On Hasse–Schmidt Derivations: The Action of Substitution Maps

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Dedicated to Antonio Campillo on the ocassion of his 65th birthday

Abstract We study the action of substitution maps between power series rings as an additional algebraic structure on the groups of Hasse–Schmidt derivations. This structure appears as a counterpart of the module structure on classical derivations.

1 Introduction

For any commutative algebra A over a commutative ring k, the set $\operatorname{Der}_k(A)$ of k-derivations of A is an ubiquous object in Commutative Algebra and Algebraic Geometry. It carries an A-module structure and a k-Lie algebra structure. Both structures give rise to a *Lie-Rinehart algebra* structure over (k, A). The k-derivations of A are contained in the filtered ring of k-linear differential operators $\mathscr{D}_{A/k}$, whose graded ring is commutative and we obtain a canonical map of graded A-algebras

$$\tau : \operatorname{Sym}_A \operatorname{Der}_k(A) \longrightarrow \operatorname{gr} \mathscr{D}_{A/k}.$$

If $\mathbb{Q} \subset k$ and $\operatorname{Der}_k(A)$ is a finitely generated projective A-module, the map τ is an isomorphism ([9, Corollary 2.17]) and we can deduce that the ring $\mathscr{D}_{A/k}$ is the enveloping algebra of the Lie-Rinehart algebra $\operatorname{Der}_k(A)$ (cf. [11, Proposition 2.1.2.11]).

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If we are not in characteristic 0, even if A is "smooth" (in some sense) over k, e.g. A is a polynomial or a power series ring with coefficients in k, the map τ has no chance to be an isomorphism.

In [9] we have proved that, if we denote by $\operatorname{Ider}_k(A) \subset \operatorname{Der}_k(A)$ the A-module of *integrable derivations* in the sense of Hasse–Schmidt (see Definition 11), then there is a canonical map of graded A-algebras

$$\vartheta : \Gamma_A \operatorname{Ider}_k(A) \longrightarrow \operatorname{gr} \mathscr{D}_{A/k},$$

where $\Gamma_A(-)$ denotes the *divided power algebra* functor, such that:

(i) $\tau = \vartheta$ when $\mathbb{Q} \subset k$ (in that case $\operatorname{Ider}_k(A) = \operatorname{Der}_k(A)$ and $\Gamma_A = \operatorname{Sym}_A$).
(ii) ϑ is an isomorphism whenever $\operatorname{Ider}_k(A) = \operatorname{Der}_k(A)$ and $\operatorname{Der}_k(A)$ is a finitely generated projective A-module.

The above result suggests an idea: under the "smoothness" hypothesis (ii), can be the ring $\mathscr{D}_{A/k}$ and their modules functorially reconstructed from Hasse–Schmidt derivations? To tackle it, we first need to explore the algebraic structure of Hasse–Schmidt derivations.

Hasse–Schmidt derivations of length $m \geq 1$ form a group, non-abelian for $m \geq 2$, which coincides with the (abelian) additive group of usual derivations $\operatorname{Der}_k(A)$ for $m = 1$. But $\operatorname{Der}_k(A)$ has also an A-module structure and a natural questions arises: Do Hasse–Schmidt derivations of any length have some natural structure extending the A-module structure of $\operatorname{Der}_k(A)$ for length $= 1$?

This paper is devoted to study the action of *substitution maps* (between power series rings) on Hasse–Schmidt derivations as an answer to the above question. This action plays a key role in [12].

Now let us comment on the content of the paper.

In Sect. 2 we have gathered, due to the lack of convenient references, some basic facts and constructions about rings of formal power series in an arbitrary number of variables with coefficients in a non-necessarily commutative ring. In the case of a finite number of variables many results and proofs become simpler, but we need the infinite case in order to study ∞-variate Hasse-Schmidt derivations later.

Sections 3 and 4 are devoted to the study of substitution maps between power series rings and their action on power series rings with coefficients on a (bi)module.

In Sect. 5 we study multivariate (possibly ∞-variate) Hasse–Schmidt derivations. They are a natural generalization of usual Hasse–Schmidt derivations and they provide a convenient framework to deal with Hasse–Schmidt derivations.

