Programa de Doctorado “Matemáticas”

PhD Dissertation

Dynamics of stochastic systems with delay and applications to real models

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Contents

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Abstract

In this thesis we investigate the long time behavior of random dynamical systems associated to several kinds of stochastic equations with delays in terms of stability for stationary solutions, weak pullback mean random attractors, random attractors and numerical attractors. The thesis consists of three parts, where the first part covers Chapters 1-3, the last two cover Chapters 4 and 5, respectively.

Chapters 1-3 are devoted to the random dynamics of 3D Lagrangian-averaged Navier-Stokes equations with infinite delay in three cases.

In Chapter 1 we consider the stability analysis of such systems in the case of bounded domains. We first use Galerkin's approximations to prove the existence and uniqueness of solutions when the non-delayed external force is locally integrable and the delay terms are globally Lipschitz continuous with an additional assumption. We then prove the existence of a unique stationary solution to the corresponding deterministic equation via the Lax-Milgram and the Schauder theorems. The stability and asymptotic stability of stationary solutions (equilibrium solutions) are also established. The local stability of stationary solutions for general delay terms is carried out by using a direct method and then apply the abstract results to two kinds of infinite delays. It is worth mentioning that all conditions are general enough to include several kinds of delays, where we mainly consider unbounded variable delays and infinite distributed delays. As we know, it is still an open and challenging problem to obtain sufficient conditions ensuring the exponential stability of solutions in case of unbounded variable delay. Fortunately, we obtained the exponential stability of stationary solutions in the case of infinite distributed delay. However, we are able to further investigate the asymptotic stability of stationary solutions in the case of unbounded variable delay by constructing suitable Lyapunov functionals. Besides, we proved the polynomial asymptotic stability of stationary solutions for the particular case of proportional delay.

In Chapter 2, we further discuss mean dynamics and stability analysis of stochastic systems in the case of unbounded domains. We first prove the well-posedness of systems with infinite delay when the non-delayed external force is locally integrable, the delay term is globally Lipschitz continuous and the nonlinear diffusion term is locally Lipschitz continuous, which leads to the existence of a mean random dynamical system. We then obtain that such a dynamical system possesses a unique weak pullback mean random attractor, which is a minimal, weakly compact and weakly pullback attracting set. Moreover, we prove the existence and uniqueness of stationary solutions to the corresponding deterministic equation via the classical Galerkin method, the Lax-Milgram and the Brouwer fixed theorems. We discuss in the last part of Chapter 2 with those stability results concerning stationary solutions discussed in Chapter 1.

The last case is concerned with the invariant measures for the autonomous version of stochastic

equations in Chapter 3 by using the method of generalized Banach limit. We first use Galerkin approximations, a priori estimates and the standard Gronwall lemma to show the well-posedness for the corresponding random equation, whose solution operators generate a random dynamical system. Next, the asymptotic compactness for the random dynamical system is established via the Ascoli-Arzelà theorem. Besides, we derive the existence of a global random attractor for the random dynamical system. Moreover, we prove that the random dynamical system is bounded and continuous with respect to the initial values. Eventually, we construct a family of invariant Borel probability measures, which is supported by the global random attractor.

It is well-known that lattice dynamical systems have wide applications in physics, chemistry, biology and engineering such as pattern formation, image processing, propagation of nerve pulses, electric circuits and so on. The theory of attractors for deterministic or stochastic lattice systems has been widely developed. Therefore, we focus on the asymptotical behavior of attractors for lattice dynamical systems in the last two chapters.

Two problems related to FitzHugh-Nagumo lattice systems are analyzed in Chapter 4. The first one is concerned with the asymptotic behavior of random delay FitzHugh-Nagumo lattice systems driven by nonlinear Wong-Zakai noise. We obtain a new result ensuring that such a system approximates the corresponding deterministic system when the correlation parameter of Wong-Zakai noise goes to infinity rather than to zero. We first prove the existence of tempered random attractors for the random delay lattice systems with a nonlinear drift function and a nonlinear diffusion term. The pullback asymptotic compactness of solutions is proved thanks to the Ascoli-Arzelà theorem and uniform tail-estimates. We then show the upper semi-continuity of attractors as the correlation parameter tends to infinity. As for the second problem, we consider the corresponding deterministic version of the previous model, and study the convergence of attractors when the delay approaches zero. Namely, the upper semicontinuity of attractors for the delay system to the nondelay one is proved.

Eventually, existence and connection of numerical attractors for discrete-time p-Laplace lattice systems via the implicit Euler scheme are proved in Chapter 5. So far, it remains open to obtain a numerical attractor for a non-autonomous (or stochastic) lattice system, and thus we can at least investigate numerical attractors for the deterministic and non-delayed version of p-Laplace lattice equations. The numerical attractors are shown to have an optimized bound, which leads to the continuous convergence of the numerical attractors when the graph of the nonlinearity closes to the vertical axis or when the external force vanishes. A new type of Taylor expansions without Fréchet derivatives is established and applied to show the discretization error of order two, which is crucial to prove that the numerical attractors converge upper semi-continuously to the global attractor of the original continuous-time system as the step size of the time goes to zero. It is also proved that the truncated numerical attractors for finitely dimensional systems converge upper semi-continuously to the numerical attractor and the lower semi-continuity holds in special cases.

The results of our investigation in this thesis are included in the following papers:

- S. Yang, Y. Li, Q. Zhang and T. Caraballo, Stability analysis of stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay, J. Dynam. Differential Equations, Published Online, (2023), Doi: 10.1007/s10884-022-10244-0.

- S. Yang, T. Caraballo and Y. Li, Dynamics and stability analysis for stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay on unbounded domains, Appl. Math. Optim., (submitted).

- S. Yang, T. Caraballo and Y. Li, Invariant measures for stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay, Commun. Nonlinear Sci. Numer. Simul., 118 (2023), pp. 107004, 21.
- S. Yang, Y. Li and T. Caraballo, Dynamical stability of random delayed FitzHugh-Nagumo lattice systems driven by nonlinear Wong-Zakai noise, J. Math. Phys., 63 (2022), pp. 111512, 32.
- Y. Li, S. Yang and Tomás Caraballo, Optimization and convergence of numerical attractors for discrete-time quasi-linear lattice system, SIAM J. Numer. Anal., (to appear), 2023.

Resumen

En esta tesis investigamos el comportamiento a largo plazo de sistemas dinámicos aleatorios asociados a varios tipos de ecuaciones estocásticas con retardos en términos de estabilidad para soluciones estacionarias, atractores aleatorios débiles en media de tipo pullback, atractores aleatorios y atractores numéricos. La tesis consta de tres partes, donde la primera parte cubre los capítulos 1-3, las dos últimas cubren los capítulos 4 y 5, respectivamente.

Los capítulos 1-3 están dedicados a la dinámica aleatoria de las ecuaciones Lagrangianas de Navier-Stokes en 3D con retardo infinito en tres casos.

En el Capítulo 1 consideramos el análisis de estabilidad de tales sistemas en el caso de dominios acotados. Primero usamos aproximaciones de Galerkin para demostrar la existencia y unicidad de soluciones cuando la fuerza externa no retardada es localmente integrable y los términos de retardo son globalmente Lipschitzianos con una hipótesis adicional. A continuación, demostramos la existencia de una solución estacionaria única para la ecuación determinista correspondiente mediante los teoremas de Lax-Milgram y Schauder. También se establecen la estabilidad y la estabilidad asintótica de las soluciones estacionarias (soluciones de equilibrio). La estabilidad local de las soluciones estacionarias para términos de retardo generales se lleva a cabo mediante un método directo y, a continuación, se aplican los resultados abstractos a dos tipos de retardos infinitos. Cabe mencionar que todas las condiciones son lo suficientemente generales como para incluir varios tipos de retardos, donde consideramos principalmente retardos variables no acotados y retardos distribuidos infinitos. Como sabemos, sigue siendo un problema abierto y desafiante obtener condiciones suficientes que garanticen la estabilidad exponencial de las soluciones en el caso de retardo variable infinito. Afortunadamente, hemos obtenido la estabilidad exponencial de las soluciones estacionarias en el caso de retardo distribuido infinito. Sin embargo, podemos investigar más a fondo la estabilidad asintótica de las soluciones estacionarias en el caso de retardo variable no acotado mediante la construcción de apropiados funcionales de Lyapunov. Además, demostramos la estabilidad asintótica polinómica de las soluciones estacionarias para el caso particular de retardo proporcional.

En el Capítulo 2, discutimos más a fondo la dinámica en media y el análisis de estabilidad de los sistemas estocásticos en el caso de dominios no acotados. En primer lugar, demostramos que los sistemas con retardo infinito están bien planteados cuando la fuerza externa no retardada es localmente integrable, el término de retardo es globalmente Lipschitziano y el término de difusión no lineal es también localmente Lipschitziano, lo que conduce a la existencia de un sistema dinámico aleatorio en media. Además, demostramos la existencia y unicidad de soluciones estacionarias para la ecuación determinista correspondiente mediante el método clásico de Galerkin, el teorema de Lax-Milgram y el teorema de Brouwer. Discutimos en la última parte del Capítulo 2 aquellos resultados de estabilidad

relativos a soluciones estacionarias analizados en el Capítulo 1.

El último caso se refiere a las medidas invariantes para la versión autónoma de las ecuaciones estocásticas del Capítulo 3 utilizando el método del límite de Banach generalizado. En primer lugar, utilizamos aproximaciones de Galerkin, estimaciones a priori y el lema estándar de Gronwall para demostrar que la ecuación aleatoria correspondiente, cuyos operadores de solución generan un sistema dinámico aleatorio, está bien planteada. A continuación, se establece la compacidad asintótica del sistema dinámico aleatorio mediante el teorema de Ascoli-Arzelà. Además, derivamos la existencia de un atractor aleatorio global para el sistema dinámico aleatorio. De igual forma, demostramos que el sistema dinámico aleatorio está acotado y es continuo con respecto a los valores iniciales. Finalmente, construimos una familia de medidas de probabilidad invariantes de Borel, que está soportada por el atractor aleatorio global.

Es bien sabido que los sistemas dinámicos reticulares tienen amplias aplicaciones en física, química, biología e ingeniería, en problemas tales como la formación de patrones, el procesamiento de imágenes, la propagación de impulsos nerviosos, los circuitos eléctricos, etc. La teoría de atractores para sistemas reticulares deterministas o estocásticos ha sido ampliamente desarrollada. Por ello, en los dos últimos capítulos nos centraremos en el comportamiento asintótico de los atractores para sistemas dinámicos reticulares.

En el Capítulo 4 se analizan dos problemas relacionados con los sistemas reticulares de tipo FitzHugh-Nagumo. El primero se refiere al comportamiento asintótico de los sistemas reticulares FitzHugh-Nagumo con retardo aleatorio conducidos por ruido Wong-Zakai no lineal. Obtenemos un nuevo resultado que asegura que tal sistema se aproxima al sistema determinista correspondiente cuando el parámetro de correlación del ruido Wong-Zakai tiende a infinito en lugar de a cero. En primer lugar, demostramos la existencia de atractores aleatorios atemperados para los sistemas reticulares de retardo aleatorio con una función de deriva no lineal y un término de difusión no lineal también. La compacidad asintótica de las soluciones se demuestra gracias al teorema de Ascoli-Arzelà y a las estimaciones uniformes de las colas. A continuación demostramos que la semicontinuidad superior de los atractores a medida que el parámetro de correlación tiende a infinito. En cuanto al segundo problema, consideramos la correspondiente versión determinista del modelo anterior, y estudiamos la convergencia de los atractores cuando el retardo se aproxima a cero. En concreto, se demuestra la semicontinuidad superior de los atractores del sistema con retardo al sistema sin retardo.

Finalmente, la existencia y la conexión de atractores numéricos para sistemas reticulares de p-Laplace de tiempo discreto mediante el esquema de Euler implícito se demuestran en el Capítulo 5. Hasta ahora, sigue abierta la posibilidad de obtener un atractor numérico para un sistema de celosía no autónomo (o estocástico), por lo que al menos podemos investigar atractores numéricos para la versión determinista y no retardada de las ecuaciones reticulares de tipo p-Laplace. Se demuestra que los atractores numéricos tienen un límite optimizado, que conduce a la convergencia continua de los atractores numéricos cuando la gráfica de la no linealidad se cierra al eje vertical o cuando la fuerza externa desaparece. Se establece y aplica un nuevo tipo de desarrollos de Taylor sin derivadas Fréchet para mostrar el error de discretización de orden dos, que es crucial para probar que los atractores numéricos convergen de forma semicontinua superior al atractor global del sistema original de tiempo continuo a medida que el tamaño de paso del tiempo va a cero. También se demuestra que los atractores numéricos truncados para sistemas finito-dimensionales convergen de forma semicontinua superior al atractor global del sistema original de tiempo continuo a medida que el tamaño del paso del tiempo se acerca a cero y la semi-continuidad inferior se cumple en casos especiales.

Los resultados de nuestra investigación en esta tesis forman parte de los siguientes trabajos:

- S. Yang, Y. Li, Q. Zhang and T. Caraballo, Stability analysis of stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay, J. Dynam. Differential Equations, Published Online, (2023), Doi: 10.1007/s10884-022-10244-0.
- S. Yang, T. Caraballo and Y. Li, Dynamics and stability analysis for stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay on unbounded domains, Appl. Math. Optim., (sometido).
- S. Yang, T. Caraballo and Y. Li, Invariant measures for stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay, Commun. Nonlinear Sci. Numer. Simul., 118 (2023), pp. 107004, 21.
- S. Yang, Y. Li and T. Caraballo, Dynamical stability of random delayed FitzHugh-Nagumo lattice systems driven by nonlinear Wong-Zakai noise, J. Math. Phys., 63 (2022), pp. 111512, 32.
- Y. Li, S. Yang and Tomás Caraballo, Optimization and convergence of numerical attractors for discrete-time quasi-linear lattice system, SIAM J. Numer. Anal., (por aparecer), 2023.

Introduction

Dynamical systems are derived from evolution equations, which describe the evolution in time of solutions of equations in a phase space (state space). Attractors, as sets possessing attracting and invariance properties as well as being compact in an appropriate phase space, can well characterize the asymptotic behavior of dynamical systems. Since the decade of 1960s, they have received much attention, see [7, 20, 27, 38, 82, 101, 115]. It is well known that the systems we study are affected by a variety of random factors in real life, so it is necessary to consider some kind of noise in our models. The concept of random dynamical system was first introduced by Ulam and von Neumann [117] in 1945. Due to the fact that stochastic differential equations can generate random dynamical systems, a growing number of researchers have studied random dynamical systems since 1980s. Amongst the many notable results, it is remarkable the importance of the work [11], where Bensoussan and Temam discussed the stochastic Navier-Stokes equations driven by white noise and random forces, providing a more realistic model to solve the problem.

To study the asymptotic behavior of random dynamical systems, Crauel, Debussche, Flandoli ([42, 43]) and Schmalfuß ([50, 108]) introduced the concept of random attractors and established the related theoretical framework. In the past three decades, the study of random attractors has attracted great interest from many outstanding scholars, such as Arnold ([5]); Bates, Lu, Wang ([8, 9]); Caraballo, Langa ([24]); Han ([60, 61]); Kloeden, Langa ([74]); Li, Guo ([87]); Wang ([118, 122]); Zhou, Wang ([155]) and others. They mainly discussed the existence, uniqueness, continuity, regularity, Morse decomposition, Hausdorff dimension and fractal dimension estimation of random attractors.

It is also worth pointing out that delay effects play a significant role in the modeling of physical, biological, engineering phenomena and in other real world applications. To describe better a realistic model, we should consider some hereditary characteristics such as aftereffect, time lag, memory and time delay (see [133–135]). It seems natural to impose an external force which may take into account not only the current state of the system, but also some part of its history (bounded delay), sometimes even the whole history (unbounded or infinite delay). Inspired by this fact, Caraballo and Real started the analysis of Navier-Stokes models with some hereditary features in [33], where the existence of solutions was established in both two and three-dimensional spaces. Besides, the uniqueness of solutions was proved in the two-dimensional case. Later on, the asymptotic behavior of those solutions and the existence of a pullback attractor were carried out in [34, 35].

Based on the previous discussion, we focus on the asymptotic behavior of random dynamical systems associated to several kinds of stochastic equations with delay by stability of stationary solutions, weak pullback mean random attractors, random attractors and numerical attractors. Correspondingly, we split the introduction into three parts as follows.

Part I: Lagrangian-averaged Navier-Stokes equations with infinite delay

As we know, the Lagrangian-averaged Navier-Stokes (LANS) model arises from a one-dimensional model of nonlinear shallow-water wave dynamics. It was the first time to use the Lagrangian averaging technique to deal with the turbulence closure problem. The main reason is that such a model requires lower computational cost than the usual Navier-Stokes equations (see Holm [66] for more details).

Concerning this model with finite delays or memory, the analysis was carried out by Caraballo et al. in [29, 31]. In the first paper, the authors proved the existence of a unique solution to the stochastic 3D LANS equations. In the second one, the existence and exponential stability of stationary solutions were established. To our knowledge, unbounded or infinite delay effects have been considered in other equations, such as reaction-diffusion equations, globally modified Navier-Stokes equations and usual Navier-Stokes models (see [95, 96, 99, 100, 153]). However, they still have not been thoroughly investigated for LANS models. Motivated by the above references, we may choose several state spaces to deal with the infinite delay case in the stochastic 3D LANS system. The first one is the Banach space

$$C_\gamma(H) = \{\varphi \in C((-\infty,0];H) : \lim_{r\to-\infty} e^{\gamma r}\varphi(r) \text{ exists in } H\}, \text{ where } \gamma > 0, \tag{0.0.1}$$

where H is the 3D Lebesgue-type Hilbert space. The second one is

$$C_{-\infty}(H) = \{\varphi \in C((-\infty,0];H) : \lim_{r\to-\infty} \varphi(r) \text{ exists in } H\}. \tag{0.0.2}$$

Moreover, we also use $C_\gamma(V)$ and $C_{-\infty}(V)$, where V is another Sobolev-type subspace instead of H in (0.0.1) and (0.0.2).

Based on the previous discussion, we investigate in this part the LANS equations with infinite delay in three cases, which correspond to Chapters 1-3.

In Chapter 1 we discuss the random dynamics of the following non-autonomous stochastic three-dimensional LANS equation with infinite delay and nonlinear hereditary noise:

$$\begin{cases} \partial_t(u-\alpha\Delta u) + \nu(Au - \alpha\Delta(Au)) + (u\cdot\nabla)(u-\alpha\Delta u) \\ -\alpha\nabla u^*\cdot\Delta u + \nabla p = f(t) + g_1(t,u_t) + g_2(t,u_t)\dot{W}, \ \text{ in } (\tau,+\infty)\times\mathcal{O}, \\ \nabla\cdot u = 0, \ \text{ in } (\tau,+\infty)\times\mathcal{O}, \\ u=0, \ Au = 0 \text{ on } (\tau,+\infty)\times\partial\mathcal{O}, \\ u(\tau+s,x) = \phi(s,x), s\in(-\infty,0], x\in\mathcal{O}, \end{cases} \tag{0.0.3}$$

where $\mathcal{O} \subseteq \mathbb{R}^3$ is a bounded open set with sufficiently regular boundary $\partial\mathcal{O}, \tau \in \mathbb{R}$, A is the Stokes operator, the pair (ν,α) of positive coefficients denotes the kinematic viscosity of the fluid and the square of the spatial scale at which fluid motion is filtered respectively, the symbol $*$ denotes the transpose of a matrix, $u = (u_1,u_2,u_3)$ is the averaged (or large-scale) velocity of the fluid, u_t denotes the segment of solutions up to time t, i.e. $u_t(s) = u(t+s)$ for all $s \leqslant 0$, p is the pressure of the fluid, f

is a non-delayed external force field, the terms g_1, g_2 contain some hereditary characteristics, such as memory, unbounded variable or infinite distributed delay, etc, ϕ is an initial velocity field defined in $(-\infty, 0] \times \mathcal{O}$, and $\dot{W}$ denotes the generalized derivative (white noise) of a cylindrical Wiener process, which will be introduced later.

Our first goal is to prove the existence and uniqueness of solutions to the stochastic three dimensional LANS in the Banach space $C_\gamma(V)$. As done by Liu and Caraballo [95] for the usual two-dimensional system, we need to assume that the non-delayed external force f is locally integrable (see **Hypothesis F**) and the delay forcing terms $g_i(t, u_t)$ $(i = 1, 2)$ are globally Lipschitz continuous (see **Hypothesis G**). An example is given in the last part of Section 1.1. The calculation shows that the example (corresponding to infinite distributed delay) satisfies all conditions of **Hypothesis G** in the space $C_\gamma(V)$ for $\gamma > 0$. Besides, we also need an extra assumption on the nonlinear diffusion term g_2 of noise (see **Hypothesis I**).

Under the above assumptions, we use the Galerkin method to construct an approximating sequence. We then establish a priori estimates for the approximating sequence ensuring the solutions exist for the whole time interval $[\tau, \tau + T]$ for all $T > 0$. We finally obtain the well-posedness of the LANS equation based on uniform estimates of solutions.

Another interesting and challenging topic is to consider the asymptotic behaviour of solutions for Eq. (0.0.3) towards to the stationary solutions. This issue will provide some useful information on future evolution of the system. Thanks to the Lax-Milgram and the Schauder theorems, we first prove the existence and uniqueness of stationary solutions to the corresponding deterministic equation. We then show the local stability of the stationary solution for the general delay term by using a direct method, where the general delay contains the unbounded variable delay and the infinite distributed delay in $C_{-\infty}(V)$. Next, we prove the global stability of the stationary solution. However, to obtain stability results in $C_\gamma(V)$ with $\gamma > 0$, the exponential stability in the case of unbounded variable delay fails to be proved in general (see [95] and [100] for more details). Fortunately, in the case of infinite distributed delay, we are able to prove not only stability of stationary solutions in $C_\gamma(V)$, but also exponential asymptotic stability. Since we may not analyze the exponential stability in the unbounded variable delay case in $C_\gamma(V)$, we will explore, at least, the asymptotic stability in $C_{-\infty}(V)$, by using the Lyapunov functionals construction proposed by Kolmanovskii and Shaikhet [78]. Furthermore, we eventually discuss the polynomial asymptotic stability in the particular case of proportional delay (also known as pantograph delay).

Note that all results in Chapter 1 have been published in [142] (*S. Yang, Y. Li, Q. Zhang and T. Caraballo, Stability analysis of stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay, J. Dynam. Differential Equations, Published Online, (2023), Doi: 10.1007/s10884-022-10244-0*).

Then, in Chapter 2, we are mainly interested in mean dynamics and stability analysis of the following stochastic 3D LANS equations with infinite delay on unbounded domains:

$$\begin{cases} \partial_t(u-\alpha\Delta u)+\nu(Au-\alpha\Delta(Au))+(u\cdot\nabla)(u-\alpha\Delta u) \\ -\alpha\nabla u^*\cdot\Delta u+\nabla p=f(t)+g(t,u_t)+\sigma(t,u)\dot{W},\ \text{in}\ (\tau,+\infty)\times\mathcal{O}, \\ \text{div}\, u=0,\ \ \text{in}\ (\tau,+\infty)\times\mathcal{O}, \\ u=0,\ Au=0 \text{ on } (\tau,+\infty)\times\partial\mathcal{O}, \\ u(\tau+s,x)=\phi(s,x),\, s\in(-\infty,0],\, x\in\mathcal{O}, \end{cases} \tag{0.0.4}$$

where $\tau\in\mathbb{R}$, $\mathcal{O}\subset\mathbb{R}^3$ is an unbounded open set with boundary $\partial\mathcal{O}$, which is a Poincaré domain, that is, there exists a positive number λ such that

$$\lambda\int_{\mathcal{O}}|\psi|^2dx\leqslant\int_{\mathcal{O}}|\nabla\psi|^2dx,\ \forall\,\psi\in H_0^1(\mathcal{O}), \tag{0.0.5}$$

g is a nonlinear term capturing the time delay, σ is a locally Lipschitz nonlinear diffusion coefficient, the other symbols are the same as the symbols of the model in Chapter 1.

The first aim of this chapter is to study mean dynamics of the stochastic 3D LANS equation (0.0.4) driven by infinite delay on unbounded domains. In the Banach space $C_\gamma(V)$, we first establish the well-posedness of problem (0.0.4) with infinite delay when the non-delayed external force $f(t)$ is locally integrable, the delay term $g(t,u_t)$ is globally Lipschitz continuous and the nonlinear diffusion term $\sigma(t,u)$ is locally Lipschitz continuous. To that end, we introduce a globally Lipschitz continuous cut-off function ξ_n (for every $n\in\mathbb{N}$) defined by (2.2.3) to approximate $\sigma(t,u)$. The solution operators enable us to define a mean random dynamical system instead of the usual pathwise random dynamical system, mainly because, in this case, there is no approach available to transfer the stochastic system (0.0.4) to a corresponding pathwise deterministic one. We then show the existence of a unique weak pullback random attractor for the mean random dynamical system.

Another purpose of this chapter is to prove the existence of a unique stationary solution to the corresponding deterministic equation (see Eq. (2.3.9)) and analyze the asymptotic behaviour of solutions for Eq. (0.0.4) towards to the stationary solution. Since our results no longer depend on Sobolev compact embeddings for unbounded domains, we modify some arguments in [31, Theorem 10], i.e. apply the classical Galerkin argument as well as the Lax-Milgram and the Brouwer fixed theorems to derive the existence and uniqueness of stationary solutions to system (2.3.9) on unbounded domains. Moreover, we establish different stability sufficient conditions via several methods used in subsections 1.3.2-1.3.5.

The work [137] (*S. Yang, T. Caraballo and Y. Li, Dynamics and stability analysis for stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay on unbounded domains, Appl. Math. Optim., (submitted)*) contains the results proved in Chapter 2.

In Chapter 3, we investigate invariant measures of the following stochastic 3D LANS equations driven by infinite delay and additive noise:

$$\begin{cases} \partial_t(u-\alpha\Delta u)+\nu(Au-\alpha\Delta(Au))+(u\cdot\nabla)(u-\alpha\Delta u) \\ -\alpha\nabla u^*\cdot\Delta u+\nabla p=f(x)+g(u_t)+\kappa(x)\dot{W},\ \text{in } (s,+\infty)\times\mathcal{O}, \\ \operatorname{div} u=0,\ \text{in } (s,+\infty)\times\mathcal{O}, \\ u=0,\ Au=0 \text{ on } (s,+\infty)\times\partial\mathcal{O}, \\ u(s+r,x)=\phi(r,x), r\in(-\infty,0], x\in\mathcal{O}, \end{cases} \tag{0.0.6}$$

where $\mathcal{O}\subseteq\mathbb{R}^3$ is a bounded open set with sufficiently regular boundary $\partial\mathcal{O}$, $s\in\mathbb{R}$, f and κ are given functions defined on $\mathcal{O}$, the delay function g is the autonomous form in Eq. (0.0.4), and W is a two-sided real-valued Wiener process on a complete probability space which will be specified in Chapter 3. The other symbols are the same as those in Eq. (0.0.3).

In the present chapter, our main goal is to prove the existence of invariant Borel probability measures for the stochastic 3D LANS system driven by infinite delay and additive noise. For the research of infinite dimensional evolution equations, especially for invariant probability measures, there are a number of different mathematical methods applied to ergodicity. One approach is to consider some kinds of simple linear or semi-linear first order PDEs, see [46, 83]. Although the authors discussed linear systems where solutions are able to depict chaotic or 'turbulent' behavior, the method is limited to a very specific class of equations. Another different method, namely the classical Krylov-Bogolyubov method, focuses on the study of stochastic PDEs, and one can define ergodicity via the Markov semigroup induced by the stochastic semiflow. Thanks to Krylov-Bogolyubov's method, the existence of invariant measures for stochastic PDEs has been extensively investigated, we refer the reader to [14, 15, 19, 45, 139] and [17, 18, 47, 102, 125] for bounded and unbounded domains, respectively. However, by using this approach, we find that invariant measures of the Markov semigroup are deterministic probability measures and thus they are not supported by global random attractors.

Based on the previous discussion, in this chapter we introduce a different technique to construct invariant measures, that is, the so called 'generalized Banach limit', denoted by $\mathrm{LIM}_{t\to+\infty}$, pioneered in [52] and developed in [23, 40, 51, 98, 127, 146–148]. The use of such limit allows us to relate time averages with ensemble averages in the state space.

To our knowledge, this work seems to be the first one discussing invariant Borel probability measures for the stochastic 3D LANS system with infinite delay and additive noise, even there is not any published work on this issue for stochastic delay equations. We mainly apply the abstract theory for autonomous random dynamical systems in [148, Theorem 2.1] to our model (0.0.6). To this end, we shall transform (0.0.6) into a deterministic equation with a random parameter (see Eq. (3.2.8)) and prove that the solution operators associated to Eq. (0.0.6) generate a random dynamical system φ in the state space $C_\gamma(V)$. In addition, we need to prove the following two assertions:

(1) the random dynamical system φ possesses a global random attractor $\mathcal{A}(\omega)$ in $C_\gamma(V)$;

(2) φ is bounded and continuous with respect to the initial values.

Using the classical Galerkin method, a priori estimates and the standard Gronwall lemma, we first derive the existence of a unique weak solution to system (3.2.8). We then obtain some uniform estimates of the solutions to system (3.2.8) ensuring the existence of a global random absorbing set in $C_\gamma(V)$ (see Lemma 3.3.5). Next, we establish the asymptotic compactness for φ in $C_\gamma(V)$ via the Ascoli-Arzelà theorem (see Lemma 3.3.6). Moreover, we show the existence of global random attractors for φ (see Theorem 3.3.8). Furthermore, for each $t \in \mathbb{R}$, almost all $\omega \in \Omega$ and $\phi \in C_\gamma(V)$, we prove the $C_\gamma(V)$-valued function $s \mapsto \varphi(t, s, \theta_{-t}\omega)\phi$ is bounded on $(-\infty, t]$, as shown in Lemma 3.4.1. Finally, the continuity of $\varphi(t, s, \theta_{-t}\omega)\phi$ with respect to the initial values (s, ϕ) on $(-\infty, t] \times C_\gamma(V)$ is proved in Lemma 3.4.4.

The results in Chapter 3 have been proved in the paper [136] (*S. Yang, T. Caraballo and Y. Li, Invariant measures for stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay, Commun. Nonlinear Sci. Numer. Simul., 118 (2023), pp. 107004, 21*).

Part II: FitzHugh-Nagumo lattice systems with delay

Some lattice dynamical systems can be derived from spatial discretization of continuum systems. They have wide applications in our daily life, including physics, chemistry, biology, engineering and other fields of science (see, e.g., [10, 49, 71, 72]). As far as we are aware, one of the most interesting lattice systems is FitzHugh-Nagumo model, which simulates the process of signal transmission across axons. It is known that, lattice dynamical systems with delay have been receiving much attention for many years ([32, 144, 149]).

The motivation of Chapter 4 is to study the long-time dynamics of pullback random attractors for the following delay FitzHugh-Nagumo lattice system driven by a nonlinear Wong-Zakai noise:

$$\begin{cases} \dfrac{du_i}{dt} - (u_{i-1} - 2u_i + u_{i+1}) + \lambda u_i + \alpha v_i = F_i(u_i(t)) + f_i(u_i(t - \varrho^{(\rho)}(t))) + g_i(t) + G_i(t, u_i)\mathcal{G}_\delta(t, \omega), \\ \dfrac{dv_i}{dt} + \varsigma v_i - \beta u_i = h_i(t) + f_i(v_i(t - \varrho^{(\rho)}(t))), \\ u_i(\tau + s) = \phi_i(s), \quad v_i(\tau + s) = \upsilon_i(s), \quad i \in \mathbb{Z}, \ t > \tau, \ \tau \in \mathbb{R}, \ s \in [-\rho, 0], \end{cases} \tag{0.0.7}$$

where λ, α, ς, β, γ and ρ are positive constants, $\varrho^{(\rho)}$ is a variable delay function with maximum delay ρ, F_i is a nonlinear drift function with polynomial growth of arbitrary order, f_i is an external force affected by memory during the interval of delay time $[-\rho, 0]$, the deterministic time-dependent forcings $g_i, h_i \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, ϕ_i, ψ_i are the initial data on the internal $[-\rho, 0]$, G_i is a nonlinear diffusion, $\mathcal{G}_\delta$ is the Wong-Zakai process with correlation time $\delta > 0$, which is δ-difference of a two-side scalar Wiener process W on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, given by

$$\mathcal{G}_\delta(t, \omega) := \frac{1}{\delta}(W(t + \delta, \omega) - W(t, \omega)), \ \forall \, \delta > 0, \ t \in \mathbb{R}, \ \omega \in \Omega. \tag{0.0.8}$$

This type of Wong-Zakai noise was first introduced by [131, 132] in which the authors used deterministic differential equations to approximate stochastic ones for one-dimensional Brownian motions. Later, a growing number of authors extended the idea of Wong-Zakai approximations to higher-dimensional Brownian motions, martingales and semimartingales (see [80, 81, 104–106, 110, 113, 114, 116]). Recently, the convergence of solutions and attractors for random equations with such a noise when $\delta \to 0$ has been extensively studied (see [1, 9, 16, 56, 59, 152]). Note that Wong-Zakai approximation systems in these references were only considered in the case of linear noise, that is, the sequence of diffusion functions $G = (G_i)_{i\in\mathbb{Z}}$ in (0.0.7) is $u = (u_i)_{i\in\mathbb{Z}}$ or independent of u. Moreover, so were all the results on the semicontinuity of random attractors, see [8, 86, 120, 143] for autonomous stochastic equations, and [9, 21, 84, 85, 89, 93, 123, 138, 145, 150] for non-autonomous stochastic equations.

However, there exist very few works on studying the attractors of random delay equations driven by the nonlinear Wong-Zakai noise, even in autonomous version. To our knowledge, the only two papers for such autonomous equations were published by Li et al. in [91, 92]. In this part, we investigate the dynamics of non-autonomous random delay FitzHugh-Nagumo lattice system with a nonlinear Wong-Zakai noise (0.0.7).

Chapter 4 is divided into two parts. In the first part, we prove the existence of a pullback random attractor $\mathcal{A}^\delta = \{\mathcal{A}^\delta(t,\omega)\}$ for the random delay system (0.0.7) with a nonlinear noise and its upper semicontinuity when $\delta \to +\infty$. This is different from the general situation $\delta \to 0$. To prove the existence of a pullback random attractor $\mathcal{A}^\delta$ in $\mathcal{X}_\sigma^\rho = C([-\rho,0],\mathcal{X}_\sigma)$ with $\mathcal{X}_\sigma = \ell_\sigma^2 \times \ell_\sigma^2$, where ℓ_σ^2 is weighted space for each $\delta > 0$, we must verify the random dynamical system (or cocycle) Ψ^δ, induced by Eq. (0.0.7) driven by the Wong-Zakai nonlinear noise, is pullback asymptotically compact in $\mathcal{X}_\sigma^\rho$. The ideas of uniform estimates and the Ascoli-Arzelà theorem are crucial tools to prove it. As for the upper semi-convergence of $\mathcal{A}^\delta$ as $\delta \to +\infty$, we need the help of the logarithm law of the Wiener process, which establishes $W(t,\omega)/\log(\log|t|) \to 0$ as $t \to \pm\infty$, as well as the result (4.1.6) in Lemma 4.1.1 which ensures

$$\lim_{\delta\to+\infty}\sup_{t\in[a,b]}\mathcal{G}_\delta(t,\omega) = 0,\ \mathbb{P}\text{-a.s. } \omega\in\Omega,\ a \leqslant b.$$

Based on the previous arguments, we consider the limiting system of random delay lattice model (0.0.7) when $\delta \to +\infty$ as the deterministic delay lattice system:

$$\begin{cases} \dfrac{d\hat{u}_i}{dt} - (\hat{u}_{i-1} - 2\hat{u}_i + \hat{u}_{i+1}) + \lambda\hat{u}_i + \alpha\hat{v}_i = F_i(\hat{u}_i(t)) + f_i(\hat{u}_i(t-\varrho^{(\rho)}(t))) + g_i(t), \\ \dfrac{d\hat{v}_i}{dt} + \varsigma\hat{v}_i - \beta\hat{u}_i = h_i(t) + f_i(\hat{v}_i(t-\varrho^{(\rho)}(t))), \\ \hat{u}_i(\tau+s) = \hat{\phi}_i,\ \ \hat{v}_i(\tau+s) = \hat{v}_i,\ \ i\in\mathbb{Z},\ t>\tau,\ \tau\in\mathbb{R},\ s\in[-\rho,0]. \end{cases} \tag{0.0.9}$$

Under some appropriate conditions (see Hypotheses **E, F1, F2, G1-G3** later), we find out that system (0.0.9) generates a pullback attractor denoted by $\mathcal{A}^\infty(t)$ whose existence has been established, see [2, 26, 119]. Then we need to check that the random pullback attractor $\mathcal{A}^\delta(t,\omega)$ semi-converges to $\mathcal{A}^\infty(t)$ as $\delta \to +\infty$ (see Theorem 4.3.2), that is,

$$\lim_{\delta\to+\infty} \mathrm{d}_{\mathcal{X}_\sigma^\rho}(\mathcal{A}^\delta(t,\omega),\mathcal{A}^\infty(t)) = 0,\ \ \forall\, t\in\mathbb{R},\ \omega\in\Omega, \tag{0.0.10}$$

where the distance $\mathrm{d}_{\mathcal{X}_\sigma^\rho}$ is defined for all subsets A and B of $\mathcal{X}_\sigma^\rho$ by

$$\mathrm{d}_{\mathcal{X}_\sigma^\rho}(A,B) := \sup_{a\in A}\inf_{b\in B}\sup_{\nu\in[-\rho,0]}\|a(\nu)-b(\nu)\|_{\mathcal{X}_\sigma}.$$

For this end, we prove the solutions to random system (0.0.7) converge to that of the corresponding deterministic system (0.0.9) when $\delta \to +\infty$. Note that the pullback attractor $\mathcal{A}^\infty(t)$ in (0.0.10) is written as $\mathcal{A}_\rho(t)$ to indicate its dependence on the delay ρ for later purpose.

In the second part of this chapter, our goal is to further establish the upper semicontinuity of the pullback attractor $\mathcal{A}_\rho(t)$ as $\rho \to 0$ (see Theorem 4.4.4), that is,

$$\lim_{\rho\to 0} \mathrm{d}^*_{\mathcal{X}^\rho_\sigma}(\mathcal{A}_\rho(t), \mathcal{A}_0(t)) = 0, \quad \forall\, t \in \mathbb{R}, \tag{0.0.11}$$

where $\mathcal{A}_0(t)$ is a pullback attractor for the non-delay version of Eq. (0.0.9), and the distance $\mathrm{d}^*_{\mathcal{X}^\rho_\sigma}$ is defined for all subset A of $\mathcal{X}^\rho_\sigma$ and B of $\mathcal{X}_\sigma$ by

$$\mathrm{d}^*_{\mathcal{X}^\rho_\sigma}(A, B) := \sup_{a\in A} \inf_{b\in B} \sup_{\nu\in[-\rho,0]} \|a(\nu) - b\|_{\mathcal{X}_\sigma}.$$

Due to the validity of all the estimates we obtained in Section 4.2 of the first part, especially in two cases of the non-delayed case ($\rho = 0$) and the deterministic case ($\delta \to +\infty$) for (0.0.7). Therefore, we immediately deduce the existence of the pullback attractor $\mathcal{A}_0(t)$. As for (0.0.11), the main task is to prove the convergence of solutions to system (0.0.9) as $\rho \to 0$.

Notice that all the results obtained in Chapter 4 are new and interesting, which have been published in [141] (*S. Yang, Y. Li and T. Caraballo, Dynamical stability of random delayed FitzHugh-Nagumo lattice systems driven by nonlinear Wong-Zakai noise, J. Math. Phys., 63 (2022), pp. 111512, 32*).

Part III: Numerical attractors for lattice systems

Recently, Han, Kloeden and Sonner [64] develop a subject about numerical attractors for the reaction diffusion lattice system. More precisely, they have proved the existence of a numerical attractor for the time-discrete lattice system via the implicit Euler scheme (IES), and also established the upper semi-convergence of the numerical attractor towards the global attractor when the step size tends to zero.

In Chapter 5, we will apply the idea of Han et al. [64] to the following p-Laplace lattice dynamical system (LDS):

$$\frac{du_i(t)}{dt} = \nu(A_p u(t))_i + f(u_i(t)) + g_i, \ i \in \mathbb{Z}, \tag{0.0.12}$$

where $\nu > 0$, $p \geqslant 2$, $u = (u_i)_{i\in\mathbb{Z}}$, and the discrete p-Laplace operator is defined by

$$(A_p u)_i = |u_{i+1} - u_i|^{p-2}(u_{i+1} - u_i) - |u_i - u_{i-1}|^{p-2}(u_i - u_{i-1}), i \in \mathbb{Z}.$$

As one knows, the p-Laplace LDS (0.0.12) is the space-discretization of the corresponding p-Laplace partial differential equation (defined on the real line), while the dynamics of the (deterministic or stochastic) p-Laplace PDE was studied in [37, 48, 54, 55, 70, 77, 79, 86, 88, 92, 115, 130].

As preliminaries, we show in Section 5.2 that the LDS (0.0.12) has a positively invariant ball $\mathcal{B}_{r^*}(0)$ and a global attractor $\mathcal{A}$ in ℓ^2, where the dissipative condition of f is different from those in [39, 57, 58, 129] and assumed by

$$\alpha := \inf_{s\neq 0} \frac{-f(s)}{s} > 0. \tag{0.0.13}$$

In this chapter, we mainly consider the numerical scheme in the discrete-time sense. Using the step size $\epsilon := t_{n+1} - t_n$ of the time to discretize the LDS (0.0.12), we obtain the p-Laplace IES

$$u^{\epsilon}_{n,i} = u^{\epsilon}_{n-1,i} + \epsilon\nu(A_p u^{\epsilon}_n)_i + \epsilon f(u^{\epsilon}_{n,i}) + \epsilon g_i, \ \forall n \in \mathbb{N}, \ i \in \mathbb{Z}. \tag{0.0.14}$$

As pointing out by Han, Kloeden, Sonner [64], the IES (0.0.14) with $p = 2$ is not globally solvable for a common step size (see Lemma 5.4.3 for the reason if $p > 2$). Instead the global solvability, we will prove in Theorem 5.2.1 that, for sufficiently small step sizes, the IES (0.0.14) is uniquely solvable when the initial datum belongs to the positively invariant ball $\mathcal{B}_{r^*}(0)$. In the recursive proof of Theorem 5.2.1, we use a new method of enlarging the radius to overcome the difficulty that the ball $\mathcal{B}_{r^*}(0)$ is no longer a positively invariant set under the operator defined by the right-hand side of (0.0.14). The proof is more careful and technical than those in [64] even for the case of $p = 2$.

For the later purpose, we have to consider the numerical approximation of solutions as $\epsilon \to 0$. In this respect, Kloeden and Lorenz [75] (see also Jentzen et al. [12, 67–69]) have introduced the method of the Taylor expansion by using the Fréchet derivatives of the *linear* Laplacian A_p ($p = 2$) and f. However, the nonlinear operator A_p ($p > 2$) has not a Fréchet derivative.

To overcome the above difficulty, we establish a new type of Taylor expansions without Fréchet derivatives and give the continuous-time error of solutions for the LDS (0.0.12) (see Lemma 5.3.1). Using this continuous-time error, we can show the discretization error of order two for the solutions between LDS (0.0.12) and IES (0.0.14), see Theorem 5.3.2. Our method is suitable for a wider class of discrete-time equations even if the operators are not Fréchet differential.

As for Section 5.5, our main purpose is to study the numerical scheme of attractors, which is a relative new subject (introduced by Han, Kloeden, Sonner [64], see also [140]) in both Numerical Analysis and Dynamical Systems [112]. More precisely, we study the discrete approximation of the global attractor $\mathcal{A}$ for LDS (0.0.12) in terms of numerical attractors for IES (0.0.14) and its finitely dimensional truncated system.

We prove in Theorem 5.4.2 that the discrete semigroup, generated from the IES (0.0.14), possesses a unique connected numerical attractor $\mathcal{A}_\epsilon$ for sufficiently small step sizes. In the proof, we need to recursively estimate the tail of solutions for all $n \in \mathbb{N}$, where the usual cut-off function technique (see [6, 7, 22, 25, 62, 65, 128, 151, 154]) is still valid in the discrete-time case.

Furthermore, we prove in Theorem 5.4.4 that $\mathcal{A}_\epsilon$ has an optimized bound given by $\|g\|/\alpha$. This bound is crucial to prove that the numerical attractor converges continuously (upper and lower) to zero as the graph of f closes enough to the y-axis and as $g \to 0$, respectively. This subject of optimization and convergence of numerical attractors is new in the literature.

In Theorem 5.5.1, we establish the upper semi-continuity from the numerical attractor $\mathcal{A}_\epsilon$ to the global attractor $\mathcal{A}$ as $\epsilon \to 0$, where the discretization error of solutions in Theorem 5.3.2 plays a crucial role in the proof.

In Section 5.6, we study the finite dimensional approximation of the numerical attractor. For this end, we truncate the IES (0.0.14) on the $(2m + 1)$-dimensional Euclid space to obtain the truncated numerical scheme with the periodic boundary condition, see the model (5.6.1). We then prove in Theorem 5.6.4 that the truncated IES (5.6.1) has an attractor denoted by $\mathcal{A}_{\epsilon,m}$, and that $\mathcal{A}_{\epsilon,m}$ converges upper semi-continuously to the numerical attractor $\mathcal{A}_\epsilon$ as $m \to \infty$. If the viscosity is zero, i.e. $\nu = 0$, the lower semi-continuity from $\mathcal{A}_{\epsilon,m}$ to $\mathcal{A}_\epsilon$ still holds as proved in Theorem 5.6.6.

All in all, we have established a convergence path from $\mathcal{A}_{\epsilon,m}$ to the global attractor $\mathcal{A}$ through $\mathcal{A}_{\epsilon}$. In fact, there is another convergence path from $\mathcal{A}_{\epsilon,m}$ to $\mathcal{A}$ through $\mathcal{A}_m$, where $\mathcal{A}_m$ is the global attractor for the truncated system of the LDS (0.0.12) on the $(2m+1)$-dimensional Euclid space. All convergence paths and optimized bounds of attractors are displayed in Figure 1.

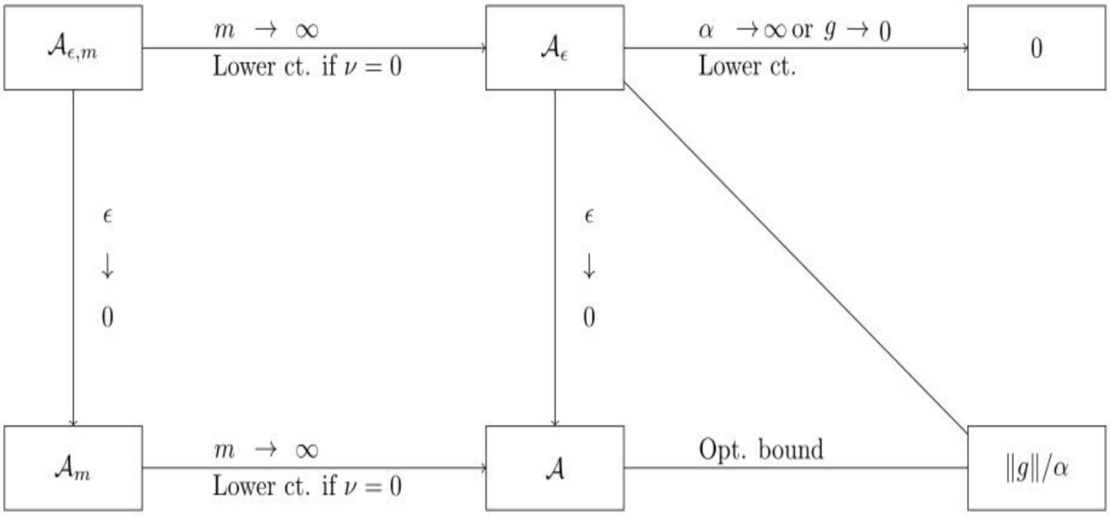

Figure 1: Convergence paths and bounds of attractors.

It is worth emphasizing that the topic we discuss in Chapter 5 on optimization and convergence of numerical attractors are challenging, and they have been published in [90] (*Y. Li, S. Yang and Tomás Caraballo, Optimization and convergence of numerical attractors for discrete-time quasi-linear lattice system, SIAM J. Numer. Anal., (to appear), 2023*).

Future work

We conclude this thesis with some comments about our future work. In general, systems are often affected by external and internal noises. In addition, if delays are also disturbed by noises, then they will be random. Recently, random delay systems have received increasing attention, especially in the area of control and optimization. One of the most important examples is the networked control system, and one of the main problems associated with it is the communication delay, which is often random, see [103, 107]. Motivated by this fact, on the one hand, we can further establish theoretical results of numerical attractors for the following p-Laplace lattice systems with random delay via IES:

$$\frac{du_i(t)}{dt} = \nu(A_p u(t))_i + f(u_i(t - \tau(t,\omega))) + g_i, \; i \in \mathbb{Z},$$

where $\tau(t,\omega)$ is a random function, which has continuous sample paths with bounded valued $\tau(t,\omega) \in [\tau_0, \tau_1]$ where $0 \leqslant \tau_0 \leqslant \tau_1$ are finite deterministic numbers and $\omega \in \Omega$ for a given probability space

$(\Omega, \mathcal{F}, \mathbb{P})$. The other symbols are the same as (0.0.12) we have studied in Chapter 5. Compared with (0.0.12), the above model can be more realistic.

On the other hand, we plan to investigate the existence, uniqueness, regularity, Morse decomposition, Hausdorff dimension and fractal dimension estimation of pullback random attractors for several kinds of non-autonomous systems with random delay. Their forward asymptotically autonomous is also considered, that is, the time-component of pullback random attractors semi-converges to global random attractors as the time-parameter tends to infinity.

Introducción

Los sistemas dinámicos se derivan de las ecuaciones de evolución, que describen la evolución de las soluciones de las ecuaciones en un espacio de fases (espacio de estados) con el tiempo. Los atractores, como conjuntos con propiedades de atracción, invarianza y compacidad en un espacio de fase, puede caracterizar bien el comportamiento asintótico de los sistemas dinámicos. Ya en la década de 1960, han recibido mucha atención, véase [7, 20, 27, 38, 82, 101, 115]. Es bien sabido que los sistemas que estudiamos se ven afectados por diversos factores aleatorios en la vida real, por lo que es necesario considerar algún tipo de ruido en nuestros modelos. El concepto de sistemas dinámicos aleatorios fue introducido por primera vez por Ulam y von Neumann [117] en 1945. Debido al hecho de que las ecuaciones diferenciales estocásticas pueden generar sistemas dinámicos aleatorios, un número creciente de investigadores han estudiado los sistemas dinámicos aleatorios desde la década de 1980. Entre los muchos resultados notables, cabe destacar la importancia del trabajo [11], donde Bensoussan y Temam discutieron las ecuaciones estocásticas de Navier-Stokes impulsadas por ruido blanco y fuerzas aleatorias, proporcionando un modelo más realista para resolver el problema.

Para estudiar el comportamiento asintótico de los sistemas dinámicos aleatorios, Crauel, Debussche, Flandoli ([42, 43]) y Schmalfuß ([50, 108]) introdujeron el concepto de atractor aleatorio y establecieron el marco teórico adecuado para su estudio. En las últimas tres décadas, el estudio de los atractores aleatorios ha suscitado un gran interés por parte de muchos destacados académicos como Arnold ([5]); Bates, Lu, Wang ([8, 9]); Caraballo, Langa ([24]); Han ([60, 61]); Kloeden, Langa ([74]); Li, Guo ([87]); Wang ([118, 122]); Zhou, Wang ([155]) y otros. Principalmente discutieron la existencia, la unicidad, la continuidad, la regularidad, la descomposición de Morse, la dimensión de Hausdorff y la estimación de la dimensión fractal de los atractores aleatorios.

También cabe señalar que los efectos de retardo desempeñan un papel importante en el modelado de fenómenos físicos, biológicos, de ingeniería y en otras aplicaciones del mundo real. Para describir mejor un modelo realista, deberíamos considerar algunas características hereditarias como el efecto de retardo, la memoria y el retardo temporal (véanse [133–135]). Parece natural imponer una fuerza externa que tenga en cuenta no sólo el estado actual del sistema, sino también una parte de su historia (retardo acotado), a veces incluso toda la historia (retardo ilimitado o infinito). Inspirados por este hecho, Caraballo y Real iniciaron el análisis de modelos de Navier-Stokes con algunas características hereditarias en [33], donde se estableció la existencia de soluciones tanto en espacios bidimensionales como tridimensionales. Además, se demostró la unicidad de soluciones en el caso bidimensional. Posteriormente, el comportamiento asintótico de dichas soluciones y la existencia de atractor de tipo pullback se llevaron a cabo en [34, 35].

Basándonos en la discusión anterior, nos centramos en el comportamiento asintótico de los sistemas dinámicos aleatorios asociados a varios tipos de ecuaciones estocásticas con retardo mediante la estabilidad para soluciones estacionarias, atractores aleatorios débiles en media de tipo pullback, atractores aleatorios y atractores numéricos. En consecuencia, dividimos la introducción en las tres partes siguientes.

Parte I: Ecuaciones Lagrangianas de Navier-Stokes con retardo infinito

Como sabemos, el modelo Lagrangiano de Navier-Stokes (LANS) surge de un modelo unidimensional de dinámica no lineal de olas en aguas poco profundas. Fue el primero en utilizar la técnica de promediado lagrangiano para tratar el problema del cierre de la turbulencia. La razón principal es que dicho modelo requiere un menor coste computacional que las ecuaciones de Navier-Stokes habituales (véase Holm [66] para más detalles).

Respecto a este modelo con retardos o memoria finitos, el análisis fue realizado por Caraballo et al. en [29, 31]. En el primer trabajo, los autores demostraron la existencia de una solución única para las ecuaciones estocásticas 3D LANS. En el segundo, se estableció la existencia y estabilidad exponencial de soluciones estacionarias. Hasta donde sabemos, se han considerado efectos de retardo ilimitado o infinito en otras ecuaciones, como las ecuaciones de reacción-difusión, las ecuaciones de Navier-Stokes modificadas globalmente y los modelos habituales de Navier-Stokes (véanse [95, 96, 99, 100, 153]). Sin embargo, aún no se han investigado a fondo para los modelos LANS. Motivados por las referencias anteriores, podemos elegir varios espacios de fases para tratar el caso de retardo infinito en el sistema estocástico 3D LANS. El primero es el espacio de Banach

$$C_\gamma(H) = \{\varphi \in C((-\infty, 0]; H) : \lim_{r\to-\infty} e^{\gamma r}\varphi(r) \text{ exists in } H\}, \text{ donde } \gamma > 0, \tag{0.0.15}$$

y donde H es un espacio de Hilbert. El segundo es

$$C_{-\infty}(H) = \{\varphi \in C((-\infty, 0]; H) : \lim_{r\to-\infty} \varphi(r) \text{ exists in } H\}. \tag{0.0.16}$$

Además, también utilizamos $C_\gamma(V)$ y $C_{-\infty}(V)$, donde V es otro subespacio de tipo Sobolev en lugar de H en (0.0.15) y (0.0.16).

Basándonos en la discusión anterior, investigamos en esta parte las ecuaciones LANS con retardo infinito en tres casos, que corresponden a los capítulos 1-3.

En el Capítulo 1 discutimos la dinámica aleatoria de la siguiente ecuación estocástica tridimensional no autónoma LANS con retardo infinito y ruido hereditario no lineal:

$$\begin{cases} \partial_t(u - \alpha\Delta u) + \nu(Au - \alpha\Delta(Au)) + (u\cdot\nabla)(u - \alpha\Delta u) \\ -\alpha\nabla u^* \cdot \Delta u + \nabla p = f(t) + g_1(t, u_t) + g_2(t, u_t)\dot{W}, \text{ in } (\tau, +\infty)\times\mathcal{O}, \\ \nabla\cdot u = 0, \text{ in } (\tau, +\infty)\times\mathcal{O}, \\ u = 0, \ Au = 0 \text{ on } (\tau, +\infty)\times\partial\mathcal{O}, \\ u(\tau + s, x) = \phi(s, x), s \in (-\infty, 0], x \in \mathcal{O}, \end{cases} \tag{0.0.17}$$

donde $\mathcal{O} \subseteq \mathbb{R}^3$ es un conjunto abierto acotado con frontera suficientemente regular $\partial\mathcal{O}$, $\tau \in \mathbb{R}$, A es el operador de Stokes, el par (ν, α) de coeficientes positivos denota la viscosidad cinemática del fluido y el cuadrado de la escala espacial a la que se filtra el movimiento del fluido respectivamente, el símbolo $*$ denota la transpuesta de una matriz, $u = (u_1, u_2, u_3)$ es la velocidad promediada (o a gran escala) del fluido, u_t denota el segmento de solución hasta el tiempo t, es decir $u_t(s) = u(t+s)$ para todo $s \leqslant 0$, p es la presión del fluido, f es un campo de fuerza externo no retardado, los términos g_1, g_2 contienen algunas características hereditarias, como memoria, variable no limitada o retardo distribuido infinito, etc, ϕ es un campo de velocidad inicial definido en $(-\infty, 0] \times \mathcal{O}$, y $\dot{W}$ denota la derivada generalizada (ruido blanco) de un proceso cilíndrico de Wiener, que se introducirá más adelante.

Nuestro primer objetivo es demostrar la existencia y unicidad de soluciones para el sistema estocástico tridimensional LANS en el espacio de Banach $C_\gamma(V)$. Como hicieron Liu y Caraballo [95] para el sistema bidimensional habitual, tenemos que suponer que la fuerza externa no retardada f es localmente integrable (véase **Hipótesis F**) y los términos de fuerza retardados $g_i(t, u_t)(i = 1, 2)$ son globalmente Lipschitz continuos (véase **Hipótesis G**). Un ejemplo se proporciona en la última parte de la Sección 1.1. El cálculo muestra que el ejemplo (correspondiente a un retardo distribuido infinito) satisface todas las condiciones de la **Hipótesis G** en el espacio $C_\gamma(V)$ para $\gamma > 0$. Además, también necesitamos una hipótesis adicional sobre el término de difusión no lineal g_2 del ruido (véase **Hipótesis I**).

Bajo los supuestos anteriores, utilizamos el método de Galerkin para construir una sucesión aproximante. A continuación obtenemos estimaciones a priori para la sucesión aproximante asegurando que las soluciones existen para todo el intervalo de tiempo $[\tau, \tau + T]$ para todo $T > 0$. Finalmente, obtenemos el buen planteamiento del problema para el modelo LANS basado en estimaciones uniformes de las soluciones.

Otro tema interesante y desafiante es considerar el comportamiento asintótico de las soluciones de la Ec. (0.0.17) hacia la solución estacionaria. Este tema proporcionará información útil sobre la evolución futura del sistema. Gracias a los teoremas de Lax-Milgram y de Schauder, demostramos primero la existencia y unicidad de soluciones estacionarias para la ecuación determinista correspondiente. A continuación, mostramos la estabilidad local de la solución estacionaria para el término de retardo general mediante un método directo, donde el retardo general contiene el retardo variable no acotado y el retardo distribuido infinito en $C_{-\infty}(V)$. A continuación, probamos la estabilidad global de la solución estacionaria. Sin embargo, para obtener resultados de estabilidad en $C_\gamma(V)$ con $\gamma > 0$, la estabilidad exponencial en el caso de retardo variable ilimitado no se puede demostrar en general (véanse [95] y [100] para más detalles). Afortunadamente, en el caso de retardo distribuido infinito, somos capaces de demostrar no sólo la estabilidad de las soluciones estacionarias en $C_\gamma(V)$, sino también la estabilidad asintótica exponencial. Puesto que no podemos analizar la estabilidad exponencial en el caso de retardo variable ilimitado en $C_\gamma(V)$, exploraremos, al menos, la estabilidad asintótica en $C_{-\infty}(V)$, utilizando la construcción de funciones de Lyapunov propuesta por Kolmanovskii y Shaikhet [78]. Además, discutimos finalmente la estabilidad asintótica polinómica en el caso particular de retardo proporcional (también conocido como retardo pantográfico).

Nótese que todos los resultados del Capítulo 1 han sido publicados en [142] (*S. Yang, Y. Li, Q. Zhang y T. Caraballo, Stability analysis of stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay, J. Dynam. Differential Equations, Published Online, (2023), Doi:*

10.1007/s10884-022-10244-0).

A continuación, en el Capítulo 2, nos interesamos principalmente por la dinámica en media y el análisis de estabilidad de las siguientes ecuaciones estocásticas 3D LANS con retardo infinito en dominios no acotados:

$$\begin{cases} \partial_t(u-\alpha\Delta u)+\nu(Au-\alpha\Delta(Au))+(u\cdot\nabla)(u-\alpha\Delta u) \\ -\alpha\nabla u^*\cdot\Delta u+\nabla p=f(t)+g(t,u_t)+\sigma(t,u)\dot{W}, \text{ in } (\tau,+\infty)\times\mathcal{O}, \\ \operatorname{div} u=0, \text{ in } (\tau,+\infty)\times\mathcal{O}, \\ u=0,\ Au=0 \text{ on } (\tau,+\infty)\times\partial\mathcal{O}, \\ u(\tau+s,x)=\phi(s,x), s\in(-\infty,0], x\in\mathcal{O}, \end{cases} \tag{0.0.18}$$

donde $\tau\in\mathbb{R}, \mathcal{O}\subset\mathbb{R}^3$ es un conjunto abierto no acotado con frontera $\partial\mathcal{O}$, que es un dominio Poincaré, es decir, existe un número positivo λ tal que

$$\lambda\int_{\mathcal{O}}|\psi|^2dx\leqslant\int_{\mathcal{O}}|\nabla\psi|^2dx,\ \forall\,\psi\in H_0^1(\mathcal{O}), \tag{0.0.19}$$

g es un término no lineal que captura el retardo temporal, σ es un coeficiente de difusión no lineal localmente Lipschitz, los otros símbolos son los mismos que los de la Ec. (0.0.17).

El primer objetivo de este capítulo es estudiar la dinámica en media de la ecuación estocástica 3D LANS (0.0.18) conducida por retardo infinito en dominios no acotados. En el espacio de Banach $C_\gamma(V)$, primero establecemos el buen planteamiento del problema (0.0.18) con retardo infinito cuando la fuerza externa no retardada $f(t)$ es localmente integrable, el término de retardo $g(t,u_t)$ es globalmente Lipschitziana y el término de difusión no lineal $\sigma(t,u)$ es localmente Lipschitziano. Con este fin, introducimos una función de truncamiento globalmente Lipschitziana ξ_n (para cada $n\in\mathbb{N}$) definida por (2.2.3) para aproximar $\sigma(t,u)$. Los operadores de solución nos permiten definir un sistema dinámico aleatorio en media en lugar del habitual sistema dinámico aleatorio trayectorial, principalmente porque, en este caso, no hay ningún método disponible para transferir el sistema estocástico (0.0.18) a uno aleatorio que sea conjugado. A continuación, demostramos la existencia de un único atractor aleatorio débil en media para el sistema dinámico aleatorio en media.

Otro propósito de este trabajo es demostrar la existencia de una solución estacionaria única para la ecuación determinista correspondiente (ver Ec. (2.3.9)) y analizar el comportamiento asintótico de las soluciones de la Ec. (0.0.18) hacia la solución estacionaria. Dado que nuestros resultados ya no dependen de las inclusiones compactas de Sobolev para dominios no acotados, modificamos algunos argumentos del [31, Teorema 10], es decir, aplicamos el argumento clásico de Galerkin así como los teoremas de Lax-Milgram y del punto fijo de Brouwer para derivar la existencia y unicidad de soluciones estacionarias del sistema (2.3.9) en dominios no acotados. Además, establecemos diferentes condiciones suficientes de estabilidad a través de varios métodos utilizados en las subsecciones 1.3.2-1.3.5.

El trabajo [137] (*S. Yang, T. Caraballo y Y. Li, Dynamics and stability analysis for stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay on unbounded domains, Appl. Math. Optim., (sometido)*) contiene los resultados demostrados en el Capítulo 2.

En el Capítulo 3, investigamos las medidas invariantes de las siguientes ecuaciones estocásticas 3D LANS conducidas por retardo infinito y ruido aditivo:

$$\begin{cases} \partial_t(u - \alpha\Delta u) + \nu(Au - \alpha\Delta(Au)) + (u \cdot \nabla)(u - \alpha\Delta u) \\ -\alpha\nabla u^* \cdot \Delta u + \nabla p = f(x) + g(u_t) + \kappa(x)\dot{W}, \text{ in } (s, +\infty) \times O, \\ \operatorname{div} u = 0, \text{ in } (s, +\infty) \times O, \\ u = 0, \ Au = 0 \text{ on } (s, +\infty) \times \partial O, \\ u(s + r, x) = \phi(r, x), r \in (-\infty, 0], x \in O, \end{cases} \tag{0.0.20}$$

donde $O \subseteq \mathbb{R}^3$ es un conjunto abierto acotado con frontera suficientemente regular ∂O, $s \in \mathbb{R}$, f y κ son funciones dadas definidas en O, la función de retardo g es de forma autónoma en la Ec. (0.0.18), y W es un proceso de Wiener de dos caras con valores reales en un espacio de probabilidad completo que se especificará en el capítulo 3. Los demás símbolos son los mismos que los de la Ec. (0.0.17).

En el presente capítulo, nuestro principal objetivo es demostrar la existencia de medidas de probabilidad invariantes de Borel para el sistema estocástico 3D LANS conducido por retardo infinito y ruido aditivo. Para la investigación de ecuaciones de evolución de dimensión infinita, especialmente para medidas de probabilidad invariantes, hay varios métodos matemáticos aplicados a la ergodicidad. Un enfoque consiste en considerar algunos tipos de EDP lineales o semilineales sencillas, véase [46, 83]. Aunque los autores analizaron sistemas lineales cuyas soluciones pueden presentar un comportamiento caótico o 'turbulento', el método se limita a una clase muy específica de ecuaciones. Otro método diferente, a saber, el método clásico de Krylov-Bogolyubov, se centra en el estudio de las EDP estocásticas, y se puede definir la ergodicidad a través del semigrupo de Markov inducido por el semiflujo estocástico. Gracias al método de Krylov-Bogolyubov, se ha investigado ampliamente la existencia de medidas invariantes para EDP estocásticas, remitimos al lector a [14, 15, 19, 45, 139] y [17, 18, 47, 102, 125] para dominios acotados y no acotados, respectivamente. Sin embargo, utilizando este enfoque, encontramos que las medidas invariantes del semigrupo de Markov son medidas de probabilidad deterministas y, por tanto, no están soportadas por atractores aleatorios globales.

Basándonos en la discusión anterior, en este capítulo introducimos una técnica diferente para construir medidas invariantes, es decir, usaremos el llamado 'límite de Banach generalizado', denotado por $\text{LIM}_{t\to+\infty}$, iniciado en [52] y desarrollado en [23, 40, 51, 98, 127, 146–148]. El uso de dicho límite nos permite relacionar los promedios temporales con los promedios de conjunto en el espacio de estados.

Hasta donde sabemos, este trabajo parece ser el primero que discute las medidas de probabilidad invariantes de Borel para el sistema estocástico 3D LANS con retardo infinito y ruido aditivo, incluso no hay ningún trabajo publicado sobre este tema para ecuaciones estocásticas de retardo. Principalmente aplicamos la teoría abstracta para sistemas dinámicos aleatorios autónomos en [148, Teorema 2.1] a nuestro modelo (0.0.20). Para ello transformaremos (0.0.20) en una ecuación determinista con un parámetro aleatorio (ver Ec. (3.2.8)) y probaremos que los operadores solución asociados a la Ec. (0.0.20) generan un sistema dinámico aleatorio φ en el espacio de estados $C_\gamma(V)$. Además, necesitamos demostrar las dos afirmaciones siguientes:

(1) el sistema dinámico aleatorio φ posee un atractor aleatorio global $\mathcal{A}(\omega)$ en $C_\gamma(V)$;

(2) φ es acotado y continuo con respecto a los valores iniciales.

Utilizando el método clásico de Galerkin, estimaciones a priori y el lema estándar de Gronwall, deducimos primero la existencia de una solución débil única del sistema (3.2.8). A continuación, obtenemos algunas estimaciones uniformes de las soluciones del sistema (3.2.8) asegurando la existencia de un conjunto absorbente aleatorio global en $C_\gamma(V)$ (ver Lemma 3.3.5). A continuación, establecemos la compacidad asintótica para φ en $C_\gamma(V)$ a través del teorema de Ascoli-Arzelà (ver Lemma 3.3.6). Además, demostramos la existencia de atractores aleatorios globales para φ (ver Teorema 3.3.8). Más aún, para cada $t \in \mathbb{R}$, casi todo $\omega \in \Omega$ y $\phi \in C_\gamma(V)$, probamos que la función $s \mapsto \varphi(t, s, \theta_{-t}\omega)\phi$ está acotada en $(-\infty, t]$, como se muestra en el Lemma 3.4.1. Por último, la continuidad de $\varphi(t, s, \theta_{-t}\omega)\phi$ con respecto a los valores iniciales (s, ϕ) en $(-\infty, t] \times C_\gamma(V)$ se demuestra en el Lemma 3.4.4.

Los resultados del Capítulo 3 se han demostrado en el artículo [136] (*S. Yang, T. Caraballo y Y. Li, Invariant measures for stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay, Commun. Nonlinear Sci. Numer. Simul., 118 (2023), pp. 107004, 21*).

Parte II: Sistemas reticulares FitzHugh-Nagumo con retardo

Algunos sistemas dinámicos reticulares pueden derivarse de la discretización espacial de sistemas continuos. Tienen amplias aplicaciones en nuestra vida cotidiana, incluyendo la física, la química, la biología, la ingeniería y otros campos de la ciencia (véase, por ejemplo, [10, 49, 71, 72]). Por lo que sabemos, uno de los sistemas reticulares más interesantes es el modelo FitzHugh-Nagumo, que simula el proceso de transmisión de señales a través de los axones. Se sabe que, los sistemas dinámicos reticulares con retardo han recibido mucha atención durante muchos años ([32, 144, 149]).

La motivación del Capítulo 4 es estudiar la dinámica a largo plazo de los atractores aleatorios para el siguiente sistema reticular FitzHugh-Nagumo con retardo conducido por un ruido de tipo Wong-Zakai no lineal:

$$\begin{cases} \dfrac{du_i}{dt} - (u_{i-1} - 2u_i + u_{i+1}) + \lambda u_i + \alpha v_i = F_i(u_i(t)) + f_i(u_i(t - \varrho^{(\rho)}(t))) + g_i(t) + G_i(t, u_i)\mathcal{G}_\delta(t, \omega), \\ \dfrac{dv_i}{dt} + \varsigma v_i - \beta u_i = h_i(t) + f_i(v_i(t - \varrho^{(\rho)}(t))), \\ u_i(\tau + s) = \phi_i(s), \;\; v_i(\tau + s) = \upsilon_i(s), \;\; i \in \mathbb{Z}, \; t > \tau, \; \tau \in \mathbb{R}, \; s \in [-\rho, 0], \end{cases} \tag{0.0.21}$$

donde λ, α, ς, β, γ y ρ son constantes positivas, $\varrho^{(\rho)}$ es una función de retardo variable con retardo máximo ρ, F_i es una función de deriva no lineal con crecimiento polinómico de orden arbitrario, f_i es una fuerza externa afectada por la memoria durante el intervalo de tiempo de retardo $[-\rho, 0]$, las fuerzas deterministas dependientes del tiempo $g_i, h_i \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, ϕ_i, ψ_i son los datos iniciales en el intervalo $[-\rho, 0]$, G_i es una difusión no lineal, $\mathcal{G}_\delta$ es el proceso de Wong-Zakai con parámetro de correlación $\delta > 0$, que es δ-diferencia de un proceso escalar de Wiener de dos caras W en un espacio de probabilidad $(\Omega, \mathcal{F}, \mathbb{P})$, dado por

$$\mathcal{G}_\delta(t, \omega) := \frac{1}{\delta}(W(t + \delta, \omega) - W(t, \omega)), \; \forall\, \delta > 0, \; t \in \mathbb{R}, \; \omega \in \Omega. \tag{0.0.22}$$

Este tipo de ruido Wong-Zakai fue introducido por primera vez por [131, 132] en el que los autores utilizaron ecuaciones diferenciales deterministas para aproximar las estocásticas para movimientos brownianos unidimensionales. Posteriormente, un número creciente de autores extendió la idea de las aproximaciones de Wong-Zakai a movimientos brownianos, martingales y semimartingales de mayor dimensión (véanse [80, 81, 104–106, 110, 113, 114, 116]). Recientemente, se ha estudiado ampliamente la convergencia de soluciones y atractores para ecuaciones aleatorias con tal ruido cuando $\delta \to 0$ (véase [1, 9, 16, 56, 59, 152]). Nótese que los sistemas de aproximación de Wong-Zakai en estas referencias sólo se consideraron en el caso de ruido lineal, es decir, la sucesión de funciones de difusión $G = (G_i)_{i\in\mathbb{Z}}$ en (0.0.21) es $u = (u_i)_{i\in\mathbb{Z}}$ o independiente de u. Además, también lo eran en todos los resultados sobre la semicontinuidad de los atractores aleatorios, véanse [8, 86, 120, 143] para ecuaciones estocásticas autónomas, y [9, 21, 84, 85, 89, 93, 123, 138, 145, 150] para ecuaciones estocásticas no autónomas.

Sin embargo, existen muy pocos trabajos sobre el estudio de los atractores de ecuaciones de retardo aleatorio conducidas por el ruido no lineal de Wong-Zakai, incluso en versión autónoma. Hasta donde sabemos, los dos únicos trabajos para tales ecuaciones autónomas fueron publicados por Li et al. en [91, 92]. En esta parte, investigamos la dinámica del sistema reticular de FitzHugh-Nagumo con retardo aleatorio no autónomo con un ruido de tipo Wong-Zakai no lineal (0.0.21).

El Capítulo 4 se divide en dos partes. En la primera demostramos la existencia de un atractor aleatorio $\mathcal{A}^\delta = \{\mathcal{A}^\delta(t,\omega)\}$ para el sistema aleatorio de retardo (0.0.21) con un ruido no lineal y analizamos su semicontinuidad superior cuando $\delta \to +\infty$. Esto es diferente de la situación general $\delta \to 0$. Para probar la existencia de un atractor aleatorio $\mathcal{A}^\delta$ en $\mathcal{X}^\rho_\sigma = C([-\rho,0],\mathcal{X}_\sigma)$ con $\mathcal{X}_\sigma = \ell^2_\sigma \times \ell^2_\sigma$, donde ℓ^2_σ es un espacio ponderado para cada $\delta > 0$, debemos verificar que el sistema dinámico aleatorio (o cociclo) Ψ^δ, inducido por la Ec. (0.0.21) impulsado por el ruido no lineal de Wong-Zakai, es pullback asintóticamente compacto en $\mathcal{X}^\rho_\sigma$. Las ideas de las estimaciones uniformes y el teorema de Ascoli-Arzelà son las herramientas cruciales para demostrarlo. En cuanto a la semiconvergencia superior de $\mathcal{A}^\delta$ cuando $\delta \to +\infty$, necesitamos la ayuda de la ley del logaritmo del proceso de Wiener, que establece $W(t,\omega)/\log(\log|t|) \to 0$ cuando $t \to \pm\infty$, así como el resultado (4.1.6) del Lemma 4.1.1 que asegura

$$\lim_{\delta\to+\infty} \sup_{t\in[a,b]} \mathcal{G}_\delta(t,\omega) = 0, \ \mathbb{P}\text{-a.s. } \omega \in \Omega, \ a \leqslant b.$$

Basándonos en los argumentos anteriores, consideramos el sistema límite del modelo reticular de retardo aleatorio (0.0.21) cuando $\delta \to +\infty$ como el sistema reticular de retardo determinista:

$$\begin{cases} \dfrac{d\hat{u}_i}{dt} - (\hat{u}_{i-1} - 2\hat{u}_i + \hat{u}_{i+1}) + \lambda\hat{u}_i + \alpha\hat{v}_i = F_i(\hat{u}_i(t)) + f_i(\hat{u}_i(t-\varrho^{(\rho)}(t))) + g_i(t), \\ \dfrac{d\hat{v}_i}{dt} + \varsigma\hat{v}_i - \beta\hat{u}_i = h_i(t) + f_i(\hat{v}_i(t-\varrho^{(\rho)}(t))), \\ \hat{u}_i(\tau+s) = \hat{\phi}_i, \ \ \hat{v}_i(\tau+s) = \hat{\upsilon}_i, \ \ i\in\mathbb{Z}, \ t>\tau, \ \tau\in\mathbb{R}, \ s\in[-\rho,0]. \end{cases} \tag{0.0.23}$$

Bajo algunas condiciones apropiadas (véanse más adelante las Hipótesis **E, F1, F2, G1-G3**), encontramos que el sistema (0.0.23) genera un atractor de pullback denotado por $\mathcal{A}^\infty(t)$ cuya existencia ha sido ya establecida, véanse [2, 26, 119]. Entonces tenemos que comprobar que el atractor aleatorio pullback $\mathcal{A}^{\delta}(t,\omega)$ semi-converge superiormente a $\mathcal{A}^{\infty}(t)$ cuando $\delta \to +\infty$ (véase Teorema 4.3.2), es decir,

$$\lim_{\delta\to+\infty} \mathrm{d}_{\mathcal{X}_\sigma^\rho}(\mathcal{A}^{\delta}(t,\omega), \mathcal{A}^{\infty}(t)) = 0, \quad \forall\, t \in \mathbb{R},\ \omega \in \Omega, \tag{0.0.24}$$

donde la distancia $\mathrm{d}_{\mathcal{X}_\sigma^\rho}$ se define para todos los subconjuntos A y B de $\mathcal{X}_\sigma^\rho$ mediante

$$\mathrm{d}_{\mathcal{X}_\sigma^\rho}(A,B) := \sup_{a\in A}\inf_{b\in B}\sup_{\nu\in[-\rho,0]} \|a(\nu)-b(\nu)\|_{\mathcal{X}_\sigma}.$$

Para ello, demostramos que las soluciones del sistema aleatorio (0.0.21) convergen a la del correspondiente sistema determinista (0.0.23) cuando $\delta \to +\infty$. Nótese que el atractor pullback $\mathcal{A}^{\infty}(t)$ de (0.0.24) se escribe como $\mathcal{A}_\rho(t)$ para indicar su dependencia del retardo ρ para propósitos posteriores.

En la segunda parte de este capítulo, nuestro objetivo es establecer también la semicontinuidad superior del atractor pullback $\mathcal{A}_\rho(t)$ cuando $\rho \to 0$ (ver Teorema 4.4.4), es decir,

$$\lim_{\rho\to 0} \mathrm{d}^*_{\mathcal{X}_\sigma^\rho}(\mathcal{A}_\rho(t), \mathcal{A}_0(t)) = 0, \quad \forall\, t \in \mathbb{R}, \tag{0.0.25}$$

donde $\mathcal{A}_0(t)$ es un atractor de pullback para la versión sin retardo de la Ec. (0.0.23), y la distancia $\mathrm{d}^*_{\mathcal{X}_\sigma^\rho}$ se define para todo subconjunto A de $\mathcal{X}_\sigma^\rho$ y B de $\mathcal{X}_\sigma$ por

$$\mathrm{d}^*_{\mathcal{X}_\sigma^\rho}(A,B) := \sup_{a\in A}\inf_{b\in B}\sup_{\nu\in[-\rho,0]} \|a(\nu)-b\|_{\mathcal{X}_\sigma}.$$

Debido a la validez de todas las estimaciones que obtuvimos en la Sección 4.2 de la primera parte, especialmente en el caso sin retardo ($\rho = 0$) y el caso determinista ($\delta \to +\infty$) para (0.0.21). Por tanto, deducimos inmediatamente la existencia del atractor pullback $\mathcal{A}_0(t)$. En cuanto a (0.0.25), la tarea principal es demostrar la convergencia de las soluciones al sistema (0.0.23) cuando $\rho \to 0$.

Nótese que todos los resultados obtenidos en el Capítulo 4 son nuevos e interesantes, y han sido publicados en [141] (*S. Yang, Y. Li y T. Caraballo, Dynamical stability of random delayed FitzHugh-Nagumo lattice systems driven by nonlinear Wong-Zakai noise, J. Math. Phys., 63 (2022), pp. 111512, 32*).

Parte III: Atractores numéricos para sistemas reticulares

Recientemente, Han, Kloeden y Sonner [64] desarrollaron un trabajo sobre atractores numéricos para el sistema reticular de difusión de reacción. Más concretamente, han demostrado la existencia de un atractor numérico para el sistema de retículo discreto en el tiempo mediante el esquema de Euler implícito (IES), y también han establecido la semiconvergencia superior del atractor numérico hacia el atractor global cuando el tamaño de paso tiende a cero.

En el Capítulo 5, aplicaremos la idea de Han et al. [64] al siguiente sistema dinámico reticular de tipo p-Laplace (LDS):

$$\frac{du_i(t)}{dt} = \nu(A_p u(t))_i + f(u_i(t)) + g_i, \quad i \in \mathbb{Z}, \tag{0.0.26}$$

donde $\nu > 0$, $p \geqslant 2$, $u = (u_i)_{i\in\mathbb{Z}}$, y el operador discreto p-Laplace está definido por

$$(A_p u)_i = |u_{i+1} - u_i|^{p-2}(u_{i+1} - u_i) - |u_i - u_{i-1}|^{p-2}(u_i - u_{i-1}), i \in \mathbb{Z}.$$

Como es sabido, la EDL p-Laplace (0.0.26) es la discretización espacial de la correspondiente ecuación en derivadas parciales de tipo p-Laplace (definida en la recta real), mientras que la dinámica de la EDP de tipo p-Laplace (determinista o estocástica) se estudió en [37, 48, 54, 55, 70, 77, 79, 86, 88, 92, 115, 130].

Como preliminares, mostramos en la Sección 5.2 que la LDS (0.0.26) tiene una bola positivamente invariante $\mathcal{B}_{r^*}(0)$ y un atractor global $\mathcal{A}$ en ℓ^2, donde la condición disipativa de f es diferente de las de [39, 57, 58, 129] y asumida por

$$\alpha := \inf_{s\neq 0} \frac{-f(s)}{s} > 0. \tag{0.0.27}$$

En este capítulo, consideramos principalmente el esquema numérico en el sentido de tiempo discreto. Utilizando el tamaño de paso $\epsilon := t_{n+1} - t_n$ del tiempo para discretizar la LDS (0.0.26), obtenemos la IES de p-Laplace.

$$u^\epsilon_{n,i} = u^\epsilon_{n-1,i} + \epsilon\nu(A_p u^\epsilon_n)_i + \epsilon f(u^\epsilon_{n,i}) + \epsilon g_i, \ \forall n \in \mathbb{N}, \ i \in \mathbb{Z}. \tag{0.0.28}$$

Como señalan Han, Kloeden, Sonner [64], la IES (0.0.28) con $p = 2$ no es globalmente resoluble para un tamaño de paso común (véase el Lemma 5.4.3 para la razón si $p > 2$). En lugar de la resolubilidad global, demostraremos en el Teorema 5.2.1 que, para tamaños de paso suficientemente pequeños, la IES (0.0.28) es únicamente resoluble cuando el dato inicial pertenece a la bola positivamente invariante $\mathcal{B}_{r^*}(0)$. En la prueba recursiva del Teorema 5.2.1, utilizamos un nuevo método de ampliación del radio para superar la dificultad de que la bola $\mathcal{B}_{r^*}(0)$ ya no es un conjunto positivamente invariante bajo el operador definido por el lado derecho de (0.0.28). La demostración es más cuidadosa y técnica que las de [64] incluso para el caso $p = 2$.

Para el propósito posterior, tenemos que considerar la aproximación numérica de soluciones cuando $\epsilon \to 0$. A este respecto, Kloeden y Lorenz [75] (véase también Jentzen et al. [12, 67–69]) han introducido el método del desarrollo de Taylor mediante el uso de derivadas Fréchet de la parte *lineal* Laplaciana A_p ($p = 2$) y f. Sin embargo, el operador no lineal A_p ($p > 2$) no tiene derivadas Fréchet.

Para superar la dificultad anterior, establecemos un nuevo tipo de desarrollo de Taylor sin derivadas Fréchet y damos el error en tiempo continuo de las soluciones para la LDS (0.0.26) (ver Lemma 5.3.1). Utilizando este error de tiempo continuo, podemos mostrar el error de discretización de orden dos para las soluciones entre LDS (0.0.26) e IES (0.0.28), ver Teorema 5.3.2. Nuestro método es adecuado para una clase más amplia de ecuaciones en tiempo discreto, incluso si los operadores no son derivables Fréchet.

A partir de la Sección 5.5, nuestro principal propósito es estudiar el esquema numérico de atractores, que es un tema relativamente nuevo (introducido por Han, Kloeden, Sonner [64], véase también [140]) tanto en análisis numérico como en sistemas dinámicos [112]. Más concretamente, estudiamos la aproximación discreta del atractor global $\mathcal{A}$ para LDS (0.0.26) en términos de atractores numéricos para IES (0.0.28) y su sistema truncado finitamente dimensional.

Demostramos en el Teorema 5.4.2 que el semigrupo discreto, generado a partir de la IES (0.0.28), posee un único atractor numérico conexo $\mathcal{A}_\epsilon$ para tamaños de paso suficientemente pequeños. En la demostración, necesitamos estimar recursivamente las colas de soluciones para todo $n \in \mathbb{N}$, donde la técnica usual de función de truncamiento (ver [6, 7, 22, 25, 62, 65, 128, 151, 154]) sigue siendo válida en el caso de tiempo discreto.

Además, demostramos en el Teorema 5.4.4 que $\mathcal{A}_\epsilon$ tiene una cota optimizada dada por $\|g\|/\alpha$. Esta cota es crucial para probar que el atractor numérico converge de forma continua (superior e inferior) a cero a medida que la gráfica de f se acerca lo suficiente al eje y y a medida que $g \to 0$, respectivamente. Este tema de la optimización y la convergencia de los atractores numéricos es nuevo en la literatura.

En el Teorema 5.5.1, establecemos la semicontinuidad superior del atractor numérico $\mathcal{A}_\epsilon$ al atractor global $\mathcal{A}$ a medida que $\epsilon \to 0$, donde el error de discretización de las soluciones en el Teorema 5.3.2 juega un papel crucial en la demostración.

En la Sección 5.6, estudiamos la aproximación finito dimensional del atractor numérico. Para ello, truncamos el IES (0.0.28) en el espacio $(2m + 1)$-dimensional de Euclides para obtener el esquema numérico truncado con la condición de contorno periódica, véase el modelo (5.6.1). Luego demostramos en el Teorema 5.6.4 que la IES truncada (5.6.1) tiene un atractor denotado por $\mathcal{A}_{\epsilon,m}$, y que $\mathcal{A}_{\epsilon,m}$ converge de forma semicontinua superior al atractor numérico $\mathcal{A}_\epsilon$ a medida que $m \to \infty$. Si la viscosidad es cero, es decir, $\nu = 0$, la semi-continuidad inferior de $\mathcal{A}_{\epsilon,m}$ a $\mathcal{A}_\epsilon$ sigue siendo válida como se demuestra en el Teorema 5.6.6.

En definitiva, hemos establecido una senda de convergencia desde $\mathcal{A}_{\epsilon,m}$ al atractor global $\mathcal{A}$ a través de $\mathcal{A}_\epsilon$. De hecho, hay otro camino de convergencia de $\mathcal{A}_{\epsilon,m}$ a $\mathcal{A}$ a través de $\mathcal{A}_m$, donde $\mathcal{A}_m$ es el atractor global para el sistema truncado de la LDS (0.0.26) en el espacio de Euclides $(2m + 1)$-dimensional. Todos los caminos de convergencia y los límites optimizados de los atractores se muestran en la Figura 2.

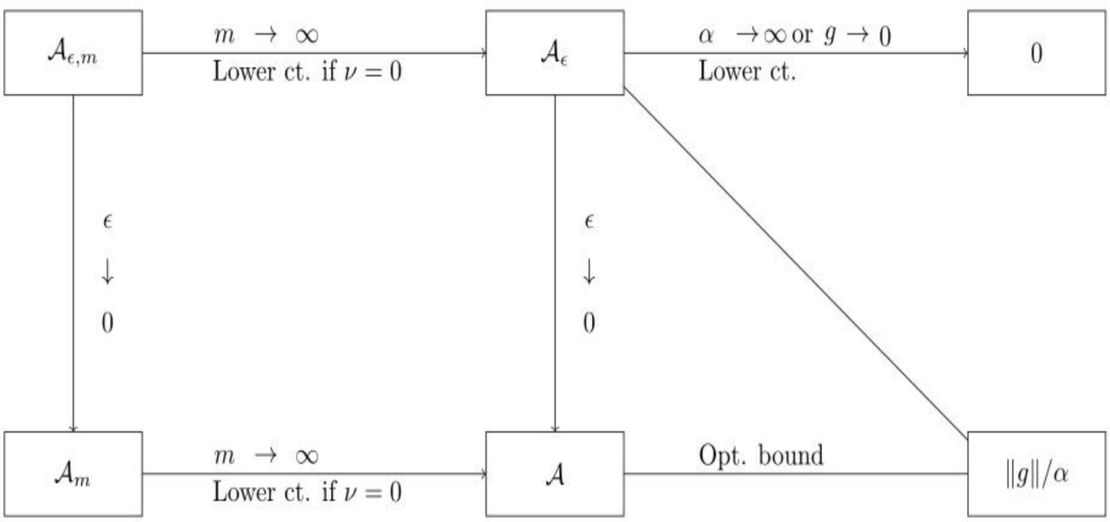

Figure 2: Trayectorias de convergencia y límites para los atractores.

