QUALITATIVE MODELLING AND SIMULATION BY PIECEWISE LINEAR ANALYSIS

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Abstract Since applications of expert systems were typical in domains with no well defined models, qualitative methods for modelling and reasoning were soon developed. Most current qualitative reasoning programs derive the qualitative behaviour of a system by simulating a hand-crafted *qualitative version* of the differential equation that caracterizes the model of a system. This paper describes a method to construct a family of piecewise linear dynamical systems from the qualitative information contained in a model. We apply the dynamical system theory to deduce results about the behaviour of the family of dynamical systems constructed as a consequence of the qualitative model.

Keywords Artificial intelligence; qualitative modelling; nonlinear systems.

1 Introduction

A lot of work is being published at present with the name of Qualitative Modelling, Qualitative Simulation and Qualitative Reasoning (Kuipers 1986 and Sacks 1987c). They all have one thing in common, which is to obtain conclusions from models of systems, where the information is either qualitative or incomplete. The systematic gathering of all qualitative information about a system makes a qualitative model. That qualitative information can be structured as a set of constraints that join together show the evolution of the different system sections. These constraints can have a derived form, an arithmetic form, a functional form, etc.

Once a qualitative model has been constructed, the question is to get obtain the maximum information from it. A set of theoretic tools and some practical implements have been developed. Qualitative Simulation is one of them. With some differences, the qualitative information contained in the qualitative model that we take into consideration in this paper, is similar to Qualitative Simulation, but the method to obtain conclusions will be very different. At this point we construct a family of piecewise linear dynamical systems containing the qualitative information and later we will apply the dynamical system theory to obtain conclusions from the model.

The research of constant piecewise linear dynamical systems has received enough attention in different fields at the last decade (Chua 1983 and Sacks 1987c), but the same hasn't happened to the family of piecewise linear dynamical systems. To contribute to the study of the latter, it is possible to apply the theory of bifurcation to obtain interesting qualitative results.

2 Qualitative Model and Analysis

Let's assume in this paper that a qualitative model has a set of variables, constants and a set of constraints among them. The variables and the constants take their values in a ***quantity space.*** Each quantity space is defined by an ordered set of landmark values. We are going to consider the landmarks that determine the quantity space as variables, with a partial order extracted from the specifications. The value of the variables, as well as the constants, is described qualitatively in terms of their quantity space. Every variable or constant may take a value in a landmark of its quantity space, or between two landmarks.

The constraints among variables can be: *Arithmetic*, where $+$, $-$, $*$, $/$, represent the arithmetic operators respectively. *Functional*, where $M+$, $M-$, Nm, they are representable by continuous functions, $f : R \rightarrow R$ and so that $M+$ is a monotonic increasing, $M-$ monotonic decreasing and

Nm no monotonic. And *Derivative* with reference to time. This is represented by D.
The associated information to the functional constraints can be extended with a list of pairs of points. In the event Nm joined with the values of the points that the function takes, we designate with the signs $+$, $-$, 0 if in this point the function is increasing, decreasing or it takes an extreme respectively. The list of Nm would contain the points where the function takes the extremes.
A qualitative model about the evolution of preys, predators and their interactions it is given in the following equations:

$$\begin{aligned} \dot{x} &= x(f(x) - g(y)) \\ \dot{y} &= y\, h(x) \end{aligned} \tag{1}$$

where the number of preys are representable by x, the number of predators are representable by y, the rate of births of the preys is representable by f, the rate of mortality of the preys by predators is representable by g and the rate of births of the predators is representable by h. We assume that:

> The consumption of prey by predator is proportional to the number of preys, and it depends on the number of predators by an increasing monotonic function. In the absence of predator this consumption is null ($g(0) = 0$).
>
> Birth of predators depends on the prey number, by an increasing monotonic function. In the absence of preys ($x = 0$) predators die out ($h(0) < 0$, or equivalent $h(h_1) = 0$ with $h_1 > 0$).
>
> In the absence of predation,preys grow without limit,for small x and when x is medium the birth rate decreases becoming negative (f takes a maximum in h_0 and $f(h_2) = 0$ with $0 < h_0 < h_2$

