University of Seville
Degree in Physics

Numerical simulation of Alfvén waves in non uniform plasmas

Daniil Kabirov Leontieva

Supervisors:
Eleonora Viezzer and Haotian Chen

Departamento de Física Atómica, Molecular y Nuclear
Facultad de Física

Plasma Science and Fusion Technology

Abstract

Nuclear fusion can provide a sustainable and nearly unlimited source of energy, but for it to become a viable alternative, the fuel must reach very high temperatures, which inevitably leads to the creation of a plasma medium inside a fusion reactor. Thus, a thorough understanding of this state of matter is essential for this goal.

One of the best known models of the plasma is the magneto-hydrodinamic model, or MHD, which describes it as particular fluid, subjected to equations analogous to Naver-Stokes' equations, where the Maxwell equations also play an important role in defining the behaviour of the plasma.

Using this model, it is possible to predict certain instabilities that hinder the containment of the plasma in a nuclear fusion reactor. One of the most relevant instabilities of this kind is the Shear Alfvén Wave, or SAW, the object of study of this thesis.

For this project a simple plasma system is assumed, from which a compact equation for the Shear Alfvén Wave is deduced. However, the equation features a singular point, which is treated performing a Fourier Transform and solving it through numerical methods, returning a profile for the SAW in real space.

Contents

Chapter 1

Introduction

Our modern society, despite all its advancements in our general quality of life, is faced with a grave challenge: a constantly increasing energy demand for an ever growing human population, which is currently supplied, in large part, by fossil fuels. This energy source is not only threatening the long term survival of Earth's ecosphere, but can also lead to undesirable geopolitical tensions due to its in inhomogeneous distribution around the globe.

Thus, humanity is in dire need of alternative energy sources, one of them being the main motivator behind this thesis: nuclear fusion. The nuclei that make up all the elements around us, as we know, are composed of neutrons and protons. However, it is well known that the mass of a certain nucleus isn't exactly the sum of the masses its components: there is a certain mass difference associated in the whole nucleus which is due to the strong interactions between the nucleons. Using Einstein's mass energy-equivalence $E = mc^2$, this mass excess can be associated to a binding energy of a system, which is the sum of the rest energy all its components minus its total rest energy $B_E = \sum m_i c^2 - Mc^2$.

Thus, by managing to transmute a system of protons and neutrons bound in a certain configuration, into another configuration of the same protons and neutrons which is more bound (less massive), then there is an energy excess which translates into kinetic energy of the final system that can be eventually repurposed for energy generation.

As it is shown in the following graph (Figure 1.1), one of those energetically viable processes is to combine two light nuclei into a heavier nucleus which is more bound than its original components.

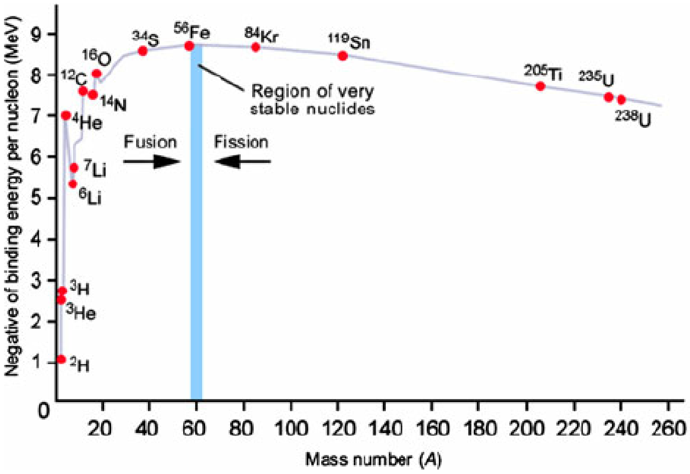

Figure 1.1: Chart of binding energies per nucleon for stable nuclei. From: *https://www.researchgate.net/figure/Binding-energy-per-nucleon-B-Z-N-A-as-a-function-of-the-mass-number_fig1_234065870*

However, to make that possible the two nuclei must overcome the Coulomb barrier due to the electrostatic interaction of the protons, which requires a certain amount of kinetic energy. In the Sun's core, where the pressure is in the order of hundreds of billions of atmospheres, this process starts at proton level, eventually producing ${}^4_2\mathrm{He}$. Meanwhile, on Earth it is impossible to reach such pressures, so a different reaction is used to achieve fusion:

$$ {}^2_1\mathrm{D} + {}^3_1\mathrm{T} \longrightarrow {}^4_2\mathrm{He} + {}^1_0\mathrm{n} + 17.6\,\mathrm{MeV} $$

Note that other reactions are also possible in a nuclear fusion reactor, but are considerably less frequent. In this reaction, ${}^2_1\mathrm{D}$ and ${}^3_1\mathrm{T}$ denote deuterium and tritium, respectively, which can be easily and cheaply obtained from Earth's oceans: directly, in the case of deuterium, or indirectly, by obtaining lithium and transmuting it into tritium in the same nuclear fusion reactor.

However, in order to achieve this nuclear reaction, a temperature in the order of 100 millions of degrees Celsius is needed, which will lead to the complete ionization of the atoms in the reactor:

thus a plasma is created. In addition, due to the extremely high temperatures, the particles have very large velocities (of the order of 10^7 m/s), easily escaping containment.

One of the proposed solutions to this problem are Tokamak fusion reactors, consisting of a toroidal field coil that generates a a magnetic field perpendicular to the solenoid surfaces. Due to the fact that the plasma is composed of charged particles, they experience a Lorentz force which makes them follow the magnetic field lines in a helicoidal orbit. However, due to the weakening of the magnetic field the further we go from the rotational axis of symmetry, there is a drift of the particle towards the outer walls of the solenoid. Thus, a secondary magnetic field in the poloidal direction is created, twisting the total magnetic field in a way that the particles are stuck in closed orbits due to the conservation of energy and magnetic momentum. A schematic representation of a Tokamak is shown in the next figure:

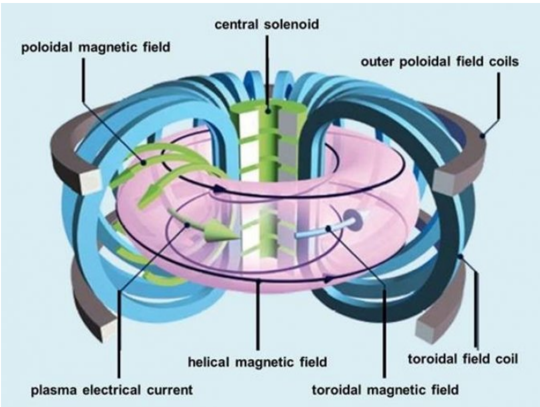

Figure 1.2: A Tokamak with its magnetic fields. From:
https://www.energy.gov/science/doe-explainsTokamaks

There is an additional challenge for containment: the plasma is subject to instabilities in a macroscopic scale, which are hard to control phenomena that eventually lead to the loss of containment of the plasma. These disruptions can be characterized using the MHD (Magneto-hydrodinamic) model.

As it name implies, it assumes that the plasma can be modeled as a fluid governed Maxwell equations. As it will be seen soon, one of the most well studied instabilities is the Shear Alfvén Wave, which will be the main topic of this thesis.

Chapter 2

Theoretical background

2.1 Introduction to plasma physics and MHD model

A plasma is defined formally as an ionized gas which is is **quasi-neutral** and displays **collective effects**. In more detail:

- Quasi-neutrality: despite the fact that the plasma is composed of positive (ions) and negative particles (electrons), for scales much larger than to a coefficient called Debye Length, λ_D, the positive charge and negative charge densities can be considered the same in modulus, thus cancelling out. This coefficient has the following expression:

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_b T_e}{e^2 n_0}} \tag{2.1}$$

  where k_b is the Boltzmann constant, n_0 the mean density of electrons and T_e the temperature associated to electrons, which is generally different from the temperature associated to the ions. Note that these temperatures aren't defined as in Thermodynamics, as that would require a system in equilibrium.

- Collective effects: the behaviour of the plasma is defined by organized displacements of many particles, as, for example, waves in fluids, instead of one-to-one particle interactions and collisions. This condition is verified when the number of particles in a cube of area λ_D^3 is very large.

In order to characterize a plasma it is modeled as a set of distribution functions for each species of particle, $f_i(x,v,t)$ at time t in the phase space, where x is a 3D point in ordinary space and v is a 3D point in the velocity space. This distribution function is such that $f_i(x,v,t)\,dx\,dv$ is the probability of finding a particle of that type at a time interval $t+dt$ in a physical volume $[x,\ x+dx]$ and with a velocity in the $[v,\ v+dv]$ velocity 3D vector space. Focusing on all the particles that enter or leave this 6D box at a time interval dt and adding a term related to the creation or annihilation of particles of a certain position and speed due to collisions, leads to the following equation, where $a(x,v,t)$ denotes the acceleration:

$$\begin{aligned}\frac{\partial f_i(x,v,t)}{\partial t}dx\,dv = &- f_i(d+dx,v,t)\,vdv + f_i(x,v,t)\,vdv \\ &- f_i(x,v+dv,t)\,a(x,v+dv,t)dx \\ &+ f_i(x,v,t)\,a(x,v,t)dx + \left(\frac{\partial f}{\partial t}\right)_C dxdv\end{aligned} \tag{2.2}$$

Performing a Taylor expansion of the terms leads to the Boltzmann equation. From now on, a variable in **bold** will denote a 3D vector, while the normal font denotes its modulus.

$$\frac{\partial f_i}{\partial t} + \mathbf{v}\cdot\frac{\partial f_i}{\partial \mathbf{x}} + \frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{a}f_i) = \left(\frac{\partial f_i}{\partial t}\right)_C \tag{2.3}$$

Note that the partial vector derivative indicates a gradient in $\mathbf{x}$ or $\mathbf{v}$ space respectively. Taking into account the Lorentz force, the equation can be written more explicitly as:

$$\frac{\partial f_i}{\partial t} + \mathbf{v}\cdot\frac{\partial f_i}{\partial \mathbf{x}} + \frac{q}{m}\left[\mathbf{E}+\mathbf{v}\times\mathbf{B}\right]\frac{\partial}{\partial \mathbf{v}}\cdot f_i = \left(\frac{\partial f_i}{\partial t}\right)_C \tag{2.4}$$

The following step is modeling the plasma as a single fluid and find the equations governing it. This framework is called the magnetohydrodinamic (MHD) model. The MHD equations can be obtained by multiplying the Boltzmann equation for each particle species (2.4) by a certain power of $\mathbf{v}$ and a relevant constant and integrating it over all 3D velocity vector space. Thus 3D real space equations for the velocity averages can obtained.

Firstly, the average species' density at a certain point is defined as:

$$n_i(\mathbf{x}, \mathbf{t}) = \int_{\mathcal{R}^3} f_i(x, \mathbf{v}, t) d\mathbf{v} \tag{2.5}$$

while the average velocity of a species is:

$$\mathbf{u}_i(\mathbf{x}, \mathbf{t}) = \frac{1}{n_i(x,t)} \int_{\mathcal{R}^3} \mathbf{v} f_i(x, \mathbf{v}, t) d\mathbf{v} \tag{2.6}$$

Now the 0th order momentum of each species' equation can is taken, which means that (2.4) is multiplied by $m_i v^0 = m_i$ and integrated it over all velocity space. Summing over the all the species leads the fluid continuity equation analogue, where ρ is the mass density of the plasma at a point (x, t):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{2.7}$$

Note that $\mathbf{u}$ is the total average velocity, which takes into account the mass of each species:

$$\mathbf{u} = \frac{\sum_i m_i \mathbf{u_i}}{\sum_i m_i} \tag{2.8}$$

Usually $m_{ion} \gg m_e \implies \mathbf{u} \approx \mathbf{u_i}$. The first order moment is calculated by multiplying by each equation $m_i \mathbf{v}$ before integrating over all 3D space. After summing over all the species, the momentum conservation equation is obtained:

$$\rho \frac{d\mathbf{u}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{P} \tag{2.9}$$

The different notation for the time derivative denotes the convective derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \tag{2.10}$$

where ∇ is taken to be a vector differential operator. This derivative can be thought as the change of a field along a trajectory of a particle moving with average velocity.

The first left hand side term is the Lorentz force term, where $\mathbf{J}$ is the average total current density of all the species. The absence of the electric field is due to the fact that the plasma is assumed to be quasi-neutral by its definition, so the sum of all charges in a certain spatial volume is 0.