Cabe destacar que el tema que tratamos en el Capítulo 5 sobre optimización y convergencia de atractores numéricos son un reto, y han sido publicados en [90] (*Y. Li, S. Yang y Tomás Caraballo, Optimization and convergence of numerical attractors for discrete-time quasi-linear lattice system, SIAM J. Numer. Anal., (por aparecer), 2023*).

Cuestiones para trabajar en el futuro

Concluimos esta tesis mostrando nuestro plan de trabajo futuro. En general, los sistemas suelen verse afectados por ruidos externos e internos. Además, si los retardos también se ven perturbados por ruidos, entonces serán aleatorios. Recientemente, los sistemas de retardo aleatorio han recibido una atención creciente, especialmente en el área del control y la optimización. Uno de los ejemplos más importantes es el sistema de control en red, y uno de los principales problemas asociados es el retardo de la comunicación, que a menudo es aleatorio, véase [103, 107]. Motivados por este hecho, por un lado, podemos establecer resultados teóricos de atractores numéricos para los siguientes sistemas reticulares p-Laplace con retardo aleatorio vía IES:

$$\frac{du_i(t)}{dt} = \nu(A_p u(t))_i + f(u_i(t - \tau(t, \omega))) + g_i, \ i \in \mathbb{Z},$$

donde $\tau(t, \omega)$ es una función aleatoria, que tiene trayectorias muestrales continuas con valor acotado $\tau(t, \omega) \in [\tau_0, \tau_1]$ donde $0 \leqslant \tau_0 \leqslant \tau_1$ son números deterministas finitos y $\omega \in \Omega$ para un espacio de probabilidad dado $(\Omega, \mathcal{F}, \mathbb{P})$. Los demás símbolos son los mismos que en (0.0.26) y que hemos estudiado en el Capítulo 5. Comparado con (0.0.26), el modelo anterior puede ser más realista.

Por otra parte, investigamos la existencia, unicidad, regularidad, descomposición de Morse, dimensión de Hausdorff y estimación de la dimensión fractal de los atractores aleatorios pullback para varios tipos de sistemas no autónomos con retardo aleatorio. También se considera su avance asintóticamente autónomo, es decir, el componente temporal de los atractores aleatorios pullback semi-converge a atractores aleatorios globales cuando el parámetro temporal tiende a infinito.

Part I

Lagrangian-averaged Navier-Stokes equations with infinite delay

Preliminaries

We focus on the notation used throughout the Part I and the cylindrical Wiener process needed for the first two chapters.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ be a complete filtered probability space such that $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing right continuous family of sub σ-algebras of $\mathcal{F}$, which contains all $\mathbb{P}$-null sets, and further $\mathcal{F}_t = \mathcal{F}_0$ for all $t \leqslant 0$.

Let $\{\beta_t^j, t \geqslant 0, j = 1, 2, 3, \ldots\}$ be a sequence of mutually independent standard real valued $\mathcal{F}_t$-Wiener processes and K a separable Hilbert space with an orthonormal basis $\{e_j;\ j = 1, 2, 3, \ldots\}$. Suppose that $\{W(t); t \geqslant 0\}$ be a K-valued cylindrical Wiener process (with the covariance operator $Q : K \to K$) given by

$$W(t) = \sum_{j=1}^{\infty} \beta^j(t) e_j, \quad t \geqslant 0. \tag{0.0.29}$$

Given a separable Hilbert space H_0, we denote by $\mathcal{L}^2(K, H_0)$ the space of Hilbert-Schmidt operators from K into H_0 with following norm

$$\|S\|^2_{\mathcal{L}^2(K,H_0)} = \mathrm{tr}(SQS^*), \ \forall S \in \mathcal{L}^2(K, H_0), \tag{0.0.30}$$

where tr denotes the trace of an operator and S^* is the adjoint operator of S.

For any separable Banach space X, interval $(a, b) \subset \mathbb{R}$ and $p \geqslant 1$, we denote by $I^p(a, b; X)$ the Banach space of all processes $\varphi \in L^p(\Omega \times (a, b), \mathcal{F} \otimes \mathcal{B}((a, b)), dP \otimes dt; X)$ such that $\varphi(t)$ is $\mathcal{F}_t$-progressively measurable for a.e. $t \in (a, b)$, where $\mathcal{B}(\cdot)$ denotes the Borel σ-algebra. We also denote by $L^p(\Omega, \mathcal{F}, dP; C(a, b; X))$ with $p \geqslant 1$ the space of all continuous and $\mathcal{F}_t$-progressively measurable X-valued processes φ such that $\mathbb{E}(\sup_{a\leqslant t\leqslant b} \|\varphi(t)\|_X^p) < \infty$, where $C(a, b; X)$ is the Banach space of all continuous functions from $[a, b]$ into X. For convenience, we write $L^p(\Omega, \mathcal{F}, dP; C(a, b; X))$ as $L^p(\Omega; C(a, b; X))$.

For any positive constant $T > 0$, process $\Phi \in I^2(\tau, \tau + T; \mathcal{L}^2(K, H_0))$ and $t \in [\tau, \tau + T]$, the stochastic integral $\int_\tau^t \Phi(s) dW(s)$ is defined by the unique continuous H_0-valued $\mathcal{F}_t$-martingale such that

$$\Big(\int_\tau^t \Phi(s) dW(s), w \Big)_{H_0} = \sum_{j=1}^{\infty} \int_\tau^t \Big(\Phi(s) e_j, w \Big)_{H_0} d\beta^j(s), \ \forall w \in H_0, \tag{0.0.31}$$

where the integral with respect to $\beta^j(s)$ is the Ito integral. By [45], if $\Phi \in I^2(\tau, \tau + T; \mathcal{L}^2(K, H_0))$ and $\phi \in L^2(\Omega, L^\infty(\tau, \tau + T; H_0))$ is $\mathcal{F}_t$-progressively measurable, then

$$\sum_{j=1}^{\infty} \int_\tau^t \Big(\Phi(s) e_j, \phi(s) \Big)_{H_0} d\beta^j(s) =: \int_\tau^t \Big(\Phi(s) dW(s), \phi(s) \Big), \ \tau \leqslant t \leqslant \tau + T,$$

converges in $L^1(\Omega, C(\tau, \tau + T))$.

Recall that given a function $\phi : (-\infty, \tau + T] \to X$, for each $t \in (\tau, \tau + T)$, the segment ϕ_t of ϕ is defined by

$$\phi_t(s) = \phi(t+s), \ \forall\ s \in (-\infty, 0]. \tag{0.0.32}$$

Next, we introduce some notations and linear operators, recall some properties with respect to the nonlinear term $(u \cdot \nabla)(u - \alpha\Delta u) - \alpha\nabla u^* \cdot \Delta u$, and impose some suitable assumptions.

Recall that $\mathcal{O} \subseteq \mathbb{R}^3$ is a bounded open set in Chapters 1 and 3, an unbounded open set in Chapter 2. Denote by $\mathbb{L}^2(\mathcal{O}) := (L^2(\mathcal{O}))^3$, $\mathbb{H}^1_0(\mathcal{O}) := (H^1_0(\mathcal{O}))^3$, $\mathbb{C}^\infty_0(\mathcal{O}) = (C^\infty_0(\mathcal{O}))^3$ and

$$\mathcal{V} = \{u \in \mathbb{C}^\infty_0(\mathcal{O}) : \nabla \cdot u = 0 \ \text{ in } \ \mathcal{O}\}. \tag{0.0.33}$$

Let H be the closure of $\mathcal{V}$ in $\mathbb{L}^2(\mathcal{O})$. Then H is a Hilbert space with the inner product and norm

$$(u, v) = \sum_{j=1}^{3} \int_{\mathcal{O}} u_j(x) v_j(x) dx, \ |u|^2 = (u, u), \ \forall u, v \in H. \tag{0.0.34}$$

Let V be the closure of $\mathcal{V}$ in $\mathbb{H}^1_0(\mathcal{O})$. Then V is a Hilbert space with the inner product

$$((u, v)) = (u, v) + \alpha(\nabla u, \nabla v) = (u, v) + \alpha \sum_{i,j=1}^{3} \int_{\mathcal{O}} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx, \ \forall u, v \in V, \tag{0.0.35}$$

and the norm $\|u\|^2 = ((u, u))$. We have $V \subset H \subset V^*$, where V^* is the dual space of V, the injections are dense, continuous and compact.

Denote by $\mathcal{P}$ the Leray projector from $\mathbb{L}^2(\mathcal{O})$ onto H and define the Stokes operator A by

$$Aw = -\mathcal{P}(\Delta w), \quad \forall\ w \in D(A) = \mathbb{H}^2(\mathcal{O}) \cap V, \tag{0.0.36}$$

where $\mathbb{H}^2(\mathcal{O}) = (H^2(\mathcal{O}))^3$. We deduce

$$(Au, v) = (\nabla u, \nabla v), \quad \|u\|_{\mathbb{H}^2(\mathcal{O})} \leqslant C_1 |Au|, \quad \forall u \in D(A), v \in V, \tag{0.0.37}$$

where C_1 is a positive constant. In particular, $D(A)$ is a Hilbert space.

Denote by $\langle \cdot, \cdot \rangle$ the duality product between $(D(A))^*$ and $D(A)$, and define a continuous linear operator $\widetilde{A} \in \mathcal{L}(D(A), (D(A))^*)$ by

$$\langle \widetilde{A}u, v \rangle = \nu(Au, v) + \nu\alpha(Au, Av), \quad \forall u, v \in D(A) =: D(\widetilde{A}). \tag{0.0.38}$$

It is well-known that the Stokes operator A has a sequence $\{\lambda_k : k \in \mathbb{N}\}$ of eigenvalues satisfying

$$0 < \inf_{v \in V \setminus \{0\}} \frac{\|v\|^2}{|v|^2} = \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \quad \lambda_k \to \infty \tag{0.0.39}$$

and a sequence $\{\xi_k \in D(A) : k \in \mathbb{N}\}$ of eigenvectors which is orthonormal in H. From (0.0.38) we have

$$\langle \widetilde{A}\xi_k, v \rangle = \nu\lambda_k((\xi_k, v)) \tag{0.0.40}$$

and the eigenvalues of the operator $\widetilde{A}$ are given by $\widetilde{\lambda}_k := \nu\lambda_k$. By (0.0.38)-(0.0.40), the operator $\widetilde{A} \in \mathcal{L}(D(A), (D(A))^*)$ satisfies the following conditions:

(A1) $\widetilde{A}$ is self-adjoint;

(A2) For all $u \in D(A)$, $2\langle \widetilde{A}u, u\rangle \geqslant \widetilde{\alpha}(Au, Au)$, where $\widetilde{\alpha} = 2\nu\alpha$;

(A3) $\widetilde{A}\xi_k = \widetilde{\lambda}_k\xi_k$ with $\widetilde{\lambda}_k = \nu\lambda_k$.

Denote by $D(\widetilde{A}) = \{u \in D(A) : \widetilde{A}u \in V\}$ the domain of the operator $\widetilde{A}$, it is a subspace of $D(A)$ with the inner product $(u, v)_{D(\widetilde{A})} = ((\widetilde{A}u, \widetilde{A}v))$, for all $u, v \in D(\widetilde{A})$, and norm $|u|_{D(\widetilde{A})} = \|\widetilde{A}u\|$. Note that $D(\widetilde{A})$ is a Hilbert space, and the injection $D(\widetilde{A}) \subset D(A)$ is continuous and

$$\widetilde{\lambda}_1\|u\|^2_{D(A)} \leqslant |u|^2_{D(\widetilde{A})}, \ \forall u \in D(\widetilde{A}). \tag{0.0.41}$$

As in [31], we associate another inner product on $D(A) = D(\widetilde{A})$, defined by

$$(u, v)_{D(A)} := \langle \widetilde{A}u, v\rangle, \text{ and so } \widetilde{\lambda}_1\|u\|^2 \leqslant \|u\|^2_{D(A)}, \ \forall u, v \in D(A). \tag{0.0.42}$$

By (0.0.37), the above is equivalent to the original inner product $((u, v)) + (Au, Av)$ for $u, v \in D(A)$.

For $u \in D(A)$ and $v \in \mathbb{L}^2(\mathcal{O})$, we regard $(u \cdot \nabla)v$ as the element of $(H^{-1}(\mathcal{O}))^3 =: \mathbb{H}^{-1}(\mathcal{O})$ given by

$$\langle (u \cdot \nabla)v, w\rangle_{-1} = \sum_{i,j=1}^{3} \langle \partial_i v_j, u_i w_j\rangle_{-1}, \quad \forall\, w \in \mathbb{H}^1_0(\mathcal{O}), \tag{0.0.43}$$

where $\langle \cdot, \cdot\rangle_{-1}$ denotes the duality product between $\mathbb{H}^{-1}(\mathcal{O})$ and $\mathbb{H}^1_0(\mathcal{O})$ or between $H^{-1}(\mathcal{O})$ and $H^1_0(\mathcal{O})$, and $u_i w_j \in H^1_0(\mathcal{O})$ due to the continuous injections of $H^2(\mathcal{O}) \subset L^\infty(\mathcal{O})$ and $H^1_0(\mathcal{O}) \subset L^6(\mathcal{O})$. Hence, there exists a positive constant $C_2 := C_2(\mathcal{O})$ such that

$$|\langle (u \cdot \nabla)v, w\rangle_{-1}| \leqslant C_2|Au||v|\|w\|, \quad \forall\, (u, v, w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times \mathbb{H}^1_0(\mathcal{O}). \tag{0.0.44}$$

If $u \in D(A)$, then $\nabla u^* \in (\mathbb{H}^1(\mathcal{O}))^3 \subset (\mathbb{L}^6(\mathcal{O}))^3$, where $\mathbb{H}^1(\mathcal{O}) = (H^1(\mathcal{O}))^3$ and $\mathbb{L}^6(\mathcal{O}) = (L^6(\mathcal{O}))^3$. For all $v \in \mathbb{L}^2(\mathcal{O})$, we have $\nabla u^* \cdot v \in (L^{3/2}(\mathcal{O}))^3 \subset \mathbb{H}^{-1}(\mathcal{O})$ satisfying

$$\langle \nabla u^* \cdot v, w\rangle_{-1} = \sum_{i,j=1}^{3} \int_{\mathcal{O}} (\partial_j u_i) v_i w_j dx, \quad \forall\, w \in \mathbb{H}^1_0(\mathcal{O}), \tag{0.0.45}$$

which implies that there exists a positive constant $C_3 := C_3(\mathcal{O})$ such that

$$|\langle \nabla u^* \cdot v, w\rangle_{-1}| \leqslant C_3|Au||v|\|w\|, \quad \forall\, (u, v, w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times \mathbb{H}^1_0(\mathcal{O}). \tag{0.0.46}$$

Now, we introduce the trilinear operator as follows:

$$b^\#(u, v, w) = \langle (u \cdot \nabla)v, w\rangle_{-1} + \langle \nabla u^* \cdot v, w\rangle_{-1}, \quad \forall\, (u, v, w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times \mathbb{H}^1_0(\mathcal{O}). \tag{0.0.47}$$

By [36, Proposition 2.2], we obtain

$$b^\#(u, v, w) = -b^\#(w, v, u), \quad \forall\, (u, v, w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times D(A), \tag{0.0.48}$$

which implies that $b^{\#}(u,v,u) = 0, \forall\ (u,v) \in D(A) \times \mathbb{L}^2(\mathcal{O})$. Moreover, there exists a positive constant $c^{\#} := c^{\#}(\mathcal{O})$ such that

$$|b^{\#}(u,v,w)| \leqslant c^{\#}|Au|\|v\|\|w\|, \quad \forall\ (u,v,w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times \mathbb{H}_0^1(\mathcal{O}), \tag{0.0.49}$$

and

$$|b^{\#}(u,v,w)| \leqslant c^{\#}\|u\|\|v\||Aw|, \quad \forall\ (u,v,w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times D(A). \tag{0.0.50}$$

We then define a bilinear mapping $\widetilde{B} : D(A) \times D(A) \to (D(A))^*$, denoted by

$$\langle \widetilde{B}(u,v), w\rangle = b^{\#}(u, v - \alpha\Delta v, w), \quad \forall\ (u,v,w) \in D(A) \times D(A) \times D(A), \tag{0.0.51}$$

and $\widetilde{B}(u) := \widetilde{B}(u,u)$ for all $u \in D(A)$. By the definition and properties of $b^{\#}$, we find that there exists a positive constant $\widetilde{c} := \widetilde{c}(\mathcal{O})$ such that

(B1) $\langle \widetilde{B}(u,v), u\rangle = 0$ and $\langle \widetilde{B}(u), v\rangle = -\langle \widetilde{B}(v,u), u\rangle, \quad \forall\ (u,v) \in D(A) \times D(A)$;

(B2) $\|\widetilde{B}(u,v)\|_{(D(A))^*} \leqslant \widetilde{c}\|u\|\|v\|_{D(A)}, \quad \forall\ (u,v) \in D(A) \times D(A)$;

(B3) $|\langle \widetilde{B}(u,v), w\rangle| \leqslant \widetilde{c}\|u\|_{D(A)}\|v\|_{D(A)}\|w\|, \quad \forall\ (u,v,w) \in D(A) \times D(A) \times D(A)$.

Recall the phase space

$$C_\gamma(V) = \{\varphi \in C((-\infty,0];V) : \lim_{s\to-\infty} e^{\gamma s}\varphi(s) \text{ exists in } V\}, \text{ where } \gamma > 0, \tag{0.0.52}$$

which is a Banach space with the sup norm

$$\|\varphi\|_{C_\gamma(V)} = \sup_{s\in(-\infty,0]} e^{\gamma s}\|\varphi(s)\|. \tag{0.0.53}$$

Chapter 1

Non-autonomous stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay on bounded domains

In the chapter we investigate the asymptotic behaviour of stochastic three-dimensional LANS equations with infinite delay and nonlinear hereditary noise. First, using Galerkin's approximations, we prove the existence and uniqueness of solutions when the non-delayed external force is locally integrable and the delay terms are globally Lipschitz continuous with an additional assumption. Next, we show the existence and uniqueness of stationary solutions to the corresponding deterministic equation via the Lax-Milgram and the Schauder theorems. Later, we focus on the stability properties of stationary solutions. To begin with, we discuss the local stability of stationary solutions for general delay terms by using a direct method and then apply the abstract results to two kinds of infinite delays. Besides, the exponential stability of stationary solutions is also established in the case of unbounded distributed delay. Moreover, we investigate the asymptotic stability of stationary solutions in the case of unbounded variable delay by constructing appropriate Lyapunov functionals. Eventually, we establish criteria on the polynomial asymptotic stability of stationary solutions for the special case of proportional delay.

1.1 Hypotheses and an example

We now impose some assumptions on the non-delayed external force and delay terms respectively.

Hypothesis F. $f \in I^2(\tau, \tau + T; \mathbb{H}^{-1}(O))$ for any $\tau \in \mathbb{R}$ and $T > 0$.

Hypothesis G. Let $g_1 : \mathbb{R} \times C_\gamma(V) \to \mathbb{H}^{-1}(O)$ and $g_2 : \mathbb{R} \times C_\gamma(V) \to \mathcal{L}^2(K, \mathbb{L}^2(O))$ satisfy the following conditions.

(G11) For any $\eta \in C_\gamma(V)$, $g_i(\cdot, \eta)$ are measurable, $i = 1, 2$.

(G12) $g_i(\cdot, 0) = 0,\ i = 1, 2$.

(G13) There exists $L_{g_i} > 0$ ($i = 1, 2$) such that for all $t \in \mathbb{R}$ and $\eta, \zeta \in C_\gamma(V)$,

$$\|g_1(t,\eta) - g_1(t,\zeta)\|_{\mathbb{H}^{-1}(O)} \leqslant L_{g_1}\|\eta - \zeta\|_{C_\gamma(V)},$$
$$\|g_2(t,\eta) - g_2(t,\zeta)\|_{\mathcal{L}^2(K,\mathbb{L}^2(O))} \leqslant L_{g_2}\|\eta - \zeta\|_{C_\gamma(V)};$$

(G14) There exists $C_{g_i} > 0$ ($i = 1, 2$) such that for all $t \in \mathbb{R}$ and $\eta, \zeta \in C_\gamma(V)$,

$$\int_\tau^t \|g_1(s,u_s) - g_1(s,v_s)\|^2_{\mathbb{H}^{-1}(O)} ds \leqslant C^2_{g_1} \int_{-\infty}^t \|u(s) - v(s)\|^2 ds,$$
$$\int_\tau^t \|g_2(s,u_s) - g_2(s,v_s)\|^2_{\mathcal{L}^2(K,\mathbb{L}^2(O))} ds \leqslant C^2_{g_2} \int_{-\infty}^t \|u(s) - v(s)\|^2 ds;$$

(G15) There exists $\widetilde{C}_{g_i} > 0$ ($i = 1, 2$) such that for all $\tau \in \mathbb{R}, t \geqslant \tau$, all decreasing function $\varpi \in C^0([\tau,t])$ and $u, v \in C^0((-\infty,t];V)$,

$$\int_\tau^t \varpi(s)\|g_1(s,u_s) - g_1(s,v_s)\|^2_{\mathbb{H}^{-1}(O)} ds \leqslant \widetilde{C}_{g_1} \int_\tau^t \varpi(s)\|u(s) - v(s)\|^2 ds,$$
$$\int_\tau^t \varpi(s)\|g_2(s,u_s) - g_2(s,v_s)\|^2_{\mathcal{L}^2(K,\mathbb{L}^2(O))} ds \leqslant \widetilde{C}_{g_2} \int_\tau^t \varpi(s)\|u(s) - v(s)\|^2 ds.$$

We infer from (G12)-(G13) that, for all $\eta \in C_\gamma(V)$,

$$\|g_1(t,\eta)\|_{\mathbb{H}^{-1}(O)} \leqslant L_{g_1}\|\eta\|_{C_\gamma(V)}, \ \|g_2(t,\eta)\|_{\mathcal{L}^2(K,\mathbb{L}^2(O))} \leqslant L_{g_2}\|\eta\|_{C_\gamma(V)}.$$

Next, let us define $\widetilde{f}(t)$ as

$$((\widetilde{f}(t), w)) = \langle f(t), w\rangle_{-1}, \ \forall\ (t,w) \in \mathbb{R} \times V.$$

By the hypothesis **F**, $\widetilde{f} \in I^2(\tau, \tau + T; (D(A))^*)$ for any $\tau \in \mathbb{R}$ and $T > 0$.

In addition, we define $\widetilde{g}_1 : \mathbb{R} \times C_\gamma(V) \to V$ such that

$$((\widetilde{g}_1(t,\eta), w)) = \langle g_1(t,\eta), w\rangle_{-1}, \ \forall\ (t,\eta,w) \in \mathbb{R} \times C_\gamma(V) \times V.$$

Finally, we define $\widetilde{g}_2 : \mathbb{R} \times C_\gamma(V) \to \mathcal{L}^2(K,V)$ such that

$$\widetilde{g}_2(t,\eta) = (I + \alpha A)^{-1} \circ \mathcal{P} \circ g_2(t,\eta), \ \forall\ (t,\eta) \in \mathbb{R} \times C_\gamma(V),$$

where I is the identity operator in H and $I + \alpha A : D(A) \to H$ is bijective, moreover,

$$(((I + \alpha A)^{-1}u, w)) = (u, w), \ \forall\ u \in H, \ w \in V.$$

Hence, for the orthonormal basis $\{e_j\}$ of K, we have

$$(g_2(t,\eta)e_j, w) = ((I + \alpha A)\widetilde{g}_2(t,\eta)e_j, w) = ((\widetilde{g}_2(t,\eta)e_j, w)),$$

for all $j \geqslant 1$ and $(t, \eta, w) \in \mathbb{R} \times C_\gamma(V) \times D(A)$, by (0.0.31), we further obtain that

$$\begin{aligned}\Big(\int_\tau^t g_2(s,\eta)dW(s), w\Big) &= \sum_{j=1}^\infty \int_\tau^t (g_2(s,\eta)e_j, w)d\beta^j(s)\\ &= \sum_{j=1}^\infty \int_\tau^t ((\widetilde{g}_2(s,\eta)e_j, w))d\beta^j(s)\\ &= \Big(\Big(\int_\tau^t \widetilde{g}_2(s,\eta)dW(s), w\Big)\Big).\end{aligned} \tag{1.1.1}$$

By the same method as in [30], one can prove that $\widetilde{g}_1 : \mathbb{R} \times C_\gamma(V) \to V$ and $\widetilde{g}_2 : \mathbb{R} \times C_\gamma(V) \to \mathcal{L}^2(K, V)$ satisfy the following conditions:

(H11) For any $\eta \in C_\gamma(V)$, $\widetilde{g}_i(\cdot, \eta)$ are measurable, $i = 1, 2$;

(H12) $\widetilde{g}_i(\cdot, 0) = 0,\ i = 1, 2$;

(H13) Taking $L_{\widetilde{g}_1} = L_{g_1}, L_{\widetilde{g}_2} = L_{g_2}/\sqrt{1+\alpha\lambda_1}$, we deduce, for all $t \in \mathbb{R}$ and $\eta, \zeta \in C_\gamma(V)$,

$$\begin{aligned}&\|\widetilde{g}_1(t,\eta) - \widetilde{g}_1(t,\zeta)\| \leqslant L_{\widetilde{g}_1}\|\eta-\zeta\|_{C_\gamma(V)},\\ &\|\widetilde{g}_2(t,\eta) - \widetilde{g}_2(t,\zeta)\|_{\mathcal{L}^2(K,V)} \leqslant L_{\widetilde{g}_2}\|\eta-\zeta\|_{C_\gamma(V)};\end{aligned}$$

(H14) Setting $C_{\widetilde{g}_1} = C_{g_1}, C_{\widetilde{g}_2} = C_{g_2}/\sqrt{1+\alpha\lambda_1}$, we obtain, for all $t \in \mathbb{R}$ and $\eta, \zeta \in C_\gamma(V)$,

$$\begin{aligned}&\int_\tau^t \|\widetilde{g}_1(s,u_s) - \widetilde{g}_1(s,v_s)\|^2 ds \leqslant C_{\widetilde{g}_1}^2 \int_{-\infty}^t \|u(s)-v(s)\|^2 ds,\\ &\int_\tau^t \|\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)} ds \leqslant C_{\widetilde{g}_2}^2 \int_{-\infty}^t \|u(s)-v(s)\|^2 ds;\end{aligned}$$

(H15) Letting $\widetilde{C}_{\widetilde{g}_1} = \widetilde{C}_{g_1}, \widetilde{C}_{\widetilde{g}_2} = \widetilde{C}_{g_2}/\sqrt{1+\alpha\lambda_1}$ such that for all $\tau \in \mathbb{R}, t \geqslant \tau$, and all decreasing function $\varpi \in C^0([\tau,t])$ and $u, v \in C^0((-\infty,t]; V)$,

$$\begin{aligned}&\int_\tau^t \varpi(s)\|\widetilde{g}_1(s,u_s) - \widetilde{g}_1(s,v_s)\|^2 ds \leqslant \widetilde{C}_{\widetilde{g}_1} \int_\tau^t \varpi(s)\|u(s)-v(s)\|^2 ds,\\ &\int_\tau^t \varpi(s)\|\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)} ds \leqslant \widetilde{C}_{\widetilde{g}_2} \int_\tau^t \varpi(s)\|u(s)-v(s)\|^2 ds.\end{aligned}$$

It follows from (H12)-(H13) that for all $t \in \mathbb{R}$ and $\eta \in C_\gamma(V)$,

$$\|\widetilde{g}_1(t,\eta)\| \leqslant L_{\widetilde{g}_1}\|\eta\|_{C_\gamma(V)},\ \|\widetilde{g}_2(t,\eta)\|_{\mathcal{L}^2(K,V)} \leqslant L_{\widetilde{g}_2}\|\eta\|_{C_\gamma(V)}. \tag{1.1.2}$$

An example of the delayed terms with (H11)-(H15) is given as follows.

Example 1: For all $t \in \mathbb{R}$ and $\xi \in C_\gamma(V)$, let

$$\widetilde{g}_i(t,\xi) = \int_{-\infty}^0 \widetilde{G}_i(t,s,\xi(s))ds,\ i = 1, 2, \tag{1.1.3}$$

where $\widetilde{G}_1 : \mathbb{R} \times (-\infty, 0] \times V \to V$ with $\widetilde{G}_1(t, s, 0) = 0$, and $\widetilde{G}_2 : \mathbb{R} \times (-\infty, 0] \times V \to \mathcal{L}^2(K, V)$ with $\widetilde{G}_2(t, s, 0) = 0$, and both are measurable. Assume that there exist $L_{\widetilde{G}_i} \in L^2(-\infty, 0)$ $(i = 1, 2)$ with $L_{\widetilde{G}_i}(\cdot)e^{-(\gamma+\theta)\cdot} \in L^2(-\infty, 0)$ for certain $\theta > 0$ such that for all $t \in \mathbb{R}$, $s \in (-\infty, 0]$ and $\eta, \zeta \in V$,

$$\|\widetilde{G}_1(t, s, \eta) - \widetilde{G}_1(t, s, \zeta)\| \leqslant L_{\widetilde{G}_1}(s)\|\eta - \zeta\|,$$
$$\|\widetilde{G}_2(t, s, \eta) - \widetilde{G}_2(t, s, \zeta)\|_{\mathcal{L}^2(K,V)} \leqslant L_{\widetilde{G}_2}(s)\|\eta - \zeta\|.$$

Thus, we can rewrite the delay terms $\widetilde{g}_i$ $(i = 1, 2)$ in our problem as $\widetilde{g}_i(t, u_t) = \int_{-\infty}^{0} \widetilde{G}_i(t, s, u(t+s))ds$ $(i = 1, 2)$. It follows that the example is within our framework, and $\widetilde{g}_i (i = 1, 2)$ fulfill conditions (H11)-(H15) (e.g., see [100] for more details).

1.2 Well-posedness of stochastic 3D LANS equations with infinite delay

In this section, we prove the well-posedness of the stochastic Eq. (0.0.3), which can be transferred into the following abstract equation:

$$\begin{cases} \dfrac{du}{dt} + \widetilde{A}u(t) + \widetilde{B}(u(t)) = \widetilde{f}(t) + \widetilde{g}_1(t, u_t) + \widetilde{g}_2(t, u_t)\dfrac{dW}{dt}, \quad \forall\, t > \tau, \\ u(\tau + s) = \phi(s), \quad s \in (-\infty, 0]. \end{cases} \tag{1.2.1}$$

Definition 1.2.1. *Suppose that $\phi \in L^2(\Omega, C_\gamma(V))$ (which is a $\mathcal{F}_t$-progressively measurable V-valued processes, $t \leqslant 0$) and $\tau \in \mathbb{R}$. A stochastic process u defined on $\mathbb{R}$ is called a solution to system (1.2.1) if*

$$u \in I^2(\tau, \tau + T; D(A)) \cap L^2(\Omega, L^\infty(\tau, \tau + T; V)), \ \forall\, T > 0,$$

$u_\tau = \phi$ and $\mathbb{P}$-almost surely

$$\begin{aligned} &((u(t), w)) + \int_\tau^t \langle \widetilde{A}u(s), w\rangle ds + \int_\tau^t \langle \widetilde{B}(u(s)), w\rangle ds \\ &\quad = ((\phi(0), w)) + \int_\tau^t \Big(\Big(\widetilde{f}(s) + \widetilde{g}_1(s, u_s), w\Big)\Big) ds + \Big(\Big(w, \int_\tau^t \widetilde{g}_2(s, u_s) dW(s)\Big)\Big) \end{aligned} \tag{1.2.2}$$

for all $t \geqslant \tau$ and $w \in D(A)$.

Lemma 1.2.2. *For all $u, v \in D(A)$, we have*

$$\langle -\widetilde{A}\overline{w} - 2(\widetilde{B}(u) - \widetilde{B}(v)), \overline{w}\rangle \leqslant \sigma\|\overline{w}\|^2\|v\|^2_{D(A)}, \tag{1.2.3}$$

where $\overline{w} = u - v$ and $\sigma = \widetilde{c}^2$.

Proof. Note that

$$\langle -\widetilde{A}\overline{w}, \overline{w}\rangle = -\|\overline{w}\|^2_{D(A)}. \tag{1.2.4}$$

By the property (B1) of the operator $\widetilde{B}$, we have

$$\langle \widetilde{B}(u), \overline{w}\rangle = -\langle \widetilde{B}(\overline{w}, u), u\rangle = -\langle \widetilde{B}(\overline{w}, u), v\rangle, \tag{1.2.5}$$

and similarly

$$\langle \widetilde{B}(v), \overline{w}\rangle = -\langle \widetilde{B}(\overline{w}, v), v\rangle. \tag{1.2.6}$$

Subtracting (1.2.6) from (1.2.5),

$$\langle \widetilde{B}(u) - \widetilde{B}(v), \overline{w}\rangle = -\langle \widetilde{B}(\overline{w}), v\rangle, \tag{1.2.7}$$

which, together with (B2), implies that

$$\begin{aligned} |\langle \widetilde{B}(u) - \widetilde{B}(v), \overline{w}\rangle| &= |\langle \widetilde{B}(\overline{w}), v\rangle| \\ &\leqslant \|\widetilde{B}(\overline{w})\|_{(D(A))^*}\|v\|_{D(A)} \\ &\leqslant \widetilde{c}\|\overline{w}\|\|\overline{w}\|_{D(A)}\|v\|_{D(A)} \\ &\leqslant \frac{1}{2}\|\overline{w}\|^2_{D(A)} + \frac{\widetilde{c}^2}{2}\|\overline{w}\|^2\|v\|^2_{D(A)}. \end{aligned} \tag{1.2.8}$$

Combining (1.2.4) and (1.2.8), we obtain (1.2.3) as desired. □

In the following, we present the well-posedness of problem (1.2.1). For this end, we further assume

Hypothesis I. For all $u, v \in L^2(-\infty, \tau + T; D(A))$ and $t \in [\tau, \tau + T]$, Eq. (1.2.1) satisfies

$$\begin{aligned} &\int_\tau^t \|\widetilde{g}_2(s, u_s) - \widetilde{g}_2(s, v_s)\|^2_{\mathcal{L}^2(K,V)} ds + 2\int_\tau^t ((\widetilde{g}_1(s, u_s) - \widetilde{g}_1(s, v_s), u(s) - v(s)))ds \\ &\leqslant \sigma \int_\tau^t \|v(s)\|^2_{D(A)}\|u(s) - v(s)\|^2 ds \\ &\quad + 2\int_\tau^t \langle \widetilde{A}(u(s) - v(s)) + \widetilde{B}(u(s)) - \widetilde{B}(v(s)), u(s) - v(s)\rangle ds, \end{aligned} \tag{1.2.9}$$

where σ is given by (1.2.3) in Lemma 1.2.2.

Remark 1.2.3. *Let*

$$C^2_{\widetilde{g}_2} + \frac{2}{\widetilde{\lambda}_1}C^2_{\widetilde{g}_1} \leqslant \frac{\widetilde{\lambda}_1}{2} \text{ and } u(s + \tau) = v(s + \tau) = \phi(s), \ \ s \leqslant 0. \tag{1.2.10}$$

Then (1.2.9) in hypothesis I is satisfied. Indeed, by (0.0.42) and Lemma 1.2.2, we only need to prove that the following inequality holds:

$$\int_\tau^t \|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds + 2\int_\tau^t ((\widetilde{g}_1(s,u_s)-\widetilde{g}_1(s,v_s),u(s)-v(s)))ds \leqslant \widetilde{\lambda}_1 \int_\tau^t \|u(s)-v(s)\|^2 ds. \tag{1.2.11}$$

The Young inequality, (H14) and (1.2.10) imply

$$\begin{aligned}
&\int_\tau^t \|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds + 2\int_\tau^t ((\widetilde{g}_1(s,u_s)-\widetilde{g}_1(s,v_s),u(s)-v(s)))ds\\
&\leqslant C^2_{\widetilde{g}_2}\int_{-\infty}^t \|u(s)-v(s)\|^2 ds + \frac{2}{\widetilde{\lambda}_1}\int_\tau^t \|\widetilde{g}_1(s,u_s)-\widetilde{g}_1(s,v_s)\|^2 ds + \frac{\widetilde{\lambda}_1}{2}\int_\tau^t \|u(s)-v(s)\|^2 ds\\
&\leqslant \Big(C^2_{\widetilde{g}_2} + \frac{2}{\widetilde{\lambda}_1}C^2_{\widetilde{g}_1}\Big)\int_{-\infty}^t \|u(s)-v(s)\|^2 ds + \frac{\widetilde{\lambda}_1}{2}\int_\tau^t \|u(s)-v(s)\|^2 ds\\
&= \Big(C^2_{\widetilde{g}_2} + \frac{2}{\widetilde{\lambda}_1}C^2_{\widetilde{g}_1} + \frac{\widetilde{\lambda}_1}{2}\Big)\int_\tau^t \|u(s)-v(s)\|^2 ds\\
&\leqslant \widetilde{\lambda}_1 \int_\tau^t \|u(s)-v(s)\|^2 ds,
\end{aligned} \tag{1.2.12}$$

which implies (1.2.11) as desired.

Theorem 1.2.4. *Suppose that hypotheses* **F, G, I** *hold, moreover, $\phi \in L^4(\Omega, C_\gamma(V))$ and $\widetilde{f} \in I^4(\tau,\tau+T;V)$, then there exists a unique solution u to (1.2.1), which satisfies in addition,*

$$u \in I^4(\tau,\tau+T;V)\cap L^4(\Omega, L^\infty(\tau,\tau+T;V)). \tag{1.2.13}$$

In fact, there exists a positive constant R depending on T, $\mathbb{E}(\|\phi\|^4_{C_\gamma(V)})$ and $\mathbb{E}(\int_\tau^{\tau+T}\|\widetilde{f}(s)\|^4 ds)$ such that

$$\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant \tau+T}\|u_r\|^4_{C_\gamma(V)}\Big) + \mathbb{E}\Big(\int_\tau^{\tau+T}\|u(s)\|^4 ds\Big) \leqslant R. \tag{1.2.14}$$

Proof. We split the proof into several steps as follows.

Step 1: *We use the Galerkin method to construct an approximating sequence*. Consider the Hilbert basis $\{w_j; j\in\mathbb{N}\}\subset D(A)$ of V such that $\widetilde{A}w_j = \widetilde{\lambda}_j w_j$, $\forall\ j\geqslant 1$, denote by V_m the linear space spanned by $\{w_1, w_2, \ldots, w_m\}$ for $m\in\mathbb{N}$, and then put

$$u^m(t) = \sum_{j=1}^m a_{m,j}(t)w_j, \tag{1.2.15}$$

where $a_{m,j}(t)$ $(j = 1, \cdots, m)$ will be obtained as the solution of the following finite dimensional system:

$$\begin{cases} ((u^m(t), w_j)) + \int_\tau^t \langle \widetilde{A} u^m(s), w_j \rangle ds + \int_\tau^t \langle \widetilde{B}(u^m(s)), w_j \rangle ds \\ \quad = ((u^m(\tau), w_j)) + \int_\tau^t \Big(\big(\widetilde{f}(s) + \widetilde{g}_1(s, u_s^m), w_j \big) \Big) ds \\ \qquad + \Big(\Big(w_j, \int_\tau^t \widetilde{g}_2(s, u_s^m) dW(s) \Big) \Big), \quad \forall\, t \in [\tau, \tau + T], \ j \in [1, m], \ \mathbb{P}\text{-a.s.} \\ u^m(\tau + s) = \mathcal{P}_m \phi(s), \ \forall\, s \in (-\infty, 0], \end{cases} \tag{1.2.16}$$

where $\mathcal{P}_m : V \to V_m$ is the projector.

By the similar argument in [13], for each $m \in \mathbb{N}$, the stochastic ODE (1.2.16) possesses a (local) solution $\{a_{m,j}(\cdot)\}_{j=1}^m$ in $[\tau, t_m)$ with $\tau < t_m$ (by the initial condition, the value of $a_{m,j}(\cdot)$ in $(-\infty, \tau]$ is well defined). From this, $u^m(\cdot)$ is well-defined in $[\tau, t_m)$ (and thus in $(-\infty, t_m)$). Next, we will give a priori estimates to ensure that the solutions u^m is global, i.e. $t_m = +\infty$.

Step 2: *We give a-priori estimates for the approximating sequence.* We first claim that, for any $T > 0$, the following inequality holds:

$$\mathbb{E}\Big(\sup_{\tau \leqslant r \leqslant \tau + T} \|u_r^m\|_{C_\gamma(V)}^2 \Big) + \mathbb{E}\Big(\int_\tau^{\tau+T} \|u^m(s)\|_{D(A)}^2 ds \Big) \leqslant R_1, \tag{1.2.17}$$

where R_1 is a positive constant depending on T, $\mathbb{E}(\|\phi\|_{C_\gamma(V)}^2)$ and $\mathbb{E}(\int_\tau^{\tau+T} \|\widetilde{f}(s)\|^2 ds)$.

Indeed, multiplying (1.2.16) by $a_{m,j}$, summing those relations for $j = 1, \cdots, m$ and applying Ito's formula to $\|u^m(t)\|^2$, we obtain that

$$\begin{aligned} \|u^m(t)\|^2 + 2\int_\tau^t \|u^m(s)\|_{D(A)}^2 ds = \|u^m(\tau)\|^2 + 2\int_\tau^t \Big(\big(\widetilde{f}(s) + \widetilde{g}_1(s, u_s^m), u^m(s) \big) \Big) ds \\ + \int_\tau^t \|\widetilde{g}_2(s, u_s^m)\|_{\mathcal{L}^2(K,V)}^2 ds + 2\int_\tau^t \Big(\big(u^m(s), \widetilde{g}_2(s, u_s^m) dW(s) \big) \Big). \end{aligned} \tag{1.2.18}$$

By (1.1.2), we can rewrite (1.2.18) as

$$\begin{aligned} \|u^m(t)\|^2 + 2\int_\tau^t \|u^m(s)\|_{D(A)}^2 ds &\leqslant \|\phi(0)\|^2 + 2\int_\tau^t \Big(\big(\widetilde{f}(s) + \widetilde{g}_1(s, u_s^m), u^m(s) \big) \Big) ds \\ &\quad + \int_\tau^t \|\widetilde{g}_2(s, u_s^m)\|_{\mathcal{L}^2(K,V)}^2 ds + 2\Big(\Big(\int_\tau^t u^m(s), \widetilde{g}_2(s, u_s^m) dW(s) \Big) \Big) \\ &\leqslant \|\phi(0)\|^2 + 2\int_\tau^t \|\widetilde{f}(s) + \widetilde{g}_1(s, u_s^m)\| \|u^m(s)\| ds \\ &\quad + L_{\widetilde{g}_2}^2 \int_\tau^t \|u_s^m\|_{C_\gamma(V)}^2 ds + 2\Big| \Big(\Big(\int_\tau^t u^m(s), \widetilde{g}_2(s, u_s^m) dW(s) \Big) \Big) \Big|. \end{aligned} \tag{1.2.19}$$

By (0.0.42), the Young inequality and (1.1.2), we find

$$
\begin{aligned}
&2\int_\tau^t \|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|\|u^m(s)\|ds\\
&\leqslant 2\widetilde{\lambda}_1^{-\frac{1}{2}}\int_\tau^t \|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|\|u^m(s)\|_{D(A)}ds\\
&\leqslant 2\widetilde{\lambda}_1^{-1}\int_\tau^t \|\widetilde{f}(s)\|^2ds+2\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}^2\int_\tau^t \|u_s^m\|_{C_\gamma(V)}^2ds+\int_\tau^t \|u^m(s)\|_{D(A)}^2ds.
\end{aligned}
\tag{1.2.20}
$$

Substituting (1.2.20) into (1.2.19), we obtain

$$
\begin{aligned}
\|u^m(t)\|^2+\int_\tau^t \|u^m(s)\|_{D(A)}^2ds &\leqslant \|\phi(0)\|^2+2\widetilde{\lambda}_1^{-1}\int_\tau^t \|\widetilde{f}(s)\|^2ds+c_1\int_\tau^t \|u_s^m\|_{C_\gamma(V)}^2ds\\
&\quad+2\Big|\int_\tau^t \Big(\Big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|,
\end{aligned}
\tag{1.2.21}
$$

where $c_1=L_{\widetilde{g}_2}^2+2\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}^2$. The above inequality implies

$$
\begin{aligned}
\|u_t^m\|_{C_\gamma(V)}^2 &\leqslant \max\Big\{\sup_{\theta\leqslant\tau-t} e^{2\gamma\theta}\|u^m(t+\theta)\|^2,\ \sup_{\tau-t\leqslant\theta\leqslant 0} e^{2\gamma\theta}\|u^m(t+\theta)\|^2\Big\}\\
&\leqslant \max\Big\{\sup_{\theta\leqslant\tau-t} e^{2\gamma\theta}\|\phi(t+\theta-\tau)\|^2,\ \sup_{\tau-t\leqslant\theta\leqslant 0} e^{2\gamma\theta}\|u^m(t+\theta)\|^2\Big\}\\
&\leqslant \max\Bigg\{\sup_{\theta\leqslant 0} e^{2\gamma(\theta-t+\tau)}\|\phi(\theta)\|^2,\ \sup_{\tau-t\leqslant\theta\leqslant 0} e^{2\gamma\theta}\Big(\|\phi(0)\|^2+2\widetilde{\lambda}_1^{-1}\int_\tau^{t+\theta}\|\widetilde{f}(s)\|^2ds\\
&\quad+c_1\int_\tau^{t+\theta}\|u_s^m\|_{C_\gamma(V)}^2ds+2\Big|\int_\tau^{t+\theta}\Big(\Big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|\Big)\Bigg\}\\
&\leqslant e^{-2\gamma(t-\tau)}\|\phi\|_{C_\gamma(V)}^2+\|\phi\|_{C_\gamma(V)}^2+2\widetilde{\lambda}_1^{-1}\int_\tau^t\|\widetilde{f}(s)\|^2ds+c_1\int_\tau^t\|u_s^m\|_{C_\gamma(V)}^2ds\\
&\quad+2\sup_{\tau-t\leqslant\theta\leqslant 0} e^{2\gamma\theta}\Big|\int_\tau^{t+\theta}\Big(\Big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|.
\end{aligned}
\tag{1.2.22}
$$

Taking supremum and expectation of (1.2.22), we find

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|_{C_\gamma(V)}^2\Big) &\leqslant 2\mathbb{E}\Big(\|\phi\|_{C_\gamma(V)}^2\Big)+2\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^t\|\widetilde{f}(s)\|^2ds\Big)+c_1\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant s}\|u_r^m\|_{C_\gamma(V)}^2\Big)ds\\
&\quad+2\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\sup_{\tau-r\leqslant\theta\leqslant 0} e^{2\gamma\theta}\Big|\int_\tau^{r+\theta}\Big(\Big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|\Big).
\end{aligned}
\tag{1.2.23}
$$

By the Burkholder-Davis-Gundy inequality and (1.1.2), the last term of (1.2.23) is bounded by

$$
2\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\sup_{\tau-r\leqslant\theta\leqslant 0} e^{2\gamma\theta}\Big|\int_\tau^{r+\theta}\Big(\Big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\Big)\Big)\Big|\Big)
$$

$$
\begin{aligned}
&\leqslant 2\mathbb{E}\Big(\sup_{\tau\leqslant r+\theta\leqslant t}\Big|\int_\tau^{r+\theta}\Big(\big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\big)\Big)\Big|\Big)\\
&\leqslant 2c_1\mathbb{E}\Big(\Big(\int_\tau^t\|u^m(s)\|^2\|\widetilde{g}_2(s,u_s^m)\|^2_{\mathcal{L}^2(K,V)}ds\Big)^{\frac12}\Big)\\
&\leqslant 2c_1\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|_{C_\gamma(V)}\Big(\int_\tau^t\|\widetilde{g}_2(s,u_s^m)\|^2ds\Big)^{\frac12}\Big)\\
&\leqslant \frac12\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|^2_{C_\gamma(V)}\Big)+2c_1^2L^2_{\widetilde{g}_2}\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant s}\|u_r^m\|^2_{C_\gamma(V)}\Big)ds.
\end{aligned}
\tag{1.2.24}
$$

It follows from (1.2.23)-(1.2.24) that, for all $t\in[\tau,\tau+T]$,

$$
\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|^2_{C_\gamma(V)}\Big)\leqslant 4\mathbb{E}\Big(\|\phi\|^2_{C_\gamma(V)}\Big)+4\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^t\|\widetilde{f}(s)\|^2ds\Big)+c_2\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant s}\|u_r^m\|^2_{C_\gamma(V)}\Big)ds, \tag{1.2.25}
$$

where $c_2=2c_1(1+2c_1L^2_{\widetilde{g}_2})$. Set

$$
c_T:=4\mathbb{E}\Big(\|\phi\|^2_{C_\gamma(V)}\Big)+4\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^{\tau+T}\|\widetilde{f}(s)\|^2ds\Big), \tag{1.2.26}
$$

which is finite due to $\phi\in L^4(\Omega,C_\gamma(V))$ and $\widetilde{f}\in I^4(\tau,\tau+T;V)$. Applying the Gronwall lemma to (1.2.25), we find, for all $t\in[\tau,\tau+T]$,

$$
\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|^2_{C_\gamma(V)}\Big)\leqslant c_Te^{c_2T}=:R_{11}. \tag{1.2.27}
$$

Finally, we infer from (1.2.21) and (1.2.24) that, for all $t\in[\tau,\tau+T]$,

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\int_\tau^r\|u^m(s)\|^2_{D(A)}ds\Big)&\leqslant\mathbb{E}\Big(\|\phi(0)\|^2\Big)+2\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^{\tau+T}\|\widetilde{f}(s)\|^2ds\Big)\\
&\quad+\frac{c_2}{2}\int_\tau^{\tau+T}\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant s}\|u_r^m\|^2_{C_\gamma(V)}\Big)ds+\frac12\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|^2_{C_\gamma(V)}\Big),
\end{aligned}
\tag{1.2.28}
$$

which, together with (1.2.27), implies that there exists a positive constant R_{12},

$$
\mathbb{E}\Big(\int_\tau^t\|u^m(s)\|^2_{D(A)}ds\Big)\leqslant R_{12},\ \forall\, m. \tag{1.2.29}
$$

Combining (1.2.27) and (1.2.29), we obtain (1.2.17) for $R_1=R_{11}+R_{12}$.

We also need to give the following estimate:

$$
\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant\tau+T}\|u_r^m\|^4_{C_\gamma(V)}\Big)+\mathbb{E}\Big(\int_\tau^{\tau+T}\|u^m(s)\|^4ds\Big)\leqslant R, \tag{1.2.30}
$$

where R depends on T, $\mathbb{E}(\|\phi\|^4_{C_\gamma(V)})$ and $\mathbb{E}(\int_\tau^{\tau+T}\|\widetilde{f}(s)\|^4ds)$.

Indeed, by (1.2.18) and Ito's formula, we infer from (1.1.2) that

$$
\begin{aligned}
&\|u^m(t)\|^4 + 4\int_\tau^t \|u^m(s)\|^2\|u^m(s)\|^2_{D(A)}ds \\
&\leqslant \|\phi(0)\|^4 + 4\int_\tau^t \|u^m(s)\|^2\Big(\big(\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m),u^m(s)\big)\Big)ds + 2\int_\tau^t \|u^m(s)\|^2\|\widetilde{g}_2(s,u_s^m)\|^2_{\mathcal{L}^2(K,V)}ds \\
&\quad + 4\int_\tau^t \|\widetilde{g}_2^*(s,u_s^m)u^m(s)\|_K^2 ds + 4\int_\tau^t \|u^m(s)\|^2\Big(\big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\big)\Big) \\
&\leqslant \|\phi(0)\|^4 + 4\int_\tau^t \|u^m(s)\|^3\|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|ds \\
&\quad + 6\int_\tau^t \|\widetilde{g}_2(s,u_s^m)\|^2_{\mathcal{L}^2(K,V)}\|u^m(s)\|^2ds + 4\int_\tau^t \|u^m(s)\|^2\Big(\big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\big)\Big) \\
&\leqslant \|\phi(0)\|^4 + 4\int_\tau^t \|u^m(s)\|^3\|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|ds \\
&\quad + 3L^4_{\widetilde{g}_2}\int_\tau^t \|u_s^m\|^4_{C_\gamma(V)}ds + 3\int_\tau^t \|u^m(s)\|^4ds + 4\int_\tau^t \|u^m(s)\|^2\Big(\big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\big)\Big), \qquad (1.2.31)
\end{aligned}
$$

where $\widetilde{g}_2^*$ is the adjoint operator of $\widetilde{g}_2$. By (0.0.42), the Young inequality and (1.1.2), we obtain

$$
\begin{aligned}
&4\int_\tau^t \|u^m(s)\|^3\|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|ds \\
&\leqslant 4\widetilde{\lambda}_1^{-\frac{1}{2}}\int_\tau^t \|u^m(s)\|^2\|u^m(s)\|_{D(A)}\|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|ds \\
&\leqslant 2\widetilde{\lambda}_1^{-1}\int_\tau^t \|u^m(s)\|^2\|\widetilde{f}(s)+\widetilde{g}_1(s,u_s^m)\|^2ds + 2\int_\tau^t \|u^m(s)\|^2\|u^m(s)\|^2_{D(A)}ds \\
&\leqslant 4\widetilde{\lambda}_1^{-1}\int_\tau^t \|u^m(s)\|^2\|\widetilde{f}(s)\|^2ds + 4\widetilde{\lambda}_1^{-1}L^2_{\widetilde{g}_1}\int_\tau^t \|u^m(s)\|^2\|u_s^m\|^2_{C_\gamma(V)}ds \\
&\quad + 2\int_\tau^t \|u^m(s)\|^2\|u^m(s)\|^2_{D(A)}ds \\
&\leqslant 2\widetilde{\lambda}_1^{-1}\int_\tau^t \|\widetilde{f}(s)\|^4ds + c_3\int_\tau^t \|u^m(s)\|^4ds + 2\widetilde{\lambda}_1^{-1}L^2_{\widetilde{g}_1}\int_\tau^t \|u_s^m\|^4_{C_\gamma(V)}ds \\
&\quad + 2\int_\tau^t \|u^m(s)\|^2\|u^m(s)\|^2_{D(A)}ds, \qquad (1.2.32)
\end{aligned}
$$

where $c_3 = 2\widetilde{\lambda}_1^{-1}(1+L^2_{\widetilde{g}_1})$. Substituting (1.2.32) into (1.2.31),

$$
\begin{aligned}
&\|u^m(t)\|^4 + 2\int_\tau^t \|u^m(s)\|^2\|u^m(s)\|^2_{D(A)}ds \\
&\leqslant \|\phi(0)\|^4 + 2\widetilde{\lambda}_1^{-1}\int_\tau^t \|\widetilde{f}(s)\|^4ds + c_4\int_\tau^t \|u^m(s)\|^4ds + c_5\int_\tau^t \|u_s^m\|^4_{C_\gamma(V)}ds
\end{aligned}
$$

$$+4\int_\tau^t \|u^m(s)\|^2\Big(\big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\big)\Big), \tag{1.2.33}$$

where $c_4 = c_3 + 3, c_5 = 3L_{\widetilde{g}_2}^4 + 2\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}^2$. By (1.2.33), we find

$$\begin{aligned}
\|u_t^m\|_{C_\gamma(V)}^4 &\leqslant \max\Big\{\sup_{\theta\leqslant\tau-t} e^{4\gamma\theta}\|u^m(t+\theta)\|^4, \sup_{\tau-t\leqslant\theta\leqslant 0} e^{4\gamma\theta}\|u^m(t+\theta)\|^4\Big\}\\
&\leqslant \max\Bigg\{\sup_{\theta\leqslant\tau-t} e^{4\gamma\theta}\|\phi(t+\theta-\tau)\|^4, \sup_{\tau-t\leqslant\theta\leqslant 0} e^{4\gamma\theta}\Big(\|\phi(0)\|^4 + 2\widetilde{\lambda}_1^{-1}\int_\tau^{t+\theta}\|\widetilde{f}(s)\|^4 ds\\
&\quad + c_4\int_\tau^{t+\theta}\|u^m(s)\|^4 ds + c_5\int_\tau^{t+\theta}\|u_s^m\|_{C_\gamma(V)}^4 ds\\
&\quad + 4\Big|\int_\tau^{t+\theta}\|u^m(s)\|^2\Big(\big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\big)\Big)\Big|\Big)\Bigg\}\\
&\leqslant e^{-4\gamma(t-\tau)}\|\phi\|_{C_\gamma(V)}^4 + \|\phi\|_{C_\gamma(V)}^4 + 2\widetilde{\lambda}_1^{-1}\int_\tau^t\|\widetilde{f}(s)\|^4 ds + c_4\int_\tau^t\|u^m(s)\|^4 ds\\
&\quad + c_5\int_\tau^t\|u_s^m\|_{C_\gamma(V)}^4 ds + 4\sup_{\tau-t\leqslant\theta\leqslant 0} e^{4\gamma\theta}\Big|\int_\tau^{t+\theta}\|u^m(s)\|^2\Big(\big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\big)\Big)\Big|.
\end{aligned} \tag{1.2.34}$$

Taking supremum and expectation of (1.2.34), we infer

$$\begin{aligned}
\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|_{C_\gamma(V)}^4\Big) &\leqslant 2\mathbb{E}\Big(\|\phi\|_{C_\gamma(V)}^4\Big) + 2\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^t\|\widetilde{f}(s)\|^4 ds\Big) + c_6\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant s}\|u_r^m\|_{C_\gamma(V)}^4\Big)ds\\
&\quad + 4\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\sup_{\tau-r\leqslant\theta\leqslant 0} e^{4\gamma\theta}\Big|\int_\tau^{r+\theta}\|u^m(s)\|^2\Big(\big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\big)\Big)\Big|\Big),
\end{aligned} \tag{1.2.35}$$

where $c_6 = c_4 + c_5$. By the Burkholder-Davis-Gundy inequality and (1.1.2), the last term of (1.2.35) satisfies

$$\begin{aligned}
&4\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\sup_{\tau-r\leqslant\theta\leqslant 0} e^{4\gamma\theta}\Big|\int_\tau^{r+\theta}\|u^m(s)\|^2\Big(\big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\big)\Big)\Big|\Big)\\
&\leqslant 4\mathbb{E}\Big(\sup_{\tau\leqslant r+\theta\leqslant t}\Big|\int_\tau^{r+\theta}\|u^m(s)\|^2\Big(\big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\big)\Big)\Big|\Big)\\
&\leqslant 4\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|_{C_\gamma(V)}^2\sup_{\tau\leqslant r+\theta\leqslant t}\Big|\int_\tau^{r+\theta}\Big(\big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\big)\Big)\Big|\Big)\\
&\leqslant \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|_{C_\gamma(V)}^4\Big) + 8\mathbb{E}\Big(\sup_{\tau\leqslant r+\theta\leqslant t}\Big|\int_\tau^{r+\theta}\Big(\big(u^m(s),\widetilde{g}_2(s,u_s^m)dW(s)\big)\Big)\Big|^2\Big)\\
&\leqslant \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|_{C_\gamma(V)}^4\Big) + 8c_7\mathbb{E}\Big(\int_\tau^t\|u^m(s)\|^2\|\widetilde{g}_2(s,u_s^m)\|_{\mathcal{L}^2(K,V)}^2 ds\Big)\\
&\leqslant \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|_{C_\gamma(V)}^4\Big) + 4c_7\mathbb{E}\Big(\int_\tau^t\|u^m(s)\|^4 ds\Big) + 4c_7L_{\widetilde{g}_2}^4\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant s}\|u_r^m\|_{C_\gamma(V)}^4\Big)ds
\end{aligned}$$

$$\leqslant \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|^4_{C_\gamma(V)}\Big) + c_8\int_\tau^t \mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant s}\|u_r^m\|^4_{C_\gamma(V)}\Big)ds, \tag{1.2.36}$$

where $c_8 = 4c_7(1 + L^4_{g_2})$. Substituting (1.2.36) into (1.2.35), we obtain

$$\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|^4_{C_\gamma(V)}\Big) \leqslant 4\mathbb{E}\Big(\|\phi\|^4_{C_\gamma(V)}\Big) + 4\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^t \|\widetilde{f}(s)\|^4 ds\Big) + c_9\int_\tau^t \mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant s}\|u_r^m\|^4_{C_\gamma(V)}\Big)ds, \tag{1.2.37}$$

where $c_9 = 2(c_6 + c_8)$. Setting

$$c^* := 4\mathbb{E}\Big(\|\phi\|^4_{C_\gamma(V)}\Big) + 4\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^{\tau+T} \|\widetilde{f}(s)\|^4 ds\Big), \tag{1.2.38}$$

then applying the Gronwall lemma to (1.2.37), we deduce that

$$\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|^4_{C_\gamma(V)}\Big) \leqslant c^* e^{c_9 T} =: R_{21},\ \forall\, t\in[\tau,\tau+T]. \tag{1.2.39}$$

It follows from (0.0.42), (1.2.33) and (1.2.36) that for all $t\in[\tau,\tau+T]$ such that

$$\begin{aligned}
2\widetilde{\lambda}_1\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\int_\tau^r \|u^m(s)\|^4 ds\Big) &\leqslant 2\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\int_\tau^r \|u^m(s)\|^2\|u^m(s)\|^2_{D(A)} ds\Big)\\
&\leqslant \mathbb{E}\Big(\|\phi(0)\|^4\Big) + 2\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_\tau^t \|\widetilde{f}(s)\|^4 ds\Big)\\
&\quad + \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|^4_{\mathbb{C}_\gamma(V)}\Big) + \frac{c_9}{2}\int_\tau^t \mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant s}\|u_r^m\|^4_{\mathbb{C}_\gamma(V)}\Big)ds,
\end{aligned} \tag{1.2.40}$$

which, together with (1.2.39), $\phi \in L^4(\Omega, C_\gamma(V))$ and $\widetilde{f} \in I^4(\tau,\tau+T;V)$, implies that there exists a positive constant R_{22} such that

$$\mathbb{E}\Big(\int_\tau^t \|u^m(s)\|^4 ds\Big) \leqslant R_{22},\ \forall\, t\in[\tau,\tau+T]. \tag{1.2.41}$$

Combining (1.2.39) and (1.2.41), we obtain

$$\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|u_r^m\|^4_{C_\gamma(V)}\Big) + \mathbb{E}\Big(\int_\tau^t \|u^m(s)\|^4 ds\Big) \leqslant R := R_{21} + R_{22},\ \forall\, t\in[\tau,\tau+T], \tag{1.2.42}$$

which implies (1.2.30) as desired.

Step 3: *We prove the existence of solutions to Eq.* (1.2.1). Indeed, by Step 2,

$$\begin{aligned}
&u^m \text{ is bounded in } L^4(\Omega, L^\infty(\tau,\tau+T;V)) \cap I^4(\tau,\tau+T;V) \cap I^2(\tau,\tau+T;D(A)),\\
&u^m(\tau+T) \text{ is bounded in } L^2(\Omega;V).
\end{aligned}$$

By (B2), (1.2.39) and (1.2.40), $\widetilde{B}(u^m)$ is bounded in $I^2(\tau,\tau+T;(D(A))^*)$. Moreover, by (1.1.2) and Step 2,

$$\widetilde{g}_1(t,u_t^m) \text{ is bounded in } I^2(\tau,\tau+T;V),$$

$$\widetilde{g}_2(t, u_t^m) \text{ is bounded in } I^2(\tau, \tau + T; \mathcal{L}^2(K, V)).$$

Thus, there exists a subsequence u^m (still denoted by itself) and five elements

$$u \in L^4(\Omega, L^\infty(\tau, \tau + T; V)) \cap I^4(\tau, \tau + T; V) \cap I^2(\tau, \tau + T; D(A)),$$

$\mu \in L^2(\Omega; V)$, $\iota \in I^2(\tau, \tau + T; (D(A))^*)$, $\kappa_1 \in I^2(\tau, \tau + T; V)$ and $\kappa_2 \in I^2(\tau, \tau + T; \mathcal{L}^2(K; V))$ such that

$$\begin{aligned}
&u^m \overset{*}{\rightharpoonup} u \text{ in } L^4(\Omega, L^\infty(\tau, \tau + T; V)),\\
&u^m \rightharpoonup u \text{ in } I^4(\tau, \tau + T; V),\\
&u^m \rightharpoonup u \text{ in } I^2(\tau, \tau + T; D(A)),\\
&u^m(\tau + T) \rightharpoonup \mu \text{ in } L^2(\Omega; V),\\
&-\widetilde{A}u^m - \widetilde{B}(u^m) \rightharpoonup \iota \text{ in } I^2(\tau, \tau + T; (D(A))^*),\\
&\widetilde{g}_1(t, u_t^m) \rightharpoonup \kappa_1 \text{ in } I^2(\tau, \tau + T; V),\\
&\widetilde{g}_2(t, u_t^m) \rightharpoonup \kappa_2 \text{ in } I^2(\tau, \tau + T; \mathcal{L}^2(K, V)).
\end{aligned}$$

As in [41], we extend Eq. (1.2.16) to an open interval $(-\delta + \tau, \tau + T + \delta)$ for any $\delta > 0$ such that all terms are equal to 0 outside of the interval $[\tau, \tau + T]$.

Let $\psi(t)$ be a function in $W^{1,4/3}(-\delta + \tau, \tau + T + \delta)$ with $\psi(\tau) = 1$. Put $w_j(t) = \psi(t)w_j$ for all integers $j \geqslant 1$, where we recall that $\{w_j\}$ is the Hilbert basis of V such that $\{w_j; j \geqslant 1\} \subset D(A)$. Applying the Ito formula to the function $(u^m(t), w_j(t))$, we obtain

$$\begin{aligned}
(u^m(\tau + T), w_j(\tau + T)) = (u^m(\tau), w_j) &+ \int_\tau^{\tau+T} \Big(u^m(s), \frac{dw_j(s)}{ds}\Big)ds + \int_\tau^{\tau+T} \langle -\widetilde{A}u^m(s) - \widetilde{B}(u^m(s)), w_j(s)\rangle ds\\
&+ \int_\tau^{\tau+T} ((\widetilde{f}(s) + \widetilde{g}_1(s, u_s^m), w_j(s)))ds + \int_\tau^{\tau+T} ((w_j(s), \widetilde{g}_2(s, u_s^m)dW(s))).
\end{aligned} \tag{1.2.43}$$

Taking limit of (1.2.43) as $m \to \infty$, we refer to the similar calculation as in [111, Theorem 2.6], then

$$\begin{aligned}
-\int_\tau^{\tau+T} \Big(u(s), \frac{dw_j(s)}{ds}\Big)ds = (\phi(0), w_j) &+ \int_\tau^{\tau+T} \langle \iota, w_j\rangle \psi(s)ds + \int_\tau^{\tau+T} ((\widetilde{f}(s) + \kappa_1(s), w_j))\psi(s)ds\\
&+ \int_\tau^{\tau+T} \psi(s)((w_j, \kappa_2(s)dW(s))) - (\mu, w_j)\psi(\tau + T).
\end{aligned} \tag{1.2.44}$$

Consider a sequence of functions $\{\psi_k\}$ such that $\psi_k \to 1_{[\tau,\tau+T]}$ and the time derivative of ψ_k tends to υ_t weakly as $k \to \infty$. We use ψ_k in (1.2.44) to replace ψ and then let $k \to \infty$, we find that

$$(u(t), w_j) = (\phi(0), w_j) + \int_\tau^t \langle \iota(s), w_j\rangle ds + \int_\tau^t ((\widetilde{f}(s) + \kappa_1(s), w_j))ds + \int_\tau^t ((w_j, \kappa_2(s)dW(s))) \tag{1.2.45}$$

for all $t < \tau + T$ with $(u(\tau + T), w_j) = (\mu, w_j)$ for all $j \geqslant 1$, then

$$u(t) = \phi(0) + \int_\tau^t (\iota(s) + \widetilde{f}(s) + \kappa_1(s))ds + \int_\tau^t \kappa_2(s)dW(s) \tag{1.2.46}$$

with $u(\tau + T) = \mu$.

Let $\varrho(t) = \int_\tau^t \|y(s)\|^2_{D(A)} ds$, where $y \in I^2(\tau, \tau + T; D(A))$ with $y(\tau + s) = \phi(s),\ s \leqslant 0$. Applying Ito's formula to the process $e^{-\sigma\varrho(t)}\|u(t)\|^2$ and $e^{-\sigma\varrho(t)}\|u^m(t)\|^2$ respectively, where σ is the same as in Lemma 1.2.2,

$$\begin{aligned}
\mathbb{E}\Big(e^{-\sigma\varrho(t)}\|u(t)\|^2\Big) = &\ \mathbb{E}\Big(\|\phi(0)\|^2\Big) - \mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|^2_{D(A)}\|u(s)\|^2 ds\Big) \\
&+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle \iota(s), u(s)\rangle ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\big(\widetilde{f}(s) + \kappa_1, u(s)\big)\Big) ds\Big) \\
&+ \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\kappa_2\|^2_{\mathcal{L}^2(K,V)} ds\Big),
\end{aligned} \tag{1.2.47}$$

and

$$\begin{aligned}
\mathbb{E}\Big(e^{-\sigma\varrho(t)}\|u^m(t)\|^2\Big) = &\ \mathbb{E}\Big(\|u^m(\tau)\|^2\Big) - \mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|^2_{D(A)}\|u^m(s)\|^2 ds\Big) \\
&+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}u^m(s) - \widetilde{B}(u^m(s)), u^m(s)\rangle ds\Big) \\
&+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\big(\widetilde{f}(s) + \widetilde{g}_1(s, u^m_s), u^m(s)\big)\Big) ds\Big) \\
&+ \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\widetilde{g}_2(s, u^m_s)\|^2_{\mathcal{L}^2(K,V)} ds\Big).
\end{aligned} \tag{1.2.48}$$

Define three elements X_m, Y_m and Z_m by

$$\begin{aligned}
X_m = &- \mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|^2_{D(A)}\|u^m(s) - y(s)\|^2 ds\Big) \\
&+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}u^m(s) - \widetilde{B}(u^m(s)), u^m(s) - y(s)\rangle ds\Big) \\
&- 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}y(s) - \widetilde{B}(y(s)), u^m(s) - y(s)\rangle ds\Big) \\
&+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\big(\widetilde{g}_1(s, u^m_s) - \widetilde{g}_1(s, y_s), u^m(s) - y(s)\big)\Big) ds\Big) \\
&+ \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\widetilde{g}_2(s, u^m_s) - \widetilde{g}_2(s, y_s)\|^2_{\mathcal{L}^2(K,V)} ds\Big). \\
Y_m = &- \mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|^2_{D(A)}\|u^m(s)\|^2 ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}u^m(s) - \widetilde{B}(u^m(s)), u^m(s)\rangle ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\big(\widetilde{g}_1(s,u_s^m),u^m(s)\big)\Big)ds\Big) + \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\widetilde{g}_2(s,u_s^m)\|^2_{\mathcal{L}^2(K,V)}ds\Big).\\
Z_m = &-\mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|^2_{D(A)}\Big(\|y(s)\|^2 - 2(u^m(s),y(s))\Big)ds\Big)\\
&+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}u^m(s) - \widetilde{B}(u^m(s)), -y(s)\rangle ds\Big)\\
&- 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}y(s) - \widetilde{B}(y(s)), u^m(s) - y(s)\rangle ds\Big)\\
&- 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\big(\widetilde{g}_1(s,y_s),u^m(s)-y(s)\big)\Big)ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\big(\widetilde{g}_1(s,u_s^m),-y(s)\big)\Big)ds\Big)\\
&+ \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\big(\widetilde{g}_2(s,y_s) - 2\widetilde{g}_2(s,u_s^m),\widetilde{g}_2(s,y_s)\big)\Big)_{\mathcal{L}^2(K,V)}ds\Big).
\end{aligned}
$$

We infer from the above equalities that $X_m = Y_m + Z_m$. By (1.2.3) in Lemma 1.2.2 and (1.2.9), we have $X_m \leqslant 0$,

$$
\begin{aligned}
0 &\geqslant \liminf_{m\to\infty} X_m\\
&\geqslant -\mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|^2_{D(A)}\|u(s)-y(s)\|^2 ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle \iota, u(s)-y(s)\rangle ds\Big)\\
&- 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle -\widetilde{A}y(s) - \widetilde{B}(y(s)), u(s)-y(s)\rangle ds\Big)\\
&+ 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\big(\kappa_1 - \widetilde{g}_1(s,y_s), u(s)-y(s)\big)\Big)ds\Big) + \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\kappa_2 - \widetilde{g}_2(s,y_s)\|^2_{\mathcal{L}^2(K,V)}ds\Big).
\end{aligned}
\tag{1.2.49}
$$

Taking $y(t) = u(t)$ in (1.2.49), since $e^{-\sigma\varrho(t)}$ is bounded with respect to $t \in [\tau, \tau+T]$, we find that $\kappa_2 = \widetilde{g}_2(t,u_t),\ t \in [\tau,\tau+T]$. By (1.2.48), we derive

$$
Y_m = \mathbb{E}\Big(e^{-\sigma\varrho(s)}\|u^m(t)\|^2\Big) - \mathbb{E}\Big(\|u^m(\tau)\|^2\Big) - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\big(\widetilde{f}(s),u^m(s)\big)\Big)ds\Big), \tag{1.2.50}
$$

which, together with (1.2.47), implies

$$
\begin{aligned}
\liminf_{m\to\infty} Y_m &\geqslant \mathbb{E}\Big(e^{-\sigma\varrho(s)}\|u(s)\|^2\Big) - \mathbb{E}\Big(\|\phi(0)\|^2\Big) - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\big(\widetilde{f}(s),u(s)\big)\Big)ds\Big)\\
&= -\mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|^2_{D(A)}\|u(s)\|^2 ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle \iota, u(s)\rangle ds\Big)\\
&\quad + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\Big(\big(\kappa_1,u(s)\big)\Big)ds\Big) + \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\|\widetilde{g}_2(s,u_s)\|^2_{\mathcal{L}^2(K,V)}ds\Big).
\end{aligned}
\tag{1.2.51}
$$

Besides,

$$
\liminf_{m\to\infty} Z_m \geqslant -\mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)}\|y(s)\|^2_{D(A)}\Big(\|y(s)\|^2 - 2(u(s),y(s))\Big)ds\Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)}\langle \iota, -y(s)\rangle ds\Big)
$$

$$
\begin{aligned}
& - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \langle -\widetilde{A}y(s) - \widetilde{B}(y(s)), u(s) - y(s) \rangle ds \Big) \\
& - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \Big(\big(\widetilde{g}_1(s, y_s), u(s) - y(s) \big) \Big) ds \Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \Big(\big(\kappa_1, -y(s) \big) \Big) ds \Big) \\
& + \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \Big(\big(\widetilde{g}_2(s, y_s) - 2\widetilde{g}_2(s, u_s), \widetilde{g}_2(s, y_s) \big) \Big)_{\mathcal{L}^2(K,V)} ds \Big). \qquad (1.2.52)
\end{aligned}
$$

Therefore, by (1.2.49), (1.2.51) and (1.2.52), we have

$$
\begin{aligned}
0 & \geqslant \liminf_{m\to\infty} X_m = \liminf_{m\to\infty} Y_m + \liminf_{m\to\infty} Z_m \\
& \geqslant -\mathbb{E}\Big(\int_\tau^t \sigma e^{-\sigma\varrho(s)} \|y(s)\|^2_{D(A)} \|u(s) - y(s)\|^2 ds \Big) + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \langle \iota, u(s) - y(s) \rangle ds \Big) \\
& + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \Big(\big(\kappa_1 - \widetilde{g}_1(s, y_s), u(s) - y(s) \big) ds \Big) \Big) - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \langle -\widetilde{A}y(s) - \widetilde{B}(y(s)), u(s) - y(s) \rangle ds \Big) \\
& + \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \|\widetilde{g}_2(s, u_s) - \widetilde{g}_2(s, y_s)\|^2_{\mathcal{L}^2(K,V)} ds \Big). \qquad (1.2.53)
\end{aligned}
$$

We further obtain

$$
\begin{aligned}
0 & \leqslant \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \|\widetilde{g}_2(s, u_s) - \widetilde{g}_2(s, y_s)\|^2_{\mathcal{L}^2(K,V)} ds \Big) \\
& \leqslant 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \langle -\widetilde{A}y(s) - \widetilde{B}(y(s)), u(s) - y(s) \rangle ds \Big) \\
& \quad + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \Big(\big(\widetilde{g}_1(s, y_s) - \kappa_1, u(s) - y(s) \big) \Big) ds \Big) \\
& \quad - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \langle \iota, u(s) - y(s) \rangle ds \Big) + \sigma\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \|y(s)\|^2_{D(A)} \|u(s) - y(s)\|^2 ds \Big). \qquad (1.2.54)
\end{aligned}
$$

Let $y(t) = u(t) - \vartheta z(t)$ with for any $z \in I^2(\tau, \tau + T; D(A)) \cap I^4(\tau, \tau + T; V)$ and $\vartheta \in [0, 1]$, then

$$
\begin{aligned}
0 & \leqslant 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \langle -\widetilde{A}(u - \vartheta z) - \widetilde{B}(u - \vartheta z), \vartheta z \rangle ds \Big) \\
& + 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \Big(\big(\widetilde{g}_1(s, u_s - \vartheta z) - \kappa_1, \vartheta z \big) \Big) ds \Big) \qquad (1.2.55) \\
& - 2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \langle \iota, \vartheta z \rangle ds \Big) + \sigma\vartheta^2\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \|y(s)\|^2_{D(A)} \|z\|^2 ds \Big).
\end{aligned}
$$

Dividing by ϑ on both sides of (1.2.55), and then letting $\vartheta \to 0$, we have

$$
\mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \langle \iota + \widetilde{A}u(s) + \widetilde{B}(u(s)), z \rangle ds \Big) + \mathbb{E}\Big(\int_\tau^t e^{-\sigma\varrho(s)} \Big(\big(\kappa_1 - \widetilde{g}_1(s, u_s), z \big) \Big) ds \Big) \leqslant 0. \qquad (1.2.56)
$$

Since $I^2(\tau, \tau + T; D(A)) \cap I^4(\tau, \tau + T; V)$ is dense in $I^2(\tau, \tau + T; V)$, we find

$$
e^{-\sigma\varrho(s)} \Big(\iota + \widetilde{A}u(s) + \widetilde{B}(u(s)) + \kappa_1 - \widetilde{g}_1(s, u_s) \Big) = 0, \;\; a.e.\ t \in [\tau, \tau + T], \;\; \omega \in \Omega. \qquad (1.2.57)
$$

Note that $\kappa_2 = \widetilde{g}_2(t, u_t),\ t \in [\tau, \tau + T]$, we can rewrite (1.2.46) as

$$
\begin{aligned}
&u(t) + \int_\tau^t \widetilde{A}u(s)ds + \int_\tau^t \widetilde{B}(u(s))ds \\
&\quad = \phi(0) + \int_\tau^t (\widetilde{f}(s) + \widetilde{g}_1(s, u_s))ds + \int_\tau^t \widetilde{g}_2(s, u_s)dW(s),\ a.e.\ t \in [\tau, \tau + T],\ \omega \in \Omega.
\end{aligned} \tag{1.2.58}
$$

Therefore, the existence of a weak solution has been proved.

Step 4: *We derive the estimate* (1.2.14). For each $n \in \mathbb{N}$ and $T > 0$, we can define a *stopping time* τ_n^m as follows.

$$
\tau_n^m = \inf\Big\{t \leqslant \tau + T : \|u^m(t)\|^2 + \int_\tau^t \|u^m(s)\|^2_{D(A)}ds \geqslant n\Big\}. \tag{1.2.59}
$$

For fixed m, the sequence $\{\tau_n^m; n \geqslant 1\}$ is increasing to $\tau + T$. By (1.2.42) and (1.2.59), we obtain

$$
\mathbb{E}\Big(\sup_{r \in [\tau, t\wedge\tau_n^m]} \|u_r^m\|^4_{C_\gamma(V)}\Big) + \mathbb{E}\Big(\int_\tau^{t\wedge\tau_n^m} \|u^m(s)\|^4 ds\Big) \leqslant R,\ \forall\, t \in [\tau, \tau + T]. \tag{1.2.60}
$$

Thanks to (1.2.60) and Fatou's lemma, we deduce that (1.2.14) holds for every $T > 0$.

Step 5: *We prove the uniqueness of solutions to Eq.* (1.2.1). Let u, v be two solutions of Eq. (1.2.1) with the same initial condition $u(s) = v(s) = \phi(s - \tau),\ s \leqslant \tau$, and let $\overline{w} := u - v$. For every $n \in \mathbb{N}$ and $T > 0$, we can define a *stopping time* T_n by

$$
T_n = \inf\Big\{t \leqslant \tau + T : \int_\tau^t \|v(s)\|^2_{D(A)}ds \geqslant n\Big\}. \tag{1.2.61}
$$

In addition, let $\varsigma(t) := e^{-\sigma\int_\tau^t \|v(s)\|^2_{D(A)}ds}$, where σ is given by (1.2.3) in Lemma 1.2.2 and $\mathbb{E}(\int_\tau^t \|v(s)\|^2_{D(A)}ds)$ is finite due to the steps 2-3. Applying Ito's formula to the process $\varsigma(t)\|\overline{w}(t)\|^2$, we infer from (1.2.3) in Lemma 1.2.2 that

$$
\begin{aligned}
\varsigma(t\wedge T_n)\|\overline{w}(t\wedge T_n)\|^2 &= -\sigma\int_\tau^{t\wedge T_n} \varsigma(s)\|v(s)\|^2_{D(A)}\|\overline{w}(s)\|^2 ds \\
&\quad + 2\int_\tau^{t\wedge T_n} \varsigma(s)\langle -\widetilde{A}\overline{w}(s) - \widetilde{B}(u) + \widetilde{B}(v), \overline{w}(s)\rangle ds \\
&\quad + 2\int_\tau^{t\wedge T_n} \varsigma(s)\Big(\big(\widetilde{g}_1(s, u_s) - \widetilde{g}_1(s, v_s), \overline{w}(s)\big)\Big)ds \\
&\quad + 2\int_\tau^{t\wedge T_n} \varsigma(s)\Big(\big(\overline{w}(s), (\widetilde{g}_2(s, u_s) - \widetilde{g}_2(s, v_s))dW(s)\big)\Big) \\
&\quad + \int_\tau^{t\wedge T_n} \varsigma(s)\|\widetilde{g}_2(s, u_s) - \widetilde{g}_2(s, v_s)\|^2_{\mathcal{L}^2(K,V)}ds \\
&\leqslant -\int_\tau^{t\wedge T_n} \varsigma(s)\|\overline{w}(s)\|^2_{D(A)}ds + 2\int_\tau^{t\wedge T_n} \varsigma(s)\Big(\big(\widetilde{g}_1(s, u_s) - \widetilde{g}_1(s, v_s), \overline{w}(s)\big)\Big)ds
\end{aligned}
$$

$$+ 2\int_\tau^{t\wedge T_n} \varsigma(s)\Big(\big(\overline{w}(s), (\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,v_s))dW(s)\big)\Big)$$

$$+ \int_\tau^{t\wedge T_n} \varsigma(s)\|\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds. \tag{1.2.62}$$

Taking supremum and expectation of (1.2.62), we find

$$\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t} \varsigma(r\wedge T_n)\|\overline{w}(r\wedge T_n)\|^2\Big) + \mathbb{E}\Big(\int_\tau^t \varsigma(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|^2_{D(A)}ds\Big)$$
$$\leqslant 2\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t\wedge T_n}\Big|\int_\tau^r \varsigma(s)\Big(\big(\widetilde{g}_1(s,u_s) - \widetilde{g}_1(s,v_s), \overline{w}(s)\big)\Big)ds\Big|\Big)$$
$$+ 2\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t\wedge T_n}\Big|\int_\tau^r \varsigma(s)\Big(\big(\overline{w}(s), (\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,v_s))dW(s)\big)\Big)\Big|\Big)$$
$$+ \mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t\wedge T_n}\Big|\int_\tau^r \varsigma(s)\|\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds\Big|\Big). \tag{1.2.63}$$

The Young inequality and (H15) imply

$$2\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t\wedge T_n}\Big|\int_\tau^r \varsigma(s)\Big(\big(\widetilde{g}_1(s,u_s) - \widetilde{g}_1(s,v_s), \overline{w}(s)\big)\Big)ds\Big|\Big)$$
$$\leqslant \mathbb{E}\Big(\int_\tau^{t\wedge T_n} \varsigma(s)\|\widetilde{g}_1(s,u_s) - \widetilde{g}_1(s,v_s)\|^2 ds\Big) + \mathbb{E}\Big(\int_\tau^{t\wedge T_n} \varsigma(s)\|\overline{w}(s)\|^2 ds\Big)$$
$$\leqslant (\widetilde{C}_{\widetilde{g}_1} + 1)\mathbb{E}\Big(\int_\tau^t \varsigma(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|^2 ds\Big)$$
$$\leqslant (\widetilde{C}_{\widetilde{g}_1} + 1)\mathbb{E}\Big(\int_\tau^t \sup_{\tau\leqslant\theta\leqslant s} \varsigma(\theta\wedge T_n)\|\overline{w}(\theta\wedge T_n)\|^2 ds\Big). \tag{1.2.64}$$

By the Burkholder-Davis-Gundy inequality and (H15), we have

$$2\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t\wedge T_n}\Big|\int_\tau^r \varsigma(s)\Big(\big(\overline{w}(s), (\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,v_s))dW(s)\big)\Big)\Big|\Big)$$
$$\leqslant 2c_{10}\mathbb{E}\bigg(\Big(\int_\tau^{t\wedge T_n}\Big(\varsigma^2(s)\|\overline{w}(s)\|^2\|\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}\Big)ds\Big)^{\frac{1}{2}}\bigg)$$
$$\leqslant 2c_{10}\mathbb{E}\bigg(\sup_{\tau\leqslant s\leqslant t} \varsigma^{1/2}(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|\Big(\int_\tau^{t\wedge T_n} \varsigma(s)\|\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds\Big)^{\frac{1}{2}}\bigg)$$
$$\leqslant \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leqslant s\leqslant t} \varsigma(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|^2\Big) + c_{11}\mathbb{E}\Big(\int_\tau^{t\wedge T_n} \varsigma(s)\|\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds\Big)$$
$$\leqslant \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\leqslant s\leqslant t} \varsigma(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|^2\Big) + c_{11}\widetilde{C}_{\widetilde{g}_2}\int_\tau^t \mathbb{E}\Big(\sup_{\tau\leqslant\theta\leqslant s} \varsigma(\theta\wedge T_n)\|\overline{w}(\theta\wedge T_n)\|^2\Big)ds, \tag{1.2.65}$$

where $c_{11} = 2c_{10}^2$. By (H15), the last line of (1.2.63) is bounded by

$$\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t\wedge T_n}\Big|\int_\tau^r \varsigma(s)\|\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds\Big|\Big)$$

$$\begin{aligned}
&\leqslant \mathbb{E}\Big(\int_{\tau}^{t\wedge T_n} \varsigma(s)\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,v_s)\|^2_{\mathcal{L}^2(K,V)}ds\Big)\\
&\leqslant \widetilde{C}_{\widetilde{g}_2}\int_{\tau}^{t}\mathbb{E}\Big(\sup_{\tau\leqslant\theta\leqslant s}\varsigma(\theta\wedge T_n)\|\overline{w}(\theta\wedge T_n)\|^2\Big)ds. \qquad (1.2.66)
\end{aligned}$$

It follows from (1.2.63)-(1.2.66) that

$$\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\varsigma(r\wedge T_n)\|\overline{w}(r\wedge T_n)\|^2\Big)\leqslant c_{12}\int_{\tau}^{t}\mathbb{E}\Big(\sup_{\tau\leqslant\theta\leqslant s}\varsigma(\theta\wedge T_n)\|\overline{w}(\theta\wedge T_n)\|^2\Big)ds,\ \forall\, t\in[\tau,\tau+T],$$

where $c_{12}=2\big(\widetilde{C}_{\widetilde{g}_1}+1+c_{11}\widetilde{C}_{\widetilde{g}_2}+\widetilde{C}_{\widetilde{g}_2}\big)$. The Gronwall Lemma, together with $0<\varsigma\leqslant 1$, implies

$$\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|\overline{w}(r\wedge T_n)\|^2\Big)=0,\ \forall\, t\in[\tau,\tau+T],$$

and thus,

$$u(r\wedge T_n)=v(r\wedge T_n),\ \text{a.e.},\ \omega\in\Omega.$$

Furthermore, by Markov's inequality,

$$P(T_n<\tau+T)=P\Big(\int_{\tau}^{t}\|v(s)\|^2_{D(A)}ds\geqslant n\Big)\leqslant\frac{\mathbb{E}\Big(\int_{\tau}^{t}\|v(s)\|^2_{D(A)}ds\Big)}{n}.$$

We infer from $\mathbb{E}\Big(\int_{\tau}^{t}\|v(s)\|^2_{D(A)}ds\Big)<\infty$ that $T_n\to\tau+T$ as $n\to\infty$. Therefore, $u(r)=v(r)$, a.e., $\omega\in\Omega$ for all $r\leqslant\tau+T$. The proof is concluded. □

1.3 Stationary solutions and their stability results

In this section, we are concerned with existence, uniqueness and stability properties of the stationary solutions to (0.0.3). For this end, we need to assume that $\widetilde{f}(t)\equiv\widetilde{f}\in(D(A))^*$ (i.e. $f(t)\equiv f\in\mathbb{H}^{-1}(O)$), which is independent of the time.

1.3.1 Existence and uniqueness of stationary solutions

We now consider the abstract equation associated to Eq. (0.0.3):

$$\begin{cases}\dfrac{du}{dt}+\widetilde{A}u(t)+\widetilde{B}(u(t))=\widetilde{f}+\widetilde{g}_1(t,u_t)+\widetilde{g}_2(t,u_t)\dfrac{dW}{dt}, & \forall\, t>0,\\ u(t)=\phi(t),t\in(-\infty,0].\end{cases} \qquad (1.3.1)$$

We denote by $u(t):=u(t;\phi)$ the solution of (0.0.3) with $\tau=0$, where $\phi=u_0$.