In Sect. 6 we see how substitution maps act on Hasse–Schmidt derivations and we study some compatibilities on this action with respect to the group structure.

In Sect. 7 we show how the action of substitution maps allows us to express any HS-derivation in terms of a fixed one under some natural hypotheses. This result generalizes Theorem 2.8 in [3] and provides a conceptual proof of it.

2 Rings and (Bi)modules of Formal Power Series

From now on R will be a ring, k will be a commutative ring and A a commutative k-algebra. A general reference for some of the constructions and results of this section is [2, §4].

Let $\mathbf{s}$ be a set and consider the free commutative monoid $\mathbb{N}^{(\mathbf{s})}$ of maps $\alpha : \mathbf{s} \to \mathbb{N}$ such that the set $\operatorname{supp} \alpha := \{s \in \mathbf{s} \mid \alpha(s) \neq 0\}$ is finite. If $\alpha \in \mathbb{N}^{(\mathbf{s})}$ and $s \in \mathbf{s}$ we will write α_s instead of $\alpha(s)$. The elements of the canonical basis of $\mathbb{N}^{(\mathbf{s})}$ will be denoted by $\mathbf{s}^t$, $t \in \mathbf{s}$: $\mathbf{s}^t_u = \delta_{tu}$ for $t, u \in \mathbf{s}$. For each $\alpha \in \mathbb{N}^{(\mathbf{s})}$ we have $\alpha = \sum_{t \in \mathbf{s}} \alpha_t \mathbf{s}^t$.

The monoid $\mathbb{N}^{(\mathbf{s})}$ is endowed with a natural partial ordering. Namely, for $\alpha, \beta \in \mathbb{N}^{(\mathbf{s})}$, we define

$$\alpha \leq \beta \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \exists \gamma \in \mathbb{N}^{(\mathbf{s})} \text{ such that } \beta = \alpha + \gamma \quad \Leftrightarrow \quad \alpha_s \leq \beta_s \quad \forall s \in \mathbf{s}.$$

Clearly, $t \in \operatorname{supp} \alpha \Leftrightarrow \mathbf{s}^t \leq \alpha$. The partial ordered set $(\mathbb{N}^{(\mathbf{s})}, \leq)$ is a directed ordered set: for any $\alpha, \beta \in \mathbb{N}^{(\mathbf{s})}$, $\alpha, \beta \leq \alpha \vee \beta$ where $(\alpha \vee \beta)_t := \max\{\alpha_t, \beta_t\}$ for all $t \in \mathbf{s}$. We will write $\alpha < \beta$ when $\alpha \leq \beta$ and $\alpha \neq \beta$.

For a given $\beta \in \mathbb{N}^{(\mathbf{s})}$ the set of $\alpha \in \mathbb{N}^{(\mathbf{s})}$ such that $\alpha \leq \beta$ is finite. We define $|\alpha| := \sum_{s \in \mathbf{s}} \alpha_s = \sum_{s \in \operatorname{supp} \alpha} \alpha_s \in \mathbb{N}$. If $\alpha \leq \beta$ then $|\alpha| \leq |\beta|$. Moreover, if $\alpha \leq \beta$ and $|\alpha| = |\beta|$, then $\alpha = \beta$. The $\alpha \in \mathbb{N}^{(\mathbf{s})}$ with $|\alpha| = 1$ are exactly the elements $\mathbf{s}^t$, $t \in \mathbf{s}$, of the canonical basis.

A *formal power series* in $\mathbf{s}$ with coefficients in R is a formal expression $\sum_{\alpha \in \mathbb{N}^{(\mathbf{s})}} r_\alpha \mathbf{s}^\alpha$ with $r_\alpha \in R$ and $\mathbf{s}^\alpha = \prod_{s \in \mathbf{s}} s^{\alpha_s} = \prod_{s \in \operatorname{supp} \alpha} s^{\alpha_s}$. Such a formal expression is uniquely determined by the family of coefficients a_α, $\alpha \in \mathbb{N}^{(\mathbf{s})}$.