The information contained in the qualitative model above is given in a more formal way in the Figure 1.
As we can observe, the relevant information has been defined for every variable in the quantity space and for every constraint of the form $M+$, $M-$ and Nm. So we can see that g takes through $(0, 0)$, the maximun of f is in (h_0, k_1), and takes through zero in $(h_2, 0)$, h takes through zero in $(h_1, 0)$ with $h_0 < h_2$, etc.
The problem now is what information we can extract regarding trasitory behaviour, and for long-term, from the system whose qualitative information we have specified.
One way to study the problem is to consider the qualitative information as a set of constraints

```
(quantity-spaces
   (x      (0  h0 h2  inf) (0 h1 h2))
   (y      (0   inf))
   (f      (minf  0 k0 k1 inf))
   (h      (minf  0  inf))
   (g      (minf  0 inf))
   (tnx    (minf  0  inf))
   (fx     (minf  0  inf))
   (fy     (minf  0  inf)))

(constraints
   ((Ma  x  f)    (0 k0 +)
                  (h0 k1 0) (h2 0 -))
     ((M+  y  g)     (0 0))
     ((M+  x  h)     (h1 0)
     ((- f  g  tnx))
     ((* x  tnx  fx))
     ((* y  h  fy))
     ((D  x    fx))
     ((D  y    fy)))
```

Figure 1: Equations of the Model

that bind the variables in a symmetric way. This is Kuiper's Qualitative Simulation approach (Kuipers 1986). However, we will consider an asymmetric causal order and the constraints as some bonds between a cause and its effect. This allows us to approach the concepts and the methods of the dynamical systems in the sense it is understood in Mathematics.
In this paper, we apply tecniques from the theory of dynamical system to obtain information from a qualitative model. The use of these techniques consists of two stages: first, in the construction and regularization of a family of piecewise linear dynamical systems and then in their analysis. The first stage consists of a sequence of events that holds up the qualitative information of the model. The second one is the application of the methods of qualitative analysis to the dynamical system that has been constructed.

3 Piecewise Linear Functions

A continuous function

$$f : R^n \rightarrow R^m$$

is piecewise linear if R^n can be divided into a finite number of polyhedral regions by a finite number of hyperplanes and in each region i, f is an afin funtion f^i:

$$f^i(x) = A^i_f x + b^i_f$$

where A^i_f is a constant $n \times n$ matrix and b^i_f is a constant n-vector. At the following the set of regions associated to the piecewise linear function f are representable by Ω_f. Ω^i_f is a region of Ω_f and f^i the afin function associated to f at region Ω^i_f. The set Ω_f and the set of afin functions f^i have to be given to specify the piecewise linear function f. If f is continuous the equation:

$$f^i(x) = f^j(x) \tag{2}$$

must be held for each x in the boundary between the regions Ω^i_f, Ω^j_f.
We show now the way to associate a piecewise linear function or an operator on piecewise linear functions to each constraint in our qualitative models. In the constraints of monotony $M+$, $M-$ and Nm,the polihedral regions are being defined by:

$$\Omega^i_{M+} = (h_i, h_{i+1})$$

where $h_0, h_1, ..., h_n$ are the landmarks of the corresponding function. In each region i,the corresponding afin function has the form $ax + b$ where $a > 0$ if the constraint is $M+$, $a < 0$ if it is $M-$ and the sign of a depends on the region if the constraint is Nm.
To the arithmetic constraints + or −, we associate an operator of piecewise linear function defined in this way. Given the continuous and piecewise linear functions:

$$f : R^n \to R, g : R^m \to R$$

and $h = f \oplus g$ with $\oplus = +$ or $\oplus = -$ where $h : R^n \times R^m \to R$, then the function h is also a piecewise linear function.
If Ω^*_f and Ω^*_g are the canonical partitions of $R^n \times R^m$ induced by Ω_f and Ω_g respectively. Each region of Ω_h is formed with the no empty intersection of a region of Ω^*_f with another one of Ω^*_g:

