PIECEWISE LINEAR ANALYSIS OF AN INFLUENCE DIAGRAM

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Abstract To each causal diagram, and the structure that it represents, a dynamical system can be associated. From its qualitative analysis, the behaviours associated to the structure can be deduced. This paper introduces a piecewise linear dynamical system associated to a causal diagram. Some interesting results on the qualitative behaviour of the system can be obtained from this dynamical system. In this paper a method is proposed to implement automatically the construction of a piecewise linear dynamical system to each causal diagram, the study of its equilibria and its stability. This allows us to obtain ,automatically, the behaviour modes associated to a causal diagram.

Keywords Artificial intelligence; qualitative modelling; nonlinear systems.

1 Introduction

Models in system dynamics are re-elaborations of verbal descriptions, where the qualitative aspects are dominant on the other, most of the conclusions obtained from a system dynamics model are mainly qualitative.

A lot of work is being published at present with the name of Qualitative Simulation Modelling and Analysis (Fishwick 91,Toro and Aracil 88, Kuipers 86). They all have one thing in common, which is to obtain conclusions from models of systems,where the information is either qualitative or incomplete. System dynamics can be included in this area of work.

In system dynamics, a model is constructed from a set of variables and parameters, a set of causal influences between them and a set of operators to quantify these influences. The variables are classified into levels, rates, auxiliary variables and parameters. The operators used are arithmetic, functional (tables), derivative and delays. Giving numerical values to the parameters and tables a nonlinear dynamical system is constructed and after its simulation conclusions regarding the model can be obtained. It is very difficult to obtain a global perspective about the behaviour's modes that the model can show. A global perspective can be obtained applying techniques to nonlinear dynamical systems.

In the above process, parameters and tables must be quantified. If the quantified process is not made a qualitative model can be obtained. It is possible to obtain conclusions directly from a qualitative model.

In this paper we show a method to apply systematically these techniques, and then we can analyse a qualitative model. The way is to obtain from the qualitative model a piecewise linear dynamical system and to analyse it systematically. Interesting conclusions can be obtained from its global analysis (Guckenheimer and Holmes 1982). Piecewise linear dynamical systems have received enough attention in different fields at the past decade (Lum and Chua 1991, Sacks 1987). This class of systems is suitable for an global analysis. The singular points of these piecewise linear system provide us the qualitative information regarding the long time behaviour of variables of the model.

The paper is organized in three parts. First a qualitative model of the growth in a finite world is presented. Second, the main concepts about piecewise linear functions are introduced and last the method is applied to the qualitative model in the first part.

2 Qualitative Model and Analysis

Let's assume in this paper that a qualitative model has a set of variables, constants, relations among them and operators.The operators define the relations.

The operators can be:

- *Arithmetic*, where $+$, $-$, $*$, $/$, represent the arithmetic operators respectively.
- *Functional*, where $M+$, $M-$, Nm, represent continuous functions, $f : R \rightarrow R$ and so that $M+$ is a monotonic increasing, $M-$ monotonic decreasing and Nm no monotonic.
- *Derivative* with reference to time, this is represented by D,and
- *Delay* which is represented by Dl.

They are the most used operators in system dynamics.

The functional operators are the tables but without quantitative information. The associated information to the functional operators can be extended with a

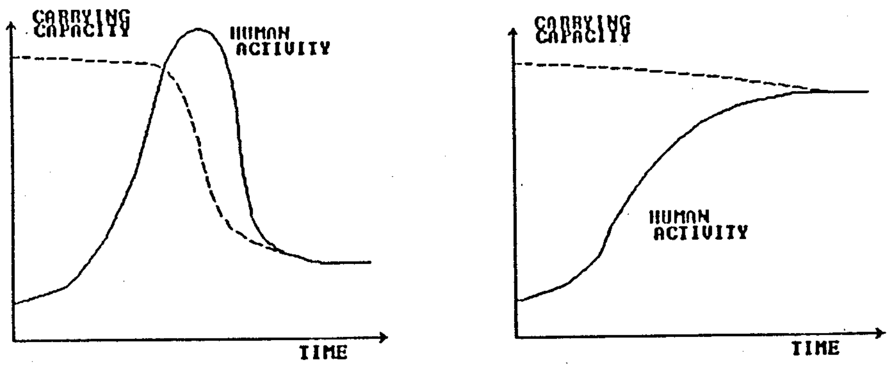

Figure 1: Basic behaviour mode of material growth in a finite world

list of pairs of points. The signs +, −, 0 can be used to show if in this point the function is increasing, decreasing or it takes an extreme respectively.

