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Nonlocal limits in the study of linear elliptic systems arising in periodic homogenization ☆

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Abstract

In the present paper, we obtain the two-scale limit system of a sequence of linear elliptic periodic problems with varying coefficients. We show that this system has not the same structure than the classical one, obtained when the coefficients are fixed. This is due to the apparition of nonlocal effects. Our results give an example showing that the homogenization of elliptic problems with varying coefficients, depending on one parameter, gives in general a nonlocal limit problem.

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1. Introduction

It is well known (see [10,8]) that given a bounded open subset $\Omega \subset \mathbb{R}^N$ and a sequence of matrices $A_\varepsilon \in L^\infty(\Omega)^{N\times N}$, which are uniformly elliptic and bounded, there exist a matrix A (homogenized matrix) in the same conditions that A_ε, and a subsequence of ε, still denoted by ε, such that for every f_ε which converges strongly in $H^{-1}(\Omega)$ to a distribution f, and every u_ε which converges in $H_0^1(\Omega)$ and satisfies

$$-\mathrm{div}\, A_\varepsilon \nabla u_\varepsilon = f_\varepsilon \quad \text{in } \Omega, \tag{1}$$

the limit of u_ε is a solution of the analogue equation, where A_ε is substituted by A, and f_ε by f. The aim of the present paper is to show that the analogue of this result is not true when the matrices A_ε measurably depend on a parameter, i.e., when at the place of (1), we have

$$-\mathrm{div}_y\, A_\varepsilon(x,y)\nabla_y u_\varepsilon(x,y) = f_\varepsilon(x,y) \quad \text{in } \Omega \text{ a.e. } x \in \Theta, \tag{2}$$

where (Θ, σ, μ) is a given space of measure. Indeed, for this type of problems the limit operator is in general nonlocal in y. This is due to the fact that the set of solutions of (2) is not compact in general in $L^2_\mu(\Theta; L^2(\Omega))$.

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The homogenization problem (2) appears, for example, in the study of the asymptotic behavior of partial differential problems with varying coefficients, which depend on an aleatory parameter $x \in \Theta$ (stochastic homogenization problems). Another interesting situation is the study of some systems which usually appear in periodic homogenization. In this way, let us consider in the present paper the homogenization problem

$$-\mathrm{div}\left(A_\varepsilon\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon - G\left(x, \frac{x}{\varepsilon}\right)\right) = 0 \quad \text{in } \Omega, \quad u_\varepsilon = 0 \text{ on } \partial\Omega, \tag{3}$$

where as above Ω is a bounded open subset of $\mathbb{R}^N$, A_ε and G are, respectively, continuous matrices and vectorial functions in $\Omega \times \mathbb{R}^N$, which are periodic in the second variable, of period Y^N, $Y = (-\frac{1}{2}, \frac{1}{2})$. The matrices A_ε are uniformly elliptic and bounded. We remark that (3) is not a particular case of (1) because $-\mathrm{div}\, G(x, \frac{x}{\varepsilon})$ does not converges strongly, in general, in $H^{-1}(\Omega)$ (it converges in $W^{-1,\infty}(\Omega)^N$ weakly-$*$). When A_ε is a constant matrix A, the two-scale convergence theory of Nguetseng and Allaire (see e.g. [1,9]) shows that denoting by $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega; H^1(Y^N)/\mathbb{R})$ the solutions of

$$-\mathrm{div}_x \int_{Y^N} (A(x, y)(\nabla_x u_0(x) + \nabla_y u_1(x, y)) - G(x, y))\, \mathrm{d}y = 0 \quad \text{in } \Omega, \tag{4}$$

$$-\mathrm{div}_y (A(x, y)(\nabla_x u_0(x) + \nabla_y u_1(x, y)) - G(x, y)) = 0 \quad \text{in } \mathbb{R}^N \text{ a.e. } x \in \Omega, \tag{5}$$