$$\Omega^k_h = \Omega^{*i}_f \cap \Omega^{*j}_g \tag{3}$$

where

$$\Omega^{*i}_f = \Omega^i_f \times R^m, \Omega^{*j}_g = R^n \times \Omega^j_g$$

In the case f, g have identical domain

$$f, g : R^n \to R$$

each non null region of Ω_h is formed simply by intersection

$$\Omega^k_h = \Omega^i_f \cap \Omega^j_g$$

The piecewise linear function that results in each region is:

$$(f \oplus g)^k = f^i \oplus g^j \tag{4}$$

This introduces ambiguity about the constraints that may support the element of the corresponding A, b. If $a_1 > 0$ and $a_2 > 0$ the sign of $a_1 - a_2$ is unknown. The opposite happens if $a_1 > 0$ and $a_2 < 0$
We can observe the composition of function, now. If

$$f : R^n \to R^m, g : R^m \to R$$

are piecewise linear functions, the composition :

$$g \circ f : R^n \to R$$

is a piecewise linear function too.
The partition $\Omega_{g \circ f}$ of R^n that will be associated to $g \circ f$ is formed with the no empty intersections of regions of Ω_f with the inverse image by f of a region of Ω_g:

$$\Omega^k_{g \circ f} = \Omega^i_f \cap (f^i)^{-1}(\Omega^j_g)$$

and the associated afin function is

$$(g \circ f)^k = g^j \circ f^i$$

If $prod = f * g$ is defined with f, g piecewise linear functions. We want to find a piecewise linear function that is a piecewise linear approach of the $prod$. We can define the piecewise linear function

$$\begin{aligned} S &= f + g \\ D &= f - g \end{aligned} \tag{5}$$

then

$$prod_L = \left\| \begin{array}{ll} g & \text{if } S \geq 0 \text{ and } D \geq 0 \\ f & \text{if } S \geq 0 \text{ and } D < 0 \\ -f & \text{if } S < 0 \text{ and } D \geq 0 \\ -g & \text{if } S < 0 \text{ and } D < 0 \end{array} \right. \tag{6}$$

The selected method assures that the zeros of $prod$ are the same as the $prod_L$. The regions of $prod_L$ are obtained following the rule in (6). So to obtain Ω_{prod_L}, each region of $\Omega_S = \Omega_D$ is divided into subregions where the sign of $S = f + g$ and the sign of $D = f - g$ are preserved.
The operator division / will be considered in a forthcoming paper.

4 The construction of the family of piecewise linear dynamical systems

The construction process consists of rules for building a methodical way. As we have established,the constraints are asymmetric, this means that if the system is well specified, the set of constraints defines implicity a directed acyclic graph that has as the initial node the state variables and the ending nodes in the variables that specify the field. This graph, without loops, specifies a partial order which is the one we follow to construct the result.

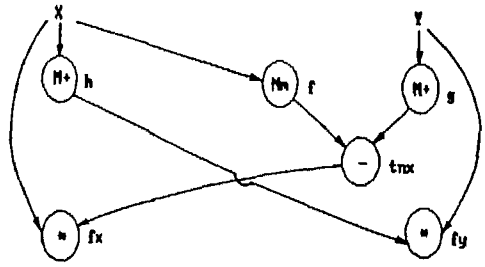

Figure 2: Directed acyclic graph associated to the example

Each node has no more than two inmediate predecessors. Every node of the graph specifies an allowed constraint. The directed acyclic graph associated to the example above is shown in Figure 2. Therefore, the constructive method consists of:

> First, for each node to obtain the piecewise linear function or operator on piecewise linear function that is associated to constraint as the rules of the previous paragraph.
>
> Second, to reduce this graph by succesive application of one step that consists to replace a function node and their inmediates predecessors nodes by another one whose associated function is the composition of associated functions to them.