By a stationary solution to (1.3.1), we mean a constant solution (in other words, an equilibrium point) of (1.3.1). Therefore, $u_\infty \in D(A)$ will be a stationary solution if formally

$$\widetilde{A}u_\infty + \widetilde{B}(u_\infty) = \widetilde{f} + \widetilde{g}_1(t, u_\infty) + \widetilde{g}_2(t, u_\infty)\frac{dW}{dt}, \quad \forall\, t \geqslant 0. \tag{1.3.2}$$

However, this equation depends on t and a noisy term. Therefore, we would need to assume that $\widetilde{g}_1$ and $\widetilde{g}_2$ would not depend on t, moreover, to get rid of the noise, we must assume that $\widetilde{g}_2(t, u_\infty) = 0$.

Consequently, we will focus on the existence of stationary solutions for the deterministic equation (i.e. $\widetilde{g}_2 = 0$ in (1.3.1)) which will be any $u_\infty \in D(A)$ such that

$$\widetilde{A}u_\infty + \widetilde{B}(u_\infty) = \widetilde{f} + \widetilde{g}_1(t, u_\infty), \quad \forall\, t \geqslant 0, \tag{1.3.3}$$

and then analyze the behavior of the solutions to (1.3.1) around these stationary solutions of (1.3.3).

Now, in order to study the existence of solutions to (1.3.3), we have to restrict ourselves to assume that for constant elements $\xi \in C_\gamma(V)$, $\widetilde{g}_i(t, \xi)$ $(i = 1, 2)$ can be rewritten as

$$\widetilde{g}_i(t, \xi) = \widetilde{\mathcal{G}}_i(\xi^*) \text{ if } \xi(s) = \xi^*, \forall s \leqslant 0, \tag{1.3.4}$$

where $\widetilde{\mathcal{G}}_1 : V \to V$ with $\widetilde{\mathcal{G}}_1(0) = 0$, $\widetilde{\mathcal{G}}_2 : V \to \mathcal{L}^2(K, V)$ with $\widetilde{\mathcal{G}}_2(0) = 0$, they are Lipschitz continuous, that is, there exist $L_{\widetilde{\mathcal{G}}_i} > 0$ $(i = 1, 2)$, for all $\eta, \zeta \in V$,

$$\|\widetilde{\mathcal{G}}_1(\eta) - \widetilde{\mathcal{G}}_1(\zeta)\| \leqslant L_{\widetilde{\mathcal{G}}_1}\|\eta - \zeta\|, \tag{1.3.5}$$

$$\|\widetilde{\mathcal{G}}_2(\eta) - \widetilde{\mathcal{G}}_2(\zeta)\|_{\mathcal{L}^2(K,V)} \leqslant L_{\widetilde{\mathcal{G}}_2}\|\eta - \zeta\|. \tag{1.3.6}$$

For example, if $\widetilde{g}_i$ $(i = 1, 2)$ are driven by unbounded variable delay, defined by

$$\widetilde{g}_i(t, \xi) = \widetilde{\mathcal{G}}_i(\xi(-h(t))), \ i = 1, 2, \tag{1.3.7}$$

with $\widetilde{\mathcal{G}}_i$ satisfying conditions (1.3.4)-(1.3.6), where $h \in C^1([0, +\infty)), h(t) \geqslant 0$ and $h^* = \sup_{t\geqslant 0} h'(t) < 1$. In this case, the delay terms $\widetilde{g}_i$ $(i = 1, 2)$ in our problem become $\widetilde{g}_i(t, u_t) = \widetilde{\mathcal{G}}_i(u(t - h(t)))$.

Another example is the case of infinite distributed delay, that is, the delay terms $\widetilde{g}_i$ $(i = 1, 2)$ are defined by

$$\widetilde{g}_i(t, \xi) = \int_{-\infty}^{0} \widetilde{\mathcal{H}}_i(s, \xi(s))ds, \tag{1.3.8}$$

where $\widetilde{\mathcal{H}}_1 : (-\infty, 0] \times V \to V$ with $\widetilde{\mathcal{H}}_1(s, 0) = 0$, and $\widetilde{\mathcal{H}}_2 : (-\infty, 0] \times V \to \mathcal{L}^2(K, V)$ with $\widetilde{\mathcal{H}}_2(s, 0) = 0$ are measurable, and they are Lipschitz continuous with respect to their second variable, that is, there exist $L_{\widetilde{\mathcal{H}}_i}(s) \in L^2(-\infty, 0)$ $(i = 1, 2)$ with $L_{\widetilde{\mathcal{H}}_i}(\cdot)e^{-(\gamma+\theta)\cdot} \in L^2(-\infty, 0)$, for certain $\theta > 0$, such that for all $s \in (-\infty, 0], \eta, \zeta \in V$,

$$\|\widetilde{\mathcal{H}}_1(s, \eta) - \widetilde{\mathcal{H}}_1(s, \zeta)\| \leqslant L_{\widetilde{\mathcal{H}}_1}(s)\|\eta - \zeta\|, \tag{1.3.9}$$

$$\|\widetilde{\mathcal{H}}_2(s, \eta) - \widetilde{\mathcal{H}}_2(s, \zeta)\|_{\mathcal{L}^2(K,V)} \leqslant L_{\widetilde{\mathcal{H}}_2}(s)\|\eta - \zeta\|. \tag{1.3.10}$$

In this case, we can rewrite the delay terms $\widetilde{g}_i$ $(i = 1, 2)$ in our problem as $\widetilde{g}_i(t, u_t) = \int_{-\infty}^{0} \widetilde{\mathcal{H}}_i(s, u(t + s))ds$ $(i = 1, 2)$.

The above both situations are within our framework, the conditions (H11)-(H15) are fulfilled for the infinite distributed delay in $C_\gamma(V)$ for $\gamma > 0$, but not necessarily for the unbounded variable delay. However, conditions (H11)-(H15) are satisfied for both delays in $C_{-\infty}(V)$.

Now, we are interested in studying the existence and uniqueness of a stationary solution to Eq. (1.3.3).

Theorem 1.3.1. *Assume that the above assumptions and notation hold. If $\widetilde{\lambda}_1 > L_{\widetilde{g}_1}$, then:*

(a) For all $\widetilde{f} \in (D(A))^*$, then there exists at least one stationary solution to (1.3.3), which belongs to $D(A)$ if $\widetilde{f} \in V$;

(b) If $(1 - \widetilde{\lambda}_1^{-1} L_{\widetilde{g}_1})^2 > \widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|$, then the stationary solution to (1.3.3) is unique.

Proof. We can prove this result by using the same method as in [31, Theorem 10] (or [30, Lemma 3.2]), which is based on the Lax-Milgram, the Schauder theorems. Therefore, we omit the details. □

1.3.2 Local stability of stationary solutions

In this subsection, we will prove the local stability of stationary solutions to (1.3.3) for general delay terms by using a direct method and then apply the abstract results to two specific situations.

Theorem 1.3.2. *Suppose that the same hypotheses and notations in Theorem 1.2.4 and Theorem 1.3.1 hold. In addition, let*

$$2\widetilde{\lambda}_1 \geqslant \frac{2\widetilde{c}\|\widetilde{f}\|}{1 - \widetilde{\lambda}_1^{-1} L_{\widetilde{g}_1}} + 2C_{\widetilde{g}_1} + C_{\widetilde{g}_2}^2. \tag{1.3.11}$$

If $u(\cdot)$ is any solution of Eq. (1.3.1), u_∞ is the unique stationary solution of Eq. (1.3.3) and $w(t) = u(t) - u_\infty$, then

$$\mathbb{E}\Big(\|w(t)\|^2\Big) \leqslant \mathbb{E}\Big(\|w(0)\|^2\Big) + (C_{\widetilde{g}_1} + C_{\widetilde{g}_2}^2) \int_{-\infty}^{0} \mathbb{E}\Big(\|\phi(s) - u_\infty\|^2\Big)ds. \tag{1.3.12}$$

Proof. Applying Ito's formula to $\|w(t)\|^2$, we obtain

$$\begin{aligned}\|w(t)\|^2 &= \|w(0)\|^2 + 2\int_0^t \langle -\widetilde{A}w(s) - \widetilde{B}(u(s)) + \widetilde{B}(u_\infty), w(s)\rangle ds \\ &\quad + 2\int_0^t \Big(\big(\widetilde{g}_1(s, u_s) - \widetilde{g}_1(s, u_\infty), w(s)\big)\Big)ds + 2\int_0^t \Big(\big(\widetilde{g}_2(s, u_s) - \widetilde{g}_2(s, u_\infty), w(s)\big)\Big)dW(s) \\ &\quad + \int_0^t \|\widetilde{g}_2(s, u_s) - \widetilde{g}_2(s, u_\infty)\|^2_{\mathcal{L}^2(K,V)}ds. \end{aligned} \tag{1.3.13}$$

Taking expectation of (1.3.13), thanks to Fubini's theorem,

$$\mathbb{E}\Big(\|w(t)\|^2\Big) + 2\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds = \mathbb{E}\Big(\|w(0)\|^2\Big) - 2\int_0^t \mathbb{E}\Big(\langle \widetilde{B}(u(s)) - \widetilde{B}(u_\infty), w(s)\rangle\Big)ds$$

$$+ 2\mathbb{E}\Big(\int_0^t \big((\widetilde{g}_1(s,u_s) - \widetilde{g}_1(s,u_\infty), w(s))\big)ds\Big)$$
$$+ \int_0^t \mathbb{E}\Big(\|\widetilde{g}_2(s,u_s) - \widetilde{g}_2(s,u_\infty)\|^2_{\mathcal{L}^2(K,V)}\Big)ds. \tag{1.3.14}$$

By (0.0.42), (B2) and (1.2.7), we deduce

$$\begin{aligned}-2\int_0^t \mathbb{E}\Big(\langle \widetilde{B}(u(s)) - \widetilde{B}(u_\infty), w(s)\rangle\Big)ds &= -2\int_0^t \mathbb{E}\Big(\langle \widetilde{B}(u_\infty, w(s)), w(s)\rangle\Big)ds\\ &\leqslant 2\widetilde{c}\int_0^t \mathbb{E}(\|u_\infty\|\|w(s)\|^2_{D(A)})ds\\ &\leqslant 2\widetilde{c}\widetilde{\lambda}_1^{-\frac{1}{2}}\int_0^t \mathbb{E}(\|u_\infty\|_{D(A)}\|w(s)\|^2_{D(A)})ds.\end{aligned} \tag{1.3.15}$$

By (0.0.42), (1.1.2) and (1.3.3), we find

$$\begin{aligned}\|u_\infty\|^2_{D(A)} &= \langle \widetilde{A}u_\infty, u_\infty\rangle\\ &= ((\widetilde{f}, u_\infty)) + ((\widetilde{g}_1(t,u_\infty), u_\infty))\\ &\leqslant \widetilde{\lambda}_1^{-\frac{1}{2}}\|\widetilde{f}\|\|u_\infty\|_{D(A)} + \widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}\|u_\infty\|^2_{D(A)},\end{aligned} \tag{1.3.16}$$

which, together with $\widetilde{\lambda}_1 > L_{\widetilde{g}_1}$, implies that

$$\|u_\infty\|_{D(A)} \leqslant \frac{\widetilde{\lambda}_1^{-\frac{1}{2}}\|\widetilde{f}\|}{1 - \widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}. \tag{1.3.17}$$

Thanks to (1.3.16)-(1.3.17), we can rewrite (1.3.15) as

$$-2\int_0^t \mathbb{E}\Big(\langle \widetilde{B}(u(s)) - \widetilde{B}(u_\infty), w(s)\rangle\Big)ds \leqslant \frac{2\widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|}{1 - \widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}\int_0^t \mathbb{E}(\|w(s)\|^2_{D(A)})ds. \tag{1.3.18}$$

We now estimate the last two terms of (1.3.14) respectively. On the one hand, by (H14), (0.0.42) and the Young inequality, with $\epsilon_0 > 0$ to be specified later on, we deduce

$$\begin{aligned}&2\mathbb{E}\Big(\int_0^t \big((\widetilde{g}_1(s,u_s) - \widetilde{g}_1(s,u_\infty), w(s))\big)ds\Big)\\ &\leqslant 2\widetilde{\lambda}_1^{-\frac{1}{2}}\int_0^t \mathbb{E}\Big(\|\widetilde{g}_1(s,u_s) - \widetilde{g}_1(s,u_\infty)\|\|w(s)\|_{D(A)}\Big)ds\\ &\leqslant \frac{1}{\epsilon_0}\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds + \epsilon_0\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_1}\int_{-\infty}^t \mathbb{E}\Big(\|w(s)\|^2\Big)ds\\ &\leqslant \frac{1}{\epsilon_0}\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds + \epsilon_0\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_1}\Big(\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s) - u_\infty\|^2\Big)ds\\ &\quad + \widetilde{\lambda}_1^{-1}\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds\Big).\end{aligned} \tag{1.3.19}$$

On the other hand, by (H14) and (0.0.42), we find

$$\int_0^t \mathbb{E}\Big(\|\widetilde{g}_2(s,u_s)-\widetilde{g}_2(s,u_\infty)\|^2_{\mathcal{L}^2(K,V)}\Big)ds$$
$$\leqslant C^2_{\widetilde{g}_2}\Big(\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds+\widetilde{\lambda}_1^{-1}\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds\Big). \tag{1.3.20}$$

It follows from the above inequalities that

$$\begin{aligned}\mathbb{E}\Big(\|w(t)\|^2\Big) &\leqslant \mathbb{E}\Big(\|w(0)\|^2\Big)+\Big(\frac{2\widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}+\frac{1}{\epsilon_0}+\epsilon_0\widetilde{\lambda}_1^{-2}C^2_{\widetilde{g}_1}+\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_2}-2\Big)\times\\ &\quad\Big(\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds\Big)+(\epsilon_0\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_1}+C^2_{\widetilde{g}_2})\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds\\ &\leqslant \mathbb{E}\Big(\|w(0)\|^2\Big)+\widetilde{\lambda}_1^{-1}\Big(\frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}+\frac{\widetilde{\lambda}_1}{\epsilon_0}+\epsilon_0\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_1}+C^2_{\widetilde{g}_2}-2\widetilde{\lambda}_1\Big)\times\\ &\quad\Big(\int_0^t \mathbb{E}\Big(\|w(s)\|^2_{D(A)}\Big)ds\Big)+(\epsilon_0\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_1}+C^2_{\widetilde{g}_2})\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds.\end{aligned} \tag{1.3.21}$$

In order to minimize the right-hand side of (1.3.21), we choose $\epsilon_0=\widetilde{\lambda}_1 C^{-1}_{\widetilde{g}_1}$ such that $\frac{\widetilde{\lambda}_1}{\epsilon_0}+\epsilon_0\widetilde{\lambda}_1^{-1}C^2_{\widetilde{g}_1}$ achieves its minimum value $2C_{\widetilde{g}_1}$. Then, by (1.3.11), we have (1.3.12) as desired. □

In what follows, we will discuss the local stability of stationary solutions to (1.3.3) when the delay terms have particular forms in $C_{-\infty}(V)$, and establish some sufficient conditions in the next corollaries. In this way, it is much easier for us to check the conditions than (1.3.11) in practical application.

Corollary 1.3.3. *Under the same hypotheses and notations in Theorem 1.2.4 and Theorem 1.3.1, let the delay terms* $\widetilde{g}_i(t,u_t)=\widetilde{\mathcal{G}}_i(u(t-h(t)))(i=1,2)$ *satisfy* (1.3.5)-(1.3.7)*, moreover,*

$$2\widetilde{\lambda}_1 \geqslant \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}+\frac{2(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1}+L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*} \tag{1.3.22}$$

is satisfied. If $u(\cdot)$ is any solution of Eq. (1.3.1)*, u_∞ is the unique stationary solution of Eq.* (1.3.3) *and $w(t)=u(t)-u_\infty$, then*

$$\mathbb{E}\Big(\|w(t)\|^2\Big)\leqslant \mathbb{E}\Big(\|w(0)\|^2\Big)+\frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1}+L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*}\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds. \tag{1.3.23}$$

Proof. Taking $\widetilde{h}=s-h(s)$, we obtain $ds=1/(1-h'(s))d\widetilde{h}\leqslant 1/(1-h^*)d\widetilde{h}$. Then, by (1.3.5), it follows

$$\begin{aligned}\int_0^t \|\widetilde{g}_1(s,u_s)-\widetilde{g}_1(s,v_s)\|^2 ds &= \int_0^t \|\widetilde{\mathcal{G}}_1(u(s-h(s)))-\widetilde{\mathcal{G}}_1(v(s-h(s)))\|^2 ds\\ &\leqslant L^2_{\widetilde{\mathcal{G}}_1}\int_0^t \|u(s-h(s))-v(s-h(s))\|^2 ds\end{aligned}$$

$$\leqslant \frac{L^2_{\widetilde{\mathcal{G}}_1}}{1-h^*}\int_{-\infty}^{t}\|u(s)-v(s)\|^2 ds, \tag{1.3.24}$$

Let $C_{\widetilde{g}_1} := \frac{L_{\widetilde{\mathcal{G}}_1}}{\sqrt{1-h^*}}$, we deduce that, there exists $C_{\widetilde{g}_1} > 0$ such that the first inequality in (H14) holds. Similarly, there exists $C_{\widetilde{g}_2} = \frac{L_{\widetilde{\mathcal{G}}_2}}{\sqrt{1-h^*}} > 0$, such that the last one in (H14) holds. Thanks to (1.3.22), we find

$$\begin{aligned} 2\widetilde{\lambda}_1 &\geqslant \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}} + \frac{2(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*} \\ &\geqslant \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}} + 2C_{\widetilde{g}_1} + C^2_{\widetilde{g}_2}, \end{aligned} \tag{1.3.25}$$

which implies (1.3.11). Therefore, by Theorem 1.3.2, we obtain (1.3.12) as desired, and thus,

$$\begin{aligned} \mathbb{E}\Big(\|w(t)\|^2\Big) &\leqslant \mathbb{E}\Big(\|w(0)\|^2\Big) + (C_{\widetilde{g}_1} + C^2_{\widetilde{g}_2})\int_{-\infty}^{0}\mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds \\ &\leqslant \mathbb{E}\Big(\|w(0)\|^2\Big) + \frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*}\int_{-\infty}^{0}\mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds. \end{aligned} \tag{1.3.26}$$

The proof is concluded. □

Corollary 1.3.4. *Assume that the same hypotheses and notations in Theorems 1.2.4 and 1.3.1 hold. Let the delay terms* $\widetilde{g}_i(t,u_t) = \int_{-\infty}^{0}\widetilde{\mathcal{H}}_i(s,u(t+s))ds$ $(i=1,2)$ *satisfy* (1.3.8)-(1.3.10)*, moreover,*

$$2\widetilde{\lambda}_1 \geqslant \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}} + 2\|L_{\widetilde{\mathcal{H}}_1}\|_{L^2(-\infty,0)} + \|L_{\widetilde{\mathcal{H}}_2}\|^2_{L^2(-\infty,0)} \tag{1.3.27}$$

holds. If $u(\cdot)$ *is any solution of Eq.* (1.3.1)*,* u_∞ *is the unique stationary solution of Eq.* (1.3.3) *and* $w(t) = u(t) - u_\infty$*, then*

$$\mathbb{E}\Big(\|w(t)\|^2\Big) \leqslant \mathbb{E}\Big(\|w(0)\|^2\Big) + \Big(\|L_{\widetilde{\mathcal{H}}_1}\|_{L^2(-\infty,0)} + \|L_{\widetilde{\mathcal{H}}_2}\|^2_{L^2(-\infty,0)}\Big)\int_{-\infty}^{0}\mathbb{E}\Big(\|\phi(s)-u_\infty\|^2\Big)ds. \tag{1.3.28}$$

Proof. The proof is similar to the one of Corollary 1.3.3. It follows from (1.3.9) and (1.3.10) that there exist $C_{\widetilde{g}_i} = \|L_{\widetilde{\mathcal{H}}_i}\|_{L^2(-\infty,0)} > 0$ $(i=1,2)$ such that (H14) hold, and then by (1.3.27), we obtain (1.3.11). By Theorem 1.3.2, we deduce (1.3.28) as desired. □

Remark 1.3.5. *In the case of infinite distributed delay, we can prove not only stability of stationary solutions in* $C_{-\infty}(V)$ *(see Corollary 1.3.4) even in* $C_\gamma(V)$*, but also their exponential asymptotic stability will be established as follows.*

1.3.3 Exponential convergence of stationary solutions

Under suitable assumptions, we prove that the solution $u(t)$ to problem (1.3.1) with infinite distributed delay converges exponentially to the unique stationary solution u_∞ of Eq. (1.3.3) in $C_\gamma(V)$ for $\gamma > 0$.

Theorem 1.3.6. *Assume that the same hypotheses and notations in Theorems 1.2.4 and 1.3.1 hold. Let the delay terms $\widetilde{g}_i(t, u_t) = \int_{-\infty}^0 \widetilde{\mathcal{H}}_i(s, u(t+s))ds$ $(i = 1, 2)$ satisfy (1.3.9)-(1.3.10), and moreover, there exists a constant $0 < \rho < 2\gamma$ such that for all $t \geqslant 0$,*

$$2\widetilde{\lambda}_1 \geqslant \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}} + 2(2\rho)^{-\frac{1}{2}}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)} + \frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)} + \rho \tag{1.3.29}$$

is satisfied. If $u(\cdot)$ is any solution of Eq. (1.3.1), u_∞ is the unique stationary solution of Eq. (1.3.3) and $w(t) = u(t) - u_\infty$, then

$$\begin{aligned}\mathbb{E}\Big(\|w(t)\|^2\Big) \leqslant e^{-\rho t}\Big(1 &+ \frac{1}{2\rho(2\gamma-\rho)}((2\rho)^{\frac{1}{2}}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)} \\ &+ \|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)})\Big)\mathbb{E}\Big(\|\phi-u_\infty\|^2_{C_\gamma(V)}\Big),\end{aligned} \tag{1.3.30}$$

and

$$\begin{aligned}\mathbb{E}\Big(\|w_t\|^2_{C_\gamma(V)}\Big) \leqslant e^{-\rho t}\Big(2 &+ \frac{1}{2\rho(2\gamma-\rho)}((2\rho)^{\frac{1}{2}}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)} \\ &+ \|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)})\Big)\mathbb{E}\Big(\|\phi-u_\infty\|^2_{C_\gamma(V)}\Big).\end{aligned} \tag{1.3.31}$$

Proof. Applying Ito's formula to $e^{\rho t}\|w(t)\|^2$ with $0 < \rho < 2\gamma$, we find, for all $t \geqslant 0$,

$$\begin{aligned}e^{\rho t}\|w(t)\|^2 = \|w(0)\|^2 &+ \rho\int_0^t e^{\rho s}\|w(s)\|^2 ds \\ &+ 2\int_0^t e^{\rho s}\langle -\widetilde{A}(w(s)) - \widetilde{B}(u(s)) + \widetilde{B}(u_\infty), w(s)\rangle ds \\ &+ 2\int_0^t e^{\rho s}\Big(\Big(\int_{-\infty}^0 (\widetilde{\mathcal{H}}_1(r, u(s+r)) - \widetilde{\mathcal{H}}_1(r, u_\infty))dr, w(s)\Big)\Big)ds \\ &+ \int_0^t e^{\rho s}\Big\|\int_{-\infty}^0 (\widetilde{\mathcal{H}}_2(r, u(s+r)) - \widetilde{\mathcal{H}}_2(r, u_\infty))dr\Big\|^2_{\mathcal{L}^2(K,V)} ds \\ &+ 2\int_0^t e^{\rho s}\Big(\Big(w(s), \Big(\int_{-\infty}^0 \widetilde{\mathcal{H}}_2(r, u(s+r)) - \widetilde{\mathcal{H}}_2(r, u_\infty)dr\Big)dW\Big)\Big).\end{aligned} \tag{1.3.32}$$

Taking expectation of (1.3.32), then using (0.0.42), we obtain

$$\begin{aligned}&\mathbb{E}\Big(e^{\rho t}\|w(t)\|^2\Big) + 2\int_0^t \mathbb{E}\Big(e^{\rho s}\|w(s)\|^2_{D(A)}\Big)ds \\ &\leqslant \mathbb{E}\Big(\|\phi-u_\infty\|^2_{C_\gamma(V)}\Big) + \rho\widetilde{\lambda}_1^{-1}\int_0^t \mathbb{E}\Big(e^{\rho s}\|w(s)\|^2_{D(A)}\Big)ds\end{aligned}$$

$$
\begin{aligned}
&-2\int_0^t \mathbb{E}\Big(e^{\rho s}\langle \widetilde{B}(u(s))-\widetilde{B}(u_\infty), w(s)\rangle\Big)ds\\
&+2\mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\Big(\int_{-\infty}^0 (\widetilde{\mathcal{H}}_1(r,u(s+r))-\widetilde{\mathcal{H}}_1(r,u_\infty))dr, w(s)\Big)\Big)ds\Big)\\
&+\mathbb{E}\Big(\int_0^t e^{\rho s}\Big\|\int_{-\infty}^0 (\widetilde{\mathcal{H}}_2(r,u(s+r))-\widetilde{\mathcal{H}}_2(r,u_\infty))dr\Big\|^2_{\mathcal{L}^2(K,V)}ds\Big).
\end{aligned}
\tag{1.3.33}
$$

Thanks to (1.3.18), we deduce

$$
-2\int_0^t \mathbb{E}\Big(e^{\rho s}\langle \widetilde{B}(u(s))-\widetilde{B}(u_\infty), w(s)\rangle\Big)ds \leqslant \frac{2\widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}\int_0^t \mathbb{E}\Big(e^{\rho s}\|w(s)\|^2_{D(A)}\Big)ds. \tag{1.3.34}
$$

By (0.0.42), (1.3.9) and the Young inequality with $\hat{\epsilon}>0$ to be specified later on, the fourth line of (1.3.33) is bounded by

$$
\begin{aligned}
&2\mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\Big(\int_{-\infty}^0 (\widetilde{\mathcal{H}}_1(r,u(s+r))-\widetilde{\mathcal{H}}_1(r,u_\infty))dr, w(s)\Big)\Big)ds\Big)\\
&\leqslant 2\widetilde{\lambda}_1^{-\frac{1}{2}}\mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_1}(r)\|w(s+r)\|dr\Big)\cdot\|w(s)\|_{D(A)}ds\Big)\\
&\leqslant \hat{\epsilon}\widetilde{\lambda}_1^{-1}\mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_1}(r)\|w(s+r)\|dr\Big)^2ds\Big)+\frac{1}{\hat{\epsilon}}\mathbb{E}\Big(\int_0^t e^{\rho s}\|w(s)\|^2_{D(A)}ds\Big)\\
&=: \hat{\epsilon}\widetilde{\lambda}_1^{-1}I+\frac{1}{\hat{\epsilon}}\mathbb{E}\Big(\int_0^t e^{\rho s}\|w(s)\|^2_{D(A)}ds\Big),
\end{aligned}
\tag{1.3.35}
$$

where I is estimated as follows. By the Hölder inequality,

$$
\begin{aligned}
I &= \mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_1}(r)\|w(s+r)\|dr\Big)^2ds\Big)\\
&\leqslant \mathbb{E}\Big(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_1}(r)e^{-\gamma r}\|w_s\|_{C_\gamma(V)}dr\Big)^2ds\Big)\\
&= \mathbb{E}\Big(\int_0^t e^{\rho s}\|w_s\|^2_{C_\gamma(V)}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_1}(r)e^{-(\gamma+\rho)r}e^{\rho r}dr\Big)^2ds\Big)\\
&\leqslant \mathbb{E}\Big(\int_0^t e^{\rho s}\|w_s\|^2_{C_\gamma(V)}\Big(\int_{-\infty}^0 L^2_{\widetilde{\mathcal{H}}_1}(r)e^{-2(\gamma+\rho)r}dr\int_{-\infty}^0 e^{2\rho r}dr\Big)ds\Big)\\
&\leqslant \frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}\mathbb{E}\Big(\int_0^t e^{\rho s}\|w_s\|^2_{C_\gamma(V)}ds\Big)\\
&\leqslant \frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}\times\\
&\quad\mathbb{E}\Big(\int_0^t e^{\rho s}\max\Big\{\sup_{\theta\leqslant -s}e^{2\gamma\theta}\|w(s+\theta)\|^2, \sup_{\theta\in[-s,0]}e^{2\gamma\theta}\|w(s+\theta)\|^2\Big\}ds\Big)\\
&\leqslant \frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}\mathbb{E}\Big(\int_0^t \Big(e^{-(2\gamma-\rho)s}\|\phi-u_\infty\|^2_{C_\gamma(V)}
\end{aligned}
$$

$$+\widetilde{\lambda}_1^{-1}\sup_{\theta\in[-s,0]}e^{(2\gamma-\rho)\theta}e^{\rho(s+\theta)}\|w(s+\theta)\|^2_{D(A)}\Big)ds\bigg). \tag{1.3.36}$$

Thanks to (1.3.10), we obtain the following result by using the same method in (1.3.36):

$$\begin{aligned}
&\mathbb{E}\bigg(\int_0^t e^{\rho s}\Big\|\int_{-\infty}^0(\widetilde{\mathcal{H}}_2(r,u(s+r))-\widetilde{\mathcal{H}}_2(r,u_\infty))dr\Big\|^2_{\mathcal{L}^2(K,V)}ds\bigg)\\
&\leqslant \mathbb{E}\bigg(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_2}(r)\|w(s+r)\|dr\Big)^2ds\bigg)\\
&\leqslant \mathbb{E}\bigg(\int_0^t e^{\rho s}\Big(\int_{-\infty}^0 L_{\widetilde{\mathcal{H}}_2}(r)e^{-\gamma r}\|w_s\|_{C_\gamma(V)}dr\Big)^2ds\bigg)\\
&\leqslant \frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}\mathbb{E}\bigg(\int_0^t\Big(e^{-(2\gamma-\rho)s}\|\phi-u_\infty\|^2_{C_\gamma(V)}\\
&\quad+\widetilde{\lambda}_1^{-1}\sup_{\theta\in[-s,0]}e^{(2\gamma-\rho)\theta}e^{\rho(s+\theta)}\|w(s+\theta)\|^2_{D(A)}\Big)ds\bigg).
\end{aligned}\tag{1.3.37}$$

Substituting (1.3.34)-(1.3.37) into (1.3.33), then by $0<\rho<2\gamma$, we have

$$\begin{aligned}
\mathbb{E}\Big(e^{\rho t}\|w(t)\|^2\Big) &\leqslant \mathbb{E}\Big(\|\phi-u_\infty\|^2_{C_\gamma(V)}\Big)+\widetilde{\lambda}_1^{-1}\bigg(\frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}_1}}+\frac{\hat{\epsilon}}{2\rho\widetilde{\lambda}_1}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}+\frac{\widetilde{\lambda}_1}{\hat{\epsilon}}\\
&\quad+\frac{1}{2\rho}\|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}+\rho-2\widetilde{\lambda}_1\bigg)\int_0^t\mathbb{E}\Big(\max_{r\in[0,s]}\{e^{\rho r}\|w(r)\|^2_{D(A)}\}\Big)ds\\
&\quad+\frac{1}{2\rho}\Big(\hat{\epsilon}\widetilde{\lambda}_1^{-1}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}+\|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}\Big)\times\\
&\quad\mathbb{E}\Big(\|\phi-u_\infty\|^2_{C_\gamma(V)}\Big)\int_0^t e^{-(2\gamma-\rho)s}ds.
\end{aligned}\tag{1.3.38}$$

Notice that

$$\min_{\hat{\epsilon}>0}\Big\{\frac{\hat{\epsilon}}{2\rho\widetilde{\lambda}_1}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)}+\frac{\widetilde{\lambda}_1}{\hat{\epsilon}}\Big\}=2(2\rho)^{-\frac{1}{2}}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}, \tag{1.3.39}$$

which is achieved by $\hat{\epsilon}=(2\rho)^{\frac{1}{2}}\widetilde{\lambda}_1\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|^{-1}_{L^2(-\infty,0)}$. Then, we infer from (1.3.38), (1.3.39) and (1.3.29) that (1.3.30) holds.

By (1.3.30), and by $0<\rho<2\gamma$, and thus $e^{(2\gamma-\rho)\theta}\leqslant 1$ when $\theta\leqslant 0$, we find, for all $t\geqslant 0$,

$$\begin{aligned}
\mathbb{E}\Big(\|w_t\|^2_{C_\gamma(V)}\Big)&=\mathbb{E}\Big(\sup_{\theta\leqslant 0}e^{2\gamma\theta}\|w(t+\theta)\|^2\Big)\\
&=\mathbb{E}\Big(\max\Big\{\sup_{\theta\in(-\infty,-t]}e^{2\gamma\theta}\|\phi(t+\theta)-u_\infty\|^2,\ \sup_{\theta\in[-t,0]}e^{2\gamma\theta}\|w(t+\theta)\|^2\Big\}\Big)\\
&=\mathbb{E}\Big(\max\Big\{e^{-2\gamma t}\|\phi-u_\infty\|^2_{C_\gamma(V)},\ \sup_{\theta\in[-t,0]}e^{2\gamma\theta}\|w(t+\theta)\|^2\Big\}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \max\Big\{e^{-\rho t}\mathbb{E}\Big(\|\phi - u_\infty\|^2_{C_\gamma(V)}\Big), e^{-\rho t}\Big(1 + \frac{1}{2\rho(2\gamma-\rho)}((2\rho)^{\frac{1}{2}}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)} \\
&\quad + \|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)})\Big)\mathbb{E}\Big(\|\phi - u_\infty\|^2_{C_\gamma(V)}\Big)\Big\} \\
&\leqslant e^{-\rho t}\Big(2 + \frac{1}{2\rho(2\gamma-\rho)}((2\rho)^{\frac{1}{2}}\|L_{\widetilde{\mathcal{H}}_1}(\cdot)e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)} \\
&\quad + \|L_{\widetilde{\mathcal{H}}_2}(\cdot)e^{-(\gamma+\rho)\cdot}\|^2_{L^2(-\infty,0)})\Big)\mathbb{E}\Big(\|\phi - u_\infty\|^2_{C_\gamma(V)}\Big).
\end{aligned} \tag{1.3.40}
$$

Therefore, the proof is complete. □

Remark 1.3.7. *In Section 1.3.2, we only analyzed the stability (rather than asymptotic stability) in the case of unbounded variable delay and proved in the current subsection the exponential stability in the particular case of distributed delay in $C_\gamma(V)$. Therefore, next, we are interested in studying the asymptotic stability for such variable delay in $C_{-\infty}(V)$. More precisely, on the one hand, we will prove the asymptotic stability by the method of Lyapunov functionals construction. On the other hand, we will prove the polynomial asymptotic stability with proportional delay, which is a particular case of unbounded variable delay.*

1.3.4 Asymptotic stability: the Lyapunov functional method

In this subsection, we first investigate the asymptotic stability of the trivial solution of the following abstract nonlinear stochastic partial functional differential systems by constructing suitable Lyapunov functionals. In the last part of this subsection, we will apply the abstract results to Eq. (0.0.3).

Now, we consider the following problem:

$$
\begin{cases}
du(t) = (\hat{A}(t,u(t)) + \hat{F}_1(t,u_t))dt + \hat{F}_2(t,u_t)dW(t), \quad \forall\, t \in [0,T], \\
u(t) = \phi(t), \ t \in (-\infty, 0],
\end{cases} \tag{1.3.41}
$$

where $\hat{A}(t,\cdot) : D(A) \to (D(A))^*$ satisfies $\langle \hat{A}(t,u), u\rangle \leqslant 0$, for all $u \in D(A)$, $\hat{F}_1(t,\cdot) : C_{-\infty}(V) \to V$ and $\hat{F}_2(t,\cdot) : C_{-\infty}(V) \to \mathcal{L}^2(K,V)$ satisfy the following conditions: $\hat{F}_1(t,0) = \hat{F}_2(t,0) = 0$ and they are Lipschitz continuous, that is, there exist $L_{\hat{F}_i} > 0$ $(i = 1,2)$ such that for all $t \geqslant 0$ and $\eta, \zeta \in C_{-\infty}(V)$,

$$
\begin{aligned}
&\|\hat{F}_1(t,\eta) - \hat{F}_1(t,\zeta)\| \leqslant L_{\hat{F}_1}\|\eta - \zeta\|_{C_{-\infty}(V)}, \\
&\|\hat{F}_2(t,\eta) - \hat{F}_2(t,\zeta)\|_{\mathcal{L}^2(K,V)} \leqslant L_{\hat{F}_2}\|\eta - \zeta\|_{C_{-\infty}(V)}.
\end{aligned} \tag{1.3.42}
$$

By the similar estimates as in Section 1.2, the well-posedness of (1.3.41) can be proved. Fixed $T > 0$ and given an initial value $\phi \in L^2(\Omega, C_{-\infty}(V))$, a solution to (1.3.41) is a stochastic process $u \in I^2(0,T;D(A)) \cap L^2(\Omega, L^\infty(0,T;V))$ satisfying

$$
\begin{cases}
u(t) = \phi(0) + \displaystyle\int_0^t \hat{A}(s,u(s))ds + \int_0^t \hat{F}_1(s,u_s))ds \\
\qquad\qquad + \displaystyle\int_0^t \hat{F}_2(s,u_s)dW(s), \ P-a.s., \ \forall\, t \in [0,T], \\
u(t) = \phi(t), \ t \in (-\infty, 0],
\end{cases} \tag{1.3.43}
$$

where the first equation is understood in $(D(A))^*$.

We denote by $u(\cdot;\phi)$ the solution of Eq. (1.3.41) corresponding to the initial condition ϕ.

Definition 1.3.8. *The trivial solution of Eq.* (1.3.41) *is said to be p-stable, with $p > 0$, if for any $\epsilon > 0$, there exists $\delta > 0$ such that $\mathbb{E}(\|u(t;\phi)\|^p) < \epsilon$, for all $t \geqslant 0$, provided that $\|\phi\|_1^p := \sup_{\theta\leqslant 0}\mathbb{E}(\|\phi(\theta)\|^p) < \delta$. If, besides, $\lim_{t\to+\infty}\mathbb{E}(\|u(t;\phi)\|^p) = 0$ for every initial function ϕ, then the trivial solution of Eq.* (1.3.41) *is called asymptotically p-stable. In particular, if $p = 2$, then the trivial solution of the system* (1.3.41) *is called asymptotically mean square stable.*

Consider the stochastic differential of the process $\eta(t) = x(t, u(t))$, where $u(t)$ is a solution of the system (1.3.41) and the function $x : [0,\infty)\times D(A) \to \mathbb{R}_+$ has continuous partial derivatives:

$$x_t' = \frac{\partial x(t,u)}{\partial t},\ x_u' = \frac{\partial x(t,u)}{\partial u},\ x_{uu}'' = \frac{\partial^2 x(t,u)}{\partial u^2}.$$

Applying Ito's formula to $\eta(t)$ we obtain

$$d\eta(t) = Lx(t,u(t))dt + \langle x_u', \hat{F}_2(t,u_t)dW(t)\rangle, \tag{1.3.44}$$

where $\langle\cdot,\cdot\rangle$ denotes inner products in Hilbert spaces, and L is called the generator of Eq. (1.3.41), defined by

$$Lx(t,u_t) = x_t'(t,u(t)) + \langle x_u'(t,u(t)), \hat{A}(t,u(t)) + \hat{F}_1(t,u_t)\rangle + \frac{1}{2}\mathrm{tr}\Big(x_{uu}''(t,u(t))\hat{F}_2(t,u_t)Q\hat{F}_2^*(t,u_t)\Big).$$

We then apply the generator L to some functionals $U(t,\xi) : [0,\infty)\times L^2(\Omega, C_{-\infty}(V)) \to \mathbb{R}_+$. In addition, assume that $U(t,\xi) = U(t,\xi(0),\xi(\theta)), \theta < 0$, and for $\xi = u_t$, then set

$$\begin{aligned} &U_\xi(t,u) = U(t,\xi) = U(t,u_t) = U(t,u,u(t+\theta)),\ \theta < 0,\\ &u = \xi(0) = u(t). \end{aligned} \tag{1.3.45}$$

Let D be the universe of functionals which satisfy conditions (1.3.45). Any functional $U_\xi(t,u) \in D$ has a continuous derivative with respect to t and two continuous derivatives with regard to u. Then,

$$\begin{aligned} LU(t,u_t) = {}& \frac{\partial U_\xi(t,u(t))}{\partial t} + \Big\langle \frac{\partial U_\xi(t,u(t))}{\partial u}, \hat{A}(t,u(t)) + \hat{F}_1(t,u_t)\Big\rangle\\ &+ \frac{1}{2}\mathrm{tr}\Big(\frac{\partial^2 U_\xi(t,u(t))}{\partial u^2}\hat{F}_2(t,u_t)Q\hat{F}_2^*(t,u_t)\Big). \end{aligned}$$

Thanks to Ito's formula, we obtain, for functionals from D,

$$\mathbb{E}\Big(U(t,u_t) - U(s,u_s)\Big) = \int_s^t \mathbb{E}\Big(LU(r,u_r)\Big)dr,\quad t \geqslant s. \tag{1.3.46}$$

In the next proposition, we generalize the idea of Shaikhet in [109, Theorem 2.1] to the infinite delay version of stochastic partial differential equations. Let us now prove the following result which plays an important role in our stability investigation.

Proposition 1.3.9. *Suppose that there exists a continuous functional $U(t,\xi) : [0,\infty)\times L^p(\Omega, C_{-\infty}(V)) \to \mathbb{R}_+$ such that for the solution $u(t)$ of problem (1.3.41) and $p \geqslant 2$, the following inequalities hold for some positive constants μ_1, μ_2 and μ_3,*

$$\begin{aligned}
&\mathbb{E}(U(t,u_t)) \geqslant \mu_1 \mathbb{E}(\|u(t)\|^p),\ \forall\, t \geqslant 0,\\
&\mathbb{E}(U(0,\phi)) \leqslant \mu_2 \|\phi\|_1^p,\\
&\mathbb{E}(U(t,u_t) - U(0,\phi)) \leqslant -\mu_3 \int_0^t \mathbb{E}(\|u(s)\|^p)ds,\ \forall\, t \geqslant 0.
\end{aligned} \tag{1.3.47}$$

Then the trivial solution of equation (1.3.41) is asymptotically p-stable, that is,

$$\lim_{t\to+\infty} \mathbb{E}(\|u(t)\|^p) = 0. \tag{1.3.48}$$

Proof. We infer from (1.3.47) that

$$\mu_1 \mathbb{E}(\|u(t)\|^p) \leqslant \mathbb{E}(U(t,u_t)) \leqslant \mathbb{E}(U(0,\phi)) \leqslant \mu_2 \|\phi\|_1^p = \mu_2 \sup_{\theta\leqslant 0} \mathbb{E}(\|\phi(\theta)\|^p), \tag{1.3.49}$$

which proves the trivial solution of equation (1.3.41) is p-stable. Taking supremum of (1.3.49) with respect to t, we find

$$\sup_{t\geqslant 0} \mathbb{E}(\|u(t)\|^p) \leqslant \frac{\mu_2}{\mu_1} \|\phi\|_1^p. \tag{1.3.50}$$

Thanks to the last two lines of (1.3.47), we obtain

$$\int_0^\infty \mathbb{E}(\|u(s)\|^p)ds \leqslant \frac{1}{\mu_3} \mathbb{E}(U(0,\phi)) \leqslant \frac{\mu_2}{\mu_3} \|\phi\|_1^p < \infty. \tag{1.3.51}$$

Applying the generator L to the function $U(t,u_t) = \|u(t)\|^p$, by the Young inequality and (1.3.42), we have

$$\begin{aligned}
LU(t,u_t) = L\|u(t)\|^p &= p\|u\|^{p-2}\langle \hat{A}(u), u\rangle + p\|u\|^{p-2}((\hat{F}_1(t,u_t), u))\\
&\quad + \frac{p}{2}\|u(t)\|^{p-2}\|\hat{F}_2(t,u_t)\|^2_{\mathcal{L}^2(K,V)} + \frac{p(p-2)}{2}\|u(t)\|^{p-2}\|\hat{F}_2(t,u_t)\|^2_{\mathcal{L}^2(K,V)}\\
&\leqslant p\|u\|^{p-2}((\hat{F}_1(t,u_t), u)) + \frac{p(p-1)}{2}\|u(t)\|^{p-2}\|\hat{F}_2(t,u_t)\|^2_{\mathcal{L}^2(K,V)}\\
&\leqslant p\|u\|^{p-2}(\|\hat{F}_1(t,u_t)\| \cdot \|u\|) + \frac{p(p-1)}{2}\|u(t)\|^{p-2}\|\hat{F}_2(t,u_t)\|^2_{\mathcal{L}^2(K,V)}\\
&\leqslant \frac{p}{2}\|u(t)\|^p + \frac{p}{2}L^2_{\hat{F}_1}\|u(t)\|^{p-2}\|u_t\|^2_{C_{-\infty}(V)} + \frac{p(p-1)}{2}L^2_{\hat{F}_2}\|u(t)\|^{p-2}\|u_t\|^2_{C_{-\infty}(V)}\\
&= \frac{p}{2}\|u(t)\|^p + \widehat{c}_1\|u(t)\|^{p-2}\|u_t\|^2_{C_{-\infty}(V)}\\
&\leqslant \Big(\frac{p}{2} + \widehat{c}_2\Big)\|u(t)\|^p + \widehat{c}_3\|u_t\|^p_{C_{-\infty}(V)}\\
&\leqslant \Big(\frac{p}{2} + \widehat{c}_2 + \widehat{c}_3\Big)\|u_t\|^p_{C_{-\infty}(V)} =: \widehat{c}_4\|u_t\|^p_{C_{-\infty}(V)},
\end{aligned} \tag{1.3.52}$$

where $\widehat{c}_1 = \frac{p}{2}L^2_{\hat{F}_1} + \frac{p(p-1)}{2}L^2_{\hat{F}_2}, \widehat{c}_2 = \frac{\widehat{c}_1(p-2)}{p}$ and $\widehat{c}_3 = \frac{2\widehat{c}_1}{p}$. The above inequality implies

$$\mathbb{E}(LU(t,u_t)) \leqslant \widehat{c}_4\mathbb{E}(\|u_t\|^p_{C_{-\infty}(V)}) =: \widehat{c}_5 < \infty. \tag{1.3.53}$$

Combining (1.3.46) and (1.3.53) , we obtain, for any $t \geqslant s \geqslant 0$,

$$\left|\mathbb{E}(\|u(t)\|^p) - \mathbb{E}(\|u(s)\|^p)\right| \leqslant \widehat{c}_5(t-s), \tag{1.3.54}$$

which implies that $\mathbb{E}(|u(r)|^p)$ is Lipschitz continuous, together with (1.3.50) and (1.3.51), shows that $\mathbb{E}(\|u(t)\|^p) \to 0$ as $t \to +\infty$. This completes the proof. □

We state our asymptotic stability result by applying the previous abstract results to our model in the next theorem.

Theorem 1.3.10. *Assume that the same hypotheses and notations in Theorems 1.2.4 and 1.3.1 hold. In addition, let the delay terms $\widetilde{g}_i(t,u_t) = \widetilde{\mathcal{G}}_i(u(t-h(t)))(i=1,2)$ satisfy (1.3.5)-(1.3.7), $\widetilde{f} = 0$ and*

$$2\widetilde{\lambda}_1 \geqslant \frac{2(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*}. \tag{1.3.55}$$

Then $u_\infty = 0$ is the unique stationary solution to problem (1.3.3). Moreover, the trivial solution of (1.3.1) is asymptotically mean square stable.

Proof. We first infer from the assumption $\widetilde{f} = 0$ and Theorem 1.3.1 that $u_\infty = 0$ is the unique stationary solution to Eq. (1.3.3). We then let

$$U(t,\xi) = \|\xi(0)\|^2 + \frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*}\int_{-h(t)}^0 \|\xi(s)\|^2 ds, \tag{1.3.56}$$

if ξ is replaced by u_t, then

$$U(t,u_t) = \|u(t)\|^2 + \frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*}\int_{t-h(t)}^t \|u(s)\|^2 ds, \tag{1.3.57}$$

and then let $\hat{A}(t,u) = -\widetilde{A}u(t) - \widetilde{B}(u(t))$, $\hat{F}_1(t,u_t) = \widetilde{g}_1(t,u_t) = \widetilde{\mathcal{G}}_1(u(t-h(t)))$, $\hat{F}_2(t,u_t) = \widetilde{g}_2(t,u_t) = \widetilde{\mathcal{G}}_2(u(t-h(t)))$ in (1.3.41), by (0.0.39), (0.0.42), (1.3.5) and (1.3.6), we obtain

$$\begin{aligned}
L\|u(t)\|^2 &= 2\langle -\widetilde{A}(u) - \widetilde{B}(u), u\rangle + 2((\widetilde{\mathcal{G}}_1(u(t-h(t))), u)) + \|\widetilde{\mathcal{G}}_2(u(t-h(t)))\|^2_{\mathcal{L}^2(K,V)}\\
&\leqslant -2\|u\|^2_{D(A)} + 2\|\widetilde{\mathcal{G}}_1(u(t-h(t)))\|\|u\| + L^2_{\widetilde{\mathcal{G}}_2}\|u(t-h(t))\|^2\\
&\leqslant -2\widetilde{\lambda}_1\|u\|^2 + \frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1}}{(1-h^*)}\|u\|^2 + (1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1}\|u(t-h(t))\|^2 + L^2_{\widetilde{\mathcal{G}}_2}\|u(t-h(t))\|^2\\
&= \Big(-2\widetilde{\lambda}_1 + \frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1}}{(1-h^*)}\Big)\|u\|^2 + \Big((1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}\Big)\|u(t-h(t))\|^2,
\end{aligned} \tag{1.3.58}$$

then

$$
\begin{aligned}
LU(t,u_t) &= L\Big(\|u(t)\|^2 + \frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*}\int_{t-h(t)}^{t}\|u(s)\|^2 ds\Big) \\
&\leqslant L\|u(t)\|^2 + \frac{(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*}\|u(t)\|^2 - \Big((1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}\Big)\|u(t-h(t))\|^2 \\
&\leqslant \Big(-2\widetilde{\lambda}_1 + \frac{2(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}_1} + L^2_{\widetilde{\mathcal{G}}_2}}{1-h^*}\Big)\|u(t)\|^2, \qquad (1.3.59)
\end{aligned}
$$

which, on account of (1.3.55), implies $LU(t,u_t) \leqslant 0$. Moreover, the functional $U(t,u_t)$ defined in (1.3.57) satisfies the conditions in Proposition 1.3.9, and thus the trivial solution of (1.3.1) is asymptotically mean square stable in the sense of Definition 1.3.8. □

Remark 1.3.11. *By using the method of Lyapunov functionals construction, we obtain the asymptotic stability of the trivial solution to* (1.3.1) *with unbounded variable delay. Notice that condition* (1.3.55) *becomes exactly condition* (1.3.22) *when* $\tilde{f} = 0$. *Therefore, Theorem 1.3.10 ensures asymptotic stability under the same sufficient conditions which ensures only stability in Corollary 1.3.3, which means that the construction of Lyapunov functionals may provide better stability results. Furthermore, our analysis is also valid to study the asymptotic stability for the general case, that is, if the stationary solution is not the origin, in this case, we can shift it to the origin by a coordinate transformation.*

1.3.5 Polynomial asymptotic stability for a particular case of unbounded variable delay

In this subsection, we study the polynomial asymptotic behaviour of solutions to deterministic pantograph equations. In the particular case of proportional delay, we not only prove asymptotic stability but we can determine that the rate of convergence is at least polynomial. Now, let us consider the following deterministic pantograph equation:

$$
\begin{cases} X'(t) = a_1X(t) + a_2X(\theta t), \ \forall\, t \geqslant 0, \\ X(0) = X_0, \end{cases} \qquad (1.3.60)
$$

where $a_1, a_2 \in \mathbb{R}$, and $\theta \in (0,1)$.

Recall that the Dini derivative D^+F, where F is a continuous real-valued function of a real variable defined by

$$
D^+F = \limsup_{\delta\downarrow 0}\frac{F(t+\delta)-F(t)}{\delta}.
$$

Thanks to [3, Lemma 3.4], we present the following result which is useful to obtain the polynomial asymptotic stability of stationary solutions to (1.3.60).

Lemma 1.3.12. *Let $a_1 \in \mathbb{R}, a_2 > 0$ and $\theta \in (0,1)$. Assume that X satisfies (1.3.60) with $X_0 > 0$. If there exists a continuous non-negative function $t \mapsto Y(t) : \mathbb{R}_+ \mapsto \mathbb{R}_+$,*

$$D^+Y(t) \leqslant a_1 Y(t) + a_2 Y(\theta t), \quad t \geqslant 0 \tag{1.3.61}$$

with $0 < Y(0) < X_0$, then $Y(t) \leqslant X(t)$ for all $t \geqslant 0$.

Lemma 1.3.13. *Assume that X is the solution of (1.3.60). If $a_1 < 0$ and $a_2 \in \mathbb{R}$, there exists a constant $M_0 = M_0(a_1, a_2, \theta) > 0$,*

$$\limsup_{t\to+\infty} \frac{|X(t)|}{t^\beta} = M_0|X_0|, \tag{1.3.62}$$

where $\beta \in \mathbb{R}$ satisfies

$$a_1 + |a_2|\theta^\beta = 0. \tag{1.3.63}$$

Then, for some $M = M(a_1, a_2, \theta) > 0$,

$$|X(t)| \leqslant M|X_0|(1+t)^\beta, \quad t \geqslant 0. \tag{1.3.64}$$

Proof. The proof is similar to [3, Lemma 3.5], thus the details are omitted here. □

Note that the polynomial asymptotic stability of the trivial solution to (1.3.60) is presented in the above Lemma when $\beta < 0$. In the following, we apply the idea to derive the polynomial asymptotic stability of stationary solution to (1.3.1).

Theorem 1.3.14. *Assume that the same hypotheses and notations in Theorems 1.2.4 and 1.3.1 hold. In addition, let the system (1.3.1) satisfy $\widetilde{f} = 0$, the delay terms $\widetilde{g}_i(t, u_t) = L_{\widetilde{g}_i} u(\theta t)$ $(i = 1, 2)$ with $\theta \in (0,1)$ and $2\widetilde{\lambda}_1 > 2|L_{\widetilde{g}_1}| + L^2_{\widetilde{g}_2}$, then the origin is the unique stationary solution of Eq. (1.3.3), moreover, any solution $u(t)$ of Eq. (1.3.1) converges to zero polynomially, that is, there exist $\widetilde{M} = \widetilde{M}(L_{\widetilde{g}_1}, L_{\widetilde{g}_2}, \widetilde{\lambda}_1, \theta) > 0$ and $\beta < 0$,*

$$\mathbb{E}(\|u(t;\phi)\|^2) \leqslant \widetilde{M}\mathbb{E}(\|\phi\|^2_{C_{-\infty}(V)})(1+t)^\beta, \quad t \geqslant 0, \tag{1.3.65}$$

where β satisfies $-2\widetilde{\lambda}_1 + |L_{\widetilde{g}_1}| + (|L_{\widetilde{g}_1}| + L^2_{\widetilde{g}_2})\theta^\beta = 0$.

Proof. The conclusion that the origin is the unique stationary solution of Eq. (1.3.3) follows from $\widetilde{f} = 0$ and Theorem 1.3.1. Applying Ito's formula to $\|u(t)\|^2$, then taking expectation, we obtain

$$\begin{aligned}
&\mathbb{E}(\|u(t)\|^2) - \mathbb{E}(\|u(0)\|^2) \\
&\leqslant -2\mathbb{E}\Big(\int_0^t \|u(s)\|^2_{D(A)} ds\Big) + |L_{\widetilde{g}_1}|\mathbb{E}\Big(\int_0^t \|u(s)\|^2 ds\Big) + (|L_{\widetilde{g}_1}| + L^2_{\widetilde{g}_2})\mathbb{E}\Big(\int_0^t \|u(\theta s)\|^2 ds\Big) \\
&\leqslant (-2\widetilde{\lambda}_1 + |L_{\widetilde{g}_1}|)\mathbb{E}\Big(\int_0^t \|u(s)\|^2 ds\Big) + (|L_{\widetilde{g}_1}| + L^2_{\widetilde{g}_2})\mathbb{E}\Big(\int_0^t \|u(\theta s)\|^2 ds\Big), \ \forall\, t > 0,
\end{aligned}$$

where we used (0.0.42). Let $v(t) = \mathbb{E}(\|u(t)\|^2)$, then

$$v'(t) \leqslant (-2\widetilde{\lambda}_1 + |L_{\widetilde{g}_1}|)v(t) + (|L_{\widetilde{g}_1}| + L^2_{\widetilde{g}_2})v(\theta t). \tag{1.3.66}$$

By Lemmas 1.3.12-1.3.13, we obtain that there exist $\widetilde{M} = \widetilde{M}(L_{\widetilde{g}_1}, L_{\widetilde{g}_2}, \widetilde{\lambda}_1, \theta) > 0$ and $\beta \in \mathbb{R}$,

$$v(t) \leqslant \widetilde{M}v(0)(1+t)^\beta. \tag{1.3.67}$$

Since $-2\widetilde{\lambda}_1 + 2|L_{\widetilde{g}_1}| + L^2_{\widetilde{g}_2} < 0$, we deduce $\beta < 0$ and

$$\mathbb{E}(\|u(t)\|^2) \leqslant \widetilde{M}\mathbb{E}(\|\phi\|^2)(1+t)^\beta \leqslant \widetilde{M}\mathbb{E}(\|\phi\|^2_{C_{-\infty}(V)})(1+t)^\beta.$$

The proof is complete. □

Remark 1.3.15. *As a matter of fact, we can take into account a more general case in the form of* $\widetilde{g}_i(t,\xi) = \widetilde{G}_i(\xi(-(1-\theta)t))$, *where* $\widetilde{G}_i(\cdot)$ *is Lipschitz continuous.*

Chapter 2

Non-autonomous stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay on unbounded domains

This chapter is devoted to investigating mean dynamics and stability analysis for stochastic 3D LANS equations driven by infinite delay on unbounded domains. We first prove the existence of a unique solution to stochastic 3D LANS equations with infinite delay when the non-delayed external force is locally integrable, the delay term is globally Lipschitz continuous and the nonlinear diffusion term is locally Lipschitz continuous. This enables us to define a mean random dynamical system. Besides, we find that such a dynamical system possesses a unique weak pullback mean random attractor. Furthermore, we prove the existence and uniqueness of stationary solutions to the corresponding deterministic equation via the classical Galerkin method, the Lax-Milgram and the Brouwer fixed theorems. The stability results of stationary solutions are also considered, including local stability, exponential stability, asymptotic stability via Lyapunov method and polynomial asymptotic stability.

2.1 Some suitable hypotheses

In order to analyze our problem, we need to establish some suitable assumptions. We first suppose that the non-delayed external force is locally integrable, that is, for all $\tau \in \mathbb{R}, T > 0$,

$$f \in I^2(\tau, \tau + T; \mathbb{H}^{-1}(O)). \tag{2.1.1}$$

Then, we need some assumptions on delay term.

Let $g : \mathbb{R} \times C_\gamma(V) \to \mathbb{H}^{-1}(O)$ satisfy the following conditions:

(G21) For any $\eta \in C_\gamma(V)$, $g(\cdot, \eta)$ is measurable;

(G22) $g(\cdot, 0) = 0$;

(G23) There exists a constant $L_g > 0$ such that for all $t \in \mathbb{R}$ and $\eta, \zeta \in C_\gamma(V)$,

$$\|g(t,\eta) - g(t,\zeta)\|_{\mathbb{H}^{-1}(\mathcal{O})} \leqslant L_g \|\eta - \zeta\|_{C_\gamma(V)};$$

(G24) There exists a constant $C_g > 0$ such that, for all $\tau \in \mathbb{R}, t \geqslant \tau$ and $u, v \in C^0((-\infty, t]; V)$,

$$\int_\tau^t \|g(s,u_s) - g(s,v_s)\|^2_{\mathbb{H}^{-1}(\mathcal{O})} ds \leqslant C_g^2 \int_{-\infty}^t \|u(s) - v(s)\|^2 ds;$$

(G25) There exists a constant $\widetilde{C}_g > 0$ such that, for all $\tau \in \mathbb{R}, t \geqslant \tau$, all decreasing function $\varpi \in C^0([\tau, t])$ and $u, v \in C^0((-\infty, t]; V)$,

$$\int_\tau^t \varpi(s)\|g(s,u_s) - g(s,v_s)\|^2_{\mathbb{H}^{-1}(\mathcal{O})} ds \leqslant \widetilde{C}_g^2 \int_\tau^t \varpi(s)\|u(s) - v(s)\|^2 ds.$$

For the diffusion coefficient, we will impose some suitable assumptions as follows.

Let $\sigma : \mathbb{R} \times V \to \mathcal{L}^2(K, \mathbb{L}^2(\mathcal{O}))$ be measurable, and locally Lipschitz continuous, that is, for every $r > 0$, there exists a positive constant L_r depending on r such that for all $t \in \mathbb{R}, w_1, w_2 \in V$ with $\|w_1\| \leqslant r$ and $\|w_2\| \leqslant r$,

$$\|\sigma(t,w_1) - \sigma(t,w_2)\|_{\mathcal{L}^2(K,\mathbb{L}^2(\mathcal{O}))} \leqslant L_r \|w_1 - w_2\|. \tag{2.1.2}$$

Besides, suppose that there exists a constant $L_\sigma > 0$ such that, for all $(t,w) \in \mathbb{R} \times V$,

$$\|\sigma(t,w)\|_{\mathcal{L}^2(K,\mathbb{L}^2(\mathcal{O}))} \leqslant L_\sigma(1 + \|w\|). \tag{2.1.3}$$

Now, let us define $\widetilde{f}(t)$ as

$$((\widetilde{f}(t), w)) = \langle f(t), w\rangle_{-1}, \ \forall\, (t,w) \in \mathbb{R} \times V. \tag{2.1.4}$$

Define $\widetilde{g} : \mathbb{R} \times C_\gamma(V) \to V$ as

$$((\widetilde{g}(t,\eta), w)) = \langle g(t,\eta), w\rangle_{-1}, \ \forall\, (t,\eta,w) \in \mathbb{R} \times C_\gamma(V) \times V. \tag{2.1.5}$$

By (2.1.1) and (2.1.4), we obtain $\widetilde{f} \in I^2(\tau, \tau + T; (D(A))^*)$ for any $\tau \in \mathbb{R}$ and $T > 0$. It follows from (G21)-(G25) and (2.1.5) that $\widetilde{g} : \mathbb{R} \times C_\gamma(V) \to V$ satisfies the following conditions:

(H21) For any $\eta \in C_\gamma(V)$, $\widetilde{g}(\cdot, \eta)$ is measurable;

(H22) $\widetilde{g}(\cdot, 0) = 0$;

(H23) Setting $L_{\widetilde{g}} = L_g$, we obtain, for all $t \in \mathbb{R}$ and $\eta, \zeta \in C_\gamma(V)$,

$$\|\widetilde{g}(t,\eta) - \widetilde{g}(t,\zeta)\| \leqslant L_{\widetilde{g}} \|\eta - \zeta\|_{C_\gamma(V)}.$$

It follows from (H22) and (H23) that, for all $\eta \in C_\gamma(V)$,

$$\|\widetilde{g}(t,\eta)\| \leqslant L_{\widetilde{g}} \|\eta\|_{C_\gamma(V)}. \tag{2.1.6}$$

(H24) Letting $C_{\widetilde{g}} = C_g$, for all $\tau \in \mathbb{R}, t \geqslant \tau$ and $u, v \in C^0((-\infty, t]; V)$,

$$\int_\tau^t \|\widetilde{g}(s, u_s) - \widetilde{g}(s, v_s)\|^2 ds \leqslant C_{\widetilde{g}}^2 \int_{-\infty}^t \|u(s) - v(s)\|^2 ds;$$

(H25) Taking $\widetilde{C}_{\widetilde{g}} = \widetilde{C}_g$, for all $\tau \in \mathbb{R}, t \geqslant \tau$, all decreasing function $\varpi \in C^0([\tau, t])$ and $u, v \in C^0((-\infty, t]; V)$,

$$\int_\tau^t \varpi(s)\|\widetilde{g}(s, u_s) - \widetilde{g}(s, v_s)\|^2 ds \leqslant \widetilde{C}_{\widetilde{g}} \int_\tau^t \varpi(s)\|u(s) - v(s)\|^2 ds.$$

Finally, we define $\widetilde{\sigma} : \mathbb{R} \times V \to \mathcal{L}^2(K, V)$ as

$$\widetilde{\sigma}(t, \upsilon) = (I + \alpha A)^{-1} \circ \mathcal{P} \circ \sigma(t, \upsilon), \ \forall\ (t, \upsilon) \in \mathbb{R} \times V,$$

where I is the identity operator in H and $I + \alpha A : D(A) \to H$ is bijective. Moreover,

$$(((I + \alpha A)^{-1} u, w)) = (u, w), \ \forall\ u \in H, \ w \in V.$$

Hence, for the orthonormal basis $\{e_j\}$ of K, we have

$$(\sigma(t, \eta)e_j, w) = ((I + \alpha A)\widetilde{\sigma}(t, \eta)e_j, w) = ((\widetilde{\sigma}(t, \eta)e_j, w)),$$

for all $j \geqslant 1$ and $(t, \upsilon, w) \in \mathbb{R} \times V \times D(A)$, by (0.0.31), we further obtain that

$$\begin{aligned}
\Big(\int_\tau^t \sigma(s, \eta) dW(s), w\Big) &= \sum_{j=1}^\infty \int_\tau^t (\sigma(s, \eta)e_j, w) d\beta^j(s) \\
&= \sum_{j=1}^\infty \int_\tau^t ((\widetilde{\sigma}(s, \eta)e_j, w)) d\beta^j(s) \\
&= \Big(\Big(\int_\tau^t \widetilde{\sigma}(s, \eta) dW(s), w\Big)\Big). \qquad (2.1.7)
\end{aligned}$$

It follows from (2.1.2), (2.1.3) and (2.1.7) that $\widetilde{\sigma} : \mathbb{R} \times V \to \mathcal{L}^2(K, V)$ is measurable, and locally Lipschitz continuous, that is, for every $\widetilde{r} > 0$, there exists a positive constant $L_{\widetilde{r}}$ depending on $\widetilde{r}$ such that, for all $t \in \mathbb{R}, w_1, w_2 \in V$ with $\|w_1\| \leqslant \widetilde{r}$ and $\|w_2\| \leqslant \widetilde{r}$,

$$\|\widetilde{\sigma}(t, w_1) - \widetilde{\sigma}(t, w_2)\|_{\mathcal{L}^2(K,V)} \leqslant L_{\widetilde{r}}\|w_1 - w_2\|. \qquad (2.1.8)$$

In addition, there exists a constant $L_{\widetilde{\sigma}} > 0$ such that, for all $(t, w) \in \mathbb{R} \times V$,

$$\|\widetilde{\sigma}(t, w)\|_{\mathcal{L}^2(K,V)} \leqslant L_{\widetilde{\sigma}}(1 + \|w\|). \qquad (2.1.9)$$

2.2 Well-posedness and mean dynamics

In this section, we investigate the well-posedness and mean dynamics of the stochastic 3D LANS system (0.0.4) with infinite delay and locally Lipschitz nonlinear noise.

2.2.1 Well-posedness of stochastic 3D LANS equations

In this subsection, our main aim is to prove the well-posedness of the stochastic 3D LANS equation (0.0.4). Based on the previous operators and assumptions, we consider the following abstract equation:

$$\begin{cases} \dfrac{du}{dt} + \widetilde{A}u(t) + \widetilde{B}(u(t)) = \widetilde{f}(t) + \widetilde{g}(t, u_t) + \widetilde{\sigma}(t, u)\dfrac{dW(t)}{dt}, \quad \forall\, t > \tau, \\ u(\tau + s) = \phi(s),\, s \in (-\infty, 0]. \end{cases} \tag{2.2.1}$$

Definition 2.2.1. *Suppose that $\phi \in L^2(\Omega, C_\gamma(V))$ (which is a $\mathcal{F}_0$-progressively measurable V-valued processes) and $\tau \in \mathbb{R}$. A stochastic process u defined on $\mathbb{R}$ is called a solution to system (2.2.1) if*

$$u \in I^2(\tau, \tau + T; D(A)) \cap L^2(\Omega, L^\infty(\tau, \tau + T; V)), \ \forall\, T > 0,$$

$u_\tau = \phi$, and the system (2.2.1) is satisfied in $(D(A))^*$, that is, for almost all $\omega \in \Omega$,

$$\begin{aligned} &((u(t), w)) + \int_\tau^t \langle \widetilde{A}u(s), w\rangle ds + \int_\tau^t \langle \widetilde{B}(u(s)), w\rangle ds \\ &\quad = ((\phi(0), w)) + \int_\tau^t \Big(\big(\widetilde{f}(s) + \widetilde{g}(s, u_s), w\big)\Big) ds + \Big(\Big(w, \int_\tau^t \widetilde{\sigma}(s, u) dW(s)\Big)\Big), \end{aligned} \tag{2.2.2}$$

for all $t \geqslant \tau$ and $w \in D(A)$.

In order to prove the well-posedness of problem (2.2.1), for every $n \in \mathbb{N}$, we introduce a sequence of cut-off functions $\xi_n : V \to V$ ($n \in \mathbb{N}$) defined by

$$\xi_n(\upsilon) = \begin{cases} \upsilon, & \text{for } \|\upsilon\| \leqslant n, \\ \frac{n\upsilon}{\|\upsilon\|}, & \text{for } \|\upsilon\| > n. \end{cases} \tag{2.2.3}$$

Then, $\xi_n : V \to V$ is globally Lipschitz continuous:

$$\|\xi_n(\upsilon_1) - \xi_n(\upsilon_2)\| \leqslant \|\upsilon_1 - \upsilon_2\|, \ \ \forall\, \upsilon_1, \upsilon_2 \in V, \tag{2.2.4}$$

and

$$\|\xi_n(\upsilon)\| \leqslant n, \ \ \forall\, \upsilon \in V. \tag{2.2.5}$$

Given $n \in \mathbb{N}$, let $\widetilde{\sigma}_n(t, \upsilon) = \widetilde{\sigma}(t, \xi_n(\upsilon))$, then we infer from (2.1.8), (2.1.9), (2.2.4) and (2.2.5) that, for every $n \in \mathbb{N}$, there exists a positive constant L_n such that for all $t \in \mathbb{R}, \upsilon_1, \upsilon_2 \in V$,

$$\|\widetilde{\sigma}_n(t, \upsilon_1) - \widetilde{\sigma}_n(t, \upsilon_2)\|_{\mathcal{L}^2(K,V)} \leqslant L_n\|\upsilon_1 - \upsilon_2\|, \tag{2.2.6}$$

and

$$\|\widetilde{\sigma}_n(t, \upsilon)\|_{\mathcal{L}^2(K,V)} \leqslant L_{\widetilde{\sigma}}(1 + \|\upsilon\|). \tag{2.2.7}$$

Next, we consider the approximating system ($n \in \mathbb{N}$):

$$\begin{cases} \dfrac{du^n}{dt} + \widetilde{A}u^n(t) + \widetilde{B}(u^n(t)) = \widetilde{f}(t) + \widetilde{g}(t, u^n_t) + \widetilde{\sigma}_n(t, u^n(t))\dfrac{dW(t)}{dt}, \quad \forall\, t > \tau,\ \tau \in \mathbb{R}, \\ u^n_\tau(s) = \phi(s),\ s \in (-\infty, 0]. \end{cases} \tag{2.2.8}$$

For every $n \in \mathbb{N}$, $\tau \in \mathbb{R}$ and $\phi \in L^2(\Omega, C_\gamma(V))$, the approximating system (2.2.8) has a unique solution u^n by using the same method as in [4]. Therefore, one can define *random stopping times* by

$$\iota_n = \inf\{t \geqslant \tau : \|u^n(t)\| > n\},\ \forall\, n \in \mathbb{N}, \tag{2.2.9}$$

where $\iota_n = +\infty$ if $\{t \geqslant \tau : \|u^n(t)\| > n\} = \emptyset$.

We will need the following lemma when proving the well-posedness of problem (2.2.1), which can be obtained in a similar manner as Lemma 1.2.2.

Lemma 2.2.2. *For all $u, v \in D(A)$, we have*

$$-2\langle \widetilde{A}\overline{w} + \widetilde{B}(u) - \widetilde{B}(v), \overline{w}\rangle \leqslant \frac{\widetilde{c}}{2}\|\overline{w}\|^2\|v\|^2_{D(A)}, \tag{2.2.10}$$

where $\overline{w} = u - v$.

Theorem 2.2.3. *Suppose that* (H21)-(H25), (2.2.6) *and* (2.2.7) *hold. Let $\phi \in L^2(\Omega, C_\gamma(V))$ and $\widetilde{f} \in L^2(\tau, \tau+T; V)$, then the stochastic system* (2.2.1) *has a unique solution u in the sense of Definition 2.2.1 such that for every $T > 0$, there exists a positive constant R depending on T, $\mathbb{E}(\|\phi\|^2_{C_\gamma(V)})$ and $\mathbb{E}(\int_\tau^{\tau+T}\|\widetilde{f}(s)\|^2 ds)$,*

$$\mathbb{E}\Big(\sup_{\tau \leqslant r \leqslant \tau+T}\|u_r\|^2_{C_\gamma(V)}\Big) + \mathbb{E}\Big(\int_\tau^{\tau+T}\|u(s)\|^2_{D(A)}ds\Big) \leqslant R. \tag{2.2.11}$$

Proof. We split the proof into the following four steps.

Step 1: *We prove $\iota_{n+1} \geqslant \iota_n$ almost surely*. In fact, we only need to prove the following equality holds:

$$u^{n+1}(t \wedge \iota_n) = u^n(t \wedge \iota_n),\ \ \mathbb{P}\text{-a.s.}\ \ \forall\, t \geqslant \tau,\ n \in \mathbb{N}. \tag{2.2.12}$$

Without loss of generality, we may assume that $\mathbb{E}(\int_\tau^t \|u^n(s)\|^2_{D(A)}ds) < \infty$. Actually, this is a direct consequence of the Step 2. Set Let $\overline{u}^n(t) := u^{n+1}(t) - u^n(t), \varsigma(t) := e^{-\frac{\widetilde{c}}{2}\int_\tau^t \|u^n(s)\|^2_{D(A)}ds}$. Applying Ito's formula to the process $\varsigma(t)\|\overline{u}^n(t)\|^2$, by (2.2.10) in Lemma 2.2.2, we deduce, for all $t \in [\tau, \tau+T]$,

$$\begin{aligned} \varsigma(t \wedge \iota_n)\|\overline{u}^n(t \wedge \iota_n)\|^2 = &-\frac{\widetilde{c}}{2}\int_\tau^{t\wedge\iota_n}\varsigma(s)\|u^n(s)\|^2_{D(A)}\|\overline{u}^n(s)\|^2 ds \\ &- 2\int_\tau^{t\wedge\iota_n}\varsigma(s)\langle \widetilde{A}\overline{u}^n(s) + \widetilde{B}(u^{n+1}(s)) - \widetilde{B}(u^n(s)), \overline{u}^n(s)\rangle ds \end{aligned}$$

$$
\begin{aligned}
&+ 2\int_\tau^{t\wedge\iota_n} \varsigma(s)((\widetilde{g}(s,u_s^{n+1}) - \widetilde{g}(s,u_s^n), \overline{u}^n(s)))ds \\
&+ \int_\tau^{t\wedge\iota_n} \varsigma(s)\|\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)}ds \\
&+ 2\int_\tau^{t\wedge\iota_n} \varsigma(s)\Big(\Big(\overline{u}^n(s), (\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n))dW(s)\Big)\Big) \\
\leqslant\ & 2\int_\tau^{t\wedge\iota_n} \varsigma(s)((\widetilde{g}(s,u_s^{n+1}) - \widetilde{g}(s,u_s^n), \overline{u}^n(s)))ds \\
&+ \int_\tau^{t\wedge\iota_n} \varsigma(s)\|\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)}ds \\
&+ 2\int_\tau^{t\wedge\iota_n} \varsigma(s)\Big(\Big(\overline{u}^n(s), (\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n))dW(s)\Big)\Big). \qquad (2.2.13)
\end{aligned}
$$

By (H25) and the Young inequality, we find

$$
\begin{aligned}
&2\int_\tau^{t\wedge\iota_n} \varsigma(s)((\widetilde{g}(s,u_s^{n+1}) - \widetilde{g}(s,u_s^n), \overline{u}^n(s)))ds \\
&\leqslant \int_\tau^{t\wedge\iota_n} \varsigma(s)\|\widetilde{g}(s,u_s^{n+1}) - \widetilde{g}(s,u_s^n)\|^2 ds + \int_\tau^{t\wedge\iota_n} \varsigma(s)\|\overline{u}^n(s)\|^2 ds \\
&\leqslant \widetilde{C}_{\widetilde{g}}^2 \int_\tau^{t\wedge\iota_n} \varsigma(s)\|\overline{u}^n(s)\|^2 ds + \int_\tau^{t\wedge\iota_n} \varsigma(s)\|\overline{u}^n(s)\|^2 ds \\
&\leqslant (\widetilde{C}_{\widetilde{g}}^2 + 1)\int_\tau^t \sup_{\tau\leqslant\vartheta\leqslant s} \varsigma(\vartheta\wedge\iota_n)\|\overline{u}^n(\vartheta\wedge\iota_n)\|^2 ds. \qquad (2.2.14)
\end{aligned}
$$

Thanks to (2.2.6), and the fact that $\widetilde{\sigma}_n(s,u^n) = \widetilde{\sigma}_{n+1}(s,u^n)$ for all $s\in[\tau,\iota_n]$, we obtain

$$
\begin{aligned}
&\int_\tau^{t\wedge\iota_n} \varsigma(s)\|\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)}ds \\
&= \int_\tau^{t\wedge\iota_n} \varsigma(s)\|\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_{n+1}(s,u^n)\|^2_{\mathcal{L}^2(K,V)}ds \\
&\leqslant L_{n+1}^2 \int_\tau^t \sup_{\tau\leqslant\vartheta\leqslant s} \varsigma(\vartheta\wedge\iota_n)\|\overline{u}^n(\vartheta\wedge\iota_n)\|^2 ds. \qquad (2.2.15)
\end{aligned}
$$

Substituting (2.2.14) and (2.2.15) into (2.2.13), we find

$$
\begin{aligned}
\varsigma(t\wedge\iota_n)\|\overline{u}^n(t\wedge\iota_n)\|^2 \leqslant\ & \tilde{c}_1 \int_\tau^t \sup_{\tau\leqslant\vartheta\leqslant s} \varsigma(\vartheta\wedge\iota_n)\|\overline{u}^n(\vartheta\wedge\iota_n)\|^2 ds \\
&+ 2\int_\tau^{t\wedge\iota_n} \varsigma(s)\Big(\Big(\overline{u}^n(s), (\widetilde{\sigma}_{n+1}(s,u^{n+1}) - \widetilde{\sigma}_n(s,u^n))dW(s)\Big)\Big),
\end{aligned}
$$

where $\tilde{c}_1 = \widetilde{C}_{\widetilde{g}}^2 + 1 + L_{n+1}^2$. Taking supremum and expectation of the above inequality, we find

$$
\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t} \varsigma(r\wedge\iota_n)\|\overline{u}^n(r\wedge\iota_n)\|^2\Big) \leqslant \tilde{c}_1 \int_\tau^t \mathbb{E}\Big(\sup_{\tau\leqslant\vartheta\leqslant s} \varsigma(\vartheta\wedge\iota_n)\|\overline{u}^n(\vartheta\wedge\iota_n)\|^2\Big)ds \qquad (2.2.16)
$$

$$+ 2\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\Big|\int_\tau^{r\wedge\iota_n}\varsigma(s)\Big(\Big(\overline{u}^n(s),(\widetilde{\sigma}_{n+1}(s,u^{n+1})-\widetilde{\sigma}_n(s,u^n))dW(s)\Big)\Big)\Big|\Big).$$

By the Burkholder-Davis-Gundy inequality and (2.2.15), the last line of (2.2.16) is bounded by

$$\begin{aligned}
&2\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\Big|\int_\tau^{r\wedge\iota_n}\varsigma(s)\Big(\Big(\overline{u}^n(s),(\widetilde{\sigma}_{n+1}(s,u^{n+1})-\widetilde{\sigma}_n(s,u^n))dW(s)\Big)\Big)\Big|\Big)\\
&\leqslant 2c_2\mathbb{E}\Big(\Big(\int_\tau^{t\wedge\iota_n}\Big(\varsigma^2(s)\|\overline{u}^n(s)\|^2\|\widetilde{\sigma}_{n+1}(s,u^{n+1})-\widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)}\Big)ds\Big)^{\frac12}\Big)\\
&\leqslant 2c_2\mathbb{E}\Big(\sup_{\tau\leqslant s\leqslant t}\varsigma^{\frac12}(s\wedge\iota_n)\|\overline{u}^n(s\wedge\iota_n)\|\Big(\int_\tau^{t\wedge\iota_n}\varsigma(s)\|\widetilde{\sigma}_{n+1}(s,u^{n+1})-\widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)}ds\Big)^{\frac12}\Big)\\
&\leqslant \frac12\mathbb{E}\Big(\sup_{\tau\leqslant s\leqslant t}\varsigma(s\wedge\iota_n)\|\overline{u}^n(s\wedge\iota_n)\|^2\Big)+2c_2^2\mathbb{E}\Big(\int_\tau^{t\wedge\iota_n}\varsigma(s)\|\widetilde{\sigma}_{n+1}(s,u^{n+1})-\widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)}ds\Big)\\
&\leqslant \frac12\mathbb{E}\Big(\sup_{\tau\leqslant s\leqslant t}\varsigma(s\wedge\iota_n)\|\overline{u}^n(s\wedge\iota_n)\|^2\Big)+\tilde{c}_3\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leqslant\vartheta\leqslant s}\varsigma(\vartheta\wedge\iota_n)\|\overline{u}^n(\vartheta\wedge\iota_n)\|^2\Big)ds,
\end{aligned}\tag{2.2.17}$$

where $\tilde{c}_3 = 2\tilde{c}_2^2L_{n+1}^2$. Combining (2.2.16) and (2.2.17), we obtain

$$\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\varsigma(r\wedge\iota_n)\|\overline{u}^n(r\wedge\iota_n)\|^2\Big)\leqslant\tilde{c}_4\int_\tau^t\mathbb{E}\Big(\sup_{\tau\leqslant\vartheta\leqslant s}\varsigma(\vartheta\wedge\iota_n)\|\overline{u}^n(\vartheta\wedge\iota_n)\|^2\Big)ds,\ \forall\, t\in[\tau,\tau+T],\tag{2.2.18}$$

where $\tilde{c}_4 = 2(\tilde{c}_1+\tilde{c}_3)$. By the Gronwall Lemma, together with $0<\varsigma\leqslant 1$, implies

$$\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant t}\|\overline{u}^n(r\wedge\iota_n)\|^2\Big)=0,\ \ \forall\, t\in[\tau,\tau+T],$$

thus,

$$u^{n+1}(t\wedge\iota_n)=u^n(t\wedge\iota_n),\ \mathbb{P}\text{-a.s. }\forall\, t\geqslant\tau,\ n\in\mathbb{N},\tag{2.2.19}$$

which implies (2.2.12) as desired.