In order to introduce the concepts about piecewise linear analysis of a qualitative model, we will go through an elementary modelling exercise: the basic World 3 model of growth in a finite world. The question to be dilucidated in this model is: Will human material activity adjust smoothly to the global carrying capacity or go through a period of overshoot and collapse? (figure 1).

Two levels are considered sufficient to describe the system under study: human activity (x) and carrying capacity (y). Another important variable is the pressure on the physical environment (PPE). The following group of processes are responsible for these modes of behaviour:

1. The level of human activity increases when the pressure on the resources is low.

2. A sufficiently high level of human activity erodes the carrying capacity of the global environment.

3. When the pressure on the physical environment is high an involuntary downward pressure on the human activity appears and after a delay spent in data gathering and institutional change a deliberate reduction of human activity is performed.

The qualitative model defined by these interactions is shown in figure 2 and it is represented by the equations:

$$\begin{aligned} \dot{x} &= nx(z + f_1(x/y)) \\ \dot{y} &= b(f_3(y) - y f_4(x/a)) \\ z &= \text{delay}(f_2(x/y), T) \end{aligned} \quad (1)$$

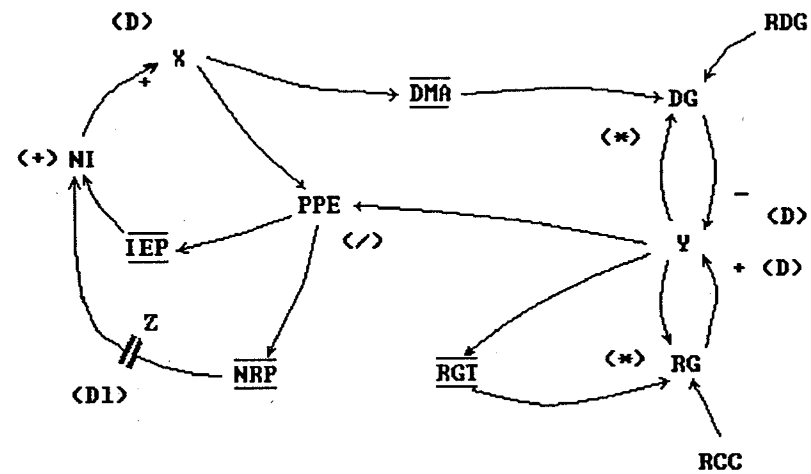

Figure 2: The qualitative model

The levels are x = human activity, y = carrying capacity, x/y represents the pressure on the physical environment, a, b, n and T are positive parameters and the functional operator f_i are defined by the equations:

function	type	corresponding values
f_1	M^-	$(0,1)$
f_2	M^-	$(0,1)$
f_3	Nm	$(0,0.5,+),(0.5,1,0),(1,0,-)$
f_4	M^+	$(0,0)$

It can be observed, the relevant information has been defined for each variable and for each functional operator of the form M^+, M^- and Nm. So f_1 and f_2 take through $(1,0)$ and are decreasing functions, the maximun of f_3 is in $(0.5,1)$ and takes through zero in $(1,0)$, and f_4 takes through zero in $(0,0)$ and it is an increasing function. It is assumed that the values of the variable pressure on the physical environment changes around 1. So a value higher than 1 represents a high value of this variable. It is assumed also that regeneration of the carrying capacity can be modelled by a non monotonic function similar to f_3.

The problem now is what information can be extracted about the transitory behaviour and for long-term, from the system whose qualitative information we have specified.

In this paper, we apply techniques from the theory of dynamical system to get information from a qualitative model. The use of these techniques consists of two stages: first, the construction of a family of piecewise linear dynamical systems and then in their analysis. The first stage consists of a sequence of events that holds up the qualitative information of the model. The second one is the application of the methods of qualitative analysis to the dynamical system that has been constructed. Both stages can be constructed systematically.

3 Piecewise Linear Function

A continuous function

$$f : R^n \to R^m$$

is piecewise linear if R^n can be divided into a finite number of polyhedral regions by a finite number of hyperplanes and in each region i, f is an afin function f^i:

$$f^i(x) = A^i_f x + b^i_f$$

where A^i_f is a constant $n \times n$ matrix and b^i_f is a constant n-vector. At the following the set of regions associated to the piecewise linear function f are representable by Ω_f. Ω^i_f is a region of Ω_f and f^i the afin function associated to f at region Ω^i_f. The set Ω_f and the set of afin functions f^i have to be given to specify the piecewise linear function f. If f is continuous the equation:

$$f^i(x) = f^j(x) \tag{2}$$

must be held for each x in the boundary between the regions Ω^i_f, Ω^j_f.