then the solutions of (3) converge weakly in $H_0^1(\Omega)$ to u_0, while ∇u_ε two-scale converges to $\nabla_x u_0 + \nabla_y u_1$ (the idea is to approximate $u_\varepsilon(x)$ of the type $u_0(x) + \varepsilon u_1(x, x/\varepsilon)$, which is better than to approximate u_ε just by u). When $\mathrm{div}_y\, G(x, y) = 0$, Eq. (5) permits to calculate u_1 from u_0 and then, substituting in (4) we get the homogenized problem (see e.g. [1,9]) for u_0. When this condition is not satisfied we can still calculate u_1 from u_0 and then to obtain an equation which only contains u_0, but their coefficients depend on G and thus, it is better to remain with the system (4), (5). In the present paper, given u_ε the solution of (3), let us search for the system satisfied by the limit of u_ε in $H_0^1(\Omega)$ and the two-scale limit of ∇u_ε. Clearly, this must contain in particular the limit of a system like (4), (5), where the second equation has a structure similar to (2). As we announced above, we will obtain a nonlocal limit system (see Theorem 4 and Proposition 5). In particular there does not exist in general a matrix A such that the corresponding system is (4), (5), as it happens when A_ε is constant. Related results have been obtained in [4] for a different problem, the asymptotic behavior of thin structures. Other results, about the apparition of nonlocal terms in the homogenization of linear elliptic problems, can be found by using the theory of Dirichlet forms (see [7]), where it is assumed strong convergence in L^2, which as we said above does not hold in our context.

2. Homogenization of periodic problems

We take $Y = (-\frac{1}{2}, \frac{1}{2})$, and $\Omega \subset \mathbb{R}^N$ a bounded open subset of $\mathbb{R}^N$.

$\mathscr{L}(L^2(Y^N)^N, L^2(Y^N)^N)$ is the space of lineal continuous functions from $L^2(Y^N)^N$ into itself.

As it is usual, we use the index $\sharp$ to mean periodicity. For example, $L^p_\sharp(Y^N)$ is the space of functions of $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ which are periodic, of period Y^N.

For a sequence of matrices $A_\varepsilon \in C^0(\bar{\Omega}; C^0_\sharp(Y^N))^{N\times N}$, such that there exist $\alpha, \gamma > 0$, with

$$A_\varepsilon(x, y)\xi\xi \geqslant \min\{\alpha|\xi|^2, \gamma|A_\varepsilon(x, y)\xi|^2\} \quad \forall \xi \in \mathbb{R}^N \;\; \forall (x, y) \in \Omega \times \mathbb{R}^N, \tag{6}$$

let us study the homogenization of (3), with $G \in C^0(\bar{\Omega}; C^0_\sharp(Y^N))^N$.

Remark 1. We recall (see [8]) that (6) is equivalent to the existence of $\alpha, \beta > 0$, such that

$$A_\varepsilon(x, y)\xi\xi \geqslant \alpha|\xi|^2, \quad |A_\varepsilon(x, y)\xi| \leqslant \beta|\xi| \quad \forall \xi \in \mathbb{R}^N \;\; \forall (x, y) \in \Omega \times \mathbb{R}^N. \tag{7}$$

To study the asymptotic behavior of u_ε, let us apply the Arbogast et al. method's [2], strongly related to the two-scale theory (see [1,3,5,6,9]). For this purpose, we define $\kappa : \mathbb{R}^N \to \mathbf{Z}^N$ by the following rule: assuming $\mathbb{R}^N$ decomposed as the union of the cubes $k + Y^N$, with $k \in \mathbf{Z}^N$, then, for a.e. $x \in \mathbb{R}^N$, $\kappa(x)$ gives the center k of the cube which contains x.

We remark that if we decompose $\mathbb{R}^N$ as the union of the cubes $\varepsilon k + \varepsilon Y^N$, with $k \in \mathbf{Z}^N$, then the center of the cube which contains x is $\varepsilon\kappa(x/\varepsilon)$.