Step 2: *We prove* $\iota := \lim_{n\to\infty}\iota_n = \sup_{n\in\mathbb{N}}\iota_n = \infty$*, almost surely*. Applying Ito's formula to (2.2.8), we find, for all $t\in[\tau,\tau+T]$,

$$\begin{aligned}
&\|u^n(t\wedge\iota_n)\|^2+2\int_\tau^{t\wedge\iota_n}\|u^n(s)\|^2_{D(A)}ds\\
&=\|\phi(0)\|^2+2\int_\tau^{t\wedge\iota_n}((\widetilde{f}(s)+\widetilde{g}(s,u_s^n),u^n(s)))ds\\
&\quad+\int_\tau^{t\wedge\iota_n}\|\widetilde{\sigma}_n(s,u^n(s))\|^2_{\mathcal{L}^2(K,V)}ds+2\int_\tau^{t\wedge\iota_n}((u^n(s),\widetilde{\sigma}_n(s,u^n)dW(s))).
\end{aligned}\tag{2.2.20}$$

By (2.1.6) and the Young inequality, we deduce

$$2\int_\tau^{t\wedge\iota_n}((\widetilde{f}(s)+\widetilde{g}(s,u_s^n),u^n(s)))ds\leqslant 2\int_\tau^{t\wedge\iota_n}(\|\widetilde{f}(s)\|^2+\|\widetilde{g}(s,u_s^n)\|^2)ds+\int_\tau^{t\wedge\iota_n}\|u^n(s)\|^2ds$$

$$\leqslant 2\int_\tau^{t\wedge\iota_n}\Big(\|\widetilde{f}(s)\|^2 + L_{\widetilde{g}}^2\int_\tau^{t\wedge\iota_n}\|u_s^n\|_{C_\gamma(V)}^2\Big)ds + \int_\tau^{t\wedge\iota_n}\|u^n(s)\|^2 ds$$
$$\leqslant 2\int_\tau^{t\wedge\iota_n}\|\widetilde{f}(s)\|^2 ds + \tilde{c}_5\int_\tau^{t\wedge\iota_n}\|u_s^n\|_{C_\gamma(V)}^2 ds, \tag{2.2.21}$$

where $\tilde{c}_5 = 2L_{\widetilde{g}}^2 + 1$. Thanks to (2.2.7), we find

$$\int_\tau^{t\wedge\iota_n}\|\widetilde{\sigma}_n(s,u^n(s))\|_{\mathcal{L}^2(K,V)}^2 ds \leqslant 2L_{\widetilde{\sigma}}^2\int_\tau^{t\wedge\iota_n}(1+\|u^n(s)\|^2)ds$$
$$\leqslant 2L_{\widetilde{\sigma}}^2\int_\tau^{t\wedge\iota_n}(1+\|u_s^n\|_{C_\gamma(V)}^2)ds,\ \forall\, t\in[\tau,\tau+T]. \tag{2.2.22}$$

By (2.2.21) and (2.2.22), we can rewrite (2.2.20) as

$$\begin{aligned}&\|u^n(t\wedge\iota_n)\|^2 + 2\int_\tau^{t\wedge\iota_n}\|u^n(s)\|_{D(A)}^2 ds\\ &\leqslant \|\phi(0)\|^2 + 2\int_\tau^{t\wedge\iota_n}\|\widetilde{f}(s)\|^2 ds + \tilde{c}_6\int_\tau^{t\wedge\iota_n}\|u_s^n\|_{C_\gamma(V)}^2 ds\\ &\quad + 2\Big|\int_\tau^{t\wedge\iota_n}\Big(\big(u^n(s),\widetilde{\sigma}_n(s,u^n)dW(s)\big)\Big)\Big| + 2L_{\widetilde{\sigma}}^2 T,\end{aligned} \tag{2.2.23}$$

where $\tilde{c}_6 = \tilde{c}_5 + 2L_{\widetilde{\sigma}}^2$. We infer from (2.2.23) that

$$\begin{aligned}\|u_{t\wedge\iota_n}^n\|_{C_\gamma(V)}^2 &\leqslant \max\Big\{\sup_{\vartheta\in(-\infty,\tau-t\wedge\iota_n]} e^{2\gamma\vartheta}\|u^n(t\wedge\iota_n+\vartheta)\|^2,\ \sup_{\vartheta\in[\tau-t\wedge\iota_n,0]} e^{2\gamma\vartheta}\|u^n(t\wedge\iota_n+\vartheta)\|^2\Big\}\\ &\leqslant \max\Big\{\sup_{\vartheta\in(-\infty,\tau-t\wedge\iota_n]} e^{2\gamma\vartheta}\|\phi(t\wedge\iota_n+\vartheta-\tau)\|^2,\ \sup_{\vartheta\in[\tau-t\wedge\iota_n,0]} e^{2\gamma\vartheta}\|u^n(t\wedge\iota_n+\vartheta)\|^2\Big\}\\ &\leqslant \max\Bigg\{\sup_{\vartheta\in(-\infty,0]} e^{2\gamma(\vartheta-t\wedge\iota_n+\tau)}\|\phi(\vartheta)\|^2,\ \sup_{\vartheta\in[\tau-t\wedge\iota_n,0]} e^{2\gamma\vartheta}\Big(\|\phi(0)\|^2 + 2\int_\tau^{t\wedge\iota_n+\vartheta}\|\widetilde{f}(s)\|^2 ds\\ &\quad + \tilde{c}_6\int_\tau^{t\wedge\iota_n+\vartheta}\|u_s^n\|_{C_\gamma(V)}^2 ds + 2\Big|\int_\tau^{t\wedge\iota_n+\vartheta}\Big(\big(u^n(s),\widetilde{\sigma}_n(s,u^n)dW(s)\big)\Big)\Big| + 2L_{\widetilde{\sigma}}^2 T\Big)\Bigg\}\\ &\leqslant e^{-2\gamma(t\wedge\iota_n-\tau)}\|\phi\|_{C_\gamma(V)}^2 + \|\phi\|_{C_\gamma(V)}^2 + 2\int_\tau^{t\wedge\iota_n}\|\widetilde{f}(s)\|^2 ds + \tilde{c}_6\int_\tau^{t\wedge\iota_n}\|u_s^n\|_{C_\gamma(V)}^2 ds\\ &\quad + 2\sup_{\vartheta\in[\tau-t\wedge\iota_n,0]} e^{2\gamma\vartheta}\Big|\int_\tau^{t\wedge\iota_n+\vartheta}\Big(\big(u^n(s),\widetilde{\sigma}_n(s,u^n)dW(s)\big)\Big)\Big| + 2L_{\widetilde{\sigma}}^2 T\\ &\leqslant 2\|\phi\|_{C_\gamma(V)}^2 + 2\int_\tau^{t\wedge\iota_n}\|\widetilde{f}(s)\|^2 ds + \tilde{c}_6\int_\tau^{t}\|u_{s\wedge\iota_n}^n\|_{C_\gamma(V)}^2 ds\\ &\quad + 2\sup_{\vartheta\in[\tau-t\wedge\iota_n,0]} e^{2\gamma\vartheta}\Big|\int_\tau^{t\wedge\iota_n+\vartheta}\Big(\big(u^n(s),\widetilde{\sigma}_n(s,u^n)dW(s)\big)\Big)\Big| + 2L_{\widetilde{\sigma}}^2 T.\end{aligned} \tag{2.2.24}$$

Taking supremum and expectation of (2.2.24), we obtain

$$\mathbb{E}\Big(\sup_{r\in[\tau,t]}\|u_{r\wedge\iota_n}^n\|_{C_\gamma(V)}^2\Big) \leqslant 2\mathbb{E}\Big(\|\phi\|_{C_\gamma(V)}^2\Big) + 2\mathbb{E}\Big(\int_\tau^{t\wedge\iota_n}\|\widetilde{f}(s)\|^2 ds\Big)$$

$$+ \tilde{c}_6 \int_\tau^t \mathbb{E}\Big(\sup_{r\in[\tau,s]} \|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big) ds + 2L^2_{\widetilde{\sigma}} T$$
$$+ 2\mathbb{E}\Big(\sup_{r\in[\tau,t]} \sup_{\vartheta\in[\tau-r\wedge\iota_n,0]} e^{2\gamma\vartheta} \Big| \int_\tau^{r\wedge\iota_n+\vartheta} \Big(\big(u^n(s), \widetilde{\sigma}_n(s,u^n)dW(s)\big)\Big)\Big|\Big). \tag{2.2.25}$$

By the Burkholder-Davis-Gundy inequality and (2.2.22), the last term of (2.2.25) is bounded by

$$\begin{aligned}
&2\mathbb{E}\Big(\sup_{r\in[\tau,t]} \sup_{\vartheta\in[\tau-r\wedge\iota_n,0]} e^{2\gamma\vartheta} \Big| \int_\tau^{r\wedge\iota_n+\vartheta} \Big(\big(u^n(s), \widetilde{\sigma}_n(s,u^n)dW(s)\big)\Big)\Big|\Big) \\
&\leqslant 2\mathbb{E}\Big(\sup_{r\wedge\iota_n+\vartheta\in[\tau,t\wedge\iota_n]} \Big| \int_\tau^{r\wedge\iota_n+\vartheta} \Big(\big(u^n(s), \widetilde{\sigma}_n(s,u^n)dW(s)\big)\Big)\Big|\Big) \\
&\leqslant 2\tilde{c}_7 \mathbb{E}\Big(\Big(\int_\tau^{t\wedge\iota_n} \|u^n_s\|^2_{C_\gamma(V)} \|\widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)} ds\Big)^{\frac{1}{2}}\Big) \\
&\leqslant 2\tilde{c}_7 \mathbb{E}\Big(\sup_{r\in[\tau,t]} \|u^n_{r\wedge\iota_n}\|_{C_\gamma(V)} \Big(\int_\tau^{t\wedge\iota_n} \|\widetilde{\sigma}_n(s,u^n)\|^2_{\mathcal{L}^2(K,V)} ds\Big)^{\frac{1}{2}}\Big) \\
&\leqslant \frac{1}{2}\mathbb{E}\Big(\sup_{r\in[\tau,t]} \|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big) + 2\tilde{c}_7^2 L^2_{\widetilde{\sigma}} \Big(T + \int_\tau^t \mathbb{E}\Big(\sup_{r\in[\tau,s]} \|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big) ds\Big).
\end{aligned} \tag{2.2.26}$$

It follows from (2.2.25) and (2.2.26) that, for all $t \in [\tau, \tau + T]$,

$$\mathbb{E}\Big(\sup_{r\in[\tau,t]} \|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big) \leqslant 4\mathbb{E}\Big(\|\phi\|^2_{C_\gamma(V)}\Big) + 4\mathbb{E}\Big(\int_\tau^t \|\widetilde{f}(s)\|^2 ds\Big)$$
$$+ \tilde{c}_9 \int_\tau^t \mathbb{E}\Big(\sup_{r\in[\tau,s]} \|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big) ds + \tilde{c}_8, \tag{2.2.27}$$

where $\tilde{c}_8 = 4L^2_{\widetilde{\sigma}} T(1 + \tilde{c}_7^2)$, $\tilde{c}_9 = 2(\tilde{c}_6 + 2\tilde{c}_7^2 L^2_{\widetilde{\sigma}})$. Set

$$\tilde{c}_{10} := 4\mathbb{E}\Big(\|\phi\|^2_{C_\gamma(V)}\Big) + 4\mathbb{E}\Big(\int_\tau^{\tau+T} \|\widetilde{f}(s)\|^2 ds\Big) + \tilde{c}_8, \tag{2.2.28}$$

which is finite due to $\phi \in L^2(\Omega, C_\gamma(V))$ and $\widetilde{f} \in I^2(\tau, \tau + T; V)$. Applying the Gronwall lemma to (2.2.27), we find, for all $t \in [\tau, \tau + T]$,

$$\mathbb{E}\Big(\sup_{r\in[\tau,t]} \|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big) \leqslant \tilde{c}_{10} e^{\tilde{c}_9 T} =: R_1. \tag{2.2.29}$$

Finally, we infer from (2.2.23) and (2.2.26) that, for all $t \in [\tau, \tau + T]$,

$$2\mathbb{E}\Big(\sup_{r\in[\tau,t]} \int_\tau^{r\wedge\iota_n} \|u^n(s)\|^2_{D(A)} ds\Big) \leqslant \mathbb{E}\Big(\|\phi(0)\|^2\Big) + 2\mathbb{E}\Big(\int_\tau^{\tau+T} \|\widetilde{f}(s)\|^2 ds\Big)$$
$$+ \frac{\tilde{c}_9}{2} \int_\tau^{\tau+T} \mathbb{E}\Big(\sup_{r\in[\tau,s]} \|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big) ds$$

$$+\frac{1}{2}\mathbb{E}\Big(\sup_{r\in[\tau,t]}\|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big)+\frac{\tilde{c}_8}{2}, \tag{2.2.30}$$

which, together with (2.2.29), implies that there exists a positive constant R_2,

$$\mathbb{E}\Big(\int_\tau^{t\wedge\iota_n}\|u^n(s)\|^2_{D(A)}ds\Big)\leqslant R_2. \tag{2.2.31}$$

Combining (2.2.29) and (2.2.31), we show that, for $R = R_1 + R_2 > 0$, the following inequality holds:

$$\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant\tau+T}\|u^n_{r\wedge\iota_n}\|^2_{C_\gamma(V)}\Big)+\mathbb{E}\Big(\int_\tau^{\tau+T}\|u^n(s\wedge\iota_n)\|^2_{D(A)}ds\Big)\leqslant R. \tag{2.2.32}$$

Next, we use (2.2.32) to prove Step 2 holds. It follows from (2.2.9) that

$$\{\iota_n<\tau+T\}\subseteq\{\sup_{\tau\leqslant r\leqslant\tau+T}\|u^n(r\wedge\iota_n)\|\geqslant n\},$$

which, together with Chebychev's inequality and (2.2.32), shows

$$\mathbb{P}\{\iota_n<\tau+T\}\leqslant\mathbb{P}\{\sup_{\tau\leqslant r\leqslant\tau+T}\|u^n(r\wedge\iota_n)\|\geqslant n\}\leqslant\frac{1}{n^2}\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant\tau+T}\|u^n(r\wedge\iota_n)\|^2\Big)\leqslant\frac{R}{n^2},$$

and thus,

$$\sum_{n=1}^{\infty}\mathbb{P}\{\iota_n<\tau+T\}\leqslant R\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty, \tag{2.2.33}$$

which, together with the Borel-Cantelli lemma, implies from (2.2.33) that

$$\mathbb{P}\Big(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{\iota_n<\tau+T\}\Big)=0.$$

Therefore, there exists a subset $\Omega_T = \cap_{m=1}^{\infty}\cup_{n=m}^{\infty}\{\iota_n<\tau+T\}$ of Ω with $\mathbb{P}(\Omega_T) = 0$ such that for each $\omega\in\Omega\setminus\Omega_T$, there exists $n_0 = n_0(\omega) > 0$ such that $\iota_n(\omega)\geqslant\tau+T$ for all $n\geqslant n_0$. Since ι_n is increasing in n, we obtain $\iota(\omega)\geqslant\tau+T$ for all $\omega\in\Omega\setminus\Omega_T$. Let $\Omega_0=\bigcup_{T=1}^{\infty}\Omega_T$, then $\mathbb{P}(\Omega_0) = 0$ and

$$\iota(\omega)\geqslant\tau+T\ \text{ for all } \omega\in\Omega\setminus\Omega_0 \text{ and } T\in\mathbb{N}.$$

Consequently, we deduce that $\iota(\omega) = \infty$ for all $\omega\in\Omega\setminus\Omega_0$ as desired.

Step 3: *We prove the existence of solutions to system* (2.2.1). By Step 1 and Step 2, there exists $\Omega_1\subseteq\Omega$ with $\mathbb{P}(\Omega\setminus\Omega_1) = 0$ such that

$$u^{n+1}(t\wedge\iota_n,\omega)=u^n(t\wedge\iota_n,\omega)\ \forall\, n\in\mathbb{N},\ t\geqslant\tau, \tag{2.2.34}$$

$$\iota(\omega)=\lim_{n\to\infty}\iota_n(\omega)=\infty,\ \ \forall\,\omega\in\Omega_1. \tag{2.2.35}$$

By (2.2.34) and (2.2.35), for every $\omega \in \Omega_1$ and $t \geqslant \tau$, there exists an $n_1 = n_1(t,\omega) \geqslant 1$ such that for all $n \geqslant n_1$,

$$\iota_n(\omega) > t, \text{ thus } u^n(t,\omega) = u^{n_1}(t,\omega). \tag{2.2.36}$$

Define a mapping $u : [\tau,\infty) \times \Omega \to V$ by

$$u(t,\omega) = \begin{cases} u^n(t,\omega), & \text{if } \omega \in \Omega_1 \text{ and } t \in [\tau, \iota_n(\omega)], \\ \phi(0,\omega), & \text{if } \omega \in \Omega \setminus \Omega_1 \text{ and } t \in [\tau,\infty). \end{cases} \tag{2.2.37}$$

Since u^n is a continuous V-valued process, it follows from (2.2.37) that u is also almost surely continuous with respect to t in $C_\gamma(V)$. Besides, we further infer from (2.2.37) that

$$\lim_{n\to\infty} u^n(t,\omega) = u(t,\omega), \forall \omega \in \Omega_1,\ t \geqslant \tau. \tag{2.2.38}$$

Note that u^n is $\mathcal{F}_t$-adapted, it follows from (2.2.38) that u is also $\mathcal{F}_t$-adapted. Thanks to (2.2.38), (2.2.32) and Fatou's lemma, we obtain that, for every $T > 0$,

$$\mathbb{E}\Big(\sup_{\tau\leqslant r\leqslant \tau+T} \|u_r\|^2_{C_\gamma(V)}\Big) + \mathbb{E}\Big(\int_\tau^{\tau+T} \|u(s)\|^2_{D(A)} ds\Big) \leqslant R, \tag{2.2.39}$$

where R is the same positive constant as in (2.2.32). Therefore, (2.2.11) holds.

We then prove that u is a solution to problem (2.2.1). Since u^n is the solution to (2.2.8), we deduce, for all $t \in [\tau,\tau+T]$,

$$\begin{aligned} u^n(t\wedge\iota_n) &+ \int_\tau^{t\wedge\iota_n} (\widetilde{A}u^n(s) + \widetilde{B}(u^n(s)))ds \\ &= \phi(0) + \int_\tau^{t\wedge\iota_n} (\widetilde{f}(s) + \widetilde{g}(s,u^n_s))ds + \int_\tau^{t\wedge\iota_n} \widetilde{\sigma}_n(s,u^n(s))dW(s). \end{aligned} \tag{2.2.40}$$

By (2.2.37), we find that, $u^n(t\wedge\iota_n) = u(t\wedge\iota_n)$ for all $t \in [\tau,\tau+T]$, $\mathbb{P}$-a.s., which together with the definition of $\widetilde{\sigma}_n$ implies $\widetilde{\sigma}_n(s,u^n(s)) = \widetilde{\sigma}(s,u(s))$ for all $s \in [\tau,\iota_n]$, $\mathbb{P}$-a.s., it follows from (2.2.40) that $\mathbb{P}$-a.s.,

$$u(t\wedge\iota_n) + \int_\tau^{t\wedge\iota_n} (\widetilde{A}u(s) + \widetilde{B}(u(s)))ds = \phi(0) + \int_\tau^{t\wedge\iota_n} (\widetilde{f}(s) + \widetilde{g}(s,u_s))ds + \int_\tau^{t\wedge\iota_n} \widetilde{\sigma}(s,u(s))dW(s), \tag{2.2.41}$$

By $\iota_n \uparrow \infty$ a.s., we can rewrite (2.2.41) as

$$u(t) + \int_\tau^t (\widetilde{A}u(s) + \widetilde{B}(u(s)))ds = \phi(0) + \int_\tau^t (\widetilde{f}(s) + \widetilde{g}(s,u_s))ds + \int_\tau^t \widetilde{\sigma}(s,u(s))dW(s),$$

and thus, we prove that u is a solution to (2.2.1) as desired.

Step 4: We show the uniqueness of solutions to (2.2.1). Let u and v be two solutions to system (2.2.1) with the same initial condition $u(s) = v(s) = \phi(s-\tau),\ s \leqslant \tau$. For every $n \in \mathbb{N}$ and $T > 0$, we can define a *stopping time* T_n as follows:

$$T_n = \inf\{t \geqslant \tau : \|u(t)\| > n \text{ or } \|v(t)\| > n\} \wedge (\tau + T). \tag{2.2.42}$$

Let $\overline{w} = u - v$, applying Ito's formula to the process $\widetilde{\varsigma}(t)\|\overline{w}(t)\|^2$ where $\widetilde{\varsigma}(t) = e^{-\frac{\widetilde{c}}{2}\int_\tau^t \|v(s)\|^2_{D(A)} ds}$, we infer from (2.2.10) in Lemma 2.2.2 that, for all $t \in [\tau, \tau + T]$,

$$\begin{aligned}
\widetilde{\varsigma}(t\wedge T_n)\overline{w}(t\wedge T_n) = &-\frac{\widetilde{c}}{2}\int_\tau^{t\wedge t_n} \widetilde{\varsigma}(s)\|v(s)\|^2_{D(A)}\|\overline{w}(s)\|^2 ds \\
&- 2\int_\tau^{t\wedge T_n} \widetilde{\varsigma}(s)\langle \widetilde{A}\overline{w}(s) + \widetilde{B}(u(s)) - \widetilde{B}(v(s)), \overline{w}(s)\rangle ds \\
&+ 2\int_\tau^{t\wedge T_n} \widetilde{\varsigma}(s)((\overline{g}(s,u_s) - \overline{g}(s,v_s), \overline{w}(s)))ds \\
&+ \int_\tau^{t\wedge T_n} \widetilde{\varsigma}(s)\|\widetilde{\sigma}(s,u(s)) - \widetilde{\sigma}(s,v(s))\|^2_{\mathcal{L}^2(K,V)} ds \\
&+ 2\int_\tau^{t\wedge T_n} \widetilde{\varsigma}(s)\Big(\Big(\overline{w}(s), (\widetilde{\sigma}(s,u(s)) - \widetilde{\sigma}(s,v(s)))dW(s)\Big)\Big) \\
\leqslant &\ 2\int_\tau^{t\wedge T_n} \widetilde{\varsigma}(s)((\overline{g}(s,u_s) - \overline{g}(s,v_s), \overline{w}(s)))ds \\
&+ \int_\tau^{t\wedge T_n} \widetilde{\varsigma}(s)\|\widetilde{\sigma}(s,u(s)) - \widetilde{\sigma}(s,v(s))\|^2_{\mathcal{L}^2(K,V)} ds \\
&+ 2\int_\tau^{t\wedge T_n} \widetilde{\varsigma}(s)\Big(\Big(\overline{w}(s), (\widetilde{\sigma}(s,u(s)) - \widetilde{\sigma}(s,v(s)))dW(s)\Big)\Big).
\end{aligned} \tag{2.2.43}$$

By the similar calculation as in Step 1, we deduce from (H25) and (2.2.42) that there exists $\tilde{c}_{11} > 0$ such that

$$\mathbb{E}\Big(\sup_{\tau\leqslant s\leqslant t} \widetilde{\varsigma}(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|^2\Big) \leqslant \tilde{c}_{11}\int_\tau^t \mathbb{E}\Big(\sup_{\tau\leqslant s\leqslant r} \widetilde{\varsigma}(s\wedge T_n)\|\overline{w}(s\wedge T_n)\|^2\Big)dr.$$

By the Gronwall lemma and $0 < \widetilde{\varsigma} \leqslant 1$, we obtain

$$\mathbb{E}\Big(\sup_{\tau\leqslant s\leqslant t} \|\overline{w}(s\wedge T_n)\|^2\Big) = 0,$$

which implies $\|\overline{w}(t\wedge T_n)\| = \|u(t\wedge T_n) - v(t\wedge T_n)\| = 0$ for all $t \in [\tau, \tau+T]$ almost surely. Since u and v are continuous with respect to t, we show $T_n = \tau + T$ for large enough n. We then obtain that $\|\overline{w}(t)\| = 0$ for all $t \in [\tau, \tau+T]$ almost surely. Therefore, for every $T > 0$,

$$\mathbb{P}\Big(\|\overline{w}(t)\| = 0,\ \text{for all}\ \ t \in [\tau, \tau+T]\Big) = 1.$$

Since T is an arbitrary number, we further imply

$$\mathbb{P}\Big(\|\overline{w}(t)\| = 0, \text{ for all } t \geqslant \tau\Big) = 1,$$

which proves the uniqueness of the solutions. This completes the proof. □

By Theorem 2.2.3, one can define the following mapping:

$$\begin{aligned} &\Phi(t,\tau) : L^2(\Omega, \mathcal{F}_\tau; C_\gamma(V)) \to L^2(\Omega, \mathcal{F}_{\tau+t}; C_\gamma(V)), \\ &\Phi(t,\tau)\phi = u_{t+\tau}(\cdot,\tau,\phi), \ \forall\, t \geqslant 0, \ \tau \in \mathbb{R}, \phi \in L^2(\Omega, \mathcal{F}_\tau; C_\gamma(V)). \end{aligned} \tag{2.2.44}$$

Thanks to [76, 124], we deduce that Φ is a *mean random dynamical system* generated by (2.2.1) on $L^2(\Omega, \mathcal{F}; C_\gamma(V))$.

2.2.2 Weak pullback mean random attractors

In this subsection, we are interested in proving the existence of a unique *weak pullback mean random attractor* for the stochastic system (2.2.1) in $L^2(\Omega, C_\gamma(V))$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$. To this end, we need to further assume

$$\int_{-\infty}^{\tau} e^{\widetilde{\lambda}_1 r}\mathbb{E}(\|\widetilde{f}(r)\|^2)dr < \infty, \quad \forall\, \tau \in \mathbb{R}, \tag{2.2.45}$$

$$\lim_{t\to+\infty} e^{-\zeta t}\int_{-\infty}^{0} e^{\widetilde{\lambda}_1 r}\mathbb{E}(\|\widetilde{f}(r-t)\|^2)dr = 0, \quad \forall\, \zeta > 0. \tag{2.2.46}$$

A family $\mathcal{D} = \{\mathcal{D}(\tau) : \tau \in \mathbb{R}\}$ is a family of nonempty bounded sets such that $\mathcal{D}(\tau) \in L^2(\Omega, \mathcal{F}_\tau; C_\gamma(V))$ and further tempered if

$$\lim_{t\to+\infty} e^{-\zeta t}\|\mathcal{D}(\tau - t)\|^2_{L^2(\Omega,\mathcal{F}_{\tau-t};C_\gamma(V))} = 0, \quad \forall\, \zeta > 0. \tag{2.2.47}$$

We will use $\mathfrak{D}$ to denote the collection (or universe) of the above tempered families.

Lemma 2.2.4. *Suppose that* (H21)-(H25), (2.2.45) *and* (2.2.46) *hold. If* $2\gamma > \widetilde{\lambda}_1$, *then for each* $\tau \in \mathbb{R}$, *it follows*

$$\mathbb{E}(\|u_\tau(\cdot,\tau - t,\phi)\|^2_{C_\gamma(V)}) \leqslant R_0(\tau), \tag{2.2.48}$$

for all $\phi \in C_\gamma(V)$, *where*

$$R_0(\tau) = \widetilde{L} + \widetilde{L}e^{-\widetilde{\lambda}_1\tau}\int_{-\infty}^{\tau} e^{\widetilde{\lambda}_1 r}\mathbb{E}(\|\widetilde{f}(r)\|^2)dr \tag{2.2.49}$$

with $\widetilde{L}$ *a positive constant independent of* τ.

Proof. By Ito's formula, we obtain from (2.2.1) that

$$\frac{d}{dt}\|u(t)\|^2 + 2\|u(t)\|^2_{D(A)} = 2\Big(\Big(\widetilde{f}(t) + \widetilde{g}(t,u_t), u(t)\Big)\Big) + \|\widetilde{\sigma}(t,u)\|^2_{\mathcal{L}^2(K,V)} + 2\Big(\Big(u(t), \widetilde{\sigma}(t,u(t))dW(t)\Big)\Big). \tag{2.2.50}$$

By the Young inequality, (0.0.42) and (2.1.6), we deduce

$$\begin{aligned} 2\Big(\Big(\widetilde{f}(t) + \widetilde{g}(t,u_t), u(t)\Big)\Big) &\leqslant 2\widetilde{\lambda}_1^{-\frac{1}{2}}\|\widetilde{f}(t) + \widetilde{g}(t,u_t)\|\|u(t)\|_{D(A)} \\ &\leqslant \widetilde{\lambda}_1^{-1}\|\widetilde{f}(t) + \widetilde{g}(t,u_t)\|^2 + \|u(t)\|^2_{D(A)} \\ &\leqslant 2\widetilde{\lambda}_1^{-1}\|\widetilde{f}(t)\|^2 + 2\widetilde{\lambda}_1^{-1}L^2_{\widetilde{g}}\|u_t\|^2_{C_\gamma(V)} + \|u(t)\|^2_{D(A)}. \end{aligned} \tag{2.2.51}$$

Thanks to (2.1.9), we find

$$\|\widetilde{\sigma}(t,u)\|^2_{\mathcal{L}^2(K,V)} \leqslant 2L^2_{\widetilde{\sigma}}(1 + \|u_t\|^2_{C_\gamma(V)}). \tag{2.2.52}$$

Substituting (2.2.51) and (2.2.52) into (2.2.50), then by (0.0.42), we obtain

$$\frac{d}{dt}\|u(t)\|^2 + \widetilde{\lambda}_1\|u(t)\|^2 \leqslant 2\widetilde{\lambda}_1^{-1}\|\widetilde{f}(t)\|^2 + \tilde{c}_1\|u_t\|^2_{C_\gamma(V)} + 2L^2_{\widetilde{\sigma}} + 2\Big(\Big(u(t), \widetilde{\sigma}(t,u(t))dW(t)\Big)\Big), \tag{2.2.53}$$

where $\tilde{c}_1 = 2\widetilde{\lambda}_1^{-1}L^2_{\widetilde{g}} + 2L^2_{\widetilde{\sigma}}$. Taking the expectation of (2.2.53), we have

$$\frac{d}{dt}\mathbb{E}(\|u(t)\|^2) + \widetilde{\lambda}_1\mathbb{E}(\|u(t)\|^2) \leqslant 2\widetilde{\lambda}_1^{-1}\mathbb{E}(\|\widetilde{f}(t)\|^2) + \tilde{c}_1\mathbb{E}(\|u_t\|^2_{C_\gamma(V)}) + 2L^2_{\widetilde{\sigma}}. \tag{2.2.54}$$

Multiplying (2.2.53) by $e^{\lambda_1 t}$ and integrating over $(\tau - t, s)$, we obtain

$$\mathbb{E}(\|u(s,\tau-t,\phi)\|^2) \leqslant e^{\widetilde{\lambda}_1(\tau-t-s)}\mathbb{E}(\|\phi(0)\|^2) + 2\widetilde{\lambda}_1^{-1}\int_{\tau-t}^{s} e^{\widetilde{\lambda}_1(r-s)}\mathbb{E}(\|\widetilde{f}(r)\|^2)dr + \tilde{c}_1\int_{\tau-t}^{s} e^{\widetilde{\lambda}_1(r-s)}\mathbb{E}(\|u_r\|^2_{C_\gamma(V)})dr + 2L^2_{\widetilde{\sigma}}\widetilde{\lambda}_1^{-1}. \tag{2.2.55}$$

Since the fact that $2\gamma > \widetilde{\lambda}_1$, we deduce

$$\begin{aligned} \mathbb{E}(\|u_s\|^2_{C_\gamma(V)}) &\leqslant \max\Big\{\sup_{\vartheta\leqslant\tau-t-s} e^{2\gamma\vartheta}\mathbb{E}(\|u(s+\vartheta)\|^2), \sup_{\tau-t-s\leqslant\vartheta\leqslant 0} e^{2\gamma\vartheta}\mathbb{E}(\|u(s+\vartheta)\|^2)\Big\} \\ &\leqslant \max\Big\{\sup_{\vartheta\leqslant\tau-t-s} e^{2\gamma\vartheta}\mathbb{E}(\|\phi(\vartheta-\tau+t+s)\|^2), \sup_{\tau-t-s\leqslant\vartheta\leqslant 0} e^{2\gamma\vartheta}\mathbb{E}(\|u(s+\vartheta)\|^2)\Big\} \\ &\leqslant \max\Bigg\{\sup_{\vartheta\leqslant 0} e^{2\gamma(\vartheta+\tau-t-s)}\mathbb{E}(\|\phi(\vartheta)\|^2), \sup_{\tau-t-s\leqslant\vartheta\leqslant 0} e^{2\gamma\vartheta}\Big(e^{\widetilde{\lambda}_1(\tau-t-s-\vartheta)}\mathbb{E}(\|\phi(0)\|^2) \\ &\quad + 2\widetilde{\lambda}_1^{-1}\int_{\tau-t}^{s+\vartheta} e^{\widetilde{\lambda}_1(r-s-\vartheta)}\mathbb{E}(\|\widetilde{f}(r)\|^2)dr + \tilde{c}_1\int_{\tau-t}^{s+\vartheta} e^{\widetilde{\lambda}_1(r-s-\vartheta)}\mathbb{E}(\|u_r\|^2_{C_\gamma(V)})dr + 2L^2_{\widetilde{\sigma}}\widetilde{\lambda}_1^{-1}\Big)\Bigg\} \\ &\leqslant 2\mathbb{E}(\|\phi\|^2_{C_\gamma(V)}) + 2\widetilde{\lambda}_1^{-1}\int_{\tau-t}^{s} e^{\widetilde{\lambda}_1(r-s)}\mathbb{E}(\|\widetilde{f}(r)\|^2)dr + \tilde{c}_1\int_{\tau-t}^{s} e^{\widetilde{\lambda}_1(r-s)}\mathbb{E}(\|u_r\|^2_{C_\gamma(V)})dr + 2L^2_{\widetilde{\sigma}}\widetilde{\lambda}_1^{-1}. \end{aligned} \tag{2.2.56}$$

Let $s = \tau$ in (2.2.56), we find

$$\mathbb{E}(\|u_\tau\|^2_{C_\gamma(V)}) \leqslant 2\mathbb{E}(\|\phi\|^2_{C_\gamma(V)}) + 2\widetilde{\lambda}_1^{-1} \int_{\tau-t}^{\tau} e^{\widetilde{\lambda}_1(r-\tau)} \mathbb{E}(\|\widetilde{f}(r)\|^2) dr$$
$$+ \tilde{c}_1 \int_{\tau-t}^{\tau} e^{\widetilde{\lambda}_1(r-\tau)} \mathbb{E}(\|u_r\|^2_{C_\gamma(V)}) dr + 2L^2_{\widetilde{\sigma}} \widetilde{\lambda}_1^{-1}. \tag{2.2.57}$$

Set

$$c_2 := 2\mathbb{E}(\|\phi\|^2_{C_\gamma(V)}) + 2\widetilde{\lambda}_1^{-1} e^{-\widetilde{\lambda}_1 \tau} \int_{-\infty}^{\tau} e^{\widetilde{\lambda}_1 r} \mathbb{E}(\|\widetilde{f}(r)\|^2) dr + 2L^2_{\widetilde{\sigma}} \widetilde{\lambda}_1^{-1}.$$

Applying the Gronwall lemma to (2.2.57), we deduce

$$\mathbb{E}(\|u_\tau\|^2_{C_\gamma(V)}) \leqslant \tilde{c}_2 e^{\tilde{c}_1 \widetilde{\lambda}_1^{-1}}. \tag{2.2.58}$$

This yields (2.2.48) as desired. □

Recall that the definition of a weak pullback mean random attractor $\mathcal{A}^{\mathrm{w}} = \{\mathcal{A}^{\mathrm{w}}(s) : s \in \mathbb{R}\} \in \mathfrak{D}$ is introduced by [124], that is, it is the minimum among all weakly compact and $\mathfrak{D}$-pullback w-attracting sets, where $\mathcal{A}^{\mathrm{w}}(\cdot)$ is called **$\mathfrak{D}$-pullback w-attracting** if for each $\mathcal{D}(\cdot) \in \mathfrak{D}$, $s \in \mathbb{R}$ and each neighborhood $N(\mathcal{A}^{\mathrm{w}}(s))$ under the weak topology of $L^2(\Omega, \mathcal{F}_s; C_\gamma(V))$, there is $T > 0$ such that

$$\Phi(t, s-t)\mathcal{D}(s-t) \subset N(\mathcal{A}^{\mathrm{w}}(s)), \ \forall t \geqslant T.$$

Theorem 2.2.5. *Assume that the same hypotheses and notations in Lemma 2.2.4 hold. Then the mean random dynamical system Φ, induced by system (2.2.1), possesses a unique weak $\mathfrak{D}$-pullback mean random attractor $\mathcal{A}^w = \{\mathcal{A}^w(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{D}$ in $L^2(\Omega, \mathcal{F}; C_\gamma(V))$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$.*

Proof. Let $\mathcal{K} = \{\mathcal{K}(\tau) : \tau \in \mathbb{R}\}$ with

$$\mathcal{K}(\tau) = \{\xi \in C_\gamma(V) : \mathbb{E}(\|\xi\|^2_{C_\gamma(V)}) \leqslant R_0(\tau)\}, \ \forall \, \tau \in \mathbb{R}, \tag{2.2.59}$$

where $R_0(\tau)$ is given by (2.2.49). Since $\mathcal{K}(\tau)$ is a bounded closed convex set in $L^2(\Omega, \mathcal{F}_\tau; C_\gamma(V))$, we obtain it is weakly compact in $L^2(\Omega, \mathcal{F}_\tau; C_\gamma(V))$.

Finally, we prove $\mathcal{K} \in \mathfrak{D}$. For every $\zeta > 0$, we infer from (2.2.46) that

$$\lim_{t\to+\infty} e^{-\zeta t} \|\mathcal{K}(\tau - t)\|^2_{L^2(\Omega, \mathcal{F}_{\tau-t}, C_\gamma(V))} = \lim_{t\to+\infty} e^{-\zeta t} R_0(\tau - t)$$
$$= \widetilde{L} \lim_{t\to+\infty} e^{-\zeta t} + \widetilde{L} \lim_{t\to+\infty} e^{-\zeta t} e^{-\widetilde{\lambda}_1(\tau-t)} \int_{-\infty}^{\tau-t} e^{\widetilde{\lambda}_1 r} \mathbb{E}(\|\widetilde{f}(r)\|^2) dr$$
$$= \widetilde{L} \lim_{t\to+\infty} e^{-\zeta t} + \widetilde{L} \lim_{t\to+\infty} e^{-\zeta t} \int_{-\infty}^{0} e^{\widetilde{\lambda}_1 r} \mathbb{E}(\|\widetilde{f}(r + \tau - t)\|^2) dr$$
$$= \widetilde{L} \lim_{t\to+\infty} e^{-\zeta t} + \widetilde{L} e^{-\zeta \tau} \lim_{t\to+\infty} e^{-\zeta t} \int_{-\infty}^{0} e^{\widetilde{\lambda}_1 r} \mathbb{E}(\|\widetilde{f}(r - t)\|^2) dr = 0,$$

which implies $\mathcal{K}$ is tempered, that is, $\mathcal{K} \in \mathfrak{D}$. This together with Lemma 2.2.4, shows $\mathcal{K}$ is a weakly compact $\mathfrak{D}$-pullback random absorbing set for Φ. By [124, Theorem 2.13], we prove the existence of a unique weak $\mathfrak{D}$-pullback mean random attractor $\mathcal{A}^{\mathrm{w}}$ for Φ. □

2.3 Stationary solutions and their stability results

In this section, we are concerned with existence, uniqueness and stability properties of the stationary solutions to (0.0.4). For this end, we need to assume that $\widetilde{f}(t) \equiv \widetilde{f} \in (D(A))^*$, $\widetilde{\sigma}(t, \cdot) \equiv \widetilde{\sigma}(\cdot) : V \to \mathcal{L}^2(K, V)$ with $\widetilde{\sigma}(0) = 0$, i.e., they are independent of time. Besides, suppose that $\widetilde{\sigma}$ is globally Lipschitz continuous, that is, there exists a positive constant $C_{\widetilde{\sigma}}$ such that for all $\upsilon_1, \upsilon_2 \in V$, $\|\widetilde{\sigma}(\upsilon_1) - \widetilde{\sigma}(\upsilon_2)\|_{\mathcal{L}^2(K,V)} \leqslant C_{\widetilde{\sigma}}\|\upsilon_1 - \upsilon_2\|$.

2.3.1 Existence and uniqueness of stationary solutions

We now consider the abstract equation associated to Eq. (0.0.4):

$$\begin{cases} \dfrac{du}{dt} + \widetilde{A}u(t) + \widetilde{B}(u(t)) = \widetilde{f} + \widetilde{g}(t, u_t) + \widetilde{\sigma}(u)\dfrac{dW}{dt}, & \forall\, t > 0, \\ u(t) = \phi(t), t \in (-\infty, 0]. \end{cases} \tag{2.3.1}$$

We denote by $u(t) := u(t; \phi)$ the solution of Eq. (0.0.4) (or (2.2.1)) with $\tau = 0$, where $\phi = u_0$.

A stationary solution (an equilibrium solution) to problem (2.3.1) is an element $u_\infty \in D(A)$ satisfying

$$\widetilde{A}u_\infty + \widetilde{B}(u_\infty) = \widetilde{f} + \widetilde{g}(t, u_\infty) + \widetilde{\sigma}(u_\infty)\frac{dW}{dt}, \quad \forall\, t \geqslant 0. \tag{2.3.2}$$

However, the above equation depends on t and a noisy term. Therefore, on the one hand, to get rid of the noise, we must assume that $\widetilde{\sigma}(u_\infty) = 0$. Consequently, we will focus on the existence and uniqueness of stationary solutions for the deterministic equation (i.e. $\widetilde{\sigma} = 0$ in (2.3.1)) which will be any $u_\infty \in D(A)$ such that

$$\widetilde{A}u_\infty + \widetilde{B}(u_\infty) = \widetilde{f} + \widetilde{g}(t, u_\infty), \quad \forall\, t \geqslant 0. \tag{2.3.3}$$

On the other hand, we would need to assume that the delay term $\widetilde{g}$ would not depend on the time t, that is, there exists a function $\widetilde{g}_0 : V \to V$ such that

$$\widetilde{g}(t, \xi) = \widetilde{g}_0(\hat{\xi}) \text{ if } \xi(s) = \hat{\xi}, \ \forall\, (s, t, \xi) \in \mathbb{R}^- \times \mathbb{R}^+ \times C_\gamma(V), \tag{2.3.4}$$

and it is Lipschitz continuous (with the same Lipschitz constant $L_{\widetilde{g}}$) and $\widetilde{g}_0(0) = 0$.

For example, if $\widetilde{g}$ is driven by unbounded variable delay, defined by

$$\widetilde{g}(t, \xi) = \widetilde{\mathcal{G}}(\xi(-h(t))), \tag{2.3.5}$$

where $h \in C^1([0, +\infty))$, $h(t) \geqslant 0$, $h^* = \sup_{t \geqslant 0} h'(t) < 1$, and $\widetilde{\mathcal{G}} : V \to V$ satisfies $\widetilde{\mathcal{G}}(0) = 0$, assume that there exists $L_{\widetilde{\mathcal{G}}} > 0$, for all $\eta, \zeta \in V$,

$$\|\widetilde{\mathcal{G}}(\eta) - \widetilde{\mathcal{G}}(\zeta)\| \leqslant L_{\widetilde{\mathcal{G}}}\|\eta - \zeta\|. \tag{2.3.6}$$

In this case, the delay term $\widetilde{g}$ in our problem becomes $\widetilde{g}(t, u_t) = \widetilde{\mathcal{G}}(u(t - h(t)))$.

Another example is the case of infinite distributed delay, that is, the delay term $\widetilde{g}$ is defined by

$$\widetilde{g}(t, \xi) = \int_{-\infty}^{0} \widetilde{\mathcal{H}}(s, \xi(s))ds, \tag{2.3.7}$$

where $\widetilde{\mathcal{H}} : (-\infty, 0] \times V \to V$ with $\widetilde{\mathcal{H}}(s, 0) = 0$ is measurable, and it is Lipschitz continuous with respect to its second variable, that is, there exists $L_{\widetilde{\mathcal{H}}}(s) \in L^2(-\infty, 0)$ with $L_{\widetilde{\mathcal{H}}}(\cdot)e^{-(\gamma+\theta)\cdot} \in L^2(-\infty, 0)$, for certain $\theta > 0$, such that for all $s \in (-\infty, 0], \eta, \zeta \in V$,

$$\|\widetilde{\mathcal{H}}(s, \eta) - \widetilde{\mathcal{H}}(s, \zeta)\| \leqslant L_{\widetilde{\mathcal{H}}}(s)\|\eta - \zeta\|. \tag{2.3.8}$$

In this case, we can rewrite the delay term $\widetilde{g}$ in our problem as $\widetilde{g}(t, u_t) = \int_{-\infty}^{0} \widetilde{\mathcal{H}}(s, u(t + s))ds$.

Observe that the above both situations are within our framework, the conditions (H21)-(H25) are fulfilled for the infinite distributed delay in $C_\gamma(V)$ for $\gamma > 0$, but not necessarily for the unbounded variable delay. However, conditions (H21)-(H25) are satisfied for both delays in $C_{-\infty}(V)$.

Now, we are interested in studying the existence and uniqueness of solutions to Eq. (2.3.3), more precisely, we will prove the existence of a unique $u_\infty \in D(A)$ such that

$$\widetilde{A}u_\infty + \widetilde{B}(u_\infty) = \widetilde{f} + \widetilde{g}_0(u_\infty). \tag{2.3.9}$$

By non-compactness of Sobolev embeddings on unbounded domains, we need to modify some arguments of [31, Theorem 10]. More precisely, we use the idea of the classical Galerkin approximations as well as the Lax-Milgram and the Brouwer fixed theorems to establish the existence and uniqueness of a stationary solution to Eq. (2.3.9).

Theorem 2.3.1. *Assume that the above hypotheses and notations hold. If* $\widetilde{\lambda}_1 > L_{\widetilde{g}}$, *then:*

(1) For all $\widetilde{f} \in (D(A))^*$, *there exists at least one stationary solution to* (2.3.9);

(2) If $(1 - \widetilde{\lambda}_1^{-1}L_{\widetilde{g}})^2 > \widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|$, *the stationary solution to* (2.3.9) *is unique.*

Proof. (1) Let an orthonormal basis $B = \{w_j; j \in \mathbb{N}\} \subset \mathcal{V}$ of V such that linear combinations of elements of B are dense in $D(A)$. Denote $V_m = \text{span}[w_1, w_2, \ldots, w_m]$ for $m \in \mathbb{N}$ with the norm $\|\cdot\|_{D(A)}$ of $D(A)$.

Step 1: *We use the Lax-Milgram Theorem to find a unique solution to Eq.* (2.3.10). More precisely, fixed $x^m \in V_m$, it suffices to find $u^m \in V_m$, which solves the equation

$$\langle \widetilde{A}u^m, w^m \rangle + \langle \widetilde{B}(u^m, x^m), w^m \rangle = ((\widetilde{f} + \widetilde{g}_0(x^m), w^m)), \quad \forall\, w^m \in V_m. \tag{2.3.10}$$

Note that for each $x^m \in V_m$, the functional $(u, w) \mapsto \langle \widetilde{A}u, w \rangle + \langle \widetilde{B}(u, x^m), w \rangle$ is bilinear continuous and coercive in $V_m \times V_m$, besides, the functional $w \mapsto ((\widetilde{f} + \widetilde{g}_0(x^m), w))$ is linear continuous in V_m and thus by the Lax-Milgram Theorem, for each fixed $x^m \in V_m$, there exists a unique solution u^m to Eq. (2.3.10). Thus, we can define an operator $T_m : V_m \to V_m$, given by

$$T_m(x^m) = u^m. \tag{2.3.11}$$

Step 2: *We apply the Brouwer fixed point theorem to the mapping* T_m (restricted to a suitable subset of V_m) to ensure that there exists $u^m \in V_m$ such that

$$\langle \widetilde{A}u^m, w^m\rangle + \langle \widetilde{B}(u^m), w^m\rangle = ((\widetilde{f} + \widetilde{g}_0(u^m), w^m)), \quad \forall\, w^m \in V_m. \tag{2.3.12}$$

Taking $w^m = u^m$ in (2.3.10), by (0.0.42), we deduce

$$\begin{aligned}
\widetilde{\lambda}_1\|u^m\|^2 &\leqslant \|u^m\|^2_{D(A)} \\
&= ((\widetilde{f} + \widetilde{g}_0(x^m), u^m)) \\
&\leqslant \|\widetilde{f}\|\|u^m\| + L_{\widetilde{g}}\|x^m\|\|u^m\|.
\end{aligned} \tag{2.3.13}$$

Since $\widetilde{\lambda}_1 > L_{\widetilde{g}}$, one can take $k > 0$ such that $k(\widetilde{\lambda}_1 - L_{\widetilde{g}}) \geqslant \|\widetilde{f}\|$, then

$$\widetilde{\lambda}_1\|u^m\| \leqslant k\widetilde{\lambda}_1 - kL_{\widetilde{g}} + L_{\widetilde{g}}\|x^m\|. \tag{2.3.14}$$

Define $C_m = \{x \in V_m : \|x\|_{D(A)} \leqslant k\}$, which is a convex set of $D(A)$, and compact in $D(A)$. Note that the mapping T_m maps C_m into itself.

Let us now apply the Brouwer fixed point theorem to $T_m|C_m$. To that end, we only need to prove T_m is continuous. Indeed, taking $x_1^m, x_2^m \in V_m$, we then denote by $u^m = T(x_1^m)$ and $v^m = T(x_2^m)$ the solutions to Eq. (2.3.10), next, taking the difference, we obtain

$$\langle \widetilde{A}(u^m - v^m), w^m\rangle + \langle \widetilde{B}(u^m, x_1^m), w^m\rangle - \langle \widetilde{B}(v^m, x_2^m), w^m\rangle = ((\widetilde{g}_0(x_1^m) - \widetilde{g}_0(x_2^m), w^m)), \tag{2.3.15}$$

for all $w^m \in V_m$. Setting $w^m = u^m - v^m$ in the above equality, by (0.0.42), (B1), (B3) and $u^m \in C_m$, we find

$$\begin{aligned}
\|u^m - v^m\|^2_{D(A)} &= \langle \widetilde{B}(v^m, x_2^m), u^m - v^m\rangle - \langle \widetilde{B}(u^m, x_1^m), u^m - v^m\rangle + ((\widetilde{g}_0(x_1^m) - \widetilde{g}_0(x_2^m), u^m - v^m)) \\
&= -\langle \widetilde{B}(u^m - v^m, x_2^m), v^m\rangle + \langle \widetilde{B}(u^m - v^m, x_1^m), u^m\rangle + ((\widetilde{g}_0(x_1^m) - \widetilde{g}_0(x_2^m), u^m - v^m)) \\
&= -\langle \widetilde{B}(u^m - v^m, x_2^m), u^m\rangle + \langle \widetilde{B}(u^m - v^m, x_1^m), u^m\rangle + ((\widetilde{g}_0(x_1^m) - \widetilde{g}_0(x_2^m), u^m - v^m)) \\
&\leqslant \widetilde{\lambda}_1^{-\frac{1}{2}}\|x_1^m - x_2^m\|_{D(A)}\|u^m - v^m\|_{D(A)}\|u^m\|_{D(A)} + \widetilde{\lambda}_1^{-1}L_{\widetilde{g}}\|x_1^m - x_2^m\|_{D(A)}\|u^m - v^m\|_{D(A)} \\
&\leqslant (\widetilde{\lambda}_1^{-\frac{1}{2}}k + \widetilde{\lambda}_1^{-1}L_{\widetilde{g}})\|x_1^m - x_2^m\|_{D(A)}\|u^m - v^m\|_{D(A)},
\end{aligned} \tag{2.3.16}$$

which implies the continuity of T_m.

Step 3: *We take limit of the solutions* proved in Step 2 to derive the existence of solutions to Eq. (2.3.9). Taking $w^m = u^m$ in (2.3.12), we have

$$\begin{aligned}
\|u^m\|^2_{D(A)} &= ((\widetilde{f} + \widetilde{g}_0(u^m), u^m)) \\
&\leqslant \widetilde{\lambda}_1^{-\frac{1}{2}}\|\widetilde{f}\|\|u^m\|_{D(A)} + \widetilde{\lambda}_1^{-1}L_{\widetilde{g}}\|u^m\|^2_{D(A)},
\end{aligned} \tag{2.3.17}$$

which implies all solutions obtained in Step 2 are bounded by $\|u^m\|_{D(A)} \leqslant \widetilde{\lambda}_1^{-\frac{1}{2}}\|\widetilde{f}\|/(1 - \widetilde{\lambda}_1^{-1}L_{\widetilde{g}})$. Thus, u^m has a weakly convergent subsequence (not relabeled) such that $u^m \rightharpoonup u$ in $D(A)$. Moreover, for any regular bounded set $Q \subset \mathcal{O}$, we derive the same uniform bounds of $u^m|_Q$, which together with the compact injection, yields $u^m|_Q \to u|_Q$ in $\mathbb{H}_0^1(Q)$.

For (2.3.12), we fix any $w_j \in B$, there exists a subsequence of u^m (relabeled the same) such that

$$\langle \widetilde{A}u^m, w_j\rangle + \langle \widetilde{B}(u^m), w_j\rangle = ((\widetilde{f} + \widetilde{g}_0(u^m), w_j)), \quad \forall\, m \geqslant j. \tag{2.3.18}$$

Taking limit in (2.3.18), we find

$$\langle \widetilde{A}u, w_j\rangle + \langle \widetilde{B}(u), w_j\rangle = ((\widetilde{f} + \widetilde{g}_0(u), w_j)). \tag{2.3.19}$$

In fact, the first term is obtained due to the weak convergence of $u^m \rightharpoonup u$ in $D(A)$. The bilinear mapping converges as long as they are defined on the support of w_j which is compact, so we denote by $Q_j \subset \mathcal{O}$ a bounded open set with smooth boundary containing it. Therefore, we not only deduce the weak convergence $u^m \rightharpoonup u$ in $D(A)$, but the strong convergence $u^m \to u$ in $\mathbb{H}_0^1(Q_j)$ (see [53, Lemma 2.4] for more details). For the last term,

$$\begin{aligned}\|((\widetilde{g}_0(u^m), w_j)) - ((\widetilde{g}_0(u), w_j))\| &\leqslant \|\widetilde{g}_0(u^m) - \widetilde{g}_0(u)\|_{\mathbb{H}_0^1(Q_j)}\|w_j\| \\ &\leqslant L_{\widetilde{g}}\|u^m - u\|_{\mathbb{H}_0^1(Q_j)}\|w_j\|,\end{aligned} \tag{2.3.20}$$

which converges to zero due to the strong convergence in $\mathbb{H}_0^1(Q_j)$. Consequently, (2.3.19) is satisfied for each w_j. Since the linear combinations of elements for $B = \{w_j; j \in \mathbb{N}\}$ are dense in $D(A)$, we deduce that (2.3.9) holds at least by $u_\infty = u$.

(2) Let $u_\infty^{(1)}$ and $u_\infty^{(2)}$ be two solutions to system (2.3.9). Then,

$$\widetilde{A}(u_\infty^{(1)} - u_\infty^{(2)}) + \widetilde{B}(u_\infty^{(1)}) - \widetilde{B}(u_\infty^{(2)}) = \widetilde{g}_0(u_\infty^{(1)}) - \widetilde{g}_0(u_\infty^{(2)}). \tag{2.3.21}$$

Taking the inner product of (2.3.21) with $u_\infty^{(1)} - u_\infty^{(2)}$, we find

$$\|u_\infty^{(1)} - u_\infty^{(2)}\|_{D(A)}^2 + \langle \widetilde{B}(u_\infty^{(1)}) - \widetilde{B}(u_\infty^{(2)}), u_\infty^{(1)} - u_\infty^{(2)}\rangle = ((\widetilde{g}_0(u_\infty^{(1)}) - \widetilde{g}_0(u_\infty^{(2)}), u_\infty^{(1)} - u_\infty^{(2)})). \tag{2.3.22}$$

Thanks to the fact that $\langle \widetilde{B}(u) - \widetilde{B}(v), u - v\rangle = -\langle \widetilde{B}(u - v), v\rangle$ for all $u, v \in D(A)$, then by (0.0.42) and (B3), we obtain

$$|\langle \widetilde{B}(u_\infty^{(1)}) - \widetilde{B}(u_\infty^{(2)}), u_\infty^{(1)} - u_\infty^{(2)}\rangle| \leqslant \widetilde{c}\widetilde{\lambda}_1^{-\frac{1}{2}}\|u_\infty^{(1)} - u_\infty^{(2)}\|_{D(A)}^2\|u_\infty^{(2)}\|_{D(A)}. \tag{2.3.23}$$

Multiplying (2.3.9) by u_∞, we infer from (0.0.42) and (2.3.6) that

$$\begin{aligned}\|u_\infty\|_{D(A)}^2 &= \langle \widetilde{A}u_\infty, u_\infty\rangle \\ &= ((\widetilde{f}, u_\infty)) + ((\widetilde{g}_0(u_\infty), u_\infty)) \\ &\leqslant \widetilde{\lambda}_1^{-\frac{1}{2}}\|\widetilde{f}\|\|u_\infty\|_{D(A)} + \widetilde{\lambda}_1^{-1}L_{\widetilde{g}}\|u_\infty\|_{D(A)}^2,\end{aligned} \tag{2.3.24}$$

which, together with $\widetilde{\lambda}_1 > L_{\widetilde{g}}$, implies that

$$\|u_\infty\|_{D(A)} \leqslant \frac{\widetilde{\lambda}_1^{-\frac{1}{2}}\|\widetilde{f}\|}{1 - \widetilde{\lambda}_1^{-1}L_{\widetilde{g}}}. \tag{2.3.25}$$

Actually, all solutions to (2.3.9) must satisfy the above bound. Therefore, combining (2.3.23) and (2.3.25), we have

$$|\langle \widetilde{B}(u_\infty^{(1)}) - \widetilde{B}(u_\infty^{(2)}), u_\infty^{(1)} - u_\infty^{(2)} \rangle| \leqslant \widetilde{c}\widetilde{\lambda}_1^{-1} \frac{\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}} \|u_\infty^{(1)} - u_\infty^{(2)}\|_{D(A)}^2. \tag{2.3.26}$$

By (0.0.42), the last term of (2.3.22) is bounded by

$$\begin{aligned}((\widetilde{g}_0(u_\infty^{(1)}) - \widetilde{g}_0(u_\infty^{(2)}), u_\infty^{(1)} - u_\infty^{(2)})) &\leqslant L_{\widetilde{g}}\|u_\infty^{(1)} - u_\infty^{(2)}\|^2 \\ &\leqslant \widetilde{\lambda}_1^{-1} L_{\widetilde{g}}\|u_\infty^{(1)} - u_\infty^{(2)}\|_{D(A)}^2. \end{aligned} \tag{2.3.27}$$

Substituting (2.3.26)-(2.3.27) into (2.3.22), we deduce

$$(1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}})^2\|u_\infty^{(1)} - u_\infty^{(2)}\|_{D(A)}^2 \leqslant \widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|\|u_\infty^{(1)} - u_\infty^{(2)}\|_{D(A)}^2, \tag{2.3.28}$$

which implies the uniqueness follows as long as we assume $(1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}})^2 > \widetilde{c}\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|$. □

Next, using the same method as 1.3, one can obtain the stability results of stationary solutions to (2.3.9).

2.3.2 Stability of stationary solutions

We first prove the local stability of stationary solutions to (2.3.9) for general delay terms by using a direct method.

Theorem 2.3.2. *Assume the same hypotheses and notations in Theorem 2.2.3 and Theorem 2.3.1 hold. In addition,*

$$2\widetilde{\lambda}_1 \geqslant \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}} + 2C_{\widetilde{g}} + C_{\widetilde{\sigma}}^2 \tag{2.3.29}$$

is satisfied. If $u(\cdot)$ is any solution of Eq. (2.3.1), u_∞ is the unique stationary solution of Eq. (2.3.9) and $w(t) = u(t) - u_\infty$, then

$$\mathbb{E}\Big(\|w(t)\|^2\Big) \leqslant \mathbb{E}\Big(\|w(0)\|^2\Big) + C_{\widetilde{g}}\int_{-\infty}^0 \mathbb{E}\Big(\|\phi(s) - u_\infty\|^2\Big)ds. \tag{2.3.30}$$

Proof. The proof can be derived in the same manner as Theorem 1.3.2 and thus omitted. □

As an immediate consequence of Theorem 2.3.2, we derive some sufficient conditions ensuring the local stability of stationary solutions to (2.3.9) when the delay term has particular forms in $C_{-\infty}(V)$.

Corollary 2.3.3. *Assume the same hypotheses and notations in Theorem 2.2.3 and Theorem 2.3.1 hold. Let the delay term $\widetilde{g}(t, u_t) = \widetilde{\mathcal{G}}(u(t-h(t)))$ satisfy (2.3.5) and (2.3.6), moreover,*

$$2\widetilde{\lambda}_1 \geqslant \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}} + \frac{2(1-h^*)^{\frac{1}{2}}L_{\widetilde{\mathcal{G}}}}{1-h^*} + C_{\widetilde{\sigma}}^2 \tag{2.3.31}$$

is satisfied. If $u(\cdot)$ is any solution of Eq. (2.3.1), u_∞ is the unique stationary solution of Eq. (2.3.9) and $w(t) = u(t) - u_\infty$, then

$$\mathbb{E}\Big(\|w(t)\|^2\Big) \leqslant \mathbb{E}\Big(\|w(0)\|^2\Big) + \frac{(1-h^*)^{\frac{1}{2}} L_{\widetilde{\mathcal{G}}}}{1-h^*} \int_{-\infty}^{0} \mathbb{E}\Big(\|\phi(s) - u_\infty\|^2\Big) ds. \tag{2.3.32}$$

Corollary 2.3.4. *Assume the same hypotheses and notations in Theorem 2.2.3 and Theorem 2.3.1 hold. Let the delay term $\widetilde{g}(t, u_t) = \int_{-\infty}^{0} \widetilde{\mathcal{H}}(s, u(t+s))ds$ satisfy (2.3.7) and (2.3.8), moreover,*

$$2\widetilde{\lambda}_1 \geqslant \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1} L_{\widetilde{g}}} + 2\|L_{\widetilde{\mathcal{H}}}\|_{L^2(-\infty,0)} + C_{\widetilde{\sigma}}^2 \tag{2.3.33}$$

holds. If $u(\cdot)$ is any solution of Eq. (2.3.1), u_∞ is the unique stationary solution of Eq. (2.3.9) and $w(t) = u(t) - u_\infty$, then

$$\mathbb{E}\Big(\|w(t)\|^2\Big) \leqslant \mathbb{E}\Big(\|w(0)\|^2\Big) + \|L_{\widetilde{\mathcal{H}}}\|_{L^2(-\infty,0)} \int_{-\infty}^{0} \mathbb{E}\Big(\|\phi(s) - u_\infty\|^2\Big) ds. \tag{2.3.34}$$

Next, we consider the exponential stability of stationary solutions in the case of unbounded distributed delay, that is, we prove that the solution $u(t)$ to problem (2.3.1) with infinite distributed delay converges exponentially to the unique stationary solution u_∞ of Eq. (2.3.9) in $C_\gamma(V)$ for $\gamma > 0$.

Theorem 2.3.5. *Assume the same hypotheses and notations in Theorem 2.2.3 and Theorem 2.3.1 hold. Let the delay term $\widetilde{g}(t, u_t) = \int_{-\infty}^{0} \widetilde{\mathcal{H}}(s, u(t+s))ds$ satisfy (2.3.7) and (2.3.8), moreover, there exists a constant $0 < \rho < 2\gamma$ such that for all $t \geqslant 0$,*

$$2\widetilde{\lambda}_1 \geqslant \frac{2\widetilde{c}\|\widetilde{f}\|}{1-\widetilde{\lambda}_1^{-1} L_{\widetilde{g}}} + 2(2\rho)^{-\frac{1}{2}} \|L_{\widetilde{\mathcal{H}}}(\cdot) e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)} + C_{\widetilde{\sigma}}^2 + \rho \tag{2.3.35}$$

is satisfied. If $u(\cdot)$ is any solution of Eq. (2.3.1), u_∞ is the unique stationary solution of Eq. (2.3.9) and $w(t) = u(t) - u_\infty$, then

$$\mathbb{E}\Big(\|w(t)\|^2\Big) \leqslant e^{-\rho t}\Big(1 + \frac{(2\rho)^{\frac{1}{2}}}{2\rho(2\gamma-\rho)} \|L_{\widetilde{\mathcal{H}}}(\cdot) e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}\Big) \mathbb{E}\Big(\|\phi - u_\infty\|_{C_\gamma(V)}^2\Big), \tag{2.3.36}$$

and

$$\mathbb{E}\Big(\|w_t\|_{C_\gamma(V)}^2\Big) \leqslant e^{-\rho t}\Big(2 + \frac{(2\rho)^{\frac{1}{2}}}{2\rho(2\gamma-\rho)} \|L_{\widetilde{\mathcal{H}}}(\cdot) e^{-(\gamma+\rho)\cdot}\|_{L^2(-\infty,0)}\Big) \mathbb{E}\Big(\|\phi - u_\infty\|_{C_\gamma(V)}^2\Big). \tag{2.3.37}$$

Proof. Using the same arguments as in the proof of Theorem 1.3.6, we imply (2.3.36) and (2.3.37) as desired.

□

Besides, we state our asymptotic stability result for our original problem via the Lyapunov functional method.

Theorem 2.3.6. *Assume that the same hypotheses and notations in Theorems 2.2.3 and 2.3.1 hold. In addition, let the delay term* $\widetilde{g}(t, u_t) = \widetilde{\mathcal{G}}(u(t-h(t)))$ *satisfy* (2.3.5), (2.3.6), $\widetilde{f} = 0$ *and*

$$2\widetilde{\lambda}_1 \geqslant \frac{2(1-h^*)^{\frac{1}{2}} L_{\widetilde{\mathcal{G}}}}{1-h^*} + C^2_{\widetilde{\sigma}}. \tag{2.3.38}$$

Then $u_\infty = 0$ *is the unique stationary solution to problem* (2.3.9). *Moreover, the trivial solution of* (2.3.1) *is asymptotically mean square stable, that is,*

$$\lim_{t\to\infty} \mathbb{E}(\|u(t;\phi)\|^2) = 0. \tag{2.3.39}$$

Proof. The proof of this theorem can be done directly by using the same method in Theorem 1.3.10 and thus omitted. □

Finally, the polynomial asymptotic stability of stationary solution to problem (2.3.1) in the particular case of proportional delay is derived.

Theorem 2.3.7. *Assume the same hypotheses and notations in Theorem 2.2.3 and Theorem 2.3.1 hold. In addition, let the system* (2.3.1) *satisfy* $\widetilde{f} = 0$, *the delay term* $\widetilde{g}(t, u_t) = L_{\widetilde{g}} u(\theta t)$ *with* $\theta \in (0, 1)$ *and* $2\widetilde{\lambda}_1 > 2|L_{\widetilde{g}}| + C^2_{\widetilde{\sigma}}$, *then the origin is the unique stationary solution to Eq.* (2.3.9), *moreover, any solution* $u(t)$ *of Eq.* (2.3.1) *converges to zero polynomially, that is,* $\widetilde{\varrho} = \widetilde{\varrho}(L_{\widetilde{g}}, C_{\widetilde{\sigma}}, \widetilde{\lambda}_1, \theta) > 0$ *and* $\beta < 0$,

$$\mathbb{E}(\|u(t;\phi)\|^2) \leqslant \widetilde{\varrho}\mathbb{E}(\|\phi\|^2_{C_{-\infty}(V)})(1+t)^\beta, \quad t \geqslant 0, \tag{2.3.40}$$

where β *satisfies* $-2\widetilde{\lambda}_1 + |L_{\widetilde{g}}| + C^2_{\widetilde{\sigma}} + |L_{\widetilde{g}}|\theta^\beta = 0$.

Proof. One can prove (2.3.40) by using the same method as in Theorem 1.3.14, which is based on Lemmas 1.3.12-1.3.13. Therefore, we omit the details. □

Remark 2.3.8. *In fact, we can take into account a more general case in the form of* $\widetilde{g}(t, \xi) = \widetilde{G}(\xi(-(1-\theta)t))$, *where* $\widetilde{G}(\cdot)$ *is Lipschitz continuous.*

Chapter 3

Invariant measures for autonomous stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay and additive noise

In this chapter we focus on random dynamics and invariant measures for stochastic 3D LANS equations driven by infinite delay and additive noise. In Section 3.1, we describe some preliminaries, including some definitions related to random dynamical systems, some notation and linear operators, some suitable assumptions about the non-delayed external force f, delay term g and additive noise κ. In Section 3.2, we prove the well-posedness of the stochastic 3D LANS equations with infinite delay and additive noise (0.0.6). Section 3.3 is devoted to the existence of a global random attractor in $C_\gamma(V)$ for the stochastic equation (0.0.6). In the last section, we construct a family of invariant Borel probability measures of Eq. (0.0.6) by using the method of generalized Banach limit.

3.1 Random dynamical systems and hypotheses

3.1.1 Random dynamical systems

Let $(X, \|\cdot\|_X)$ be a separable Banach space equipped with its Borel σ-algebra $\mathcal{B}(X)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a group $\{\theta_t\}_{t\in\mathbb{R}}$ such that $\mathbb{P}$ is the Wiener distribution, Ω is identified with the subset $\{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ via the relationship $W(t, \omega) = \omega(t)$, $\mathcal{F}$ is a σ-algebra, and each $\theta_t : \Omega \to \Omega$ is measure-preserving. If $\{\theta_t\}_{t\in\mathbb{R}}$ fulfills the group property and the mapping $(t, \omega) \mapsto \theta_t\omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$ measurable, then $(\Omega, \mathcal{F}, \mathbb{P}; \{\theta_t\}_{t\in\mathbb{R}})$ is called a measurable dynamical system and $\{\theta_t\}_{t\in\mathbb{R}}$ is said to be a metric dynamical system over the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

For the reader convenience, we need to introduce the following definitions related to random dynamical systems (see [42, 148]).

Definition 3.1.1. *A family of mappings $\varphi(t,s,\omega) : X \mapsto X$, $-\infty < s < t < +\infty$, parameterized by $\omega \in \Omega$, is said to be a random dynamical system over the measurable dynamical system $(\Omega, \mathcal{F}, \mathbb{P}; \{\theta_t\}_{t\in\mathbb{R}})$, if it holds, for almost all $\omega \in \Omega$,*

(i) $\varphi(t,r,\omega)\varphi(r,s,\omega)x = \varphi(t,s,\omega)x$ for all $s \leqslant r \leqslant t$ and $x \in X$;

(ii) $\varphi(t,s,\omega)\cdot$ is continuous on X;

(iii) for all $t \in \mathbb{R}, x \in X$, the mapping $(s,\omega) \mapsto \varphi(t,s,\omega)x$ is measurable from $((-\infty,t] \times \Omega, \mathfrak{B}((-\infty,t]) \otimes \mathcal{F})$ to $(X, \mathfrak{B}(X))$;

(iv) for all $s < t, x \in X$, the mapping $\omega \mapsto \varphi(t,s,\omega)x$ is measurable from $(\Omega, \mathcal{F})$ to $(X, \mathfrak{B}(X))$.

Set

$$\Psi(t-s, \theta_s\omega) = \varphi(t,s,\omega) \text{ and } \Phi(t) : (\omega, \phi) \mapsto (\theta_t\omega, \Psi(t,\omega)\phi).$$

Notice that if $\Psi(t,\omega)$ satisfies the cocycle property, that is, $\Psi(t+s,\omega) = \Psi(t,\theta_s\omega)\Psi(s,\omega)$, then $\{\Phi(t)\}_{t\in\mathbb{R}}$ fulfills the semigroup property $\Phi(t+s) = \Phi(t)\Phi(s)$. The mapping $\{\Phi(t)\}_{t\in\mathbb{R}}$ is a skew product flow on $\Omega \times X$.

A family $\mathcal{D} = \{\mathcal{D}(\omega) \subset X : \omega \in \Omega\}$ of closed subset is said to be random (or measurable) if the mapping $\omega \to \text{dist}_X(x, \mathcal{D}(\omega))$ is $(\mathcal{F}, \mathfrak{B}(\mathbb{R}^+))$ measurable for each $x \in X$.

Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X. Let $\mathfrak{D}$ be a family of nonempty random sets $\mathcal{D} = \{\mathcal{D}(\omega) : \omega \in \Omega\} \subseteq \mathcal{P}(X)$. The class $\mathfrak{D}$ is said to be a universe in $\mathcal{P}(X)$.

Definition 3.1.2. *The random dynamical system φ is said to be $\mathfrak{D}$-asymptotically compact if, for any $(t,\omega,\mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}$ and any sequences $\{s_n\}, \{x_n\} \subset X$ with $s_n \leqslant t$, $\lim_{n\to+\infty} s_n = -\infty$ and $x_n \in \mathcal{D}(\theta_{s_n-t}\omega)$, the sequence $\{\varphi(t, s_n, \theta_{-t}\omega)x_n\}$ has a convergent subsequence in X.*

Definition 3.1.3. *Let $\{\varphi(t,s,\omega)\}_{t\geqslant s,\omega\in\Omega}$ be a random dynamical system with a universe $\mathfrak{D}$ over the measurable dynamical system $(\Omega, \mathcal{F}, \mathbb{P}; \{\theta_t\}_{t\in\mathbb{R}})$. A random subset $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ of X is called a global $\mathfrak{D}$-random attractor for $\{\varphi(t,s,\omega)\}_{t\geqslant s,\omega\in\Omega}$, if*

(i) $\mathcal{A}$ is compact, that is, each $\mathcal{A}(\omega)$ is compact in X;

(ii) $\mathcal{A}$ is invariant, that is, for all $(t,\omega) \in \mathbb{R} \times \Omega$,

$$\varphi(t,s,\omega)\mathcal{A}(\theta_s\omega) = \mathcal{A}(\theta_t\omega), \ \forall s \leqslant t;$$

(iii) $\mathcal{A}$ is $\mathfrak{D}$-attracting, that is, for each $(t,\omega,\mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}$,

$$\lim_{s\to-\infty} \text{dist}_X(\varphi(t,s,\theta_{-t}\omega)\mathcal{D}(\theta_{s-t}\omega), \mathcal{A}(\omega)) = 0. \tag{3.1.1}$$

3.1.2 Hypotheses

In order to analyze our problem, we need to establish some suitable assumptions. We first suppose that there exists a constant μ such that

$$0 < \mu < 1 \text{ and } (1-\mu)\widetilde{\lambda}_1 < \gamma. \tag{3.1.2}$$

Let $a := 2(1-\mu)\widetilde{\lambda}_1$, then $0 < a < 2\gamma$.

Recall that $\mathcal{P}(C_\gamma(V))$ is the family of all subsets of $C_\gamma(V)$. Denote by $\mathfrak{D}_a$ the tempered universe of nonempty random subsets $\mathcal{D} = \{\mathcal{D}(\omega) : \omega \in \Omega\} \subseteq \mathcal{P}(C_\gamma(V))$, that is, $\mathcal{D} \in \mathfrak{D}_a$ if and only if,

$$\lim_{s\to-\infty} e^{as}\|\mathcal{D}(\theta_{s-t}\omega)\|^2_{C_\gamma(V)} = 0, \text{ for all } (t,\omega) \in \mathbb{R}\times\Omega. \tag{3.1.3}$$

We then assume that the non-delayed external force f and the additive noise κ satisfy:

$$f \text{ and } \kappa \in \mathbb{H}^{-1}(O). \tag{3.1.4}$$

We also require some assumptions on the delay term g. Namely, let $g : C_\gamma(V) \to \mathbb{H}^{-1}(O)$ satisfy the following conditions:

(G31) For any $\eta \in C_\gamma(V)$, $g(\eta)$ is measurable;

(G32) $g(0) = 0$;

(G33) There exists a constant $L_g > 0$ such that for all $\eta, \zeta \in C_\gamma(V)$,

$$\|g(\eta) - g(\zeta)\|_{\mathbb{H}^{-1}(O)} \leqslant L_g\|\eta - \zeta\|_{C_\gamma(V)};$$

(G34) There exists a constant $C_g > 0$ such that, for all $s \in \mathbb{R}, t \geqslant s$ and $u, v \in C^0((-\infty,t);V)$,

$$\int_s^t \|g(u_r) - g(v_r)\|^2_{\mathbb{H}^{-1}(O)}dr \leqslant C_g^2\int_{-\infty}^t \|u(r) - v(r)\|^2dr;$$

(G35) There exists a constant $\widetilde{C}_g > 0$ such that, for all $s \in \mathbb{R}, t \geqslant s$, all decreasing function $\varpi \in C^0([s,t])$,

$$\int_s^t \varpi(r)\|g(u_r) - g(v_r)\|^2_{\mathbb{H}^{-1}(O)}dr \leqslant \widetilde{C}_g^2\int_s^t \varpi(r)\|u(r) - v(r)\|^2dr;$$

(G6) If the sequence $\{v^m\}$ converges weakly to v in $L^2(-\infty,T;D(A))$, weakly star in $L^\infty(s,T;V)$ and strongly in $L^2(-\infty,T;V)$, then $g(v^m_\cdot)$ converges weakly to $g(v_\cdot)$ in $L^2(s,T;\mathbb{H}^{-1}(O))$, $\forall\ T > s$.

Next, let us define $\widetilde{f}, \widetilde{\kappa} \in V$ and $\widetilde{g} : C_\gamma(V) \to V$ as

$$((\widetilde{f}, w)) = \langle f, w\rangle_{-1},\ \forall\ w \in V, \tag{3.1.5}$$

$$((\widetilde{\kappa}, w)) = \langle \kappa, w\rangle_{-1},\ \forall\ w \in V, \tag{3.1.6}$$

$$((\widetilde{g}(\eta), w)) = \langle g(\eta), w\rangle_{-1},\ \forall\ (\eta, w) \in C_\gamma(V)\times V. \tag{3.1.7}$$

Besides, $\widetilde{g} : C_\gamma(V) \to V$ also satisfies the following conditions:

(H31) For any $\eta \in C_\gamma(V)$, $\widetilde{g}(\eta)$ is measurable;

(H32) $\widetilde{g}(0) = 0$;

(H33) Setting $L_{\widetilde{g}} = L_g$, it follows, for all $\eta, \zeta \in C_\gamma(V)$,

$$\|\widetilde{g}(\eta) - \widetilde{g}(\zeta)\| \leqslant L_{\widetilde{g}}\|\eta - \zeta\|_{C_\gamma(V)};$$

It follows from (H32) and (H33) that, for all $\eta \in C_\gamma(V)$,

$$\|\widetilde{g}(\eta)\| \leqslant L_{\widetilde{g}}\|\eta\|_{C_\gamma(V)}. \tag{3.1.8}$$

(H34) Letting $C_{\widetilde{g}} = C_g$, for all $s \in \mathbb{R}, t \geqslant s$ and $u, v \in C^0((-\infty, t); V)$,

$$\int_s^t \|\widetilde{g}(u_r) - \widetilde{g}(v_r)\|^2 dr \leqslant C_{\widetilde{g}}^2 \int_{-\infty}^t \|u(r) - v(r)\|^2 dr;$$

(H35) Taking $\widetilde{C}_{\widetilde{g}} = \widetilde{C}_g$, for all $s \in \mathbb{R}, t \geqslant s$ and all decreasing function $\varpi \in C^0([s,t])$,

$$\int_s^t \varpi(r)\|\widetilde{g}(u_r) - \widetilde{g}(v_r)\|^2 dr \leqslant \widetilde{C}_{\widetilde{g}} \int_s^t \varpi(r)\|u(r) - v(r)\|^2 dr;$$

(H36) If the sequence $\{v^m\}$ converges weakly to v in $L^2(-\infty, T; D(A))$, weakly star in $L^\infty(s, T; V)$ and strongly in $L^2(-\infty, T; V)$, then $\widetilde{g}(v^m_\cdot)$ converges weakly to $\widetilde{g}(v_\cdot)$ in $L^2(s, T; V)$, $\forall\, T > s$.

In what follows, the letters c and c_i $(i \in \mathbb{Z})$ represent positive constants whose values may vary from line to line even in the same line.

3.2 Well-posedness of autonomous stochastic 3D LANS equations

Based on the previous operators and assumptions, we focus on the random dynamics and invariant measures of the following stochastic 3D LANS equations with infinite delay and additive noise:

$$\begin{cases} \dfrac{du}{dt} + \widetilde{A}u(t) + \widetilde{B}(u(t)) = \widetilde{f} + \widetilde{g}(u_t) + \widetilde{\kappa}\dfrac{dW}{dt}, \quad \forall\, t > s, \\ u_s = \phi, \end{cases} \tag{3.2.1}$$

which is satisfied in $(D(A))^*$, a.s. for all $t > s$.

In order to define a random dynamical system for Eq. (3.2.1), we need to transform the stochastic equation into a random system. As usual, let $z(\theta_t\omega) = -\int_{-\infty}^0 e^s(\theta_t\omega)(s)ds$, with $t \in \mathbb{R}$, be the Ornstein-Uhlenbeck processes, which is the stationary solution of the stochastic Langevin equation $dz + zdt = dW(t)$. Thanks to [5], we obtain that there exists a θ_t-invariant set $\widetilde{\Omega}$ of full measure such that $z(\theta_t\omega)$ is continuous with respect to t, and the following results hold:

$$\lim_{t\to\pm\infty} \frac{z(\theta_t w)}{t} = \lim_{t\to\pm\infty} \frac{1}{t}\int_{-t}^0 z(\theta_s w)ds = 0, \tag{3.2.2}$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_{-t}^0 |z(\theta_s w)|^m ds = \frac{\Gamma(\frac{1+m}{2})}{\sqrt{\pi}}, \forall m > 0, \tag{3.2.3}$$

for all $\omega \in \widetilde{\Omega}$, where Γ denotes the Gamma function. Note that $t \to z(\theta_t\omega)$ is continuous and tempered for all $\omega \in \widetilde{\Omega}$, where $\widetilde{\Omega}$ is a θ-invariant full-measure subspace of Ω, but we will not distinguish them below. Therefore, it follows from [5, Proposition 4.3.3] that there exists a tempered function $r(\omega) > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$|z(\omega)|^2 \leqslant r(\omega), \tag{3.2.4}$$

where $r(\omega)$ satisfies, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$r(\theta_t\omega) \leqslant e^{\frac{\alpha}{2}|t|} r(\omega), \ t \in \mathbb{R}. \tag{3.2.5}$$

Combining (3.2.4) and (3.2.5), for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$|z(\theta_t\omega)|^2 \leqslant e^{\frac{\alpha}{2}|t|} r(\omega), \ t \in \mathbb{R}. \tag{3.2.6}$$

Let

$$v(t, s, \omega, \psi) = u(t, s, \omega, \phi) - \widetilde{\kappa} z(\theta_t\omega), \ t \geqslant s. \tag{3.2.7}$$

Then, by (3.2.1) and (3.2.7), we deduce

$$\begin{cases} \dfrac{dv}{dt} + \widetilde{A}v(t) + \widetilde{B}(v(t)) = -z(\theta_t\omega)\widetilde{A}\widetilde{\kappa} - z(\theta_t\omega)\widetilde{B}(\widetilde{\kappa}) \\ \quad + \widetilde{f} + \widetilde{g}(u_t) + z(\theta_t\omega)\widetilde{\kappa}, \quad \forall\, t > s, \\ v_s = \psi, \end{cases} \tag{3.2.8}$$

where $\psi(\iota) = \phi(\iota) - \widetilde{\kappa} z(\theta_{\iota+s}\omega)$ with $\iota \leqslant 0$.

Definition 3.2.1. *Suppose that $\psi \in C_\gamma(V)$. A stochastic process v defined on $\mathbb{R}$ is called a solution to system (3.2.8) if*

$$v \in L^2(s, T; D(A)) \cap L^\infty(s, T; V), \ \forall\, T > s,$$

$v_s = \psi$ and the system (3.2.8) is satisfied in $(D(A))^*$, that is, for almost all $\omega \in \Omega$,

$$\begin{aligned} ((v(t), w)) + \int_s^t \langle \widetilde{A}v(r) + \widetilde{B}(v(r)), w\rangle dr + \int_s^t z(\theta_r\omega)\langle \widetilde{A}\widetilde{\kappa} + \widetilde{B}(\widetilde{\kappa}), w\rangle dr \\ = ((\psi(0), w)) + \int_s^t \Big(\big(\widetilde{f} + \widetilde{g}(u_r), w\big)\Big) dr + \int_s^t z(\theta_r\omega)((\widetilde{\kappa}, w)) dr, \end{aligned} \tag{3.2.9}$$

for all $t \geqslant s$ and $w \in D(A)$.

Theorem 3.2.2. *Suppose* (H31)-(H36) *and* (3.1.5)-(3.1.7) *hold. For each $(\omega, s, \psi) \in \Omega \times \mathbb{R} \times C_\gamma(V)$, system* (3.2.8) *possesses a unique weak solution $v(\cdot, s, \omega, \psi)$ in the sense of Definition 3.2.1 defined on $[s, +\infty)$.*

Proof. Using a Galerkin method and a priori estimates given in [33], one can similarly prove the existence of weak solutions to Eq. (3.2.8), while, the uniqueness follows from a standard Gronwall lemma. □

The following result shows that the solution to system (3.2.8) is continuous with respect to initial data.

Theorem 3.2.3. *Suppose that* (H31)-(H36) *and* (3.1.5)-(3.1.7) *are satisfied. Let* $\psi, \widetilde{\psi} \in C_\gamma(V)$ *be two initial values to problem* (3.2.8)*. Let* $v(\cdot) = v(\cdot, s, \omega, \psi)$ *and* $\widetilde{v}(\cdot) = \widetilde{v}(\cdot, s, \omega, \widetilde{\psi})$ *be two solutions to system* (3.2.8) *at the initial time s, respectively. Then, for all* $\iota \geqslant s$,

$$\max_{r\in[s,\iota]} \|v(r,s,\omega,\psi) - \widetilde{v}(r,s,\omega,\widetilde{\psi})\|^2 \leqslant \Big(1 + \frac{L_g^2}{2\gamma}\Big)\|\psi - \widetilde{\psi}\|^2_{C_\gamma(V)} \exp\Big(\int_s^\iota \Big(\frac{\widetilde{c}^2}{2}\|v(\sigma)\|^2_{D(A)} + 1 + L_g^2\Big)d\sigma\Big). \tag{3.2.10}$$

Proof. Setting $u(\iota) = u(\iota, s, \omega, \phi) = v(\iota) + \widetilde{\kappa} z(\theta_\iota\omega)$, $\widetilde{u}(\iota) = \widetilde{u}(\iota, s, \omega, \widetilde{\phi}) = \widetilde{v}(\iota) + \widetilde{\kappa} z(\theta_\iota\omega)$, we have $v(\iota) - \widetilde{v}(\iota) = u(\iota) - \widetilde{u}(\iota)$ and $\|v_\iota - \widetilde{v}_\iota\|^2_{C_\gamma(V)} = \|u_\iota - \widetilde{u}_\iota\|^2_{C_\gamma(V)}$. Then it follows from (3.2.8) that

$$\frac{d}{d\iota}\|v(\iota) - \widetilde{v}(\iota)\|^2 = -2\langle \widetilde{A}(v(\iota) - \widetilde{v}(\iota)) + \widetilde{B}(v) - \widetilde{B}(\widetilde{v}), v(\iota) - \widetilde{v}(\iota)\rangle + 2((\widetilde{g}(u_\iota) - \widetilde{g}(\widetilde{u}_\iota), v(\iota) - \widetilde{v}(\iota))), \tag{3.2.11}$$

where $\iota \geqslant s$. By (B3), (H33) and the fact that $\langle \widetilde{B}(u) - \widetilde{B}(v), u - v\rangle = \langle \widetilde{B}(v, u - v), u - v\rangle$ for all $u, v \in D(A)$, we easily obtain

$$\frac{d}{d\iota}\|v(\iota) - \widetilde{v}(\iota)\|^2 = \Big(\frac{\widetilde{c}^2}{2}\|v\|^2_{D(A)} + 1\Big)\|v(\iota) - \widetilde{v}(\iota)\|^2 + L_g^2\|v_\iota - \widetilde{v}_\iota\|^2_{C_\gamma(V)}. \tag{3.2.12}$$

Note that for $\sigma \in [s, \iota]$, we deduce

$$\begin{aligned}
\|v_\sigma - \widetilde{v}_\sigma\|^2_{C_\gamma(V)} &\leqslant \max\Big\{\sup_{\vartheta\leqslant s-\sigma} e^{2\gamma\vartheta}\|v(\sigma+\vartheta) - \widetilde{v}(\sigma+\vartheta)\|^2, \sup_{s-\sigma\leqslant\vartheta\leqslant 0} e^{2\gamma\vartheta}\|v(\sigma+\vartheta) - \widetilde{v}(\sigma+\vartheta)\|^2\Big\}\\
&\leqslant \max\Big\{\sup_{\vartheta\leqslant s-\sigma} e^{2\gamma\vartheta}\|\psi(\sigma+\vartheta-s) - \widetilde{\psi}(\sigma+\vartheta-s)\|^2, \sup_{s-\sigma\leqslant\vartheta\leqslant 0} e^{2\gamma\vartheta}\|v(\sigma+\vartheta) - \widetilde{v}(\sigma+\vartheta)\|^2\Big\}\\
&\leqslant \max\Big\{\sup_{\vartheta\leqslant 0} e^{2\gamma(\vartheta-\sigma+s)}\|\psi(\vartheta) - \widetilde{\psi}(\vartheta)\|^2, \sup_{s\leqslant\vartheta\leqslant\sigma} \|v(\vartheta) - \widetilde{v}(\vartheta)\|^2\Big\}\\
&\leqslant \max\Big\{e^{2\gamma(s-\sigma)}\|\psi - \widetilde{\psi}\|^2_{C_\gamma(V)}, \max_{s\leqslant\vartheta\leqslant\sigma}\|v(\vartheta) - \widetilde{v}(\vartheta)\|^2\Big\}.
\end{aligned} \tag{3.2.13}$$

Combining (3.2.12) and (3.2.13),

$$\begin{aligned}
\|v(\iota) - \widetilde{v}(\iota)\|^2 &\leqslant \|\psi(0) - \widetilde{\psi}(0)\|^2 + \int_s^\iota \Big(\frac{\widetilde{c}^2}{2}\|v(\sigma)\|^2_{D(A)} + 1\Big)\|v(\sigma) - \widetilde{v}(\sigma)\|^2 d\sigma\\
&\quad + L_g^2\|\psi - \widetilde{\psi}\|^2_{C_\gamma(V)}\int_s^\iota e^{2\gamma(s-\sigma)}d\sigma + L_g^2\int_s^\iota \max_{s\leqslant\vartheta\leqslant\sigma}\|v(\vartheta) - \widetilde{v}(\vartheta)\|^2 d\sigma.
\end{aligned} \tag{3.2.14}$$

Taking supremum of (3.2.14),

$$\begin{aligned}
&\max_{r\in[s,\iota]}\|v(r,s,\omega,\psi) - \widetilde{v}(r,s,\omega,\widetilde{\psi})\|^2\\
&\leqslant \|\psi(0) - \widetilde{\psi}(0)\|^2 + \int_s^\iota \Big(\frac{\widetilde{c}^2}{2}\|v(\sigma)\|^2_{D(A)} + 1\Big)\|v(\sigma) - \widetilde{v}(\sigma)\|^2 d\sigma
\end{aligned}$$

$$+ L_{\widetilde{g}}^2\|\psi - \widetilde{\psi}\|^2_{C_\gamma(V)} \int_s^t e^{2\gamma(s-\sigma)} d\sigma + L_{\widetilde{g}}^2 \int_s^t \max_{s\leqslant\vartheta\leqslant\sigma} \|v(\vartheta) - \widetilde{v}(\vartheta)\|^2 d\sigma \tag{3.2.15}$$
$$\leqslant \Big(1 + \frac{L_{\widetilde{g}}^2}{2\gamma}\Big)\|\psi - \widetilde{\psi}\|^2_{C_\gamma(V)} + \int_s^t \Big(\frac{\widetilde{c}^2}{2}\|v(\sigma)\|^2_{D(A)} + 1 + L_{\widetilde{g}}^2\Big) \max_{s\leqslant\vartheta\leqslant\sigma} \|v(\vartheta) - \widetilde{v}(\vartheta)\|^2 d\sigma,$$

which, together with the Gronwall inequality, yields (3.2.10) as desired. □

Let $v(t, s, \omega, \psi)$ be the solution to system (3.2.8) with $\psi(\iota) = \phi(\iota) - \widetilde{\kappa}z(\theta_{\iota+s}\omega)$, where $\iota \leqslant 0$, and s is the initial time. Then $u(t, s, \omega, \phi) = v(t, s, \omega, \psi) + \widetilde{\kappa}z(\theta_t\omega)$ is the solution to Eq. (3.2.1) corresponding to the initial value ϕ. Next, we can define the family of operators $\{\varphi(t, s, \omega)\}_{t\geqslant s,\omega\in\Omega}$ by

$$\begin{aligned}\varphi(t, s, \omega)\phi &= u_t(\cdot, s, \omega, \phi)\\ &= v_t(\cdot, s, \omega, \psi) + \widetilde{\kappa}z(\theta_{t+\cdot}\omega).\end{aligned}$$

By Theorems 3.2.2 and 3.2.3, one can prove the above mapping is a continuous random dynamical system over the measurable dynamical system $(\Omega, \mathcal{F}, \mathbb{P}; \{\theta_t\}_{t\in\mathbb{R}})$ with state space $C_\gamma(V)$ by using the same method in [42]. Moreover, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\varphi(t, s, \omega)\phi = \varphi(t - s, 0, \theta_s\omega)\phi, \text{ for all } s < t, \ \phi \in C_\gamma(V).$$

3.3 Existence of global random attractors in $C_\gamma(V)$

In this section, we first obtain uniform estimates on the solutions of problem (3.2.8). Then, we prove that the random dynamical system φ associated with problem (0.0.6) has a global $\mathfrak{D}_a$-random absorbing set in $C_\gamma(V)$, and further prove it is $\mathfrak{D}_a$-asymptotically compact in $C_\gamma(V)$ via the Ascoli-Arzelà theorem. Finally, the existence of global $\mathfrak{D}_a$-random attractors for φ is proved.

Uniform estimates of solutions

Lemma 3.3.1. *Let* (H31)-(H36), (3.1.2) *and* (3.1.5)-(3.1.7) *and* $\widetilde{\kappa} \in D(\widetilde{A})$ *hold. Then, for each* $(t, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}_a$ *and* $\psi \in \mathcal{D}(\theta_{s-t}\omega)$, *there exists an* $s_0 := s_0(t, \omega, \mathcal{D}) < t$ *such that for all* $s \leqslant s_0$,

$$\|v_t(\cdot, s, \theta_{-t}\omega, \psi)\|^2_{C_\gamma(V)} \leqslant R(\omega), \tag{3.3.1}$$

where $R(\omega) = c + cr(\omega)$ *with the positive constant* c *being independent of* t, s *and* ω.