We show now the way to associate a piecewise linear function or an operator on piecewise linear functions to each constraint in our qualitative models. In the constraints of monotony $M+$, $M-$ and Nm the polihedral regions are being defined by:

$$\Omega^i_{M+} = (h_i, h_{i+1})$$

where $h_0, h_1, ..., h_n$ are the landmarks of the corresponding function. In each region i the corresponding afin function has the form $ax + b$ where $a > 0$ if the constraint is $M+$, $a < 0$ if it is $M-$ and the sign of a depends on the region if the constraint is Nm.

To the arithmetic constraints $+$, $-$ we associate an operator of piecewise linear function defined in this way. Given the continuous and piecewise linear functions:

$$f : R^n \to R, g : R^m \to R$$

and $h = f \oplus g$ with $\oplus = +$ or $\oplus = -$ where $h : R^n \times R^m \to R$, then the function h is also a piecewise linear function.

If Ω^*_f and Ω^*_g are the canonical partition of the $R^n \times R^m$ induced by Ω_f and Ω_g respectively. Each region of Ω_h is formed with the no empty intersection of a region of Ω^*_f with another one of Ω^*_g:

$$\Omega^k_h = \Omega^{*i}_f \cap \Omega^{*j}_g \tag{3}$$

where

$$\Omega_f^{*i} = \Omega_f^i \times R^m, \Omega_g^{*j} = R^n \times \Omega_g^j$$

In the case f, g have identical domain

$$f, g : R^n \to R$$

each non null region of Ω_h is formed simply by intersection

$$\Omega_h^k = \Omega_f^i \cap \Omega_g^j$$

The piecewise linear function that results in each region is:

$$(f \oplus g)^k = f^i \oplus g^j \qquad (4)$$

This introduces ambiguity about the constraints that may support the element of the corresponding A, b. If $a_1 > 0$ and $a_2 > 0$ the sign of $a_1 - a_2$ is unknown. And the opposite happens if $a_1 > 0$ and $a_2 < 0$

We can observe the composition of function, now. If

$$f : R^n \to R^m, g : R^m \to R$$

are piecewise linear functions, the composition :

$$g \circ f : R^n \to R$$

is a piecewise linear function too.

The partition Ω_{gof} of R^n that will be associated to $g \circ f$ is formed with the no empty intersections of regions of Ω_f with the inverse image by f of a region of Ω_g:

$$\Omega_{gof}^k = \Omega_f^i \cap (f^i)^{-1}(\Omega_g^j)$$

and the associated afin function is

$$(g \circ f)^k = g^j \circ f^i$$

If $prod = f * g$ is defined with f, g piecewise linear functions. We want to find a piecewise linear function that is a piecewise linear approach of the *prod*. We can define the piecewise linear function

$$\begin{aligned} S &= f + g \\ D &= f - g \end{aligned} \qquad (5)$$

then

$$prod_L = \left\| \begin{array}{ll} g & \text{if } S \geq 0 \text{ and } D \geq 0 \\ f & \text{if } S \geq 0 \text{ and } D < 0 \\ -f & \text{if } S < 0 \text{ and } D \geq 0 \\ -g & \text{if } S < 0 \text{ and } D < 0 \end{array} \right. \qquad (6)$$

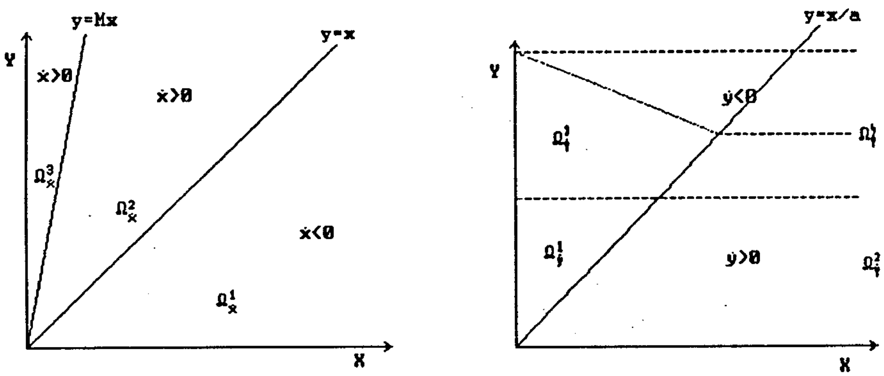

Figure 3: Signs of the field in evey regions

The selected method assures that the zeros of $prod$ are the same that the $prod_L$. The regions of $prod_L$ are obtained following the rule in (6). So to obtain Ω_{prod_L}, each region of $\Omega_S = \Omega_D$ is divided into subregions where $S = f + g$ and where $D = f - g$ preserve their signs in them.