The second right hand side term is the divergence of the pressure term, $\mathbf{P}$, is defined as a second order tensor, such as taking divergence yields a vectorial quantity. This tensor can be understood as an anisotropic generalization of hydro-static scalar pressure. Still, if we assume that there are enough collisions for the plasma distribution function to be approximated as a Maxwell-Boltzmann distribution, then the pressure can be assumed isotropic (scalar). This condition, along with the collective effect condition of the plasma, put an upper and lower limit to the prevalence of the collisions in a plasma. Thus, equation (2.9) becomes:

$$\rho \frac{d\mathbf{u}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla P \tag{2.11}$$

Another equation needed to describe the relationship between $\mathbf{J}$ and $\mathbf{u}$. Starting with the first order momentum (momentum conservation) equation for the electrons:

$$m_e \frac{d\mathbf{u_e}}{dt} = -e(\mathbf{E} - \mathbf{u}_e \times \mathbf{B}) - \frac{1}{n_e} \nabla (n_e \, k_b \, T_e) \tag{2.12}$$

Note that the integral of the collision term from the Boltzmann equation (2.4) has been neglected due to the assumption that the collective effects dominate over the particle collisions, which means that the resistivity is zero. The MHD model with this condition is called the ideal MHD model. Taking into account the expression for the total current density and that $\mathbf{u} \approx \mathbf{u}_{ion}$ leads to:

$$\frac{m_e}{e} \frac{d\mathbf{u_e}}{dt} = -\mathbf{E} - \mathbf{u} \times \mathbf{B} + \frac{1}{n_e e} \mathbf{J} \times \mathbf{B} - \frac{1}{n_e e} \nabla (n_e \, k_b \, T_e) \tag{2.13}$$

The electron mass is negligibly small, so the derivative term can be neglected. Now, going to (2.11) and assuming that the pressure term is negligible compared to the other terms, which is true for Earth plasmas, $\mathbf{J} \sim \omega \rho \frac{u}{B} \sim \frac{\omega}{\omega_{ci}} u$, where ω is the characteristic frequency of the phenomenon and $\omega_{ci} = \frac{ZeB}{m_i}$ the ion cyclotron frequency. Considering a slow phenomena is studied, then $\omega / \omega_{ci} \gg 1$ and the $\mathbf{J} \times \mathbf{B}$ term can be dropped.

Now, if the electron cyclotron radius $r_{ce} = v_e \omega_{ce}$ is small compared to the typical length of the system, the thermal term can be neglected in favour of the $\mathbf{u} \times \mathbf{B}$ term, leading to the ideal Ohm's law for plasmas:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \tag{2.14}$$

Now, by using Maxwell equations for low frequencies, as ω must be small, in principle the plasma would be already characterized as a magnetohydrodinamic fluid. However, a closure equation to obtain a well defined system is still needed. Thus, the thermodynamical polytropic evolution equation is added, $PV^{\Gamma} = constant$, which can be redifined in terms of the density:

$$P\rho^{\Gamma} = constant \tag{2.15}$$

where Γ depends on the type of thermodynamical characteristics and the heat transfer inside the plasma. Usually, the plasma is considered to behave like an ideal gas, so $\Gamma = 5/3$.

In summary, the ideal MD equations are the following:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \cdot \mathbf{v}) = 0 \tag{2.16a}$$

$$\rho_o \frac{\partial \mathbf{v}}{\partial t} = -\nabla P + \frac{1}{c}\mathbf{J} \times \mathbf{B} \tag{2.16b}$$

$$\frac{dP}{dt} = -\Gamma P \cdot \nabla \mathbf{v} = 0 \tag{2.16c}$$

$$\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B} = 0 \tag{2.16d}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{2.16e}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \tag{2.16f}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.16g}$$

2.2 SAW in homogeneous plasma

The first approach to the phenomena of SAW should be a very simplified case, which will give an explanation to the numerical results later on. The starting point are the ideal MHD equation, deduced in the previous section.

For this simplified example, a homogeneous stationary plasma slab at equilibrium is assumed, where the magnetic field is parallel to the z-axis. This plasma thus has the following background constants:

$$\begin{aligned} \rho &= \rho_0 \\ \mathbf{v} &= 0 \\ P &= P_0 \\ \mathbf{B} &= B_0 \mathbf{e_z} \\ \mathbf{J} &= 0 \end{aligned} \tag{2.17}$$

In order to proceed with the analysis, the linearization method will be used. It consists on considering that the studied phenomenon, in our case the SAW, is a small, time dependent perturbation of the equilibrium plasma. Thus, any plasma variable can be expressed as:

$$f = f_0 + f_1 \tag{2.18}$$

- f_0 being the value of the function at static, steady state equilibrium (no time dependence).
- f_1 being the small independent perturbation from said equilibrium, verifying $|f_1| \ll |f_0|$

Applying this method to ideal MHD equation for this simplified case leads to the following system of equations:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v_1} = 0 \tag{2.19a}$$

$$\rho_o \frac{\partial \mathbf{v_1}}{\partial t} = -\nabla P_1 + \frac{1}{c} \mathbf{J_1} \times \mathbf{B_0} \tag{2.19b}$$

$$\frac{\partial P_1}{\partial t} + \Gamma P_0 \nabla \cdot \mathbf{v_1} = 0 \tag{2.19c}$$

$$\mathbf{E_1} + \frac{1}{c}\mathbf{v_1} \times \mathbf{B_0} = 0 \tag{2.19d}$$

$$\nabla \times \mathbf{E_1} = -\frac{1}{c}\frac{\partial \mathbf{B_1}}{\partial t} \tag{2.19e}$$

$$\nabla \times \mathbf{B_1} = \frac{4\pi}{c}\mathbf{J_1} \tag{2.19f}$$

$$\nabla \cdot \mathbf{B_1} = 0 \tag{2.19g}$$

Now, each one of the perturbed quantities can be expressed in terms of a Fourier series, both for the spatial and temporal coordinates:

$$f_1(\mathbf{r}, t) = \sum_{\omega}\sum_{\mathbf{k}} \hat{f} e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})} \tag{2.20}$$

Owing to the fact that the equation is linear, each Fourier term will verify the system of equations separately.

For simplicity, and without loosing physical meaning due to symmetry, $\mathbf{k}$ is considered to be contained in the YZ plane. Its components are also redefined in terms of its relative orientation to the magnetic field: $k_{\parallel} \equiv k_z$ and $k_{\perp} \equiv k_y$.

After some manipulations, one can express the system in terms of the fluid velocity $\mathbf{v_1} = (v_{1x}, v_{1y}, v_{1z})$ and three equations:

$$\begin{aligned} &(\omega^2 - k_{\parallel}^2 v_A^2)\, v_{1x} = 0 \\ &(\omega^2 - k_{\perp}^2 v_S^2 - k^2 v_A^2)\, v_{1y} - k_{\perp} k_{\parallel} v_S^2\, v_{1z} = 0 \\ &- k_{\parallel} k_{\perp} v_S^2\, v_{1y} + (\omega^2 - k_{\parallel}^2 v_S^2)\, v_{1z} = 0 \end{aligned} \tag{2.21}$$

where $v_A = \sqrt{B_0^2/(4\pi\rho_0)}$ is the Alfvén velocity, related to the SAW propagation speed, and $v_S = \sqrt{\Gamma P_0/\rho_0}$ is the sound speed at which ion sound waves propagate, similarly to the sound waves in air.

For this system to be solvable, its determinant must be zero. Thus, 3 different roots for ω^2 are obtained:

$$\omega^2 = k_{\parallel}^2 v_A^2 \tag{2.22a}$$

$$\omega^2 = \frac{1}{2}k^2(v_A^2 + v_S^2)(1 \pm \sqrt{1-\alpha^2}); \quad \alpha = 4\frac{k_{\parallel}^2}{k^2}\frac{v_A^2 v_S^2}{(v_A^2 + v_S^2)^2} \tag{2.22b}$$

The first root (2.22a) corresponds to the Shear Alfvén Wave (SAW). Substituting it into the system of equations (2.21), v_{1x} becomes undetermined, while the remaining two equations are trivial, giving us $v_{1y} = v_{1x} = 0$. Therefore, v_{1x} is non-zero.

It can be easily seen that the Shear Alfvén Wave is incompressible, as $\nabla \cdot \mathbf{v_1} = 0$. It can be proven that when an arbitrary plasma displacement compresses the fluid, it must perform work, decreasing its kinetic energy and increasing the potential energy, stabilizing the perturbation. As this effect is not present in SAW waves, they tend to be more unstable, making their study to be of special interest.

Recalling the definition of the group velocity of a 3D wave: $\mathbf{v_g} = \nabla_k \omega$, the group velocity of the SAW is the following:

$$\mathbf{v_G} = v_A \frac{\mathbf{B_0}}{B_0} \tag{2.23}$$

Thus, the collective motion of the perturbation is parallel to the magnetic field. However, from Ohm's Law (2.29d) and Faraday's law (2.29e), it is deduced that $\mathbf{B_1}$ is parallel to the perturbed velocity $\mathbf{V_1}$ and perpendicular to the equilibrium magnetic field $\mathbf{B_0}$:

$$\mathbf{v_1} = \pm v_A \frac{\mathbf{B_1}}{B_0} \tag{2.24}$$

Recalling that the perturbed quantities have an oscillatory element to them $f_1(\mathbf{r}, t) = \hat{f} e^{-i(\omega t - \mathbf{k} \cdot \mathbf{r})}$, the modulus of $\mathbf{B_1}$ oscillates along z. That means that there is a field line bending, that leads to a perturbed current $\mathbf{J_1}$ due to Ampere's Law (2.29f), which in turn leads to a time variation in $\mathbf{v_1}$ due to the momentum equation (2.29b). Thus, the SAW is transverse wave with an oscillation mechanism analogous to a string vibrating under tension. A figure representing the bending can be seen in the following page.

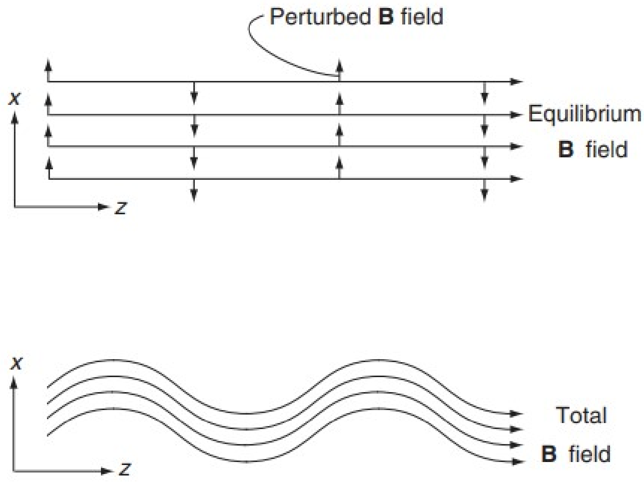

Figure 2.1: Diagram of the Shear Alfvén Wave in a homogeneus plasma. From [7].

The second mode, which is the positive root in (2.22b), is called the fast magneto-acoustic wave, as it can easily be seen that its characteristic frequency, given by the equation in (2.22b), is greater than the Alfvén frequency. There is an interesting limit which is relevant to a plasma in a Tokamak. It is given by the β parameter, defined as the ration between the plasma scalar pressure P we have defined in the previous section, and the magnetic pressure $P_{mag} = \frac{B_0^2}{2\mu_0}$ associated to the magnetic field, which behaves similarly.

$$\beta = \frac{P}{P_{mag}} = \frac{2\mu P}{B_0^2} \tag{2.25}$$

Usually Tokamaks function in the low β limit, that is, the pressure associated to the magnetic field is much bigger than the fluid pressure, which means that that $v_s^2/v_a^2 \ll 1$, Then, the fast magneto-acoustic wave, which is then called fast compressional wave, has the following dispersion relation:

$$\omega^2 = k^2 v_A^2 \tag{2.26}$$

The group velocity would be:

$$\mathbf{v_G} = v_A \frac{\mathbf{k}}{k} \tag{2.27}$$

The perturbed fluid velocity $\mathbf{v_1}$ components are now $v_{1x} = 0$ and $v_{1y},\ v_{1z} \neq 0$, thus $\nabla \cdot \mathbf{v_1} \neq 0$: in this mode of oscillation the plasma is compressible, and has a longitudinal and transversal component. This leads to this mode being described by a magnetic tension force, as in the last case, plus a magnetic compression force, as in a sound wave.

For the negative root in (2.22b), the slow magneto-sonic wave appears, whose frequency is smaller that the SAW frequency. In the low beta limit of interest:

$$\omega^2 = k_{\|}^2 v_S^2 \tag{2.28}$$

In this mode the results for the perturbed velocity are the same, but in the low beta limit case: $v_{1y}/v_{1z} \gg 1$, which means that the main force at play here is the plasma compression.

As a final note, let's make another analogy with the elastic string under tension for SAW and fast compressional waves: recalling the relationship between the elastic constant of a spring and its frequency, it turns out that the terms $k_{\|} v_A$ for SAW and $k\, v_A$ for the fast compressional wave act as the analogue for said elastic constants in the plasma. Usually, the bigger the elastic constant, the more kinetic energy is needed to produce a perturbation of the same amplitude. This is another argument by which compressional waves are more stable than SAW.

Energetically speaking, a $k_{\|} v_A$ elastic constant implies kinetic energy absorption through the bending of the magnetic field lines, while a $k_{\perp} v_A$ elastic constant corresponds to compressional energy absorption. For SAW waves, the modes of interest are the more dangerous and unstable ones that can get the plasma out of confinement, so when studying SAW, $|k_{\|}|$ is made as small as possible, of the order of the inverse of the typical length of the Tokamak.