Proof. Let $r \in \mathbb{R}$ be fixed. Taking the inner product of the first equation in (3.2.8) with $v(r) := v(r, s, \omega, \psi)$, $s \leqslant r$, we obtain

$$\frac{d}{dr}\|v\|^2 + 2\|v\|^2_{D(A)} = -2z(\theta_r\omega)\langle\widetilde{A}\widetilde{\kappa} + \widetilde{B}(\widetilde{\kappa}), v\rangle + 2((\widetilde{f} + \widetilde{g}(u_r), v)) + 2z(\theta_r\omega)((\widetilde{\kappa}, v)). \tag{3.3.2}$$

Consider μ, which is given by (3.1.2). Since $\widetilde{\kappa} \in D(\widetilde{A}) \hookrightarrow D(A)$, (0.0.42) and (B3), we deduce that, there exists a positive constant $\overline{c}_1$ such that

$$2|z(\theta_r\omega)\langle\widetilde{A}\widetilde{\kappa} + \widetilde{B}(\widetilde{\kappa}), v\rangle| \leqslant 2|z(\theta_r\omega)||\langle\widetilde{A}\widetilde{\kappa} + \widetilde{B}(\widetilde{\kappa}), v\rangle|$$

$$\begin{aligned}&\leqslant 2|z(\theta_r\omega)|(|\widetilde{\kappa}\|_{D(A)}\|v\|_{D(A)}+\widetilde{\lambda}_1^{-\frac{1}{2}}\widetilde{c}|\widetilde{\kappa}\|_{D(A)}^2\|v\|_{D(A)})\\&\leqslant \overline{c}_1|z(\theta_r\omega)|^2+\frac{1}{4}\mu\|v\|_{D(A)}^2.\end{aligned}\tag{3.3.3}$$

Thanks to (0.0.42) and (3.1.8), we derive

$$\begin{aligned}2((\widetilde{f}+\widetilde{g}(u_r),v))&\leqslant 2\widetilde{\lambda}_1^{-\frac{1}{2}}(\|\widetilde{f}\|+\|\widetilde{g}(u_r)\|)\|v\|_{D(A)}\\&\leqslant \overline{c}_2\|\widetilde{f}\|^2+\overline{c}_2\|\widetilde{g}(u_r)\|^2+\frac{1}{2}\mu\|v\|_{D(A)}^2\\&\leqslant \overline{c}_2\|\widetilde{f}\|^2+\overline{c}_3\|u_r\|_{C_\gamma(V)}^2+\frac{1}{2}\mu\|v\|_{D(A)}^2,\end{aligned}\tag{3.3.4}$$

where $\overline{c}_2=4\widetilde{\lambda}_1^{-1}\mu^{-1},\overline{c}_3=\overline{c}_2L_{\widetilde{g}}^2$. Since $v(r,s,\omega,\psi)=u(r,s,\omega,\phi)-\widetilde{\kappa}z(\theta_r\omega)$ with $s\leqslant r$, we deduce

$$\begin{aligned}\|u_r\|_{C_\gamma(V)}^2&=\sup_{\iota\leqslant 0}e^{2\gamma\iota}\|u(r+\iota)\|^2\\&=\sup_{\iota\leqslant 0}e^{2\gamma\iota}\|v(r+\iota)+\widetilde{\kappa}z(\theta_{r+\iota}\omega)\|^2\\&\leqslant 2\|v_r\|_{C_\gamma(V)}^2+2\|\widetilde{\kappa}\|^2\sup_{\iota\leqslant 0}e^{2\gamma\iota}|z(\theta_{r+\iota}\omega)|^2.\end{aligned}\tag{3.3.5}$$

By (0.0.42), we obtain

$$\begin{aligned}2|z(\theta_r\omega)||((\widetilde{\kappa},v))|&\leqslant 2\widetilde{\lambda}_1^{-\frac{1}{2}}|z(\theta_r\omega)|\|\widetilde{\kappa}\|\|v\|_{D(A)}\\&\leqslant \overline{c}_4|z(\theta_r\omega)|^2+\frac{1}{4}\mu\|v\|_{D(A)}^2,\end{aligned}\tag{3.3.6}$$

where $\overline{c}_4=\overline{c}_2\|\widetilde{\kappa}\|^2<\infty$ on account of the fact that $\widetilde{\kappa}\in D(\widetilde{A})\hookrightarrow V$. Substituting (3.3.3)-(3.3.6) into (3.3.2), we have

$$\frac{d}{dr}\|v\|^2+(2-\mu)\|v\|_{D(A)}^2\leqslant \overline{c}_2\|\widetilde{f}\|^2+2\overline{c}_3\|v_r\|_{C_\gamma(V)}^2+\overline{c}_5\sup_{\iota\leqslant 0}e^{2\gamma\iota}|z(\theta_{r+\iota}\omega)|^2,$$

where $\overline{c}_5=\overline{c}_1+2\overline{c}_3\|\widetilde{\kappa}\|^2+\overline{c}_4$. By $a=2(1-\mu)\widetilde{\lambda}_1\in(0,2\gamma)$ and (0.0.42), we can rewrite the above inequality as

$$\frac{d}{dr}\|v\|^2+a\|v\|^2+\mu\|v\|_{D(A)}^2\leqslant \overline{c}_2\|\widetilde{f}\|^2+2\overline{c}_3\|v_r\|_{C_\gamma(V)}^2+\overline{c}_5\sup_{\iota\leqslant 0}e^{2\gamma\iota}|z(\theta_{r+\iota}\omega)|^2.\tag{3.3.7}$$

Multiplying (3.3.7) by e^{ar} and integrating the inequality on $[s,r]$, we obtain

$$\begin{aligned}\|v(r)\|^2+\mu\int_s^r e^{a(\sigma-r)}\|v(\sigma,s,\omega,\psi)\|_{D(A)}^2d\sigma&\leqslant e^{a(s-r)}\|\psi\|_{C_\gamma(V)}^2+\overline{c}_2a^{-1}\|\widetilde{f}\|^2+2\overline{c}_3\int_s^r e^{a(\sigma-r)}\|v_\sigma\|_{C_\gamma(V)}^2d\sigma\\&\quad+\overline{c}_5\int_s^r e^{a(\sigma-r)}\sup_{\iota\leqslant 0}e^{2\gamma\iota}|z(\theta_{\sigma+\iota}\omega)|^2d\sigma.\end{aligned}\tag{3.3.8}$$

For all $t \in \mathbb{R}$ with $t \geqslant s$, we replace ω by $\theta_{-t}\omega$ in (3.3.8). Then, thanks to (3.2.6),

$$
\begin{aligned}
&\|v(r,s,\theta_{-t}\omega,\psi)\|^2 + \mu \int_s^r e^{a(\sigma-r)}\|v(\sigma,s,\theta_{-t}\omega,\psi)\|^2_{D(A)}d\sigma \\
&\quad \leqslant e^{a(s-r)}\|\psi\|^2_{C_\gamma(V)} + \overline{c}_2 a^{-1}\|\widetilde{f}\|^2 + 2\overline{c}_3 \int_s^r e^{a(\sigma-r)}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma + 2\overline{c}_5 a^{-1} r(\omega) e^{\frac{a}{2}|r-t|}.
\end{aligned} \tag{3.3.9}
$$

By (3.3.9) and $0 < a < 2\gamma$, we deduce that, for all $s \leqslant r$,

$$
\begin{aligned}
\|v_r\|^2_{C_\gamma(V)} &\leqslant \max\Big\{ \sup_{\vartheta \leqslant s-r} e^{2\gamma\vartheta}\|v(r+\vartheta)\|^2, \sup_{s-r\leqslant\vartheta\leqslant 0} e^{2\gamma\vartheta}\|v(r+\vartheta)\|^2 \Big\} \\
&\leqslant \max\Big\{ \sup_{\vartheta \leqslant s-r} e^{2\gamma\vartheta}\|\psi(r+\vartheta-s)\|^2, \sup_{s-r\leqslant\vartheta\leqslant 0} e^{2\gamma\vartheta}\|v(r+\vartheta)\|^2 \Big\} \\
&\leqslant \max\Bigg\{ \sup_{\vartheta\leqslant 0} e^{2\gamma(\vartheta+s-r)}\|\psi(\vartheta)\|^2, \sup_{s-r\leqslant\vartheta\leqslant 0} e^{2\gamma\vartheta}\Big(e^{a(s-r-\vartheta)}\|\psi\|^2_{C_\gamma(V)} \\
&\qquad + \overline{c}_2 a^{-1}\|\widetilde{f}\|^2 + 2\overline{c}_3 \int_s^{r+\vartheta} e^{a(\sigma-r-\vartheta)}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma + 2\overline{c}_5 a^{-1} r(\omega) e^{\frac{a}{2}|r+\vartheta-t|}\Big)\Bigg\} \\
&\leqslant 2e^{a(s-r)}\|\psi\|^2_{C_\gamma(V)} + \overline{c}_2 a^{-1}\|\widetilde{f}\|^2 + 2\overline{c}_3 \int_s^r e^{a(\sigma-r)}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma + 2\overline{c}_5 a^{-1} r(\omega) e^{\frac{a}{2}|r-t|}.
\end{aligned} \tag{3.3.10}
$$

Since $s \leqslant r$, (3.3.10) can be rewritten as

$$
\begin{aligned}
\|v_r\|^2_{C_\gamma(V)} &\leqslant 2\|\psi\|^2_{C_\gamma(V)} + \overline{c}_2 a^{-1}\|\widetilde{f}\|^2 + 2\overline{c}_3 \int_s^r e^{a(\sigma-r)}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma + 2\overline{c}_5 a^{-1} r(\omega) e^{\frac{a}{2}|r-t|} \\
&=: \beta(r) + 2\overline{c}_3 \int_s^r e^{a(\sigma-r)}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma,
\end{aligned} \tag{3.3.11}
$$

where $\beta(r) = 2\|\psi\|^2_{C_\gamma(V)} + \overline{c}_2 a^{-1}\|\widetilde{f}\|^2 + 2\overline{c}_5 a^{-1} r(\omega) e^{\frac{a}{2}|r-t|}$. Applying the Gronwall lemma to (3.3.11), we deduce, for all $s \leqslant r$,

$$
\begin{aligned}
\|v_r(\cdot,s,\theta_{-t}\omega,\psi)\|^2_{C_\gamma(V)} &\leqslant \beta(r) + \int_s^r \beta(\sigma) e^{a(\sigma-r)} e^{\int_\sigma^r e^{a(\vartheta-r)}d\vartheta} d\sigma \\
&\leqslant c + c e^{\frac{a}{2}|r-t|} r(\omega).
\end{aligned} \tag{3.3.12}
$$

Letting $r = t$ in (3.3.10), by $\psi \in \mathcal{D}(\theta_{s-t}\omega)$, then there exists an $s_0 = s_0(\omega, \mathcal{D}) < t$ such that for all $s \leqslant s_0$, we derive

$$
\|v_t\|^2_{C_\gamma(V)} \leqslant 2 + \overline{c}_2 a^{-1}\|\widetilde{f}\|^2 + 2\overline{c}_3 \int_s^t e^{a(\sigma-t)}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma + 2\overline{c}_5 a^{-1} r(\omega). \tag{3.3.13}
$$

Using the Gronwall lemma to (3.3.13) or taking $r = t$ in (3.3.12), they both imply

$$
\|v_t(\cdot,s,\theta_{-t}\omega,\psi)\|^2_{C_\gamma(V)} \leqslant R(\omega), \tag{3.3.14}
$$

where $R(\omega)$ is the same number as in (3.3.1). This proof is concluded. □

Remark 3.3.2. *For each $(t, \omega, \psi) \in \mathbb{R} \times \Omega \times C_\gamma(V)$ and for all $s \in (-\infty, t]$, the proof of Lemma 3.3.1 implies (3.3.1) also holds.*

Let us recall now the uniform Gronwall lemma as in [115, Lemma 1.1] which is the key to prove the asymptotic compactness of solutions.

Lemma 3.3.3. *Let $t_0 \in \mathbb{R}$, assume that y, h_1, h_2 are three nonnegative, locally integrable functions on $[t_0, \infty)$ such that y' is locally integrable on $[t_0, \infty)$ and*

$$\frac{dy}{dr} \leqslant h_1 y + h_2, \text{ for } r \geqslant t_0. \tag{3.3.15}$$

In addition,

$$\int_r^{r+T} h_1(\sigma)d\sigma \leqslant b_1, \int_r^{r+T} h_2(\sigma)d\sigma \leqslant b_2, \int_r^{r+T} y(\sigma)d\sigma \leqslant b_3, \ \forall r \geqslant t_0, \tag{3.3.16}$$

where b_1, b_2, b_3, T are positive constants, then

$$y(r+T) \leqslant e^{b_1}\Big(b_2 + \frac{b_3}{T}\Big), \ \forall r \geqslant t_0. \tag{3.3.17}$$

Thanks to Lemmas 3.3.1 and 3.3.3, we derive the uniform estimates as follows.

Lemma 3.3.4. *Assume the same hypotheses and notation in Lemma 3.3.1 hold. Then, for each $\omega \in \Omega$, for all $s \leqslant t_0$ for some fixed $t_0 \in \mathbb{R}$, for all $r \geqslant t_0$ and $T > 0$,*

$$\|v(r, s, \theta_{-t}\omega, \psi)\|^2_{D(A)} \leqslant e^{R(\omega)}, \ \forall\, t \in [r, t_0 + T], \tag{3.3.18}$$

and

$$\int_r^{r+T} \|\widetilde{A}v(\sigma, s, \theta_{-t}\omega, \psi)\|^2 d\sigma \leqslant e^{R(\omega)}, \ \forall\, t \in [r, t_0 + T], \tag{3.3.19}$$

where $R(\omega) = c + cr(\omega)$ with the positive constant c being independent of r, t, s and ω.

Proof. Replacing ω by $\theta_{-t}\omega$ in (3.3.7) and integrating the inequality between r and $r + 1$ for $r \geqslant t_0$, we deduce from (3.2.6), (3.3.1) and (3.3.12) in Lemma 3.3.1 that, for $t \in [r, t_0 + T]$,

$$\begin{aligned}
&\|v(r+T, s, \theta_{-t}\omega, \psi)\|^2 + a\int_r^{r+T} \|v(\sigma, s, \theta_{-t}\omega, \psi)\|^2 d\sigma + \mu \int_r^{r+T} \|v(\sigma, s, \theta_{-t}\omega, \psi)\|^2_{D(A)} d\sigma \\
&\leqslant \|v(r, s, \theta_{-t}\omega, \psi)\|^2 + c\|\widetilde{f}\|^2 + c\int_r^{r+T} \|v_\sigma(\cdot, s, \theta_{-t}\omega, \psi)\|^2_{C_\gamma(V)} d\sigma + c\int_r^{r+T} \sup_{\iota\leqslant 0} e^{2\gamma\iota}|z(\theta_{\sigma+\iota-t}\omega)|^2 d\sigma \\
&\leqslant c + ce^{\frac{a}{2}|r-t|}r(\omega) + c\|\widetilde{f}\|^2 + cT + c\int_r^{r+T} e^{\frac{a}{2}|\sigma-t|} r(\omega) d\sigma + ce^{\frac{a}{2}T} r(\omega) \\
&\leqslant c + ce^{\frac{a}{2}T} r(\omega) + c\|\widetilde{f}\|^2 + cT + cTe^{\frac{a}{2}T} r(\omega) + ce^{\frac{a}{2}T} r(\omega) \\
&\leqslant R(\omega).
\end{aligned} \tag{3.3.20}$$

Dropping the first two terms on the left-hand side of (3.3.20), we obtain

$$\int_r^{r+T} \|v(\sigma, s, \theta_{-t}\omega, \psi)\|^2_{D(A)} d\sigma \leqslant \mu^{-1} R(\omega), \ \forall\ r \geqslant t_0,\ t \in [r, t_0 + T]. \tag{3.3.21}$$

Taking the inner product of the first equation in (3.2.8) with $\widetilde{A}v(r) := \widetilde{A}v(r, s, \theta_{-t}\omega, \psi)$, we obtain, for all $r \geqslant t_0$,

$$\begin{aligned}
&\frac{d}{dr}\|v(r)\|^2_{D(A)} + 2\|\widetilde{A}v(r)\|^2 + 2\langle \widetilde{B}(v(r)), \widetilde{A}v(r)\rangle \\
&\quad = -2z(\theta_{r-t}\omega)((\widetilde{A}\widetilde{\kappa}, \widetilde{A}v(r))) - 2z(\theta_{r-t}\omega)\langle \widetilde{B}(\widetilde{\kappa}), \widetilde{A}v(r)\rangle \\
&\qquad + 2((\widetilde{f} + \widetilde{g}(u_r), \widetilde{A}v(r))) + 2z(\theta_{r-t}\omega)((\widetilde{\kappa}, \widetilde{A}v(r))).
\end{aligned} \tag{3.3.22}$$

Thanks to (B3), it follows

$$\begin{aligned}
2\langle \widetilde{B}(v(r)), \widetilde{A}v(r)\rangle &\leqslant 2\widetilde{c}\|v(r)\|^2_{D(A)}\|\widetilde{A}v(r)\| \\
&\leqslant \frac{1}{4}\mu\|\widetilde{A}v(r)\|^2 + c\|v(r)\|^4_{D(A)}.
\end{aligned}$$

By $\widetilde{\kappa} \in D(\widetilde{A})$ and (B3), we have

$$\begin{aligned}
&- 2z(\theta_{r-t}\omega)((\widetilde{A}\widetilde{\kappa}, \widetilde{A}v(r))) - 2z(\theta_{r-t}\omega)\langle \widetilde{B}(\widetilde{\kappa}), \widetilde{A}v(r)\rangle \\
&\quad \leqslant 2|z(\theta_{r-t}\omega)|\|\widetilde{A}\widetilde{\kappa}\|\|\widetilde{A}v(r)\| + 2\widetilde{c}|z(\theta_{r-t}\omega)|\|\widetilde{\kappa}\|^2_{D(A)}\|\widetilde{A}v(r)\| \\
&\quad \leqslant c|z(\theta_{r-t}\omega)|^2 + \frac{1}{4}\mu\|\widetilde{A}v(r)\|^2.
\end{aligned} \tag{3.3.23}$$

Thanks to (3.2.7) and (3.3.12), we deduce, for $t \in [r, t_0 + T]$,

$$\begin{aligned}
2((\widetilde{f} + \widetilde{g}(u_r), \widetilde{A}v(r))) &\leqslant 2(\|\widetilde{f}\| + \|\widetilde{g}(u_r)\|)\|\widetilde{A}v(r)\| \\
&\leqslant c\|\widetilde{f}\|^2 + c\|v_r\|^2_{C_\gamma(V)} + c\sup_{\iota\leqslant 0} e^{2\gamma\iota}|z(\theta_{r+\iota-t}\omega)|^2 + \frac{1}{4}\mu\|\widetilde{A}v(r)\|^2 \\
&\leqslant c\|\widetilde{f}\|^2 + c + ce^{\frac{a}{2}|r-t|}r(\omega) + c\sup_{\iota\leqslant 0} e^{2\gamma\iota}|z(\theta_{r+\iota-t}\omega)|^2 + \frac{1}{4}\mu\|\widetilde{A}v(r)\|^2 \\
&\leqslant c\|\widetilde{f}\|^2 + c + ce^{\frac{a}{2}T}r(\omega) + c\sup_{\iota\leqslant 0} e^{2\gamma\iota}|z(\theta_{r+\iota-t}\omega)|^2 + \frac{1}{4}\mu\|\widetilde{A}v(r)\|^2.
\end{aligned} \tag{3.3.24}$$

Noticing that $\widetilde{\kappa} \in D(\widetilde{A})$, we have

$$\begin{aligned}
2z(\theta_{r-t}\omega)((\widetilde{\kappa}, \widetilde{A}v(r))) &\leqslant 2|z(\theta_{r-t}\omega)|\|\widetilde{\kappa}\|\|\widetilde{A}v(r)\| \\
&\leqslant 4\mu^{-1}\|\widetilde{\kappa}\|^2|z(\theta_{r-t}\omega)|^2 + \frac{1}{4}\mu\|\widetilde{A}v(r)\|^2.
\end{aligned} \tag{3.3.25}$$

It follows from (3.3.22)-(3.3.25) that

$$\frac{d}{dr}\|v(r)\|^2_{D(A)} + (2-\mu)\|\widetilde{A}v(r)\|^2 \leqslant c + cr(\omega) + c\sup_{\iota\leqslant 0} e^{2\gamma\iota}|z(\theta_{r+\iota-t}\omega)|^2 + c\|v(r)\|^2_{D(A)}\|v(r)\|^2_{D(A)}, \tag{3.3.26}$$

where we recall that $0 < \mu < 1$. By (3.2.6), it yields, for $t \in [r, t_0 + T]$,

$$c \sup_{\iota \leqslant 0} e^{2\gamma\iota} |z(\theta_{r+\iota-t}\omega)|^2 \leqslant c e^{\frac{\alpha}{2}|r-t|} r(\omega) \leqslant c e^{\frac{\alpha}{2}T} r(\omega).$$

Thus, one can rewrite (3.3.26) as

$$\frac{d}{dr}\|v(r)\|^2_{D(A)} + (2-\mu)\|\widetilde{A}v(r)\|^2 \leqslant R(\omega) + c\|v(r)\|^2_{D(A)}\|v(r)\|^2_{D(A)}. \tag{3.3.27}$$

Applying the uniform Gronwall lemma introduced in Lemma 3.3.3 to (3.3.27), we have, for $r \geqslant t_0$ and $t \in [r, t_0 + T]$,

$$\|v(r+T, s, \theta_{-t}\omega, \psi)\|^2_{D(A)} \leqslant e^{b_1}\Big(b_2 + \frac{b_3}{T}\Big),$$

where $b_1 = cb_3,\ b_3 = \mu^{-1}R(\omega)$ on account of (3.3.21), $b_2 = R(\omega)$. Since $t_0 \in \mathbb{R}$ is arbitrary, then for all $r \geqslant t_0$,

$$\|v(r, s, \theta_{-t}\omega, \psi)\|^2_{D(A)} \leqslant e^{b_1}\Big(b_2 + \frac{b_3}{T}\Big),\ t \in [r, t_0 + T], \tag{3.3.28}$$

which implies (3.3.18) holds as desired. Combining (3.3.27) and (3.3.28), we imply, for all $r \geqslant t_0$ and $t \in [r, t_0 + T]$,

$$\begin{aligned}\int_r^{r+T} \|\widetilde{A}v(r)\|^2 dr &\leqslant (2-\mu)\int_r^{r+T} \|\widetilde{A}v(r)\|^2 dr \\ &\leqslant TR(\omega) + cTe^{2b_1}\Big(b_2 + \frac{b_3}{T}\Big)^2 + \|v(r, s, \theta_{-t}\omega, \psi)\|^2_{D(A)} \\ &\leqslant TR(\omega) + cTe^{2b_1}\Big(b_2 + \frac{b_3}{T}\Big)^2 + e^{b_1}\Big(b_2 + \frac{b_3}{T}\Big),\end{aligned} \tag{3.3.29}$$

which shows (3.3.19) holds. The proof is complete. □

3.3.1 Existence of global random absorbing sets

We now prove the existence of a global $\mathfrak{D}_a$-random absorbing set in $C_\gamma(V)$

Lemma 3.3.5. *Suppose that* (H31)-(H36), (3.1.2), (3.1.5)-(3.1.7) *and* $\widetilde{\kappa} \in D(\widetilde{A})$ *are satisfied. Let* φ *be the random dynamical system generated by problem* (0.0.6). *For each* $(t, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}_a$ *and* $\phi \in \mathcal{D}(\theta_{s-t}\omega)$, *there exists an* $s_0 := s_0(t, \omega, \mathcal{D}) < t$ *such that for all* $s \leqslant s_0$,

$$\|u_t(\cdot, s, \theta_{-t}\omega, \phi)\|^2_{C_\gamma(V)} \leqslant R(\omega), \tag{3.3.30}$$

where we recall that $R(\omega) = c + cr(\omega)$ *with the positive constant* c *being independent of* t, s *and* ω. *Moreover, the random dynamical system* φ *has a global* $\mathfrak{D}_a$*-random absorbing set in* $C_\gamma(V)$.

Proof. Given $\mathcal{D} = \{\mathcal{D}(\omega) : \omega \in \Omega\} \in \mathfrak{D}_a$, we define

$$\widetilde{\mathcal{D}}(\omega) = \{\xi \in C_\gamma(V) : \|\xi\|^2_{C_\gamma(V)} \leqslant 2\|\mathcal{D}(\omega)\|^2_{C_\gamma(V)} + 2|\widetilde{\kappa}|^2 r(\omega)\}. \tag{3.3.31}$$

Suppose that $\widetilde{\mathcal{D}}$ is a family corresponding to $\mathcal{D}$ which consists of the sets defined by (3.3.31), that is,

$$\widetilde{\mathcal{D}} = \{\widetilde{\mathcal{D}}(\omega) : \widetilde{\mathcal{D}}(\omega) \text{ satisfies (3.3.31)},\ \omega \in \Omega\}. \tag{3.3.32}$$

Since $\mathcal{D} \in \mathfrak{D}_a$, we infer from (3.2.5), for all $t \in \mathbb{R}$,

$$\begin{aligned} e^{as}\|\widetilde{\mathcal{D}}(\theta_{s-t}\omega)\|^2_{C_\gamma(V)} &\leqslant 2e^{as}\|\mathcal{D}(\theta_{s-t}\omega)\|^2_{C_\gamma(V)} + 2e^{as}\|\widetilde{\kappa}\|^2_{C_\gamma(V)} e^{\frac{a}{2}|s-t|} r(\omega) \\ &\leqslant 2e^{as}\|\mathcal{D}(\theta_{s-t}\omega)\|^2_{C_\gamma(V)} + 2e^{\frac{a}{2}s} e^{\frac{a}{2}|t|}\|\widetilde{\kappa}\|^2_{C_\gamma(V)} r(\omega) \to 0, \text{ as } s \to -\infty. \end{aligned}$$

This shows $\widetilde{\mathcal{D}} \in \mathfrak{D}_a$. Since $\psi(\iota) = \phi(\iota) - \widetilde{\kappa} z(\theta_{\iota+s-t}\omega)$ with $\iota \leqslant 0$, it follows from (3.2.4), (3.2.5), $\phi \in \mathcal{D}(\theta_{s-t}\omega)$ and $0 < a < 2\gamma$ that

$$\begin{aligned} \|\psi\|^2_{C_\gamma(V)} &= \sup_{\iota\leqslant 0} e^{2\gamma\iota}\|\psi(\iota)\|^2 \\ &= \sup_{\iota\leqslant 0} e^{2\gamma\iota}\|\phi(\iota) - \widetilde{\kappa} z(\theta_{\iota+s-t}\omega)\|^2 \\ &\leqslant 2\sup_{\iota\leqslant 0} e^{2\gamma\iota}\|\phi(\iota)\| + 2|\widetilde{\kappa}|^2 \sup_{\iota\leqslant 0} e^{2\gamma\iota}|z(\theta_{\iota+s-t}\omega)|^2 \\ &\leqslant 2\|\phi\|_{C_\gamma(V)} + 2|\widetilde{\kappa}|^2 \sup_{\iota\leqslant 0} e^{2\gamma\iota} e^{\frac{a}{2}|\iota|} r(\theta_{s-t}\omega) \\ &\leqslant 2\|\mathcal{D}(\theta_{s-t}\omega)\|^2_{C_\gamma(V)} + 2|\widetilde{\kappa}|^2 r(\theta_{s-t}\omega), \end{aligned}$$

which, together with (3.3.31), yields $\psi \in \widetilde{\mathcal{D}}(\theta_{s-t}\omega)$. Since $\widetilde{\mathcal{D}}$ is tempered, it follows from (3.3.1) in Lemma 3.3.1 that, there exists an $s_0 := s_0(t,\omega,\mathcal{D}) < t$ such that for all $s \leqslant s_0$,

$$\|v_t(\cdot, s, \theta_{-t}\omega, \psi)\|^2_{C_\gamma(V)} \leqslant R(\omega). \tag{3.3.33}$$

Thanks to (3.2.6), (3.2.7), $0 < a < 2\gamma$ and (3.3.33), we deduce

$$\begin{aligned} \|u_t\|^2_{C_\gamma(V)} &= \sup_{\iota\leqslant 0} e^{2\gamma\iota}\|u(t+\iota, s, \theta_{-t}\omega, \phi)\|^2 \\ &= \sup_{\iota\leqslant 0} e^{2\gamma\iota}\|v(t+\iota, s, \theta_{-t}\omega, \psi) + \widetilde{\kappa} z(\theta_\iota\omega)\|^2 \\ &\leqslant 2\|v_t\|^2_{C_\gamma(V)} + 2|\widetilde{\kappa}|^2 \sup_{\iota\leqslant 0} e^{2\gamma\iota}|z(\theta_\iota\omega)|^2 \\ &\leqslant 2\|v_t\|^2_{C_\gamma(V)} + 2|\widetilde{\kappa}|^2 r(\omega) \\ &\leqslant R(\omega). \end{aligned} \tag{3.3.34}$$

This implies (3.3.30) as desired. We then define the family $\mathcal{K} = \{\mathcal{K}(\omega) : \omega \in \Omega\}$,

$$\mathcal{K}(\omega) = \{\zeta \in C_\gamma(V) : \|\zeta\|^2_{C_\gamma(V)} \leqslant R(\omega)\}. \tag{3.3.35}$$

Then $\mathcal{K}$ is a random absorbing set for φ in $C_\gamma(V)$.

Finally, it suffices to prove that $\mathcal{K}$ is tempered, that is, $\mathcal{K} \in \mathfrak{D}_a$. Indeed, by (3.2.5), we deduce, for each $t \in \mathbb{R}$,

$$\begin{aligned} e^{as}R(\theta_{s-t}\omega) &= ce^{as} + ce^{as}r(\theta_{s-t}\omega) \\ &\leqslant ce^{as} + ce^{as}e^{\frac{a}{2}|s-t|}r(\omega) \\ &\leqslant ce^{as} + ce^{\frac{a}{2}s}e^{\frac{a}{2}|t|}r(\omega) \to 0 \text{ as } s \to -\infty, \end{aligned} \tag{3.3.36}$$

which implies

$$e^{as}\|\mathcal{K}(\theta_{s-t}\omega)\|^2_{C_\gamma(V)} \leqslant e^{as}R(\theta_{s-t}\omega) \to 0 \text{ as } s \to -\infty.$$

This shows $\mathcal{K} \in \mathfrak{D}_a$ as desired. Therefore, the proof is complete. □

3.3.2 Asymptotic compactness of solutions in $C_\gamma(V)$

In this subsection, we establish the $\mathfrak{D}_a$-asymptotic compactness of solutions to problem (0.0.6) in $C_\gamma(V)$ by using the Ascoli-Arzelà theorem. To this end, we require the asymptotic compactness of solutions to problem (3.2.8) in $C_\gamma(V)$ as stated below.

Lemma 3.3.6. *Let* (H31)-(H36)*,* (3.1.2)*,* (3.1.5)*-*(3.1.7) *and* $\widetilde{\kappa} \in D(\widetilde{A})$ *hold. For each* $(t, s_0, \omega, \mathcal{D}) \in \mathbb{R} \times \mathbb{R} \times \Omega \times \mathfrak{D}_a$*, assume that* $\{s_n\}_{n\geqslant 1}$ *is a decreasing sequence satisfying* $s_n \to -\infty$ *as* $n \to +\infty$ *and* $s_n \leqslant s_0$*. Besides,* ψ_n *is a sequence of functions such that* $\psi_n \in \mathcal{D}(\theta_{s_n-t}\omega)$ *for each positive integer n. Denote by* $v^{(n)}(\cdot) = v(\cdot, s_n, \theta_{-t}\omega, \psi_n)$ *the solutions to system* (3.2.8) *corresponding to the initial data* ψ_n *at the initial time* s_n*. Then the sequence* $\{v^{(n)}_{s_0}(\cdot)\}$ *has a convergent subsequence in* $C_\gamma(V)$*.*

Proof. Let $\omega \in \Omega$ be fixed, and take an arbitrary sequence $s_n \to -\infty$ such that $s_n \leqslant s_0$ for some fixed $s_0 \in \mathbb{R}$. Let T be the same number as in Lemma 3.3.4. We infer from (3.3.12) in Lemma 3.3.1 that there exists $n_0 \in \mathbb{Z}^+$ satisfying $s_n \leqslant s_0 - T$ for all $n \geqslant n_0$, and

$$\begin{aligned} \|v^{(n)}_r\|^2_{C_\gamma(V)} &\leqslant c + c \sup_{r\in[s_0-T,s_0],t\in[r,s_0]} e^{\frac{a}{2}|r-t|}r(\omega) \\ &\leqslant R(\omega),\ \forall\, r \in [s_0 - T, s_0], t \in [r, s_0],\ \forall\, n \geqslant n_0, \end{aligned} \tag{3.3.37}$$

where we recall that $R(\omega) = c + cr(\omega)$ with the positive constant c being independent of $r,\ t,\ s,\ s_0, n$ and ω. Using (3.3.18) in Lemma 3.3.4, we obtain

$$\|v^{(n)}(r)\|^2_{D(A)} \leqslant e^{R(\omega)},\ \forall\, r \in [s_0 - T, s_0], t \in [r, s_0],\ \forall\, n \geqslant n_0, \tag{3.3.38}$$

For the proof of the lemma, we will proceed in the following three steps.

Step 1: *We show that* $\{v^{(n)}(r)\}_{n\geqslant n_0,\omega\in\Omega}$ *is pre-compact in V for all* $r \in [s_0 - T, s_0], t \in [r, s_0]$. By (3.3.38), we deduce that $\{v^{(n)}(\cdot)\}_{n\geqslant n_0,\omega\in\Omega}$ is bounded in $L^\infty(s_0 - T, s_0; D(A))$. Due to the compactness of the embedding $D(A) \hookrightarrow V$, it follows that the pre-compactness of $\{v^{(n)}(r)\}_{n\geqslant n_0,\omega\in\Omega}$ in V is satisfied for all $r \in [s_0 - T, s_0], t \in [r, s_0]$.

Step 2: *We establish the equi-continuity of* $\{v^{(n)}(r)\}_{n\geqslant n_0,\omega\in\Omega}$ *in* V*, for all* $r\in[s_0-T,s_0], t\in[r,s_0]$*,* $s_n\leqslant s_0-T$ *for all* $n\geqslant n_0$ *and* $\psi_n\in\mathcal{D}(\theta_{s_n-t}\omega)$ *by contradiction.* Assume that the equi-continuity does not hold true, then there would exist a positive constant ϵ_0 and two sequences $\{r_n^{(1)}\}$ and $\{r_n^{(2)}\}$ such that $s_0-T\leqslant r_n^{(1)}\leqslant r_n^{(2)}\leqslant s_0$ and $|r_n^{(1)}-r_n^{(2)}|\leqslant\frac{1}{n}$,

$$\|v^{(n)}(r_n^{(1)})-v^{(n)}(r_n^{(2)})\|\geqslant\epsilon_0. \tag{3.3.39}$$

By Step 1, we obtain $\{v^{(n)}(r)\}_{n\geqslant n_0,\omega\in\Omega}$ is pre-compact in V. Thus, we can assume that $r_n^{(1)}\to r^*$, $v^{(n)}(r^*)\to z^*$ and $v^{(n)}(r_n^{(i)})\to z^{(i)}$ $(i=1,2)$ in V as $n\to+\infty$. Then, it immediately follows $r_n^{(2)}\to r^*$ as $n\to+\infty$. Moreover,

$$\|z^{(1)}-z^{(2)}\|\geqslant\epsilon_0. \tag{3.3.40}$$

Let $y^{(n)}(r):=v^{(n)}(r)-v^{(n)}(r^*)=v(r,s_n,\theta_{-t}\omega,\psi_n)-v(r^*,s_n,\theta_{-t}\omega,\psi_n)$ with $r\in[s_0-T,s_0], t\in[r,s_0]$ for all $n\geqslant n_0$ and $\omega\in\Omega$. We infer from (3.2.8) that

$$\begin{aligned}&\frac{d}{dr}\|y^{(n)}(r)\|^2+2\|y^{(n)}(r)\|^2_{D(A)}+2\langle\widetilde{A}v^{(n)}(r^*),y^{(n)}(r)\rangle+2\langle\widetilde{B}(v^{(n)}(r)),y^{(n)}(r)\rangle\\&\quad=-2z(\theta_{r-t}\omega)\langle\widetilde{A}\widetilde{\kappa}+\widetilde{B}(\widetilde{\kappa}),y^{(n)}(r)\rangle+2((\widetilde{f}+\widetilde{g}(u_r^{(n)}),y^{(n)}(r)))+2z(\theta_{r-t}\omega)((\widetilde{\kappa},y^{(n)}(r))).\end{aligned} \tag{3.3.41}$$

The Young inequality implies

$$\begin{aligned}2|\langle\widetilde{A}v^{(n)}(r^*),y^{(n)}(r)\rangle|&=2|((v^{(n)}(r^*),y^{(n)}(r)))_{D(A)}|\\&\leqslant6\|v^{(n)}(r^*)\|^2_{D(A)}+\frac{1}{6}\|y^{(n)}(r)\|^2_{D(A)}.\end{aligned} \tag{3.3.42}$$

By (0.0.42), (B3) and (3.3.38), we obtain

$$\begin{aligned}2|\langle\widetilde{B}(v^{(n)}(r)),y^{(n)}(r)\rangle|&\leqslant2\widetilde{\lambda}_1^{-\frac{1}{2}}\widetilde{c}\|v^{(n)}(r)\|^2_{D(A)}\|y^{(n)}(r)\|_{D(A)}\\&\leqslant6\widetilde{\lambda}_1^{-1}\widetilde{c}^2\|v^{(n)}(r)\|^4_{D(A)}+\frac{1}{6}\|y^{(n)}(r)\|^2_{D(A)}\\&\leqslant e^{2R(\omega)}+\frac{1}{6}\|y^{(n)}(r)\|^2_{D(A)}.\end{aligned} \tag{3.3.43}$$

Taking into account (0.0.42), (B3) and $\widetilde{\kappa}\in D(\widetilde{A})\hookrightarrow D(A)$, we have

$$\begin{aligned}2|z(\theta_{r-t}\omega)\langle\widetilde{A}\widetilde{\kappa}+\widetilde{B}(\widetilde{\kappa}),y^{(n)}(r)\rangle|&\leqslant2|z(\theta_{r-t}\omega)|(\|\widetilde{\kappa}\|_{D(A)}\|y^{(n)}(r)\|_{D(A)}+\widetilde{\lambda}_1^{-\frac{1}{2}}\widetilde{c}\|\widetilde{\kappa}\|^2_{D(A)}\|y^{(n)}(r)\|_{D(A)})\\&\leqslant\bar{c}_1|z(\theta_{r-t}\omega)|^2+\frac{1}{6}\|y^{(n)}(r)\|^2_{D(A)}.\end{aligned} \tag{3.3.44}$$

Thanks to (3.2.6), we deduce, for all $t\in[r,s_0]$,

$$\begin{aligned}|z(\theta_{r-t}\omega)|^2&\leqslant r(\omega)\sup_{r\in[s_0-T,s_0],t\in[r,s_0]}e^{\frac{a}{2}|r-t|}\\&\leqslant e^{\frac{a}{2}T}r(\omega).\end{aligned} \tag{3.3.45}$$

Thanks to (0.0.42), (3.2.6), (3.2.7), $0 < a < 2\gamma$ and (3.3.37), we deduce

$$
\begin{aligned}
&2((\widetilde{f} + \widetilde{g}(u_r^{(n)}), y^{(n)}(r)))\\
&\quad \leqslant 2\widetilde{\lambda}_1^{-\frac{1}{2}}(\|\widetilde{f}\| + \|\widetilde{g}(u_r^{(n)})\|)\|y^{(n)}(r)\|_{D(A)}\\
&\quad \leqslant 6\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|^2 + 6\widetilde{\lambda}_1^{-1}\|\widetilde{g}(u_r^{(n)})\|^2 + \frac{1}{3}\|y^{(n)}(r)\|^2_{D(A)}\\
&\quad \leqslant 6\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|^2 + 12\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}^2\|v_r^{(n)}\|^2_{C_\gamma(V)} + 12\widetilde{\lambda}_1^{-1}L_{\widetilde{g}}^2\|\widetilde{\kappa}\|^2\sup_{\iota\leqslant 0} e^{2\gamma\iota}|z(\theta_{r+\iota-t}\omega)|^2 + \frac{1}{3}\|y^{(n)}(r)\|^2_{D(A)}\\
&\quad \leqslant 6\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|^2 + R(\omega) + \overline{c}_2 r(\omega) + \frac{1}{3}\|y^{(n)}(r)\|^2_{D(A)}.
\end{aligned}
\tag{3.3.46}
$$

By (0.0.42), (3.3.45) and $\widetilde{\kappa} \in D(\widetilde{A})$, we obtain

$$
\begin{aligned}
2|z(\theta_{r-t}\omega)||((\widetilde{\kappa}, y^{(n)}(r)))| &\leqslant 2\widetilde{\lambda}_1^{-\frac{1}{2}}|z(\theta_{r-t}\omega)|\|\widetilde{\kappa}\|\|y^{(n)}(r)\|_{D(A)}\\
&\leqslant c_3 r(\omega) + \frac{1}{6}\|y^{(n)}(r)\|^2_{D(A)}.
\end{aligned}
\tag{3.3.47}
$$

Substituting (3.3.42)-(3.3.47) into (3.3.41),

$$
\frac{d}{dr}\|y^{(n)}(r)\|^2 + \|y^{(n)}(r)\|^2_{D(A)} \leqslant 6\|v^{(n)}(r^*)\|^2_{D(A)} + e^{2R(\omega)} + 2R(\omega). \tag{3.3.48}
$$

Integrating (3.3.48) from r^* to $r_n^{(i)}$, we have

$$
\|y^{(n)}(r_n^{(i)})\|^2 \leqslant \Big(\widetilde{\rho}_0 + e^{2R(\omega)} + 2R(\omega)\Big)|r_n^{(i)} - r^*|, \tag{3.3.49}
$$

where $\widetilde{\rho}_0 = 6\sup_{r\in[s_0-T,s_0]}\{\|v^{(n)}(r)\|^2_{D(A)} : n \geqslant n_0, \omega \in \Omega\}$ is bounded by $e^{R(\omega)}$ due to (3.3.38). Letting $n \to +\infty$ in (3.3.49), we derive

$$
\|z^{(i)} - z^*\|^2 = \lim_{n\to+\infty}\|v^{(n)}(r_n^{(i)}) - v^{(n)}(r^*)\|^2 = 0, \; i = 1, 2,
$$

which contradicts (3.3.40).

Step 3: *We establish the asymptotic compactness of solutions to problem* (3.2.8) *in* $C_\gamma(V)$.

Recall that $v^{(n)}(\cdot) = v(\cdot, s_n, \theta_{-t}\omega, \psi_n)$. By steps 1-2, it follows from the Ascoli-Arzelà theorem that $\{v^{(n)}(\cdot)\}_{n\in\mathbb{N}^+,\omega\in\Omega}$ is pre-compact in $C([s_0 - T, s_0]; V)$ with each $T > 0$, and thus there exists a function $\xi(\cdot) \in C([-T, 0]; V)$ and a subsequence of $v_{s_0}^{(n)}(\cdot)$ such that $v_{s_0}^{(n)}(\cdot)\big|_{[-T,0]} \to \xi(\cdot)$ in $C([-T, 0]; V)$. Repeating the procedure for nT with $n = 2, 3, \cdots$, and using the diagonal procedure (relabeled the same), we can obtain a function $\xi(\cdot) \in C((-\infty, 0]; V)$ satisfying $v_{s_0}^{(n)}(\cdot)\big|_{[-T,0]} \to \xi(\cdot)$ in $C([-T, 0]; V)$. Moreover, by the estimate (3.3.37), we obtain

$$
\|\xi(r)\|^2 \leqslant R(\omega), \; \forall\, r \in [-T, 0], \text{ for any } T > 0. \tag{3.3.50}
$$

In the following, we prove that in fact $v_{s_0}^{(n)}(\cdot) \to \xi(\cdot)$ in $C_\gamma(V)$. It suffices to prove that, for every $\epsilon > 0$, there exists some integer $n_\epsilon > 0$ such that

$$
\sup_{r\in(-\infty,0]} e^{2\gamma r}\|v_{s_0}^{(n)}(r) - \xi(r)\|^2 < \epsilon, \; \forall\, t \in [r, s_0], \; n \geqslant n_\epsilon. \tag{3.3.51}
$$

Let $T_\epsilon > 0$ be fixed such that $\max\{ce^{-2\gamma T_\epsilon}, ce^{-2\gamma T_\epsilon} r(\omega), ce^{-(2\gamma-\frac{a}{2})T_\epsilon} e^{\frac{a}{2}T} r(\omega)\} < \frac{\epsilon}{8}$. Taking $n_\epsilon \geqslant n_0$ such that $e^{2\gamma r}\|v_{s_0}^{(n)}(r) - \xi(r)\|^2 < \epsilon$ for all $r \in [-T_\epsilon, 0]$, and $s_n \leqslant s_0 - T_\epsilon$ for all $n \geqslant n_\epsilon$. Therefore, to prove (3.3.51), we only need to check the following conclusion holds:

$$\sup_{r\in(-\infty,-T_\epsilon]} e^{2\gamma r}\|v_{s_0}^{(n)}(r) - \xi(r)\|^2 < \epsilon, \ \forall\, t \in [r, s_0],\ n \geqslant n_\epsilon.$$

By (3.3.50) and the fact that $0 < a < 2\gamma$, we imply, for all $m \geqslant 0$, and $r \in [-(T_\epsilon + m + 1), -(T_\epsilon + m)]$,

$$\begin{aligned} e^{2\gamma r}\|\xi(r)\|^2 &\leqslant ce^{-2\gamma(T_\epsilon+m)}(1 + r(\omega)) \\ &\leqslant ce^{-2\gamma T_\epsilon} + ce^{-2\gamma T_\epsilon} r(\omega) \\ &< \frac{\epsilon}{4}. \end{aligned}$$

Note that

$$v_{s_0}^{(n)}(r) = \begin{cases} \psi_n(r + s_0 - s_n), & \text{if } r \in (-\infty, s_n - s_0), \\ v^{(n)}(r + s_0), & \text{if } r \in [s_n - s_0, 0]. \end{cases}$$

Therefore, this proof is finished if we prove that

$$\max\left\{ \sup_{r\in(-\infty,s_n-s_0)} e^{2\gamma r}\|\psi_n(r + s_0 - s_n)\|^2, \sup_{r\in[s_n-s_0,-T_\epsilon]} e^{2\gamma r}\|v^{(n)}(r + s_0)\|^2 \right\} < \frac{\epsilon}{4}.$$

On the one hand, by $\psi_n \in \mathcal{D}(\theta_{s_n-t}\omega)$ and (3.3.37), we deduce

$$\begin{aligned} \sup_{r\leqslant s_n-s_0} e^{2\gamma r}\|\psi_n(r + s_0 - s_n)\|^2 &= \sup_{r\leqslant s_n-s_0} e^{2\gamma(r+s_0-s_n)} e^{2\gamma(s_n-s_0)}\|\psi_n(r + s_0 - s_n)\|^2 \\ &\leqslant e^{2\gamma(s_n-s_0)}\|\psi_n\|^2_{C_\gamma(V)} \\ &= e^{2\gamma(s_n-s_0)} R(\omega) \\ &\leqslant ce^{2\gamma(s_n-s_0)} + ce^{2\gamma(s_n-s_0)} r(\omega) \\ &\leqslant ce^{-2\gamma T_\epsilon} + ce^{-2\gamma T_\epsilon} r(\omega) \\ &\leqslant \frac{\epsilon}{8} + \frac{\epsilon}{8} \\ &= \frac{\epsilon}{4}. \end{aligned}$$

On the other hand, by (3.3.37) with $T = T_\epsilon$, we deduce

$$\begin{aligned} \sup_{r\in[s_n-s_0,-T_\epsilon]} e^{2\gamma r}\|v^{(n)}(r + s_0)\|^2 &= \sup_{r\in[s_n-s_0+T_\epsilon,0]} e^{2\gamma(r-T_\epsilon)}\|v^{(n)}(r - T_\epsilon + s_0)\|^2 \\ &\leqslant e^{-2\gamma T_\epsilon}\|v^{(n)}_{-T_\epsilon+s_0}\|^2_{C_\gamma(V)} \\ &\leqslant e^{-2\gamma T_\epsilon} R(\omega) \\ &= ce^{-2\gamma T_\epsilon} + ce^{-2\gamma T_\epsilon} r(\omega) \\ &\leqslant \frac{\epsilon}{4}. \end{aligned}$$

Therefore, the proof of Lemma 3.3.6 is complete. □

Let us now prove the $\mathfrak{D}_a$-asymptotic compactness of solutions to problem (0.0.6) in $C_\gamma(V)$.

Lemma 3.3.7. *Assume that the same hypotheses and notation in Lemma 3.3.5 hold, then the sequence $\varphi(t, s_n, \theta_{-t}\omega)\phi_n$ has a convergent subsequence in $C_\gamma(V)$. In other words, φ is $\mathfrak{D}_a$-asymptotically compact in $C_\gamma(V)$.*

Proof. By (3.2.7), it follows, for each $s_n \leqslant t$,

$$\begin{aligned}\varphi(t, s_n, \theta_{-t}\omega)\phi_n &= u_t(\cdot, s_n, \theta_{-t}\omega, \phi_n)\\ &= v_t(\cdot, s_n, \theta_{-t}\omega, \psi_n) + \widetilde{\kappa}z(\theta.\omega).\end{aligned}$$

Then $\psi_n \in \mathfrak{D}_a$ on account of $\phi_n \in \mathcal{D}(\theta_{s_n-t}\omega)$. Thanks to Lemma 3.3.6, we derive that the sequence $v_t(\cdot, s_n, \theta_{-t}\omega, \psi_n)$ of solutions to problem (3.2.8) has a convergent subsequence in $C_\gamma(V)$, which together with the continuity of $z(\theta.\omega)$, proves that $\varphi(t, s_n, \theta_{-t}\omega)\phi_n$ has also a convergent subsequence in $C_\gamma(V)$. □

3.3.3 Existence of global random attractors

Based on the previous results, one can derive the existence of a global $\mathfrak{D}_a$-random attractor for φ in $C_\gamma(V)$ as stated below.

Theorem 3.3.8. *Suppose that the same hypotheses and notation in Lemmas 3.3.5 and 3.3.7 hold. Then, the random dynamical system φ possesses a global $\mathfrak{D}_a$-random attractor $\mathcal{A} = \{\mathcal{A}(\omega) : \omega \in \Omega\}$ in $C_\gamma(V)$.*

Proof. It follows from Lemma 3.3.5 that the random dynamical system φ has a global $\mathfrak{D}_a$-random absorbing set in $C_\gamma(V)$. By Lemma 3.3.7, φ is $\mathfrak{D}_a$-asymptotically compact in $C_\gamma(V)$. Thanks to [28, Theorem 7], we finally obtain the existence of a global $\mathfrak{D}_a$-random attractor $\mathcal{A} = \{\mathcal{A}(\omega) : \omega \in \Omega\}$. □

3.4 Existence of invariant measures in $C_\gamma(V)$

In the rest of this chapter, we prove the existence of invariant measures in $C_\gamma(V)$. To this end, we need to show that, for each $t \in \mathbb{R}$, $\omega \in \Omega$ and $\phi \in C_\gamma(V)$, the $C_\gamma(V)$-valued function $s \mapsto \varphi(t, s, \theta_{-t}\omega)\phi$ is bounded $(-\infty, t]$, and further derive the continuity of $(s, \phi) \mapsto \varphi(t, s, \theta_{-t}\omega)\phi$ on $(-\infty, t] \times C_\gamma(V)$ (see, e.g., [148]).

Lemma 3.4.1. *Let* (H31)-(H36), (3.1.2), (3.1.5)-(3.1.7) *and $\widetilde{\kappa} \in D(\widetilde{A})$ hold. Then, for each $(t, \omega, \phi) \in \mathbb{R} \times \Omega \times C_\gamma(V)$, the $C_\gamma(V)$-valued function $s \mapsto \varphi(t, s, \theta_{-t}\omega)\phi$ is bounded on $(-\infty, t]$.*

Proof. Let $t \in \mathbb{R}$, $\omega \in \Omega$ and $\phi \in C_\gamma(V)$ be given. By Remark 3.3.2 and (3.3.34), we deduce that, for all $s \in (-\infty, t]$

$$\|\varphi(t, s, \theta_{-t}\omega)\phi\|^2_{C_\gamma(V)} = \|u_t(\cdot, s, \theta_{-t}\omega, \phi)\|^2_{C_\gamma(V)} \leqslant R(\omega), \tag{3.4.1}$$

where we recall that $R(\omega) = c + cr(\omega)$ with the positive constant c being independent of t, s and ω. □

Lemma 3.4.2. *If $\psi \in C_\gamma(V)$ is given, then for each $\epsilon > 0$, there exists $\delta = \delta(\psi, \epsilon) > 0$ such that, for all $s_1, s_2 \in (-\infty, 0]$ with $|s_1 - s_2| < \delta$,*

$$e^{\gamma s_2}\|\psi(s_1) - \psi(s_2)\| < \epsilon. \tag{3.4.2}$$

Proof. Let $\psi_\infty := \lim_{s\to-\infty} e^{\gamma s}\psi(s) \in V$. By the definition of $C_\gamma(V)$, we deduce that, for any $\epsilon > 0$ there exists an $s_0 < 0$ such that

$$\|e^{\gamma s}\psi(s) - \psi_\infty\| < \frac{\epsilon}{4}, \ \forall\ s \leqslant s_0, \tag{3.4.3}$$

which implies

$$\|e^{\gamma s_1}\psi(s_1) - e^{\gamma s_2}\psi(s_2)\| \leqslant \|e^{\gamma s_1}\psi(s_1) - \psi_\infty\| + \|e^{\gamma s_2}\psi(s_2) - \psi_\infty\| < \frac{\epsilon}{2}, \ \forall\ s_1, s_2 \leqslant s_0. \tag{3.4.4}$$

Due to the uniform continuity of the V-valued function $s \mapsto e^{\gamma s}\psi(s)$ on the interval $[s_0, 0]$, there exists $\delta' \in (0, 1)$ such that, for all $s_1, s_2 \in [s_0, 0]$ with $|s_1 - s_2| < \delta'$,

$$\|e^{\gamma s_1}\psi(s_1) - e^{\gamma s_2}\psi(s_2)\| < \frac{\epsilon}{2}. \tag{3.4.5}$$

Let $\delta = \min\left\{\delta', \frac{1}{\gamma}\ln(1 + \frac{\epsilon}{2\|\psi\|_{C_\gamma(V)}})\right\}$, we infer from (3.4.4) and (3.4.5) that, for all $s_1, s_2 \in (-\infty, 0]$ with $|s_1 - s_2| < \delta$,

$$\begin{aligned} e^{\gamma s_2}\|\psi(s_1) - \psi(s_2)\| &\leqslant \|e^{\gamma s_1}\psi(s_1) - e^{\gamma s_2}\psi(s_2)\| + |e^{\gamma s_1} - e^{\gamma s_2}|\|\psi(s_1)\| \\ &\leqslant \frac{\epsilon}{2} + \left|e^{\gamma(s_2 - s_1)} - 1\right|e^{\gamma s_1}\|\psi(s_1)\| \\ &\leqslant \frac{\epsilon}{2} + \left|e^{\gamma(s_2 - s_1)} - 1\right|\|\psi\|_{C_\gamma(V)} < \epsilon. \end{aligned} \tag{3.4.6}$$

This yields (3.4.2) as desired. □

Lemma 3.4.3. *Let* (H31)-(H36), (3.1.2), (3.1.5)-(3.1.7) *and $\widetilde{\kappa} \in D(\widetilde{A})$ hold. For given $s_* \in \mathbb{R}$, $\psi \in C_\gamma(V)$, for all $T_0 > 0$, almost all $\omega \in \Omega$, and for any $\epsilon > 0$, there exists $\delta_0 = \delta_0(s_*, \psi, T_0, \omega, \epsilon) > 0$ such that for all $s \in (s_* - \delta_0, s_*)$, $r \in [s, s_*]$ and $t \in [s_*, s_* + T_0]$,*

$$\|v(r, s, \theta_{-t}\omega, \psi) - \psi(0)\| < \epsilon. \tag{3.4.7}$$

Proof. Firstly, we prove that there exists a constant $\chi > 0$ such that

$$\int_s^{s_*} \left\|\frac{d}{dr}v(r, s, \theta_{-t}\omega, \psi)\right\|^2_{(D(A))^*} dr \leqslant \chi, \ \forall\ s \in [s_* - 1, s_*], \ t \in [s_*, s_* + T_0]. \tag{3.4.8}$$

Indeed, we infer from (3.2.8), $V \hookrightarrow (D(A))^*$, (B2), (0.0.42) and $\widetilde{\kappa} \in D(\widetilde{A})$ that

$$\left\|\frac{d}{dr}v(r, s, \theta_{-t}\omega, \psi)\right\|^2_{(D(A))^*} \leqslant c\|\widetilde{A}v\|^2 + c\|\widetilde{B}(v)\|^2_{(D(A))^*} + c|z(\theta_{r-t}\omega)|^2(\|\widetilde{A}\widetilde{\kappa}\|^2 + \|\widetilde{B}(\widetilde{\kappa})\|^2_{(D(A))^*})$$

$$+ c\|\widetilde{f}\|^2 + c\|\widetilde{g}(u_r)\|^2 + c|z(\theta_{r-t}\omega)|^2\|\widetilde{\kappa}\|^2$$
$$\leqslant c\|\widetilde{A}v\|^2 + c\|v\|^2_{D(A)} + c|z(\theta_{r-t}\omega)|^2 + c\|\widetilde{f}\|^2 + c\|u_r\|^2_{C_\gamma(V)}. \tag{3.4.9}$$

Integrating (3.4.9) over $[s, s_*]$, we deduce, for all $s \in [s_* - 1, s_*],\ t \in [s_*, s_* + T_0]$,

$$\int_s^{s_*} \left\|\frac{d}{dr}v(r, s, \theta_{-t}\omega, \psi)\right\|^2_{(D(A))^*} dr \leqslant c\int_{s_*-1}^{s_*} \|\widetilde{A}v(r)\|^2 dr + c\int_{s_*-1}^{s_*} \|v(r)\|^2_{D(A)} dr$$
$$+ c\|\widetilde{f}\|^2 + \int_{s_*-1}^{s_*} \|u_r\|^2_{C_\gamma(V)} dr + c\int_{s_*-1}^{s_*} |z(\theta_{r-t}\omega)|^2 dr. \tag{3.4.10}$$

By similar arguments to those in (3.3.19) and (3.3.21), we deduce the first two terms on the right-hand side of (3.4.10) are bounded, that is,

$$c\int_{s_*-1}^{s_*} \|\widetilde{A}v(r)\|^2 dr + c\int_{s_*-1}^{s_*} \|v(r)\|^2_{D(A)} dr < \infty.$$

Thanks to (3.3.5) and (3.3.12) in Lemma 3.3.1, for all $s \in [s_* - 1, s_*],\ t \in [s_*, s_* + T_0]$, we have

$$\int_{s_*-1}^{s_*} \|u_r\|^2_{C_\gamma(V)} dr \leqslant c + c\int_{s_*-1}^{s_*} e^{\frac{\alpha}{2}|r-t|} r(\omega) dr$$
$$\leqslant c + ce^{\frac{\alpha}{2}(T_0+1)} r(\omega).$$

By (3.2.6), the last term in (3.4.10) is bounded by

$$\int_{s_*-1}^{s_*} |z(\theta_{r-t}\omega)|^2 dr \leqslant r(\omega)\int_{s_*-1}^{s_*} e^{\frac{\alpha}{2}|r-t|} dr$$
$$\leqslant e^{\frac{\alpha}{2}(T_0+1)} r(\omega),\ \forall\ s \in [s_* - 1, s_*],\ t \in [s_*, s_* + T_0], \tag{3.4.11}$$

which, together with $\widetilde{f} \in V$, shows (3.4.10) is finite. Therefore, (3.4.8) holds.

Note that, for all $s_* - 1 \leqslant s \leqslant r \leqslant s_* \leqslant t$, it follows,

$$\|v(r, s, \theta_{-t}\omega, \psi) - \psi(0)\|^2$$
$$= \|v(r, s, \theta_{-t}\omega, \psi)\|^2 - \|\psi(0)\|^2 - 2((v(r, s, \theta_{-t}\omega, \psi) - \psi(0), \psi(0)))$$
$$= \int_s^r \frac{d}{d\sigma}\left\|v(\sigma, s, \theta_{-t}\omega, \psi)\right\|^2 d\sigma - 2((v(r, s, \theta_{-t}\omega, \psi) - \psi(0), \psi(0))). \tag{3.4.12}$$

We now estimate the last two terms of (3.4.12). On the one hand, by (3.2.6), (3.2.7), (3.3.7) and (3.3.12), we have

$$\left|\int_s^r \frac{d}{d\sigma}\left\|v(\sigma, s, \theta_{-t}\omega, \psi)\right\|^2 d\sigma\right| \leqslant c(s_* - s)\|\widetilde{f}\|^2 + c\int_s^{s_*} \|v_\sigma\|^2_{C_\gamma(V)} d\sigma + c\int_s^{s_*} \sup_{\iota\leqslant 0} e^{2\gamma\iota}|z(\theta_{\sigma+\iota-t}\omega)|^2 d\sigma$$
$$\leqslant c(s_* - s)\|\widetilde{f}\|^2 + c\int_s^{s_*} \|v_\sigma\|^2_{C_\gamma(V)} d\sigma + c\int_s^{s_*} e^{\frac{\alpha}{2}|\sigma-t|} r(\omega) d\sigma$$

$$\begin{aligned}
&\leqslant c(s_*-s)\|\widetilde{f}\|^2 + c(s_*-s)\Big(1+\sup_{\sigma\in[s_*-1,s_*]} e^{\frac{a}{2}|\sigma-t|}r(\omega)\Big)\\
&\leqslant c(s_*-s)\|\widetilde{f}\|^2 + c(s_*-s)\Big(1+e^{\frac{a}{2}(T_0+1)}r(\omega)\Big),\ t\in[s_*,s_*+T_0].
\end{aligned}\tag{3.4.13}$$

As $\widetilde{f}\in V$, there exists some $\delta_0'=\delta_0'(s_*,\psi,T_0,\omega,\epsilon)\in(0,1)$ such that

$$\Big|\int_s^r \frac{d}{d\sigma}\Big\|v(\sigma,s,\theta_{-t}\omega,\psi)\Big\|^2 d\sigma\Big| < \frac{\epsilon^2}{2},\ s_*-\delta_0' < s < r \leqslant s_* \leqslant t \leqslant s_*+T_0. \tag{3.4.14}$$

On the other hand, since $\psi(0)\in V$ and $D(A)$ is dense in V, there exists some $\psi^*\in D(A)$ such that

$$\|\psi^*-\psi(0)\| < \frac{\epsilon^2}{16\sqrt{R(\omega)}}. \tag{3.4.15}$$

Thanks to (3.3.12) in Lemma 3.3.1, we imply, for all $s\in[s_*-1,s_*],\ r\in[s,s_*],\ t\in[s_*,s_*+T_0]$,

$$\begin{aligned}
\|v(r,s,\theta_{-t}\omega,\psi)\|^2 &\leqslant c + ce^{\frac{a}{2}|r-t|}r(\omega)\\
&\leqslant c + ce^{\frac{a}{2}(T_0+1)}r(\omega)\\
&\leqslant R(\omega).
\end{aligned}\tag{3.4.16}$$

By (3.4.8), (3.4.15) and (3.4.16), we deduce

$$\begin{aligned}
&2\Big|((v(r,s,\theta_{-t}\omega,\psi)-\psi(0),\psi(0)))\Big|\\
&\quad\leqslant 2\Big|\langle v(r,s,\theta_{-t}\omega,\psi)-\psi(0),\psi^*\rangle\Big| + 2\Big|((v(r,s,\theta_{-t}\omega,\psi)-\psi(0),\psi(0)-\psi^*))\Big|\\
&\quad\leqslant 2\Big|\Big\langle \int_s^r \frac{d}{d\sigma}v(\sigma,s,\theta_{-t}\omega,\psi)d\sigma,\psi^*\Big\rangle\Big| + 4\sqrt{R(\omega)}\|\psi(0)-\psi^*\|\\
&\quad\leqslant 2\sqrt{s_*-s}\|\psi^*\|_{D(A)}\Big(\int_s^{s_*}\Big\|\frac{d}{d\sigma}v(\sigma,s,\theta_{-t}\omega,\psi)\Big\|^2_{(D(A))^*}d\sigma\Big)^{\frac12} + \frac{\epsilon^2}{4}\\
&\quad\leqslant 2\sqrt{\chi(s_*-s)}\|\psi^*\|_{D(A)} + \frac{\epsilon^2}{4},\ \forall\, r\in[s,s_*],\ t\in[s_*,s_*+T_0].
\end{aligned}\tag{3.4.17}$$

This implies that there exists some $\delta_0''=\delta_0''(s_*,\psi,T_0,\omega,\epsilon)\in(0,1)$ such that

$$2\Big|((v(r,s,\theta_{-t}\omega,\psi)-\psi(0),\psi(0)))\Big| \leqslant \frac{\epsilon^2}{2},\ s_*-\delta_0'' < s\leqslant r\leqslant s_*\leqslant t\leqslant s_*+T_0. \tag{3.4.18}$$

Letting $\delta_0=\min\{\delta_0',\delta_0''\}$, we infer from (3.4.12), (3.4.14) and (3.4.18) that (3.4.7) holds. □

Lemma 3.4.4. *Let* (H31)-(H36), (3.1.2), (3.1.5)-(3.1.7) *and* $\widetilde{\kappa}\in D(\widetilde{A})$ *hold. Then, for every* $t\in\mathbb{R}$, *almost all* $\omega\in\Omega$ *and* $\phi\in C_\gamma(V)$, *the* $C_\gamma(V)$*-valued function* $(s,\phi)\mapsto\varphi(t,s,\theta_{-t}\omega)\phi$ *is continuous on* $(-\infty,t]\times C_\gamma(V)$.

Proof. Note that

$$\begin{aligned}(\varphi(t,s,\theta_{-t}\omega)\phi)(\iota) &= u_t(\iota,s,\theta_{-t}\omega,\phi)\\ &= v_t(\iota,s,\theta_{-t}\omega,\psi)+\widetilde{\kappa}z(\theta_\iota\omega),\ \iota\leqslant 0,\end{aligned}\tag{3.4.19}$$

where $\psi\in C_\gamma(V)$ due to $\phi\in C_\gamma(V)$. The above equality, together with the continuity of $|z(\theta_\iota\omega)|$ with respect to $\iota\in\mathbb{R}$ for $\mathbb{P}$-a.s. $\omega\in\Omega$, shows that we only need to prove that $v_t(\iota,\cdot,\theta_{-t}\omega,\cdot)$ is continuous on $(-\infty,t]\times C_\gamma(V)$. For given $s_*\in(-\infty,t]$, $\psi_*\in C_\gamma(V)$, for all $T_0>0$ and almost all $\omega\in\Omega$, we just need to prove that for any $\epsilon>0$, there exists a positive constant $\delta=\delta(s_*,\psi_*,T_0,\omega,\epsilon)>0$ such that for all $|s-s_*|\vee\|\psi-\psi_*\|_{C_\gamma(V)}<\delta$ and $t\in[s_*,s_*+T_0]$,

$$\sup_{\iota\in(-\infty,0]} e^{\gamma\iota}\|v_t(\iota,s,\theta_{-t}\omega,\psi)-v_t(\iota,s_*,\theta_{-t}\omega,\psi_*)\|<\epsilon.\tag{3.4.20}$$

Notice that the above inequality satisfies

$$\begin{aligned}\sup_{\iota\in(-\infty,0]} e^{\gamma\iota}\|v_t(\iota,s,\theta_{-t}\omega,\psi)-v_t(\iota,s_*,\theta_{-t}\omega,\psi_*)\| &\leqslant \sup_{\iota\in(-\infty,0]} e^{\gamma\iota}\|v_t(\iota,s,\theta_{-t}\omega,\psi)-v_t(\iota,s,\theta_{-t}\omega,\psi_*)\|\\ &\quad+\sup_{\iota\in(-\infty,0]} e^{\gamma\iota}\|v_t(\iota,s,\theta_{-t}\omega,\psi_*)-v_t(\iota,s_*,\theta_{-t}\omega,\psi_*)\|.\end{aligned}\tag{3.4.21}$$

Using similar arguments as Theorem 3.2.3, one can derive that there exists $\delta_1>0$ such that $|s-s_*|\vee\|\psi-\psi_*\|_{C_\gamma(V)}<\delta_1$, for all $t\in[s_*,s_*+T_0]$ and almost all $\omega\in\Omega$, the first term on the right-hand side of (3.4.21) is bounded by

$$\sup_{\iota\in(-\infty,0]} e^{\gamma\iota}\|v_t(\iota,s,\theta_{-t}\omega,\psi)-v_t(\iota,s,\theta_{-t}\omega,\psi_*)\|<\frac{\epsilon}{2}.\tag{3.4.22}$$

In the following, we only need to prove the last line of (3.4.21) is also bounded by $\frac{\epsilon}{2}$, that is, we need to show that $v_t(\iota,\cdot,\theta_{-t}\omega,\psi_*)$ is both left and right continuous on $(-\infty,t]$. We start with the left continuity.

By Lemma 3.4.2, we deduce that, for every $\epsilon>0$, there exists some $\delta_2=\delta_2(\psi_*,\epsilon)>0$ such that for all $s_1,s_2\in(-\infty,0]$ with $|s_1-s_2|<\delta_2$,

$$e^{\gamma s_2}\|\psi_*(s_1)-\psi_*(s_2)\|<\frac{\epsilon}{4}.\tag{3.4.23}$$

Thanks to Lemma 3.4.3, there exists $\delta_3=\delta_3(s_*,\psi_*,T_0,\omega,\epsilon)>0$ such that, for all $s\in(s_*-\delta_3,s_*)$, $r\in[s,s_*]$, $t\in[s_*,s_*+T_0]$ and almost all $\omega\in\Omega$, it follows

$$\|v(r,s,\theta_{-t}\omega,\psi_*)-\psi_*(0)\|<\frac{\epsilon}{4}.\tag{3.4.24}$$

By (3.4.23) and (3.4.24), for the above ϵ, there exists some $\delta_4=\delta_4(\epsilon,s_*,\omega,\psi_*)=\min\{\delta_2,\delta_3\}>0$ such that, for all $s\in(s_*-\delta_4,s_*)$, $r\in[s,s_*]$, $t\in[s_*,s_*+T_0]$ and almost all $\omega\in\Omega$,

$$\|\widetilde{v}(r,s_*,\theta_{-t}\omega,\psi_*)-v(r,s,\theta_{-t}\omega,\psi_*)\|=\|\psi_*(r-s_*)-v(r,s,\theta_{-t}\omega,\psi_*)\|$$

$$\leqslant \|\psi_*(r-s_*)-\psi_*(0)\| + \|\psi_*(0) - v(r,s,\theta_{-t}\omega,\psi_*)\|$$
$$< \frac{\epsilon}{2}, \tag{3.4.25}$$

where $\widetilde{v}(\cdot, s_*, \theta_{-t}\omega, \psi_*)$ is the solution with the initial datum ψ_* at the initial time s_*. The above inequality implies

$$\max_{r\in[s,s_*]} \|\widetilde{v}(r,s_*,\theta_{-t}\omega,\psi_*) - v(r,s,\theta_{-t}\omega,\psi_*)\| \leqslant \frac{\epsilon}{2}, \ s\in(s_*-\delta_4, s_*),\ t\in[s_*, s_*+T_0],\ \omega\in\Omega. \tag{3.4.26}$$

Combining (3.4.23) and (3.4.26), we deduce that, for all $s \in (s_* - \delta_4, s_*)$, $t \in [s_*, s_* + T_0]$ and almost all $\omega \in \Omega$,

$$\begin{aligned}\|\psi_* - v_{s_*}\|_{C_\gamma(V)} &\leqslant \max\Big\{ \sup_{r\leqslant s-s_*} e^{\gamma r}\|\psi_*(r) - v(s_*+r)\|, \sup_{r\in[s-s_*,0]} e^{\gamma r}\|\psi_*(r)-v(s_*+r)\|\Big\}\\
&\leqslant \max\Big\{ \sup_{r\leqslant s-s_*} e^{\gamma r}\|\psi_*(r) - \psi_*(s_*+r-s)\|, \sup_{r\in[s,s_*]} \|\psi_*(r-s_*)-v(r)\|\Big\}\\
&\leqslant \max\Big\{ \sup_{r\leqslant 0} e^{\gamma(r-s_*+s)}\|\psi_*(r-s_*+s) - \psi_*(r)\|,\\
&\qquad \sup_{r\in[s,s_*]} \|\widetilde{v}(r,s_*,\theta_{-t}\omega,\psi_*) - v(r,s,\theta_{-t}\omega,\psi_*)\|\Big\} \leqslant \frac{\epsilon}{2}.\end{aligned} \tag{3.4.27}$$

Note that $v(r,s,\omega,\psi_*) = v(r,s_*,\omega,v_{s_*}(\cdot,s,\omega,\psi_*))$, by (3.2.10) in Theorem 3.2.3, we deduce

$$\begin{aligned}&\max_{r\in[s_*,t]} \|\widetilde{v}(r,s_*,\theta_{-t}\omega,\psi_*) - v(r,s,\theta_{-t}\omega,\psi_*)\|\\
&\quad= \max_{r\in[s_*,t]} \|\widetilde{v}(r,s_*,\theta_{-t}\omega,\psi_*) - v(r,s_*,\theta_{-t}\omega,v_{s_*}(\cdot,s,\theta_{-t}\omega,\psi_*))\|\\
&\quad\leqslant \Big(1+\frac{L_{\widetilde{g}}^2}{2\gamma}\Big)\|\psi_* - v_{s_*}\|^2_{C_\gamma(V)} \exp\Big(\int_{s_*}^t \Big(\frac{\widetilde{c}^2}{2}\|\widetilde{v}(r,s_*,\theta_{-t}\omega,\psi_*)\|^2_{D(A)} + 1 + L_{\widetilde{g}}^2\Big)dr\Big),\end{aligned} \tag{3.4.28}$$

which, together with (3.4.27), yields for all $s \in (s_* - \delta_4, s_*)$, $t \in [s_*, s_* + T_0]$ and almost all $\omega \in \Omega$,

$$\max_{r\in[s_*,t]} \|\widetilde{v}(r,s_*,\theta_{-t}\omega,\psi_*) - v(r,s,\theta_{-t}\omega,\psi_*)\| \leqslant \frac{\epsilon}{2}. \tag{3.4.29}$$

We deduce from (3.4.26) and (3.4.29) that

$$\max_{r\in[s,t]} \|\widetilde{v}(r,s_*,\theta_{-t}\omega,\psi_*) - v(r,s,\theta_{-t}\omega,\psi_*)\| \leqslant \frac{\epsilon}{2}. \tag{3.4.30}$$

By (3.4.23) and (3.4.30), we obtain that, for all $s \in (s_* - \delta_4, s_*)$, $t \in [s_*, s_* + T_0]$ and almost all $\omega \in \Omega$,

$$\begin{aligned}\sup_{\iota\in(-\infty,0]} e^{\gamma\iota}\|v_t(\iota,s,\theta_{-t}\omega,\psi_*) - v_t(\iota,s_*,\theta_{-t}\omega,\psi_*)\| &= \sup_{\iota\leqslant 0} e^{\gamma\iota}\Big(\|v(\iota+t,s,\theta_{-t}\omega,\psi_*) - \widetilde{v}(\iota+t,s_*,\theta_{-t}\omega,\psi_*)\|\Big)\\
&\leqslant \max\Big\{\sup_{\iota\leqslant s-t} e^{\gamma\iota}\|\psi_*(\iota+t-s) - \psi_*(\iota+t-s_*)\|,\\
&\qquad \sup_{s-t\leqslant\iota\leqslant 0} e^{\gamma\iota}\|v(\iota+t,s,\theta_{-t}\omega,\psi_*) - \widetilde{v}(\iota+t,s_*,\theta_{-t}\omega,\psi_*)\|\Big\}\end{aligned}$$

$$\leqslant \max\Big\{\sup_{\iota\leqslant 0} e^{\gamma(\iota+s-t)}\|\psi_*(\iota)-\psi_*(\iota+s-s_*)\|,$$
$$\sup_{s\leqslant\iota\leqslant t}\|v(\iota,s,\theta_{-t}\omega,\psi_*)-\widetilde{v}(\iota,s_*,\theta_{-t}\omega,\psi_*)\|\Big\}\leqslant\frac{\epsilon}{2}. \tag{3.4.31}$$

This implies that the $C_\gamma(V)$-valued function $s\mapsto\varphi(t,s,\theta_{-t}\omega)\phi$ is left continuous at $s=s_*$. It is similar to prove the right continuity of $\varphi(t,s,\theta_{-t}\omega)\phi$ at $s=s_*$, and we omit the details.

For the above ϵ, there exists $\delta=\min\{\delta_1,\delta_4\}>0$ such that for all $|s-s_*|\vee\|\psi-\psi_*\|_{C_\gamma(V)}<\delta$ and $t\in[s_*,s_*+T_0]$, we deduce from (3.4.21) and (3.4.22) that (3.4.20) holds as desired. □

Recall the definition of generalized Banach limit, which plays an important role in constructing the invariant measures for φ (see [51, 52, 98, 148] for more details).

Definition 3.4.5. *A generalized Banach limit is any linear functional, denoted by $LIM_{t\to+\infty}$, defined on the space of all bounded real-valued functions on $[0,+\infty)$ and satisfying*

(1) $LIM_{t\to+\infty}\xi(t)\geqslant 0$ for nonnegative functions $\xi(\cdot)$ on $[0,+\infty)$;

(2) $LIM_{t\to+\infty}\xi(t)=\lim_{t\to+\infty}\xi(t)$ if the usual limit $\lim_{t\to+\infty}\xi(t)$ exists.

Remark 3.4.6. *Note that we will discuss the asymptotic behavior $s\to-\infty$ of $\varphi(t,s,\omega)\bullet$, and thus, we require generalized limits as $s\to-\infty$. For a given real-valued function ξ defined on $(-\infty,0]$ and a given Banach limit $LIM_{t\to+\infty}$, we define $LIM_{t\to-\infty}\xi(t)$=$LIM_{t\to+\infty}\xi(-t)$.*

Theorem 3.4.7. *Let* (H31)-(H36), (3.1.2), (3.1.5)-(3.1.7) *and $\widetilde{\kappa}\in D(\widetilde{A})$ hold. Let φ be the random dynamical system associated with problem* (0.0.6) *over the measurable dynamical system $(\Omega,\mathcal{F},\mathbb{P};\{\theta_t\}_{t\in\mathbb{R}})$ with the state space $C_\gamma(V)$. Let $\mathcal{A}(\omega)$ be the global $\mathfrak{D}_a$-random attractor obtained in Theorem 3.3.8. Then for a given continuous mapping $\zeta_s:\mathbb{R}\mapsto C_\gamma(V)$ with $\zeta_s(\cdot)\in\mathfrak{D}_a$ and a generalized Banach limit* $\mathrm{LIM}_{t\to+\infty}$*, there exists for almost all $\omega\in\Omega$, a family of Borel probability measures $\{\mu_{\theta_t\omega}\}_{t\in\mathbb{R}}$ on $C_\gamma(V)$ such that the support of $\mu_{\theta_t\omega}$ is contained in $\mathcal{A}(\theta_t\omega)$ and*

$$\begin{aligned}\mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^t\Upsilon(\varphi(t,r,\omega)\zeta_r)dr&=\int_{\mathcal{A}(\theta_t\omega)}\Upsilon(u)d\mu_{\theta_t\omega}(u)\\&=\int_{C_\gamma(V)}\Upsilon(u)d\mu_{\theta_t\omega}(u)\\&=\mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^t\int_{C_\gamma(V)}\Upsilon(\varphi(t,r,\omega)u)d\mu_{\theta_r\omega}(u)dr,\end{aligned}$$

for every real-valued continuous functional Υ on $C_\gamma(V)$. Moreover, $\mu_{\theta_t\omega}$ is invariant in the sense that

$$\int_{\mathcal{A}(\theta_t\omega)}\Upsilon(u)d\mu_{\theta_t\omega}(u)=\int_{\mathcal{A}(\theta_s\omega)}\Upsilon(\varphi(t,s,\omega)u)d\mu_{\theta_s\omega}(u),\ \forall\, t\geqslant s.$$

Proof. For the random dynamical system φ on the space $C_\gamma(V)$, we need to verify the conditions (i) and (ii) in [148, Theorem 2.1].

By Theorem 3.3.8, we obtain that φ possesses a global $\mathfrak{D}_a$-random attractor $\mathcal{A}(\omega)$ in $C_\gamma(V)$. Thus, (i) has been proved. Moreover, by Lemmas 3.4.1 and 3.4.4, we deduce that φ is continuous and bounded with respect to the initial values, and thus (ii) holds true. Therefore, we obtain the results of Theorem 3.4.7. □

Part II

FitzHugh-Nagumo lattice systems with delay

Chapter 4

Dynamical stability of random delayed FitzHugh-Nagumo lattice systems driven by nonlinear Wong-Zakai noise

In this chapter, we first introduce Wong-Zakai process, weighted spaces and some notations, impose some suitable assumptions, and define a family of continuous cocycles in the next section. In Section 4.2, we then prove the existence of pullback random attractors for problem (0.0.7). In Section 4.3, we further establish its upper semicontinuity as $\delta \to +\infty$. The last section is devoted to the upper semicontinuity of pullback attractors for problem (0.0.9) as $\rho \to 0$.

4.1 Random delayed FitzHugh-Nagumo lattice systems with Wong-Zakai noise

In this section, we first prove some useful results on Wong-Zakai processes and weighted spaces. We then define a continuous cocycle (non-autonomous random dynamical system) Ψ^{δ} associated with the random delay FitzHugh-Nagumo lattice system (0.0.7) for all $\delta > 0$, and establish some suitable assumptions.

4.1.1 Wong-Zakai process

As usual, we identify the Wiener process $W(t,\omega)$ with the path $\omega(t)$ on the metric dynamical system $(\Omega, \mathfrak{F}, \mathbb{P}, \theta\}$, i.e., $W(t,\omega) = \omega(t)$, where $\Omega = \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$ with the compact-open topology, $\mathfrak{F}$ is the Borel σ-algebra, $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathfrak{F})$, $\theta = \{\theta_t : t \in \mathbb{R}\}$ is a group on $(\Omega, \mathfrak{F}, \mathbb{P})$ denoted by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$, and there is a θ-invariant full-measure set $\Omega_0 \subset \Omega$ satisfying

$$\lim_{t\to\pm} \frac{\omega(t)}{t} = 0, \ \forall\, \omega \in \Omega_0. \tag{4.1.1}$$

For convenience, we write Ω_0 as Ω. For each $\delta > 0$, define a random variable $\mathcal{G}_\delta$ by

$$\mathcal{G}_\delta(\omega) := \mathcal{G}_\delta(0, \omega) = \frac{\omega(\delta)}{\delta}, \ \forall\, \delta > 0, \ \omega \in \Omega, \tag{4.1.2}$$

which implies that the Wong-Zakai process has another form:

$$\mathcal{G}_\delta(t, \omega) = \frac{1}{\delta}(W(t+\delta, \omega) - W(t, \omega)) = \mathcal{G}_\delta(\theta_t \omega), \ \forall\, \delta > 0, \ t \in \mathbb{R}, \ \omega \in \Omega. \tag{4.1.3}$$

The following Lemma gives several conclusions on $\mathcal{G}_\delta$.

Lemma 4.1.1. *For each $(\delta, \omega) \in \mathbb{R}^+ \times \Omega$, we obtain the following results*

(i) The mapping $t \to \mathcal{G}_\delta(\theta_t \omega)$ is continuous such that

$$\lim_{\delta \to 0} \sup_{t \in [a,b]} \left| \int_0^t \mathcal{G}_\delta(\theta_s \omega) ds - \omega(t) \right| = 0; \tag{4.1.4}$$

(ii) The mapping $t \to \mathcal{G}_\delta(\theta_t \omega)$ is of sublinear growth, i.e.,

$$\lim_{t \to \pm\infty} \frac{\mathcal{G}_\delta(\theta_t \omega)}{t} = 0; \tag{4.1.5}$$

(iii) The mapping $\delta \to \mathcal{G}_\delta(\theta_t \omega)$ is continuous on $(0, +\infty)$ and uniformly continuous on $[\delta_0, +\infty)$ for all $\delta_0 > 0$ such that

$$\lim_{\delta \to +\infty} \sup_{t \in [a,b]} |\mathcal{G}_\delta(\theta_t \omega)| = 0; \tag{4.1.6}$$

(iv) For any $\varsigma_1, \varsigma_2 > 0$ and $\omega \in \Omega$ such that for all $\delta > 0$,

$$\int_{-\infty}^0 e^{\varsigma_1 t} |\mathcal{G}_\delta(\theta_t \omega)|^{\varsigma_2} dt < +\infty, \ and \ \lim_{\delta \to +\infty} \int_{-\infty}^0 e^{\varsigma_1 t} |\mathcal{G}_\delta(\theta_t \omega)|^{\varsigma_2} dt = 0. \tag{4.1.7}$$

Proof. (i) It follows from [97, lemma 2.1] that (4.1.4) holds true.

(ii) According to (4.1.2), we obtain

$$\lim_{t \to \pm\infty} \frac{\mathcal{G}_\delta(\theta_t \omega)}{t} = \lim_{t \to \pm\infty} \frac{\omega(t+\delta) - \omega(t)}{\delta t} = \lim_{t \to \pm\infty} \frac{\omega(t+\delta)}{t+\delta} \cdot \frac{t+\delta}{\delta t} - \frac{1}{\delta} \lim_{t \to \pm\infty} \frac{\omega(t)}{t} = 0. \tag{4.1.8}$$

(iii) Since $t \to \omega(t)$ is continuous, one can imply that $\delta \to \mathcal{G}_\delta(\theta_t \omega)$ is continuous on $(0, +\infty)$. We now prove that it is uniformly continuous on $[\delta_0, +\infty)$ for all $\delta_0 > 0$. And thus we need to imply that

$$\begin{aligned} \lim_{\delta \to +\infty} \sup_{t \in [a,b]} \mathcal{G}_\delta(\theta_t \omega) &= \lim_{\delta \to +\infty} \sup_{t \in [a,b]} \frac{\omega(t+\delta) - \omega(t)}{\delta} \\ &= \lim_{\delta \to +\infty} \sup_{t \in [a,b]} \frac{\omega(t+\delta)}{\delta} - \lim_{\delta \to +\infty} \inf_{t \in [a,b]} \frac{\omega(t)}{\delta} = 0. \end{aligned} \tag{4.1.9}$$

On the one hand, for given $\epsilon > 0$ and $\omega \in \Omega$, note that $\frac{\omega(t)}{t} \to 0$ as $t \to +\infty$, so there exists $T_1 := T_1(\epsilon, \omega) > 0$ such that $|\omega(t)| \leqslant \epsilon t$ for all $t \geqslant T_1$. For each $a, b \in \mathbb{R}$ and $a \leqslant b$, then $[a, b]$ is compact. Then, for all $\delta \geqslant T_1 - a$, and so $t + \delta \geqslant T_1 > 0$ whenever $t \in [a, b]$,

$$\sup_{t\in[a,b]} \frac{\omega(t+\delta)}{t+\delta} \leqslant \sup_{t\in[a,b]} \frac{|\omega(t+\delta)|}{t+\delta} \leqslant \sup_{t\in[a,b]} \frac{\epsilon(t+\delta)}{t+\delta} = \epsilon. \tag{4.1.10}$$

We then easily check that $0 \leqslant \frac{t+\delta}{\delta} \leqslant 2$ for all $\delta \geqslant \max\{|a|, |b|\}$. Let $\delta_0 = \max\{T_1 - a, |a|, |b|\}$, then for all $\delta \geqslant \delta_0$ and $t \in [a, b]$ such that

$$\sup_{t\in[a,b]} \frac{\omega(t+\delta)}{\delta} = \sup_{t\in[a,b]} \frac{\omega(t+\delta)}{t+\delta} \cdot \frac{t+\delta}{\delta} \leqslant 2\epsilon,$$

which implies

$$\lim_{\delta\to+\infty} \sup_{t\in[a,b]} \frac{\omega(t+\delta)}{\delta} = 0. \tag{4.1.11}$$

On the other hand, since the minimum of $\omega(\cdot)$ over $[a, b]$ exists and is finite, we deduce

$$\lim_{\delta\to+\infty} \inf_{t\in[a,b]} \frac{\omega(t)}{\delta} = 0. \tag{4.1.12}$$

Combining (4.1.11) and (4.1.12), we obtain we obtain (4.1.9). This implies (4.1.6), which together with the continuity of $\delta \to \mathcal{G}_\delta(\theta_t\omega)$, yields the uniform continuity on $[\delta_0, +\infty)$.

(iv) By (ii), $t \to \mathcal{G}_\delta(\theta_t\omega)$ is of sublinear growth as $t \to -\infty$, which along with the continuity of $\delta \to \mathcal{G}_\delta(\theta_t\omega)$ shows that $e^{\varsigma_1 t}|\mathcal{G}_\delta(\theta_t\omega)|^{\varsigma_2}$ is integrable with respect to $t \in (-\infty, 0]$ for any $\varsigma_1, \varsigma_2 > 0$. By $\frac{\omega(t)}{t} \to 0$ as $t \to \pm\infty$, there is a $T := T(\omega) > 0$ such that $\left|\frac{\omega(t)}{t}\right| \leqslant 1$ for all $|t| \geqslant T$,

$$|\omega(t)| \leqslant |t| + C(\omega), \ \forall\, t \in \mathbb{R}, \tag{4.1.13}$$

where $C(\omega) = \sup\limits_{t\in[-T,T]} |\omega(t)| < +\infty$.

For all $\delta \geqslant 1$, we then proves the following inequality holds true.

$$|\mathcal{G}_\delta(\theta_t\omega)| \leqslant 2C(\omega) - 2t + 1, \ \forall\, t \leqslant 0, \ \omega \in \Omega. \tag{4.1.14}$$

Case A: If $t \in [-\delta, 0]$, then for all $\delta \geqslant 1$,

$$\begin{aligned}
|\mathcal{G}_\delta(\theta_t\omega)| =& \frac{1}{\delta}|\omega(t+\delta) - \omega(t)| \\
\leqslant& \frac{1}{\delta}(|\omega(t+\delta)| + |\omega(t)|) \\
\leqslant& \frac{1}{\delta}\Big((t+\delta+C(\omega)) + (-t+C(\omega))\Big) \\
=& 1 + \frac{2}{\delta}C(\omega)
\end{aligned}$$

$$\leqslant 2C(\omega) - 2t + 1. \tag{4.1.15}$$

Case B: If $t \in (-\infty, -\delta]$, then for all $\delta \geqslant 1$,

$$\begin{aligned}|\mathcal{G}_\delta(\theta_t\omega)| =& \frac{1}{\delta}|\omega(t+\delta) - \omega(t)| \\ \leqslant& \frac{1}{\delta}(|\omega(t+\delta)| + |\omega(t)|) \\ \leqslant& \frac{1}{\delta}\Big((-t-\delta+C(\omega)) + (-t+C(\omega))\Big) \\ \leqslant& 2C(\omega) - 2t + 1. \end{aligned}\tag{4.1.16}$$

Combining two cases **A** and **B**, we have (4.1.14) as desired. Thus, we easily show

$$\int_{-\infty}^{0} e^{\varsigma_1 t}|\mathcal{G}_\delta(\theta_t\omega)|^{\varsigma_2}dt \leqslant \int_{-\infty}^{0} e^{\varsigma_1 t}(2C(\omega) - 2t + 1)^{\varsigma_2}dt < +\infty, \ \forall \ \varsigma_1, \varsigma_2 > 0. \tag{4.1.17}$$

According to the Lebesgue control convergence theorem and (4.1.6), we deduce

$$\lim_{\delta\to+\infty}\int_{-\infty}^{0} e^{\varsigma_1 t}|\mathcal{G}_\delta(\theta_t\omega)|^{\varsigma_2}dt = \int_{-\infty}^{0} e^{\varsigma_1 t}\lim_{\delta\to+\infty}|\mathcal{G}_\delta(\theta_t\omega)|^{\varsigma_2}dt = 0, \tag{4.1.18}$$

which proves (4.1.7) as desired. All proofs are complete.