For the proof of the following theorem we refer to [2,5,6].

Theorem 2. *Consider a sequence u_ε which is bounded in $H_0^1(\Omega)$ and define $\hat{u}_\varepsilon \in L^2(\mathbb{R}^N; H^1(Y^N))$ by (u_ε is extended by zero outside of Ω)*

$$\hat{u}_\varepsilon(x, y) = u_\varepsilon\left(\varepsilon\kappa\left(\frac{x}{\varepsilon}\right) + \varepsilon y\right), \quad \textit{a.e. } (x, y) \in \mathbb{R}^N \times Y^N. \tag{8}$$

then there exists a subsequence of ε, still denoted by ε, and there exist $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega; H_\sharp^1(Y^N)/\mathbb{R})$, such that

$$u_\varepsilon \rightharpoonup u \quad \textit{in } H_0^1(\Omega), \tag{9}$$

$$\frac{1}{\varepsilon}\nabla_y \hat{u}_\varepsilon \rightharpoonup \nabla_x u_0 + \nabla_y u_1 \quad \textit{in } L^2(\mathbb{R}^N \times Y^N). \tag{10}$$

Remark 3. For $k \in \mathbf{Z}^N$, $\hat{u}_\varepsilon(x, y)$ restricted to $(\varepsilon k + \varepsilon Y^N) \times Y^N$ does not depend on x, and as a function of y, it is obtained from u_ε by using the change of variables $y = (x - \varepsilon k)/\varepsilon$ which transforms the small cube $\varepsilon k + \varepsilon Y^N$ on Y^N. Statement (10) is equivalent to ∇u_ε two-scale converges to $\nabla_x u_0 + \nabla_y u_1$.

The homogenization of (3) is given by the following theorem. Its proof is based on the one of the classical result of F. Murat and L. Tartar for the compactness of the H-convergence [8].

Theorem 4. *There exist a subsequence of ε, still denoted by ε, and an operator $\mathscr{A} \in L^\infty(\Omega; \mathscr{L}(L^2(Y^N)^N, L^2(Y^N)^N))$ such that*:

For every $\xi \in \mathbb{R}^N$, every $w \in H^1(\mathbb{R}^N)$ and a.e. $x \in \Omega$, we have

$$\int_{Y^N} \mathscr{A}(x)(\xi + \nabla w) \cdot (\xi + \nabla w)\,\mathrm{d}y$$
$$\geqslant \min\left\{\alpha\left(|\xi|^2 + \int_{Y^N} |\nabla w|^2\,\mathrm{d}y\right), \gamma\int_{Y^N} |\mathscr{A}(x)(\xi + \nabla w)|^2\,\mathrm{d}y\right\}. \tag{11}$$

For every $G \in C^0(\bar{\Omega}; C_\sharp^0(Y^N))^N$, the solution u_ε of (3) *satisfies* (9) *and* (10), *where $\hat{u}_\varepsilon$ is given by* (8) *and $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega; H_\sharp^1(Y^N)/\mathbb{R})$ are the unique solutions of*

$$-\mathrm{div}_x\left(\int_{Y^N} (\mathscr{A}(x)(\nabla_x u_0(x) + \nabla_y u_1(x, .)) - G)\,\mathrm{d}y\right) = 0 \quad \textit{in } \Omega, \tag{12}$$

$$-\mathrm{div}_y(\mathscr{A}(x)(\nabla_x u_0(x) + \nabla_y u_1(x, .)) - G) = 0 \quad \textit{in } \mathbb{R}^N \textit{ a.e. } x \in \Omega. \tag{13}$$

Proof. We divide the proof in several steps:

Step 1: For $G \in C^0(\bar{\Omega}; C_\sharp^0(Y^N))^N$, the solutions u_ε of (3) are bounded in $H_0^1(\Omega)$. So, by Theorem 2, up to a subsequence, there exist $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega; H^1(Y^N)/\mathbb{R})$ such that (9) and (10) hold. Moreover, using that $\hat{A}_\varepsilon$ defined by (take an extension of A_ε outside of Ω)

$$\hat{A}_\varepsilon(x, y) = A_\varepsilon\left(\varepsilon\kappa\left(\frac{x}{\varepsilon}\right) + \varepsilon y, y\right) \quad \text{a.e. } (x, y) \in \mathbb{R}^N \times Y^N \tag{14}$$

is bounded in $L^\infty(\Omega \times Y^N)^{N\times N}$ and that $(1/\varepsilon)\,\nabla_y \hat{u}_\varepsilon$ is bounded in $L^2(\Omega \times Y^N)^N$, we can also assume that there exists $\sigma \in L^2(\Omega \times Y^N)^N$, such that

$$\frac{1}{\varepsilon}\hat{A}_\varepsilon \nabla_y \hat{u}_\varepsilon \rightharpoonup \sigma \quad \text{in } L^2(\Omega \times Y^N)^N. \tag{15}$$

For $\varphi_0,\ \varphi_1 \in C_0^1(\Omega)$, $\psi \in C_\sharp^1(Y)$, we take $v_\varepsilon(x) = \varphi_0(x) + \varepsilon\psi(x/\varepsilon)\varphi_1(x)$ as test function in (3). Decomposing $\mathbb{R}^N$ as the union of the cubes $\varepsilon k + \varepsilon Y^N$, $k \in \mathbf{Z}^N$ and using in each cube the change of variables $y = (x - \varepsilon k)/\varepsilon$, we get

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0} \int_\Omega \left(A_\varepsilon\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon - G\left(x, \frac{x}{\varepsilon}\right)\right)\cdot \nabla v_\varepsilon \,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \int_{\Omega\times Y^N} \left(\hat{A}_\varepsilon \frac{1}{\varepsilon}\nabla_y \hat{u}_\varepsilon - G\right)\cdot(\nabla_x\varphi_0(x) + \varphi_1(x)\nabla_y\psi(y))\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\Omega\times Y^N} (\sigma - G)\cdot(\nabla_x\varphi_0(x) + \varphi_1(x)\nabla_y\psi(y))\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Since $\varphi_0, \varphi_1, \psi$ are arbitrary, this means that σ satisfies the equations

$$-\mathrm{div}_x\left(\int_Y (\sigma(x,y) - G(x,y))\,\mathrm{d}y\right) = 0 \quad \text{a.e. in } \Omega, \tag{16}$$

$$-\mathrm{div}_y\,(\sigma(x,y) - G(x,y)) = 0 \quad \text{in } \mathbb{R}^N \ \text{ a.e. } x \in \Omega. \tag{17}$$

For $\varphi \in C_0^1(\Omega)$, we take $u_\varepsilon\varphi$ as test function in (3), using then the Rellich–Kondrachov compactness theorem, (16) and (17), we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega A_\varepsilon\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon\cdot\nabla u_\varepsilon\varphi\,\mathrm{d}x \\
&\quad = -\lim_{\varepsilon\to 0}\int_\Omega \left(A_\varepsilon\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon\cdot\nabla\varphi\, u_\varepsilon - G\left(x, \frac{x}{\varepsilon}\right)\cdot(\nabla u_\varepsilon\varphi + u_\varepsilon\nabla\varphi)\right)\mathrm{d}x \\
&\quad = -\lim_{\varepsilon\to 0}\int_{\Omega\times Y^N}\left(\hat{A}_\varepsilon\,\frac{1}{\varepsilon}\,\nabla_y\hat{u}_\varepsilon\cdot\nabla_x\varphi\, u_0 - G\cdot\left(\frac{1}{\varepsilon}\,\nabla_y\hat{u}_\varepsilon\varphi + u_0\nabla_x\varphi\right)\right)\mathrm{d}x\,\mathrm{d}y \\
&\quad = -\int_{\Omega\times Y^N}(\sigma\cdot\nabla_x\varphi u_0 - G\cdot((\nabla_x u_0 + \nabla_y u_1)\varphi + u_0\nabla_x\varphi))\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_{\Omega\times Y^N}\sigma\cdot(\nabla_x u_0 + \nabla_y u_1)\varphi\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}\tag{18}$$