Moreover, there is another energy absorbing mechanism we want to minimize, which happens for large $|k_{\|}|$: damping through wave-particle resonance, where the wave energy is transferred to the resonant particles, also called Landau Damping. Essentially, what happens is that particles moving with a similar speed an direction as an MHD wave, get accelerated if they are moving slightly slower than the wave and decelerated in the opposite case. Thus, if the number of particles that go slower is greater than the ones that go faster, the wave cedes energy to the slow particles, losing energy.

This mechanism can't be explained by the MHD fluid model, these phenomena won't play a role in this case. Still, it is another justification for the approximation $|k_{\parallel}| \ll |k_{\perp}|$ that can be often found in the bibliography, as in references [3] and [2]

2.3 Linear equations for a non-homogeneous plasma

The main topic of this thesis is the characterization of the SAW in a non-homogeneous plasma slab, where the equilibrium quantities may depend on a single coordinate. Before performing the linearization, some simplifications can be made: due to the quasi-neutraility condition, $\mathbf{E_0} = 0$ and it will be assumed that $\nabla \cdot \mathbf{v_1} = 0$, which is equivalent to saying that the perturbed mode will be in-compressible. There are two reasons behind this simplification:

- There are multiple types of perturbations, or plasma modes, that can be obtained in these equations. While the Shear Alfvén Mode is the one which is in-compressible, it isn't the case for other modes. Thus, it is a way to isolate the study of SAW. Still, fast compressional modes will still play a part in explaining the the behaviour of the wave later.
- The fact that a perturbation is compressible has been proven to make the system more stable. The phenomena of interest are the ones that could destabilize the system, as they are of relevance for plasma containment in Tokamaks.

Thus, the linearized MHD equations are the following:

$$\frac{\partial \rho_1}{\partial t} + \nabla \rho_o \cdot \mathbf{v_1} = 0 \tag{2.29a}$$

$$\rho_o \frac{\partial \mathbf{v_1}}{\partial t} = -\nabla P_1 + \frac{1}{c}\mathbf{J_1} \times \mathbf{B_0} + \frac{1}{c}\mathbf{J_0} \times \mathbf{B_1} \tag{2.29b}$$

$$\frac{\partial P_1}{\partial t} + \mathbf{v_1} \cdot \nabla P_0 = 0 \tag{2.29c}$$

$$\mathbf{E_1} + \frac{1}{c}\mathbf{v_1} \times \mathbf{B_0} = 0 \tag{2.29d}$$

$$\nabla \times \mathbf{E_1} = -\frac{1}{c}\frac{\partial \mathbf{B_1}}{\partial t} \tag{2.29e}$$

$$\nabla \times \mathbf{B_1} = \frac{4\pi}{c}\mathbf{J_1} \tag{2.29f}$$

$$\nabla \cdot \mathbf{B_1} = 0 \tag{2.30}$$

2.4 Alfvén-Interchange-Equation

Having obtained the linearized MHD equations, the next objective is to obtaining a single equation that can describe the SAW. Thus, all the perturbed quantities will be expressed in terms of the perturbed potential ϕ_1, which makes it easier to add the aformentioned wave-particle interaction phenomenon for a future project.

The first step is multiplying vectorially the equation (2.29b) $\mathbf{B_0}$ from the right side:

$$\mathbf{B_0} \times \rho_o \frac{\partial \mathbf{v_1}}{\partial t} = -\mathbf{B_0} \times \nabla P_1 + \frac{1}{c}\mathbf{B_0} \times (\mathbf{J_1} \times \mathbf{B_0}) + \frac{1}{c}\mathbf{B_0} \times (\mathbf{J_0} \times \mathbf{B_1}) \tag{2.31}$$

However, $\mathbf{B_0}$ doesn't depend explicitly on t, so it can be inserted into the time derivative. Invoking (2.29d) along with the vector property A1, from *Appendix A*, leads to:

$$\rho_o c \frac{\partial \mathbf{E_1}}{\partial t} = -\mathbf{B_0} \times \nabla P_1 + \frac{1}{c}\mathbf{B_0} \times (\mathbf{J_1} \times \mathbf{B_0}) + \frac{1}{c}\mathbf{B_0} \times (\mathbf{J_0} \times \mathbf{B_1}) \tag{2.32}$$

The second right-hand side term (RHT) can be transformed by applying (2.29f) and vector property A2:

$$\mathbf{B_0} \times (\mathbf{J_1} \times \mathbf{B_0}) = \frac{cB_0^2}{4\pi}\left[(\nabla \times \mathbf{B_1}) - (\nabla \times \mathbf{B_1}) \cdot \mathbf{bb}\right] \equiv \frac{cB_0^2}{4\pi}(\nabla \times \mathbf{B_1})_\perp \tag{2.33}$$

where $\mathbf{b} \equiv \frac{\mathbf{B_0}}{|\mathbf{B_0}|}$ is the unitary vector parallel to the magnetic field line, $\mathbf{bb}$ is the tensor diad that verifies $\mathbf{v} \cdot \mathbf{bb} \equiv (\mathbf{v} \cdot \mathbf{b})\mathbf{b}$, $\mathbf{v}$ being a vector in 3D space. With this notation, $\mathbf{v} - \mathbf{v} \cdot \mathbf{bb} \equiv \mathbf{v}_\perp$ is defined as the projection of the vector $\mathbf{v}$ perpendicular to the magnetic field line.

Substituting the new expression (2.33) in (2.32):

$$\rho_o c \frac{\partial \mathbf{E_1}}{\partial t} = -\mathbf{B_0} \times \nabla P_1 + \frac{B_0^2}{4\pi}(\nabla \times \mathbf{B_1})_\perp + \frac{1}{c}\mathbf{B_0} \times (\mathbf{J_0} \times \mathbf{B_1}) \tag{2.34}$$

Differentiating the equation by $\frac{\partial}{\partial t}$ and using the relations given by (2.29c) and (2.29f) leads to the following expression:

$$\rho_o c \frac{\partial^2 \mathbf{E_1}}{\partial t^2} = \mathbf{B_0} \times \nabla(\nabla P_0 \cdot \mathbf{v_1}) - \frac{cB_0^2}{4\pi}(\nabla \times \nabla \times \mathbf{E_1})_\perp + \mathbf{B_0} \times [\mathbf{J_0} \times (\nabla \times \mathbf{E_1})] \quad (2.35)$$

Assuming that $\mathbf{v_1} \perp \mathbf{B_0}$, which is true for SAW waves in an homogeneous plasma, Ohm's Law (2.29d) can be inverted:

$$\mathbf{v_1} = -\frac{c}{B_0^2}\mathbf{E_1} \times \mathbf{B_0} \quad (2.36)$$

This result can be used in the first RHS term in (2.35):

$$\rho_o \frac{\partial^2 \mathbf{E_1}}{\partial t^2} = \mathbf{B_0} \times \nabla \left[\frac{(\mathbf{b} \times \nabla P_0) \cdot \mathbf{E_1}}{B_0}\right] - \frac{B_0^2}{4\pi}(\nabla \times \nabla \times \mathbf{E_1})_\perp + \frac{1}{c}\mathbf{B_0} \times [\mathbf{J_0} \times (\nabla \times \mathbf{E_1})] \quad (2.37)$$

where the triple product has been reorganized and the c constant has been cleared.

Now it is necessary to review the electromagnetic potential formulation, by which the perturbed electromagnetic fields can be expressed using the scalar and vector perturbed potentials:

$$\mathbf{E_1} = -\nabla \phi_1 - \frac{1}{c}\frac{\partial \mathbf{A_1}}{\partial t} \quad (2.38a)$$

$$\mathbf{B_1} = \nabla \times \mathbf{A_1} \quad (2.38b)$$

Coulomb's Gauge has also been used: $\nabla \cdot \mathbf{A_1} = 0$.

Taking into account that the equation is not only linear for any of the three Cartesian coordinates, but also that the equilibrium properties of the plasma will vary along one of those coordinates, which will be x. Therefore, as in the last section, every perturbed quantity can be expressed as $f_1 = \tilde{f}_1(x)e^{-i(k_y y + k_z z)}$.

Moreover, the del operator into two parts: one parallel and one perpendicular to the magnetic field lines: $\nabla = \nabla_\| + \nabla_\perp$. Using Coulomb's Gauge along with said decomposition:

$$\nabla \cdot \mathbf{A_1} = \nabla_\| \cdot \mathbf{A_1} + \nabla_\perp \cdot \mathbf{A_1} = k_\| v_{1\|} + k_\perp v_{1\perp} \stackrel{!}{=} 0 \quad (2.39)$$

As it has been assumed that $|k_{\|}| \ll |k_{\perp}|$, the following relation holds true:

$$|A_{1\|}| \gg |A_{1\perp}| \implies \mathbf{A_1} \approx A_{1\|}\mathbf{b} \tag{2.40}$$

However, going back to Ohm's law (2.29d), the perturbed electric field is perpendicular to the magnetic equilibrium field due to the properties of the vector product, so $\mathbf{E_1}_{\|} = 0$. Using the electrostatic potential equation (2.38a) for said parallel component:

$$\nabla_{\|}\phi_1 = -\frac{1}{c}\frac{\partial A_{1\|}}{\partial t} \tag{2.41}$$

By inserting that condition into the $\mathbf{E_1}$ (2.38a), taking the curl of the equation with the approximation (2.40):

$$\nabla \times \mathbf{E_1} = \cancel{-\nabla \times \nabla\phi_1} - \frac{1}{c}\nabla \times \left(\frac{\partial A_{1\|}}{\partial t}\mathbf{b}\right) = \nabla \times (\nabla_{\|}\phi_1\mathbf{b}) \tag{2.42}$$

However, the approximation (2.40) also leads to:

$$\mathbf{E_1} = \mathbf{E_1}_{\perp} = \nabla_{\perp}\phi_1 \tag{2.43}$$

These results can be inserted into the previous equation (2.37), leading to:

$$\begin{aligned} \rho\frac{\partial^2\mathbf{E_1}}{\partial^2 t} = &\underbrace{\mathbf{B_0} \times \nabla\left[\frac{(\mathbf{b} \times \nabla P_0)\cdot\nabla_{\perp}\phi_1}{B_0}\right]}_{\text{I}} + \underbrace{\frac{B_0^2}{4\pi}[\nabla \times \nabla \times (\nabla_{\|}\phi_1\mathbf{b})]_{\perp}}_{\text{II}} + \\ &+ \frac{1}{c}\underbrace{\mathbf{B_0} \times \mathbf{J_0} \times [\nabla \times (\nabla_{\|}\boldsymbol{\phi_1}\mathbf{b})]}_{\text{III}} \end{aligned} \tag{2.44}$$

The equation is now going to acquire a lot of terms, so each of the terms has been labeled with a roman numeral as they will be treated separately.

Term I:

$$\mathbf{B_0} \times \nabla \left[\frac{(\mathbf{b} \times \nabla P_0) \cdot \nabla_\perp \phi_1}{B_0} \right] \tag{2.45}$$

In order to continue, the following expression must be proven:

$$\nabla \times \frac{\mathbf{b}}{B_0} = \frac{4\pi}{cB_0^2}(\mathbf{J}_\parallel - \mathbf{J}_\perp) + \frac{2}{B}\mathbf{b} \times \boldsymbol{\kappa} \tag{2.46}$$

where $\boldsymbol{\kappa} \equiv (\mathbf{b} \cdot \nabla)\mathbf{b}$ is the curvature vector of the magnetic field.

Proof (2.46): *First, the LHT (left hand term) is expanded by virtue of the successive application of vector calculus property A4*:

$$\nabla \times \frac{\mathbf{b}}{B_0} = \frac{1}{B_0}\nabla \times \mathbf{b} - \frac{1}{B_0^2}\nabla B_0 \times \mathbf{b} = \frac{2}{B_0}\nabla \times \mathbf{b} - \frac{1}{B_0^2}\nabla \times \mathbf{B_0} \tag{2.47}$$

For the first RHT in the original equation (2.46), (2.29f) and the definition of the the vectors parallel $\mathbf{v}_\parallel$ *and perpendicular* $\mathbf{v}_\perp$ *to the equilibrium magnetic field are used, and the obtained expression is simplified by using A4 where* $\mathbf{B_0} = B_0\mathbf{b}$*:*

$$\frac{4\pi}{cB_0^2}(\mathbf{J}_\parallel - \mathbf{J}_\perp) = \frac{2}{B_0^2}(\nabla \times \mathbf{b})\mathbf{b}\mathbf{b} - \frac{1}{B_0^2}\nabla \times \mathbf{B_0} \tag{2.48}$$

For the second RHT in (2.46), the alternative definition of $\boldsymbol{\kappa}$ *which is* $\boldsymbol{\kappa} = -\mathbf{b} \times (\nabla \times \mathbf{b})$ *is applied:*

$$\frac{2}{B_0}\mathbf{b} \times \boldsymbol{\kappa} = -\frac{2}{B_0}\mathbf{b} \times \mathbf{b} \times (\nabla \times \mathbf{b}) = -\frac{2}{B_0}(\nabla \times \mathbf{b})\mathbf{b}\mathbf{b} + \frac{2}{B_0}(\nabla \times \mathbf{b}) \tag{2.49}$$

By adding (2.48) to (2.49) we get (2.47), thus proving (2.46). **End of proof**.