□

4.1.2 Weighted spaces and continuous cocycles

Given $p \geqslant 1$ and $\sigma > \frac{1}{2}$, we define the weighted p-times summation space by

$$\ell_\sigma^p = \left\{ u = \{u_i\}_{i\in\mathbb{Z}} : \|u\|_{\sigma,p} = (\sum_{i\in\mathbb{Z}} \xi_i |u_i|^p)^{\frac{1}{p}} \right\}, \tag{4.1.19}$$

where $\xi_i = (1+i^2)^{-\sigma}$ for $i \in \mathbb{Z}$, and so $\xi = (\xi_i)_{i\in\mathbb{Z}} \in \ell^p$ for any $p \geqslant 1$. Thanks to [9, 63], $(\ell_\sigma^p, \|\cdot\|_{\sigma,p})$ is a separable Banach space. In particular, ℓ_σ^2 is a Hilbert space with inner product and norm, respectively:

$$(u,v)_\sigma = \sum_{i\in\mathbb{Z}} \xi_i u_i v_i, \ \ \|u\|_\sigma = (u,u)_\sigma^{\frac{1}{2}}, \ \ \forall \ u, v \in \ell_\sigma^2. \tag{4.1.20}$$

By the Hölder inequality, for $p > q \geqslant 1$, we have $\|\varpi\|_{\sigma,q}^q \leqslant \|\xi\|_{\ell^1}^{\frac{p-q}{p}} \|\varpi\|_{\sigma,p}^q$, $\forall \ \varpi \in \ell_\sigma^p$. More precisely,

$$\begin{aligned}\|\varpi\|_{\sigma,q}^q =& \sum_{i\in\mathbb{Z}} \xi_i|\varpi|^q = \sum_{i\in\mathbb{Z}} \xi_i^{\frac{p-q}{p}}\left(\xi_i^{\frac{q}{p}}|\varpi|^q\right) \\ \leqslant& \Big(\sum_{i\in\mathbb{Z}} (\xi_i^{\frac{p-q}{p}})^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}} \Big(\sum_{i\in\mathbb{Z}} \left(\xi_i^{\frac{q}{p}}|\varpi_i|^q\right)^{\frac{p}{q}}\Big)^{\frac{q}{p}}\end{aligned}$$

$$=\|\xi\|_{\ell^1}^{\frac{p-q}{p}}\|\varpi\|_{\sigma,p}^q. \tag{4.1.21}$$

Taking into account the delay, let $X_\sigma^\rho = C([-\rho,0],\ell_\sigma^2)$, which is the space of all continuous functions from $[-\rho,0]$ to ℓ_σ^2 with the following norm

$$\|\upsilon\|_{X_\sigma^\rho} = \sup_{s\in[-\rho,0]}\|\upsilon(s)\|_\sigma = \sup_{s\in[-\rho,0]}\Big(\sum_{i\in\mathbb{Z}}\xi_i|\upsilon_i(s)|^2\Big)^{\frac{1}{2}},\ \forall\ \upsilon\in X_\sigma^\rho. \tag{4.1.22}$$

For convenience, the delay shift of $\varphi=(u,v):\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2$ is defined by

$$\varphi_t=(u_t,v_t):[-\rho,0]\times[-\rho,0]\to\mathbb{R}^2,\ \varphi_t(s,x)=(u(t+s),v(t+s)) \tag{4.1.23}$$

for all $s\in[-\rho,0]$.

Let $\mathcal{X}_\sigma=\ell_\sigma^2\times\ell_\sigma^2$ and $\mathcal{X}_\sigma^\rho=C([-\rho,0],\mathcal{X}_\sigma)$ be equipped by the norms

$$\|\varphi\|_{\mathcal{X}_\sigma}^2=\beta\|u\|_\sigma^2+\alpha\|v\|_\sigma^2,\ \forall\varphi=(u,v), \tag{4.1.24}$$

and

$$\|\varphi_t\|_{\mathcal{X}_\sigma^\rho}^2=\beta\|u_t\|_{X_\sigma^\rho}^2+\alpha\|v_t\|_{X_\sigma^\rho}^2=\beta\sup_{s\in[-\rho,0]}\|u_t(s)\|_\sigma^2+\alpha\|v_t(s)\|_\sigma^2,\ \forall\varphi=(u,v), \tag{4.1.25}$$

where α and β are as in (0.0.7). Then, we introduce the discrete Laplace and gradient operators by

$$(Au)_i=-u_{i-1}+2u_i-u_{i+1},\quad (Bu)_i=u_{i+1}-u_i,\quad (B^*u)_i=u_{i-1}-u_i, \tag{4.1.26}$$

which shows that $A=BB^*=B^*B$, see [62]. Note that for all $u,v\in l^2$ such that $(Bu,v)=(u,B^*v),(Au,v)=(Bu,Bv)$. It is simple to obtain that for all $i\in\mathbb{Z}$,

$$0.4^\sigma\leqslant\frac{\xi_{i+1}}{\xi_i}=\Big(\frac{1+i^2}{1+(i+1)^2}\Big)^\sigma\leqslant 2.5^\sigma,\ \text{and } 0.4^\sigma\leqslant\frac{\xi_{i-1}}{\xi_i}=\Big(\frac{1+(i-1)^2}{1+i^2}\Big)^\sigma\leqslant 2.5^\sigma, \tag{4.1.27}$$

which implies that

$$\xi_{i\pm1}\leqslant 2.5^\sigma\xi_i,\ \ |(B\xi)_i|=|\xi_{i+1}-\xi_i|\leqslant 2.5^\sigma\xi_i,\ \ |(B^*\xi)_i|\leqslant 2.5^\sigma\xi_i. \tag{4.1.28}$$

Let $F(x,u(t))=(F_i(u_i(t)))_{i\in\mathbb{Z}}$, $f(u(t-\varrho^{(\rho)}(t)))=(f_i(u_i(t-\varrho^{(\rho)}(t))))_{i\in\mathbb{Z}}$, $f(v(t-\varrho^{(\rho)}(t)))=(f_i(v_i(t-\varrho^{(\rho)}(t))))_{i\in\mathbb{Z}}$, $g(x,t)=(g_i(t))_{i\in\mathbb{Z}}$, $G(t,u)=(G_i(t,u_i))_{i\in\mathbb{Z}}$, and $h(x,t)=(h_i(t))_{i\in\mathbb{Z}}$. Then system (0.0.7) can be rewritten as

$$\begin{cases}\dfrac{du}{dt}+Au+\lambda u+\alpha v=F(x,u(t))+f(u(t-\varrho^{(\rho)}(t)))+g(x,t)+G(t,u)\mathcal{G}_\delta(\theta_t\omega),\\ \dfrac{dv}{dt}+\varsigma v-\beta u=h(x,t)+f(v(t-\varrho^{(\rho)}(t))),\\ u(\tau+s)=\phi(s),\ \ v(\tau+s)=\upsilon(s),\ t>\tau,\ \tau\in\mathbb{R},\ s\in[-\rho,0],\ \rho>0.\end{cases} \tag{4.1.29}$$

Hypothesis E. The delay function $\varrho^{(\rho)}(\cdot)$ is a positive continuously differentiable function satisfying

$$\rho := \sup_{t\in\mathbb{R}} \varrho^{(\rho)}(t) < +\infty,\ \rho_* := \sup_{\rho\in(0,\rho_0]} \sup_{t\in\mathbb{R}} \frac{d}{dt}\varrho^{(\rho)}(t) < 1. \tag{4.1.30}$$

Therefore, the memory time $\rho \in (0, \rho_0]$ for some $\rho_0 > 0$.

Hypothesis F1. For the nonlinear drift function $F_i \in C^1(\mathbb{R}, \mathbb{R})$, we assume that for all $s \in \mathbb{R}$ and $i \in \mathbb{Z}$,

$$F_i(s)s \leqslant -\alpha_1|s|^p + \mu_{1,i}, \qquad \mu_1 = (\mu_{1,i})_{i\in\mathbb{Z}} \in \ell_\sigma^{\frac{2p-2}{p}}, \tag{4.1.31}$$

$$\left|F_i(s)\right| \leqslant \alpha_2|s|^{p-1} + \mu_{2,i}, \qquad \mu_2 = (\mu_{2,i})_{i\in\mathbb{Z}} \in \ell_\sigma^2, \tag{4.1.32}$$

$$\frac{\partial F_i}{\partial s}(s) \leqslant -\alpha_3|s|^{p-2} + \mu_{3,i}, \qquad \mu_3 = (\mu_{3,i})_{i\in\mathbb{Z}} \in \ell^\infty, \tag{4.1.33}$$

where $p \geqslant 2, \alpha_1, \alpha_2$ and α_3 are positive constants.

Hypothesis F2. The nonlinear delay term f_i is continuous such that for all $s_1, s_2 \in \mathbb{R}$,

$$\begin{aligned} &f_i(0) = 0,\ \ \forall\, i \in \mathbb{Z}, \\ &\sup_{i\in\mathbb{Z}} \sup_{s_1,s_2\in\mathbb{R}} |f_i(s_1) - f_i(s_2)| \leqslant L_f|s_1 - s_2|, \end{aligned} \tag{4.1.34}$$

where $L_f > 0$ is constant.

From now on, let $\kappa = \min\{\lambda, \varsigma\}$ and $\sigma_0 := 4 \times 2.5^{2\sigma} + \frac{4}{3}(2.5^{3\sigma} + 2\|\mu_3\|_{\ell^\infty})$. Besides, we assume $\sigma_0 + \frac{4L_f^2}{\kappa(1-\rho_*)} < \kappa$. In this case, there exists $m_0 > 0$ small enough such that for all $m \in (0, m_0)$,

$$m + \sigma_0 - \kappa + \frac{4L_f^2 e^{m\rho_0}}{\kappa(1 - \rho_*)} < 0. \tag{4.1.35}$$

In particular, $m - \kappa + \frac{4L_f^2 e^{m\rho_0}}{\kappa(1-\rho_*)} < 0$.

Hypothesis G1. Let $G_i(\cdot, \cdot)$ be continuous from $\mathbb{R}^2$ to $\mathbb{R}$ satisfying

$$|G_i(t, s)| \leqslant \alpha_4|s|^{q-1} + \mu_{4,i}(t), \qquad \mu_4 = (\mu_{4,i})_{i\in\mathbb{Z}} \in L^\infty(\mathbb{R}, \ell_\sigma^p), \tag{4.1.36}$$

where $2 \leqslant q < p,\ \alpha_4 > 0$.

We further impose the following assumptions.

Hypothesis G2. The forces g and h are backward tempered:

$$\Upsilon(\tau) := \sup_{r\leqslant\tau} \int_{-\infty}^0 e^{m\nu}(\|g(\nu + r)\|_\sigma^2 + \|h(\nu + r)\|_\sigma^2)d\nu < +\infty,\ \forall\, \tau \in \mathbb{R}. \tag{4.1.37}$$

Hypothesis G3. The forces g and h are backward tail-small:

$$\lim_{k\to\infty} \sup_{r\leqslant\tau} \int_{-\infty}^0 e^{m\nu} \sum_{|i|\geqslant k} \xi_i(|g_i(\nu + r)|^2 + |h_i(\nu + r)|^2)d\nu = 0,\ \forall\, \tau \in \mathbb{R}. \tag{4.1.38}$$

Under the assumptions (4.1.30)-(4.1.38), similarly to the Galerkin method, we can show that for each $\delta > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\psi_\tau = (u_\tau, v_\tau) \in \mathcal{X}_\sigma^\rho = C([-\rho, 0], \ell_\sigma^2 \times \ell_\sigma^2)$, the random delay FitzHugh-Nagumo lattice system (4.1.29) possesses a unique solution $\varphi^\delta(\cdot, \tau, \omega, \psi_\tau) = (u^\delta(\cdot, \tau, \omega, u_\tau), v^\delta(\cdot, \tau, \omega, v_\tau))$ such that

$$\varphi^\delta \in C([\tau - \rho, +\infty), \ell_\sigma^2 \times \ell_\sigma^2). \tag{4.1.39}$$

Besides, the solution φ^δ is continuous with respect to the initial data ψ_τ in $\mathcal{X}_\sigma^\rho$. By the same method as in [44], one can prove that $\varphi^\delta(t, \tau, \omega, \psi_\tau)$ is $(\mathfrak{F}, \mathfrak{B}(\mathcal{X}_\sigma^\rho))$-measurable in $\omega \in \Omega$. Then, for each $\delta > 0$, we can define a family of continuous cocycles $\Psi^\delta : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathcal{X}_\sigma^\rho \mapsto \mathcal{X}_\sigma^\rho$ given by

$$\Psi^\delta(t, \tau, \omega)\psi_\tau = \varphi_{t+\tau}^\delta(\cdot, \tau, \theta_{-\tau}\omega, \psi_\tau). \tag{4.1.40}$$

Let $\mathfrak{D}$ be the universe of all backward tempered bi-parametric sets in $\mathcal{X}_\sigma^\rho$, where a bi-parametric set $\mathcal{D} := \{\mathcal{D}(\tau, \omega) : (\tau, \omega) \in \mathbb{R} \times \Omega\}$ in $\mathcal{X}_\sigma^\rho$ is called backward tempered, that is, $\mathcal{D} \in \mathfrak{D}$ if and only if

$$\lim_{t \to +\infty} e^{-\gamma t} \sup_{r \leqslant \tau} \|\mathcal{D}(r - t, \theta_{-t}\omega)\|_{\mathcal{X}_\sigma^\rho}^2 = 0, \ \forall\, (\gamma, \tau, \omega) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega. \tag{4.1.41}$$

We easily check that $\mathfrak{D}$ is backward-union closed in the sense of $\hat{\mathcal{D}} \in \mathfrak{D}$ whenever $\mathcal{D} \in \mathfrak{D}$, where

$$\hat{\mathcal{D}}(\tau, \omega) = \bigcup_{r \leqslant \tau} \mathcal{D}(r, \omega), \ \forall\, (\tau, \omega) \in \mathbb{R} \times \Omega. \tag{4.1.42}$$

However, the usual universe $\widetilde{\mathfrak{D}}$ of all tempered bi-parametric sets is not backward-union closed, where $\widetilde{\mathcal{D}} \in \widetilde{\mathfrak{D}}$ if and only if

$$\lim_{t \to +\infty} e^{-\gamma t} \|\widetilde{\mathcal{D}}(\tau - t, \theta_{-t}\omega)\|_{\mathcal{X}_\sigma^\rho}^2 = 0, \ \forall\, (\gamma, \tau, \omega) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega. \tag{4.1.43}$$

4.2 Existence of pullback random attractors

This subsection is devoted to the existence of pullback random attractors for the random delay FitzHugh-Nagumo system (4.1.29). We first derive a variety of backward uniform estimates of solutions to Eq. (4.1.29), including the backward uniform absorption and tail-estimates. We then prove the pullback asymptotic compactness of the solutions via the Ascoli-Arzelà theorem in $\mathcal{X}_\sigma^\rho = C([-\rho, 0], \mathcal{X}_\sigma)$, where $\mathcal{X}_\sigma = \ell_\sigma^2 \times \ell_\sigma^2$. Finally, we prove the existence of tempered random attractors for Eq. (4.1.29).

Backward uniform absorption

Lemma 4.2.1. *Let hypotheses* **E, F1, F2, G1, G2** *and* (4.1.35) *be satisfied. Then, for each* $(\tau, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}$ *and* $\psi_{r-t} = (\phi_{r-t}, v_{r-t}) \in \mathcal{D}(r - t, \theta_{-t}\omega)$*, there exists a* $T := T(\tau, \omega, \mathcal{D}) \geqslant 3\rho + 1$ *such that for all* $t \geqslant T$*, the solution* $\varphi^\delta = (u^\delta, v^\delta)$ *to* (4.1.29) *satisfies*

$$\sup_{r \leqslant \tau} \sup_{s \in [-2\rho - 1, 0]} \|\varphi^\delta(r + s, r - t, \theta_{-r}\omega, \psi_{r-t})\|_{\mathcal{X}_\sigma}^2 \leqslant cR_\delta(\tau, \omega), \tag{4.2.1}$$

$$\sup_{r\leqslant\tau}\int_{r-t}^{r}e^{m(\nu-r)}\|\varphi^{\delta}(\nu)\|^2_{\mathcal{X}_\sigma}d\nu+\sup_{r\leqslant\tau}\int_{r-t}^{r}e^{m(\nu-r)}(\|\varphi^{\delta}(\nu)\|^2_{\mathcal{X}_\sigma}+\|u^{\delta}(\nu)\|^p_{\sigma,p})d\nu\leqslant cR_\delta(\tau,\omega),\tag{4.2.2}$$

where $R_\delta(\tau,\omega)=1+\Upsilon(\tau)+\eta_\delta(\omega)$ *with*

$$\Upsilon(\tau)=\sup_{r\leqslant\tau}\int_{-\infty}^{0}e^{m\nu}(\|g(\nu+r)\|^2_\sigma+\|h(\nu+r)\|^2_\sigma)d\nu,\ \eta_\delta(\omega)=\int_{-\infty}^{0}e^{m\nu}|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}}d\nu.\tag{4.2.3}$$

Proof. Taking the inner product of (4.1.29) with $(2\beta u^\delta,2\alpha v^\delta):=(2\beta u^\delta(\nu,r-t,\theta_{-r}\omega,\phi_{r-t}),2\alpha v^\delta(\nu,r-t,\theta_{-r}\omega,\upsilon_{r-t}))$ in $\mathcal{X}_\sigma=\ell^2_\sigma\times\ell^2_\sigma$ (when no ambiguity is possible, we delete the superscript δ below), we obtain

$$\begin{aligned}\frac{d}{d\nu}\|\varphi\|^2_{\mathcal{X}_\sigma}+2\kappa\|\varphi\|^2_{\mathcal{X}_\sigma}&=-2\beta\sum_{i\in\mathbb{Z}}\xi_i(Au)_iu_i+2\beta(F(x,u),u)_\sigma\\&\quad+2\beta(f(u(\nu-\varrho^{(\rho)}(\nu))),u)_\sigma+2\alpha(f(v(\nu-\varrho^{(\rho)}(\nu))),v)_\sigma\\&\quad+2\beta(g(x,\nu),u)_\sigma+2\alpha(h(x,\nu),v)_\sigma+2\beta\mathcal{G}_\delta(\theta_{\nu-r}\omega)(G,u)_\sigma,\end{aligned}\tag{4.2.4}$$

where we recall that $\|\varphi\|^2_{\mathcal{X}_\sigma}=\beta\|u\|^2_\sigma+\alpha\|v\|^2_\sigma$, $\kappa=\min\{\lambda,\varsigma\}$. By (4.1.28) and $(B(\xi u))_i=(B\xi)_iu_{i+1}+\xi_i(Bu)_i$, we have

$$\begin{aligned}-2\beta\sum_{i\in\mathbb{Z}}\xi_i(Au)_iu_i&=-2\beta\sum_{i\in\mathbb{Z}}(Bu)_i(B(\xi u))_i\\&=-2\beta\sum_{i\in\mathbb{Z}}(B\xi)_i(Bu)_iu_{i+1}-2\beta\sum_{i\in\mathbb{Z}}\xi_i|(Bu)_i|^2\\&\leqslant2\beta\sum_{i\in\mathbb{Z}}2.5^\sigma\xi_i|(Bu)_i||u_{i+1}|-2\beta\sum_{i\in\mathbb{Z}}\xi_i|(Bu)_i|^2\\&\leqslant\beta\sum_{i\in\mathbb{Z}}\xi_i(2.5^{2\sigma}|u_{i+1}|^2+|(Bu)_i|^2)-2\beta\sum_{i\in\mathbb{Z}}\xi_i|(Bu)_i|^2\\&\leqslant2.5^{2\sigma}\beta\sum_{i\in\mathbb{Z}}\xi_i|u_{i+1}|^2\\&\leqslant2.5^{3\sigma}\beta\sum_{i\in\mathbb{Z}}\xi_{i+1}|u_{i+1}|^2=2.5^{3\sigma}\beta\|u\|^2_\sigma.\end{aligned}\tag{4.2.5}$$

By (4.1.31) in the hypothesis **F1**, we imply

$$\begin{aligned}2\beta(F(x,u),u)_\sigma&=2\beta\sum_{i\in\mathbb{Z}}\xi_iF_i(u_i)u_i\\&\leqslant-2\alpha_1\beta\sum_{i\in\mathbb{Z}}\xi_i|u_i|^p+2\beta\sum_{i\in\mathbb{Z}}\xi_i\mu_{1,i}\\&\leqslant-2\alpha_1\beta\|u\|^p_{\sigma,p}+2\beta\|\mu_1\|_{\sigma,1}.\end{aligned}\tag{4.2.6}$$

According to the Young inequality and (4.1.34) in the hypothesis **F2**, we deduce

$$2\beta(f(u(\nu-\varrho^{(\rho)}(\nu))),u)_\sigma+2\alpha(f(v(\nu-\varrho^{(\rho)}(\nu))),v)_\sigma$$

$$
\begin{aligned}
&= 2\beta \sum_{i\in\mathbb{Z}} \xi_i f_i(u_i(\nu-\varrho^{(\rho)}(\nu)))u_i + 2\alpha \sum_{i\in\mathbb{Z}} \xi_i f_i(v_i(\nu-\varrho^{(\rho)}(\nu)))v_i \\
&\leqslant \frac{4L_f^2}{\kappa} \sum_{i\in\mathbb{Z}} \left(\beta|u_i(\nu-\varrho^{(\rho)}(\nu))|^2 + \alpha|v_i(\nu-\varrho^{(\rho)}(\nu))|^2\right) + \frac{\kappa}{4}\|\varphi\|^2_{X_\sigma} \\
&\leqslant \frac{4L_f^2}{\kappa}\|\varphi(\nu-\varrho^{(\rho)}(\nu))\|^2_{X_\sigma} + \frac{\kappa}{4}\|\varphi\|^2_{X_\sigma}. \qquad (4.2.7)
\end{aligned}
$$

The Young inequality gives

$$
2\beta(g(x,\nu),u)_\sigma + 2\alpha(h(x,\nu),v)_\sigma \leqslant \frac{\kappa}{4}\|\varphi\|^2_{X_\sigma} + \hat{c}_1(\|g(\nu)\|^2_\sigma + \|h(\nu)\|^2_\sigma), \qquad (4.2.8)
$$

where $\hat{c}_1 = \hat{c}_1(\beta,\alpha,\kappa) > 0$. By (4.1.36) in the hypothesis **G1**, we have

$$
\begin{aligned}
2\beta\mathcal{G}_\delta(\theta_{\nu-r}\omega)(G,u)_\sigma \leqslant& 2\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \xi_i|G_i(\nu,u_i)||u_i| \\
\leqslant& 2\beta\alpha_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \xi_i|u_i|^q + 2\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \xi_i|\mu_{4,i}(\nu)||u_i| \\
\leqslant& \hat{c}_2|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|u\|^q_{\sigma,q} + \frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|\mu_4(\nu)\|^{\hat{q}}_{\sigma,\hat{q}}, \qquad (4.2.9)
\end{aligned}
$$

where $\hat{c}_2 = 2\beta\alpha_4 + \frac{2}{q}\beta$ and $\frac{1}{\hat{q}} + \frac{1}{q} = 1$. Now, we estimate the last two terms in (4.2.9), respectively. On the one hand, by (4.1.21), we obtain

$$
\begin{aligned}
\hat{c}_2|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|u\|^q_{\sigma,q} &\leqslant \hat{c}_2|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|\xi\|_{\ell^1}^{\frac{p-q}{p}}\|u\|^q_{\sigma,p} \\
&\leqslant \frac{1}{2}\alpha_1\beta\|u\|^p_{\sigma,p} + \hat{c}_3|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}, \qquad (4.2.10)
\end{aligned}
$$

where $\hat{c}_3 = \hat{c}_3(\|\xi\|_{\ell^1},\alpha_1,\beta,\hat{c}_2) > 0$. On the other hand, note that $q \geqslant 2$ and so $\hat{q} \leqslant 2 \leqslant q < p$, by (4.1.21) and $\mu_4 \in L^\infty(\mathbb{R},\ell^p_\sigma)$, we imply

$$
\begin{aligned}
\frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|\mu_4(\nu)\|^{\hat{q}}_{\sigma,\hat{q}} &\leqslant \frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|\xi\|_{\ell^1}^{\frac{p-\hat{q}}{p}}\|\mu_4(\nu)\|^{\hat{q}}_{\sigma,p} \\
&\leqslant \frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|\xi\|_{\ell^1}^{\frac{p-\hat{q}}{p}}\|\mu_4\|^{\hat{q}}_{L^\infty(\mathbb{R},\ell^p_\sigma)} \\
&\leqslant \hat{c}_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} + \hat{c}_5. \qquad (4.2.11)
\end{aligned}
$$

Substituting (4.2.10)-(4.2.11) into (4.2.9),

$$
2\beta\mathcal{G}_\delta(\theta_{\nu-r}\omega)(G,u)_\sigma \leqslant \frac{1}{2}\alpha_1\beta\|u\|^p_{\sigma,p} + \hat{c}_6|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} + \hat{c}_5, \qquad (4.2.12)
$$

where $\hat{c}_6 = \hat{c}_3 + \hat{c}_4$. By (4.2.5)-(4.2.8) and (4.2.12), we can rewrite (4.2.4) as follows.

$$
\frac{d}{d\nu}\|\varphi\|^2_{X_\sigma} + \kappa\|\varphi\|^2_{X_\sigma} + \frac{\kappa}{2}\|\varphi\|^2_{X_\sigma} + \frac{3}{2}\alpha_1\beta\|u\|^p_{\sigma,p}
$$

$$\leqslant 2.5^{3\sigma}\beta\|u\|_\sigma^2 + \frac{4L_f^2}{\kappa}\|\varphi(\nu-\varrho^{(\rho)}(\nu))\|_{X_\sigma}^2 + \hat{c}_1(\|g(\nu)\|_\sigma^2 + \|h(\nu)\|_\sigma^2) + \hat{c}_6|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} + \hat{c}_7, \tag{4.2.13}$$

where $\hat{c}_7 = \hat{c}_5 + 2\beta\|\mu_1\|_{\sigma,1} < +\infty$ in view of $\mu_1 \in \ell_\sigma^{\frac{2p-2}{p}}$. Taking into account (4.1.21), we obtain

$$2.5^{3\sigma}\beta\|u\|_\sigma^2 \leqslant 2.5^{3\sigma}\beta\|\xi\|_{\ell^1}^{\frac{p-2}{p}}\|u\|_{\sigma,p}^2 \leqslant \frac{1}{2}\alpha_1\beta\|u\|_{\sigma,p}^p + \hat{c}_8, \tag{4.2.14}$$

where $\hat{c}_8 = \hat{c}_8(\|\xi\|_{\ell^1}, \sigma, \beta, \alpha_1) > 0$. Combining (4.2.13) and (4.2.14), we have

$$\frac{d}{d\nu}\|\varphi\|_{X_\sigma}^2 + \kappa\|\varphi\|_{X_\sigma}^2 + \frac{\kappa}{2}\|\varphi\|_{X_\sigma}^2 + \alpha_1\beta\|u\|_{\sigma,p}^p \tag{4.2.15}$$
$$\leqslant \frac{4L_f^2}{\kappa}\|\varphi(\nu-\varrho^{(\rho)}(\nu))\|_{X_\sigma}^2 + \hat{c}_9(1 + \|g(\nu)\|_\sigma^2 + \|h(\nu)\|_\sigma^2) + \hat{c}_6|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}.$$

Multiplying (4.2.15) by $e^{m\nu}$ and integrating it about $\nu \in [r-t, r+s]$, where $r \leqslant \tau$, $t \geqslant 3\rho + 1$ and $s \in [-2\rho - 1, 0]$, we deduce

$$\begin{aligned} &e^{m(r+s)}\|\varphi(r+s, r-t, \theta_{-r}\omega, \psi_{r-t})\|_{X_\sigma}^2 + \frac{\kappa}{2}\int_{r-t}^{r+s} e^{m\nu}\|\varphi(\nu)\|_{X_\sigma}^2 d\nu + \alpha_1\beta\int_{r-t}^{r+s} e^{m\nu}\|u(\nu)\|_{\sigma,p}^p d\nu \\ &\leqslant e^{m(r-t)}\|\psi_{r-t}(0)\|_{X_\sigma}^2 + (m-\kappa)\int_{r-t}^{r+s} e^{m\nu}\|\varphi(\nu)\|_{X_\sigma}^2 d\nu + \frac{4L_f^2}{\kappa}\int_{r-t}^{r+s} e^{m\nu}\|\varphi(\nu-\varrho^{(\rho)}(\nu))\|_{X_\sigma}^2 d\nu \\ &\quad + c_9\int_{r-t}^{r+s} e^{m\nu}(1 + \|g(\nu)\|_\sigma^2 + \|h(\nu)\|_\sigma^2) d\nu + c_6\int_{r-t}^{r+s} e^{m\nu}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} d\nu. \end{aligned} \tag{4.2.16}$$

Now, we compute the third term on the right-hand of (4.2.16):

$$\begin{aligned} &\frac{4L_f^2}{\kappa}\int_{r-t}^{r+s} e^{m\nu}\|\varphi(\nu-\varrho^{(\rho)}(\nu))\|_{X_\sigma}^2 d\nu \\ &\leqslant \frac{4L_f^2}{\kappa(1-\rho_*)}\int_{r-t-\rho}^{r+s} e^{m(\mu+\varrho^{(\rho)}(\nu))}\|\varphi(\mu)\|_{X_\sigma}^2 d\mu \\ &\leqslant \frac{4L_f^2 e^{m\rho_0}}{\kappa(1-\rho_*)}\int_{r-t-\rho}^{r-t} e^{m\mu}\|\varphi(\mu)\|_{X_\sigma}^2 d\mu + \frac{4L_f^2 e^{m\rho_0}}{\kappa(1-\rho_*)}\int_{r-t}^{r+s} e^{m\mu}\|\varphi(\mu)\|_{X_\sigma}^2 d\mu \\ &\leqslant \frac{4L_f^2 e^{m\rho_0}}{m\kappa(1-\rho_*)} e^{m(r-t)}\|\psi_{r-t}\|_{X_\sigma^\rho}^2 + \frac{4L_f^2 e^{m\rho_0}}{\kappa(1-\rho_*)}\int_{r-t}^{r+s} e^{m\mu}\|\varphi(\mu)\|_{X_\sigma}^2 d\mu. \end{aligned} \tag{4.2.17}$$

It follows from (4.1.35), (4.2.16) and (4.2.17) that

$$\|\varphi(r+s, r-t, \theta_{-r}\omega, \psi_{r-t})\|_{X_\sigma}^2 + \frac{\kappa}{2}\int_{r-t}^{r+s} e^{m(\nu-r-s)}\|\varphi(\nu)\|_{X_\sigma}^2 d\nu$$

$$
\begin{aligned}
&+\alpha_1\beta\int_{r-t}^{r+s} e^{m(\nu-r-s)}\|u(\nu)\|_{\sigma,p}^{p}d\nu\\
&\leqslant \hat{c}_{10}e^{m(-t-s)}\|\psi_{r-t}\|^2_{\mathcal{X}_\sigma^\rho}+c_9\int_{r-t}^{r+s} e^{m(\nu-r-s)}(1+\|g(\nu)\|_\sigma^2+\|h(\nu)\|_\sigma^2)d\nu\\
&+c_6\int_{r-t}^{r+s} e^{m(\nu-r-s)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}d\nu. \qquad (4.2.18)
\end{aligned}
$$

By $s\in[-2\rho-1,0]$ and $\rho\in(0,\rho_0]$, we have

$$
\begin{aligned}
&\|\varphi(r+s,r-t,\theta_{-r}\omega,\psi_{r-t})\|^2_{\mathcal{X}_\sigma}+\frac{\kappa}{2}\int_{r-t}^{r+s} e^{m(\nu-r)}\|\varphi(\nu)\|^2_{\mathcal{X}_\sigma}d\nu\\
&+\alpha_1\beta\int_{r-t}^{r+s} e^{m(\nu-r)}\|u(\nu)\|_{\sigma,p}^{p}d\nu\\
&\leqslant \hat{c}_{10}e^{m(2\rho_0+1)}e^{-mt}\|\psi_{r-t}\|^2_{\mathcal{X}_\sigma^\rho}+c_9e^{m(2\rho_0+1)}\int_{r-t}^{r} e^{m(\nu-r)}(1+\|g(\nu)\|_\sigma^2+\|h(\nu)\|_\sigma^2)d\nu\\
&+c_6e^{m(2\rho_0+1)}\int_{r-t}^{r} e^{m(\nu-r)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}d\nu. \qquad (4.2.19)
\end{aligned}
$$

Since $\psi_{r-t}\in\mathcal{D}(r-t,\theta_{-t}\omega)$ and $\mathcal{D}\in\mathfrak{D}$, we obtain that there exists a $T:=T(\tau,\omega,\mathcal{D})\geqslant 3\rho+1$ such that for all $t\geqslant T$,

$$
\sup_{r\leqslant\tau} e^{-mt}\|\psi_{r-t}\|^2_{\mathcal{X}_\sigma^\rho}\leqslant e^{-mt}\sup_{r\leqslant\tau}\|\mathcal{D}(r-t,\theta_{-t}\omega)\|^2_{\mathcal{X}_\sigma^\rho}\leqslant 1,
$$

which, together with (4.2.19), implies that for all $t\geqslant T$

$$
\sup_{r\leqslant\tau}\sup_{s\in[-2\rho-1,0]}\|\varphi(r+s,r-t,\theta_{-r}\omega,\psi_{r-t})\|^2_{\mathcal{X}_\sigma}\leqslant c(1+\Upsilon(\tau)+\eta_\delta(\omega)),
$$

which yields (4.2.1). Letting $s=0$ in (4.2.19) shows (4.2.2) as desired. □

As an immediate consequence of Lemma 4.2.1, we prove $\mathfrak{D}$-backward absorption, which means $\mathfrak{D}$-pullback absorption is uniform with respect to the past time.

Proposition 4.2.2. *Let hypotheses* **E, F1, F2, G1, G2** *and* (4.1.35) *be satisfied. The cocycle* Ψ^δ *associated with the random delay FitzHugh-Nagumo lattice system* (0.0.7) *possesses a* $\mathfrak{D}$*-pullback random absorbing set* $\mathcal{K}_\delta\in\mathfrak{D}$*, given by*

$$
\mathcal{K}_\delta(\tau,\omega)=\{\varphi^\delta=(u^\delta,v^\delta)\in\mathcal{X}_\sigma^\rho:\|\varphi^\delta\|^2_{\mathcal{X}_\sigma^\rho}\leqslant cR_\delta(\tau,\omega)\}, \qquad (4.2.20)
$$

where $R_\delta(\tau,\omega)$ *is the same as in Lemma* 4.2.1*. Moreover,* $\mathcal{K}_\delta$ *is* $\mathfrak{D}$*-backward absorbing set, that is, for each* $(\tau,\omega,\mathcal{D})\in\mathbb{R}\times\Omega\times\mathfrak{D}$*, there is a* $T:=T(\tau,\omega,\mathcal{D})\geqslant 3\rho+1$ *such that*

$$
\Psi^\delta(t,r-t,\theta_{-t}\omega)\mathcal{D}(r-t,\theta_{-t}\omega)\subset\mathcal{K}_\delta(\tau,\omega),\ \forall\, r\leqslant\tau,\ t\geqslant T. \qquad (4.2.21)
$$

Proof. It follows from (4.2.1) in Lemma 4.2.1 that $\mathcal{K}_\delta$ is $\mathfrak{D}$-backward absorbing set as in (4.2.21), which implies the $\mathfrak{D}$-pullback absorbing when $r = \tau$. By the hypothesis **G2** and (4.1.7) in Lemma 4.1.1, we easily obtain $R_\delta(\tau,\omega) = 1 + \Upsilon(\tau) + \eta_\delta(\omega) < \infty$. We then infer from the randomness of $\eta_\delta(\cdot)$ that, for each $\tau \in \mathbb{R}$, $R_\delta(\tau,\omega)$ is random in ω, and thus $\mathcal{K}_\delta(\tau,\cdot)$ is a random set in $\mathcal{X}^\rho_\sigma$.

It suffices to prove that $\mathcal{K}_\delta \in \mathfrak{D}$ for all $\delta > 0$. Since $\tau \to \mathcal{K}_\delta(\tau,\omega)$ is increasing, it follows that

$$\begin{aligned} e^{-\gamma t}\sup_{r\leqslant\tau}\|\mathcal{K}_\delta(r-t,\theta_{-t}\omega)\|^2_{\mathcal{X}^\rho_\sigma} &= e^{-\gamma t}\|\mathcal{K}_\delta(\tau-t,\theta_{-t}\omega)\|^2_{\mathcal{X}^\rho_\sigma} \\ &\leqslant ce^{-\gamma t}(1+\Upsilon(\tau-t)+\eta_\delta(\theta_{-t}\omega)). \end{aligned} \tag{4.2.22}$$

Now, we estimate the last line of (4.2.22). On the one hand, by (4.1.37) in the hypothesis **G2**,

$$\begin{aligned} ce^{-\gamma t}\Upsilon(\tau-t) &\leqslant ce^{-\gamma t}\sup_{r\leqslant\tau-t}\int_{-\infty}^0 e^{m\nu}(\|g(\nu+r)\|^2_\sigma+\|h(\nu+r)\|^2_\sigma)d\nu \\ &\leqslant ce^{-\gamma t}\sup_{r\leqslant\tau}\int_{-\infty}^0 e^{m\nu}(\|g(\nu+r)\|^2_\sigma+\|h(\nu+r)\|^2_\sigma)d\nu \\ &= ce^{-\gamma t}\Upsilon(\tau)\to 0, \end{aligned} \tag{4.2.23}$$

as $t \to +\infty$ in view of $\Upsilon(\tau) < +\infty$. On the other hand, let $\hat\gamma := \min\{\gamma, m\}$, then by (4.1.7) in Lemma 4.1.1, we deduce

$$\begin{aligned} ce^{-\gamma t}\eta_\delta(\theta_{-t}\omega) &= ce^{-\gamma t}\int_{-\infty}^0 e^{m\nu}|\mathcal{G}_\delta(\theta_{\nu-t}\omega)|^{\frac{p}{p-q}}d\nu \\ &\leqslant ce^{-\gamma t}\int_{-\infty}^0 e^{\hat\gamma\nu}|\mathcal{G}_\delta(\theta_{\nu-t}\omega)|^{\frac{p}{p-q}}d\nu \\ &= ce^{-\gamma t}\int_{-\infty}^{-t} e^{\hat\gamma(\nu+t)}|\mathcal{G}_\delta(\theta_{\nu}\omega)|^{\frac{p}{p-q}}d\nu \\ &\leqslant ce^{-(\gamma-\hat\gamma)t}\int_{-\infty}^0 e^{\hat\gamma\nu}|\mathcal{G}_\delta(\theta_{\nu}\omega)|^{\frac{p}{p-q}}d\nu \to 0,\ \text{as } t\to+\infty. \end{aligned} \tag{4.2.24}$$

Using (4.2.23) and (4.2.24) in (4.2.22), we imply

$$e^{-\gamma t}\sup_{r\leqslant\tau}\|\mathcal{K}_\delta(r-t,\theta_{-t}\omega)\|^2_{\mathcal{X}^\rho_\sigma}\to 0,\ \text{as } t\to+\infty. \tag{4.2.25}$$

The desired result is proved. □

Let us now obtain the uniform estimates of solutions in ℓ^p_σ for later purpose.

Lemma 4.2.3. *Let hypotheses* **E, F1, F2, G1, G2** *and* (4.1.35) *be satisfied. Then, for each* $(\tau,\omega,\mathcal{D}) \in \mathbb{R}\times\Omega\times\mathfrak{D}$ *and* $\psi_{r-t} = (\phi_{r-t}, v_{r-t}) \in \mathcal{D}(r-t,\theta_{-t}\omega)$, *there exists a* $T := T(\tau,\omega,\mathcal{D}) \geqslant 3\rho+1$ *such that for all* $t \geqslant T$, *the solution* $\varphi^\delta = (u^\delta, v^\delta)$ *of* (4.1.29) *satisfies for all* $s \in [-\rho, 0]$,

$$\sup_{r\leqslant\tau}\|u^\delta(r+s,r-t,\theta_{-r}\omega,\psi_{r-t})\|^p_{\sigma,p}+\sup_{r\leqslant\tau}\int_{r-\rho}^r\|u^\delta(\nu,r-t,\theta_{-r}\omega,\psi_{r-t})\|^{2p-2}_{\sigma,2p-2}d\nu\leqslant c\widetilde{R}_\delta(\tau,\omega), \tag{4.2.26}$$

where $\widetilde{R}_\delta(\tau,\omega) = R_\delta(\tau,\omega) + \widetilde{\eta}_\delta(\omega)$ *with*

$$\widetilde{\eta}_\delta(\omega) = \int_{-\infty}^{0} e^{m\nu}(|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{2p-2}{p-q}} + |\mathcal{G}_\delta(\theta_\nu\omega)|^p)d\nu, \tag{4.2.27}$$

and $R_\delta(\tau,\omega)$ *is the same as in Lemma 4.2.1.*

Proof. Taking the ℓ^2_σ-inner product of the first equation in (4.1.29) with $|u|^{p-2}u$, where $u := u(\nu, r-t, \theta_{-r}\omega, \psi_{r-t})$, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{d\nu}\|u\|^p_{\sigma,p} + \lambda\|u\|^p_{\sigma,p} + (Au,|u|^{p-2}u)_\sigma =& -\alpha(v,|u|^{p-2}u)_\sigma + (F(x,u),|u|^{p-2}u)_\sigma\\ &+ (f(u(\nu-\varrho^{(\varphi)}(\nu))),|u|^{p-2}u)_\sigma + (g,|u|^{p-2}u)_\sigma\\ &+ \mathcal{G}_\delta(\theta_{\nu-r}\omega)(G(\nu,u),|u|^{p-2}u)_\sigma.\end{aligned} \tag{4.2.28}$$

By $(B(\xi u))_i = (B\xi)_i u_{i+1} + \xi_i(Bu)_i$, (4.1.28) and the fact that $(s_1-s_2)(|s_1|^{p-2}s_1 - |s_2|^{p-2}s_2) \geqslant 0$ for $s_1, s_2 \in \mathbb{R}$, we have

$$\begin{aligned}-(Au,|u|^{p-2}u)_\sigma &= -\sum_{i\in\mathbb{Z}}\xi_i(Au)_i(|u|^{p-2}u)_i = -\sum_{i\in\mathbb{Z}}(Bu)_i(B\xi|u|^{p-2}u)_i\\ &= -\sum_{i\in\mathbb{Z}}(Bu)_i(B\xi)_i|u_{i+1}|^{p-2}u_{i+1} - \sum_{i\in\mathbb{Z}}\xi_i(Bu)_i(B|u|^{p-2}u)_i\\ &\leqslant 2.5^\sigma\sum_{i\in\mathbb{Z}}\xi_i|(Bu)_i||u_{i+1}|^{p-1} - \sum_{i\in\mathbb{Z}}\xi_i(Bu)_i(B|u|^{p-2}u)_i\\ &\leqslant 2.5^\sigma\sum_{i\in\mathbb{Z}}\xi_i(|u_{i+1}|+|u_i|)|u_{i+1}|^{p-1} - \sum_{i\in\mathbb{Z}}\xi_i(u_{i+1}-u_i)(|u_{i+1}|^{p-2}u_{i+1} - |u_i|^{p-2}u_i)\\ &\leqslant 2.5^\sigma\sum_{i\in\mathbb{Z}}\xi_i|u_{i+1}|^p + 2.5^\sigma\sum_{i\in\mathbb{Z}}\xi_i|u_i||u_{i+1}|^{p-1}\\ &\leqslant 2.5^{2\sigma}\|u\|^p_{\sigma,p} + 2.5^\sigma\Big(\frac{1}{p} + \frac{p-1}{p}\times 2.5^\sigma\Big)\|u\|^p_{\sigma,p}\\ &\leqslant 2\times 2.5^{2\sigma}\|u\|^p_{\sigma,p}.\end{aligned} \tag{4.2.29}$$

The Young inequality gives

$$-\alpha(v,|u|^{p-2}u)_\sigma \leqslant \frac{1}{16}\alpha_1\|u\|^{2p-2}_{\sigma,2p-2} + \hat{c}_1\|v\|^2_\sigma, \tag{4.2.30}$$

where α_1 is the number given by (4.1.31) in the hypothesis **F1**. Using (4.1.31) again, and by the Young inequality, we imply

$$\begin{aligned}(F(x,u),|u|^{p-2}u)_\sigma &= \sum_{i\in\mathbb{Z}}\xi_i F_i(u_i)|u_i|^{p-2}u_i\\ &\leqslant \sum_{i\in\mathbb{Z}}\xi_i(-\alpha_1|u_i|^p + \mu_{1,i})|u_i|^{p-2}\\ &= -\alpha_1\|u\|^{2p-2}_{\sigma,2p-2} + \sum_{i\in\mathbb{Z}}\xi_i|u_i|^{p-2}\mu_{1,i}\end{aligned}$$

$$
\begin{aligned}
&\leqslant -\alpha_1\|u\|^{2p-2}_{\sigma,2p-2}+\frac{\alpha_1}{2}\sum_{i\in\mathbb{Z}}\xi_i|u_i|^{2p-2}+\hat{c}_2\sum_{i\in\mathbb{Z}}\xi_i|\mu_{1,i}|^{\frac{2p-2}{p}}\\
&=-\frac{\alpha_1}{2}\|u\|^{2p-2}_{\sigma,2p-2}+\hat{c}_3,
\end{aligned}\tag{4.2.31}
$$

where $\hat{c}_3=\hat{c}_2\|\mu_1\|^{\frac{2p-2}{p}}_{\sigma,\frac{2p-2}{p}}<+\infty$. Using (4.1.34) in the hypothesis **F2** and the Young inequality again,

$$
\begin{aligned}
&(f(u(\nu-\varrho^{(\rho)}(\nu))),|u|^{p-2}u)_\sigma+(g,|u|^{p-2}u)_\sigma\\
&\quad\leqslant\sum_{i\in\mathbb{Z}}\xi_i|f(u_i(\nu-\varrho^{(\rho)}(\nu)))||u_i|^{p-1}+\sum_{i\in\mathbb{Z}}\xi_i g_i|u_i|^{p-1}\\
&\quad\leqslant\frac{1}{16}\alpha_1\sum_{i\in\mathbb{Z}}\xi_i|u_i|^{2p-2}+\hat{c}_4\sum_{i\in\mathbb{Z}}\xi_i|f(u_i(\nu-\varrho^{(\rho)}(\nu)))|^2+\hat{c}_5\|g\|^2_\sigma+\frac{1}{16}\alpha_1\|u\|^{2p-2}_{\sigma,2p-2}\\
&\quad\leqslant\frac{1}{16}\alpha_1\|u\|^{2p-2}_{\sigma,2p-2}+\hat{c}_4L_f^2\sum_{i\in\mathbb{Z}}\xi_i|u_i(\nu-\varrho^{(\rho)}(\nu))|^2+\hat{c}_5\|g\|^2_\sigma+\frac{1}{16}\alpha_1\|u\|^{2p-2}_{\sigma,2p-2}.
\end{aligned}\tag{4.2.32}
$$

According to (4.1.36) in the hypothesis **G1**, we have

$$
\begin{aligned}
\mathcal{G}_\delta(\theta_{\nu-r}\omega)(G(\nu,u),|u|^{p-2}u)_\sigma\leqslant&|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\xi_i|G_i(\nu,u_i)||u_i|^{p-1}\\
\leqslant&\alpha_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|u\|^{p+q-2}_{\sigma,p+q-2}\\
&+|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\xi_i|u_i|^{p-1}|\mu_{4,i}(\nu)|.
\end{aligned}\tag{4.2.33}
$$

By (4.1.21), the second line of (4.2.33) is bounded by

$$
\begin{aligned}
\alpha_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|u\|^{p+q-2}_{\sigma,p+q-2}\leqslant&\alpha_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|\xi\|^{\frac{p-q}{2p-2}}_{\ell^1}\|u\|^{p+q-2}_{\sigma,2p-2}\\
\leqslant&\hat{c}_6|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{2p-2}{p-q}}+\frac{1}{16}\alpha_1\|u\|^{2p-2}_{\sigma,2p-2}.
\end{aligned}\tag{4.2.34}
$$

And we can rewrite the last term in (4.2.33) by

$$
\begin{aligned}
|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\xi_i|u_i|^{p-1}|\mu_{4,i}(\nu)|\leqslant&\frac{\lambda}{2}\|u\|^p_{\sigma,p}+\hat{c}_7|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^p\|\mu_4(\nu)\|^p_{\sigma,p}\\
\leqslant&\frac{\lambda}{2}\|u\|^p_{\sigma,p}+\hat{c}_7|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^p\|\mu_4\|^p_{L^\infty(\mathbb{R},\ell^p_\sigma)}.
\end{aligned}\tag{4.2.35}
$$

Using (4.2.34) and (4.2.35) in (4.2.33), we obtain

$$
\mathcal{G}_\delta(\theta_{\nu-r}\omega)(G(\nu,u),|u|^{p-2}u)_\sigma\leqslant c_6|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{2p-2}{p-q}}+\frac{1}{16}\alpha_1\|u\|^{2p-2}_{\sigma,2p-2}+\frac{\lambda}{2}\|u\|^p_{\sigma,p}+c_8|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^p.\tag{4.2.36}
$$

It follows from (4.2.28)-(4.2.36) that

$$
\frac{d}{d\nu}\|u\|^p_{\sigma,p}+\frac{p}{2}\hat{\lambda}\|u\|^p_{\sigma,p}+\frac{\alpha_1 p}{4}\|u\|^{2p-2}_{\sigma,2p-2}\leqslant\hat{c}_1p\|v\|^2_\sigma+\hat{c}_4L_f^2p\|u(\nu-\varrho^{(\rho)}(\nu))\|^2_\sigma\tag{4.2.37}
$$

$$+\hat{c}_5 p\|g(\nu)\|_\sigma^2 + c_6 p|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{2p-2}{p-q}} + c_8 p|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^p + c_9,$$

where $\hat{\lambda} = \lambda - 4\times 2.5^{2\sigma} > \kappa - \sigma_0 > 0$ in view of (4.1.35). Let $(r,\omega)\in\mathbb{R}\times\Omega,\ \xi\in(r+s-1,r+s)$ for $s\in[-\rho,0]$. Integrating (4.2.37) over $(\xi, r+s)$, we obtain

$$\begin{aligned}\|u(r+s)\|_{\sigma,p}^p \leqslant{}& \|u(\xi)\|_{\sigma,p}^p + \hat{c}_1 p\int_{r-\rho-1}^r \|v(\nu)\|_\sigma^2 d\nu\\ &+ \hat{c}_4 L_f^2 p\int_{r-\rho-1}^r \|u(\nu-\varrho^{(\rho)}(\nu))\|_\sigma^2 d\nu + \hat{c}_5 p\int_{r-\rho-1}^r \|g(\nu)\|_\sigma^2 d\nu\\ &+ \hat{c}_{10}\int_{r-\rho-1}^r (|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{2p-2}{p-q}} + |\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^p)d\nu + \hat{c}_{11}. \qquad (4.2.38)\end{aligned}$$

Integrating (4.2.38) with respect to ξ on $(r+s-1, r+s)$, taking the supremum over $r\in(-\infty,\tau]$, we obtain for all $s\in[-\rho,0]$,

$$\begin{aligned}\sup_{r\leqslant\tau}\|u(r+s)\|_{\sigma,p}^p \leqslant{}&(1+c_1 p)\sup_{r\leqslant\tau}\int_{r-\rho-1}^r (\|u(\nu)\|_{\sigma,p}^p + \|v(\nu)\|_\sigma^2)d\nu\\ &+ \hat{c}_4 L_f^2 p\sup_{r\leqslant\tau}\int_{r-\rho-1}^r \|u(\nu-\varrho^{(\rho)}(\nu))\|_\sigma^2 d\nu + \hat{c}_5 p\sup_{r\leqslant\tau}\int_{r-\rho-1}^r \|g(\nu)\|_\sigma^2 d\nu\\ &+ \hat{c}_{10}\sup_{r\leqslant\tau}\int_{r-\rho-1}^r (|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{2p-2}{p-q}} + |\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^p)d\nu + \hat{c}_{11}. \qquad (4.2.39)\end{aligned}$$

According to (4.2.2) in Lemma 4.2.1, there exists a $T := T(\tau,\omega,\mathcal{D}) \geqslant 3\rho+1$ such that for all $t\geqslant T$,

$$\begin{aligned}&e^{-m(3\rho+1)}\sup_{r\leqslant\tau}\int_{r-\rho-1}^r (\|u(\nu)\|_{\sigma,p}^p + \|v(\nu)\|_\sigma^2)d\nu\\ &\leqslant \sup_{r\leqslant\tau}\int_{r-3\rho-1}^r e^{m(\nu-r)}(\|u(\nu)\|_{\sigma,p}^p + \|v(\nu)\|_\sigma^2)d\nu\\ &\leqslant \sup_{r\leqslant\tau}\int_{r-t}^r e^{m(\nu-r)}(\|u(\nu)\|_{\sigma,p}^p + \|v(\nu)\|_\sigma^2)d\nu \leqslant cR_\delta(\tau,\omega). \qquad (4.2.40)\end{aligned}$$

By (4.2.1) in Lemma 4.2.1, we imply for all $t\geqslant T$,

$$\begin{aligned}\sup_{r\leqslant\tau}\int_{r-\rho-1}^r \|u(\nu-\varrho^{(\rho)}(\nu))\|_\sigma^2 d\nu &\leqslant (\rho+1)\sup_{r\leqslant\tau}\sup_{r-\rho-1\leqslant\nu\leqslant r}\|u(\nu-\varrho^{(\rho)}(\nu))\|_\sigma^2\\ &\leqslant c(\rho+1)R_\delta(\tau,\omega). \qquad (4.2.41)\end{aligned}$$

The hypothesis **G2** gives

$$\begin{aligned}\sup_{r\leqslant\tau}\int_{r-\rho-1}^r \|g(\nu)\|_\sigma^2 d\nu ={}& \sup_{r\leqslant\tau}\int_{-\rho-1}^0 \|g(\nu+r)\|_\sigma^2 d\nu\\ \leqslant{}& e^{m(\rho+1)}\sup_{r\leqslant\tau}\int_{-\infty}^0 e^{m\nu}\|g(\nu+r)\|_\sigma^2 d\nu\end{aligned}$$

$$\leqslant e^{m(\rho_0+1)}\Upsilon(\tau) < +\infty. \tag{4.2.42}$$

Note that

$$\begin{aligned} &\sup_{r\leqslant\tau}\int_{r-\rho-1}^{r}(|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{2p-2}{p-q}}+|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{p})d\nu \\ &\quad\leqslant e^{m(\rho+1)}\sup_{r\leqslant\tau}\int_{-\infty}^{0}e^{m\nu}(|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{2p-2}{p-q}}+|\mathcal{G}_\delta(\theta_\nu\omega)|^{p})d\nu. \end{aligned}\tag{4.2.43}$$

It follows from (4.2.39)-(4.2.43) that for all $t\geqslant T$ and $s\in[-\rho,0]$,

$$\sup_{r\leqslant\tau}\|u(r+s)\|_{\sigma,p}^{p}\leqslant \hat{c}_{12}R_\delta(\tau,\omega)+\hat{c}_{13}\int_{-\infty}^{0}e^{m\nu}(|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{2p-2}{p-q}}+|\mathcal{G}_\delta(\theta_\nu\omega)|^{p})d\nu. \tag{4.2.44}$$

Then, integrating (4.2.37) over $(r-\rho,r)$ and taking the supremum over $r\in(-\infty,\tau]$ such that for all $t\geqslant T$,

$$\begin{aligned} &\frac{\alpha_1 p}{4}\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|u(\nu)\|_{\sigma,2p-2}^{2p-2}d\nu \\ &\quad\leqslant \sup_{r\leqslant\tau}\|u(r-\rho,r-t,\theta_{-r}\omega,\phi_{r-t})\|_{\sigma,p}^{p}+\hat{c}_1 p\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|v(\nu)\|_{\sigma}^{2}d\nu \\ &\qquad+\hat{c}_4L_f^2 p\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|u(\nu-\varrho^{(\rho)}(\nu))\|_{\sigma}^{2}d\nu+\hat{c}_5 p\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|g(\nu)\|_{\sigma}^{2}d\nu \\ &\qquad+c_6 p\sup_{r\leqslant\tau}\int_{r-\rho}^{r}|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{2p-2}{p-q}}d\nu+c_8 p\sup_{r\leqslant\tau}\int_{r-\rho}^{r}|\mathcal{G}_\delta(\theta_{\nu-r})|^{p}d\nu+c_9\rho, \end{aligned}\tag{4.2.45}$$

which, along with (4.2.40)-(4.2.44), yields (4.2.26) as desired. □

Next, we derive uniform tail-estimates of solutions.

Backward uniform tail-estimates

Assume that $\iota:\mathbb{R}^+\to[0,1]$ is a smooth function such that

$$\iota(s)=\begin{cases}0, & \text{if } 0\leqslant s\leqslant 1,\\ 1, & \text{if } s\geqslant 2.\end{cases}\tag{4.2.46}$$

Let $\iota_{k,i}:=\iota(\frac{|i|}{k})$ for each $k\geqslant 1$ and $i\in\mathbb{Z}$. It is not hard to check that $\iota_k=(\iota_{k,i})_{i\in\mathbb{Z}}\in\ell^\infty$ and for all $k\geqslant 1,\ i\in\mathbb{Z}$,

$$|\iota_{k,i+1}-\iota_{k,i}|\leqslant\frac{c_*}{k}. \tag{4.2.47}$$

Lemma 4.2.4. *Let hypotheses* **E, F1, F2, G1-G3** *and* (4.1.35) *be satisfied. Then, for each* $(\tau, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}$*, then the solution* $\varphi^\delta = (u^\delta, v^\delta)$ *to* (4.1.29) *satisfies*

$$\lim_{k,t \to +\infty} \sup_{r \leqslant \tau} \sup_{s \in [-\rho, 0]} \|\varphi^\delta(r+s, r-t, \theta_{-r}\omega, \psi_{r-t})\|^2_{X_\sigma(|x| \geqslant k)} = 0 \tag{4.2.48}$$

uniformly in $\psi_{r-t} = (\phi_{r-t}, \upsilon_{r-t}) \in \mathcal{D}(r-t, \theta_{-t}\omega)$*. Moreover, the convergence in* (4.2.48) *is uniform with respect to large* δ*, that is, there exists a* $\delta_0 := \delta_0(\omega)$ *which is independent of* $\tau, \mathcal{D}$ *such that*

$$\lim_{k,t \to +\infty} \sup_{\delta \geqslant \delta_0} \sup_{r \leqslant \tau} \sup_{s \in [-\rho, 0]} \|\varphi^\delta(r+s, r-t, \theta_{-r}\omega, \psi_{r-t})\|^2_{X_\sigma(|x| \geqslant k)} = 0. \tag{4.2.49}$$

Proof. Taking the inner product of (4.1.29) with $(2\beta\iota_{k,i}\xi_i u_i(\nu), 2\alpha\iota_{k,i}\xi_i v_i(\nu)) := (2\beta\iota_{k,i}\xi_i u_i(\nu, r-t, \theta_{-r}\omega, u_{r-t}), 2\alpha\iota_{k,i}\xi_i v_i(\nu, r-t, \theta_{-r}\omega, v_{r-t}))$ and summing up the product over $i \in \mathbb{Z}$, it follows

$$\begin{aligned}
&\frac{d}{d\nu} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(\beta|u_i|^2 + \alpha|v_i|^2) + 2\kappa \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(\beta|u_i|^2 + \alpha|v_i|^2) + 2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(Au)_i u_i \\
&\quad = 2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i F_i(u_i)u_i + 2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i f(u_i(\nu - \varrho^{(\rho)}(\nu)))u_i \\
&\qquad + 2\alpha \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i f(v_i(\nu - \varrho^{(\rho)}(\nu)))v_i + 2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i g_i(\nu)u_i \\
&\qquad + 2\alpha \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i h_i(\nu)v_i + 2\beta\mathcal{G}_\delta(\theta_{\nu-r}\omega) \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i G_i(\nu, u_i)u_i,
\end{aligned} \tag{4.2.50}$$

where we recall that $\kappa = \min\{\lambda, \varsigma\}$. By $(\iota_{k,i}\xi_i u_i, (Au)_i) = ((B\iota_k\xi u)_i, (Bu)_i) = (\iota_{k,i+1}\xi_{i+1}u_{i+1} - \iota_{k,i}\xi_i u_i, (Bu)_i)$ and $(B(\xi u))_i = (B\xi)_i u_{i+1} + \xi_i(Bu)_i$, we obtain

$$\begin{aligned}
-2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(Au)_i u_i &= 2\beta \sum_{i\in\mathbb{Z}} (\iota_{k,i}\xi_i u_i - \iota_{k,i+1}\xi_{i+1}u_{i+1})(Bu)_i \\
&= 2\beta \sum_{i\in\mathbb{Z}} (\iota_{k,i} - \iota_{k,i+1})\xi_{i+1}u_{i+1}(Bu)_i - 2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}(\xi_{i+1}u_{i+1} - \xi_i u_i)(Bu)_i \\
&= 2\beta \sum_{i\in\mathbb{Z}} (\iota_{k,i} - \iota_{k,i+1})\xi_{i+1}u_{i+1}(Bu)_i - 2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}(B\xi)_i u_{i+1}(Bu)_i \\
&\quad - 2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i|(Bu)_i|^2.
\end{aligned} \tag{4.2.51}$$

By (4.2.47) and $\xi_{i+1} \leqslant 2.5^\sigma \xi_i$ as in (4.1.28), we deduce

$$\begin{aligned}
2\beta \sum_{i\in\mathbb{Z}} (\iota_{k,i} - \iota_{k,i+1})\xi_{i+1}u_{i+1}(Bu)_i &\leqslant \frac{2\beta c_*}{k} \sum_{i\in\mathbb{Z}} \xi_{i+1}(|u_{i+1}|^2 + |u_{i+1}||u_i|) \\
&\leqslant \frac{2\beta c_*}{k} \sum_{i\in\mathbb{Z}} \xi_{i+1}\Big(\frac{3}{2}|u_{i+1}|^2 + \frac{1}{2}|u_i|^2\Big) \\
&\leqslant \frac{3\beta c_*}{k}\|u\|^2_\sigma + 2.5^\sigma \frac{\beta c_*}{k}\|u\|^2_\sigma
\end{aligned}$$

$$= (3 + 2.5^{\sigma})\frac{\beta c_*}{k}\|u\|_{\sigma}^2. \tag{4.2.52}$$

By (4.2.47) and (4.1.28) again, we have

$$\begin{aligned}
&-2\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}(B\xi)_i u_{i+1}(Bu)_i - 2\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|(Bu)_i|^2\\
&\leqslant 2\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}2.5^{\sigma}\xi_i|u_{i+1}||(Bu)_i| - 2\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|(Bu)_i|^2\\
&\leqslant \beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(2.5^{2\sigma}|u_{i+1}|^2 + |(Bu)_i|^2) - 2\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|(Bu)_i|^2\\
&\leqslant 2.5^{2\sigma}\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_{i+1}|^2 \leqslant 2.5^{3\sigma}\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_{i+1}|u_{i+1}|^2\\
&= 2.5^{3\sigma}\beta\sum_{i\in\mathbb{Z}}\iota_{k,i+1}\xi_{i+1}|u_{i+1}|^2 + 2.5^{3\sigma}\beta\sum_{i\in\mathbb{Z}}(\iota_{k,i} - \iota_{k,i+1})\xi_{i+1}|u_{i+1}|^2\\
&\leqslant 2.5^{3\sigma}\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^2 + 2.5^{3\sigma}\frac{\beta c_*}{k}\|u\|_{\sigma}^2.
\end{aligned} \tag{4.2.53}$$

Using (4.2.52) and (4.2.53) in (4.2.51), we deduce

$$-2\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(Au)_i u_i \leqslant 2.5^{3\sigma}\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^2 + \frac{\hat{c}_1}{k}\|u\|_{\sigma}^2, \tag{4.2.54}$$

where $\hat{c}_1 = \beta(3 + 2.5^{\sigma} + 2.5^{3\sigma})c_*$. By (4.1.31) in the hypothesis **F1**, we imply

$$2\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i F_i(u_i)u_i \leqslant -2\alpha_1\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^p + 2\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|\mu_{1,i}|. \tag{4.2.55}$$

Applying the Young inequality and using (4.1.34) in the hypothesis **F2**, we yield

$$\begin{aligned}
&2\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i f(u_i(\nu - \varrho^{(\rho)}(\nu)))u_i + 2\alpha\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i f(v_i(\nu - \varrho^{(\rho)}(\nu)))v_i\\
&\leqslant \frac{4L_f^2}{\kappa}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\nu - \varrho^{(\rho)}(\nu))|^2 + \alpha|v_i(\nu - \varrho^{(\rho)}(\nu))|^2\Big) + \frac{\kappa}{4}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i|^2 + \alpha|v_i|^2).
\end{aligned} \tag{4.2.56}$$

The Young inequality gives

$$\begin{aligned}
&2\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i g_i(\nu)u_i + 2\alpha\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i h_i(\nu)v_i\\
&\leqslant \hat{c}_2\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(|g_i(\nu)|^2 + |h_i(\nu)|^2) + \frac{\kappa}{4}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i|^2 + \alpha|v_i|^2).
\end{aligned} \tag{4.2.57}$$

According to (4.1.36) in the hypothesis **G1**, the last term of (4.2.50) is bounded by

$$2\beta\mathcal{G}_\delta(\theta_{\nu-r}\omega)\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i G_i(\nu, u_i)u_i$$

$$
\begin{aligned}
&\leqslant 2\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|G_i(\nu,u_i)||u_i|\\
&\leqslant 2\beta\alpha_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^q+2\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|\mu_{4,i}(\nu)||u_i|\\
&\leqslant \hat{c}_3|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^q+\frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|\mu_{4,i}(\nu)|^{\hat{q}}, \qquad (4.2.58)
\end{aligned}
$$

where $\hat{c}_3 = 2\beta\alpha_4 + \frac{2}{q}\beta$, we recall that $\frac{1}{\hat{q}} + \frac{1}{q} = 1$. Now, we estimate the last two terms in (4.2.58), respectively. On the one hand, by the Young inequality and the same method as in the proof of (4.1.21), we imply

$$
\begin{aligned}
\hat{c}_3|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^q &= \hat{c}_3\sum_{i\in\mathbb{Z}}\Big(\iota_{k,i}^{\frac{q}{p}}\xi_i^{\frac{q}{p}}|u_i|^q\Big)\Big(\iota_{k,i}^{\frac{p-q}{p}}\xi_i^{\frac{p-q}{p}}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\Big)\\
&\leqslant \frac{1}{2}\alpha_1\beta\sum_{i\in\mathbb{Z}}\Big(\iota_{k,i}^{\frac{q}{p}}\xi_i^{\frac{q}{p}}|u_i|^q\Big)^{\frac{p}{q}}+\hat{c}_4\sum_{i\in\mathbb{Z}}\Big(\iota_{k,i}^{\frac{p-q}{p}}\xi_i^{\frac{p-q}{p}}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\Big)^{\frac{p}{p-q}}\\
&= \frac{1}{2}\alpha_1\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^p+\hat{c}_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i, \qquad (4.2.59)
\end{aligned}
$$

where $\hat{c}_4 = \hat{c}_4(p, q, \hat{c}_3, \beta, \alpha_1)$. On the other hand, note that $q \geqslant 2$ and so $\hat{q} \leqslant 2 \leqslant q < p$, we have

$$
\begin{aligned}
\frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|\mu_{4,i}(\nu)|^{\hat{q}} &\leqslant \frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\Big(\iota_{k,i}^{\frac{\hat{q}}{p}}\xi_i^{\frac{\hat{q}}{p}}|\mu_{4,i}(\nu)|^{\hat{q}}\Big)\Big(\iota_{k,i}^{\frac{p-\hat{q}}{p}}\xi_i^{\frac{p-\hat{q}}{p}}\Big)\\
&\leqslant \frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\Big(\iota_{k,i}\xi_i|\mu_{4,i}(\nu)|^p\Big)^{\frac{\hat{q}}{p}}\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}\\
&\leqslant \frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|\mu_4(\nu)\|_{\sigma,p}^{\hat{q}}\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}\\
&\leqslant \hat{c}_5|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}, \qquad (4.2.60)
\end{aligned}
$$

where $\hat{c}_5 = \frac{2}{\hat{q}}\beta\|\mu_4\|^{\hat{q}}_{L^\infty(\mathbb{R},\ell^p_\sigma)} < +\infty$. Using (4.2.59) and (4.2.60) in (4.2.58), we obtain

$$
\begin{aligned}
2\beta\mathcal{G}_\delta(\theta_{\nu-r}\omega)\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_iG_i(\nu,u_i)u_i \leqslant& \frac{1}{2}\alpha_1\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^p+\hat{c}_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\\
&+\hat{c}_5|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}. \qquad (4.2.61)
\end{aligned}
$$

From the above estimates, (4.2.50) can be rewritten:

$$
\frac{d}{d\nu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i|^2+\alpha|v_i|^2)+\kappa\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i|^2+\alpha|v_i|^2)
$$

$$
\begin{aligned}
&+\frac{\kappa}{2}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i|^2+\alpha|v_i|^2)+\frac{3}{2}\alpha_1\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^p \\
&\leqslant 2.5^{3\sigma}\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^2+\frac{\hat{c}_1}{k}\|u\|_\sigma^2+\frac{4L_f^2}{\kappa}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\nu-\varrho^{(\rho)}(\nu))|^2+\alpha|v_i(\nu-\varrho^{(\rho)}(\nu))|^2\Big) \\
&\quad+\hat{c}_6\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(|\mu_{1,i}(\nu)|+|g_i(\nu)|^2+|h_i(\nu)|^2) \\
&\quad+\hat{c}_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i+\hat{c}_5|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}, \qquad (4.2.62)
\end{aligned}
$$

where $\hat{c}_6=2\beta+\hat{c}_2$. Note that

$$
\begin{aligned}
2.5^{3\sigma}\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^2&=2.5^{3\sigma}\beta\sum_{i\in\mathbb{Z}}(\iota_{k,i}^{\frac{p-2}{p}}\xi_i^{\frac{p-2}{p}})(\iota_{k,i}^{\frac{2}{p}}\xi_i^{\frac{2}{p}}|u_i|^2) \\
&\leqslant\frac{1}{2}\alpha_1\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^p+\hat{c}_7\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i. \qquad (4.2.63)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{d}{d\nu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i|^2+\alpha|v_i|^2)+\kappa\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i|^2+\alpha|v_i|^2) \\
&\quad+\frac{\kappa}{2}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i|^2+\alpha|v_i|^2)+\alpha_1\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^p \\
&\leqslant\frac{\hat{c}_1}{k}\|u\|_\sigma^2+\frac{4L_f^2}{\kappa}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\nu-\varrho^{(\rho)}(\nu))|^2+\alpha|v_i(\nu-\varrho^{(\rho)}(\nu))|^2\Big) \\
&\quad+\hat{c}_8\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(1+|\mu_{1,i}(\nu)|+|g_i(\nu)|^2+|h_i(\nu)|^2) \\
&\quad+\hat{c}_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i+\hat{c}_5|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}. \qquad (4.2.64)
\end{aligned}
$$

Multiplying (4.2.64) by $e^{m\nu}$ and integrating it about $\nu\in[r-t,r+s]$, where $r\leqslant\tau$ and $s\in[-\rho,0]$, we deduce

$$
\begin{aligned}
&e^{m(r+s)}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i(r+s)|^2+\alpha|v_i(r+s)|^2)+\frac{\kappa}{2}\int_{r-t}^{r+s}e^{m\nu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i(\nu)|^2+\alpha|v_i(\nu)|^2)d\nu \\
&\quad+\alpha_1\beta\int_{r-t}^{r+s}e^{m\nu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i(\nu)|^p d\nu \\
&\leqslant e^{m(r-t)}\Big(\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_{r-t,i}|^2+\alpha\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|v_{r-t,i}|^2\Big) \\
&\quad+(m-\kappa)\int_{r-t}^{r+s}e^{m\nu}\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i(\nu)|^2+\alpha|v_i(\nu)|^2)\Big)d\nu
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4L_f^2}{\kappa}\int_{r-t}^{r+s} e^{m\nu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\nu-\varrho^{(\rho)}(\nu))|^2+\alpha|v_i(\nu-\varrho^{(\rho)}(\nu))|^2\Big)d\nu\\
&+\frac{\hat{c}_1}{k}\int_{r-t}^{r+s} e^{m\nu}\|u(\nu)\|_\sigma^2 d\nu+\hat{c}_8\int_{r-t}^{r+s} e^{m\nu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(1+|\mu_{1,i}(\nu)|+|g_i(\nu)|^2+|h_i(\nu)|^2)d\nu\\
&+\hat{c}_4\int_{r-t}^{r+s} e^{m\nu}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}d\nu\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i+\hat{c}_5\int_{r-t}^{r+s} e^{m\nu}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|d\nu\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}.
\end{aligned}
\tag{4.2.65}
$$

For the third line of (4.2.65), we easily deduce

$$
e^{m(r-t)}\Big(\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_{r-t,i}|^2+\alpha\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|v_{r-t,i}|^2\Big)\leqslant e^{m(r-t)}\|\psi_{r-t}(0)\|_{\mathcal{X}_\sigma}^2.
\tag{4.2.66}
$$

The fifth line of (4.2.65) is bounded by

$$
\begin{aligned}
&\frac{4L_f^2}{\kappa}\int_{r-t}^{r+s} e^{m\nu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\nu-\varrho^{(\rho)}(\nu))|^2+\alpha|v_i(\nu-\varrho^{(\rho)}(\nu))|^2\Big)d\nu\\
&\leqslant\frac{4L_f^2}{\kappa(1-\rho_*)}\int_{r-t-\rho}^{r+s} e^{m(\mu+\varrho^{(\rho)}(\nu))}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\mu)|^2+\alpha|v_i(\mu)|^2\Big)d\mu\\
&\leqslant\frac{4L_f^2e^{m\rho_0}}{\kappa(1-\rho_*)}\int_{r-t-\rho}^{r-t} e^{m\mu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\mu)|^2+\alpha|v_i(\mu)|^2\Big)d\mu\\
&\quad+\frac{4L_f^2e^{m\rho_0}}{\kappa(1-\rho_*)}\int_{r-t}^{r+s} e^{m\mu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\mu)|^2+\alpha|v_i(\mu)|^2\Big)d\mu\\
&\leqslant\frac{4L_f^2e^{m\rho_0}}{m\kappa(1-\rho_*)}e^{m(r-t)}\|\psi_{r-t}\|_{\mathcal{X}_\sigma^\rho}^2\\
&\quad+\frac{4L_f^2e^{m\rho_0}}{\kappa(1-\rho_*)}\int_{r-t}^{r+s} e^{m\mu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\mu)|^2+\alpha|v_i(\mu)|^2\Big)d\mu.
\end{aligned}
\tag{4.2.67}
$$

By (4.1.35) and (4.2.67), we can rewrite (4.2.65) by

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i(r+s)|^2+\alpha|v_i(r+s)|^2)+\frac{\kappa}{2}\int_{r-t}^{r+s} e^{m(\nu-r-s)}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i(\nu)|^2+\alpha|v_i(\nu)|^2)d\nu\\
&\quad+\alpha_1\beta\int_{r-t}^{r+s} e^{m(\nu-r-s)}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i(\nu)|^p d\nu\\
&\leqslant\hat{c}_9e^{m(-t-s)}\|\psi_{r-t}\|_{\mathcal{X}_\sigma^\rho}^2+\frac{\hat{c}_1}{k}\int_{r-t}^{r+s} e^{m(\nu-r-s)}\|u(\nu)\|_\sigma^2 d\nu\\
&\quad+\hat{c}_8\int_{r-t}^{r+s} e^{m(\nu-r-s)}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(1+|\mu_{1,i}(\nu)|+|g_i(\nu)|^2+|h_i(\nu)|^2)d\nu\\
&\quad+\hat{c}_4\int_{r-t}^{r+s} e^{m(\nu-r-s)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}d\nu\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i
\end{aligned}
\tag{4.2.68}
$$

$$+\hat{c}_5\int_{r-t}^{r+s}e^{m(\nu-r-s)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|d\nu\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}.$$

By $s\in[-\rho,0]$ and $\rho\in(0,\rho_0]$, we have

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i(r+s)|^2+\alpha|v_i(r+s)|^2)+\frac{\kappa}{2}\int_{r-t}^{r+s}e^{m(\nu-r)}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i(\nu)|^2+\alpha|v_i(\nu)|^2)d\nu\\
&\quad+\alpha_1\beta\int_{r-t}^{r+s}e^{m(\nu-r)}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i(\nu)|^pd\nu\\
&\leqslant\hat{c}_9e^{m\rho_0}e^{-mt}\|\psi_{r-t}\|^2_{X^\rho_\sigma}+\hat{c}_1e^{m\rho_0}\frac{1}{k}\int_{r-t}^{r}e^{m(\nu-r)}\|u(\nu)\|^2_\sigma d\nu\\
&\quad+\hat{c}_8e^{m\rho_0}\int_{r-t}^{r}e^{m(\nu-r)}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(1+|\mu_{1,i}(\nu)|+|g_i(\nu)|^2+|h_i(\nu)|^2)d\nu\\
&\quad+\hat{c}_4e^{m\rho_0}\int_{r-t}^{r}e^{m(\nu-r)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}d\nu\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\\
&\quad+\hat{c}_5e^{m\rho_0}\int_{r-t}^{r}e^{m(\nu-r)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|d\nu\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}.
\end{aligned}\tag{4.2.69}$$

Since $\psi_{r-t}=(\phi_{r-t},\upsilon_{r-t})\in\mathcal{D}(r-t,\theta_{-t}\omega)$, we imply

$$e^{-mt}\|\psi_{r-t}\|^2_{X^\rho_\sigma}\leqslant e^{-mt}\sup_{r\leqslant\tau}\|\mathcal{D}(r-t,\theta_{-t}\omega)\|^2_{X^\rho_\sigma}\to0,\ \text{as } t\to\infty.\tag{4.2.70}$$

By (4.2.1) in Lemma 4.2.1, since $\Upsilon(\tau)$, $\eta_\delta(\omega)<+\infty$ such that for each $\delta>0$,

$$\frac{1}{k}\sup_{r\leqslant\tau}\int_{r-t}^{r}e^{m(\nu-r)}\|u(\nu)\|^2_\sigma d\nu\leqslant\frac{c}{k}R_\delta(\tau,\omega)\to0,\ \text{as } k,\ t\to+\infty.\tag{4.2.71}$$

According to (4.1.38) in the hypothesis **G3**, we obtain

$$\begin{aligned}
&\sup_{r\leqslant\tau}\int_{r-t}^{r}e^{m(\nu-r)}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(1+|\mu_{1,i}(\nu)|+|g_i(\nu)|^2+|h_i(\nu)|^2)d\nu\\
&\quad\leqslant\sup_{r\leqslant\tau}\int_{-\infty}^{0}e^{m\nu}\sum_{|i|\geqslant k}\xi_i(1+|\mu_{1,i}(\nu+r)|+|g_i(\nu+r)|^2+|h_i(\nu+r)|^2)d\nu\to0,
\end{aligned}\tag{4.2.72}$$

as $k,t\to+\infty$. It follows from (4.1.7) in Lemma 4.1.1 and $\xi\in\ell^1$ that for all $\delta>0$,

$$\sup_{r\leqslant\tau}\int_{r-t}^{r}e^{m(\nu-r)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}d\nu\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\leqslant\int_{-\infty}^{0}e^{m\nu}|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}}d\nu\sum_{|i|\geqslant k}\xi_i\to0,\tag{4.2.73}$$

as $k,t\to+\infty$. In fact, the convergence of (4.2.73) is uniform convergence for large δ. By (4.1.7) in Lemma 4.1.1, there exists a $\delta_1=\delta_1(\omega)>0$ such that

$$\eta_\delta(\omega)=\int_{-\infty}^{0}e^{m\nu}|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}}d\nu\leqslant1,\ \forall\,\delta\geqslant\delta_1,\tag{4.2.74}$$

which implies that

$$\sup_{\delta\geqslant\delta_1}\sup_{r\leqslant\tau}\int_{r-t}^{r} e^{m(\nu-r)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}d\nu\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i \leqslant \sum_{|i|\geqslant k}\xi_i \to 0, \text{ as } k,t\to+\infty. \tag{4.2.75}$$

Using the same method, we obtain for all $\delta > 0$,

$$\sup_{r\leqslant\tau}\int_{r-t}^{r} e^{m(\nu-r)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|d\nu\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}} \leqslant \sup_{r\leqslant\tau}\int_{-\infty}^{0} e^{m\nu}|\mathcal{G}_\delta(\theta_{\nu}\omega)|d\nu\Big(\sum_{|i|\geqslant k}\xi_i\Big)^{\frac{p-\hat{q}}{p}} \to 0, \tag{4.2.76}$$

as $k,t\to+\infty$. And the above convergence is also uniform convergence for large δ. More precisely, by (4.1.7) in Lemma 4.1.1 again, $\int_{-\infty}^0 e^{mr}|\mathcal{G}_\delta(\theta_r\omega)|dr \to 0$ as $\delta\to+\infty$, hence, there exists a $\delta_2=\delta_2(\omega) > 0$ such that

$$\sup_{\delta\geqslant\delta_2}\sup_{r\leqslant\tau}\int_{r-t}^{r} e^{m(\nu-r)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|d\nu\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}} \leqslant \Big(\sum_{|i|\geqslant k}\xi_i\Big)^{\frac{p-\hat{q}}{p}} \to 0, \tag{4.2.77}$$

as $k,t\to+\infty$. It follows from (4.2.69)-(4.2.77) that

$$\begin{aligned}&\sup_{r\leqslant\tau}\sup_{s\in[-\rho,0]}\sum_{|i|\geqslant 2k}\xi_i(\beta|u_i^\delta(r+s)|^2+\alpha|v_i^\delta(r+s)|^2) \\ &\leqslant \sup_{r\leqslant\tau}\sup_{s\in[-\rho,0]}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i^\delta(r+s)|^2+\alpha|v_i^\delta(r+s)|^2) \to 0, \text{ as } k,t\to+\infty,\end{aligned} \tag{4.2.78}$$

for all $\delta > 0$ and uniformly in large δ. This completes the proof. □

Backward asymptotic compactness of solutions and existence of pullback random attractors

The following lemma is useful for verifying the asymptotic compactness of solutions.

Lemma 4.2.5. *Let hypotheses* **E, F1, F2, G1, G2** *and* (4.1.35) *be satisfied. Then, for each* $(\tau,\omega,\mathcal{D})\in\mathbb{R}\times\Omega\times\mathfrak{D}$ *and* $\psi_{r-t}=(\phi_{r-t},\upsilon_{r-t})\in\mathcal{D}(r-t,\theta_{-t}\omega)$*, there exists a* $T:=T(\tau,\omega,\mathfrak{D})\geqslant 3\rho+1$ *such that for all* $t\geqslant T$*, the solution* $\varphi^\delta=(u^\delta,v^\delta)$ *to* (4.1.29) *satisfies*

$$\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\Big\|\frac{d}{d\nu}u^\delta(\nu,r-t,\theta_{-r}\omega,\psi_{r-t})\Big\|_\sigma^2 d\nu+\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\Big\|\frac{d}{d\nu}v^\delta(\nu,r-t,\theta_{-r}\omega,\psi_{r-t})\Big\|_\sigma^2 d\nu \leqslant c\widetilde{R}_\delta(\tau,\omega), \tag{4.2.79}$$

where $\widetilde{R}_\delta(\tau,\omega)$ *is given by Lemma* 4.2.3.

Proof. Multiplying the first equation in (4.1.29) with $du/d\nu$, where $u(\nu):=u(\nu,r-t,\theta_{-r}\omega,\psi_{r-t})$, we obtain

$$\begin{aligned}\Big\|\frac{du}{d\nu}\Big\|_\sigma^2 \leqslant & c\|Au\|_\sigma^2+c\|u\|_\sigma^2+c\|v\|_\sigma^2+c\|F(u(\nu))\|_\sigma^2+c\|f(u(\nu-\varrho^{(\rho)}(\nu)))\|_\sigma^2 \\ &+c\|g(\nu)\|_\sigma^2+c|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^2\|G(\nu,u)\|_\sigma^2 \\ \leqslant & c\|u\|_\sigma^2+c\|v\|_\sigma^2+c\|F(u(\nu))\|_\sigma^2+c\|f(u(\nu-\varrho^{(\rho)}(\nu)))\|_\sigma^2\end{aligned}$$

$$+ c\|g(\nu)\|_\sigma^2 + c|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^2\|G(\nu,u)\|_\sigma^2. \tag{4.2.80}$$

Integrating from $r-\rho$ to r and taking the supremum over $r \in (-\infty, \tau]$, we deduce

$$\begin{aligned}
&\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\left\|\frac{d}{d\nu}u(\nu, r-t, \theta_{-r}\omega, \psi_{r-t})\right\|_\sigma^2 d\nu \\
&\leqslant c\sup_{r\leqslant\tau}\int_{r-\rho}^{r}(\|u(\nu)\|_\sigma^2 + \|v(\nu)\|_\sigma^2)d\nu + c\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|F(u(\nu))\|_\sigma^2 d\nu \\
&\quad + c\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|f(u(\nu-\varrho^{(\rho)}(\nu)))\|_\sigma^2 d\nu + c\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|g(\nu)\|_\sigma^2 d\nu \\
&\quad + c\sup_{r\leqslant\tau}\int_{r-\rho}^{r}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^2\|G(\nu,u)\|_\sigma^2 d\nu.
\end{aligned} \tag{4.2.81}$$

By (4.2.2) in Lemma 4.2.1, there exists $T := T(\tau,\omega,\mathcal{D}) \geqslant 3\rho+1$ such that for all $t \geqslant T$, the first term on the right-hand side of (4.2.81) is bounded by

$$\begin{aligned}
\sup_{r\leqslant\tau}\int_{r-\rho}^{r}(\|u(\nu)\|_\sigma^2 + \|v(\nu)\|_\sigma^2)d\nu &\leqslant ce^{m(3\rho+1)}\sup_{r\leqslant\tau}\int_{r-3\rho-1}^{r}e^{m(\nu-r)}\|\varphi(\nu)\|_\sigma^2 d\nu \\
&\leqslant ce^{m(3\rho_0+1)}R_\delta(\tau,\omega).
\end{aligned} \tag{4.2.82}$$

By (4.1.32) in the hypothesis **F1** and (4.2.26) in Lemma 4.2.3, we have

$$\begin{aligned}
\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|F(u(\nu))\|_\sigma^2 d\nu \leqslant &\sup_{r\leqslant\tau}\int_{r-\rho}^{r}(2\alpha_2^2\|u\|_{\sigma,2p-2}^{2p-2} + 2\|\mu_2\|_\sigma^2)d\nu \\
\leqslant &2\alpha_2^2\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|u\|_{\sigma,2p-2}^{2p-2}d\nu + c \\
\leqslant &2\alpha_2^2 c\widetilde{R}_\delta(\tau,\omega) + c,
\end{aligned} \tag{4.2.83}$$

where we used $\mu_2 \in \ell_\sigma^2$. According to (4.1.34) in the hypothesis **F2** and (4.2.41), we obtain

$$\begin{aligned}
\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|f(u(\nu-\varrho^{(\rho)}(\nu)))\|_\sigma^2 d\nu \leqslant &L_f^2\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|u(\nu-\varrho^{(\rho)}(\nu))\|_\sigma^2 d\nu \\
\leqslant &cL_f^2(\rho+1)R_\delta(\tau,\omega).
\end{aligned} \tag{4.2.84}$$

As done in (4.2.42), we have

$$\begin{aligned}
\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|g(\nu)\|_\sigma^2 d\nu &\leqslant e^{m(\rho+1)}\sup_{r\leqslant\tau}\int_{-\infty}^{0}e^{m\nu}\|g(\nu+r)\|_\sigma^2 d\nu \\
&\leqslant e^{m(\rho_0+1)}\Upsilon(\tau) < +\infty.
\end{aligned} \tag{4.2.85}$$

It follows from Lemma 4.1.1 (i) that $t \to \mathcal{G}_\delta(\theta_t\omega)$ is continuous. Thus, there exists an $L_0 > 0$ such that

$$\sup_{\nu\in[-\rho,0]}|\mathcal{G}_\delta(\theta_\nu\omega)|^2 \leqslant L_0. \tag{4.2.86}$$

According to (4.1.36) in the hypothesis **G1**, we obtain

$$\begin{aligned}\sup_{r\leqslant\tau}\int_{r-\rho}^{r}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^2\|G(\nu,u)\|_\sigma^2 d\nu \leqslant & L_0\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\sum_{i\in\mathbb{Z}}\xi_i|G_i(\nu,u_i)|^2 d\nu\\ \leqslant & L_0\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\sum_{i\in\mathbb{Z}}\xi_i(2\alpha_4^2|u_i(\nu)|^{2q-2}+2|\mu_{4,i}(\nu)|^2)d\nu\\ = & 2\alpha_4^2L_0\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\sum_{i\in\mathbb{Z}}\xi_i|u_i(\nu)|^{2q-2}d\nu\\ & + 2L_0\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|\mu_4(\nu)\|_\sigma^2 d\nu.\end{aligned}\tag{4.2.87}$$

Now, we estimate the last two lines of (4.2.87) separately. On the one hand, by $2\leqslant q<p$, and so $2q-2>0$, $\frac{2p-2}{2q-2}>1$, and by (4.2.26) in Lemma 4.2.3, we deduce

$$\begin{aligned}\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\sum_{i\in\mathbb{Z}}\xi_i|u_i(\nu)|^{2q-2}d\nu \leqslant & c\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\sum_{i\in\mathbb{Z}}\xi_i(|u_i(\nu)|^{2p-2}+1)d\nu\\ = & c\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|u(\nu)\|_{\sigma,2p-2}^{2p-2}d\nu + c\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|\xi\|_{\ell^1}d\nu\\ \leqslant & c\widetilde{R}_\delta(\tau,\omega).\end{aligned}\tag{4.2.88}$$

On the other hand, by (4.1.21), $\xi\in\ell^1$ and $\mu_4\in L^\infty(\mathbb{R},\ell_\sigma^p)$,

$$\begin{aligned}\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|\mu_4(\nu)\|_\sigma^2 d\nu \leqslant & \sup_{r\leqslant\tau}\int_{r-\rho}^{r}\|\xi\|_{\ell^1}^{\frac{p-2}{p}}\|\mu_4(\nu)\|_{\sigma,p}^2 d\nu\\ \leqslant & \rho\|\xi\|_{\ell^1}^{\frac{p-2}{p}}\|\mu_4\|_{L^\infty(\mathbb{R},\ell_\sigma^p)}^2<+\infty.\end{aligned}\tag{4.2.89}$$

Using (4.2.88) and (4.2.89) in (4.2.87), we imply

$$\sup_{r\leqslant\tau}\int_{r-\rho}^{r}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^2\|G(\nu,u)\|_\sigma^2 d\nu\leqslant c\widetilde{R}_\delta(\tau,\omega).\tag{4.2.90}$$

By (4.2.81)-(4.2.90), we deduce

$$\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\left\|\frac{d}{d\nu}u^\delta(\nu,r-t,\theta_{-r}\omega,\psi_{r-t})\right\|_\sigma^2 d\nu\leqslant c\widetilde{R}_\delta(\tau,\omega).\tag{4.2.91}$$

One can similarly prove that

$$\sup_{r\leqslant\tau}\int_{r-\rho}^{r}\left\|\frac{d}{d\nu}v^\delta(\nu,r-t,\theta_{-r}\omega,\psi_{r-t})\right\|_\sigma^2 d\nu\leqslant c\widetilde{R}_\delta(\tau,\omega).\tag{4.2.92}$$

This together with (4.2.91) yields (4.2.79) as desired. □

Proposition 4.2.6. *Let hypotheses* **E, F1, F2, G1-G3** *and* (4.1.35) *be satisfied. For each* $\delta > 0$, *the cocycle* Ψ^δ *associated with the random delay FitzHugh-Nagumo lattice system* (0.0.7) *is* $\mathfrak{D}$*-backward asymptotically compact in* $\mathcal{X}_\sigma = \ell^2_\sigma \times \ell^2_\sigma$. *More precisely, for all* $s \in [-\rho, 0]$,

$$(\Psi^\delta(t_n, r_n - t_n, \theta_{-t_n}\omega)\psi_n)(s) = \varphi^\delta(r_n + s, r_n - t_n, \theta_{-r_n}\omega, \psi_n)$$

has a convergent subsequence in $\mathcal{X}_\sigma$ *whenever* $r_n \leqslant \tau, t_n \uparrow +\infty$ *and* $\psi_n = (\phi_n, \upsilon_n) \in \mathcal{D}(r_n - t_n, \theta_{-t_n}\omega)$.