On the other hand, from (6) and the lower semicontinuity of the weak convergence, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_\Omega A_\varepsilon\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon\cdot\nabla u_\varepsilon\varphi\,\mathrm{d}x &\geqslant \liminf_{\varepsilon\to 0}\int_\Omega \min\left\{\alpha|\nabla u_\varepsilon|^2, \gamma\left|A_\varepsilon\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right|^2\right\}\varphi\,\mathrm{d}x \\
&= \liminf_{\varepsilon\to 0}\int_{\Omega\times Y^N}\min\left\{\alpha\left|\frac{1}{\varepsilon}\,\nabla_y\hat{u}_\varepsilon\right|^2, \gamma\left|\frac{1}{\varepsilon}\,\hat{A}_\varepsilon\nabla_y\hat{u}_\varepsilon\right|^2\right\}\varphi\,\mathrm{d}x\,\mathrm{d}y \\
&\geqslant \int_{\Omega\times Y^N}\min\{\alpha|\nabla_x u_0 + \nabla_y u_1|^2, \gamma|\sigma|^2\}\varphi\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Thus, from (18) and φ arbitrary, for a.e. $x \in \Omega$ we get

$$\int_{Y^N}\sigma(\nabla_x u_0 + \nabla_y u_1)\,\mathrm{d}y \geqslant \int_{Y^N}\min\{\alpha|\nabla_x u_0 + \nabla_y u_1|^2, \gamma|\sigma|^2\}\,\mathrm{d}y. \tag{19}$$

Step 2: We consider a countable subset D of $C_0^1(\Omega)\times C_0^1(\Omega; C_\sharp^1(Y^N))$ which is dense in $H_0^1(\Omega)\times L^2(\Omega; H_\sharp^1(Y^N)/\mathbb{R})$. By a diagonal argument, there exists a subsequence of ε, still denoted by ε (this will be the subsequence which appears in the statement of Theorem 4) such that for every $(w_0, w_1) \in \mathrm{Span}(D)$, the solution u_ε of (3), with $G = \nabla_x w_0 + \nabla_y w_1$, is such that there exist $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega; H^1_\sharp(Y^N)/\mathbb{R})$, $\sigma \in L^2(\Omega \times Y^N)^N$ such that (9), (10), (15)–(19) hold. From (19), (16) and (17), we have

$$\begin{aligned}
&\alpha\left(\int_\Omega |\nabla_x u_0|^2\,\mathrm{d}x + \int_{\Omega\times Y^N} |\nabla_y u_1|^2\,\mathrm{d}x\,\mathrm{d}y\right)\\
&\quad\leqslant \alpha\int_{\Omega\times Y^N} |\nabla_x u_0 + \nabla_y u_1|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\leqslant \int_{\Omega\times Y^N} \sigma\cdot(\nabla_x u_0 + \nabla_y u_1)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad= \int_{\Omega\times Y^N} (\nabla_x w_0 + \nabla_y w_1)\cdot(\nabla_x u_0 + \nabla_y u_1)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}\tag{20}$$

and thus, we deduce

$$\alpha\|(u_0, u_1)\|_{H_0^1(\Omega)\times L^2(\Omega; H^1_\sharp(Y^N)/\mathbb{R})} \leqslant \|(w_0, w_1)\|_{H_0^1(\Omega)\times L^2(\Omega; H^1_\sharp(Y^N)/\mathbb{R})}. \tag{21}$$