Using curl chain rule A4 in the term I (2.45) multiple times, while writing $\mathbf{B_0}$ as $\mathbf{B_0} = B_0^2 \frac{\mathbf{b}}{B_0}$

$$\begin{aligned} &\mathbf{B_0} \times \nabla \left[\frac{(\mathbf{b} \times \nabla P_0) \cdot \nabla_\perp \phi_1}{B_0} \right] = \\ &= B_0^2 \left\{ -\nabla \times \left[\frac{(\mathbf{b} \times \nabla P_0) \cdot \nabla_\perp \phi_1}{B_0^2} \mathbf{b} \right] + \left[\nabla \times \left(\frac{\mathbf{b}}{B_0} \right) \right] \left[\frac{(\mathbf{b} \times \nabla P_0) \cdot \nabla_\perp \phi_1}{B_0} \right] \right\} \end{aligned} \tag{2.50}$$

This way, applying the proof at (2.46), term 1 (2.45) is decomposed into the following expansion:

$$\begin{aligned} \mathbf{B_0} \times \nabla \left[\frac{(\mathbf{b} \times \nabla P_0) \cdot \nabla_\perp \phi_1}{B_0} \right] &= \underbrace{\frac{4\pi}{c} \mathbf{J}_\parallel \left[\frac{(\mathbf{b} \times \nabla P_0) \cdot \nabla_\perp \phi_1}{B_0} \right]}_{\text{I.A}} \\ &- \underbrace{B_0^2 \nabla \times \left[\frac{(\mathbf{b} \times \nabla P_0) \cdot \nabla_\perp \phi_1}{B_0^2} \mathbf{b} \right]}_{\text{I.B}} - \underbrace{\frac{4\pi}{c} \left[\mathbf{J}_\perp - \mathbf{B_0} \times \boldsymbol{\kappa} \right] \left[\frac{(\mathbf{b} \times \nabla P_0) \cdot \nabla_\perp \phi_1}{B_0} \right]}_{\text{I.C}} \end{aligned} \tag{2.51}$$

Term II:

Ignoring the prefactor, the term II would be:

$$[\nabla \times \nabla \times (\nabla_\parallel \phi_1 \mathbf{b})]_\perp \equiv [\nabla \times \nabla \times (\nabla_\parallel \phi_1 \mathbf{b})] - [\nabla \times \nabla \times (\nabla_\parallel \phi_1 \mathbf{b})]\mathbf{b}\mathbf{b} \tag{2.52}$$

In order to perform a similar expansion of the terms, the following expression must be proven:

$$[\nabla \times \nabla \times (\nabla_\parallel \phi_1 \mathbf{b})]\mathbf{b}\mathbf{b} = \{(\nabla_\parallel \phi_1)(\nabla \cdot \boldsymbol{\kappa}) - \nabla \cdot [\nabla_\perp (\nabla_\parallel \phi_1)] + (\nabla_\parallel \phi_1) |\nabla \times \mathbf{b}|^2 \} \mathbf{b} \tag{2.53}$$

Proof (2.53): *For* $(\nabla_\parallel \phi_1)$*, which is a scalar, the notation* $(\nabla_\parallel \phi_1) \equiv F$ *will be used. First, use* $\boldsymbol{\kappa} = -\mathbf{b} \times (\nabla \times \mathbf{b})$ *on the first RHT:*

$$F \, \nabla \cdot \boldsymbol{\kappa} = -F \, \nabla \cdot [\mathbf{b} \times (\nabla \times \mathbf{b})] = -F |\nabla \times \mathbf{b}|^2 + F(\nabla \times \nabla \times \mathbf{b}) \cdot \mathbf{b} \tag{2.54}$$

Then, applying A4 to the second RHT of the immediately previous equation (2.54) two times:

$$F(\nabla \times \nabla \times \mathbf{b}) \cdot \mathbf{b} = \{\nabla \times [\nabla \times (F\mathbf{b})] - \nabla \times (\nabla F \times \mathbf{b}) - \nabla F \times (\nabla \times \mathbf{b})\} \cdot \mathbf{b} \tag{2.55}$$

Now, by using A5 on the second RHT of the original equation (2.53):

$$\begin{aligned} \nabla \cdot [\nabla_\perp F] &= -\nabla \cdot \{\mathbf{b} \times (\mathbf{b} \times \nabla F)\} = \\ &= [\nabla \times (\mathbf{b} \times \nabla F)] \cdot \mathbf{b} - (\mathbf{b} \times \nabla F) \cdot (\nabla \times \mathbf{b}) \end{aligned} \tag{2.56}$$

Now (2.54), (2.55), (2.56) have been substituted in (2.53). Some terms will go away, resulting in:

$$\begin{aligned} &F\nabla \cdot \boldsymbol{\kappa} - \nabla \cdot (\nabla_\perp F) + F|\nabla \times \mathbf{b}|^2 = \\ &= (\mathbf{b} \times \nabla F) \cdot (\nabla \times \mathbf{b}) + \{\nabla \times [\nabla \times (F\mathbf{b})] - \nabla F \times (\nabla \times \mathbf{b})\} \cdot \mathbf{b} \end{aligned} \tag{2.57}$$

By using the cyclic property of the triple product in the third RHT it goes away with the first term:

$$F\nabla \cdot \boldsymbol{\kappa} - \nabla \cdot (\nabla_\perp F) + F|\nabla \times \mathbf{b}|^2 = \{\nabla \times [\nabla \times (F\mathbf{b})]\} \cdot \mathbf{b} \tag{2.58}$$

The equation we get is precisely (2.53), but in scalar form. Multiplying by $\mathbf{b}$ *on both sides gives the original equation.* **End of proof.**

Now, the term II (2.52) can be expressed in the following way:

$$\begin{aligned} &\frac{B_0^2}{4\pi}[\nabla \times \nabla \times (\nabla_\parallel \phi_1 \mathbf{b})]_\perp \equiv \underbrace{\frac{B_0^2}{4\pi}[\nabla \times \nabla \times (\nabla_\parallel \phi_1 \mathbf{b})]}_{\text{II.A}} - \frac{B_0^2}{4\pi}[\nabla \times \nabla \times (\nabla_\parallel \phi_1 \mathbf{b})]\mathbf{b}\mathbf{b}; \\ &[\nabla \times \nabla \times (\nabla_\parallel \phi_1 \mathbf{b})]\mathbf{b}\mathbf{b} = \{\underbrace{(\nabla_\parallel \phi_1)(\nabla \cdot \boldsymbol{\kappa})}_{\text{II.B}} - \underbrace{\nabla \cdot [\nabla_\perp (\nabla_\parallel \phi_1)]}_{\text{II.C}} + \underbrace{(\nabla_\parallel \phi_1)|\nabla \times \mathbf{b}|^2}_{\text{II.D}}\}\mathbf{b} \end{aligned} \tag{2.59}$$

Note that the terms II.B, II.C, II.D also include the $\mathbf{b}$ and the $\frac{B_0^2}{4\pi}$ prefactors.

Term III:

The term in question is:

$$\mathbf{B_0} \times \mathbf{J_0} \times [\nabla \times (\nabla_\parallel \boldsymbol{\phi_1} \mathbf{b})] \tag{2.60}$$

Decomposing the vecorial product into two separate terms and using the definitions of perpendicular and parallel components of a vector:

$$\mathbf{B_0} \times \mathbf{J_0} \times [\nabla \times (\nabla_{\parallel}\phi_{\mathbf{1}}\mathbf{b})] = B_0[\nabla \times (\nabla_{\parallel}\phi_{\mathbf{1}}\mathbf{b})]_{\parallel}\mathbf{J_{0\perp}} - B_0[\nabla \times (\nabla_{\parallel}\phi_{\mathbf{1}}\mathbf{b})]_{\perp}\mathbf{J_{0\parallel}} \tag{2.61}$$

Taking the first term in the previous equation and applying the property A2 and (2.29f):

$$\begin{aligned} &B_0[\nabla \times (\nabla_{\parallel}\phi_{\mathbf{1}}\mathbf{b})]_{\parallel}\mathbf{J_{0\perp}} \equiv \mathbf{B_0} \cdot [\nabla \times (\nabla_{\parallel}\phi_{\mathbf{1}}\mathbf{b})]\mathbf{J_{0\perp}} = \\ &= \{\nabla \times \cancel{[\mathbf{B_0} \times \nabla_{\parallel}\phi_1\mathbf{b}]} + (\nabla_{\parallel}\phi_{\mathbf{1}}\mathbf{b}) \cdot \nabla \times \mathbf{B_0}\}\mathbf{J_{0\perp}} = \frac{4\pi}{c}\nabla_{\parallel}\phi_1 J_{0\parallel}\mathbf{J_{0\perp}} \end{aligned} \tag{2.62}$$

The final form will be:

$$\frac{1}{c}\mathbf{B_0} \times \mathbf{J_0} \times [\nabla \times (\nabla_{\parallel}\phi_{\mathbf{1}}\mathbf{b})] = \underbrace{\frac{4\pi}{c^2}\nabla_{\parallel}\phi_1 J_{0\parallel}\mathbf{J_{0\perp}}}_{\text{III.A}} - \underbrace{B_0[\nabla \times (\nabla_{\parallel}\phi_{\mathbf{1}}\mathbf{b})]_{\perp}\mathbf{J_{0\parallel}}}_{\text{III.B}} \tag{2.63}$$

Restructuring

After expanding all the terms in (2.44), they can be rearranged into a set of physically significant terms. But before that, the proof for the equilibrium balance equation is needed:

$$c\mathbf{b} \times \nabla P_0 = B_0\mathbf{J_{0\perp}} \tag{2.64}$$

Proof (2.64): *Starting from the momentum conservation equation (2.16b):*

$$\nabla P_0 = \frac{1}{c}\mathbf{J_0} \times \mathbf{B_0} = \frac{1}{c}(\mathbf{J_{0\perp}} + \cancel{\mathbf{J_{0\parallel}}}) \times \mathbf{B_0} \tag{2.65}$$

It can be multiplied vectorially by $\mathbf{b}$*, and by use the BAC-CAB property:*

$$\mathbf{b} \times \nabla P_0 = \frac{1}{c}\mathbf{b} \times \mathbf{J_{0\perp}} \times \mathbf{B_0} = \frac{1}{c}B_0\mathbf{J_{0\perp}} - \cancel{\frac{1}{c}\mathbf{B_0}(\mathbf{b} \cdot \mathbf{J_{0\perp}})} \tag{2.66}$$

End of proof.

Then, adding I.A from (2.51) to II.D from (2.59) leads to the following expresison:

$$\frac{4\pi}{c}\left[\frac{(\mathbf{b}\times\nabla P_0)\cdot\nabla_\perp\phi_1}{B_0}\right]\mathbf{J_{0\parallel}} - \frac{B_0^2}{4\pi}(\nabla_\parallel\phi_1)|\nabla\times\mathbf{b}|^2\mathbf{b} \tag{2.67}$$

By using property A3, it can be easily proven that:

$$\kappa^2 = [\mathbf{b}\times(\nabla\times\mathbf{b})]^2 = |\nabla\times\mathbf{b}|^2 - [\mathbf{b}\cdot(\nabla\times\mathbf{b})]^2 = |\nabla\times\mathbf{b}|^2 - \frac{4\pi}{cB_0}\mathbf{J_{0\parallel}} \tag{2.68}$$

Thus, (2.67) becomes:

$$\begin{aligned}&\frac{4\pi}{c}\left[\frac{(\mathbf{b}\times\nabla P_0)\cdot\nabla_\perp\phi_1}{B_0}\right]\mathbf{J_{0\parallel}} - \frac{B_0^2}{4\pi}(\nabla_\parallel\phi_1)|\nabla\times\mathbf{b}|^2\mathbf{b} = \\ &= \frac{4\pi}{c^2}\mathbf{J_{0\parallel}}(\mathbf{J_{0\parallel}} - \mathbf{J_{0\perp}})\cdot\nabla\phi_1 - \frac{B_0^2}{4\pi}\kappa^2(\nabla_\parallel\phi_1)\mathbf{b}\end{aligned} \tag{2.69}$$

On the other hand, adding term III.A from (2.63) to the component proportional to $\mathbf{J_{0\perp}}$ in I.C (2.51) leads to:

$$\begin{aligned}&\frac{4\pi}{c^2}\nabla_\parallel\phi_1 J_{0\parallel}\mathbf{J_{0\perp}} - \frac{4\pi}{c}\mathbf{J_{0\perp}}\left[\frac{(\mathbf{b}\times\nabla P_0)\cdot\nabla_\perp\phi_1}{B_0}\right] = \\ &= \frac{4\pi}{c^2}\mathbf{J_{0\perp}}(\mathbf{J_{0\parallel}} - \mathbf{J_{0\perp}})\cdot\nabla\phi_1\end{aligned} \tag{2.70}$$

Coming back to (2.44) with all the expanded terms, using (2.69) and (2.70), multiplying by $4\pi/B_0^2$ taking the divergence leads to:

$$\begin{aligned}0 = &\underbrace{\nabla\cdot\left[\frac{1}{v_A^2}\nabla_\perp\frac{\partial^2}{\partial t^2}\phi_1\right]}_{ICU} - \underbrace{\nabla\cdot\left[\frac{8\pi}{B_0^2}(\mathbf{b}\times\mathbf{k})(\mathbf{b}\times\nabla P_0)\cdot\nabla_\perp\phi_1\right]}_{BIC} - \\ &- \underbrace{\nabla\cdot\left[\mathbf{b}\nabla\cdot\nabla_\perp(\nabla_\parallel\phi_1)\right]}_{FLB} + \underbrace{\nabla\cdot\left[\frac{4\pi J_{0\parallel}}{cB}\nabla\times(\mathbf{b}\nabla_\parallel\phi_1)\right]_\perp}_{KINK} + \\ &+ \underbrace{\nabla\cdot\left[(\nabla\cdot\boldsymbol{\kappa}+\kappa^2)\mathbf{b}(\nabla_\parallel\phi_1)\right] + \nabla\cdot\left[\frac{16\pi^2}{c^2B_0^2}(\mathbf{J_{\parallel 0}} - \mathbf{J_{0\perp}})(\mathbf{J_{\parallel 0}} - \mathbf{J_{0\perp}})\cdot\nabla\phi_1\right]}_{corrections}\end{aligned} \tag{2.71}$$

with $v_A \equiv \sqrt{\frac{B_0^2}{4\pi\rho_0}}$ being the Alfvén velocity.