After we have applied the succesive constraints in the determined order we will obtain a family of the piecewise linear functions and a set of constraints for their parameters. This family, after to regularize it in the boundary of each regions is the field of the dynamical system that we wanted to construct.
The application of the qualitative analysis of the dynamical system and the bifurcations theory to the resultant family of dynamical systems provide us the relevant information from the qualitative point of view.

5 Qualitative Analysis of Dynamical Systems

Given the family of dynamical systems:

$$\dot{x} = f(x, p) \tag{7}$$

The qualitative analysis of (7) is the long term study of its solutions for a fixed value of its parameters $p = p_0$. The long term behaviour is related to the class of attractor that the system (7) has for $p = p_0$.
The stable equilibrium is the most known of these attractors. It is related to a long term where $\dot{x} = 0$. Another attractor is the Limit Cycle. This is related to the periodic oscillations.
The bifurcation theory is the study about the change in number and kind of attractors when the parameters change.
If the system (7) is a family of piecewise linear dynamical system its qualitative analysis is simplified. So the equilibria ($\bar{x}$) must verify:

$$A^i_f \bar{x} + b^i_f = 0, \ \bar{x} \in \Omega^i_f \tag{8}$$

and its stability is given by the A^i_f matrix.
When the parameters change the number and kind of attractors changes too. The bifurcation theory can help us to find new kind of atractors. For example, the Hopf bifurcation (a couple of eigenvalues of Jacobian matrix cross the imaginary axis) can be used to find limit cycles.
If $x \in R^2$ the characteristic polynomial of the Jacobian Matrix evaluated in the equilibrium $\bar{x}(p)$ is

$$\lambda^2 + a_1(p)\lambda + a_2(p) = 0$$

A Hopf bifurcation appears if $a_1(p)$ crosses the zero and $a_2(p) > 0$ when parameters change (Guckenheiner 1982).

6 Applications to the example

The application of the above method to example give the following conclusions (see the appendix for intermediate steps). The differentiable field fx, fy is null in three points:

- E1: $(0,0)$ in the region $\Omega^2_{fx} \cap \Omega^1_{fy}$
- E2: $(h_2, 0)$ in the region $\Omega^4_{fx} \cap \Omega^3_{fy}$
- E3:
  - $(h_1, (a^2_f h_1 + b^2_f)/a^1_g)$ in the region $\Omega^4_{fx} \cap \Omega^2_{fy}$, if $h_0 < h_1$
  - $(h_1, (a^1_f h_1 + b^1_f)/a^1_g)$ in the region $\Omega^1_{fx} \cap \Omega^2_{fy}$, if $h_1 < h_0$.

E1: In the region $\Omega^2_{fx} \cap \Omega^1_{fy}$ the field is

$$\dot{x} = x \tag{9}$$
$$\dot{y} = -y \tag{10}$$

eigenvalues: $\lambda_1 = 1 > 0, \lambda_2 = -1 > 0$. So $(0,0)$ is an unstable equilibrium.
E2: In the region $\Omega^4_{fx} \cap \Omega^3_{fy}$ the field is:

$$\dot{x} = a^2_f x - a^1_g y + b^2_f \tag{11}$$
$$\dot{y} = y \tag{12}$$

and the eigenvalues : $\lambda_1 = a_f^2 < 0, \lambda_2 = 1 > 0$. So $(h_2, 0)$ is an unstable equilibrium.
E3: In the region $\Omega_{fx}^4 \cap \Omega_{fy}^2$, if $h_0 < h_1$, the field is:

$$\dot{x} = a_f^2 x - a_g^1 y + b_f^2 \quad (13)$$
$$\dot{y} = a_h^1 x + b_h^1 \quad (14)$$