The two first lines of (20) also show

$$\alpha\|(u_0, u_1)\|_{H_0^1(\Omega)\times L^2(\Omega; H^1_\sharp(Y^N)/\mathbb{R})} \leqslant \|\sigma\|_{L^2(\Omega\times Y^N)^N}, \tag{22}$$

while from (16) and (17), we have

$$\int_\Omega |\nabla_x w_0|^2\,\mathrm{d}x + \int_{\Omega\times Y^N} |\nabla_y w_1|^2\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega\times Y^N} \sigma\cdot(\nabla_x w_0 + \nabla_y w_1)\,\mathrm{d}x\,\mathrm{d}y. \tag{23}$$

So, taking into account (19) and (22) we get

$$\|(w_0, w_1)\|_{H_0^1(\Omega)\times L^2(\Omega; H^1_\sharp(Y^N)/\mathbb{R})} \leqslant \frac{1}{\gamma^2}\|(u_0, u_1)\|_{H_0^1(\Omega)\times L^2(\Omega; H^1_\sharp(Y^N)/\mathbb{R})}. \tag{24}$$

Following (21) and (24) we can extend the linear application $(w_0, w_1) \in D \mapsto (u_0, u_1)$, to a linear application Q on $H_0^1(\Omega) \times L^2(\Omega; H^1_\sharp(Y^N)/\mathbb{R})$ which still satisfies (21), (24). From Lax–Milgram's theorem, Q is bijective and has a continuous inverse. From (19) and (21) we can also extend the linear application $(w_0, w_1) \in D \mapsto \sigma$ to a continuous application R from $H_0^1(\Omega)\times L^2(\Omega; H^1_\sharp(Y^N)/\mathbb{R})$ into $L^2(\Omega\times Y^N)^N$. In particular, we can define $S = RQ^{-1}$: $H_0^1(\Omega) \times L^2(\Omega; H^1_\sharp(Y^N)/\mathbb{R}) \to L^2(\Omega \times Y^N)^N$, which extends the application $(u_0, u_1) \in QD \mapsto \sigma$.

Now, given an increasing sequence of compact subsets K_n of Ω such that $\bigcup_{n\in\mathbb{N}} K_n = \Omega$, $K_0 = \emptyset$, we consider $\psi_n \in C_0^1(\Omega)$, such that $\psi_n(x) = x$, for every $x \in K_n$. Then, we define $\mathscr{A} \in L^\infty(\Omega; \mathscr{L}(L^2(Y^N)^N, L^2(Y^N)^N))$ by

$$\mathscr{A}(x)(g) = S\left(\int_{Y^N} g(y)\,\mathrm{d}y\cdot\psi_n, w\right), \tag{25}$$

$\forall g \in L^2(Y^N)^N$, a.e. $x \in K_n\backslash K_{n-1}$, with $w \in H^1_\sharp(Y^N)/\mathbb{R}$, $-\Delta w = -\operatorname{div} g$ in $\mathbb{R}^N$. From (19), $\mathscr{A}$ satisfies (11).

Step 3: Let us take $G \in C^0(\bar\Omega; C^0_\sharp(Y^N))^N$, and define u_ε as the solution of (3) (with ε the subsequence of ε given by the previous step). By Theorem 2, and Step 1, there exist a subsequence of ε, still denoted by ε, and $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega; H^1_\sharp(Y^N)/\mathbb{R})$, $\sigma \in L^2(\Omega \times Y^N)^N$ such that (9), (10), (15) hold. Let us prove that u_0, u_1 satisfy (12), (13), and then, by uniqueness that there is not necessary to extract any subsequence. Applying Step 1 to G replaced by $G - \nabla_x w_0 - \nabla_y w_1$, $(w_0, w_1) \in Span(D)$, we deduce from (19)

$$\gamma^2\int_{Y^N} |\sigma - R(w_0, w_1)|^2\,\mathrm{d}y \leqslant \int_{Y^N} |\nabla_x u_0 + \nabla_y u_1 - Q(w_0, w_1)|\,\mathrm{d}y \quad \text{a.e. in } \Omega.$$