This equation, called the Alfvén-Interchange equation, is very important in MHD theory and also physically rich, as every term is due to specific forces acting on the plasma, and may lead to interesting instabilities.

- ICU (Inertia-charge uncovering): It is related to the inertial mass of the plasma through the mass density, a little bit analogous to the classic $m\,a$ term in Newtons equation.

- BIC (Balloon-interchange contribution): Due to the combined effects of a pressure gradient with the equilibrium magnetic field curvature. Gives rise to the balloon-interchange instability in Tokamak plasmas.

- FLB (Field line bending): Responsible of the recovering forces which appear when the perturbed magnetic field is bent, creating the Shear Alfvén Waves.

- Kink (Kink-term): Related to the kink instability, which can be explained as the constantly increasing bending of a plasma column due to differences in the magnetic field in both sides.

- The last two two terms are second order contributions of the curvature and can be ignored for low β plasmas, which is the case for this thesis.

2.5 Simplifications of the deduced equation

Plasmas are usually subjected to a myriad of different phenomena which span a great range of time and spatial scales. Thus, a scale ordering of those phenomena is needed in order to focus exclusively on the studied phenomenon.

The initial, and most drastic approximation, is working with a plasma slab where the equilibrium magnetic field isn't curved, so $\boldsymbol{\kappa} = 0$. Thus, the Alfvén-Interchange equation becomes:

$$0 = \underbrace{\nabla \cdot \left[\frac{1}{v_a^2}\nabla_\perp \frac{\partial^2}{\partial t^2}\phi_1\right]}_{ICU} - \underbrace{\nabla \cdot \Big[\mathbf{b}\nabla \cdot \nabla_\perp(\nabla_\parallel \phi_1)\Big]}_{FLB} + \underbrace{\nabla \cdot \left[\frac{4\pi J_{0\parallel}}{cB}\nabla \times (\mathbf{b}\nabla_\parallel \phi_1)\right]_\perp}_{Kink} + \\ + \underbrace{\nabla \cdot \left[\frac{16\pi^2}{c^2 B_0^2}(\mathbf{J}_{\parallel\mathbf{0}} - \mathbf{J}_{\mathbf{0}\perp})(\mathbf{J}_{\parallel\mathbf{0}} - \mathbf{J}_{\mathbf{0}\perp}) \cdot \nabla\phi_1\right]}_{correction} \tag{2.72}$$

Before proceeding, an explanation of the Larmor precession is required. In its simplest form, particles of a charge q, under a constant magnetic field B, will describe a helicoidal movement perpendicular to the magnetic field of a frequency Ω and radius ρ, given by:

$$\Omega = \frac{|q|B}{m} \tag{2.73a}$$

$$\rho_i = \frac{v_\perp B}{mc} \tag{2.73b}$$

m being its mass and $v_\perp$ it's velocity perpendicular to the magnetic field.

The scale ordering that is used for plasmas inside a Tokamak is the called the gyrokinetic framework: it comes from the experimental reasoning that some phenomena of interest, which are unstable kinetic effects in Tokamaks, have a characteristic frequency ω and characteristic inverse lengths k, which in relation to the equilibrium magnetic field is decomposed in $k_\parallel$ and $k_\perp$, obey the criteria described in the following equations:

$$\left|\frac{\omega}{\Omega_i}\right| \sim \epsilon \ll 1 \tag{2.74a}$$

$$k_\perp \rho_i \sim 1 \tag{2.74b}$$

$$\frac{k_\parallel}{k_\perp} \sim \epsilon \tag{2.74c}$$

where Larmor parameters expressed here refer to ions.

Strictly speaking, this exact framework isn't necessary as the focus will be the Shear Alfvén Waves, which are MHD phenomena, but in order to further incorporate kinetic effects, as to study the Landau Damping and the associated Kinetic Alfvén Waves resulting from this interaction, these assumptions must be made beforehand.

Still, the condition (2.74a) is already present in the necessary conditions for an MHD fluid, as discussed in Section 2.1, while the condition (2.74c) has been justified in *Section 2.2* for SAW waves. These assumptions can be expressed as:

$$k_{\parallel} \sim \frac{\epsilon}{a} \ll \frac{1}{a} \tag{2.75}$$

$$k_{\perp} \sim \frac{1}{a} \tag{2.76}$$

Where a is the characteristic length of the system, which tends to be of the order of meters. The condition which characterizes the Shear Alfvén Waves is the following:

$$\omega \sim k_{\parallel} v_A \tag{2.77}$$

Using (2.76) in (2.77):

$$\omega \sim \frac{\epsilon}{a} v_A \tag{2.78}$$

Recalling the Fourier expansion of the perturbed quantities $f(x)e^{-i(\omega t - k_y y - k_z z)}$, means that using the operators $\nabla_{\parallel}$ and $\nabla_{\perp}$ on the perturbed potential ϕ_1 leads to terms proportional $k_{\parallel}$ and $k_{\perp}$ respectively. Thus, the vectorial operators will be of the following order:

$$\nabla_{\parallel} \sim \frac{\epsilon}{a} \tag{2.79}$$

$$\nabla_{\perp} \sim \nabla \sim \frac{1}{a} \tag{2.80}$$

The same happens for the time derivative:

$$\frac{\partial}{\partial t} \sim \omega \tag{2.81}$$

Recalling the low β approximation from Section 2.2:

$$\beta = \frac{8\pi P_0}{B_0^2} \sim \epsilon^2 \tag{2.82}$$

Now, particular scale ordering of the equilibrium magnetic field is imposed:

$$\nabla_\perp B_0 \sim \nabla B_0 \sim \frac{\epsilon}{a} B_0 \tag{2.83}$$

$$\nabla_\parallel B_0 \sim \frac{\epsilon^2}{a} B_0 \tag{2.84}$$

With the low β approximation, using (2.64) leads to the following estimate for $J_{0\perp}$

$$J_{0\perp} \sim \frac{c\nabla P_0}{B_0} \sim \frac{cB_0\beta}{a} \sim \frac{cB_0\epsilon^2}{a} \tag{2.85}$$

For the approximation of $J_{0\parallel}$, Ampere's law can be applied:

$$\nabla \times \mathbf{B_0} \sim \nabla B_0 \sim \frac{1}{c} J_0 \implies J_{0\parallel} \sim \frac{cB_0\epsilon}{a} \tag{2.86}$$

After all these simplifications, it is possible to see how each of the terms in equation (2.72) relate to each other. Taking into account that the nabla operator decomposes into two terms $\nabla = \nabla_\parallel + \nabla_\perp$, which in our case are "perpendicular" to each other. For example $\nabla_\parallel \cdot \nabla_\perp f = 0$, where f is a scalar function.

Starting with the ICU term: here the divergence acts as $\nabla_\perp$, as the other component yields zero:

$$\nabla \cdot \left[\frac{1}{v_a^2} \nabla_\perp \frac{\partial^2}{\partial t^2} \phi_1 \right] \sim \frac{k_\perp^2 \omega^2}{v_a^2} \sim \frac{\epsilon^2}{a^4} \tag{2.87}$$

However, for the FLB term, the divergence inside the brackets acts as $\nabla_\perp$ while the one outside the brackets acts as $\nabla_\parallel$:

$$\nabla \cdot \left[\mathbf{b} \nabla \cdot \nabla_\perp (\nabla_\parallel \phi_1) \right] \sim k_\perp^2 k_\parallel^2 \epsilon \sim \frac{\epsilon^2}{a^4} \tag{2.88}$$

In the kink term, we have the perpendicular component a vector, which points perpendicularly to the magnetic field, thus the divergence acts as $\nabla_\perp$:

$$\nabla \cdot \left[\frac{4\pi J_{0\|}}{cB_0} \nabla \times (\mathbf{b}\nabla_\| \phi_1) \right]_\perp \sim k_\perp k_\|^2 \frac{\epsilon}{a} \sim \frac{\epsilon^3}{a^4} \tag{2.89}$$

The final term has a mix of both parallel and perpendicular components of $\mathbf{J_0}$, which verify $J_{0\|} J_{0\perp}$. Therefore:

$$\nabla \cdot \left[\frac{16\pi^2}{c^2 B_0^2} (\mathbf{J_{\|0}} - \mathbf{J_{0\perp}})(\mathbf{J_{\|0}} - \mathbf{J_{0\perp}}) \cdot \nabla \phi_1 \right] \sim \frac{\epsilon^4}{a^4} \tag{2.90}$$

After evaluating the order of each term, leaving only the lowest order terms leads to:

$$\nabla \cdot \left[\frac{1}{v_a^2} \nabla_\perp \frac{\partial^2}{\partial t^2} \phi_1 \right] - \nabla \cdot \left[\mathbf{b} \nabla \cdot \nabla_\perp (\nabla_\| \phi_1) \right] = 0 \tag{2.91}$$

It is possible to perform a discrete Fourier expansion for ϕ_1 in the y and z coordinates, for the only dependence of the involved variables is on the x axis, as well as a forward Laplace time transform:

$$\phi_1 = \sum_{k_y, k_z = \infty}^{\infty} \hat{\phi}(x, k_y, k_z, p)\, e^{i(k_y y + k_z z)}; \quad k_y, k_z \in \mathbb{R} \tag{2.92}$$

$$\hat{\phi} \equiv \mathcal{L}(\phi_1) = \int_0^\infty dt\, \phi_1(x, t)\, e^{-pt}; \quad p \in \mathbb{C} \tag{2.93}$$

Due to the fact that there is no dependence in y nor z, there is a separate equation for each $\hat{\phi}$ term, in which the complex exponential is also omitted. Also, the following properties of the forward Laplace transform will be used:

$$\mathcal{L}\left(\frac{\partial \phi}{\partial t} \right) = p\hat{\phi} - \phi_0; \quad \phi_0 = \phi_1(\mathbf{r}, t = 0) \tag{2.94a}$$

$$\mathcal{L}\left(\frac{\partial^2 \phi}{\partial t^2} \right) = p^2 \hat{\phi} - p\phi_0 - \dot{\phi}_0; \quad \dot{\phi}_0 \equiv \left. \frac{\partial \phi_1}{\partial t} \right|_{\mathbf{r}, t=0} \tag{2.94b}$$

Substituting these identities into (2.91):

$$\nabla \cdot \left[\frac{1}{v_a^2}\nabla_\perp(p^2\hat{\phi} - p\phi_0 - \dot{\phi}_0)\right] - \nabla \cdot \left[\mathbf{b}\nabla \cdot \nabla_\perp(\nabla_\parallel\hat{\phi})\right] = 0 \tag{2.95}$$

Now define an equilibrium magnetic field which verifies all the previous assumptions:

$$\mathbf{B_0} = B_0\frac{\mathbf{e_z} + \alpha(x)\mathbf{e_y}}{\sqrt{1+\alpha^2(x)}} \tag{2.96}$$

in which $\alpha(x) \ll 1$. Thus:

$$\nabla_\parallel f \equiv i\,\frac{k_z + k_y\alpha}{\sqrt{1+\alpha^2}}\frac{\mathbf{e_z} + \alpha\mathbf{e_y}}{\sqrt{1+\alpha^2}}f \equiv ik_\parallel f\mathbf{b} \tag{2.97}$$

$$\nabla_\perp f \equiv \nabla f - \nabla_\parallel f = \frac{\partial f}{\partial x}\mathbf{e_x} - if\frac{(k_y - \alpha k_z)(\mathbf{e_y} - \alpha\mathbf{e_z})}{1+\alpha^2} \tag{2.98}$$