Proof. Let $(\tau, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}$ be fixed and suppose that $r_n \leqslant \tau, t_n \uparrow +\infty$ and $\psi_n \in \mathcal{D}(r_n - t_n, \theta_{-t_n}\omega)$. For each $s \in [-\rho, 0]$, we define $Y^n(s) := (\Psi^\delta(t_n, r_n - t_n, \theta_{-t_n}\omega)\psi_n)(s)$. It suffices to prove that the sequence $\{Y^n(s)\}_{n=1}^\infty$ has a convergent subsequence in $\mathcal{X}_\sigma$. Besides, we write $Y^n(s) = (Y^n_i(s))_{i\in\mathbb{Z}}$ for each $n \in \mathbb{N}$. Given $\epsilon > 0$, by Lemma 4.2.4, there exist $k_1, n_1 \in \mathbb{N}$ such that

$$\sup_{r\leqslant\tau}\sup_{s\in[-\rho,0]} \|Y^n(s)\|^2_{\mathcal{X}_\sigma(|x|\geqslant k_1)} \leqslant \epsilon^2, \ \forall\, n \geqslant n_1, k \geqslant k_1. \tag{4.2.93}$$

According to (4.2.1) in Lemma 4.2.1, there exists $n_2 \geqslant n_1$ such that for all $n \geqslant n_2$,

$$\|Y^n(s)\|^2_{\mathcal{X}_\sigma} \leqslant cR_\delta(\tau, \omega) < +\infty, \tag{4.2.94}$$

which implies that the sequence $\{Y^n(s)\}_{n=1}^\infty$ is bounded in $\mathcal{X}_\sigma$. In particular, the sequence $\{(Y^n_i(s))_{|i|<k_1}\}_{n=1}^\infty$ is bounded and pre-compact in the finite-dimensional space $\mathbb{R}^{2k_1-1}$. In this case, there is a subsequence $\{(Y^{n^*}_i(s))_{|i|<k_1}\}_{n=1}^\infty$ such that it is a Cauchy sequence in $\mathbb{R}^{2k_1-1}$. Hence, there exists $n_3 \geqslant n_2$ such that for all $n^*, m^* \geqslant n_3$,

$$\sum_{|i|<k_1} \xi_i |Y^{n^*}_i(s) - Y^{m^*}_i(s)|^2 \leqslant \sum_{|i|<k_1} |Y^{n^*}_i(s) - Y^{m^*}_i(s)|^2 \leqslant \epsilon^2, \tag{4.2.95}$$

where we recall that $\xi_i = (1 + i^2)^{-\sigma} \leqslant 1$ for all $i \in \mathbb{Z}$ and $\sigma > \frac{1}{2}$. By (4.2.93) and (4.2.95), we show that for all $n^*, m^* \geqslant n_3$,

$$\begin{aligned}\|Y^{n^*}(s) - Y^{m^*}(s)\|^2_{\mathcal{X}_\sigma} &= \sum_{|i|<k_1} \xi_i |Y^{n^*}_i(s) - Y^{m^*}_i(s)|^2 + \sum_{|i|\geqslant k_1} \xi_i |Y^{n^*}_i(s) - Y^{m^*}_i(s)|^2 \\ &\leqslant \epsilon^2 + 2\sum_{|i|\geqslant k_1} \xi_i |Y^{n^*}_i(s)|^2 + 2\sum_{|i|\geqslant k_1} \xi_i |Y^{m^*}_i(s)|^2 \leqslant 5\epsilon^2,\end{aligned}$$

which shows $\|Y^{n^*}(s) - Y^{m^*}(s)\|_{\mathcal{X}_\sigma} \leqslant \sqrt{5}\epsilon$. Therefore, $\{Y^{n^*}(s)\}$ is a Cauchy subsequence of $\{Y^n(s)\}$ and convergent in $\mathcal{X}_\sigma$. □

We are now in a position to show the existence of $\mathfrak{D}$-pullback random attractors for the cocycle Ψ^δ.

Theorem 4.2.7. *Suppose all hypotheses* **E, F1, F2, G1-G3** *and* (4.1.35) *are satisfied. For each* $\delta > 0$ *and* $s \in [-\rho, 0]$, *the cocycle* Ψ^δ *associated with the random delay FitzHugh-Nagumo lattice system* (0.0.7) *has a* $\mathfrak{D}$*-pullback random attractor* $\mathcal{A}^\delta \in \mathfrak{D}$ *and a* $\widetilde{\mathfrak{D}}$*-pullback random attractor* $\widetilde{\mathcal{A}}^\delta \in \widetilde{\mathfrak{D}}$ *in* $\mathcal{X}^\rho_\sigma = C([-\rho, 0], \mathcal{X}_\sigma)$, *respectively. Moreover,* $\mathcal{A}^\delta = \widetilde{\mathcal{A}}^\delta$.

Proof. We mainly proof that Ψ^δ is $\mathfrak{D}$-backward asymptotically compact in $\mathcal{X}_\sigma^\rho$. That is, for any sequences $r_n \leqslant \tau, t_n \uparrow +\infty$ and $\psi_n = (\phi_n, \upsilon_n) \in \mathcal{D}(r_n - t_n, \theta_{-t_n}\omega)$, the sequence

$$\Psi^\delta(t_n, r_n - t_n, \theta_{-t_n}\omega)\psi_n = \varphi^\delta_{r_n}(\cdot, r_n - t_n, \theta_{-r_n}\omega, \psi_n)$$

has a convergent subsequence in $\mathcal{X}_\sigma^\rho$. For this end, we need to check the following three steps.

Step 1: For each $s \in [-\rho, 0]$, we prove $\{(\Psi^\delta(t_n, r_n - t_n, \theta_{-t_n}\omega)\psi_n)(s)\}_{n\in\mathbb{N}}$ is pre-compact in $\mathcal{X}_\sigma = \ell^2_\sigma \times \ell^2_\sigma$. The conclusion holds true on account of Proposition 4.2.6.

Step 2: We show the sequence $\{\Psi^\delta(t_n, r_n - t_n, \theta_{-t_n}\omega)\psi_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}_\sigma^\rho$ is equi-continuous from $[-\rho, 0]$ to $\mathcal{X}_\sigma$. Let $s_1, s_2 \in [-\rho, 0]$ with $s_2 > s_1$. By Lemma 4.2.5, there exists an $N \in \mathbb{N}$ such that $t_N \geqslant T$ and thus, for all $n \geqslant N$,

$$\begin{aligned}
&\|(\Psi^\delta(t_n, r_n - t_n, \theta_{-t_n}\omega)\psi_n)(s_1) - (\Psi^\delta(t_n, r_n - t_n, \theta_{-t_n}\omega)\psi_n)(s_2)\|_{\mathcal{X}_\sigma} \\
&\quad = \|\varphi^\delta(r_n + s_1, r_n - t_n, \theta_{-r_n}\omega, \psi_n) - \varphi^\delta(r_n + s_2, r_n - t_n, \theta_{-r_n}\omega, \psi_n)\|_{\mathcal{X}_\sigma} \\
&\quad \leqslant c \int_{r_n+s_1}^{r_n+s_2} \Big\|\frac{d}{dv}u^\delta(v, r_n - t_n, \theta_{-r_n}\omega, \phi_n)\Big\|_\sigma dv + c\int_{r_n+s_1}^{r_n+s_2} \Big\|\frac{d}{dv}v^\delta(v, r_n - t_n, \theta_{-r_n}\omega, \upsilon_n)\Big\|_\sigma dv \\
&\quad \leqslant c\Big(\int_{r_n-\rho}^{r_n} \Big\|\frac{d}{dv}u^\delta(v, r_n - t_n, \theta_{-r_n}\omega, \phi_n)\Big\|^2_\sigma dv\Big)^{\frac{1}{2}}|s_2 - s_1|^{\frac{1}{2}} \\
&\qquad + c\Big(\int_{r_n-\rho}^{r_n} \Big\|\frac{d}{dv}v^\delta(v, r_n - t_n, \theta_{-r_n}\omega, \upsilon_n)\Big\|^2_\sigma dv\Big)^{\frac{1}{2}}|s_2 - s_1|^{\frac{1}{2}} \\
&\quad \leqslant c\widetilde{R}_\delta(\tau, \omega)|s_2 - s_1|^{\frac{1}{2}}.
\end{aligned}$$

Hence, the sequence $\{\Psi^\delta(t_n, r_n - t_n, \theta_{-t_n}\omega)\psi_n\}_{n\geqslant N}$ in $\mathcal{X}_\sigma^\rho$ is equi-continuous from $[-\rho, 0]$ to $\mathcal{X}_\sigma$. Note that it is obvious that the finite set $\{\Psi^\delta(t_n, r_n - t_n, \theta_{-t_n}\omega)\psi_n\}_{n<N}$ in $\mathcal{X}_\sigma^\rho$ is equi-continuous, and so is the whole sequence $\{\Psi^\delta(t_n, r_n - t_n, \theta_{-t_n}\omega)\psi_n\}_{n\in\mathbb{N}}$.

Step 3: We prove the existence and equality of two pullback random attractors. By Steps 1-2, it follows from the Ascoli-Arzelà theorem that the sequence $\{\Psi^\delta(t_n, r_n - t_n, \theta_{-t_n}\omega)\psi_n\}_{n\in\mathbb{N}}$ is pre-compact in $\mathcal{X}_\sigma^\rho$. Thanks to Proposition 4.2.2, Ψ^δ has a $\mathfrak{D}$-pullback random absorbing set $\mathcal{K}_\delta = \{\mathcal{K}_\delta(\tau, \omega)\} \in \mathfrak{D}$. Using the abstract results established in [121, Theorem 2.23], we derive that Ψ^δ has a $\mathfrak{D}$-pullback random attractor $\mathcal{A}^\delta \in \mathfrak{D}$ in $\mathcal{X}_\sigma^\rho$, which is the omega-limit set of $\mathcal{K}_\delta$.

By $\mathfrak{D} \subset \widetilde{\mathfrak{D}}$, we imply that $\mathcal{K}_\delta$ is also a $\widetilde{\mathfrak{D}}$-pullback random absorbing set and $\mathcal{K}_\delta \in \widetilde{\mathfrak{D}}$. By the same argument of Proposition 4.2.6 and the above Steps 1-2, we derive that Ψ^δ is $\widetilde{\mathfrak{D}}$-pullback asymptotically compact in $\mathcal{X}_\sigma^\rho$. It follows from [94] that the existence and uniqueness of a $\widetilde{\mathfrak{D}}$-pullback random attractor $\widetilde{\mathcal{A}}^\delta \in \widetilde{\mathfrak{D}}$ are obtained, where $\widetilde{\mathcal{A}}^\delta$ is the omega-limit set of $\mathcal{K}_\delta$. Therefore, $\widetilde{\mathcal{A}}^\delta = \mathcal{A}^\delta \in \mathfrak{D}$. □

4.3 Upper semicontinuity of attractors as correlation time tends to infinity

In this section, we mainly discuss the upper semicontinuity of the pullback random attractor $\mathcal{A}^\delta$ for problem (0.0.7) as $\delta \to +\infty$. For this end, we need to verify convergence of solutions.

Lemma 4.3.1. *Suppose the hypotheses* **E, F1, F2, G1, G2** *and* (4.1.35) *hold. Let* $\varphi^\delta = (u^\delta, v^\delta)$ *and* $\hat{\varphi} = (\hat{u}, \hat{v})$ *be the solutions to* (0.0.7) *and* (0.0.9) *with initial value* $\psi^\delta = (\phi^\delta, \upsilon^\delta)$ *and* $\hat{\psi} = (\hat{\phi}, \hat{\upsilon})$*, respectively. If* $\|\psi^\delta - \hat{\psi}\|_{\mathcal{X}_\sigma^\rho} \to 0$ *as* $\delta \to +\infty$*, more precisely,*

$$\mathrm{d}_{\mathcal{X}_\sigma^\rho}(\psi^\delta, \hat{\psi}) = \sup_{\nu \in [-\rho, 0]} \|(\phi^\delta, \upsilon^\delta)(\nu) - (\hat{\phi}, \hat{\upsilon})(\nu)\|_{\mathcal{X}_\sigma} \to 0, \ as\ \delta \to +\infty, \tag{4.3.1}$$

then φ^δ *converges to* $\hat{\varphi}$ *in the following sense:*

$$\lim_{\delta \to +\infty} \sup_{s \in [-\rho, 0]} \|\varphi^\delta(t+s, \tau, \omega, \psi^\delta) - \hat{\varphi}(t+s, \tau, \hat{\psi})\|^2_{\mathcal{X}_\sigma} = 0, \ \forall\, t \geqslant \tau, \ \omega \in \Omega. \tag{4.3.2}$$

Proof. Let $U^\delta(\nu) = u^\delta(\nu, \tau, \omega, \phi^\delta) - \hat{u}(\nu, \tau, \hat{\phi})$, $V^\delta(\nu) = v^\delta(\nu, \tau, \omega, \upsilon^\delta) - \hat{v}(\nu, \tau, \hat{\upsilon})$ and $W^\delta(\nu) = \varphi^\delta(\nu, \tau, \omega, \psi^\delta) - \hat{\varphi}(\nu, \tau, \hat{\psi}) = (U^\delta(\nu), V^\delta(\nu))$, which is equipped by the norm $\|W^\delta\|^2_{\mathcal{X}_\sigma} = \beta\|U^\delta\|^2_\sigma + \alpha\|V^\delta\|^2_\sigma$. We subtract (0.0.9) from (0.0.7) to obtain $W^\delta = (U^\delta, V^\delta)$ satisfies that for $\nu \geqslant \tau$,

$$\begin{cases} \dfrac{dU_i^\delta}{d\nu} + (AU^\delta)_i + \lambda U_i^\delta + \alpha V_i^\delta = F_i(u_i^\delta(\nu)) - F_i(\hat{u}_i(\nu)) + f_i(u_i^\delta(\nu - \varrho^{(\rho)}(\nu))) \\ \quad - f_i(\hat{u}_i(\nu - \varrho^{(\rho)}(\nu))) + G_i(\nu, u_i^\delta)\mathcal{G}_\delta(\theta_\nu\omega), \\ \dfrac{dV_i^\delta}{d\nu} + \varsigma V_i^\delta - \beta U_i^\delta = f_i(v_i^\delta(\nu - \varrho^{(\rho)}(\nu))) - f_i(\hat{v}_i(\nu - \varrho^{(\rho)}(\nu))), \end{cases} \tag{4.3.3}$$

where $U^\delta = (U_i^\delta)_{i\in\mathbb{Z}}$ and $V^\delta = (V_i^\delta)_{i\in\mathbb{Z}}$. Taking the inner product of (4.3.3) with $(2\beta\xi_i U_i^\delta, 2\alpha\xi_i V_i^\delta)$ and summing up the product over $i \in \mathbb{Z}$, it follows that

$$\begin{aligned} &\frac{d}{d\nu}(\beta\|U^\delta\|^2_\sigma + \alpha\|V^\delta\|^2_\sigma) + 2\kappa(\beta\|U^\delta\|^2_\sigma + \alpha\|V^\delta\|^2_\sigma) \\ &= -2\beta\sum_{i\in\mathbb{Z}} \xi_i U_i^\delta (AU^\delta)_i + 2\beta\sum_{i\in\mathbb{Z}} \xi_i U_i^\delta(F_i(u_i^\delta) - F_i(\hat{u}_i)) + 2\beta\mathcal{G}_\delta(\theta_\nu\omega)\sum_{i\in\mathbb{Z}} \xi_i U_i^\delta G_i(\nu, u_i^\delta) \\ &\quad + 2\beta\sum_{i\in\mathbb{Z}} \xi_i U_i^\delta(f_i(u_i^\delta(\nu - \varrho^{(\rho)}(\nu))) - f_i(\hat{u}_i(\nu - \varrho^{(\rho)}(\nu)))) \\ &\quad + 2\alpha\sum_{i\in\mathbb{Z}} \xi_i V_i^\delta(f_i(v_i^\delta(\nu - \varrho^{(\rho)}(\nu))) - f_i(\hat{v}_i(\nu - \varrho^{(\rho)}(\nu)))), \end{aligned} \tag{4.3.4}$$

where we recall that $\kappa = \min\{\lambda, \varsigma\}$. As done in (4.2.5), we have

$$-2\beta\sum_{i\in\mathbb{Z}} \xi_i U_i^\delta (AU^\delta)_i \leqslant 2.5^{3\sigma}\beta\|U^\delta\|^2_\sigma. \tag{4.3.5}$$

According to the mean valued theorem and (4.1.33) in the hypothesis **F1**, there exists $a := a(u_i^\delta, \hat{u}_i) \in (0, 1)$ such that

$$\begin{aligned} 2\beta\sum_{i\in\mathbb{Z}} \xi_i U_i^\delta(F_i(u_i^\delta) - F_i(\hat{u}_i)) &\leqslant 2\beta\sum_{i\in\mathbb{Z}} \xi_i |U_i^\delta|^2 \frac{\partial F_i}{\partial s}(au_i^\delta + (1-a)\hat{u}_i) \\ &\leqslant -2\beta\alpha_3\sum_{i\in\mathbb{Z}} \xi_i |U_i^\delta|^2 |au_i^\delta + (1-a)\hat{u}_i|^{p-2} + 2\beta\sum_{i\in\mathbb{Z}} \xi_i |U_i^\delta|^2 \mu_{3,i} \end{aligned}$$

$$\leqslant 2\beta\|\mu_3\|_{\ell^\infty}\|U^\delta\|_\sigma^2. \tag{4.3.6}$$

Note that $\hat{q} \leqslant 2 \leqslant q < p$, by the Young inequality and (4.1.36) in the hypothesis **G1**, we imply

$$\begin{aligned} 2\beta\mathcal{G}_\delta(\theta_\nu\omega)\sum_{i\in\mathbb{Z}}\xi_i U_i^\delta G_i(\nu,u_i^\delta) \leqslant& 2\beta|\mathcal{G}_\delta(\theta_\nu\omega)|\sum_{i\in\mathbb{Z}}\xi_i|U_i^\delta|(\alpha_4|u_i^\delta|^{q-1}+\mu_{4,i}(\nu)) \\ \leqslant& \hat{c}_1|\mathcal{G}_\delta(\theta_\nu\omega)|\sum_{i\in\mathbb{Z}}\xi_i(|u_i^\delta|^q+|\hat{u}_i|^q) \\ &+\hat{c}_2|\mathcal{G}_\delta(\theta_\nu\omega)|\sum_{i\in\mathbb{Z}}\xi_i(|u_i^\delta|^2+|\hat{u}_i|^2+|\mu_{4,i}(\nu)|^2) \\ \leqslant& \hat{c}_3|\mathcal{G}_\delta(\theta_\nu\omega)|(\|u^\delta\|_{\sigma,p}^p+\|\hat{u}\|_{\sigma,p}^p+\|\mu_4(\nu)\|_{\sigma,p}^p+1) \\ \leqslant& \hat{c}_4|\mathcal{G}_\delta(\theta_\nu\omega)|(\|u^\delta\|_{\sigma,p}^p+\|\hat{u}\|_{\sigma,p}^p+1), \end{aligned} \tag{4.3.7}$$

where we recall that $\mu_4 \in L^\infty(\mathbb{R},\ell_\sigma^p)$. By (4.1.34) in the hypothesis **F2**, we obtain

$$\begin{aligned} &2\beta\sum_{i\in\mathbb{Z}}\xi_i U_i^\delta(f_i(u_i^\delta(\nu-\varrho^{(\rho)}(\nu)))-f_i(\hat{u}_i(\nu-\varrho^{(\rho)}(\nu)))) \\ &\quad+2\alpha\sum_{i\in\mathbb{Z}}\xi_i V_i^\delta(f_i(v_i^\delta(\nu-\varrho^{(\rho)}(\nu)))-f_i(\hat{v}_i(\nu-\varrho^{(\rho)}(\nu)))) \\ &\leqslant 2\beta L_f\sum_{i\in\mathbb{Z}}\xi_i|U_i^\delta||u_i^\delta(\nu-\varrho^{(\rho)}(\nu))-\hat{u}_i(\nu-\varrho^{(\rho)}(\nu))| \\ &\quad+2\alpha L_f\sum_{i\in\mathbb{Z}}\xi_i|V_i^\delta||v_i^\delta(\nu-\varrho^{(\rho)}(\nu))-\hat{v}_i(\nu-\varrho^{(\rho)}(\nu))| \\ &\leqslant \frac{4L_f^2}{\kappa}(\beta\|U^\delta(\nu-\varrho^{(\rho)}(\nu))\|_\sigma^2+\alpha\|V^\delta(\nu-\varrho^{(\rho)}(\nu))\|_\sigma^2)+\frac{\kappa}{4}(\beta\|U^\delta\|_\sigma^2+\alpha\|V^\delta\|_\sigma^2). \end{aligned} \tag{4.3.8}$$

We substitute (4.3.5)-(4.3.8) into (4.3.4) that

$$\begin{aligned} &\frac{d}{d\nu}\|W^\delta\|_{X_\sigma}^2+\kappa\|W^\delta\|_{X_\sigma}^2+\left(\frac{3\kappa}{4}-2.5^{3\sigma}-2\|\mu_3\|_{\ell^\infty}\right)\|W^\delta\|_{X_\sigma}^2 \\ &\quad\leqslant \frac{4L_f^2}{\kappa}\|W^\delta(\nu-\varrho^{(\rho)}(\nu))\|_{X_\sigma}^2+\hat{c}_4|\mathcal{G}_\delta(\theta_\nu\omega)|(\|u^\delta\|_{\sigma,p}^p+\|\hat{u}\|_{\sigma,p}^p+1), \end{aligned} \tag{4.3.9}$$

where $\frac{3\kappa}{4}-2.5^{3\sigma}-2\|\mu_3\|_{\ell^\infty}>0$ in view of (4.1.35), and we recall that $\|W^\delta\|_{X_\sigma}^2=\beta\|U^\delta\|_\sigma^2+\alpha\|V^\delta\|_\sigma^2$.

Integrating (4.3.9) over $[\tau,t+s]$, where $t>\tau$ and $s\in[-\rho,0]$, we deduce

$$\begin{aligned} \|W^\delta(t+s)\|_{X_\sigma}^2 \leqslant& \|W^\delta(\tau)\|_{X_\sigma}^2-\kappa\int_\tau^{t+s}\|W^\delta(\nu)\|_{X_\sigma}^2 d\nu \\ &+\frac{4L_f^2}{\kappa}\int_\tau^{t+s}\|W^\delta(\nu-\varrho^{(\rho)}(\nu))\|_{X_\sigma}^2 d\nu \\ &+\hat{c}_4\int_\tau^{t+s}|\mathcal{G}_\delta(\theta_\nu\omega)|(\|u^\delta(\nu)\|_{\sigma,p}^p+\|\hat{u}(\nu)\|_{\sigma,p}^p+1)d\nu, \end{aligned} \tag{4.3.10}$$

where $\|W^\delta(\tau)\|^2_{\mathcal{X}_\sigma} = \beta\|\phi^\delta(0) - \hat{\phi}(0)\|^2_\sigma + \alpha\|\upsilon^\delta(0) - \hat{\upsilon}(0)\|^2_\sigma \leqslant \mathrm{d}^2_{\mathcal{X}^\rho_\sigma}(\psi^\delta, \hat{\psi}) \to 0$ as $\delta \to +\infty$. The second line of (4.3.10) satisfies

$$\begin{aligned}\frac{4L_f^2}{\kappa}\int_\tau^{t+s}\|W^\delta(\nu-\varrho^{(\rho)}(\nu))\|^2_{\mathcal{X}_\sigma}d\nu &\leqslant \frac{4L_f^2}{\kappa(1-\rho_*)}\int_{\tau-\rho}^{\tau}\|W^\delta(\mu)\|^2_{\mathcal{X}_\sigma}d\mu + \frac{4L_f^2}{\kappa(1-\rho_*)}\int_\tau^{t+s}\|W^\delta(\nu)\|^2_{\mathcal{X}_\sigma}d\nu\\ &\leqslant \frac{4L_f^2\rho_0}{\kappa(1-\rho_*)}\mathrm{d}^2_{\mathcal{X}^\rho_\sigma}(\psi^\delta,\hat{\psi}) + \frac{4L_f^2}{\kappa(1-\rho_*)}\int_\tau^{t+s}\|W^\delta(\nu)\|^2_{\mathcal{X}_\sigma}d\nu,\end{aligned}$$

which, together with $\frac{4L_f^2}{\kappa(1-\rho_*)} < \kappa$, yields

$$\begin{aligned}\|W^\delta(t+s)\|^2_{\mathcal{X}_\sigma} &\leqslant \mathrm{d}^2_{\mathcal{X}^\rho_\sigma}(\psi^\delta,\hat{\psi})\Big(1 + \frac{4L_f^2\rho_0}{\kappa(1-\rho_*)}\Big)\\ &\quad + \hat{c}_4\int_\tau^{t+s}|\mathcal{G}_\delta(\theta_\nu\omega)|(\|u^\delta(\nu)\|^p_{\sigma,p} + \|\hat{u}(\nu)\|^p_{\sigma,p} + 1)d\nu.\end{aligned}\tag{4.3.11}$$

Similar to the argument as in Lemma 4.2.3, we imply $\hat{u} \in L^p_{loc}((\tau,+\infty),\ell^p_\sigma)$, which yields $\int_\tau^t \|\hat{u}(\nu)\|^p_{\sigma,p}d\nu < +\infty$. Finally, we only need to prove

$$\limsup_{\delta\to+\infty}\int_\tau^t\|u^\delta(\nu)\|^p_{\sigma,p}d\nu < +\infty.\tag{4.3.12}$$

Replacing $\theta_{-r}\omega$ by ω in the energy inequality (4.2.15), by (4.1.6) in Lemma 4.1.1, there exists a $\delta_0 := \delta_0(\omega) > 0$ such that for all $\delta \geqslant \delta_0,\ \nu \in [\tau, t]$,

$$\begin{aligned}&\frac{d}{d\nu}\|\varphi^\delta\|^2_{\mathcal{X}_\sigma} + \kappa\|\varphi^\delta\|^2_{\mathcal{X}_\sigma} + \frac{\kappa}{2}\|\varphi^\delta\|^2_{\mathcal{X}_\sigma} + \alpha_1\beta\|u^\delta\|^p_{\sigma,p}\\ &\leqslant \frac{4L_f^2}{\kappa}\|\varphi^\delta(\nu-\varrho^{(\rho)}(\nu))\|^2_{\mathcal{X}_\sigma} + \hat{c}_5(1 + \|g(\nu)\|^2_\sigma + \|h(\nu)\|^2_\sigma) + \hat{c}_6|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}}.\end{aligned}\tag{4.3.13}$$

Using (4.1.6) in Lemma 4.1.1, there exists a $\delta_1 \geqslant \delta_0$ such that for all $\delta \geqslant \delta_1$,

$$\sup_{\delta\geqslant\delta_1}\sup_{\nu\in[\tau,t]}|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}} \leqslant 1.$$

Then we can rewrite (4.3.13) as follows.

$$\begin{aligned}&\frac{d}{d\nu}\|\varphi^\delta\|^2_{\mathcal{X}_\sigma} + \kappa\|\varphi^\delta\|^2_{\mathcal{X}_\sigma} + \frac{\kappa}{2}\|\varphi^\delta\|^2_{\mathcal{X}_\sigma} + \alpha_1\beta\|u^\delta\|^p_{\sigma,p}\\ &\leqslant \frac{4L_f^2}{\kappa}\|\varphi^\delta(\nu-\varrho^{(\rho)}(\nu))\|^2_{\mathcal{X}_\sigma} + \hat{c}_5(1 + \|g(\nu)\|^2_\sigma + \|h(\nu)\|^2_\sigma) + \hat{c}_6,\end{aligned}$$

which, together with the Gronwall inequality, $\varphi^\delta(\cdot,\tau,\omega,\psi^\delta) \in C([\tau-\rho,\infty),\mathcal{X}_\sigma)$, $g,h \in L^2_{loc}(\mathbb{R},\ell^2_\sigma)$, implies that

$$\sup_{\delta\geqslant\delta_0}\int_\tau^t\|u^\delta(\nu,\tau,\omega,\phi^\delta)\|^p_{\sigma,p}d\nu \leqslant \hat{c}_7\sup_{\delta\geqslant\delta_0}\|\psi^\delta\|^2_{\mathcal{X}^\rho_\sigma} + \hat{c}_8.$$

Since $\|\psi^\delta - \hat{\psi}\|_{\mathcal{X}_\sigma^\rho} \to 0$ as $\delta \to +\infty$, we obtain that $\|\psi^\delta\|^2_{\mathcal{X}_\sigma^\rho}$ is bounded when $\delta \to +\infty$. Therefore, (4.3.12) holds true. It follows from (4.3.11) and (4.1.6) in Lemma 4.1.1 that

$$\|W^\delta(t+s)\|^2_{\mathcal{X}_\sigma} \leqslant \hat{c}_9 \mathrm{d}^2_{\mathcal{X}_\sigma^\rho}(\psi^\delta, \hat{\psi}) + \hat{c}_{10} \sup_{\nu\in[\tau,t]} |\mathcal{G}_\delta(\theta_\nu\omega)| \to 0,$$

as $\delta \to +\infty$. Therefore, we obtain (4.3.2) as desired. □

We assume that $\Psi^\infty : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{X}_\sigma^\rho \mapsto \mathcal{X}_\sigma^\rho$ is the corresponding deterministic dynamical system (or process), given by

$$\Psi^\infty(t,\tau)\hat{\psi} = \hat{\varphi}_{t+\tau}(\cdot,\tau,\hat{\psi}),\ t \geqslant 0,\ (\tau,\hat{\psi}) \in \mathbb{R}\times\mathcal{X}_\sigma^\rho, \tag{4.3.14}$$

where $\hat{\varphi} = (\hat{u}, \hat{v})$ is the unique solution to system (0.0.9). One can prove that Ψ^∞ has a $\mathfrak{D}^\infty$-pullback attractor $\mathcal{A}^\infty$ by using the same method as in Theorem 4.2.7, where $\mathfrak{D}^\infty$ is the universe of all backward tempered sets in $\mathcal{X}_\sigma^\rho$, that is, $\mathcal{D}^\infty \in \mathfrak{D}^\infty$ if and only if

$$\lim_{t\to+\infty} e^{-\gamma t} \sup_{r\leqslant\tau} \|\mathcal{D}^\infty(r-t)\|^2_{\mathcal{X}_\sigma^\rho} = 0,\ \forall\,\gamma > 0,\ \tau \in \mathbb{R}. \tag{4.3.15}$$

Theorem 4.3.2. *Let hypotheses* **E, F1, F2, G1-G3** *and* (4.1.35) *be satisfied. Suppose $\mathcal{A}^\delta$ is the $\mathfrak{D}$-pullback random attractor of random delay lattice system* (0.0.7) *with the size $\delta > 0$ and $\mathcal{A}^\infty$ is the $\mathfrak{D}^\infty$-pullback attractor of deterministic delay lattice system* (0.0.9)*. Then $\mathcal{A}^\delta$ converges to $\mathcal{A}^\infty$, i.e.*

$$\lim_{\delta\to+\infty} \mathrm{d}_{\mathcal{X}_\sigma^\rho}(\mathcal{A}^\delta(\tau,\omega), \mathcal{A}^\infty(\tau)) = 0,\ \ \forall\,\tau\in\mathbb{R},\ \omega\in\Omega. \tag{4.3.16}$$

Proof. We split the proof into the following three steps.

Step 1: We prove the cocycle Ψ^δ is uniformly absorbing in $\mathcal{X}_\sigma^\rho$ with respect to the large-size δ. Indeed, by Proposition 4.2.2, each cocycle Ψ^δ has a $\mathfrak{D}$-pullback random absorbing ball $\mathcal{K}_\delta(\cdot,\cdot) \in \mathfrak{D}$ with the radius

$$c^{\frac{1}{2}} R_\delta^{\frac{1}{2}}(\tau,\omega) = c^{\frac{1}{2}}(1 + \Upsilon(\tau) + \eta_\delta(\omega))^{\frac{1}{2}},\ \forall\,(\tau,\omega)\in\mathbb{R}\times\Omega. \tag{4.3.17}$$

By (4.1.7) in Lemma 4.1.1, we have

$$\lim_{\delta\to+\infty} \eta_\delta(\omega) = \lim_{\delta\to+\infty} \int_{-\infty}^0 e^{m\nu} |\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}} d\nu = 0,\ \forall\,\omega\in\Omega. \tag{4.3.18}$$

Since all estimates in section 3 are valid when $\delta \to +\infty$, one can show that the determinstic system Ψ^∞ has a $\mathfrak{D}^\infty$-pullback absorbing set $\mathcal{K}_\infty$ given by

$$\mathcal{K}_\infty(\tau) = \{w \in \mathcal{X}_\sigma^\rho : \|w\|^2_{\mathcal{X}_\sigma^\rho} \leqslant c(2 + \Upsilon(\tau))\},\ \forall\,\tau\in\mathbb{R}. \tag{4.3.19}$$

Using the same method as in Proposition 4.2.2, one can show $\mathcal{K}_\infty \in \mathfrak{D}^\infty$. Thus, we imply

$$\limsup_{\delta\to+\infty} \|\mathcal{K}_\delta(\tau,\omega)\|^2_{\mathcal{X}_\sigma^\rho} \leqslant \|\mathcal{K}_\infty(\tau)\|^2_{\mathcal{X}_\sigma^\rho},\ \forall\,(\tau,\omega)\in\mathbb{R}\times\Omega. \tag{4.3.20}$$

Step 2: We verify the large-size uniformness of the $\mathfrak{D}$-pullback asymptotic compactness for the cocycle Ψ^δ in $\mathcal{X}_\sigma^\rho$. By the proof of Theorem 4.2.7, we prove the conclusion as desired.

Step 3: We prove the upper semicontinuity in (4.3.16). In fact, the convergence of systems ($\Psi^\delta \to \Psi^\infty$ as $\delta \to +\infty$) has been obtained in Lemma 4.3.1. And for all large enough δ, the uniform absorbing has been proved in Step 1. Moreover, the uniform asymptotic compactness has been derived in Step 2. Using the abstract result of upper semicontinuity for random attractors as in [86, Theorem 4.1], we prove (4.3.16) as desired. □

4.4 Upper semicontinuity of attractors as delay goes to zero

The last section is devoted to the upper semicontinuity of the pullback attractor $\mathcal{A}_\rho$ for problem (0.0.9) as $\rho \to 0$. Hereafter, we write the solution and deterministic dynamical system (or process) of system (0.0.9) as $\hat{\varphi}^\rho = (\hat{u}^\rho, \hat{v}^\rho)$ and Ψ_ρ, respectively. In addition, we use $\mathfrak{D}_\rho = \{\mathcal{D}_\rho(\tau) : \tau \in \mathbb{R}\}$ to replace the notation $\mathfrak{D}^\infty$ defined by (4.3.15).

As proved in Section 4.3, Ψ_ρ has a $\mathfrak{D}_\rho$-pullback attractor $\mathcal{A}_\rho$ in $\mathcal{X}_\sigma^\rho$ and a $\mathfrak{D}_\rho$-pullback absorbing set $\mathcal{K}_\rho$ given by

$$\mathcal{K}_\rho(\tau) = \{w \in \mathcal{X}_\sigma^\rho : \|w\|^2_{\mathcal{X}_\sigma^\rho} \leqslant c\widetilde{R}(\tau)\}, \ \forall\, \tau \in \mathbb{R}, \tag{4.4.1}$$

where $\widetilde{R}(\tau) = 2 + \Upsilon(\tau)$ and $\Upsilon(\tau)$ is given by (4.2.3).

Let $\rho = 0$ in (0.0.9), we obtain

$$\begin{cases} \dfrac{d\hat{u}_i^0}{dt} + A\hat{u}_i^0 + \lambda\hat{u}_i^0 + \alpha\hat{v}_i^0 = F_i(\hat{u}_i^0(t)) + f_i(\hat{u}_i^0(t)) + g_i(t), \\ \dfrac{d\hat{v}_i^0}{dt} + \varsigma\hat{v}_i^0 - \beta\hat{u}_i^0 = h_i(t) + f_i(\hat{v}_i^0(t)), \\ \hat{u}_i^0(\tau) = \hat{\phi}_i^0, \ \ \hat{v}_i^0(\tau) = \hat{v}_i^0, \ \ t > \tau, \ \tau \in \mathbb{R}. \end{cases} \tag{4.4.2}$$

From now on, we denote by $\hat{\varphi}^0 = (\hat{u}^0, \hat{v}^0)$ the solution to Eq. (4.4.2). Assume that $\mathfrak{D}_0$ is the universe of all backward tempered sets in $\mathcal{X}_\sigma$, that is, $\mathcal{D}_0 \in \mathfrak{D}_0$ if and only if

$$\lim_{t\to+\infty} e^{-\gamma t} \sup_{r\leqslant\tau} \|\mathcal{D}_0(r-t)\|^2_{\mathcal{X}_\sigma} = 0, \ \forall\, \gamma > 0, \tau \in \mathbb{R}. \tag{4.4.3}$$

Since all estimates in Section 4.2 are valid when in both cases $\rho = 0$ and $\delta \to +\infty$ for problem (0.0.7). We deduce that the deterministic dynamical system $\Psi_0(\cdot,\cdot)$ induced by Eq. (4.4.2), possesses a $\mathfrak{D}_0$-pullback attractor $\mathcal{A}_0 = \{\mathcal{A}_0(t) : t \in \mathbb{R}\} \in \mathfrak{D}_0$ and a $\mathfrak{D}_0$-pullback absorbing set $\mathcal{K}_0$ given by

$$\mathcal{K}_0(\tau) = \{w \in \mathcal{X}_\sigma : \|w\|^2_{\mathcal{X}_\sigma} \leqslant c\widetilde{R}(\tau)\}, \ \forall\, \tau \in \mathbb{R}, \tag{4.4.4}$$

where $\widetilde{R}(\tau)$ is the same as in (4.4.1). Combining (4.4.1) and (4.4.4), we infer

$$\limsup_{\rho\to 0} \|\mathcal{K}_\rho(\tau)\|_{\mathcal{X}_\sigma^\rho} = \|\mathcal{K}_0(\tau)\|_{\mathcal{X}_\sigma}. \tag{4.4.5}$$

Thanks to Theorem 4.2.7, the following Lemma is immediate.

Lemma 4.4.1. *Suppose all hypotheses* **E, F1, F2, G1-G3**, *(4.1.35) are satisfied. Then the process Ψ_ρ associated with the deterministic delay FitzHugh-Nagumo lattice system (0.0.9) is $\mathfrak{D}_\rho$-backward asymptotically compact in $\mathcal{X}_\sigma^\rho = C([-\rho,0],\mathcal{X}_\sigma)$, that is, for each $(t,\mathcal{D}_\rho)\in\mathbb{R}\times\mathfrak{D}_\rho$, for all $\psi_n\in\mathcal{D}_\rho(\tau_n)$, and for each sequence $\{\tau_n\}\leqslant t$ with $\tau_n\to-\infty$ as $n\to\infty$, the sequence $\{\Psi_\rho(t,\tau_n)\psi_n\}_{n\in\mathbb{N}}$ is pre-compact in $\mathcal{X}_\sigma^\rho$.*

Proof. One can prove the proof by using the same method as in Theorem 4.2.7, which is based on the Ascoli-Arzelà theorem. More precisely, we can complete this proof by the following two steps.

Step 1: For each $s\in[-\rho,0]$, we prove $\{(\Psi_\rho(t,\tau_n)\psi_n)(s)\}_{n\in\mathbb{N}}$ is pre-compact in $\mathcal{X}_\sigma=\ell_\sigma^2\times\ell_\sigma^2$.

Step 2: We show the sequence $\{\Psi_\rho(t,\tau_n)\psi_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}_\sigma^\rho$ is equi-continuity from $[-\rho,0]$ to $\mathcal{X}_\sigma$. Let $s_1,s_2\in[-\rho,0]$ with $s_2>s_1$.

$$\|(\Psi_\rho(t,\tau_n)\psi_n)(s_1)-(\Psi_\rho(t,\tau_n)\psi_n)(s_2)\|_{\mathcal{X}_\sigma}\leqslant c\widetilde{R}(\tau)|s_2-s_1|^{\frac{1}{2}}.$$

□

Let us first prove the convergence of solutions as $\rho\to0$.

Lemma 4.4.2. *Suppose the hypotheses* **E, F1, F2, G1, G2** *hold. Let $\hat\varphi^\rho=(\hat u^\rho,\hat v^\rho)$ and $\hat\varphi^0=(\hat u^0,\hat v^0)$ be the solutions to (0.0.9) and (4.4.2) with initial value $\hat\psi^\rho=(\hat\phi^\rho,\hat\upsilon^\rho)$ and $\hat\psi^0=(\hat\phi^0,\hat\upsilon^0)$, respectively. If $\hat\psi^\rho$ converges to $\hat\psi^0$, i.e.,*

$$\mathrm{d}^*_{\mathcal{X}_\sigma^\rho}(\hat\psi^\rho,\hat\psi^0)=\sup_{s\in[-\rho,0]}\|(\hat\phi^\rho,\hat\upsilon^\rho)(s)-(\hat\phi^0,\hat\upsilon^0)\|_{\mathcal{X}_\sigma}\to0,\ \text{as}\ \rho\to0,\tag{4.4.6}$$

then $\hat\varphi^\rho$ converges to $\hat\varphi^0$ in the following sense:

$$\lim_{\rho\to0}\sup_{s\in[-\rho,0]}\|\hat\varphi^\rho(t+s,\tau,\hat\psi^\rho)-\hat\varphi^0(t,\tau,\hat\psi^0)\|^2_{\mathcal{X}_\sigma}=0,\ \forall\ t\geqslant\tau.\tag{4.4.7}$$

Proof. Let $U^\rho(\nu)=\hat u^\rho(\nu+s,\tau,\hat\phi^\rho)-\hat u^0(\nu,\tau,\hat\phi^0)$, $V^\rho(\nu)=\hat v^\rho(\nu+s,\tau,\hat\upsilon^\rho)-\hat v^0(\nu,\tau,\hat\upsilon^0)$ and $W^\rho(\nu)=\hat\varphi^\rho(\nu+s,\tau,\hat\psi^\rho)-\hat\varphi^0(\nu,\tau,\hat\psi^0)=(U^\rho(\nu),V^\rho(\nu))$, which is equipped by the norm $\|W^\rho\|^2_{\mathcal{X}_\sigma}=\beta\|U^\rho\|^2_\sigma+\alpha\|V^\rho\|^2_\sigma$. We subtract (4.4.2) from (0.0.9) to obtain $W^\rho=(U^\rho,V^\rho)$ satisfies that for $\nu\geqslant\tau$,

$$\begin{cases}\dfrac{dU_i^\rho}{d\nu}+(AU^\rho)_i+\lambda U_i^\rho+\alpha V_i^\rho=F_i(\hat u_i^\rho(\nu+s))-F_i(\hat u_i^0(\nu))+f_i(\hat u_i^\rho(\nu+s-\varrho^{(\rho)}(\nu+s)))\\ \quad-f_i(\hat u_i^0(\nu))+g_i(\nu+s)-g_i(\nu),\\ \dfrac{dV_i^\rho}{d\nu}+\varsigma V_i^\rho-\beta U_i^\rho=f_i(\hat v_i^\rho(\nu+s-\varrho^{(\rho)}(\nu+s)))-f_i(\hat v_i^0(\nu))+h_i(\nu+s)-h_i(\nu).\end{cases}$$

Taking the inner product of (4.4.8) with $(2\beta\xi_iU_i^\rho,2\alpha\xi_iV_i^\rho)$ and summing up the product over $i\in\mathbb{Z}$, it follows that

$$\begin{aligned}&\frac{d}{d\nu}(\beta\|U^\rho\|^2_\sigma+\alpha\|V^\rho\|^2_\sigma)+2\kappa(\beta\|U^\rho\|^2_\sigma+\alpha\|V^\rho\|^2_\sigma)\\&=-2\beta\sum_{i\in\mathbb{Z}}\xi_iU_i^\rho(AU^\rho)_i+2\beta\sum_{i\in\mathbb{Z}}\xi_iU_i^\rho(F_i(\hat u_i^\rho(\nu+s))-F_i(\hat u_i^0(\nu)))\end{aligned}$$

$$
\begin{aligned}
&+ 2\beta \sum_{i\in\mathbb{Z}} \xi_i U_i^\rho (g_i(\nu+s) - g_i(\nu)) + 2\alpha \sum_{i\in\mathbb{Z}} \xi_i V_i^\rho (h_i(\nu+s) - h_i(\nu)) \\
&+ 2\beta \sum_{i\in\mathbb{Z}} \xi_i U_i^\rho (f_i(\hat{u}_i^\rho(\nu+s-\varrho^{(\rho)}(\nu+s))) - f_i(\hat{u}_i^0(\nu))) \\
&+ 2\alpha \sum_{i\in\mathbb{Z}} \xi_i V_i^\rho (f_i(v_i^\rho(\nu+s-\varrho^{(\rho)}(\nu+s))) - f_i(\hat{v}_i^0(\nu))), \qquad (4.4.8)
\end{aligned}
$$

where we recall that $\kappa = \min\{\lambda, \varsigma\}$. Using the same arguments as in (4.2.5) and (4.3.6), we deduce

$$
\begin{aligned}
&- 2\beta \sum_{i\in\mathbb{Z}} \xi_i U_i^\rho (AU^\rho)_i + 2\beta \sum_{i\in\mathbb{Z}} \xi_i U_i^\rho (F_i(\hat{u}_i^\rho(\nu+s)) - F_i(\hat{u}_i^0(\nu))) \\
&\leqslant 2.5^{3\sigma}\beta \|U^\rho\|_\sigma^2 + 2\beta\|\mu_3\|_{\ell^\infty}\|U^\rho\|_\sigma^2. \qquad (4.4.9)
\end{aligned}
$$

The Young inequality gives

$$
\begin{aligned}
&2\beta \sum_{i\in\mathbb{Z}} \xi_i U_i^\rho (g_i(\nu+s) - g_i(\nu)) + 2\alpha \sum_{i\in\mathbb{Z}} \xi_i V_i^\rho (h_i(\nu+s) - h_i(\nu)) \qquad (4.4.10)\\
&\leqslant \hat{c}_1 (\|g(\nu+s) - g(\nu)\|_\sigma^2 + \|h(\nu+s) - h(\nu)\|_\sigma^2) + \frac{\kappa}{4}(\beta\|U^\rho\|_\sigma^2 + \alpha\|V^\rho\|_\sigma^2).
\end{aligned}
$$

According to (4.1.34) in the hypothesis **F2**, we imply

$$
\begin{aligned}
&2\beta \sum_{i\in\mathbb{Z}} \xi_i U_i^\rho (f_i(\hat{u}_i^\rho(\nu+s-\varrho^{(\rho)}(\nu+s))) - f_i(\hat{u}_i^0(\nu))) \\
&\quad + 2\alpha \sum_{i\in\mathbb{Z}} \xi_i V_i^\rho (f_i(v_i^\rho(\nu+s-\varrho^{(\rho)}(\nu+s))) - f_i(\hat{v}_i^0(\nu))) \\
&\leqslant \frac{4L_f^2}{\kappa}\Big(\|\hat{u}^\rho(\nu+s-\varrho^{(\rho)}(\nu+s)) - \hat{u}^0(\nu)\|_\sigma^2 \\
&\quad + \|\hat{v}^\rho(\nu+s-\varrho^{(\rho)}(\nu+s)) - \hat{v}^0(\nu)\|_\sigma^2\Big) + \frac{\kappa}{4}(\beta\|U^\rho\|_\sigma^2 + \alpha\|V^\rho\|_\sigma^2). \qquad (4.4.11)
\end{aligned}
$$

Substituting (4.4.9)-(4.4.11) into (4.4.8), we obtain for $\nu > \tau - s$ and $s \in [-\rho, 0]$,

$$
\begin{aligned}
&\frac{d}{d\nu}\|W^\rho(\nu)\|_{X_\sigma}^2 + \frac{3}{2}\kappa\|W^\rho(\nu)\|_{X_\sigma}^2 \qquad (4.4.12)\\
&\leqslant (2.5^{3\sigma} + 2\|\mu_3\|_{\ell^\infty})\|W^\rho(\nu)\|_{X_\sigma}^2 + \hat{c}_1(\|g(\nu+s) - g(\nu)\|_\sigma^2 + \|h(\nu+s) - h(\nu)\|_\sigma^2) \\
&+ \frac{4L_f^2}{\kappa}\Big(\|\hat{u}^\rho(\nu+s-\varrho^{(\rho)}(\nu+s)) - \hat{u}^0(\nu)\|_\sigma^2 + \|\hat{v}^\rho(\nu+s-\varrho^{(\rho)}(\nu+s)) - \hat{v}^0(\nu)\|_\sigma^2\Big).
\end{aligned}
$$

Integrating (4.4.12) over $[\tau - s, \nu]$ with $\nu \in [\tau - s, \tau + T]$ and $T > \rho$,

$$
\begin{aligned}
\|W^\rho(\nu)\|_{X_\sigma}^2 &\leqslant \|W^\rho(\tau - s)\|_{X_\sigma}^2 + (2.5^{3\sigma} + 2\|\mu_3\|_{\ell^\infty}) \int_{\tau-s}^{\nu} \|W^\rho(r)\|_{X_\sigma}^2 dr \\
&\quad + \hat{c}_1 \int_{\tau-s}^{\nu} (\|g(r+s) - g(r)\|_\sigma^2 + \|h(r+s) - h(r)\|_\sigma^2) dr \qquad (4.4.13)
\end{aligned}
$$

$$+\frac{4L_f^2}{\kappa}\int_{\tau-s}^{\nu}\Big(\|\hat{u}^\rho(r+s-\varrho^{(\rho)}(r+s))-\hat{u}^0(r)\|_\sigma^2 + \|\hat{v}^\rho(r+s-\varrho^{(\rho)}(r+s))-\hat{v}^0(r)\|_\sigma^2\Big)dr.$$

Note that

$$\begin{aligned}\|W^\rho(\tau-s)\|_{X_\sigma}^2 &= \beta\|\hat{\phi}^\rho(0)-\hat{u}^0(\tau-s,\tau,\hat{\phi}^0)\|_\sigma^2+\alpha\|\hat{v}^\rho(0)-\hat{v}^0(\tau-s,\tau,\hat{v}^0)\|_\sigma^2\\ &\leqslant 2(\mathrm{d}^*_{X^\rho_\sigma}(\hat{\psi}^\rho,\hat{\psi}^0))^2+2\beta\|\hat{\phi}^0-\hat{u}^0(\tau-s,\tau,\hat{\phi}^0)\|_\sigma^2+2\alpha\|\hat{v}^0-\hat{v}^0(\tau-s,\tau,\hat{v}^0)\|_\sigma^2. \quad (4.4.14)\end{aligned}$$

For all $r\in\mathbb{R}$, $s\in[-\rho,0]$, let $\zeta=y(r)=r+s-\varrho^{(\rho)}(r+s)$, then $y'(r)\geqslant 1-\rho_*>0$, and thus there exists an inverse function such that $r=y^{-1}(\zeta)$ for all $\zeta\in\mathbb{R}$. If let $\hat{r}=r-\varrho^{(\rho)}(r+s)$, then $r=y^{-1}(\hat{r}+s)$ and

$$\begin{aligned}&\int_{\tau-s}^{\nu}\|\hat{u}^\rho(r+s-\varrho^{(\rho)}(r+s))-\hat{u}^0(r)\|_\sigma^2dr\\ &=\int_{\tau-s}^{y^{-1}(\tau)}\|\hat{u}^\rho(r+s-\varrho^{(\rho)}(r+s))-\hat{u}^0(r)\|_\sigma^2dr+\int_{y^{-1}(\tau)}^{\nu}\|\hat{u}^\rho(r+s-\varrho^{(\rho)}(r+s))-\hat{u}^0(r)\|_\sigma^2dr\\ &\leqslant 2\int_{\tau-s}^{y^{-1}(\tau)}\|\hat{u}^\rho(r+s-\varrho^{(\rho)}(r+s))-\hat{\phi}^0\|_\sigma^2dr+2\int_{\tau-s}^{y^{-1}(\tau)}\|\hat{u}^0(r)-\hat{\phi}^0\|_\sigma^2dr\\ &\quad+\frac{1}{1-\rho_*}\int_{\tau-s}^{\nu}\|\hat{u}^\rho(\hat{r}+s)-\hat{u}^0(y^{-1}(\hat{r}+s))\|_\sigma^2d\hat{r}\\ &\leqslant\frac{2}{1-\rho_*}\int_{\tau-\varrho^{(\rho)}(\tau)}^{\tau}\|\hat{u}^\rho(r)-\hat{\phi}^0\|_\sigma^2dr+2\int_{\tau}^{\tau+2\rho}\|\hat{u}^0(r)-\hat{\phi}^0\|_\sigma^2dr\\ &\quad+\frac{2}{1-\rho_*}\int_{\tau-s}^{\nu}\|\hat{u}^\rho(r+s)-\hat{u}^0(r)\|_\sigma^2dr+\frac{2}{1-\rho_*}\int_{\tau-s}^{\nu}\|\hat{u}^0(h^{-1}(r+s))-\hat{u}^0(r)\|_\sigma^2dr\\ &\leqslant\frac{2\rho_0}{1-\rho_*}\sup_{s\in[-\rho,0]}\|\hat{\phi}^\rho(s)-\hat{\phi}^0\|_\sigma^2+2\int_{\tau}^{\tau+2\rho}\|\hat{u}^0(r)-\hat{\phi}^0\|_\sigma^2dr \quad (4.4.15)\\ &\quad+\frac{2}{1-\rho_*}\int_{\tau-s}^{\nu}\|U^\rho(r)\|_\sigma^2dr+\frac{2}{1-\rho_*}\int_{\tau}^{\tau+T}\|\hat{u}^0(h^{-1}(r+s))-\hat{u}^0(r)\|_\sigma^2dr.\end{aligned}$$

Similarly, we deduce

$$\begin{aligned}&\int_{\tau-s}^{\nu}\|\hat{v}^\rho(r+s-\varrho^{(\rho)}(r+s))-\hat{v}^0(r)\|_\sigma^2dr\\ &\leqslant\frac{2\rho_0}{1-\rho_*}\sup_{s\in[-\rho,0]}\|\hat{v}^\rho(s)-\hat{v}^0\|_\sigma^2+2\int_{\tau}^{\tau+2\rho}\|\hat{v}^0(r)-\hat{v}^0\|_\sigma^2dr \quad (4.4.16)\\ &\quad+\frac{2}{1-\rho_*}\int_{\tau-s}^{\nu}\|V^\rho(r)\|_\sigma^2dr+\frac{2}{1-\rho_*}\int_{\tau}^{\tau+T}\|\hat{v}^0(h^{-1}(r+s))-\hat{v}^0(r)\|_\sigma^2dr.\end{aligned}$$

It follows from (4.4.13)-(4.4.16) that

$$\|W^\rho(\nu)\|_{X_\sigma}^2\leqslant\hat{c}_2\int_{\tau-s}^{\nu}\|W^\rho(r)\|_{X_\sigma}^2dr+\hat{c}_3(\mathrm{d}^*_{X^\rho_\sigma}(\hat{\psi}^\rho,\hat{\psi}^0))^2$$

$$
\begin{aligned}
&+ 2\beta\|\hat{\phi}^0 - \hat{u}^0(\tau - s, \tau, \hat{\phi}^0)\|_\sigma^2 + 2\alpha\|\hat{\upsilon}^0 - \hat{v}^0(\tau - s, \tau, \hat{\upsilon}^0)\|_\sigma^2 \\
&+ \hat{c}_4 \int_\tau^{\tau+2\rho} (\|\hat{u}^0(r) - \hat{\phi}^0\|_\sigma^2 + \|\hat{v}^0(r) - \hat{\upsilon}^0\|_\sigma^2) dr \\
&+ \hat{c}_5 \int_\tau^{\tau+T} \Big(\|\hat{u}^0(h^{-1}(r+s)) - \hat{u}^0(r)\|_\sigma^2 + \|\hat{v}^0(h^{-1}(r+s)) - \hat{v}^0(r)\|_\sigma^2\Big) dr \\
&+ \hat{c}_1 \int_\tau^{\tau+T} (\|g(r+s) - g(r)\|_\sigma^2 + \|h(r+s) - h(r)\|_\sigma^2) dr. \qquad (4.4.17)
\end{aligned}
$$

Applying the Gronwall lemma to (4.4.17), we deduce, for all $\nu \in [\tau - s, \tau + T]$,

$$
\begin{aligned}
\|W^\rho(\nu)\|_{X_\sigma}^2 \leqslant\ & \hat{c}_3 e^{\hat{c}_2 T} (\mathrm{d}^*_{X_\sigma^\rho}(\hat{\psi}^\rho, \hat{\psi}^0))^2 + 2\beta e^{\hat{c}_2 T}\|\hat{\phi}^0 - \hat{u}^0(\tau - s, \tau, \hat{\phi}^0)\|_\sigma^2 + 2\alpha e^{\hat{c}_2 T}\|\hat{\upsilon}^0 - \hat{v}^0(\tau - s, \tau, \hat{\upsilon}^0)\|_\sigma^2 \\
&+ \hat{c}_4 e^{\hat{c}_2 T} \int_\tau^{\tau+2\rho} (\|\hat{u}^0(r) - \hat{\phi}^0\|_\sigma^2 + \|\hat{v}^0(r) - \hat{\upsilon}^0\|_\sigma^2) dr \\
&+ \hat{c}_5 e^{\hat{c}_2 T} \int_\tau^{\tau+T} \Big(\|\hat{u}^0(h^{-1}(r+s)) - \hat{u}^0(r)\|_\sigma^2 + \|\hat{v}^0(h^{-1}(r+s)) - \hat{v}^0(r)\|_\sigma^2\Big) dr \\
&+ \hat{c}_1 e^{\hat{c}_2 T} \int_\tau^{\tau+T} (\|g(r+s) - g(r)\|_\sigma^2 + \|h(r+s) - h(r)\|_\sigma^2) dr. \qquad (4.4.18)
\end{aligned}
$$

By (4.4.6), we imply the first term on the right-hand side of (4.4.18) $\hat{c}_3 e^{\hat{c}_2 T} (\mathrm{d}^*_{X_\sigma^\rho}(\hat{\psi}^\rho, \hat{\psi}^0))^2 \to 0$ as $\rho \to 0$. Then we infer from the continuity of $\hat{u}^0(\cdot, \tau, \hat{\phi}^0), \hat{v}^0(\cdot, \tau, \hat{\upsilon}^0)$ at τ and $s \in [-\rho, 0]$ that

$$
\begin{aligned}
&2\beta e^{\hat{c}_2 T}\|\hat{\phi}^0 - \hat{u}^0(\tau - s, \tau, \hat{\phi}^0)\|_\sigma^2 + 2\alpha e^{\hat{c}_2 T}\|\hat{\upsilon}^0 - \hat{v}^0(\tau - s, \tau, \hat{\upsilon}^0)\|_\sigma^2 \\
&\quad + \hat{c}_4 e^{\hat{c}_2 T} \int_\tau^{\tau+2\rho} (\|\hat{u}^0(r) - \hat{\phi}^0\|_\sigma^2 + \|\hat{v}^0(r) - \hat{\upsilon}^0\|_\sigma^2) dr \to 0, \text{ as } \rho \to 0. \qquad (4.4.19)
\end{aligned}
$$

Since $\hat{u}^0, \hat{v}^0$ are uniformly continuous over $[\tau, \tau + T + \rho]$, then the third line of (4.4.18) is bounded by

$$
\hat{c}_5 e^{\hat{c}_2 T} \int_\tau^{\tau+T} \Big(\|\hat{u}^0(h^{-1}(r+s)) - \hat{u}^0(r)\|_\sigma^2 + \|\hat{v}^0(h^{-1}(r+s)) - \hat{v}^0(r)\|_\sigma^2\Big) dr \to 0, \qquad (4.4.20)
$$

as $\rho \to 0$. Thanks to $g, h \in L^2_{loc}(\mathbb{R}, \ell^2_\sigma)$ and $s \in [-\rho, 0]$, the last line of (4.4.18) satisfies

$$
\hat{c}_1 e^{\hat{c}_2 T} \int_\tau^{\tau+T} (\|g(r+s) - g(r)\|_\sigma^2 + \|h(r+s) - h(r)\|_\sigma^2) dr \to 0, \text{ as } \rho \to 0. \qquad (4.4.21)
$$

Collecting the above estimations, we deduce that for all $\nu \in [\tau - s, \tau + T]$ and $s \in [-\rho, 0]$,

$$
\|W^\rho(\nu)\|_{X_\sigma}^2 \to 0, \text{ as } \rho \to 0. \qquad (4.4.22)
$$

We now consider the other case $\nu \in [\tau, \tau - s]$. Let $\mu = \nu - \tau$. Then we obtain $\nu = \mu + \tau$ and $0 \leqslant \mu \leqslant \rho$. Therefore,

$$
\|W^\rho(\nu)\|_{X_\sigma}^2 = \|\hat{\varphi}^\rho(\nu + s, \tau, \hat{\psi}^\rho) - \hat{\varphi}^0(\nu, \tau, \hat{\psi}^0)\|_{X_\sigma}^2
$$

$$\begin{aligned}&\leqslant 2\|\hat{\varphi}^{\rho}(\nu+s,\tau,\hat{\psi}^{\rho})-\hat{\psi}^0\|^2_{X_\sigma}+2\|\hat{\varphi}^0(\nu,\tau,\hat{\psi}^0)-\hat{\psi}^0\|^2_{X_\sigma}\\&\leqslant 2\sup_{s\in[-\rho,0]}\|\hat{\psi}^{\rho}(s)-\hat{\psi}^0\|^2_{X_\sigma}+2\|\hat{\varphi}^0(\mu+\tau,\tau,\hat{\psi}^0)-\hat{\psi}^0\|^2_{X_\sigma}.\end{aligned}$$

By the continuity of $\hat{\varphi}^0=(\hat{\phi}^0,\hat{\upsilon}^0)$ at τ, $\mu\in[0,-s]$ and the condition (4.4.6), we imply the above inequality goes to zero as $\rho\to 0$, which together with (4.4.22), yields that for all $\nu\in[\tau,\tau+T]$ and $s\in[-\rho,0]$, (4.4.7) holds true. □

Lemma 4.4.3. *Let hypotheses* **E, F1, F2, G1-G3** *and* (4.1.35) *be satisfied. If* $\rho_n\to 0$, $t\in\mathbb{R}$ *and* $\psi_n=(\phi_n,\upsilon_n)\in\mathcal{A}_{\rho_n}(t)\subset X_\sigma^{\rho_n}$, *then there exist* $\hat{\psi}^0=(\hat{\phi}^0,\hat{\upsilon}^0)\in X_\sigma$ *and an index subsequence* $\{n^*\}$ *of* $\{n\}$ *such that*

$$\mathrm{d}^*_{X_\sigma^{\rho_{n^*}}}(\psi_{n^*},\hat{\psi}^0)=\sup_{s\in[-\rho_{n^*},0]}\|\psi_{n^*}(s)-\hat{\psi}^0\|_{X_\sigma}\to 0,\ as\ n^*\to\infty.\tag{4.4.23}$$

Proof. Take a sequence $\tau_n\to-\infty$. By the invariance of $\mathcal{A}_{\rho_n}(\cdot)$, there exists a $\hat{\psi}_n:=(\hat{\phi}_n,\hat{\upsilon}_n)\in\mathcal{A}_{\rho_n}(\tau_n)$ such that

$$\psi_n=\Psi_{\rho_n}(t,\tau_n)\hat{\psi}_n.\tag{4.4.24}$$

By $\mathcal{A}_{\rho_n}\in\mathfrak{D}_{\rho_n}$, and using the same method as in Step 1 of Lemma 4.4.1, we deduce that $\{(\Psi_{\rho_n}(t,\tau_n)\hat{\psi}_n)(0)\}_{n\in\mathbb{N}}$ is pre-compact in $X_\sigma=\ell^2_\sigma\times\ell^2_\sigma$, and thus there exist a $\hat{\psi}^0:=(\hat{\phi}^0,\hat{\upsilon}^0)\in X_\sigma$ and an index subsequence $\{n^*\}$ of $\{n\}$ such that

$$\|(\Psi_{\rho_{n^*}}(t,\tau_{n^*})\hat{\psi}_{n^*})(0)-\hat{\psi}^0\|_{X_\sigma}\to 0,\ \text{as}\ n^*\to+\infty,$$

which implies that for given any $\epsilon>0$, there exists $N_1\geqslant 1$ such that for all $n^*\geqslant N_1$,

$$\|(\Psi_{\rho_{n^*}}(t,\tau_{n^*})\hat{\psi}_{n^*})(0)-\hat{\psi}^0\|_{X_\sigma}\leqslant\epsilon.\tag{4.4.25}$$

By the arguments as in Step 2 of Lemma 4.4.1, we imply that there exists $\iota>0$ with $|s_1-s_2|<\iota$ such that for all $\epsilon>0$,

$$\|(\Psi_{\rho_{n^*}}(t,\tau_{n^*})\hat{\psi}_{n^*})(s_1)-(\Psi_{\rho_{n^*}}(t,\tau_{n^*})\hat{\psi}_{n^*})(s_2)\|_{X_\sigma}\leqslant\epsilon.$$

Since $\rho_{n^*}\to 0$ as $n^*\to+\infty$, there exists $N_2\geqslant N_1$ such that $\rho_{n^*}<\iota$ for all $n^*\geqslant N_2$, then

$$\|(\Psi_{\rho_{n^*}}(t,\tau_{n^*})\hat{\psi}_{n^*})(s)-(\Psi_{\rho_{n^*}}(t,\tau_{n^*})\hat{\psi}_{n^*})(0)\|_{X_\sigma}\leqslant\epsilon,\tag{4.4.26}$$

for all $s\in[-\rho_{n^*},0]$. It follows from (4.4.24)-(4.4.26) that there exists $N_3\geqslant N_2$ such that

$$\begin{aligned}\|\psi_{n^*}(s)-\hat{\psi}^0\|_{X_\sigma}&=\|(\Psi_{\rho_{n^*}}(t,\tau_{n^*})\hat{\psi}_{n^*})(s)-\hat{\psi}^0\|_{X_\sigma}\\&\leqslant\|(\Psi_{\rho_{n^*}}(t,\tau_{n^*})\hat{\psi}_{n^*})(s)-(\Psi_{\rho_{n^*}}(t,\tau_{n^*})\hat{\psi}_{n^*})(0)\|_{X_\sigma}\\&\quad+\|(\Psi_{\rho_{n^*}}(t,\tau_{n^*})\hat{\psi}_{n^*})(0)-\hat{\psi}^0\|_{X_\sigma}\leqslant 2\epsilon,\end{aligned}$$

for all $n^*\geqslant N_3$ and $s\in[-\rho_{n^*},0]$, which yields (4.4.23) as desired. □

Theorem 4.4.4. *Let hypotheses* **E, F1, F2, G1-G3***, (4.1.35) be satisfied. Suppose $\mathcal{A}_\rho$ is the $\mathfrak{D}_\rho$-pullback attractor of deterministic delay lattice system (0.0.9) and $\mathcal{A}_0$ is the $\mathfrak{D}_0$-pullback attractor of deterministic non-delay lattice system (4.4.2). Then $\mathcal{A}_\rho$ converges to $\mathcal{A}_0$, i.e.*

$$\lim_{\rho\to 0} \mathrm{d}^*_{X^\rho_\sigma}(\mathcal{A}_\rho(t), \mathcal{A}_0(t)) = 0, \quad \forall\, t \in \mathbb{R}. \tag{4.4.27}$$

Proof. If (4.4.27) does not hold true, then there exist $\epsilon > 0, \rho_n \to 0$ and $\psi_n := (\phi_n, \upsilon_n) \in \mathcal{A}_{\rho_n}(t)$ such that

$$\mathrm{d}^*_{X^{\rho_n}_\sigma}(\psi_n, \mathcal{A}_0(t)) \geqslant \epsilon, \quad \forall\, n \in \mathbb{N}. \tag{4.4.28}$$

Thanks to (4.4.23) in Lemma 4.4.3, there exist a subsequence ψ_n (relabeled the same) and an element $\hat{\psi}^0 := (\hat{\phi}^0, \hat{\upsilon}^0) \in X_\sigma$ such that

$$\lim_{n\to\infty} \sup_{s\in[-\rho_n,0]} \|\psi_n(s) - \hat{\psi}^0\|_{X_\sigma} = 0. \tag{4.4.29}$$

We now prove that $\hat{\psi} \in \mathcal{A}_0(t)$. By the invariance of $\mathcal{A}_{\rho_n}$, there exists $\hat{\psi}^k_n := (\hat{\phi}^k_n, \hat{\upsilon}^k_n) \in \mathcal{A}_{\rho_n}(\tau_k)$ such that

$$\psi_n = \Psi_{\rho_n}(t, \tau_k)\hat{\psi}^k_n, \ \forall\, n, k \in \mathbb{N}, \tag{4.4.30}$$

where $\tau_k \to -\infty$ as $k \to +\infty$. By (4.4.23) in Lemma 4.4.3, there exist a subsequence of $\hat{\psi}^k_n$ and an element $\hat{\psi}^k \in X_\sigma$ such that

$$\mathrm{d}^*_{X^{\rho_{n^*}}_\sigma}(\hat{\psi}^k_{n^*}, \hat{\psi}^k) \to 0, \ \text{as } n^* \to +\infty. \tag{4.4.31}$$

It follows from a diagonal process that there exists an index subsequence (relabeled the same) of $\{n^*\}$ such that

$$\lim_{n^*\to+\infty} \sup_{s\in[-\rho_{n^*},0]} \|\hat{\psi}^k_{n^*}(s) - \hat{\psi}^k\|_{X_\sigma} \to 0, \ \forall\, k \in \mathbb{N}. \tag{4.4.32}$$

By (4.4.7) in Lemma 4.4.2, we have

$$\lim_{n^*\to+\infty} \sup_{s\in[-\rho_{n^*},0]} \|\Psi_{\rho_{n^*}}(t, \tau_k)\hat{\psi}^k_{n^*}(s) - \Psi_0(t, \tau_k)\hat{\psi}^k\|^2_{X_\sigma} = 0, \ \forall\, k \in \mathbb{N}, \tag{4.4.33}$$

which, together with (4.4.29) and (4.4.30), implies

$$\hat{\psi}^0 = \Psi_0(t, \tau_k)\hat{\psi}^k, \ \forall\, k \in \mathbb{N}. \tag{4.4.34}$$

Since $\mathcal{K}_{\rho_n}$ is a pullback $\mathfrak{D}_{\rho_n}$-absorbing set, and by the invariance of $\mathcal{A}_{\rho_n}$, there exist a $\hat{\tau}_k := \hat{\tau}_k(\tau_k, \mathcal{A}_{\rho_n}) \leqslant \tau_k$ such that

$$\begin{aligned} \mathcal{A}_{\rho_n}(\tau_k) &= \Psi_{\rho_n}(\tau_k, \hat{\tau}_k)\mathcal{A}_{\rho_n}(\hat{\tau}_k) \\ &\subset \mathcal{K}_{\rho_n}(\tau_k), \end{aligned} \tag{4.4.35}$$

which shows $\hat{\psi}_n^k \in \mathcal{K}_{\rho_n}(\tau_k)$. Combining (4.4.5) and (4.4.32), we obtain, for all $k \in \mathbb{N}$,

$$\begin{aligned}
\|\hat{\psi}^k\|_{X_\sigma}^2 &= \lim_{n\to+\infty} \|\hat{\psi}_n^k(0)\|_{X_\sigma}^2 \\
&\leqslant \limsup_{n\to+\infty} \|\hat{\psi}_n^k\|_{X_\sigma^\rho}^2 \\
&\leqslant \|\mathcal{K}_0(\tau_k)\|_{X_\sigma}^2.
\end{aligned} \tag{4.4.36}$$

As $\mathcal{A}_0(\cdot)$ is a pullback $\mathcal{D}_0$-attracting set, and by (4.4.34) and $\mathcal{K}_0 \in \mathcal{D}_0$, we deduce

$$\begin{aligned}
\mathrm{d}_{X_\sigma}^*(\hat{\psi}^0, \mathcal{A}_0(t)) &\leqslant \mathrm{d}_{X_\sigma}^*(\Psi_0(t,\tau_k)\hat{\psi}^k, \mathcal{A}_0(t)) \\
&\leqslant \mathrm{d}_{X_\sigma}^*(\Psi_0(t,\tau_k)\mathcal{K}_0(\tau_k), \mathcal{A}_0(t)) \\
&\to 0, \text{ as } n \to +\infty,
\end{aligned} \tag{4.4.37}$$

which implies $\hat{\psi}^0 \in \mathcal{A}_0(t)$. We then infer from (4.4.29) that

$$\begin{aligned}
\mathrm{d}_{X_\sigma^{\rho_{n^*}}}^*(\psi_n, \mathcal{A}_0(t)) &\leqslant \sup_{s\in[-\rho_n,0]} \|\psi_n(s) - \hat{\psi}^0\|_{X_\sigma} + \mathrm{d}_{X_\sigma}^*(\hat{\psi}^0, \mathcal{A}_0(t)) \\
&\to 0, \text{ as } n \to +\infty.
\end{aligned} \tag{4.4.38}$$

This contradicts with (4.4.28). The proof is complete. □

Part III

Numerical attractors for lattice systems

Chapter 5

Optimization and convergence of numerical attractors for discrete-time quasi-linear lattice systems

Since the nonlinear operator A_p $(p > 2)$ has not a Fréchet derivative, we first establish a new type of Taylor expansions and give the continuous-time error of solutions for the LDS (0.0.12). Next, we prove the existence and uniqueness of a numerical attractor $\mathcal{A}_\epsilon$ for sufficiently small step sizes. We then derive its optimized bound given by $\|g\|/\alpha$. Besides, we establish the upper semi-continuity from the numerical attractor $\mathcal{A}_\epsilon$ to the global attractor $\mathcal{A}$ as $\epsilon \to 0$. Eventually, we study the finitely dimensional approximation of the numerical attractor.

5.1 Positively invariant ball and global attractor for p-Laplace lattice

The discrete p-Laplace operator A_p $(p \geqslant 2)$ can be formally written as

$$A_p u = -B^*(|Bu|^{p-2}Bu),\ (Bu)_i := u_{i+1} - u_i,\ (B^*u)_i := u_{i-1} - u_i,$$

where $u = (u_i)_{i\in\mathbb{Z}}$, $|u|^q = (|u_i|^q)_{i\in\mathbb{Z}}$ and $uv = (u_i v_i)_{i\in\mathbb{Z}}$. By [57], we have $(A_p u, u) = -\|Bu\|_p^p$, where $\|\cdot\|_q$ (omitting the subscript if $q = 2$) denotes the norm in the Banach space

$$\ell^q := \{u = (u_i)_{i\in\mathbb{Z}} :\ \|u\|_q^q = \sum_{i\in\mathbb{Z}} |u_i|^q < \infty\},\ q \geqslant 1.$$

We assume that $g = (g_i)_{i\in\mathbb{Z}} \in \ell^2$ and $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschtz continuous, i.e. for each $r > 0$, there is $L_r \geqslant 0$ (increasingly in r) such that

$$|f(s_1) - f(s_2)| \leqslant L_r|s_1 - s_2|,\ \forall |s_1| \leqslant r,\ |s_2| \leqslant r, \tag{5.1.1}$$

and the dissipative condition (0.0.13) holds. Note that $f \in C^1(\mathbb{R},\mathbb{R})$ implies (5.1.1), which together with (0.0.13) implies that

$$f(s)s \leqslant -\alpha s^2,\ \forall s \in \mathbb{R} \text{ and } f(0) = 0. \tag{5.1.2}$$

Moreover, we obtain the bounded and locally Lipschtz continuous Nemytskii operator

$$F : \ell^2 \to \ell^2,\ F(u) = (f(u_i))_{i\in\mathbb{Z}},\ \forall u \in \ell^2.$$

Now, the p-Laplace LDS (0.0.12) is rewritten as an abstract form

$$\begin{cases} \dfrac{du(t)}{dt} = \nu A_p u(t) + F(u(t)) + g,\ t > 0, \\ u(0) = u_0 \in \ell^2, \end{cases} \tag{5.1.3}$$

where $\nu > 0$, $p > 2$. Although the dissipative condition ((0.0.13) or equivalently (5.1.2)) is different from those in [39, 57], one can similarly prove that the p-Laplace LDS (5.1.3) has a unique solution $u \in C([0,\infty),\ell^2)$ for each $u_0 \in \ell^2$, which generates a continuous semigroup (semi dynamical system) defined by

$$S(t) : \ell^2 \to \ell^2,\ \ S(t)u_0 = u(t;u_0),\ \forall t \geqslant 0, u_0 \in \ell^2.$$

Lemma 5.1.1. *The semigroup $S(\cdot)$ has a positively invariant absorbing ball*

$$\mathcal{B}_{r^*}(0) := \{x \in \ell^2 : \|x\| \leqslant r^* := \sqrt{1 + \|g\|^2/\alpha^2}\}.$$

Proof. By the inner product of (5.1.3) with $u(t)$, using (5.1.2) we obtain

$$\begin{aligned} &\frac{d}{dt}\|u\|^2 = 2\nu(A_p u, u) + 2(F(u), u) + 2(g, u) \\ &= -\|Bu\|_p^p - 2\alpha \sum_{i\in\mathbb{Z}} f(u_i)u_i + 2(g, u) \\ &\leqslant -2\alpha\|u\|^2 + \left(\alpha\|u\|^2 + \frac{\|g\|^2}{\alpha}\right) = -\alpha\|u\|^2 + \frac{\|g\|^2}{\alpha}. \end{aligned}$$

The Gronwall lemma yields

$$\|u(t)\|^2 \leqslant e^{-\alpha t}\|u_0\|^2 + \frac{\|g\|^2}{\alpha^2}(1 - e^{-\alpha t}),\ \forall t \geqslant 0. \tag{5.1.4}$$

For all $u_0 \in \mathcal{B}_r(0)$ with arbitrary radius $r > 0$, we have

$$\|u(t,u_0)\|^2 \leqslant e^{-\alpha t} r^2 + \frac{\|g\|^2}{\alpha^2}(1 - e^{-\alpha t}) \leqslant 1 + \frac{\|g\|^2}{\alpha^2} = (r^*)^2$$

if $t \geqslant \frac{2}{\alpha}\log r$. Hence $\mathcal{B}_{r^*}(0)$ is an absorbing ball.

By (5.1.4) again, for all $u_0 \in \mathcal{B}_{r^*}(0)$ and $t \geqslant 0$,

$$\begin{aligned}\|u(t,u_0)\|^2 &\leqslant e^{-\alpha t}\Big(1+\frac{\|g\|^2}{\alpha^2}\Big)+\frac{\|g\|^2}{\alpha^2}(1-e^{-\alpha t})\\ &= e^{-\alpha t}+\frac{\|g\|^2}{\alpha^2}\leqslant 1+\frac{\|g\|^2}{\alpha^2}=(r^*)^2.\end{aligned}$$

Hence $\mathcal{B}_{r^*}(0)$ is also positively invariant under $S(\cdot)$. □

We remark here that the bigger radius $\sqrt{1+2\|g\|^2/\alpha^2}$ was used in [64].

By the technique of a cut-off function (see e.g. [6, 151] or Lemma 5.4.1 later), one can give the uniform estimates of the tail of the solution on the ball $\mathcal{B}_{r^*}(0)$, which leads to the existence of a global attractor. The proof is similar as those in [57].

Theorem 5.1.2. *The semigroup $S(\cdot)$, generated from the p-Laplace lattice, possesses a unique global attractor $\mathcal{A} \subset \mathcal{B}_{r^*}(0)$.*

5.2 Numerical solutions and discrete semigroup on a ball

The implicit Euler scheme for the p-Laplace LDS (5.1.3) with the step size $\epsilon > 0$ can be read as

$$\begin{cases} u_n^\epsilon = u_{n-1}^\epsilon + \epsilon\nu A_p u_n^\epsilon + \epsilon F(u_n^\epsilon) + \epsilon g,\\ u_0^\epsilon = u_0 \in \ell^2. \end{cases} \tag{5.2.1}$$

Note that there does not exist a common step size such that (5.2.1) is solvable for all initial data (see Lemma 5.4.3 later or see [64, 112] in the case of $p = 2$). So, we will restrict (5.2.1) on the ball $\mathcal{B}_{r^*}(0)$ to ensure the existence of a discrete-time dynamical system for at least one size.

We need the local Lipschitz continuity of the discrete p-Laplace operator

$$\|A_p u - A_p v\| \leqslant L_{p,r}\|u-v\| \text{ and } \|A_p u\| \leqslant L_{p,r}\|u\|,\ \forall u,v \in \mathcal{B}_r(0), \tag{5.2.2}$$

where $L_{p,r} := (p-1)2^{2p}r^{p-2}$ depends increasingly on $r \geqslant 0$. The proof of (5.2.2) is similar to those in [126].

Theorem 5.2.1. *There is $\epsilon^* > 0$ such that, for each $\epsilon \in (0,\epsilon^*]$ and $u_0 \in \mathcal{B}_{r^*}(0)$, the IES (5.2.1) has a unique solution such that*

$$u_n^\epsilon(u_0) \in \mathcal{B}_{r^*}(0),\ \forall n \in \mathbb{N},\ \ \textit{where } r^* = \sqrt{1+\|g\|^2/\alpha^2}.$$

Proof. We recursively prove the theorem in four steps.

Step 1: In the case of $n = 1$, we find an $\epsilon^* > 0$ such that the IES (5.2.1) has a solution

$$u_1^\epsilon(u_0) \in \mathcal{B}_{r^*+1}(0),\ \forall \epsilon \in (0,\epsilon^*],\ u_0 \in \mathcal{B}_{r^*}(0),$$

where the radius is temporarily enlarged (from r^* to r^*+1).

For each $\epsilon > 0$ and $u_0 \in \mathcal{B}_{r^*}(0)$, we define an operator $\mathcal{M}^\epsilon_{u_0} : \ell^2 \to \ell^2$ by

$$\mathcal{M}^\epsilon_{u_0}(x) = u_0 + \epsilon\nu A_p x + \epsilon F(x) + \epsilon g, \quad \forall x \in \ell^2. \tag{5.2.3}$$

We prove that $\mathcal{M}^\epsilon_{u_0}$ maps $\mathcal{B}_{r^*+1}(0)$ into itself if ϵ is small enough. Since A_p is bounded, it follows from the second inequality of (5.2.2) that

$$\nu\|A_p x\| \leqslant \nu L_{p,r^*+1}\|x\| \leqslant (r^* + 1)\nu L_{p,r^*+1}, \quad \forall x \in \mathcal{B}_{r^*+1}(0).$$

By the local Lipschitz continuity (5.1.1) and $f(0) = 0$, we obtain

$$\|F(x)\| \leqslant L_{r^*+1}\|x\| \leqslant (r^* + 1)L_{r^*+1}, \quad \forall x \in \mathcal{B}_{r^*+1}(0).$$

Hence, for all $u_0 \in \mathcal{B}_{r^*}(0)$ and $x \in \mathcal{B}_{r^*+1}(0)$, we have

$$\begin{aligned}\|\mathcal{M}^\epsilon_{u_0}(x)\| &\leqslant \|u_0\| + \epsilon(\nu\|A_p x\| + \|F(x)\| + \|g\|) \\ &\leqslant r^* + \epsilon\Big((r^* + 1)(\nu L_{p,r^*+1} + L_{r^*+1}) + \|g\|\Big).\end{aligned}$$

We define an essential constant by

$$\epsilon^* := \frac{1}{(r^* + 1)(\nu L_{p,r^*+1} + L_{r^*+1}) + \|g\|}. \tag{5.2.4}$$

Then for all $\epsilon \in (0, \epsilon^*]$, $u_0 \in \mathcal{B}_{r^*}(0)$ and $x \in \mathcal{B}_{r^*+1}(0)$,

$$\|\mathcal{M}^\epsilon_{u_0}(x)\| \leqslant r^* + 1, \quad \text{and so } \mathcal{M}^\epsilon_{u_0}(\mathcal{B}_{r^*+1}(0)) \subset \mathcal{B}_{r^*+1}(0).$$

We then prove that, for each $\epsilon \in (0, \epsilon^*]$ and $u_0 \in \mathcal{B}_{r^*}(0)$, the mapping $\mathcal{M}^\epsilon_{u_0} : \mathcal{B}_{r^*+1}(0) \to \mathcal{B}_{r^*+1}(0)$ is contractive. Indeed, by the local Lipschitz continuity in (5.1.1) and (5.2.2), for all $x, y \in \mathcal{B}_{r^*+1}(0)$,

$$\begin{aligned}\|\mathcal{M}^\epsilon_{u_0}(x) - \mathcal{M}^\epsilon_{u_0}(y)\| &\leqslant \epsilon(\nu\|A_p x - A_p y\| + \|F(x) - F(y)\|) \\ &\leqslant \epsilon(\nu L_{p,r^*+1} + L_{r^*+1})\|x - y\|.\end{aligned} \tag{5.2.5}$$

If $\epsilon \in (0, \epsilon^*]$, where ϵ^* is the constant defined by (5.2.4), then

$$\begin{aligned}&\epsilon(\nu L_{p,r^*+1} + L_{r^*+1}) \leqslant \epsilon^*(\nu L_{p,r^*+1} + L_{r^*+1}) \\ &= \frac{\nu L_{p,r^*+1} + L_{r^*+1}}{(r^* + 1)(\nu L_{p,r^*+1} + L_{r^*+1}) + \|g\|} \leqslant \frac{1}{r^* + 1} < 1.\end{aligned}$$

By the contraction mapping principle, for each $\epsilon \in (0, \epsilon^*]$ and $u_0 \in \mathcal{B}_{r^*}(0)$, the mapping $\mathcal{M}^\epsilon_{u_0} : \mathcal{B}_{r^*+1}(0) \to \mathcal{B}_{r^*+1}(0)$ has a unique fixed point

$$u^\epsilon_1 \in \mathcal{B}_{r^*+1}(0) \text{ such that } \mathcal{M}^\epsilon_{u_0}(u^\epsilon_1) = u^\epsilon_1,$$

which is the unique solution of the IES (5.2.1) for $n = 1$ in $\mathcal{B}_{r^*+1}(0)$.