If $h_0 < h_1$ the system has an equilibrium $(h_1, (a_f^2 h_1 + b_f^2)/a_g^1)$,with characteristic polynomial:

$$\lambda^2 - a_f^2 \lambda + a_h^1 a_g^1 = 0 \quad (15)$$

Both eigenvalues have a negative real part. So this point is an attractor.
If $0 < h_1 < h_0$, the field, in region $\Omega_{fx}^1 \cap \Omega_{fy}^2$, is:

$$\dot{x} = a_f^1 x - a_g^1 y + b_f^1 \quad (16)$$
$$\dot{y} = a_h^1 x + b_h^1 \quad (17)$$

The equilibrium E3 changes. It is the point $(h_1, (a_f^1 h_1 + b_f^1)/a_g^1)$,with characteristic polynomial:

$$\lambda^2 - a_f^1 \lambda + a_h^1 a_g^1 = 0 \quad (18)$$

Now it is a repulsor.
When parameter h_1 changes from $h_0 < h_1$ to $h_1 < h_0$ the characteristic polynomial of equilibrium E3 changes from (15) to (18). Since $a_h^1 a_g^1 > 0, -a_f^2 > 0, -a_f^1 < 0$ then a Hopf bifurcation appears. If $h_1 < h_0$ a Limit Cycle exists.
We can conclude from the qualitative model in Figure 1 that the system will evolve only to the attractor. This attractor is a stable equilibrium if $h_0 < h_1$ and a Limit Cycle if $h_1 < h_0$.

7 Conclusions

This paper has described a technique for deriving the properties of nonlinear dynamic systems from their qualitative description. These dynamic systems are interesting because they pose unsolved problems in the representation of knowledge, and because they appear fundamental to commonsense knowledge of causality.
The example presented above demonstrate a representation for qualitative reasoning about causality in ecological mechanisms, and it is possible to be applicationed to others problems.
The technique developped can be used to deduce results from the qualitative information about a system. It uses the theory of dynamical systems to avoid the ambiguity that appears in other methodologies.

8 References

Javier Aracil and Miguel Toro (1991). *Qualitative methods for system dynamic models.* In Revue Internacional of Systemique.

Leon Chua and Robin Ying (1983). *Canonical Piecewise-Linear Analysis.* IEEE Transactions on circuits and Systems, Vol.Cas-30, NO 3. pp, 125-140.

J. Guckenheimer and P. Holmes (1982). *Nonlinear Oscillations, dynamical systems and bifurcations of vector fields.* Springer Verlag.

Benjamin Kuipers. *Qualitative Simulation.* In Artificial Intelligence 29, pages 289-338.

Elisha P. Sacks (1985). *Qualitative mathematical reasoning.* In Proceedings of the Ninth International Conference on Artificial Intelligence, pages 137-139.

Elisha P. Sacks (1987a). *Hierarchical reasoning about inequalities.* In Proceedings of the National Conference on Artificial Intelligence, American Association for Artificial Intelligence.

Elisha P. Sacks (1987b). *Qualitative sketching of parameterized functions.* In Proceedings of the Second International Conference on Applications of Artificial Intelligence in Engineering, August 1987.

Elisha P. Sacks (1987c). *Piecewise Linear Reasoning.* In Proceedings AAAI-87, 655-659.

Miguel Toro and Javier Aracil (1988a). *Qualitative Analysis of System Dynamic Ecological Models.* In System Dynamics Review, Vol. 4, Num. 1-2, 56-60.

Miguel Toro and Javier Aracil (1988b). *Oscillations and Chaos in Ecological Populations.* In Proceedings of the Internacional Conference of the System Dynamics Society, 391-403.