Applying these operators to the first term of (2.95):

$$\nabla \cdot \left[\frac{1}{v_a^2}\nabla_\perp(p^2\hat{\phi} - p\phi_0 - \dot{\phi}_0)\right] = \frac{1}{v_a^2}\left(\frac{\partial^2}{\partial x^2} - \hat{k}\right)(p^2\hat{\phi} - p\phi_0 - \dot{\phi}_0) \tag{2.99}$$

$$\hat{k} \equiv \frac{(k_y - \alpha k_z)^2}{1+\alpha^2}$$

In order to simplify the equation, the initial condition will be such that it verifies:

$$\left(\frac{\partial^2}{\partial x^2} - \hat{k}\right)(p\phi_0 + \dot{\phi}_0) = 0 \tag{2.100}$$

Now, the second term of (2.95) would be:

$$\begin{aligned}\nabla \cdot \left[\mathbf{b}\nabla \cdot \nabla_\perp(\nabla_\parallel\hat{\phi})\right] &= -k_\parallel(\frac{\partial^2}{\partial x^2} - \hat{k})(k_\parallel\hat{\phi}) = \\ \hat{k}k_\parallel^2\hat{\phi} - k_\parallel &\left(\frac{\partial^2 k_\parallel}{\partial x^2}\hat{\phi} + 2\frac{\partial k_\parallel}{\partial x}\frac{\partial\hat{\phi}}{\partial x} + k_\parallel\frac{\partial^2\hat{\phi}}{\partial x^2}\right)\end{aligned} \tag{2.101}$$

Putting (2.99) and (2.101) together and grouping the derivatives of the same order:

$$-\left(\frac{p^2}{v_A^2}+k_\parallel^2\right)\hat{k}\hat{\phi}+k_\parallel\frac{\partial^2 k_\parallel}{\partial x^2}\hat{\phi}+\left(\frac{p^2}{v_A^2}+k_\parallel^2\right)\frac{\partial^2\hat{\phi}}{\partial x^2}+2k_\parallel\frac{\partial k_\parallel}{\partial x}\frac{\partial\hat{\phi}}{\partial x}=0 \tag{2.102}$$

Assuming that $\alpha(x)$ is small as well as all its derivatives (very smooth change in direction), so that the second order derivative of $k_\parallel$ goes away. In addition, the last two terms can be joined together:

$$\frac{\partial}{\partial x}\left[\left(\frac{p^2}{v_A^2}+k_\parallel^2\right)\frac{\partial\hat{\phi}}{\partial x}\right]-\left(\frac{p^2}{v_A^2}+k_\parallel^2\right)\hat{k}\hat{\phi}=0 \tag{2.103}$$

The derivative of $v_A(x)^2$ can also be considered negligible compared to the derivative of $\hat{\phi}$ Thus, multiplying by v_A^2 and simplifiying some terms leads to a very simple equation:

$$\partial_x\left(P\,\partial_x\hat{\phi}\right)-Q\hat{\phi}=0 \tag{2.104}$$

$$P\equiv p^2+k_\parallel^2 v_A^2;\quad Q\equiv\hat{k}\,P$$

This equation has a singular point at $p^2=-k_\parallel^2 v_A^2$ where $P=0$. In the next section, a Fourier transform for the remaining spatial variable x will be performed to get rid of it.

Chapter 3

Numerical implementation

3.1 Obtaining a low order approximation

While the previous equation has a simple expression, if all the functions are written explicitly in terms of x, the whole equation can get quite complex. Thus, more simplifications must be made:

- k_0 is a function of x due to the $\alpha(x)$ term. We assume for now that $\alpha(x) = 0$ so $k_0(x) = k_y^2$, simplifying the calculations.
- The function is even around $x = 0$. This consideration doesn't impact the physics of the problem and it creates additional restriction which allows obtaining useful eigenvalues out of our eigenvalue equation.
- Then the Alfvén frequency is defined as $k_\parallel^2 v_A^2 \equiv \omega_A^2(x)$, which is assumed analytical and can be expanded in a Taylor series up to second order around $x = 0$.

Performing the aforementioned expansion:

$$\omega_A^2(x) = \omega_A^2(0) + \omega_A^2{}'(0)x + \frac{1}{2}\omega_A^2{}''(0)x^2 \tag{3.1}$$

In order to simplify the notation, define $\lambda = \omega_A^2(0)$ and λ_n as it's successive derivatives:

$$\omega_A^2(x) = \tilde{\lambda} + \lambda_1 x + \frac{1}{2}\lambda_2 x^2; \quad \tilde{\lambda} = p^2 + \lambda \tag{3.2}$$

Substituting in the previous equation:

$$(\lambda_1 + \lambda_2 x)\,\partial_x\hat{\phi} + (\tilde{\lambda} + \lambda_1 x + \frac{1}{2}\lambda_2 x^2)\,\partial_x^2\hat{\phi} - k_y^2\,(\tilde{\lambda} + \lambda_1 x + \frac{1}{2}\lambda_2 x^2)\hat{\phi} = 0 \tag{3.3}$$

Now, assuming that $\hat{\phi}$ is well behaved, a Fourier transform can be performed:

$$\Phi(k) \equiv \int_{-\infty}^{\infty} dx\,\hat{\phi}(x)\,e^{-ikx}; \quad k \in \mathbb{R} \tag{3.4}$$

k must be real by definition of the Fourier transform, later on, it will have to be to extended analytically to the complex plane due to convergence issues.

Let $f(x)$ be a well behaved function, $f^n(x)$ its n-th derivative, and $F[\_]$ the Fourier transform operator. Some useful properties of the Fourier transform are:

$$F[f^n(x)] = (ik)^n F[f(x)] \tag{3.5a}$$

$$F[x^n\,f(x)] = (i\partial_k)^n F[f(x)] \tag{3.5b}$$

$$F[x^n f^m(x)] = (i\partial_k)^m\{(ik)^n\,F[f(x)]\} \tag{3.5c}$$

These properties can be deduced by integration by parts assuming $f(x)$ vanishes at infinity.

Now these properties can be substituted in the equation, in which $\Phi \equiv F(\hat{\phi})$ and $\Phi^n = (\partial_k)^n F(\hat{\phi})$:

$$ik\lambda_1\Phi - \lambda_2(k\Phi' + \Phi) - k^2\tilde{\lambda}\,\Phi - i\lambda_1(k^2\Phi' + 2k\Phi) + \frac{1}{2}\lambda_2(k^2\Phi'' + 4k\Phi' + 2\Phi) - \tag{3.6}$$

$$-k_y^2(-\frac{1}{2}\lambda_2\Phi'' + i\lambda_1\Phi' + \tilde{\lambda}\Phi) = 0$$

Regrouping all the derivatives of Φ^n together:

$$-[(k^2+k_y^2)\tilde{\lambda}+ik\lambda_1]\Phi+[k\lambda_2-i(k^2+k_y^2)\lambda_1]\Phi'+\frac{1}{2}(k^2+k_y^2)\lambda_2\Phi''=0 \tag{3.7}$$

Due to the assumption ω_A^2 is even around 0, so is $\lambda_1=0$:

$$-(k^2+k_y^2)\tilde{\lambda}\Phi+k\lambda_2\Phi'+\frac{1}{2}(k^2+k_y^2)\lambda_2\Phi''=0 \tag{3.8}$$

Simplifying this equation even more by taking common factor k_y^2 and $\frac{1}{2}\lambda_2$ leads to:

$$\left\{\partial_\theta\left[(1+\theta^2)\partial_\theta\right]+\Omega^2(1+\theta^2)\right\}\Phi=0 \tag{3.9}$$

where $\theta\equiv\frac{k}{k_y}$ and $\Omega^2\equiv-\frac{2\tilde{\lambda}}{\lambda_2}$.

Thus, a second order equation in k-space is obtained, with a arbitrary complex parameter Ω^2. Now appropriate boundary conditions have to be imposed in order to solve it.

3.2 Boundary conditions

By writing the equation explicitly:

$$(1+\theta^2)\,\partial_\theta^2\Phi+2\theta\,\partial_\theta\Phi+\Omega^2(1+\theta^2)\,\Phi=0 \tag{3.10}$$

Taking the limit $\theta\gg1$ leads to the following simplified form:

$$\theta^2\,\partial_\theta^2\Phi+2\theta\,\partial_\theta\Phi+\Omega^2\theta^2\,\Phi=0 \tag{3.11}$$

Now a new variable $\Psi=\theta\Phi$ is defined, and after some algebra:

$$\partial_\theta^2\Psi=-\Omega^2\Psi \tag{3.12}$$

The two possible solutions for this trivial equations are $\Phi_\pm = C\, e^{\pm i\Omega\theta}$, with C being a complex scalar. Undoing the variable change, the asymptotic boundary condition appears:

$$\Phi_\pm = C\,\frac{e^{\pm i\Omega\theta}}{\theta} \tag{3.13}$$

The question now is which one of the solutions to select. For that goal the causality condition, which will now be explained, is used. In order to undo the Fourier transform in x and the Laplace transform in t, the following integral is done:

$$\hat{\phi}(x,t) = K \lim_{T\to\infty} \int_{\eta-iT}^{\eta+iT} d\Omega \int_{-\infty}^{\infty} d\theta\, \Phi(\theta,\Omega) e^{i(\theta\, x - \Omega t)} \tag{3.14}$$

Where K is a constant complex constant due to the variable change from k to θ and from p to Ω, while η is a real number which must be chosen in a way that the function converges through all the complex integration path.

This way the instability can be regarded as a group of plane waves propagating in a certain direction. Assuming it is possible to perform the following Taylor expansion up to first order:

$$\Omega(\theta) = \Omega_0 + \left.\frac{d\Omega}{d\theta}\right|_{\theta=0} (\theta - \theta_0) \tag{3.15}$$

What is left is the the classical expression for a wave group modulated by a perfect monochromatic wave:

$$\hat{\phi}(x,t) = K e^{i(\theta_0\, x - \Omega_0 t)} \int_\Omega d\Omega \int_\theta d\theta\, \Phi(\theta,\Omega) e^{i(\theta-\theta_0)(x - \frac{d\Omega}{d\theta} t)} \tag{3.16}$$

The exponent in the integral will oscillate periodically in the domain of θ, as the $(x - \frac{d\Omega}{d\theta} t)$ term is constant when integrating for all θ. Thus, the final result will be really small unless the exponent is approximately zero. Putting the time origin at $t = 0$, and due to symmetry, the perturbation can be considered to propagate along the right half of the plane. Thus:

$$\frac{x}{t} \approx \left.\frac{d\Omega}{d\theta}\right|_{\theta=0} > 0 \tag{3.17}$$

However, instead of Ω, it is possible to expand θ:

$$\theta(\Omega) = \theta_0 + \left.\frac{d\theta}{d\Omega}\right|_{\theta=0} (\Omega - \Omega_0) \tag{3.18}$$

This way, the integral becomes:

$$\hat{\phi}(x,t) = Ke^{i(\theta_0\, x - \Omega_0 t)} \int_\Omega d\Omega \int_\theta d\theta\, \Phi(\theta, p) e^{i(\Omega-\Omega_0)(\frac{d\theta}{d\Omega}x - t)} \tag{3.19}$$

However, Ω is complex, so the exponential will not simply oscillate, but it can increase or decrease in modulus depending on the sign of the real exponential. Thus, the integral will have a divergent or convergent factor $e^{-\mathrm{Im}(\Omega)\frac{d\theta}{d\Omega}x}$.

Assuming $\Omega(\theta)$ and $\theta(\Omega)$ are continuous and differentiable functions, the *Inverse Function Theorem* along with (3.17), can be used:

$$\frac{d\theta}{d\Omega} = \left(\frac{d\Omega}{d\theta}\right)^{-1} > 0, \text{ for } x > 0 \tag{3.20}$$

There are certain phenomena in plasma which are unstable: they start in a certain point (let's assume $x = 0$ in this case), expand from that point and then dissipate when they go further in time and space from the starting point. Therefore, the plane waves that compose it must be spatially decreasing in $t = 0$ and $\mathrm{Im}(\Omega) > 0$, otherwise the real exponential factor leads to a phenomenon that propagates at infinite speed through all space when it appears at a certain point: a violation of causality.

However, the integral is performed for all values of θ positive and negative, so unless some condition on the plane wave amplitude $\Phi(\theta, \Phi)$ is put, part of the integral would diverge for large values of θ. Thus, it is imposed that:

$$\Phi(\theta, \Phi) \approx C\, \frac{e^{\mathrm{sign}(\theta) i\Omega\theta}}{\theta} \tag{3.21}$$

Nonetheless, the function must be even in Fourier space, as the original function is even and real in x space as well. Therefore, another boundary condition must be $\partial_\theta \Phi(\theta = 0) = 0$.

