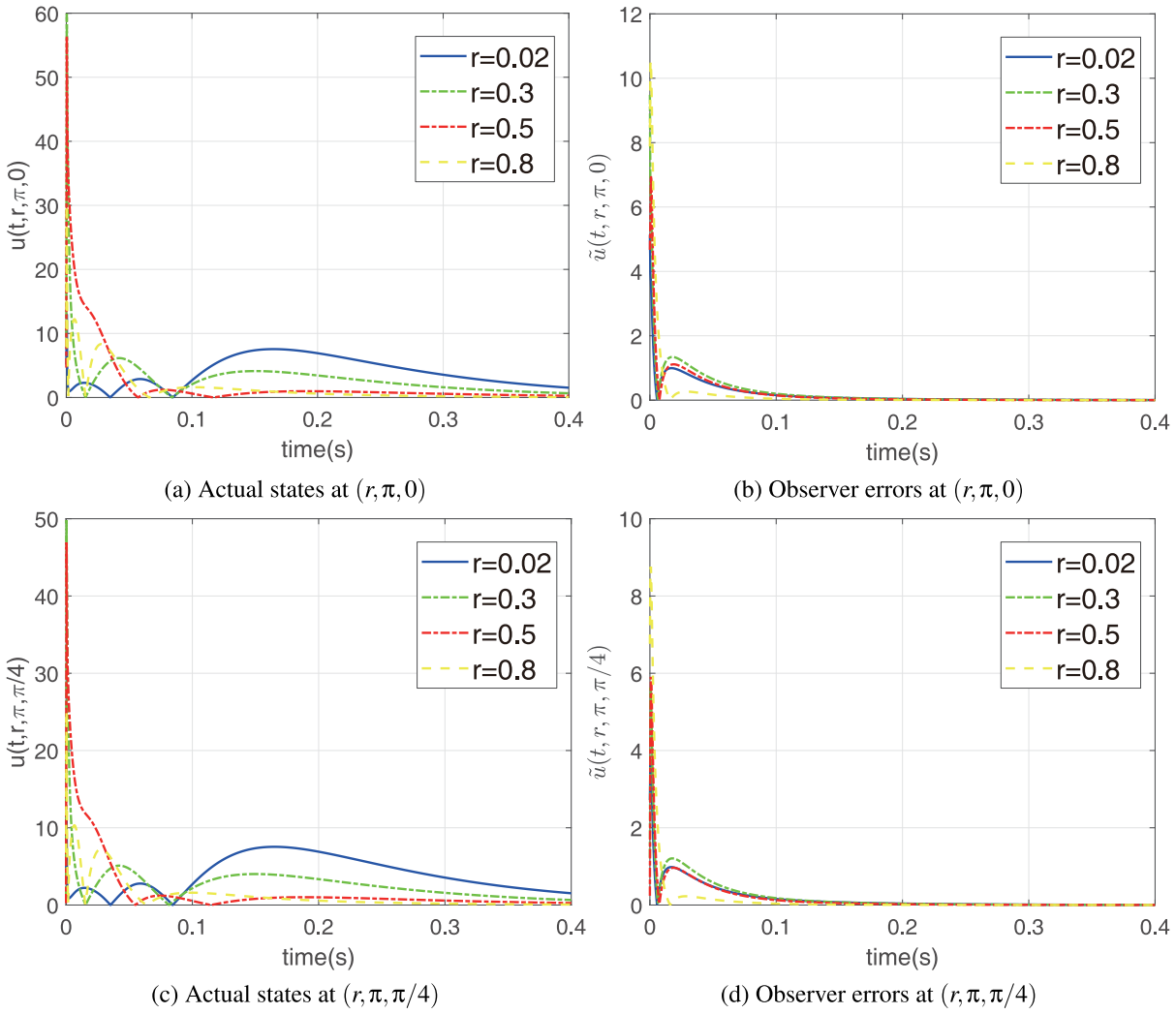
the remainder of the power series of an exponential. In that case, it is easy to see that the size of the term $\sqrt{\frac{\lambda}{\epsilon}}R$ would define the required truncation level. If we extrapolate this behavior, then, beyond the convergence rate of the series representation of $\lambda(r)$, we can say that higher values of R and $\max_{r\in[0,R]}|\lambda(r)|$ and lower values of ϵ would result in slower-converging series; coincidentally, these are exactly the same factors that would result in a more unstable open-loop plant.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

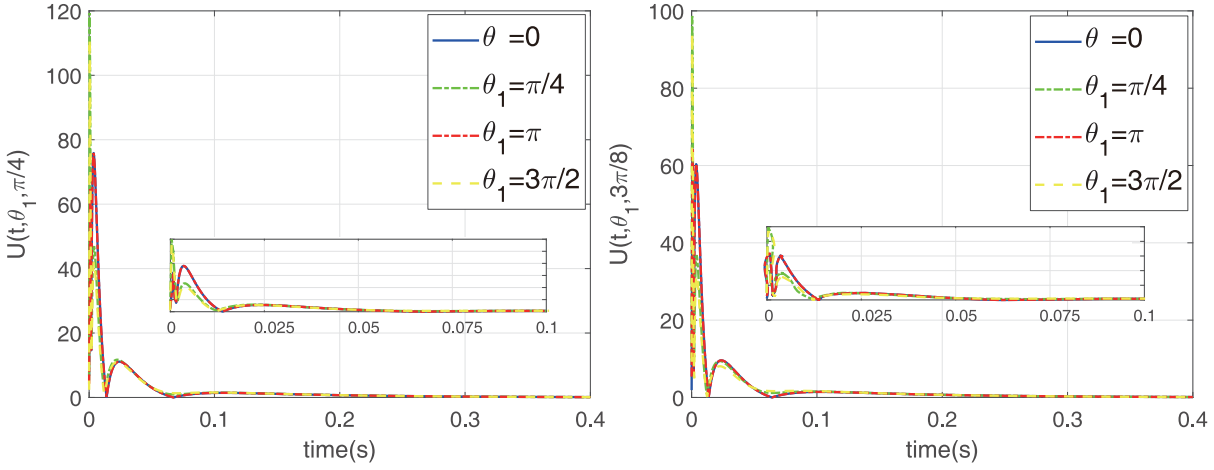
No data was used for the research described in the article.



(a) Actual states at $(r,\pi,0)$ (b) Observer errors at $(r,\pi,0)$

(c) Actual states at $(r,\pi,\pi/4)$ (d) Observer errors at $(r,\pi,\pi/4)$

Fig. 4. The details of closed-loop evolution at different r or $\vec{\theta}$. (a) (c) Actual states. (b) (d) Observer errors between the actual and estimated states.



(a) Control effort at $(\theta_1,\pi/4)$ (b) Control effort at $(\theta_1,3\pi/8)$

Fig. 5. The control effort at different $\vec{\theta}$.

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