If $r = \sum_{\alpha \in \mathbb{N}^{(\mathbf{s})}} r_\alpha \mathbf{s}^\alpha$ and $r' = \sum_{\alpha \in \mathbb{N}^{(\mathbf{s})}} r'_\alpha \mathbf{s}^\alpha$ are two formal power series in $\mathbf{s}$ with coefficients in R, their sum and their product are defined in the usual way

$$r + r' := \sum_{\alpha \in \mathbb{N}^{(\mathbf{s})}} S_\alpha \mathbf{s}^\alpha, \quad S_\alpha := r_\alpha + r'_\alpha,$$

$$r r' := \sum_{\alpha \in \mathbb{N}^{(\mathbf{s})}} P_\alpha \mathbf{s}^\alpha, \; P_\alpha := \sum_{\beta + \gamma = \alpha} r_\beta r'_\gamma.$$

The set of formal power series in $\mathbf{s}$ with coefficients in R endowed with the above internal operations is a ring called the *ring of formal power series in* $\mathbf{s}$ *with coefficients in* R and is denoted by $R[[\mathbf{s}]]$. It contains the polynomial ring $R[\mathbf{s}]$ (and so the ring R) and all the monomials $\mathbf{s}^\alpha$ are in the center of $R[[\mathbf{s}]]$. There is a natural ring epimorphism, that we call the *augmentation*, given by

$$\sum_{\alpha \in \mathbb{N}^{(\mathbf{s})}} r_\alpha \mathbf{s}^\alpha \in R[[\mathbf{s}]] \longmapsto r_0 \in R, \tag{1}$$

which is a retraction of the inclusion $R \subset R[[\mathbf{s}]]$. Clearly, the ring $R[[\mathbf{s}]]$ is commutative if and only if R is commutative and $R^{\text{opp}}[[\mathbf{s}]] = R[[\mathbf{s}]]^{\text{opp}}$.

Any ring homomorphism $f : R \to R'$ induces a ring homomorphism

$$\overline{f} : \sum_{\alpha\in\mathbb{N}^{(\mathbf{s})}} r_\alpha \mathbf{s}^\alpha \in R[[\mathbf{s}]] \longmapsto \sum_{\alpha\in\mathbb{N}^{(\mathbf{s})}} f(r_\alpha)\mathbf{s}^\alpha \in R'[[\mathbf{s}]], \qquad (2)$$

and clearly the correspondences $R \mapsto R[[\mathbf{s}]]$ and $f \mapsto \overline{f}$ define a functor from the category of rings to itself. If $\mathbf{s} = \emptyset$, then $R[[\mathbf{s}]] = R$ and the above functor is the identity.

Definition 1 A *k-algebra over A* is a (non-necessarily commutative) k-algebra R endowed with a map of k-algebras $\iota : A \to R$. A map between two k-algebras $\iota : A \to R$ and $\iota' : A \to R'$ over A is a map $g : R \to R'$ of k-algebras such that $\iota' = g \circ \iota$.

If R is a k-algebra (over A), then $R[[\mathbf{s}]]$ is also a $k[[\mathbf{s}]]$-algebra (over $A[[\mathbf{s}]]$).

If M is an $(A; A)$-bimodule, we define in a completely similar way the set of formal power series in $\mathbf{s}$ with coefficients in M, denoted by $M[[\mathbf{s}]]$. It carries an addition $+$, for which it is an abelian group, and left and right products by elements of $A[[\mathbf{s}]]$. With these operations $M[[\mathbf{s}]]$ becomes an $(A[[\mathbf{s}]]; A[[\mathbf{s}]])$-bimodule containing the polynomial $(A[\mathbf{s}]; A[\mathbf{s}])$-bimodule $M[\mathbf{s}]$. There is also a natural *augmentation* $M[[\mathbf{s}]] \to M$ which is a section of the inclusion $M \subset M[\mathbf{s}]$ and $M^{\mathrm{opp}}[[\mathbf{s}]] = M[[\mathbf{s}]]^{\mathrm{opp}}$. If $\mathbf{s} = \emptyset$, then $M[[\mathbf{s}]] = M$.