3.3 Solver code

3.3.1 Summary of the problem

The problem that needs to be solved is the following: the equation (3.9) is a second order differential ordinary equation for $\Phi(\theta)$ in the complex plane where the independent variable θ has been assumed to be real, spanning the whole set of $\mathbb{R}$. However, due to the parameter Ω, the equation becomes part of an eigenvalue problem, where the solution verifies the following boundary conditions:

1. The solution must be even, both in the Fourier Space and Real Space. This can be accomplished by imposing:
$$\partial_\theta \Phi(\theta = 0) = 0$$
2. For large $\theta \in \mathbb{R}$, the equation must tend asymptotically to:
$$\Phi^{as}(\theta) \equiv C\,\frac{e^{\text{sign}(\theta)i\Omega\theta}}{\theta}$$

Numerically, there are no strict infinities or strict zeros, so the boundary conditions must be modified in the following way:

1. The even parity condition can be approximated as $\Phi'(\theta = 0) = \epsilon \ll 1$. However, a problem arises when the value isn't exactly zero: the value $\Phi(\theta = 0)$ tends to matter. For example, the numerical derivative of $f(x) = A\cos(x)$ at $x \approx 0$ is $f'(x \approx 0) = A\epsilon$: the amplitude of the function is affecting the value of ϵ. Those, a normalized derivative is defined:
$$F \equiv \frac{\partial_k \Phi(\theta = 0)}{|\Phi(\theta = 0)|} = \epsilon \ll 1 \tag{3.22}$$
2. A large value of $|\theta| \gg 1$ is used instead of infinity, in this case $\theta = 100$. Thus, the boundary condition is transformed into a initial condition for the value of the solution

and its derivative at $\theta = 100$:

$$\Phi(\theta = 100) = \Phi^{as}(\theta = 100) = \left.\frac{e^{i\Omega\theta}}{\theta}\right|_{\theta = 100} \tag{3.23a}$$

$$\partial_\theta \Phi(\theta = 100) = \partial_\theta \Phi^{as}(\theta = 100) = (i\Omega\theta + 1)\left.\frac{e^{-i\Omega\theta}}{\theta^2}\right|_{\theta = 100} \tag{3.23b}$$

There are three three different boundary conditions for a second order ODE. That is because Ω is undetermined, so while the ODE is solved with the asymptotic initial conditions (3.23a) and (3.23b), we are going to determine Ω by using (3.22). Thus, the proposed problem becomes eigenvalue problem.

Another restriction to the solution is that stable plasma modes are considered: Im$[\Omega] < 0$. This is due to the fact that in the current model, without any kinetic effects, there are no dissipation mechanisms for the Shear Alfvén Wave, thus it won't dissipate for large values of $|x|$. This means that the imposed boundary conditions in θ space are divergent.

This is problematic, as computing a solution for divergent function gives rise to numerical complications. However, by extending θ to the complex plane and solving the problem along a straight line in the complex plane, passing through the origin, which due to the Stokes phenomenon, will give us a convergent solution.

The Stokes phenomenon is the mathematical property of complex functions by which their asymptotic behaviour changes for different regions of the complex space. For example, $f(z) = e^z$, diverges in modulus for Re$[z] \to +\infty$, but tends to zero for Re$[z] \to -\infty$. Thus, the complex plane is separated by the imaginary axis, where the function is purely oscillatory, into two halves, in which the asymptotic behaviour of this function is completely different. The line which separates two different regions of different convergence is called the Anti-Stokes line.

In the relevant case, the function tends asymptotically to $e^{i\Omega\theta}$ in the real axis. However by assuming θ is a complex number, for Re$[\theta] \gg 1$, $\Phi(\theta) \to e^{i\Omega\theta}$. In this case, the Anti-Stokes line, assuming that we are far away from the origin, would be given by a complex line that is parallel to Re$[i\Omega\theta] = 0$ line. Thus, if we assume θ goes along a straight line in complex space, it could eventually cross the Anti-Stokes line and boundary conditions would become convergent.

3.3.2 Differential equation solver

The ODE can be solved analytically, but that is not trivial, nor the goal of this thesis. Thus, an integrated MatLab solver, the ode113 solver will be used, which is based upon the variable-step, variable-order Adams-Bashforth-Moulton PECE solver of order 1 to 13.

This method is based upon the linear multistep method family of solvers. A simple example would be a first order initial condition problem in real space, which can be written in the following way:

$$\frac{dy(x)}{dx} = f(x,y); \quad y(x_0) = y_0, \quad x, y \in \mathbb{R} \tag{3.24}$$

Essentially, the mechanism of these solvers consists of the following: from the initial condition, it is possible to compute $f(x_o, y_o)$ and to obtain the derivative, which by taking a discrete step h in the real space allows us to compute $y(x+h)$. Now, the algorithm has two points from which it can compute the position of a third point after taking another step h.

Thus, for any point, by using the information from a set of n previously computed points, n being the order of the solver, it is possible to reiteratively construct the set of points of the solution. The most basic example is the first order Euler's method, where the equation that determines $y(x+h)$ is the following:

$$y(x+h) = y(x) + hf(x,y) \tag{3.25}$$

Where h is a constant through all the domain. More sophisticated methods approximate the next step by a variable linear combination of $f(x_i, y_i)$ at previous points, which is decided by an error evaluation algorithm, whose tolarence can be adjusted. Moreover, the points themselves and the number of them are variable as well, hence the "variable-step, variable-order" title.

However, the ode113 solver works on real, first order ODE, while we have a second order, ODE where both variables are complex. These restrictions can be circumvented easily. First, regarding the order of the ODE, the equation is linear, thus it can be converted into a system of equations, which the algorithm can solve as well. A similar decomposition can be performed for the complex variable Φ, dividing into a real and imaginary term, and while θ is complex as well, a change of variable to a real variable s is performed, such as $\theta = Cs; \quad C \in \mathbb{C}$.

Thus, the equation can be expressed as:

$$\partial_\theta \mathbf{X}(Cs) = \mathbf{M}(Cs)\,\mathbf{X}(Cs); \qquad \mathbf{x}(\theta) = \begin{bmatrix} \text{Re}[\Phi] \\ \text{Im}[\Phi] \\ \text{Re}[\partial_k \Phi] \\ \text{Im}[\partial_k \Phi] \end{bmatrix} \tag{3.26}$$

where $\mathbf{M}$ is a matrix only dependent on θ, as the ODE is linear. More explicitly:

$$f(\theta, \mathbf{X}) = -C^2 \left[\frac{2s}{1 + C^2 s^2}(X_3 + iX_4) + \Omega^2 (X_1 + iX_2)\right]; \quad \begin{cases} \partial X_1 = X_3 \\ \partial X_2 = X_4 \\ \partial X_3 = \text{Re}[(Cs, \mathbf{X})] \\ \partial X_4 = \text{Im}[(Cs, \mathbf{X})] \end{cases} \tag{3.27}$$

Finally, as a relevant remark, this family of algorithms has been used of the more traditional ode45 and ode23 Runge-Kutta algorithms as the ode113 solver has proven to be much faster for this particular problem while returning almost identical solutions for a given error tolerance.

3.3.3 Solving the eigenvalue problem

Given the ODE for with the (3.23a) and (3.23b) initial conditions for a certain value of Ω, the solution must also verify the (3.22) boundary condition, which generally won't be true. Thus, an algorithm is needed in order to find a Ω which returns the closest to zero value of F. This is an analogous to finding the roots of a non-elemental function, for which the Newton-Raphson algorithm can be used. This algorithm, originally devised to find roots of real functions, can also be used to compute the roots of a function in the complex plane. The procedure to follow is the following:

Let $f(z) \in \mathbb{C}$ be a continuous and differentiable function in the complex plane, where $z \in \mathbb{C}$ is also complex. The procedure to find the complex roots of $f(z)$ is as such:

1. Start by an initial guess $z_0 \in \mathbb{C}$.

2. Evaluate $f(z_0)$ and it's derivative at z_0, $f'(z_0)$,

3. Compute the point z_1 given by

$$z_1 = z_0 - \frac{f(z_0)}{f'(z_0)} \tag{3.28}$$

4. Repeat the last steps until $f(z_i)$ is small enough to consider it a root

However, when choosing the initial guess, some care must be taken, specially in the complex plane, as generally the convergence to any root, or to the root we are searching for, isn't guaranteed for an arbitrary point. Thus, it is helpful to make an initial evaluation of $f(z_0)$ for a mesh of points in the complex plane in order to see where the roots may lie and to make an appropriate guess.

In this case, the roots are of the function $F(\Omega) = \left|\frac{\partial_k \Phi(\theta=0;\Omega)}{\Phi(\theta=0;\Omega)}\right|$, which is assumed to behave as a continuous and differentiable function at least locally in the surroundings of a root. The same algorithm can be applied, but with a couple caveats: to evaluate $F(\Omega)$, the associated initial value problem is solved, while $F'(\Omega)$ can be computed by solving the same problem at $\Omega + \Delta\Omega$, where $\Delta\Omega \in \mathbb{C}$ has a really small modulus compared to Ω. Thus, we have:

$$F'(\Omega) = \frac{F(\Omega + \Delta\Omega) - F(\Omega)}{\Delta\Omega} \tag{3.29}$$

Note that it doesn't matter the direction in the complex plane for $\Delta\Omega$, as if $F(\Omega)$ is differentiable, at least locally, any direction can be chosen.

To summarize the numerical procedure used for obtaining the eigenvalue, the a flowchart is shown in the following page:

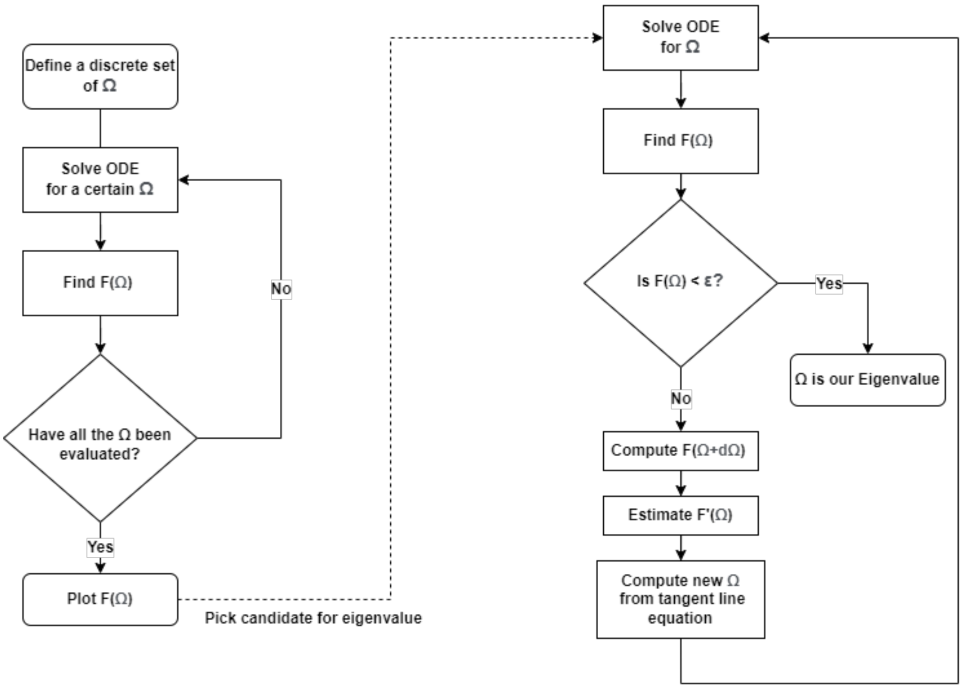

Figure 3.1: Flowchart with the relevant steps in our code. From: *https://app.diagrams.net/*

Chapter 4

Results

As per the shown flowchart (Figure 3.1), $F(\Omega)$ is evaluated for different values of Ω, in order to find a Ω_0 such as $F(\Omega_0)$ is close to 0. More concretely, $F(\Omega)$ is computed along a 50×50 grid in the following complex domain:

$$\text{Re}[\Omega] \in [0, 1], \quad \text{Im}[\Omega] \in [0, -1] \tag{4.1}$$

Regarding the integration variable $\theta = Cs$, the parameters have the following values:

$$C = 1 + 1i, \quad s \in [-100, 100] \tag{4.2}$$

From this sweep of the different values of Ω, the contour plot seen in the following page is obtained (Figure 4.1). Thus, it can be deduce that there is a single zero near $\Omega_0 = 0.68 - 0.62i$ point, where $F(\Omega)$ seems locally differentiable.