Step 2: To ensure that ϵ^* does not change in the recursive proofs, we further prove that the unique solution in Step 1 satisfies

$$u_1^\epsilon(u_0) \in \mathcal{B}_{r^*}(0), \ \forall \epsilon \in (0, \epsilon^*], \ \ u_0 \in \mathcal{B}_{r^*}(0).$$

For this purpose, we take the inner product of the equation (5.2.1) for $n = 1$ by u_1^ϵ to obtain

$$\|u_1^\epsilon\|^2 = (u_0, u_1^\epsilon) + \epsilon\nu(A_p u_1^\epsilon, u_1^\epsilon) + \epsilon(F(u_1^\epsilon), u_1^\epsilon) + \epsilon(g, u_1^\epsilon). \tag{5.2.6}$$

By (5.1.2),

$$\epsilon(F(u_1^\epsilon), u_1^\epsilon) = \epsilon \sum_{i\in\mathbb{Z}} f(u_{1,i}^\epsilon)u_{1,i}^\epsilon \leqslant -\epsilon\alpha\|u_1^\epsilon\|^2.$$

The Young inequality implies

$$(u_0, u_1^\epsilon) \leqslant \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\|u_1^\epsilon\|^2 \text{ and } \epsilon(g, u_1^\epsilon) \leqslant \frac{\epsilon}{2\alpha}\|g\|^2 + \frac{\epsilon\alpha}{2}\|u_1^\epsilon\|^2.$$

Since $(A_p u_1^\epsilon, u_1^\epsilon) = -\|Bu_1^\epsilon\|_p^p \leqslant 0$, it follows from (5.2.6) and the above estimates that

$$\|u_1^\epsilon\|^2 \leqslant \frac{1}{2}\|u_0\|^2 + \frac{\epsilon}{2\alpha}\|g\|^2 + \frac{1-\epsilon\alpha}{2}\|u_1^\epsilon\|^2,$$

which can be reorganized as

$$\|u_1^\epsilon\|^2 \leqslant \frac{1}{1+\epsilon\alpha}\Big(\|u_0\|^2 + \frac{\epsilon}{\alpha}\|g\|^2\Big). \tag{5.2.7}$$

Since $u_0 \in \mathcal{B}_{r^*}(0)$, it follows from (5.2.7) that

$$\begin{aligned}\|u_1^\epsilon\|^2 &\leqslant \frac{1}{1+\epsilon\alpha}\Big(1 + \frac{\|g\|^2}{\alpha^2} + \frac{\epsilon}{\alpha}\|g\|^2\Big)\\ &= \frac{1}{1+\epsilon\alpha} + \frac{1}{1+\epsilon\alpha}\,\frac{\|g\|^2 + \epsilon\alpha\|g\|^2}{\alpha^2} \leqslant 1 + \frac{\|g\|^2}{\alpha^2} = (r^*)^2,\end{aligned}$$

which means that $u_1^\epsilon \in \mathcal{B}_{r^*}(0)$ as desired.

Step 3: We show that the solution is unique globally. Let $\epsilon \in (0, \epsilon^*]$ and $u_0 \in \mathcal{B}_{r^*}(0)$. By Step 1, the solution $u_1^\epsilon(u_0)$ is unique in $\mathcal{B}_{r^*+1}(0)$. By Step 2, there is not a solution outside $\mathcal{B}_{r^*}(0)$ and thus the solution $u_1^\epsilon(u_0)$ is unique in ℓ^2. So far, the theorem for $n = 1$ has been proved.

Step 4: Suppose the theorem holds for a certain n, that is, for each $\epsilon \in (0, \epsilon^*]$ (where ϵ^* is still the constant given by (5.2.4)) and $u_0 \in \mathcal{B}_{r^*}(0)$, the n-th IES (5.2.1) has a unique solution $u_n^\epsilon(u_0) \in \mathcal{B}_{r^*}(0)$. We then define a mapping by

$$\mathcal{M}_{u_n^\epsilon}^\epsilon(x) = u_n^\epsilon + \epsilon\nu A_p x + \epsilon F(x) + \epsilon g, \ \ \forall x \in \mathcal{B}_{r^*+1}(0),$$

where $u_n^\epsilon \in \mathcal{B}_{r^*}(0)$ instead of $u_0 \in \mathcal{B}_{r^*}(0)$ in (5.2.3). Repeating the process in Step 1, we know that, for each $\epsilon \in (0, \epsilon^*]$, the mapping $\mathcal{M}_{u_n^\epsilon}^\epsilon : \mathcal{B}_{r^*+1}(0) \to \mathcal{B}_{r^*+1}(0)$ is well-defined and contractive, which implies the existence of a unique fixed point u_{n+1}^ϵ in $\mathcal{B}_{r^*+1}(0)$.

Repeating the estimates in Step 2, we obtain an analogue inequality of (5.2.7) as follows

$$\|u_{n+1}^{\epsilon}\|^2 \leqslant \frac{1}{1+\epsilon\alpha}\Big(\|u_n^{\epsilon}\|^2 + \frac{\epsilon}{\alpha}\|g\|^2\Big). \tag{5.2.8}$$

By the recursive hypothesis $u_n^{\epsilon} \in \mathcal{B}_{r^*}(0)$, we infer from (5.2.8) that $u_{n+1}^{\epsilon} \in \mathcal{B}_{r^*}(0)$, which is the unique solution of the $(n+1)$-th IES (5.2.1). The recursive proof is complete. □

Remark 5.2.2. *The above proof is more careful than the proof of [64, Lemma 2] even in the case of $p = 2$. In fact, $\mathcal{B}_{r^*}(0)$ may not be positively invariant under the operator $\mathcal{M}_{u_0}^{\epsilon}$ (although it is invariant under the solution mapping, see [64, Lemma 1]). To overcome this difficulty, we enlarge the radius r^* to r^*+1 such that $\mathcal{B}_{r^*+1}(0)$ is positive invariant under $\mathcal{M}_{u_0}^{\epsilon}$ with a possible maximal size ϵ^*.*

The following result shows the generation of a discrete-time dynamical system (see [73]), which has better properties than the continuous system.

Corollary 5.2.3. *For each $\epsilon \in (0, \epsilon^*]$, where ϵ^* is given by (5.2.4), the unique solution of the IES (5.2.1) in $\mathcal{B}_{r^*}(0)$ generates a discrete semigroup given by*

$$S_{\epsilon}(n) : \mathcal{B}_{r^*}(0) \to \mathcal{B}_{r^*}(0),\ S_{\epsilon}(n)u_0 = u_n^{\epsilon}(u_0),\ \forall n \in \mathbb{N}_0,\ u_0 \in \mathcal{B}_{r^*}(0).$$

Proof. The unique solution of the IES (5.2.1) for $n = 1$ defines an operator by

$$S_{\epsilon}(1) : \mathcal{B}_{r^*}(0) \to \mathcal{B}_{r^*}(0),\ S_{\epsilon}(1)u_0 = u_1^{\epsilon}(u_0).$$

Due to the same recursive relation in (5.2.1) for any n, the solution u_n^{ϵ} satisfies

$$u_n^{\epsilon}(u_0) = S_{\epsilon}(1)u_{n-1}^{\epsilon} = (S_{\epsilon}(1))^n u_0,\ \forall n \in \mathbb{N},\ u_0 \in \mathcal{B}_{r^*}(0).$$

Hence $S_{\epsilon}(\cdot)$ constitutes a discrete semigroup on $\mathcal{B}_{r^*}(0)$. □

Lemma 5.2.4. *For $\epsilon \in (0, \epsilon^*]$ and $n \in \mathbb{N}$, the operator $S_{\epsilon}(n)$ is Lipschitz continuous in $\mathcal{B}_{r^*}(0)$.*

Proof. Let $n = 1$ and $u_0, v_0 \in \mathcal{B}_{r^*}(0)$. By (5.2.5), the solutions $u_1^{\epsilon} = S_{\epsilon}(1)u_0$ and $v_1^{\epsilon} = S_{\epsilon}(1)v_0$ satisfy

$$\begin{aligned}\|u_1^{\epsilon} - v_1^{\epsilon}\| &= \|(u_0 + \epsilon\nu A_p u_1^{\epsilon} + \epsilon F(u_1^{\epsilon}) + \epsilon g) - (v_0 + \epsilon\nu A_p v_1^{\epsilon} + \epsilon F(v_1^{\epsilon}) + \epsilon g)\| \\ &\leqslant \|u_0 - v_0\| + \epsilon(\nu\|A_p u_1^{\epsilon} - A_p v_1^{\epsilon}\| + \|F(u_1^{\epsilon}) - F(v_1^{\epsilon})\|) \\ &\leqslant \|u_0 - v_0\| + \epsilon(\nu L_{p,r^*+1} + L_{r^*+1})\|u_1^{\epsilon} - v_1^{\epsilon}\|,\end{aligned}$$

which further implies that for all $\epsilon \in (0, \epsilon^*]$,

$$\|u_1^{\epsilon} - v_1^{\epsilon}\| \leqslant \frac{\|u_0 - v_0\|}{1 - \epsilon(\nu L_{p,r^*+1} + L_{r^*+1})} \leqslant \frac{\|u_0 - v_0\|}{1 - \epsilon^*(\nu L_{p,r^*+1} + L_{r^*+1})},$$

where $\epsilon^*(\nu L_{p,r^*+1} + L_{r^*+1}) < 1$ in view of (5.2.4). By the semigroup property,

$$\|S_{\epsilon}(n)u_0 - S_{\epsilon}(n)v_0\| \leqslant \frac{\|u_0 - v_0\|}{(1 - \epsilon^*(\nu L_{p,r^*+1} + L_{r^*+1}))^n}$$

for all $n \in \mathbb{N}$. The proof is complete. □

5.3 Generalized Taylor expansion and discretization error

To study the convergence of attractors, we need to estimate the discretization error of solutions, for which we need to develop a *generalized* Taylor expansion.

5.3.1 Generalized Taylor expansion for continuous-time error

According to the method in [64, 75], one must consider the Taylor expansion of LDS (5.1.3) starting from $u(t_{n+1}; u_0)$ and going back to $u(t_n; u_0)$ as follows:

$$u(t_n) = u(t_{n+1}) + (-\epsilon)\mathcal{H}_p(u(t_{n+1})) + \frac{1}{2}(-\epsilon)^2 D\mathcal{H}_p(u(\theta_\epsilon))$$

where $t_{n+1} - t_n = \epsilon$, $\theta_\epsilon \in (t_n, t_{n+1})$, the operator $\mathcal{H}_p : \ell^2 \to \ell^2$ is given by

$$\mathcal{H}_p(x) := \nu A_p x + F(x) + g, \ \forall x \in \ell^2 \tag{5.3.1}$$

and $D\mathcal{H}_p$ denotes the Fréchet derivative (perhaps formal) of $\mathcal{H}_p$. If $p = 2$, then $A := A_p$ is a bounded linear operator, which has a Fréchet derivative given by itself, and thus, by the method as in [69], one can clearly write the Fréchet derivative as

$$D\mathcal{H}(x) = (\nu A + \mathrm{diag}(f'(x_i))\mathcal{H}(x), \ \forall x \in \ell^2.$$

However, if $p > 2$, then the nonlinear operator A_p has not a Fréchet derivative (even the original function $y = |s|^{p-2}s$ is not differential in $\mathbb{R}$).

To overcome the difficulty, we give an alternative for the second order Taylor expansion of LDS (5.1.3) without Fréchet derivatives, which will be useful for estimating the discretization error in the next subsection.

Lemma 5.3.1. *Let $u(\cdot; u_0)$ be the solution of LDS (5.1.3), $t_{n+1} - t_n = \epsilon > 0$, $t_n \geqslant 0$. Then, for each $u_0 \in \mathcal{B}_r(0)$ with any radius $r > 0$, there is $\mathcal{M}_\epsilon(u_0) \in \ell^2$ such that*

$$u(t_n; u_0) = u(t_{n+1}; u_0) - \epsilon\mathcal{H}_p(u(t_{n+1}; u_0)) + \epsilon\mathcal{M}_\epsilon(u_0), \tag{5.3.2}$$

$$\|\mathcal{M}_\epsilon(u_0)\| \leqslant \epsilon C_r, \ \ \forall u_0 \in \mathcal{B}_r(0), \tag{5.3.3}$$

where C_r is increasing in r (but independent of ϵ) and the operator $\mathcal{H}_p : \ell^2 \to \ell^2$ is well-defined by (5.3.1).

Proof. The first order Taylor expansion of LDS (5.1.3) can be read as

$$\begin{aligned} u(t_n) &= u(t_{n+1}) - \epsilon\frac{du}{dt}(\theta) = u(t_{n+1}) - \epsilon\mathcal{H}_p(u(\theta)) \\ &= u(t_{n+1}) - \epsilon\mathcal{H}_p(u(t_{n+1})) + \epsilon(\mathcal{H}_p(u(t_{n+1})) - \mathcal{H}_p(u(\theta))), \end{aligned}$$

where $\theta \in (t_n, t_{n+1})$. Hence (5.3.2) follows if we put

$$\mathcal{M}_\epsilon(u_0) := \mathcal{H}_p(u(t_{n+1}; u_0)) - \mathcal{H}_p(u(\theta; u_0)).$$

To prove (5.3.3), we assume without lose of generality that $r > \|g\|/\alpha$ (otherwise, one can use $r + \|g\|/\alpha$ instead of r), and claim that $\mathcal{B}_r(0)$ is a positively invariant set for the semigroup $S(\cdot)$. Indeed, by (5.1.4), for all $t \geqslant 0$ and $u_0 \in \mathcal{B}_r(0)$,

$$\|u(t;u_0)\|^2 \leqslant e^{-\alpha t}\|u_0\|^2 + \frac{\|g\|^2}{\alpha^2}(1 - e^{-\alpha t}) \leqslant e^{-\alpha t}\Big(r^2 - \frac{\|g\|^2}{\alpha^2}\Big) + \frac{\|g\|^2}{\alpha^2} \leqslant r^2. \tag{5.3.4}$$

By the local Lipschitz continuity of A_p and F, we obtain

$$\begin{aligned}\|\mathcal{M}_\epsilon(u_0)\| &= \|\nu(A_p u(t_{n+1}) - A_p u(\theta)) + (F(u(t_{n+1})) - F(u(\theta)))\| \\ &\leqslant \nu\|A_p u(t_{n+1}) - A_p u(\theta)\| + \|F(u(t_{n+1})) - F(u(\theta))\| \\ &\leqslant (\nu L_{p,r} + L_r)\|u(t_{n+1}) - u(\theta)\|.\end{aligned}$$

By the first order Taylor expansion again, we have

$$\begin{aligned}u(t_{n+1}) - u(\theta) &= (t_{n+1} - \theta)\frac{du}{dt}(\hat{\theta}) = (t_{n+1} - \theta)\mathcal{H}_p(u(\hat{\theta})) \\ &= (t_{n+1} - \theta)(\nu A_p u(\hat{\theta}) + F(u(\hat{\theta})) + g)\end{aligned}$$

for some $\hat{\theta} \in (\theta, t_{n+1})$. By the local Lipschitz continuity of A_p and F again, it follows from (5.3.4) that

$$\|u(t_{n+1}) - u(\theta)\| \leqslant |t_{n+1} - \theta|\Big((\nu L_{p,r} + L_r)\|u(\hat{\theta})\| + \|g\|\Big) \leqslant \epsilon(r(\nu L_{p,r} + L_r) + \|g\|),$$

which further implies that for all $u_0 \in \mathcal{B}_r(0)$,

$$\|\mathcal{M}_\epsilon(u_0)\| \leqslant \epsilon\Big(r(\nu L_{p,r} + L_r)^2 + \|g\|(\nu L_{p,r} + L_r)\Big) =: \epsilon C_r,$$

where C_r is obviously increasing in r. The proof is complete. □

5.3.2 Discretization error of order two

We now use the generalized Taylor expansion in Lemma 5.3.1 to estimate the discretisation error of solutions when the initial data are restricted on the ball $\mathcal{B}_{r^*}(0)$.

Theorem 5.3.2. *Let $u(t;u_0)$ and $u_n^\epsilon(u_0)$ be the solutions of LDS (5.1.3) and IES (5.2.1) respectively, where $u_0 \in \mathcal{B}_{r^*}(0)$. We have the discretisation error of order 2:*

$$\|u(\epsilon; u_n^\epsilon(u_0)) - u_{n+1}^\epsilon(u_0)\| \leqslant \epsilon^2 C_{r^*}, \quad \forall \epsilon \in (0, \epsilon^*],\ n \in \mathbb{N}_0. \tag{5.3.5}$$

Furthermore, for each $T > 0$, there is a $C_{T,r^*} > 0$ such that

$$\|u(t_n; u_0) - u_n^\epsilon(u_0)\| \leqslant \epsilon C_{T,r^*}, \quad \forall t_n := \epsilon n \in [0, T],\ \epsilon \in (0, \epsilon^*]. \tag{5.3.6}$$

Proof. Both (5.3.2) and (5.2.1) can be rewritten as

$$\begin{aligned} u(t_{n+1}) &= u(t_n) + \epsilon\mathcal{H}_p(u(t_{n+1})) - \epsilon\mathcal{M}_\epsilon(u_0), \\ u^\epsilon_{n+1} &= u^\epsilon_n + \epsilon\mathcal{H}(u^\epsilon_{n+1}), \end{aligned}$$

where $\mathcal{H}_p = \nu A_p + F + gI$ as given in (5.3.1). From the difference between the above two equalities, we know that the discretisation error

$$\Delta^\epsilon_n(u_0) := u(t_n; u_0) - u^\epsilon_n(u_0)$$

satisfies the following equation:

$$\Delta^\epsilon_{n+1} = \Delta^\epsilon_n + \epsilon(\mathcal{H}_p(u(t_{n+1})) - \mathcal{H}_p(u^\epsilon_{n+1})) - \epsilon\mathcal{M}_\epsilon(u_0).$$

Taking the inner product with Δ^ϵ_{n+1} yields

$$\begin{aligned} \|\Delta^\epsilon_{n+1}\|^2 &= (\Delta^\epsilon_n + \epsilon(\mathcal{H}_p(u(t_{n+1})) - \mathcal{H}_p(u^\epsilon_{n+1})) - \epsilon\mathcal{M}_\epsilon(u_0), \Delta^\epsilon_{n+1}) \\ &\leqslant (\|\Delta^\epsilon_n\| + \epsilon\|\mathcal{H}_p(u(t_{n+1})) - \mathcal{H}_p(u^\epsilon_{n+1})\| + \epsilon\|\mathcal{M}_\epsilon(u_0)\|)\|\Delta^\epsilon_{n+1}\|, \end{aligned}$$

which further implies

$$\|\Delta^\epsilon_{n+1}\| \leqslant \|\Delta^\epsilon_n\| + \epsilon\|\mathcal{H}_p(u(t_{n+1})) - \mathcal{H}_p(u^\epsilon_{n+1})\| + \epsilon\|\mathcal{M}_\epsilon(u_0)\|. \tag{5.3.7}$$

Since $\mathcal{B}_{r^*}(0)$ is positively invariant under both $S(t)$ and $S_\epsilon(n)$ (see Lemma 5.1.1 and Theorem 5.2.1), it follows that $u(t_{n+1}), u^\epsilon_n \in \mathcal{B}_{r^*}(0)$, and thus we see from the local Lipschitz continuity of A_p and F that

$$\begin{aligned} &\|\mathcal{H}_p(u(t_{n+1})) - \mathcal{H}_p(u^\epsilon_{n+1})\| \\ &= \|\nu(A_p u(t_{n+1}) - A_p u^\epsilon_{n+1}) + F(u(t_{n+1})) - F(u^\epsilon_{n+1})\| \\ &\leqslant \nu\|A_p u(t_{n+1}) - A_p u^\epsilon_{n+1}\| + \|F(u(t_{n+1})) - F(u^\epsilon_{n+1})\| \\ &\leqslant (\nu L_{p,r^*} + L_{r^*})\|u(t_{n+1}) - u^\epsilon_{n+1}\| = (\nu L_{p,r^*} + L_{r^*})\|\Delta^\epsilon_{n+1}\|. \end{aligned}$$

On the other hand, by Lemma 5.3.1, we have

$$\|\mathcal{M}_\epsilon(u_0)\| \leqslant \epsilon C_{r^*}, \ \forall u_0 \in \mathbb{B}_{r^*}(0).$$

Substituting two bounds into (5.3.7) yields

$$\|\Delta^\epsilon_{n+1}\| \leqslant \|\Delta^\epsilon_n\| + \epsilon(\nu L_{p,r^*} + L_{r^*})\|\Delta^\epsilon_{n+1}\| + \epsilon^2 C_{r^*}.$$

Denote by $\hat{L}_{r^*} := \nu L_{p,r^*} + L_{r^*}$. We see from (5.2.4) that for all $\epsilon \in (0, \epsilon^*]$,

$$\epsilon\hat{L}_{r^*} \leqslant \epsilon^*\hat{L}_{r^*} = \frac{\nu L_{p,r^*} + L_{r^*}}{(r^* + 1)(\nu L_{p,r^*+1} + L_{r^*+1}) + \|g\|} < 1,$$

and thus we obtain, for all $\epsilon \in (0, \epsilon^*]$,

$$\|\Delta^\epsilon_{n+1}\| \leqslant \frac{1}{1-\epsilon\hat{L}_{r^*}}\|\Delta^\epsilon_n\| + \epsilon^2 C_{r^*}, \ \forall n \in \mathbb{N}_0, \tag{5.3.8}$$

where C_{r^*} is $1/(1-\epsilon^*\hat{L}_{r^*})$-times bigger than the original constant. Since $\Delta^\epsilon_0 = 0$, we infer from (5.3.8) that

$$\|u(\epsilon; u_0) - u^\epsilon_1(u_0)\| = \|\Delta^\epsilon_1\| \leqslant \epsilon^2 C_{r^*}.$$

Using u^ϵ_n as an initial datum in the above formula, we obtain the discretization error (5.3.5) of order 2.

On the other hand, for all $t_n = \epsilon n \in [0, T]$, by the recursive inequality (5.3.8) and $\Delta^\epsilon_0 = 0$, we have

$$\|\Delta^\epsilon_n\| \leqslant \epsilon^2 C_{r^*} \sum_{j=0}^{n-1} \frac{1}{(1-\epsilon\hat{L}_{r^*})^j}. \tag{5.3.9}$$

Since $\epsilon\hat{L}_{r^*} < 1$ and $n \leqslant T/\epsilon$, it follows that

$$\epsilon\hat{L}_{r^*} \sum_{j=0}^{n-1} \frac{1}{(1-\epsilon\hat{L}_{r^*})^j} = \frac{1-(1-\epsilon\hat{L}_{r^*})^n}{(1-\epsilon\hat{L}_{r^*})^{n-1}} \leqslant (1-\epsilon\hat{L}_{r^*})^{-(n-1)} \leqslant (1-\epsilon\hat{L}_{r^*})^{-\frac{T}{\epsilon}} \uparrow e^{T\hat{L}_{r^*}}$$

as $\epsilon \downarrow 0$, where the last limit is deduced from the basic limit $(1+1/k)^k \uparrow e$ as $k \to \infty$. By (5.3.9),

$$\|\Delta^\epsilon_n\| \leqslant \epsilon \frac{C_{r^*}}{\hat{L}_{r^*}} \epsilon\hat{L}_{r^*} \sum_{j=0}^{n-1} \frac{1}{(1-\epsilon\hat{L}_{r^*})^j} \leqslant \epsilon \frac{C_{r^*} e^{T\hat{L}_{r^*}}}{\hat{L}_{r^*}} =: \epsilon C_{T,r^*},$$

for all $\epsilon \in (0, \epsilon^*]$, $t_n = \epsilon n \in [0, T]$ and $u_0 \in \mathbb{B}_{r^*}(0)$. Hence (5.3.6) holds true. □

5.4 Numerical attractors: existence, optimized bound and continuity

In this section, we derive the existence, optimized bound and continuity of a numerical attractor for IES (5.2.1).

5.4.1 Estimates for tails of numerical solutions

We need to give the estimate of tails of the solutions in $\mathcal{B}_{r^*}(0)$.

Lemma 5.4.1. *Let $\epsilon \in (0, \epsilon^*]$. Then, for each $\delta > 0$, there are $I(\delta) \in \mathbb{N}$ (independent of ϵ) and $N_\epsilon(\delta) \in \mathbb{N}$ such that the solution of IES (5.2.1) satisfies*

$$\|S_\epsilon(n)u_0\|^2_{\ell^2(|i| \geqslant I(\delta))} = \sum_{|i| \geqslant I(\delta)} |u^\epsilon_{n,i}|^2 < \delta, \tag{5.4.1}$$

for all $n \geqslant N_\epsilon(\delta)$ and $u_0 \in \mathcal{B}_{r^*}(0)$.

Proof. As usual, we consider a cut-off function $\xi \in C^1(\mathbb{R}^+, [0,1])$ such that $\xi(s) = 0$ for all $s \in [0, 1/2]$ and $\xi(s) = 1$ for all $s \in [1, +\infty)$. For each $k > 0$, we define an element $\xi_k \in \ell^\infty$ by

$$\xi_k := (\xi_{k,i})_{i\in\mathbb{Z}} \ \text{ where } \xi_{k,i} = \xi\Big(\frac{|i|}{k}\Big), \ \forall i \in \mathbb{Z}. \tag{5.4.2}$$

Since the IES (5.2.1) is well-defined in ℓ^2 and

$$\xi_k u_n^\epsilon = (\xi_{k,i} u_{n,i}^\epsilon)_{i\in\mathbb{Z}} \in \ell^2, \ \forall k > 0, \ n \in \mathbb{N}, \ \epsilon \in (0, \epsilon^*],$$

we can use the inner product of (5.2.1) with $\xi_k u_n^\epsilon$ to obtain

$$\sum_{i\in\mathbb{Z}} \xi_{k,i} |u_{n,i}^\epsilon|^2 = (u_{n-1}^\epsilon, \xi_k u_n^\epsilon) + \epsilon(\nu A_p u_n^\epsilon + F(u_n^\epsilon) + g, \xi_k u_n^\epsilon). \tag{5.4.3}$$

We now estimate all terms on the right-hand side. First,

$$(u_{n-1}^\epsilon, \xi_k u_n^\epsilon) \leqslant \frac{1}{2}\sum_{i\in\mathbb{Z}} \xi_{k,i}|u_{n,i}^\epsilon|^2 + \frac{1}{2}\sum_{i\in\mathbb{Z}} \xi_{k,i}|u_{n-1,i}^\epsilon|^2.$$

Second, since $A_p x = -B^*(|Bx|^{p-2}Bx)$ and

$$(B^* x, y) = (x, By), \quad B(xy) = xBy + yBx$$

for two sequences $x = (x_i)_{i\in\mathbb{Z}}$ and $y = (y_i)_{i\in\mathbb{Z}}$ (need not belong to ℓ^2), it follows that

$$\begin{aligned}
&(A_p u_n^\epsilon, \xi_k u_n^\epsilon) = -(|Bu_n^\epsilon|^{p-2}Bu_n^\epsilon, B(\xi_k u_n^\epsilon)) \\
&= -(|Bu_n^\epsilon|^{p-2}Bu_n^\epsilon, \xi_k Bu_n^\epsilon) - (|Bu_n^\epsilon|^{p-2}Bu_n^\epsilon, u_n^\epsilon B\xi_k) \\
&= -\sum_{i\in\mathbb{Z}} \xi_{k,i}|(Bu_n^\epsilon)_i|^p - (|Bu_n^\epsilon|^{p-2}Bu_n^\epsilon, u_n^\epsilon B\xi_k) \leqslant |(|Bu_n^\epsilon|^{p-2}Bu_n^\epsilon, u_n^\epsilon B\xi_k)|.
\end{aligned}$$

Since $|\xi'(s)| \leqslant C$ for all $s \geqslant 0$, it follows from the mean-valued theorem that

$$|(B\xi_k)_i| = \Big|\xi\Big(\frac{|i+1|}{k}\Big) - \xi\Big(\frac{|i|}{k}\Big)\Big| \leqslant \frac{C}{k}, \ \forall k \in \mathbb{N}, \ i \in \mathbb{Z}.$$

By Theorem 5.2.1, $u_n^\epsilon \in \mathcal{B}_{r^*}(0)$ and thus $|(Bu_n^\epsilon)_i| \leqslant \|Bu_n^\epsilon\| \leqslant 2\|u_n^\epsilon\| \leqslant 2r^*$. Therefore,

$$\begin{aligned}
&\epsilon(\nu A_p u_n^\epsilon, \xi_k u_n^\epsilon) \leqslant \epsilon\nu|(|Bu_n^\epsilon|^{p-2}Bu_n^\epsilon, u_n^\epsilon B\xi_k)| \\
&\leqslant \epsilon\nu \sum_{i\in\mathbb{Z}} |(B\xi_k)_i||(Bu_n^\epsilon)_i|^{p-1}|u_{n,i}^\epsilon| \leqslant \epsilon\frac{C_p}{k}(r^*)^{p-1}\|u_{n,i}^\epsilon\| \leqslant \epsilon\frac{C_p}{k}(r^*)^p.
\end{aligned}$$

Third, by (5.1.2), we have

$$\epsilon(F(u_n^\epsilon) + g, \xi_k u_n^\epsilon) = \epsilon\sum_{i\in\mathbb{Z}} \xi_{k,i}(f(u_{n,i}^\epsilon)u_{n,i}^\epsilon + g_i u_{n,i}^\epsilon)$$

$$\leqslant -\frac{\epsilon\alpha}{2}\sum_{i\in\mathbb{Z}}\xi_{k,i}|u^{\epsilon}_{n,i}|^2 + \frac{\epsilon}{2\alpha}\sum_{i\in\mathbb{Z}}\xi_{k,i}g_i^2.$$

Substituting the three estimates into (5.4.3) we find

$$\sum_{i\in\mathbb{Z}}\xi_{k,i}|u^{\epsilon}_{n,i}|^2 \leqslant \frac{1}{1+\epsilon\alpha}\sum_{i\in\mathbb{Z}}\xi_{k,i}|u^{\epsilon}_{n-1,i}|^2 + \frac{\epsilon}{1+\epsilon\alpha}\Big(\frac{C_p}{k}(r^*)^p + \frac{1}{\alpha}\sum_{|i|\geqslant -1+k/2} g_i^2\Big). \tag{5.4.4}$$

Given $\delta > 0$, there is $I(\delta) \in \mathbb{N}$ (independent of ϵ) such that

$$\frac{C_p}{k}(r^*)^p + \frac{1}{\alpha}\sum_{|i|\geqslant -1+k/2} g_i^2 < \frac{\alpha}{2}\delta,\ \forall k \geqslant I(\delta),$$

which together with (5.4.4) implies that for all $k \geqslant I(\delta)$,

$$\sum_{i\in\mathbb{Z}}\xi_{k,i}|u^{\epsilon}_{n,i}|^2 \leqslant \frac{1}{1+\epsilon\alpha}\sum_{i\in\mathbb{Z}}\xi_{k,i}|u^{\epsilon}_{n-1,i}|^2 + \frac{\epsilon}{1+\epsilon\alpha}\frac{\alpha}{2}\delta.$$

Iterating the above inequality yields

$$\begin{aligned}\sum_{i\in\mathbb{Z}}\xi_{k,i}|u^{\epsilon}_{n,i}|^2 &\leqslant \frac{1}{(1+\epsilon\alpha)^n}\sum_{i\in\mathbb{Z}}\xi_{k,i}|u_{0,i}|^2 + \frac{\delta}{2}\sum_{j=1}^{n}\frac{\epsilon\alpha}{(1+\epsilon\alpha)^j}\\ &\leqslant \frac{\|u_0\|^2}{(1+\epsilon\alpha)^n} + \frac{\delta}{2} \leqslant \frac{(r^*)^2}{(1+\epsilon\alpha)^n} + \frac{\delta}{2}.\end{aligned}$$

Note that

$$\lim_{n\to\infty}\frac{(r^*)^2}{(1+\epsilon\alpha)^n} \to 0. \tag{5.4.5}$$

Hence, there is $N_\epsilon(\delta) \in \mathbb{N}$ such that for all $n \geqslant N_\epsilon(\delta)$ and $k \geqslant I(\delta)$,

$$\sum_{|i|\geqslant k}|u^{\epsilon}_{n,i}|^2 \leqslant \sum_{i\in\mathbb{Z}}\xi_{k,i}|u^{\epsilon}_{n,i}|^2 < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Setting $k = I(\delta)$ we obtain (5.4.1) as desired. □

5.4.2 Existence and connection of numerical attractors

Recall that a compact subset $\mathcal{A}_\epsilon$ of $\mathcal{B}_{r^*}(0)$ is call a (numerical) attractor of the discrete-time dynamical system $\{S_\epsilon(n)\}_{n\in\mathbb{N}_0}$ for the IES (5.2.1) if $\mathcal{A}_\epsilon$ is invariant and attracting

$$S_\epsilon(n)\mathcal{A}_\epsilon = \mathcal{A}_\epsilon\ (\forall n \in \mathbb{N}),\ \text{and}\ \lim_{n\to\infty}\operatorname{dist}_{\ell^2}(S_\epsilon(n)\mathcal{B}_{r^*}(0), \mathcal{A}_\epsilon) = 0.$$

Theorem 5.4.2. *For each $\epsilon \in (0, \epsilon^*]$, the discrete semigroup $\{S_\epsilon(n)\}_{n\in\mathbb{N}_0}$ on $\mathcal{B}_{r^*}(0)$ has a unique numerical attractor $\mathcal{A}_\epsilon$ such that $\mathcal{A}_\epsilon$ is topologically connected in ℓ^2.*

Proof. We prove that the semigroup $S_\epsilon(\cdot)$ is asymptotically compact on $\mathcal{B}_{r^*}(0)$. It suffices to prove that the sequence $\{S_\epsilon(n)u_0^n : n \in \mathbb{N}\}$ is relative compact for any sequence $\{u_0^n : n \in \mathbb{N}\}$ in $\mathcal{B}_{r^*}(0)$.

Given $\delta > 0$, we see from Lemma 5.4.1 that there are $N_\epsilon(\delta), I(\delta) \in \mathbb{N}$ such that

$$\|S_\epsilon(n)u_0^n\|^2_{\ell^2(|i|>I)} = \|u_n^\epsilon(u_0^n)\|^2_{\ell^2(|i|>I)} < \delta^2, \quad \forall\, n \geqslant N.$$

By Theorem 5.2.1, $\{S_\epsilon(n)u_0^n : n \in \mathbb{N}\} \subset \mathcal{B}_{r^*}(0)$, which is bounded in ℓ^2. In particular,

$$(S_\epsilon(n)u_0^n)_{|i|\leqslant I} \text{ is bounded in } \ell^2(|i| \leqslant I) \cong \mathbb{R}^{2I+1},$$

where the space is finitely dimensional. Then the sequence $\{(S_\epsilon(n)u_0^n)_{|i|\leqslant I}\}_{n\geqslant N}$ has a finite δ-net with centers $x_1, x_2, \cdots, x_{k_0} \in \mathbb{R}^{2I+1}$. We define the null-expansion $\widetilde{y}$ of an element $y \in \mathbb{R}^{2I+1}$ by

$$\widetilde{y}_i = y_i, \forall |i| \leqslant I \text{ and } \widetilde{y}_i = 0, \forall |i| > I.$$

Hence, for each $n \geqslant N$, there is $x_k \in \mathbb{R}^{2I+1}$, where $k \in \{1, 2, \cdots, k_0\}$, such that

$$\|S_\epsilon(n)u_0^n - \widetilde{x}_k\|^2 = \|S_\epsilon(n)u_0^n\|^2_{\ell^2(|i|>I)} + \|S_\epsilon(n)u_0^n - x_k\|^2_{\ell^2(|i|\leqslant I)} < 2\delta^2,$$

which means that the sequence $\{S_\epsilon(n)u_0^n : n \geqslant N\}$ has a finite $\sqrt{2}\delta$-net in ℓ^2. Since the finite set $\{S_\epsilon(n)u_0^n : n < N\}$ is compact, it follows that the whole sequence $\{S_\epsilon(n)u_0^n : n \in \mathbb{N}\}$ has a finite $\sqrt{2}\delta$-net too and thus relatively compact in ℓ^2.

Therefore, since the state space $\mathcal{B}_{r^*}(0)$ is bounded, it follows that the discrete semigroup $S_\epsilon(\cdot)$ has a unique numerical attractor denoted by $\mathcal{A}_\epsilon$.

Suppose that $\mathcal{A}_\epsilon$ is not topologically connected. Then there are two open sets $O_1, O_2 \subset \ell^2$ such that

$$O_1 \cup O_2 \supset \mathcal{A}_\epsilon \;\; O_1 \cap \mathcal{A}_\epsilon \neq \emptyset, \;\; O_2 \cap \mathcal{A}_\epsilon \neq \emptyset.$$

Let $\mathcal{K}_\epsilon$ be the closed convex hull of $\mathcal{A}_\epsilon$ in ℓ^2. Then $\mathcal{K}_\epsilon$ is pathwise connected and thus topologically connected in ℓ^2. As the ball $\mathcal{B}_{r^*}(0)$ is closed and convex, we have $\mathcal{K}_\epsilon \subset \mathcal{B}_{r^*}(0)$ and thus the set $S_\epsilon(n)\mathcal{K}_\epsilon$ is well-defined. By the invariance of $\mathcal{A}_\epsilon$, we have $\mathcal{A}_\epsilon = S_\epsilon(n)\mathcal{A}_\epsilon \subset S_\epsilon(n)\mathcal{K}_\epsilon$ and thus

$$O_1 \cap S_\epsilon(n)\mathcal{K}_\epsilon \neq \emptyset, \;\; O_2 \cap S_\epsilon(n)\mathcal{K}_\epsilon \neq \emptyset, \;\; \forall n \in \mathbb{N}. \tag{5.4.6}$$

By Lemma 5.2.4, the operator $S_\epsilon(n) : \mathcal{B}_{r^*}(0) \to \mathcal{B}_{r^*}(0)$ is (Lipschitz) continuous. Since $\mathcal{K}_\epsilon$ is topologically connected, $S_\epsilon(n)\mathcal{K}_\epsilon$ is topologically connected too, which together with (5.4.6) implies that $O_1 \cup O_2$ cannot cover $S_\epsilon(n)\mathcal{K}_\epsilon$. In particular, for each $n \in \mathbb{N}$ there is

$$x_n \in S_\epsilon(n)\mathcal{K}_\epsilon \text{ so that } x_n \notin O_1 \cup O_2.$$

Since $\mathcal{A}_\epsilon$ attracts the bounded set $\mathcal{K}_\epsilon$, it follows that

$$\lim_{n\to\infty} \operatorname{dist}_{\ell^2}(x_n, \mathcal{A}_\epsilon) = 0.$$

By the compactness of $\mathcal{A}_\epsilon$, passing to a subsequence, $x_n \to x$ for some $x \in \mathcal{A}_\epsilon$. Hence $x \in O_1 \cup O_2$, which contradicts $x_n \in \ell^2 \setminus (O_1 \cup O_2)$ (a closed set). □

5.4.3 Optimized bound and continuity of attractors on f, g

To give an optimized bound of the numerical attractors, we consider the restriction of the IES (5.2.1) on arbitrary balls.

Lemma 5.4.3. *For each $r_0 > \|g\|/\alpha$, there is $\epsilon_{r_0} > 0$, given by*

$$\epsilon_{r_0} := \frac{1}{(r_0+1)(\nu L_{p,r_0+1} + L_{r_0+1}) + \|g\|},$$

such that, for all $\epsilon \in (0, \epsilon_{r_0}]$ and $u_0 \in \mathcal{B}_{r_0}(0)$, the IES (5.2.1) has a unique solution $\{u_n^\epsilon\}_{n\in\mathbb{N}} \subset \mathcal{B}_{r_0}(0)$, which generates a discrete semigroup

$$S_{\epsilon,r_0}(n) : \mathcal{B}_{r_0}(0) \to \mathcal{B}_{r_0}(0), \quad S_{\epsilon,r_0}(n)u_0 = u_n^\epsilon(u_0), \ \forall \epsilon \in (0, \epsilon_{r_0}].$$

Proof. By the same method as in Step 1 of Theorem 5.2.1, one can prove that, for each $u_0 \in \mathcal{B}_{r_0}(0)$ and $\epsilon \in (0, \epsilon_{r_0}]$, the operator $\mathcal{M}_{u_0}^\epsilon : \mathcal{B}_{r_0+1}(0) \to \mathcal{B}_{r_0+1}(0)$ is well-defined and contractive. Hence the IES (5.2.1) with $n = 1$ has a unique solution $u_1^\epsilon \in \mathcal{B}_{r_0+1}(0)$. By the method in Step 2, we have $u_1^\epsilon \in \mathcal{B}_{r_0}(0)$. Suppose the solution $u_n^\epsilon \in \mathcal{B}_{r_0}(0)$ for some $n \in \mathbb{N}$. Then we see from (5.2.8) in Step 3 and $r_0 > \|g\|/\alpha$ that

$$\begin{aligned}
\|u_{n+1}^\epsilon\|^2 &\leqslant \frac{1}{1+\epsilon\alpha}\Big(\|u_n^\epsilon\|^2 + \frac{\epsilon}{\alpha}\|g\|^2\Big) \leqslant \frac{1}{1+\epsilon\alpha}\Big(r_0^2 + \frac{\epsilon}{\alpha}\|g\|^2\Big) \\
&= \frac{1}{1+\epsilon\alpha}\Big(r_0^2 - \frac{\|g\|^2}{\alpha^2}\Big) + \frac{1}{1+\epsilon\alpha}\Big(\frac{\|g\|^2}{\alpha^2} + \epsilon\alpha\frac{\|g\|^2}{\alpha^2}\Big) \\
&\leqslant \Big(r_0^2 - \frac{\|g\|^2}{\alpha^2}\Big) + \frac{\|g\|^2}{\alpha^2} = r_0^2.
\end{aligned}$$

Hence the recursive proof is available. □

Note that $\epsilon_{r_0} \downarrow 0$ as $r_0 \to \infty$ and $\epsilon_{r^*} = \epsilon^*$, where ϵ^* is defined by (5.2.4).

Theorem 5.4.4. *For each $r_0 > \|g\|/\alpha$, there is $\epsilon_{r_0} > 0$ such that, for each $\epsilon \in (0, \epsilon_{r_0}]$, the discrete semigroup $S_{\epsilon,r_0}(\cdot)$ has a unique attractor $\mathcal{A}_{\epsilon,r_0}$ in $\mathcal{B}_{r_0}(0)$. Moreover, the numerical attractor $\mathcal{A}_\epsilon$ in Theorem 5.4.2 fulfills*

$$\begin{aligned}
&\mathcal{A}_\epsilon = \mathcal{A}_{\epsilon,r_0}, \ \forall \epsilon \in (0, \min\{\epsilon_{r_0}, \epsilon^*\}], \\
&\|\mathcal{A}_\epsilon\| := \sup_{x\in\mathcal{A}_\epsilon} \|x\| \leqslant \frac{\|g\|}{\alpha}, \ \forall \epsilon \in (0, \epsilon^*].
\end{aligned} \tag{5.4.7}$$

Proof. By the same method as in Theorem 5.4.2, one can prove the existence of a unique attractor $\mathcal{A}_{\epsilon,r_0}$. To prove the equality between two attractors, we let $r_0 < \hat{r}_0$ and $r_0, \hat{r}_0 \in (\|g\|/\alpha, +\infty)$ (note that r^* belongs to this interval). Since $r_0 \to \epsilon_{r_0}$ is decreasing, we have $\min\{\epsilon_{r_0}, \epsilon_{\hat{r}_0}\} = \epsilon_{\hat{r}_0}$.

Next, we prove that $\mathcal{B}_{r_0}(0)$ is an absorbing set of the semigroup $S_{\epsilon,\hat{r}_0}(\cdot)$ on $\mathcal{B}_{\hat{r}_0}(0)$ for all $\epsilon \in (0, \epsilon_{\hat{r}_0}]$. Given any ball $\mathcal{B}_r(0)$ with the radius $r \in (0, \hat{r}_0]$. For each $u_0 \in \mathcal{B}_r(0) \subset \mathcal{B}_{\hat{r}_0}(0)$, it is similar to prove the recursive formula as in (5.2.8), given by

$$\|S_{\epsilon,\hat{r}_0}(n)u_0\|^2 \leqslant \frac{1}{1+\epsilon\alpha}\Big(\|S_{\epsilon,\hat{r}_0}(n-1)u_0\|^2 + \frac{\epsilon}{\alpha}\|g\|^2\Big), \ \forall n \in \mathbb{N}_0.$$

Iterating it yields

$$\|S_{\epsilon,\hat{r}_0}(n)u_0\|^2 \leqslant \frac{1}{(1+\epsilon\alpha)^n}\|u_0\|^2 + \frac{\epsilon}{\alpha}\|g\|^2 \sum_{j=1}^{n} \frac{1}{(1+\epsilon\alpha)^j} \leqslant \frac{r^2}{(1+\epsilon\alpha)^n} + \frac{\|g\|^2}{\alpha^2}.$$

Since $r^2/(1+\epsilon\alpha)^n \to 0$ as $n \to \infty$ and $r_0 > \|g\|/\alpha$, there is $N = N(r)$ such that for all $n \geqslant N$,

$$\|S_{\epsilon,\hat{r}_0}(n)u_0\|^2 \leqslant \frac{r^2}{(1+\epsilon\alpha)^n} + \frac{\|g\|^2}{\alpha^2} \leqslant (r_0^2 - \frac{\|g\|^2}{\alpha^2}) + \frac{\|g\|^2}{\alpha^2} = r_0^2.$$

Hence $\mathcal{B}_{r_0}(0)$ is a bounded absorbing set for $S_{\epsilon,\hat{r}_0}(\cdot)$.

Since an attractor is the omega-limit set of any bounded absorbing set, it follows that

$$\mathcal{A}_{\epsilon,\hat{r}_0} = \cap_{k\in\mathbb{N}} \overline{\cup_{n\geqslant k} S_{\epsilon,\hat{r}_0}(n)\mathcal{B}_{r_0}(0)} = \cap_{k\in\mathbb{N}} \overline{\cup_{n\geqslant k} S_{\epsilon,r_0}(n)\mathcal{B}_{r_0}(0)} = \mathcal{A}_{\epsilon,r_0},$$

where we have used the uniqueness of solutions to ensure $S_{\epsilon,\hat{r}_0}(n) = S_{\epsilon,r_0}(n)$ on $\mathcal{B}_{r_0}(0)$. In particular, since $\mathcal{A}_\epsilon = \mathcal{A}_{\epsilon,r^*}$, it follows that

$$\mathcal{A}_\epsilon = \mathcal{A}_{\epsilon,r_0}, \ \forall 0 < \epsilon \leqslant \min\{\epsilon_{r_0}, \epsilon^*\}, \ r_0 > \frac{\|g\|}{\alpha}.$$

If $r_0 \in (\|g\|/\alpha, r^*]$, then $\epsilon_{r_0} \geqslant \epsilon^*$. The above equality implies

$$\mathcal{A}_\epsilon \subset \mathcal{B}_{r_0}(0), \ \forall r_0 \in (\frac{\|g\|}{\alpha}, r^*], \ \epsilon \in (0, \epsilon^*].$$

Letting $r_0 \to \|g\|/\alpha$ we obtain $\mathcal{A}_\epsilon \subset \mathcal{B}_{\|g\|/\alpha}(0)$ for all $\epsilon \in (0, \epsilon^*]$. □

Example. The bound $\|g\|/\alpha$ of $\|\mathcal{A}_\epsilon\|$ in (5.4.7) seems to be optimized. Let $\nu = 0$ and $f(s) = -\alpha s$ (satisfying (5.1.2)). Then the IES (5.2.1) is read as

$$u_n = u_{n-1} - \epsilon\alpha u_n + \epsilon g.$$

It has an entire solution $u_n \equiv g/\alpha$ for all $n \in \mathbb{Z}$, which belongs to the attractor and $\|u_n\| = \|g\|/\alpha$.

To close this section, we deduce the continuity (upper and lower semi-continuity) of the numerical attractors depended on the nonlinearity f or the external force g. The Hausdorff metric between two subsets $X, Y \subset \ell^2$ is defined by

$$\text{dist}_h(X, Y) = \max(d(X, Y), \text{dist}(X, Y)), \ d(X, Y) := \sup_{x\in X} \inf_{y\in Y} \|x - y\|.$$

Corollary 5.4.5. *Denoting the numerical attractor $\mathcal{A}_\epsilon$ by $\mathcal{A}_\epsilon(\alpha, g)$, depended on the constant α in (5.1.2) and the force g, we have*

$$\lim_{\alpha\to\infty} \text{dist}_h(\mathcal{A}_\epsilon(\alpha, g), \{0\}) = 0 \ \textit{and} \ \lim_{g\to 0} \text{dist}_h(\mathcal{A}_\epsilon(\alpha, g), \{0\}) = 0.$$

In particular, if f_1 and f_2 satisfy (5.1.2) with the same constant α, then

$$\lim_{\alpha\to\infty} \text{dist}_h(\mathcal{A}_\epsilon(f_1), \mathcal{A}_\epsilon(f_2)) = 0.$$

Proof. By (5.4.7) we have

$$\mathrm{dist}_h(\mathcal{A}_\epsilon(\alpha, g), \{0\}) = \|\mathcal{A}_\epsilon(\alpha, g)\| \leqslant \frac{\|g\|}{\alpha} \to 0$$

as $\alpha \to \infty$ or $g \to 0$. By (5.4.7) again,

$$\mathrm{dist}_h(\mathcal{A}_\epsilon(f_1), \mathcal{A}_\epsilon(f_2)) \leqslant \|\mathcal{A}_\epsilon(f_1)\| + \|\mathcal{A}_\epsilon(f_2)\| \leqslant 2\frac{\|g\|}{\alpha} \to 0$$

as $\alpha \to \infty$. □

Remark 5.4.6. *A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies the dissipative condition (5.1.2) if and only if the curve $y = f(s)$ falls in the area surrounded by two straight lines $y = -\alpha s$ and $s = 0$, In particular, the graph of $y = f(s)$ closes to the vertical axis as $\alpha \to \infty$, see Figure 5.1.*

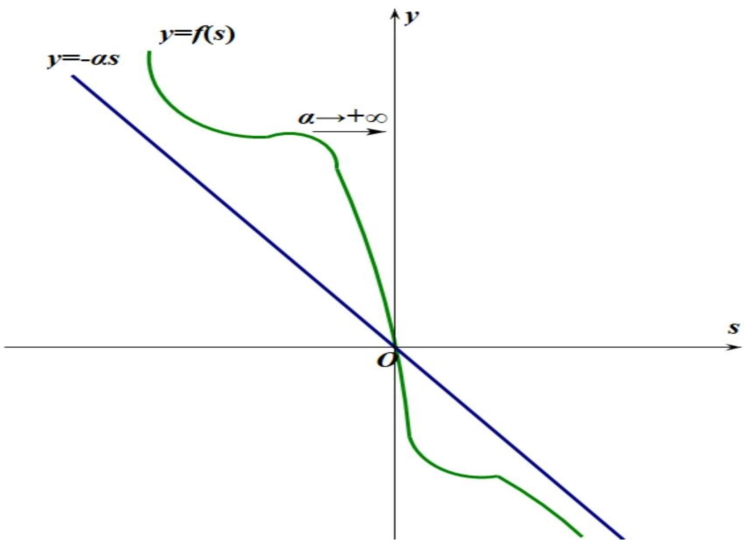

Figure 5.1: Graph and limit of f.

5.5 Convergence from numerical attractor to global attractor

We are in a position to establish the upper semi-continuity of the numerical attractors.

Theorem 5.5.1. *Let $\mathcal{A}_\epsilon$ and $\mathcal{A}$ be the numerical attractor and the global attractor for IES (5.2.1) and Eq.(5.1.3) respectively. Then*

$$\lim_{\epsilon \to 0^+} d_{\ell^2}(\mathcal{A}_\epsilon, \mathcal{A}) = 0. \tag{5.5.1}$$

Proof. Suppose (5.5.1) is false, then there are $\epsilon_k \downarrow 0$ (as $k \to +\infty$), $x_k \in \mathcal{A}_{\epsilon_k}$ and $\delta_0 > 0$ such that

$$d_{\ell^2}(x_k, \mathcal{A}) \geqslant \delta_0, \ \forall k \in \mathbb{N}. \tag{5.5.2}$$

Since the global attractor $\mathcal{A}$ attracts $\mathbb{B}_{r^*}(0)$, we can find a $T > 0$ such that

$$d_{\ell^2}(u(t; \mathbb{B}_{r^*}(0)), \mathcal{A}) < \frac{\delta_0}{2}, \ \forall t \geqslant T.$$

We can assume $\epsilon_k < 1$ for all $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $\epsilon_k n_k \in [T, T+1]$ and thus

$$d_{\ell^2}(u(\epsilon_k n_k; \mathbb{B}_{r^*}(0)), \mathcal{A}) < \frac{\delta_0}{2}, \ \forall k \in \mathbb{N}.$$

By the invariance of each attractor $\mathcal{A}_{\epsilon_k}$, we have

$$x_k = S_{\epsilon_k}(n_k) y_k \text{ for some } y_k \in \mathcal{A}_{\epsilon_k} \subset \mathbb{B}_{r^*}(0).$$

Now, the discretization error (5.3.6) in Lemma 5.3.5 implies

$$\|S_{\epsilon_k}(n_k) y_k - u(\epsilon_k n_k; y_k)\| \leqslant \epsilon_k C_{T+1, r^*},$$

where the constant depends on $T + 1$ in view of $\epsilon_k n_k \leqslant T + 1$. Since $\epsilon_k \downarrow 0$, there is an $k_0 \in \mathbb{N}$ such that

$$\|S_{\epsilon_k}(n_k) y_k - u(\epsilon_k n_k; y_k)\| < \frac{\delta_0}{2}, \ \forall k \geqslant k_0.$$

Therefore, for all $k \geqslant k_0$,

$$\begin{aligned} d_{\ell^2}(x_k, \mathcal{A}) &= \mathrm{dist}_{\ell^2}(S_{\epsilon_k}(n_k) y_k, \mathcal{A}) \\ &\leqslant \|S_{\epsilon_k}(n_k) y_k - u(\epsilon_k n_k; y_k)\| + d_{\ell^2}(u(\epsilon_k n_k; \mathbb{B}_{r^*}(0)), \mathcal{A}) < \frac{\delta_0}{2} + \frac{\delta_0}{2} = \delta_0, \end{aligned}$$

which gives a contradiction to (5.5.2). □

Corollary 5.5.2. *The union $\cup_{\epsilon \in (0, \epsilon^*]} \mathcal{A}_\epsilon$ is relatively compact in ℓ^2.*

Proof. Let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence taking from the union. Then there is $\{\epsilon_k\} \subset (0, \epsilon^*]$ such that $x_k \in \mathcal{A}_{\epsilon_k}$. We prove that $\{x_k\}_{k \in \mathbb{N}}$ has a convergent subsequence in two cases.

Case 1: $\inf \epsilon_k > 0$. Then $\epsilon_k \in [\epsilon_0, \epsilon^*]$ for some $\epsilon_0 > 0$. By Lemma 5.4.1, the tail estimate of solutions is uniform for all ϵ_k. More precisely, for $\delta > 0$, there is $N(\delta), I(\delta) \in \mathbb{N}$ such that for all $n \geqslant N$,

$$\|S_{\epsilon_k}(n) u_0\|_{\ell^2(|i| > I)} < \delta, \ \ \forall\, n \geqslant N, \ k \in \mathbb{N}, \ u_0 \in \mathcal{B}_{r^*}(0).$$

The invariance implies $x_k = S_{\epsilon_k}(N) y_k$ for some $y_k \in \mathcal{B}_{r^*}(0)$ and thus

$$\|x_k\|_{\ell^2(|i| > I)} < \delta, \ \ \forall k \in \mathbb{N}. \tag{5.5.3}$$

Since $\{x_k\}$ is bounded in ℓ^2, its truncation on $\ell^2(|i| \leqslant I) = \mathbb{R}^{2I+1}$ is also bounded. Hence the truncated sequence of $\{x_k\}$ has a finite δ-net in $\mathbb{R}^{2I+1}$, which together with (5.5.3) implies that the sequence $\{x_k\}$ has a finite 2δ-net and thus relatively compact in ℓ^2.

Case 2: $\inf \epsilon_k = 0$. Passing to a subsequence, we assume $\epsilon_k \to 0$. By the upper semi-continuity as in Theorem 5.5.1, we have

$$d_{\ell^2}(x_k, \mathcal{A}) \leqslant d_{\ell^2}(\mathcal{A}_{\epsilon_k}, \mathcal{A}) \to 0 \text{ as } k \to \infty.$$

Since $\mathcal{A}$ is compact, we see from [38, lemma 2.3] that the sequence $\{x_k\}$ has a convergent subsequence. □

Remark 5.5.3. *The uniform compactness of attractors is usually applied to prove the upper semi-continuity, see [86, 130]. For the numerical attractors, the situation is reversed.*

5.6 Finitely dimensional approximation of numerical attractors

How to truncate the IES (5.2.1) on a finite-dimensional space? For each $m \in \mathbb{N}$, the operator $F : \ell^2 \to \ell^2$ has a natural truncation given by

$$F_m : \mathbb{R}^{2m+1} \to \mathbb{R}^{2m+1}, \ F_m(x) = (f(x_i))_{|i|\leqslant m}, \ \forall x = (x_i)_{|i|\leqslant m} \in \mathbb{R}^{2m+1}.$$

However, it is not easy to truncate the discrete p-Laplace operators A_p because $A_p x$ $(x = (x_i)_{|i|\leqslant m})$ involves two unknown components x_{m+1} and x_{-m-1}. To overcome it, we may use the following periodic boundary conditions (see [7, 64])

$$x_{m+1} = x_{-m} \text{ and } x_{-m-1} = x_m.$$

So, the truncation $A_{p,m} : \mathbb{R}^{2m+1} \to \mathbb{R}^{2m+1}$ of A_p can be defined by

$$\begin{aligned}
(A_{p,m}x)_{-m} &= |x_{-m+1} - x_{-m}|^{p-2}(x_{-m+1} - x_{-m}) - |x_{-m} - x_m|^{p-2}(x_{-m} - x_m), \\
(A_{p,m}x)_i &= (A_p x)_i, \quad \forall |i| < m, \\
(A_{p,m}x)_m &= |x_{-m} - x_m|^{p-2}(x_{-m} - x_m) - |x_m - x_{m-1}|^{p-2}(x_m - x_{m-1})
\end{aligned}$$

for all $x = (x_i)_{|i|\leqslant m} \in \mathbb{R}^{2m+1}$. For $p > 2$, the truncated operator $A_{p,m}$ is nonlinear and thus it is not a matrix. But $A_{p,m}$ can be denoted by a function of a special matrix

$$B_m = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & -1 & 1 \\ 1 & 0 & \cdots & 0 & -1 \end{pmatrix} \in (\mathbb{R}^{2m+1})^2,$$

which is just the truncated matrix of the linear operator B. We denote by B_m^T the transport matrix of B_m to obtain

$$A_{p,m}x = -B_m^T(|B_m x|^{p-2} \diamond B_m x), \ \forall x \in \mathbb{R}^{2m+1}.$$

Then the IES (5.2.1) can be truncated as follows:

$$\begin{cases} u_n^{\epsilon,m} = u_{n-1}^{\epsilon,m} + \epsilon\nu A_{p,m} u_n^{\epsilon,m} + \epsilon F_m(u_n^{\epsilon,m}) + \epsilon g|_m, \\ u_0^{\epsilon,m} = u_0^m \in \mathbb{R}^{2m+1}, \end{cases} \tag{5.6.1}$$

where $g|_m := (g_i)_{|i|\leqslant m} \in \mathbb{R}^{2m+1}$ is the truncation of $g \in \ell^2$, and the unknown is denoted by $u_n^{\epsilon,m} = (u_{n,i}^{\epsilon,m})_{|i|\leqslant m} \in \mathbb{R}^{2m+1}$.

As in (5.2.2), $A_{p,m} : \mathbb{R}^{2m+1} \to \mathbb{R}^{2m+1}$ is locally Lipschitz continuous too:

$$\|A_{p,m}x - A_{p,m}y\| \leqslant L_{p,r}\|x - y\| \text{ and } \|A_{p,m}x\| \leqslant L_{p,r}\|x\| \tag{5.6.2}$$

for all $x, y \in \mathcal{B}_r^m(0)$, where $\mathcal{B}_r^m(0)$ is the ball in $\mathbb{R}^{2m+1}$ and $L_{p,r} := (p-1)2^{2p}r^{p-2}$.

5.6.1 Existence and bound of truncated numerical attractors

As in the infinite dimension case, there may not be a common step size such that the truncated IES (5.6.1) is globally solvable and thus we restrict it on some suitable balls.

Lemma 5.6.1. *For each $r_0 > \|g|_m\|/\alpha$ and $m \in \mathbb{N}$, there is $\epsilon_{r_0} > 0$, such that, for all $\epsilon \in (0, \epsilon_{r_0}]$ and $u_0^m \in \mathcal{B}_{r_0}^m(0)$, the truncated IES (5.6.1) has a unique solution $\{u_n^{\epsilon,m}\}_{n\in\mathbb{N}}$ satisfies*

$$\|u_n^{\epsilon,m}\|^2 \leqslant \frac{1}{1+\epsilon\alpha}\Big(\|u_{n-1}^{\epsilon,m}\|^2 + \frac{\epsilon}{\alpha}\|g|_m\|^2\Big). \tag{5.6.3}$$

In particular, for all $n \in \mathbb{N}$, $u_n^{\epsilon,m} \in \mathcal{B}_{r_0}^m(0)$.

Proof. We recursively prove it as done in Theorem 5.2.1. Consider the case $n = 1$. For $\epsilon > 0$ and $u_0^m \in \mathcal{B}_{r_0}^m(0)$, we denote by

$$\mathcal{M}_{u_0^m}^\epsilon(x) = u_0^m + \epsilon\nu A_{p,m}x + \epsilon F_m(x) + \epsilon g|_m, \ \ \forall x \in \mathcal{B}_{r_0+1}^m(0).$$

By the local Lipschitz continuity of $A_{p,m}$ and F_m, we have

$$\begin{aligned} \|\mathcal{M}_{u_0^m}^\epsilon(x)\| &\leqslant \|u_0^m\| + \epsilon(\nu\|A_{p,m}x\| + \|F_m(x)\| + \|g|_m\|) \\ &\leqslant r_0 + \epsilon\Big((r_0+1)(\nu L_{p,r_0+1} + L_{r_0+1}) + \|g|_m\|\Big). \end{aligned}$$

Using $\|g\|$ instead of $\|g|_m\|$ we put

$$\epsilon_{r_0} := \frac{1}{(r_0+1)(\nu L_{p,r_0+1} + L_{r_0+1}) + \|g\|}, \tag{5.6.4}$$

which is independent of m. Since $\|g|_m\| \leqslant \|g\|$, it follows that for all $\epsilon \in (0, \epsilon_{r_0}]$,

$$\|\mathcal{M}^{\epsilon}_{u_0^m}(x)\| \leqslant r_0 + \frac{(r_0+1)(\nu L_{p,r_0+1} + L_{r_0+1}) + \|g|_m\|}{(r_0+1)(\nu L_{p,r_0+1} + L_{r_0+1}) + \|g\|} \leqslant r_0 + 1,$$

which means that $\mathcal{M}^{\epsilon}_{u_0^m} : \mathcal{B}^m_{r_0+1}(0) \to \mathcal{B}^m_{r_0+1}(0)$ is well-defined. By the local Lipschitz continuity of $A_{p,m}$ and F_m again, for all $\epsilon \in (0, \epsilon_{r_0}]$ and $x, y \in \mathcal{B}^m_{r_0+1}(0)$,

$$\|\mathcal{M}^{\epsilon}_{u_0^m}(x) - \mathcal{M}^{\epsilon}_{u_0^m}(y)\| \leqslant \epsilon(\nu\|A_{p,m}x - A_{p,m}y\| + \|F_m(x) - F_m(y)\|)$$
$$\leqslant \epsilon(\nu L_{p,r_0+1} + L_{r_0+1})\|x - y\| \leqslant \frac{1}{r_0+1}\|x - y\|.$$

Then the contraction mapping principle implies that the first equation of (5.6.1) has a unique solution

$$u_1^{\epsilon,m} \in \mathcal{B}^m_{r_0+1}(0), \quad \forall \epsilon \in (0, \epsilon_{r_0}].$$

Now, we take the $\mathbb{R}^{2m+1}$-inner product of the truncted IES (5.6.1) with $u_1^{\epsilon,m}$, the result is

$$\|u_1^{\epsilon,m}\|^2 = \langle u_0^m, u_1^{\epsilon,m}\rangle + \epsilon\nu\langle A_{p,m}u_1^{\epsilon,m}, u_1^{\epsilon,m}\rangle + \epsilon\langle F_m(u_1^{\epsilon,m}), u_1^{\epsilon,m}\rangle + \epsilon\langle g|_m, u_1^{\epsilon,m}\rangle. \tag{5.6.5}$$

Since $A_{p,m}$ is the function of the matrix B_m, it follows that for all $x \in \mathbb{R}^{2m+1}$,

$$\langle A_{p,m}x, x\rangle = -\langle B_m^T(|B_m x|^{p-2} \diamond B_m x), x\rangle$$
$$= -\langle |B_m x|^{p-2} \diamond B_m x, B_m x\rangle = -\sum_{|i|\leqslant m} |(B_m x)_i|^p \leqslant 0.$$

Hence, by estimating other three terms in (5.6.5) and using the method in Theorem 5.2.1, we obtain

$$\|u_1^{\epsilon,m}\|^2 \leqslant \frac{1}{1+\epsilon\alpha}\Big(\|u_0^m\|^2 + \frac{\epsilon}{\alpha}\|g|_m\|^2\Big).$$

Since $r_0 > \|g|_m\|/\alpha$ and $u_0^m \in \mathcal{B}^m_{r_0}(0)$, it follows that

$$\|u_1^{\epsilon,m}\|^2 \leqslant \frac{1}{1+\epsilon\alpha}\Big(r_0^2 + \frac{\epsilon}{\alpha}\|g|_m\|^2\Big)$$
$$= \frac{1}{1+\epsilon\alpha}\Big(r_0^2 - \frac{\|g|_m\|^2}{\alpha^2}\Big) + \frac{1}{1+\epsilon\alpha}\Big(\frac{\|g|_m\|^2}{\alpha^2} + \frac{\epsilon}{\alpha}\|g|_m\|^2\Big)$$
$$= \frac{1}{1+\epsilon\alpha}\Big(r_0^2 - \frac{\|g|_m\|^2}{\alpha^2}\Big) + \frac{\|g|_m\|^2}{\alpha^2} \leqslant r_0^2,$$

which means $u_1^{\epsilon,m} \in \mathcal{B}^m_{r_0}(0)$ for all $\epsilon \in (0, \epsilon_{r_0}]$. Repeating the above process with $u_{n-1}^{\epsilon,m} \in \mathcal{B}^m_{r_0}(0)$ instead of $u_0^m \in \mathcal{B}^m_{r_0}(0)$, the recursive proof is available. □

Note that the radius r^* and step size ϵ^* in the previous sections satisfy

$$r^* = \sqrt{1 + \frac{\|g\|^2}{\alpha^2}} > \frac{\|g|_m\|}{\alpha}, \quad \text{and } \epsilon^* = \epsilon_{r^*},$$

where ϵ_{r^*} is defined as in (5.6.4). By Lemma 5.6.1, for each $\epsilon \in (0, \epsilon^*]$ and $m \in \mathbb{N}$, we can define a discrete semigroup by

$$S_{\epsilon,m}(n) : \mathcal{B}^m_{r^*}(0) \to \mathcal{B}^m_{r^*}(0), \; S_{\epsilon,m}(n)u_0^m = u_n^{\epsilon,m}(u_0^m), \; \forall n \in \mathbb{N}_0.$$

Theorem 5.6.2. *For each $\epsilon \in (0, \epsilon^*]$ and $m \in \mathbb{N}$, the discrete semigroup $S_{\epsilon,m}(\cdot)$ has a (numerical) attractor $\mathcal{A}_{\epsilon,m}$ such that*

$$\mathcal{A}_{\epsilon,m} \subset \mathcal{B}^m_{\|g|_m\|/\alpha}(0) \text{ and } \mathcal{A}_{\epsilon,m} \text{ is connected.} \tag{5.6.6}$$

Proof. The existence of a unique attractor $\mathcal{A}_{\epsilon,m}$ follows from the compactness of the state space $\mathcal{B}^m_{r^*}(0)$ immediately. The connection of $\mathcal{A}_{\epsilon,m}$ follows from the same method as in Theorem 5.4.2.

To prove the bound of the attractor, we put $r_0 \in (\|g|_m\|/\alpha, r^*)$ and prove that $\mathcal{B}^m_{r_0}(0)$ is an absorbing set for the semigroup $S_{\epsilon,m}(\cdot)$. It suffices to prove that $\mathcal{B}^m_{r_0}(0)$ absorbs the whole state space $\mathcal{B}^m_{r^*}(0)$. Iterating (5.6.3) in Lemma 5.6.1, we have for all $u^m_0 \in \mathcal{B}^m_{r^*}(0)$,

$$\begin{aligned}\|u^{\epsilon,m}_n\|^2 &\leqslant \frac{1}{1+\epsilon\alpha}\Big(\|u^{\epsilon,m}_{n-1}\|^2 + \frac{\epsilon}{\alpha}\|g|_m\|^2\Big)\\ &\leqslant \frac{\|u^m_0\|^2}{(1+\epsilon\alpha)^n} + \frac{\epsilon}{\alpha}\|g|_m\|^2 \sum_{j=1}^{n} \frac{1}{(1+\epsilon\alpha)^j} \leqslant \frac{(r^*)^2}{(1+\epsilon\alpha)^n} + \frac{\|g|_m\|^2}{\alpha^2}.\end{aligned}$$

Since $r_0 > \|g|_m\|/\alpha$, it follows that there is $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$\|u^{\epsilon,m}_n\|^2 \leqslant \Big(r_0^2 - \frac{\|g|_m\|^2}{\alpha^2}\Big) + \frac{\|g|_m\|^2}{\alpha^2} = r_0^2.$$

Since $r_0 < r^*$, we see from (5.6.4) that $\epsilon_{r_0} > \epsilon^*$. By Lemma 5.6.1, $\mathcal{B}^m_{r_0}(0)$ is positively invariant under $S_{\epsilon,m}(\cdot)$ for all $0 < \epsilon \leqslant \epsilon^*$ (and thus $\epsilon \leqslant \epsilon_{r_0}$). Therefore,

$$\mathcal{A}_{\epsilon,m} = \cap_{k\in\mathbb{N}} \overline{\cup_{n\geqslant k} S_{\epsilon,m}(n)\mathcal{B}^m_{r_0}(0)} \subset \mathcal{B}^m_{r_0}(0)$$

for all $\epsilon \in (0, \epsilon^*]$ and $r_0 \in (\|g|_m\|/\alpha, r^*)$. Taking the limit as $r_0 \to \|g|_m\|/\alpha$, we obtain the inclusion in (5.6.6). □

5.6.2 Convergence from the truncated attractor to the numerical attractor

The following result states that the tail of any element in $\mathcal{A}_{\epsilon,m}$ is uniformly small in $\epsilon \in (0, \epsilon^*]$ as $m \to \infty$.

Lemma 5.6.3. *For each $\delta > 0$, there exists $I(\delta) \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ and $\epsilon \in (0, \epsilon^*]$,*

$$\sum_{I(\delta)\leqslant|i|\leqslant m} |x_i|^2 < \delta, \quad \forall (x_i)_{|i|\leqslant m} \in \mathcal{A}_{\epsilon,m}, \tag{5.6.7}$$

where the sum is zero if $m < I(\delta)$.

Proof. Taking the inner product of the truncated IES (5.6.1) with $\xi^m_k \diamond u^{\epsilon,m}_n$ in $\mathbb{R}^{2m+1}$, where $\xi^m_k = (\xi_{k,i})_{|i|\leqslant m}$, we obtain

$$\sum_{|i|\leqslant m} \xi_{k,i}|u^{\epsilon}_{n,i}|^2 = (u^{\epsilon,m}_{n-1}, \xi^m_k u^{\epsilon,m}_n)$$

$$+\epsilon\nu(A_{p,m}u_n^{\epsilon,m},\xi_k^m u_n^{\epsilon,m})+\epsilon(F_m(u_n^{\epsilon,m})+g|_m,\xi_k^m u_n^{\epsilon,m}).$$

Since $A_{p,m}$ and A_p have the same local Lipschitz constants, we can similarly obtain

$$\nu(A_{p,m}u_n^{\epsilon,m},\xi_k^m\diamond u_n^{\epsilon,m})\leqslant\frac{C_p}{k}(r^*)^p. \tag{5.6.8}$$

where the constant C_p is independent of m. Hence, by other same arguments as in the proof of Lemma 5.4.1, it follows that, for each $\delta>0$ and $\epsilon\in(0,\epsilon^*]$, there are $N_\epsilon(\delta)\in\mathbb{N}$ and $I(\delta)\in\mathbb{N}$ such that for all $n\geqslant N_\epsilon(\delta)$, $k\geqslant I(\delta)$ and $m\in\mathbb{N}$,

$$\sum_{k\leqslant|i|\leqslant m}|u_{n,i}^{\epsilon,m}|^2\leqslant\sum_{|i|\leqslant m}\xi_{k,i}|u_{n,i}^{\epsilon,m}|^2<\delta,$$

which further implies (by taking $k=I(\delta)$) that

$$\sum_{I(\delta)\leqslant|i|\leqslant m}|u_{n,i}^{\epsilon,m}|^2<\delta,\quad\forall n\geqslant N_\epsilon(\delta).$$

Given now $x\in\mathcal{A}_{\epsilon,m}$ with arbitrary $\epsilon\in(0,\epsilon^*]$ and $m\in\mathbb{N}$. The invariance implies that

$$x=S_{\epsilon,m}(N_\epsilon(\delta))u_0^m=u_{N_\epsilon(\delta)}^{\epsilon,m}(u_0^m),\ \text{for some } u_0^m\in\mathcal{B}_{r^*}^m(0),$$

and thus

$$\sum_{I(\delta)\leqslant|i|\leqslant m}|x_i|^2=\sum_{I(\delta)\leqslant|i|\leqslant m}|u_{N_\epsilon(\delta),i}^{\epsilon,m}|^2<\delta,$$

which implies (5.6.7) as desired. □

Any $x\in\mathbb{R}^{2m+1}$ has a *null-extension* $\tilde{x}\in\ell^2$ defined by

$$\tilde{x}_i=0,\ \forall|i|>m,\ \tilde{x}_i=x_i,\ \forall|i|\leqslant m.$$

Then a set D in $\mathbb{R}^{2m+1}$ still has a *null-extension* set in ℓ^2 denoted by $\widetilde{D}$. In this viewpoint, both attractors $\mathcal{A}_{\epsilon,m}$ and $\mathcal{A}_\epsilon$ can be contained into the same ball $\mathcal{B}_{r^*}(0)$ of ℓ^2.

Theorem 5.6.4. *For each $\epsilon\in(0,\epsilon^*]$, the numerical attractor $\mathcal{A}_{\epsilon,m}$ of the truncated IES* (5.6.1) *upper semi-converges to the attractor $\mathcal{A}_\epsilon$ of the IES* (5.2.1)*, i.e.*

$$d_{\ell^2}(\mathcal{A}_{\epsilon,m},\mathcal{A}_\epsilon):=d_{\ell^2}(\widetilde{\mathcal{A}}_{\epsilon,m},\mathcal{A}_\epsilon)\to0,\ as\ m\to\infty. \tag{5.6.9}$$

Proof. We fix $\epsilon\in(0,\epsilon^*]$ and prove the theorem by using the idea of an entire solution. Suppose (5.6.9) is false. Then there are $\eta_0>0$, subsequence $\{m_j\}$ of $\{m\}$ and $x^{m_j}\in\mathcal{A}_{\epsilon,m_j}$ such that the null-extension $\tilde{x}^{m_j}$ of x^{m_j} satisfies

$$d(\tilde{x}^{m_j},\mathcal{A}_\epsilon)\geqslant\eta_0,\ \forall j\in\mathbb{N}. \tag{5.6.10}$$

The solution of (5.6.1) with the initial data x^{m_j} is given by

$$u_n^{\epsilon,m_j} = u_n^{\epsilon,m_j}(x^{m_j}) = S_{\epsilon,m_j}(n)x^{m_j}, \ \forall n \in \mathbb{N}_0, \ j \in \mathbb{N}.$$

Since $x^{m_j} \in \mathcal{A}_{\epsilon,m_j}$, it follows that the solution u_n^{ϵ,m_j} can be expanded as an entire solution, i.e. defined for all $n \in \mathbb{Z}$.

Since $u_n^{\epsilon,m_j} \in \mathcal{A}_{\epsilon,m_j}$ for all $n \in \mathbb{Z}$, it follows from Lemma 5.6.3 that, for each $\delta > 0$, there is $I(\delta) \in \mathbb{N}$ such that

$$\sum_{|i|\geqslant I(\delta)} |\tilde{u}_{n,i}^{\epsilon,m_j}|^2 = \sum_{I(\delta)\leqslant|i|\leqslant m_j} |u_{n,i}^{\epsilon,m_j}|^2 < \delta^2, \ \forall n \in \mathbb{Z}, \ j \in \mathbb{N}, \tag{5.6.11}$$

where $\tilde{u}$ is the null-extension of u. By the previous discussion, all attractors are contained in those balls of radius r^*, we have

$$\|\tilde{u}_n^{\epsilon,m_j}\| \leqslant r^*, \ \forall n \in \mathbb{Z}, \ j \in \mathbb{N}.$$

In particular, the double sequence

$$\{(\tilde{u}_{n,i}^{\epsilon,m_j})_{|i|<I(\delta)} : n \in \mathbb{Z}, \ j \in \mathbb{N}\}$$

is bounded in $\mathbb{R}^{2I(\delta)-1}$ and thus it has a finite δ-net, which together with (5.6.11) implies that the double sequence

$$\{\tilde{u}_n^{\epsilon,m_j} : n \in \mathbb{Z}, \ j \in \mathbb{N}\}$$

has a finite 2δ-net and thus it is relatively compact in ℓ^2. By a diagonal argument, there are $\{u_n^* : n \in \mathbb{Z}\} \subset \ell^2$ and an index subsequence (denoted by itself) of $\{j\}$ such that

$$\|\tilde{u}_n^{\epsilon,m_j} - u_n^*\| \to 0 \text{ as } j \to \infty, \ \forall n \in \mathbb{Z}. \tag{5.6.12}$$

We then prove that $\{u_n^* : n \in \mathbb{Z}\}$ is an entire solution of the IES (5.2.1). As an entire solution, $\{u_n^{\epsilon,m_j} : n \in \mathbb{Z}\}$ satisfies the truncated IES (5.6.1) for all $n \in \mathbb{Z}$:

$$u_n^{\epsilon,m_j} = u_{n-1}^{\epsilon,m_j} + \epsilon\nu A_{p,m_j}u_n^{\epsilon,m_j} + \epsilon F_m(u_n^{\epsilon,m_j}) + \epsilon g|_{m_j}.$$

We now fix $i \in \mathbb{Z}$, then there is $j_i \in \mathbb{N}$ such that for all $j \geqslant j_i$ we have $m_j \geqslant |i| + 1$, and thus

$$u_{n,i}^{\epsilon,m_j} = \tilde{u}_{n,i}^{\epsilon,m_j}, \ \ (A_{p,m_j}u_n^{\epsilon,m_j})_i = (A_p\tilde{u}_n^{\epsilon,m_j})_i, \ \ (g|_{m_j})_i = g_i$$

for all $j \geqslant j_i$ and $n \in \mathbb{Z}$. Hence, the ith-component of the entire solution u_n^{ϵ,m_j} satisfies

$$\tilde{u}_{n,i}^{\epsilon,m_j} = \tilde{u}_{n-1,i}^{\epsilon,m_j} + \epsilon\nu(A_p\tilde{u}_n^{\epsilon,m_j})_i + \epsilon f(\tilde{u}_{n,i}^{\epsilon,m_j}) + \epsilon g_i \tag{5.6.13}$$

for all $n \in \mathbb{Z}$ and $j \geqslant j_i$. By the local Lipschitz continuity of A_p and F, we have

$$|(A_p\tilde{u}_n^{\epsilon,m_j})_i - (A_pu_n^*)_i| \leqslant \|A_p\tilde{u}_n^{\epsilon,m_j} - A_pu_n^*\| \leqslant L_{p,r^*}\|\tilde{u}_n^{\epsilon,m_j} - u_n^*\|,$$

$$|f(\tilde{u}_{n,i}^{\epsilon,m_j}) - f(u_{n,i}^*)| \leqslant \|F(\tilde{u}_n^{\epsilon,m}) - F(u_n^*)\| \leqslant L_{r^*}\|\tilde{u}_n^{\epsilon,m_j} - u_n^*\|.$$

Letting $j \to \infty$ (thus $m_j \to \infty$) in (5.6.13) and using (5.6.12), we obtain

$$u_{n,i}^* = u_{n-1,i}^* + \epsilon\nu(A_p u_n^*)_i + \epsilon f(u_{n,i}^*) + \epsilon g_i, \ \forall n \in \mathbb{Z}.$$

Since $i \in \mathbb{Z}$ is arbitrary, it follows that $\{u_n^* : n \in \mathbb{Z}\}$ is a (bounded) entire solution of the truncated IES (5.6.1). Hence, $u_0^* \in \mathcal{A}_\epsilon$, which together with

$$\tilde{x}^{m_j} = \tilde{u}_0^{\epsilon,m_j} \to u_0^* \text{ as } j \to \infty$$

gives a contradiction to (5.6.10). □

5.6.3 Lower semi-continuity of numerical attractors for viscosity zero

We denote the restrictions of an element $x \in \ell^2$ and a subset $D \subset \ell^2$ on $\mathbb{R}^{2m+1}$ by

$$x|_m = (x_i)_{|i|\leqslant m} \text{ and } D|_m = \{y \in \mathbb{R}^{2m+1} : \exists x \in D, \text{ s.t. } y = x|_m\}.$$

Proposition 5.6.5. *For each $\epsilon \in (0, \epsilon^*]$, the numerical attractor $\mathcal{A}_\epsilon$ of the IES (5.2.1) satisfies the following lower semi-continuity:*

$$\lim_{m\to\infty} d_{\ell^2}(\mathcal{A}_\epsilon, \mathcal{A}_\epsilon|_m) = 0. \tag{5.6.14}$$

Proof. By Lemma 5.4.1, for each $\delta > 0$, there are $I(\delta) \in \mathbb{N}$ and $N_\epsilon(\delta) \in \mathbb{N}$ such that the solution of the IES (5.2.1) satisfies

$$\|S_\epsilon(n)u_0\|_{\ell^2(|i|\geqslant I(\delta))} < \delta, \ \forall n \geqslant N_\epsilon(\delta), \ u_0 \in \mathcal{B}_{r^*}(0).$$

Given any $x \in \mathcal{A}_\epsilon$. By the invariance, we have $x = S_\epsilon(N_\epsilon(\delta))y$ for some $y \in \mathcal{A}_\epsilon$. Hence, for all $m \geqslant I(\delta)$,

$$\|x - \widetilde{x|_m}\|^2 = \|x\|^2_{\ell^2(|i|\geqslant m)} = \|S_\epsilon(N_\epsilon(\delta))y\|^2_{\ell^2(|i|\geqslant m)} < \delta^2.$$

Since $\widetilde{x|_m} \in \widetilde{\mathcal{A}_\epsilon|_m}$, it follows that for all $m \geqslant I(\delta)$ and $x \in \mathcal{A}_\epsilon$,

$$d_{\ell^2}(x, \widetilde{\mathcal{A}_\epsilon|_m}) \leqslant \|x - \widetilde{x|_m}\| < \delta,$$

which further implies

$$d_{\ell^2}(\mathcal{A}_\epsilon, \widetilde{\mathcal{A}_\epsilon|_m}) = \sup_{x\in\mathcal{A}_\epsilon} d_{\ell^2}(x, \widetilde{\mathcal{A}_\epsilon|_m}) < \delta,$$

for all $m \geqslant I(\delta)$. Hence the lower semi-continuity (5.6.14) holds as desired. □

However, $\mathcal{A}_{\epsilon,m} \neq \mathcal{A}_\epsilon|_m$ generally, where $\mathcal{A}_{\epsilon,m}$ is the truncated numerical attractor for the truncated IES (5.6.1). We only prove the lower semi-continuity in a special case of viscosity zero.

Theorem 5.6.6. *Suppose $\nu = 0$ in both IES (5.2.1) and (5.6.1). Then, for each $\epsilon \in (0, \epsilon^*]$, we have the following lower semi-convergence:*

$$\lim_{m\to\infty} d_{\ell^2}(\mathcal{A}_\epsilon, \mathcal{A}_{\epsilon,m}) = 0. \tag{5.6.15}$$

Proof. Given $x \in \mathcal{A}_\epsilon$. We know that the solution

$$u_n := u_n^\epsilon(x) = S_\epsilon(n)x$$

can be expanded into an entire solution defined for all $n \in \mathbb{Z}$. Hence, the entire solution $\{u_n : n \in \mathbb{Z}\}$ satisfies

$$u_n = u_{n-1} + \epsilon F(u_n) + \epsilon g, \ \forall n \in \mathbb{Z}, \ \ u_0 = x. \tag{5.6.16}$$

The component form of (5.6.16) can be read as

$$u_{n,i} = u_{n-1,i} + \epsilon f(u_{n,i}) + \epsilon g_i, \ \forall n \in \mathbb{Z}, \ i \in \mathbb{Z}. \tag{5.6.17}$$

Considering the truncation of (5.6.17) for those components $|i| \leqslant m$, it follows that $u_n|_m$ satisfies

$$u_n|_m = u_{n-1}|_m + \epsilon F_m(u_n|_m) + \epsilon g^m, \ \forall n \in \mathbb{Z},$$
$$u_0|_m = x|_m \in \mathbb{R}^{2m+1},$$

which means $\{u_n|_m : n \in \mathbb{Z}\}$ is an entire solution of the truncated IES (5.6.1) with $\nu = 0$. Due to the positively invariance, we know $u_n|_m \in \mathcal{B}^m_{r^*}(0)$ and thus the entire solution is bounded in $\mathbb{R}^{2m+1}$, which implies

$$x|_m = u_0|_m \in \mathcal{A}_{\epsilon,m}.$$

Denote by $\widetilde{x|_m}$ the null-expansion of $x|_m$, by $x \in \ell^2$, we have

$$\lim_{m\to\infty} \|\widetilde{x|_m} - x\| = \lim_{m\to\infty} \sum_{|i|>m} |x_i|^2 = 0. \tag{5.6.18}$$

Suppose now the lower semi-convergence (5.6.15) is false. Then there is a subsequence $\{m_j\}$ of $\{m\}$ and $\delta_0 > 0$ such that

$$d_{\ell^2}(\mathcal{A}_\epsilon, \widetilde{\mathcal{A}_{\epsilon,m_j}}) > \delta, \ \forall j \in \mathbb{N},$$

where the tilde denote the null-expansion of the set. Furthermore, for each $j \in \mathbb{N}$, there is $y_j \in \mathcal{A}_\epsilon$ such that

$$d_{\ell^2}(y_j, \widetilde{\mathcal{A}_{\epsilon,m_j}}) > \delta, \ \forall j \in \mathbb{N}. \tag{5.6.19}$$

Since $\mathcal{A}_\epsilon$ is compact in ℓ^2, there is an index subsequence $\{j_k\}$ of $\{j\}$ such that $y_{j_k} \to x$ for some $x \in \mathcal{A}_\epsilon$.

By the previous proof, we know $x|_m \in \mathcal{A}_{\epsilon,m}$ such that (5.6.18) holds. In particular,

$$\lim_{m\to\infty} \|\widetilde{x|_{m_{j_k}}} - x\| = 0, \text{ and } \widetilde{x|_{m_{j_k}}} \in \widetilde{\mathcal{A}_{\epsilon,m_{j_k}}}.$$

Hence,

$$d_{\ell^2}(y_{j_k}, \widetilde{\mathcal{A}_{\epsilon,m_{j_k}}}) \leqslant \|y_{j_k} - x\| + \|x - \widetilde{x|_{m_{j_k}}}\|$$
$$+ d_{\ell^2}(\widetilde{x|_{m_{j_k}}}, \widetilde{\mathcal{A}_{\epsilon,m_{j_k}}}) \to 0, \text{ as } k \to \infty,$$

which contradicts (5.6.19). □

5.6.4 Final Conclusions

As displaying in Figure 1, we have established a path of upper semi-convergence from the truncated numerical attractor $\mathcal{A}_{\epsilon,m}$ to the global attractor $\mathcal{A}$ through the numerical attractor $\mathcal{A}_{\epsilon}$, see Theorems 5.6.4 and 5.5.1.

On the other hand, we can establish another path of upper semi-convergence from $\mathcal{A}_{\epsilon,m}$ to $\mathcal{A}$ through $\mathcal{A}_m$, where $\mathcal{A}_m$ is the attractor of the following truncated LDS of LDS (5.1.3):

$$\frac{du(t)}{dt} = \nu A_{p,m}u(t) + F_m(u(t)) + g|_m, \; u(0) \in \mathbb{R}^{2m+1}.$$

In fact, by the similar method as in Theorem 5.5.1, one can prove the upper semi-convergence from $\mathcal{A}_{\epsilon,m}$ to $\mathcal{A}_m$, while the upper semi-convergence from $\mathcal{A}_m$ to $\mathcal{A}$ follows from the same method as in [7].

Only in the special case of $\nu = 0$, we can establish the two classes of lower semi-convergence as in Figure 1. Lower semi-convergence in other cases remains open.

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