By density this holds for every $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega; H_\sharp^1(Y^N)/\mathbb{R})$. Taking $(w_0, w_1) = Q^{-1}(\xi \cdot \psi_n, \nabla_y w)$, with $\xi \in \mathbb{R}^N$, $w \in H_\sharp^1(Y^N)/\mathbb{R})$ we get

$$\gamma^2 \int_{Y^N} |\sigma - \mathscr{A}(x)(\xi + \nabla_y w)|^2 \, dy \leqslant \int_{Y^N} |\nabla_x u_0 - \xi + \nabla_y (u_1 - w)|^2 \, dy$$

a.e. in K_n, and then in Ω. Thus, $\sigma = \mathscr{A}(x)(\nabla_x u_0(x) + \nabla_y u_1)$ a.e. in Ω, which by (16) and (17) proves (12), (13). $\square$

The question now is if effectively there is some example with a nonlocal term. If $N = 1$, it is possible to show that the homogenized problem of (3) is local. For $N = 2$, the following result gives an example where a nonlocal term appears.

Proposition 5. *Let be $\psi_\varepsilon \in C_\sharp^0(Y)$ a sequence which converges almost everywhere to the function $\psi = \sum_{l \in \mathbf{Z}} \chi_{(l-(1/2),l)}$, and it is such that $0 \leqslant \psi \leqslant 1$ in $\mathbb{R}$. Given*

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix},$$

we define the sequence of matrices $A_\varepsilon \in C^0(\mathbb{R}^2; C_\sharp^0(Y^2))^{2\times 2}$ by $A_\varepsilon(x, y) = A_1 \psi(x_1/\sqrt{\varepsilon}) + A_2(1 - \psi(x_1/\sqrt{\varepsilon}))$, $x = (x_1, x_2) \in \Omega$, $y = (y_1, y_2) \in \mathbb{R}^2$.

Then, for every bounded open set $\Omega \subset \mathbb{R}^2$ and every $G \in C^0(\bar{\Omega}; C_\sharp^0(Y^2))^2$, the solution u_ε of (3) *is such that* (9), (10) *are satisfied, where $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega; H_\sharp^1(Y^2)/\mathbb{R})$ are the solutions of*

$$-\Delta u_0 = -\mathrm{div}_x \int_{Y^2} G(x, y) \, dy \quad \text{in } \Omega, \tag{26}$$

$$-\Delta_y u_1 + \frac{1}{4} \frac{\partial}{\partial y_2} R\left(\frac{\partial u_1}{\partial y_2}\right) = -\mathrm{div}_y G \quad \text{in } \mathbb{R}^2 \text{ a.e. in } \Omega, \tag{27}$$

where $R : L_\sharp^2(Y^2) \to L_\sharp^2(Y^2)$ is the nonlocal operator given by $R(z) = \partial w/\partial y_2$, for every $z \in L_\sharp^2(Y^2)$, with $w \in H_\sharp^1(Y^2)/\mathbb{R}$, $-\Delta w = -\partial z/\partial y_2$ in $\mathbb{R}^2$.

Proof. An easy application of the two-scale convergence theory shows that (9) and (10) hold, where taking $A \in L^\infty(Y)^{2\times 2}$ as $A_1 \chi_{(-1/2,0)} + A_2 \chi_{(0,1/2)}$, there exist $\hat{u}_0 \in L^2(\Omega, H_\sharp^1(Y)/\mathbb{R})$, $\hat{u}_1 \in L^2(\Omega; L_\sharp^2(Y; H_\sharp^1(Y^2)/\mathbb{R}))$ such that $(u_0, \hat{u}_0, \hat{u}_1)$ satisfy the variational problem