9 Appendix

$f =$

General Constrainsts
$0 < h_0 < h_1 < h_2 < \infty$
$0 < k_0 < k_1 < \infty$

Regions
$\Omega_f^1 = \{x | x \geq 0, x \leq h_0\}$
$\Omega_f^2 = \{x | x \geq h_0\}$
Afin Functions
$f^1 \equiv a_f^1 x + b_f^1, \qquad f^2 \equiv a_f^2 x + b_f^2$
Parameter Constrainsts
$a_f^1 > 0, \qquad a_f^1 h_0 + b_f^1 = k_1$
$b_f^1 = k_0 \qquad a_f^2 < 0$
$a_f^2 h_0 + b_f^2 = k_1 a_f^2 h_2 + b_f^2 = 0$

$$g = \left\| \begin{array}{l} \text{Regions} \\ \Omega_g^1 = \{y | y \geq 0\} \\ \text{Afin Functions} \\ g^1 \equiv a_g^1 y \\ \text{Parameter Constrainsts} \\ a_g^1 > 0 \end{array} \right.$$

$$h = \left\| \begin{array}{l} \text{Regions} \\ \Omega_h^1 = \{x | x \geq 0\} \\ \text{Afin Functions} \\ h^1 \equiv a_h^1 x + b_h^1 \\ \text{Parameter Constrainsts} \\ a_h^1 > 0 \\ a_h^1 h_1 + b_h^1 = 0 \end{array} \right.$$

$$tnx = \left\| \begin{array}{l} \text{Regions} \\ \Omega_{tnx}^1 = \Omega_f^1 \times \Omega_g^1 \\ \Omega_{tnx}^2 = \Omega_f^2 \times \Omega_g^1 \\ \text{Afin Functions} \\ tnx^1 \equiv a_f^1 x - a_g^1 y + b_f^1 \\ tnx^2 \equiv a_f^2 x - a_g^1 y + b_f^2 \end{array} \right.$$

The constraints of the fx and the coefficient of tnx and h involve the next definition of fx and fy, and the product rule (6):

$$fx = \left\| \begin{array}{l} \text{Regions} \\ \Omega_{fx}^1 = \{(x,y) \in \Omega_{tnx}^1 | x + tnx^1 \geq 0, \\ \qquad x - tnx^1 \geq 0\} \\ \Omega_{fx}^2 = \{(x,y) \in \Omega_{tnx}^1, x + tnx^1 \geq 0, \\ \qquad x - tnx^1 < 0\} \\ \Omega_{fx}^3 = \{(x,y) \in \Omega_{tnx}^1, x + tnx^1 < 0, \\ \qquad x - tnx^1 \geq 0\} \\ \Omega_{fx}^4 = \{(x,y) \in \Omega_{tnx}^2, x + tnx^2 \geq 0, \\ \qquad x - tnx^2 \geq 0\} \\ \Omega_{fx}^5 = \{(x,y) \in \Omega_{tnx}^2, x + tnx^2 \geq 0, \\ \qquad x - tnx^2 < 0\} \\ \Omega_{fx}^6 = \{(x,y) \in \Omega_{tnx}^2, x + tnx^2 < 0, \\ \qquad x - tnx^2 \geq 0\} \\ \text{Afin Functions} \\ fx^1 \equiv tnx^1 \\ fx^2 \equiv x \\ fx^3 \equiv -x \\ fx^4 \equiv tnx^2 \\ fx^5 \equiv x \\ fx^6 \equiv -x \end{array} \right.$$

$$fy = \left\| \begin{array}{l} \text{Regions} \\ \Omega_{fy}^1 = \{(x,y) \in \Omega_h^1, y + h^1 < 0, \\ \qquad y - h^1 \geq 0\} \\ \Omega_{fy}^2 = \{(x,y) \in \Omega_h^1, y + h^1 \geq 0, \\ \qquad y - h^1 \geq 0\} \\ \Omega_{fy}^3 = \{(x,y) \in \Omega_h^1, y + h^1 \geq 0, \\ \qquad y - h^1 < 0\} \\ \Omega_{fy}^4 = \{(x,y) \in \Omega_h^1, y + h^1 \geq 0, \\ \qquad y - h^1 < 0\} \\ \text{Afin Functions} \\ fy^1 \equiv -y \\ fy^2 \equiv h^1 \\ fy^3 \equiv y \\ fy^4 \equiv -h^1 \end{array} \right.$$