Using that point as the initial guess for Newton's method, the following eigenvalue is obtained, rounded up to four decimals, along with the eigenfunction represented in the following page (Figure):

$$\Omega = 0.6814 - 0.6178i \tag{4.3}$$

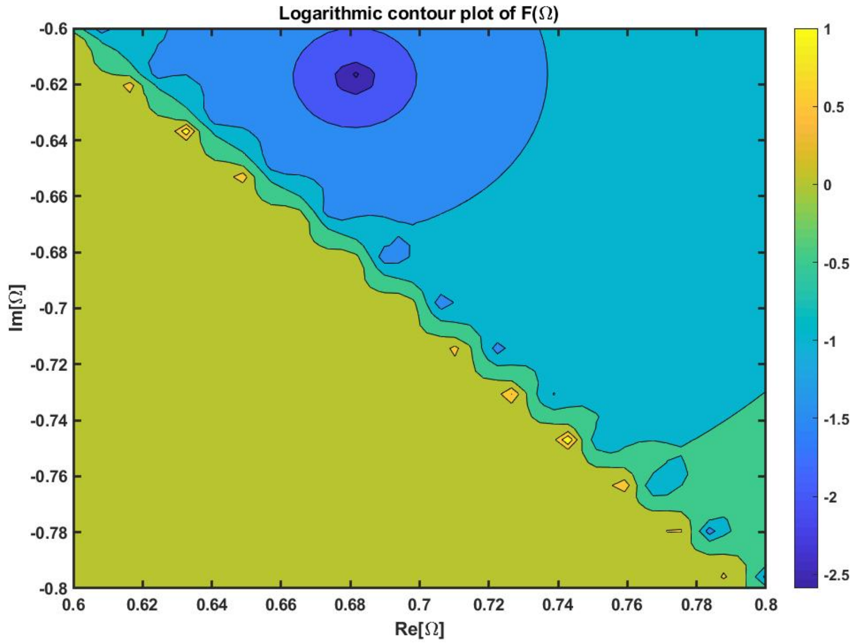

Figure 4.1: Relevant section of the complex contour plot of $\ln F(\Omega)$

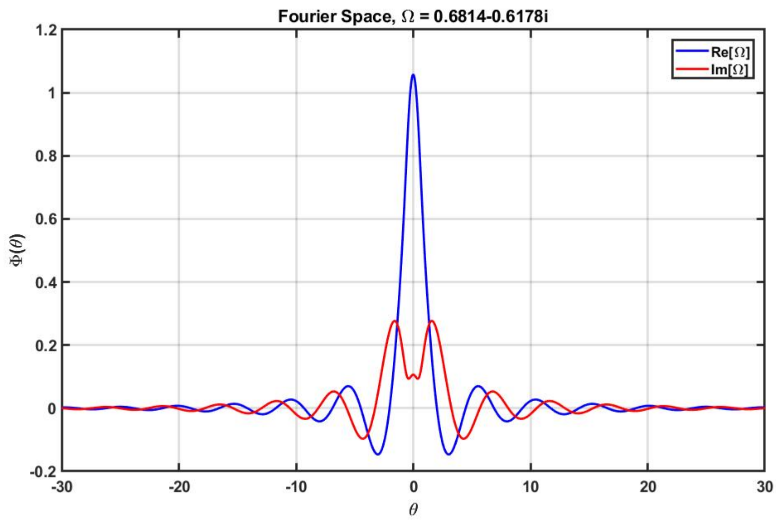

Figure 4.2: Representation of the even solution in Fourier. Notice that the horizontal axis refers to the real component of θ.

Indeed, it is easily seen that the eigenfunction is even in Fourier space. However, recalling that the Fourier Transform and Inverse Fourier Transform are only defined for real values of θ, the eigenfunction with complex θ cannot be used directly for obtaining the solution in real space. Thus, the equation will be solved again for $\theta \in \mathcal{R}$ while using the complex eigenvalue Ω computed before. The modulus of the resulting solution is shown in the following figure:

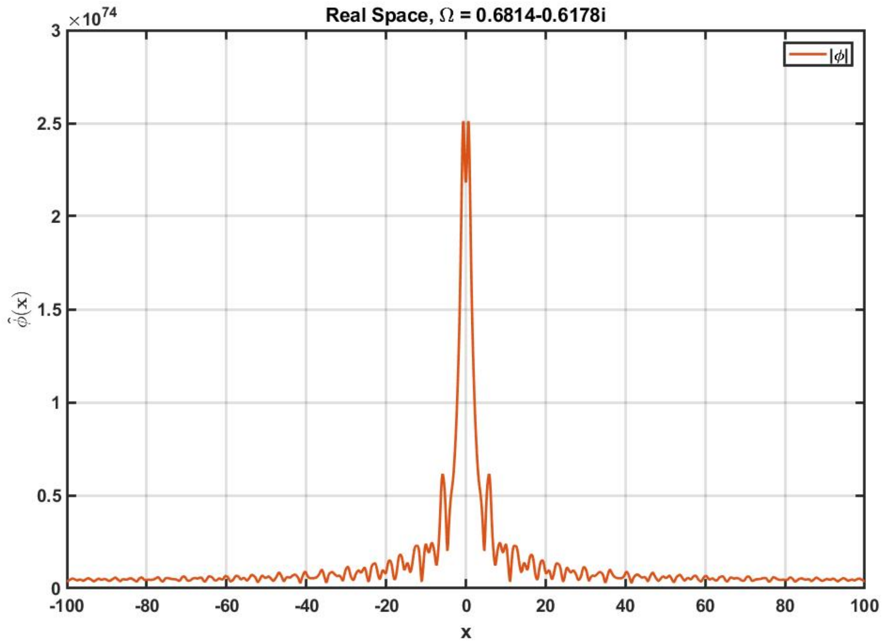

Figure 4.3: Representation of modulus of the eigenfunction in real space when assuming $\theta \in \mathcal{R}$.

In the first place, the function in real space is even, which seems to indicate that the obtained eigenvalue is also valid for the real space solution. However, solving the ODE for other arbitrary Ω also yields even functions in real space, and while not present in the figure, the solution is actually a complex function: $\hat{\phi} \in \mathcal{C}$.

These results are due to the fact that while the ODE has some intrinsic parity in real space, the parity condition $\partial_\theta \Phi(\theta = 0) = 0$ in $\theta \in \mathcal{R}$ Fourier space for our solution is violated. For a function in Fourier space to be even and real, $\overline{\Phi}(\theta) = \Phi(-\theta) = \Phi(\theta)$, the function in real space must also be even and real $\hat{\phi}(x) = \hat{\phi}(-x) \in \mathcal{R}$, which isn't the case for this eigenfunction, where $\frac{\partial_\theta \Phi(\theta=0)}{\Phi(\theta=0)} \approx -3$.

The proof of the the even-parity property is the following:

$$\hat{\phi}(x) = \hat{\phi}(-x) \in \mathbb{R} \implies \Phi(x) = \int_{-\infty}^{\infty} d\theta \, \mathrm{Re}[\phi(x)e^{i\theta x}] \stackrel{!}{=} \int_{0}^{\infty} d\theta \, \phi(\theta) \; \cos(\theta x) \in \mathcal{R} \qquad (4.4)$$

Still, by taking the modulus of the function, the solution correctly describes the behaviour of the Shear Alfvén Wave in the magnetohydrodinamic model: two peaks for the amplitude of the perturbed potential ϕ_1 can be seen at $x \approx \pm 0.68$, which correspond to the the singular points where $P = 0$ in (2.104), as there is a considerable energy accumulation compared to the rest of the plasma. This can be explained by the fact that the SAW has a local complex resonance frequency given by $p^2 = -k_{\parallel}^2(x)v_A^2(x)$ at a certain point in the plasma.

In equation (3.3), assuming $\lambda_1 = 0$ as the function is even, there is a singularity in the real part of the ODE if $x^2 = -\frac{2\tilde{\lambda}}{\lambda_2} \equiv \Omega^2 \implies x = \pm\mathrm{Re}[\Omega]$, which corresponds with the maxima of the numerical results, but instead of getting a singular point, due to a the discrete nature of the numerical method, a sharp maximum is obtained.

The harmonic oscillator analogue serves again as a good analogy: if we have a undampened harmonic oscillator of a certain frequency, and excite it with a force at the same frequency and phase, it will start absorbing energy indefinitely. This is a simple explanation of what is happening at the singularity point. In this case, the driving force analogue responsible for the energy accumulation are the fast compressional waves. Even-though the incompressible plasma assumption has been made, there is still some coupling between the SAW waves and compressional waves, which have velocities perpendicular to one another if $|k_{\parallel}| \ll |k_{\perp}|$. Thus, in the x axis, in which the SAW propagates, there is a surface which can accept an unlimited amount of energy, which is given by a perturbation coming from the y and z axes. If Figure 4.2 can be generalized as the $y = 0$ plane in a two-dimensional plot, the compressional waves would propagate along the direction perpendicular to the paper.

Chapter 5

Summary and discussion

Starting from the MHD equations, the Alfvén-Interchange equation was successfully obtained, which allows the study the different perturbations in a low-pressure plasma system. After performing some geometrical and physical simplifications in order to isolate the physical phenomenon of interest, it was shown that behaviour is explained by the magnetohydrodinamic model: the Shear Alfvén Wave, due to its spectrum, which depends on the x-spatial coordinate, has a resonance point where we can observe an energy build up.

The same numerical simulation was performed in reference [1], where the equation was also resolved for $\theta \in \mathcal{C}$ on the same straight line, and while the perturbation in the the real plane isn't shown, the eigenvalue obtained for the same boundary conditions was, up to three significant digits, $\Omega_{bib} = 0.681 - 0.616i$. Compared to the obtained value $\Omega = 0.6814 - 0.6178$, there is a very small deviation in its imaginary value of less than 1%. This deviation is expected, due to the fact that our boundary conditions at the origin and infinity are approximated, and that the numerical algorithm has some error due to its discrete nature as well. Thus, there is a certain arbitrariety in the choice of the approximated boundary conditions and algorithm which can explain this discrepancy.

Still, the obtained eigenvalue has been found to give rise to a overall complex solution when assuming $\theta \in R$, but it doesn't pose a problem for the physical interpretation of the phenomenon.

However, from references [3] and [2] it can be seen that the empirical observations don't correspond to the numerical predictions presented here. This is due to the fact that we have assume a plasma with no dissipation processes, which allows an infinite energy build up, as in a non-dampened harmonic oscillator which is driven by an external force, which with time oscillates more and more, reaching infinite amplitude.

In reality, as it can be read in [3], there are dampening mechanisms, which can be explained by the interaction of the collective MHD perturbation with the plasma particles, such as the aforementioned Landau Damping, by which the Shear Alfvén Wave to accelerates particles that move slightly slower than the wave. This phenomena creates an energy transfer from the wave to the particles, which carry away the energy built up at the singularity, making it finite. The incorporation of the terms responsible for these dissipation mechanisms is beyond the scope of this thesis, as it requires combining this fluid model with a discrete particle model. Nonetheless, the formulation of the MHD equations in terms of the perturbed potential leads nicely to these corrections, which could give place to an expansion of this thesis later on.

Chapter 6

Appendix

6.1 Vector algebra and calculus identities

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be differentiable vector fields and f a differentiable scalar field.

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{A1}$$

$$\mathbf{A} \times \mathbf{B} \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{A2}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \tag{A3}$$

$$\nabla \times (f\mathbf{v}) = \nabla f \times \mathbf{v} + f\nabla \times \mathbf{v} \tag{A4}$$

$$\nabla \cdot (\mathbf{v} \times \mathbf{u}) = \mathbf{u} \cdot (\nabla \times \mathbf{v}) - \mathbf{v} \cdot (\nabla \times \mathbf{u}) \tag{A5}$$

$$\nabla_{\perp} f = -\mathbf{b} \times \mathbf{b} \times \nabla f \tag{A6}$$

Bibliography

[1] C.Z Cheng, Liu Chen, and M.S. Chance. “High-n Ideal and Resistive Shear Alfvén Waves in Tokamaks”. In: *Annals of Physics* (1985).

[2] L. Chen and F. Zonca. “Theory of Shear Alfven Waves in Toroidal Plasmas”. In: *Physica Scripta* T60 (1995).

[3] G. Vlad, L. Chen, and F. Zonca. “Dynamics of Alfvén Waves in Tokkamaks”. In: *IL NUOVO CIMENTO* (2003).

[4] J.P. Goedbloed and S. Poedts. *Principles of Magnetohydrodinamics: With Applications to Laboratory and Astrophysical Plasmas*. Cambridge University Press, 2004. Chap. 10.

[5] P.M. Bellan. *Fundamentals of Plamsa Physics*. Cambridge University Press, 2006.

[6] R.B. White. *The Theory of Toroidally Confined Plasmas*. 2nd ed. Imperial College Press, 2006. Chap. 9.

[7] J.P. Freidberg. *Ideal MHD*. Cambridge University Press, 2014.

[8] L. Chen and F. Zonca. “Physics of Alfven waves and energetic particles in burning plasmas”. In: *Reviews of Modern Physics* 88 (2016).

[9] L. Chen and H. Chen. “Gyrokinetic theory of low-frequency electromagnetic waves in finite-b anisotropic plasmas”. In: *Phys. Plasmas* 28 (2021).

[10] Haotian Chen. “Notes on SAW in ideal MHD”. In: (2022).

[11] P. Ricci. *Lecture Notes in Plasma Physics and Application to Fusion Energy, Astrophysics and Industry*.