$$\int_{\Omega\times Y\times Y^2} \left(A(t) \left(\nabla_x u_0 + \frac{d\hat{u}_0}{dt} e_1 + \nabla_y \hat{u}_1 \right) - G \right) \left(\nabla_x v_0 + \frac{d\hat{v}_0}{dt} e_1 + \nabla_y \hat{v}_1 \right) dx \, dt \, dy = 0$$
$$\forall (v_0, \hat{v}_0, \hat{v}_1) \in H_0^1(\Omega) \times L^2(\Omega, H_\sharp^1(Y)/\mathbb{R}) \times L^2(\Omega; L_\sharp^2(Y; H_\sharp^1(Y^2)/\mathbb{R})),$$

where e_1 is the first vector of the usual basis of $\mathbb{R}^2$ and u_1 is given by

$$u_1(x, y) = \int_Y \hat{u}_1(x, t, y) \, dt \quad \text{a.e. } (x, y) \in \Omega \times \mathbb{R}^N. \tag{28}$$

Taking $v_0 = \hat{v}_0 = 0$, we deduce $\hat{u}_1(x, t, y) = \Phi_1(x, y)\chi_{(-\frac{1}{2},0)}(t) + \Phi_2(x, y)\chi_{(0,\frac{1}{2})}(t)$, with $\Phi_i \in L^2(\Omega; H_\sharp^1(Y^2)/\mathbb{R})$, $-\mathrm{div}_y (A_i \nabla_y \Phi_i - G) = 0$ and $\mathbb{R}^N$, a.e. $x \in \Omega$, $i = 1, 2$. From (28), we have $u_1 = (\Phi_1 + \Phi_2)/2$ and then, using $A_1 + A_2 = 2I$, we get $-\Delta_y \Phi_i = -\mathrm{div}_y A_j \nabla_y u_1$ and $\mathbb{R}^N$, for a.e. $x \in \Omega$, where $i, j \in \{1, 2\}$, $i \neq j$. Denoting by $\Upsilon \in L^2(\Omega; H_\sharp^1(Y^2)/\mathbb{R})$ the solution of $-\Delta_y \Upsilon = -\partial^2 u_1/\partial y_2^2$ in $\mathbb{R}^N$, for a.e. $x \in \Omega$, we have that $\Phi_i = u_1 + (-1)^{i+1}\Upsilon/2$, $i = 1, 2$, and then, using the equations satisfied by Φ_i, we conclude that u_1 satisfies (27). $\square$

Taking in the variational equation $v_0 = \hat{v}_1 = 0$, we now get $\hat{u}_0 = 0$, and then using $\hat{v}_0 = \hat{v}_1 = 0$ we conclude that u_0 is the solution of (26).

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References

[1] G. Allaire, Homogenization and two-scale convergence, SIAM J. Math. Anal. 23 (6) (1992) 1482–1518.
[2] T. Arbogast, J. Douglas, U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory, SIAM J. Math. Anal. 21 (4) (1990) 823–836.
[3] J. Casado-Díaz, Two-scale convergence for nonlinear Dirichlet problems in perforated domains, Proc. Royal Soc. Edinburgh A 130 (2000) 249–276.
[4] J. Casado-Díaz, F. Murat, A. Sili, Nonlocal Limits in the Asymptotic Behavior of Monotone Problems in thin Domains, to appear.
[5] D. Cioranescu, A. Damlamian, G. Griso, Periodic unfolding and homogenization, C. R. Acad. Sci. Paris I 335 (2002) 99–104.
[6] M. Lenczner, Homogénéisation d'un circuit électrique, C. R. Acad. Sci. Paris I 324 (1997) 537–542.
[7] U. Mosco, Composite media and asymptotic Dirichlet forms, J. Functional Anal. 123 (2) (1994) 368–421.
[8] F. Murat, L. Tartar, H-convergence, in: A. Cherkaev, R. Kohn (Eds.), Topics in the Mathematical Modelling of Composite Materials, Progress in Nonlinear Differential Equations and their Applications, Birkauser, Boston, 1997, pp. 21–43.
[9] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal. 20 (3) (1989) 608–623.
[10] S. Spagnolo, Sulla convergenza di soluzioni di equazione paraboliche ed ellittiche, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 3 (22) (1968) 577–597.