If $div = f/g$ is defined with f, g piecewise linear functions, we can define the piecewise linear function

$$div_L = \left\| \begin{array}{ll} f - g + 1 & \text{if } f \geq 0,\ g > 0 \text{ and } D \geq 0 \\ \frac{f-g}{M} - 1 & \text{if } f \geq 0,\ g > 0 \text{ and } D < 0 \end{array} \right. \qquad (7)$$

where M is a constant big positive number, only one along a model. This function is a piecewise linear approach of the operator division /, in regions where f and g are nonnegatives. If $g = 0$, it is indefinite. That functions can be extended to every values of f and g by use of signs rule: $div_L(f,g) = sign(f * g)div_L(|f|,|g|)$.

A piecewise linear dynamical system can be obtained from a qualitative model applying the rules above in the order deduced from the model.

4 Applications to the example

From the qualitative model above a piecewise linear dynamical system is obtained. In the figure 3 is shown the regions that details the dynamical system obtained. In each regions is shown the sign of the functions and the line where the functions change theirs signs. The linear forms that define the system obtained are given in the appendix.

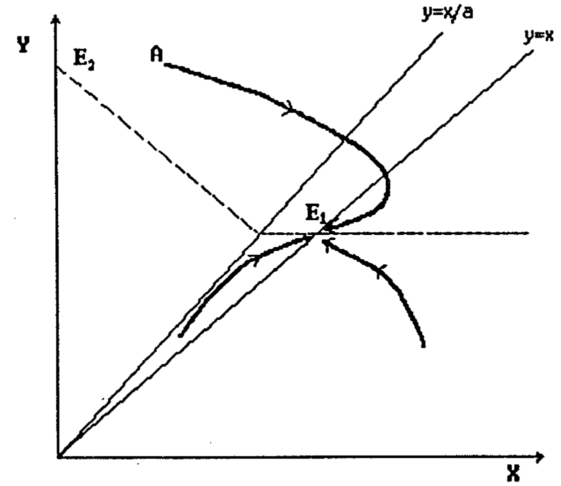

Figure 4: Trajectories in the phase plane

The application of the qualitative analysis to the resultant dynamical systems, provides us the relevant information from the qualitative point of view. A global analysis can be performed to obtain conclusions about the transitory behaviour of the model. So the phase plane with the different trajectories are shown in figure 4. It can be observed that this system has two singular points: a stable (E_1) and another unstable (E_2).

From figure 3 it is possible to conclude that starting at the point A in the phase plane the system will evolve until to reach the point attractor E_2. If the trajectories cross the line $y = x$ then the sign of $\dot{x}$ changes and the behaviour of x will show a maximum. The relative value of the parameters n and b will determine if the trajectory cross that line and so if the behaviour of x has a maximum more or les high. If $n \gg b$ then a big maximum appears. If $n \ll b$ then the maximum in the behaviour does not appear.

The above analysis was performed considering the very short time of delay. So z was substituted by $f_2(x/y)$. The analysis can be made with larger time of delay. In this case the phase plane is R^3 and the conclusions are more difficult to present in a graphical point of view, but they are similar to the former. It can be remarked that if the time of delay increases the maximum of x is higher.

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6 Appendix

The piecewise linear dynamical system obtained is the following:

$$\dot{x} = \left\| \begin{array}{l} \text{Regions} \\ \Omega^1_{\dot{x}} = \{(x,y,z)|y-x \le 0\} \\ \Omega^2_{\dot{x}} = \{(x,y,z)|y-x \ge 0, y-Mx \le 0\} \\ \Omega^3_{\dot{x}} = \{(x,y,z)|y-Mx \ge 0\} \\ \text{Afin Functions} \\ \dot{x}^1 \equiv -2n(x-y) \\ \dot{x}^2 \equiv -2n\frac{x-y}{M} \\ \dot{x}^3 \equiv nx \end{array} \right.$$

$$\dot{y} = \left\| \begin{array}{l} \text{Regions} \\ \Omega^1_{\dot{y}} = \{(x,y,z)|0 \le y \le 0.5,\ y-x/a \ge 0\} \\ \Omega^2_{\dot{y}} = \{(x,y,z)|0 \le y \le 0.5,\ y-x/a \le 0\} \\ \Omega^3_{\dot{y}} = \{(x,y,z)|y \ge 0.5,\ y-x/a \ge 0\} \\ \Omega^4_{\dot{y}} = \{(x,y,z)|y \le 0.5,\ y-x/a \le 0\} \\ \text{Afin Functions} \\ \dot{y}^1 \equiv b(0.5+y-x/a) \\ \dot{y}^2 \equiv 0.5b \\ \dot{y}^3 \equiv b(2-2y-x/a) \\ \dot{y}^4 \equiv b(2-3y) \end{array} \right.$$