The *support* of a series $m = \sum_\alpha m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]$ is $\mathrm{supp}(x) := \{\alpha \in \mathbb{N}^{(\mathbf{s})} \mid m_\alpha \neq 0\} \subset \mathbb{N}^{(\mathbf{s})}$. It is clear that $m = 0 \Leftrightarrow \mathrm{supp}(m) = \emptyset$. The *order* of a non-zero series $m = \sum_\alpha m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]$ is $\mathrm{ord}(m) := \min\{|\alpha| \mid \alpha \in \mathrm{supp}(m)\} \in \mathbb{N}$. If $m = 0$ we define $\mathrm{ord}(0) = \infty$. It is clear that for $a \in A[[\mathbf{s}]]$ and $m, m' \in M[[\mathbf{s}]]$ we have $\mathrm{supp}(m + m') \subset \mathrm{supp}(m) \cup \mathrm{supp}(m')$, $\mathrm{supp}(am), \mathrm{supp}(ma) \subset \mathrm{supp}(m) + \mathrm{supp}(a)$, $\mathrm{ord}(m + m') \geq \min\{\mathrm{ord}(m), \mathrm{ord}(m')\}$ and $\mathrm{ord}(am), \mathrm{ord}(ma) \geq \mathrm{ord}(a) + \mathrm{ord}(m)$. Moreover, if $\mathrm{ord}(m') > \mathrm{ord}(m)$, then $\mathrm{ord}(m + m') = \mathrm{ord}(m)$.

Any $(A; A)$-linear map $h : M \to M'$ between two $(A; A)$-bimodules induces in an obvious way and $(A[[\mathbf{s}]]; A[[\mathbf{s}]])$-linear map

$$\overline{h} : \sum_{\alpha\in\mathbb{N}^{(\mathbf{s})}} m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]] \longmapsto \sum_{\alpha\in\mathbb{N}^{(\mathbf{s})}} h(m_\alpha)\mathbf{s}^\alpha \in M'[[\mathbf{s}]], \qquad (3)$$

and clearly the correspondences $M \mapsto M[[\mathbf{s}]]$ and $h \mapsto \overline{h}$ define a functor from the category of $(A; A)$-bimodules to the category $(A[[\mathbf{s}]]; A[[\mathbf{s}]])$-bimodules.

For each $\beta \in M^{(\mathbf{s})}$, let us denote by $\mathfrak{n}_\beta^M(\mathbf{s})$ the subset of $M[[\mathbf{s}]]$ whose elements are the formal power series $\sum m_\alpha \mathbf{s}^\alpha$ with $m_\alpha = 0$ for all $\alpha \leq \beta$. One has $\mathfrak{n}_\beta^M(\mathbf{s}) \subset \mathfrak{n}_\gamma^M(\mathbf{s})$ whenever $\gamma \leq \beta$, and $\mathfrak{n}_{\alpha\vee\beta}^M(\mathbf{s}) \subset \mathfrak{n}_\alpha^M(\mathbf{s}) \cap \mathfrak{n}_\beta^M(\mathbf{s})$.

It is clear that the $\mathfrak{n}_\beta^M(\mathbf{s})$ are sub-bimodules of $M[[\mathbf{s}]]$ and $\mathfrak{n}_\beta^A(\mathbf{s})M[[\mathbf{s}]] \subset \mathfrak{n}_\beta^M(\mathbf{s})$ and $M[[\mathbf{s}]]\mathfrak{n}_\beta^A(\mathbf{s}) \subset \mathfrak{n}_\beta^M(\mathbf{s})$. For $\beta = 0$, $\mathfrak{n}_0^M(\mathbf{s})$ is the kernel of the augmentation $M[[\mathbf{s}]] \to M$.

In the case of a ring R, the $\mathfrak{n}_\beta^R(\mathbf{s})$ are two-sided ideals of $R[[\mathbf{s}]]$, and $\mathfrak{n}_0^R(\mathbf{s})$ is the kernel of the augmentation $R[[\mathbf{s}]] \to R$.

We will consider $R[[\mathbf{s}]]$ as a topological ring with $\{\mathfrak{n}_\beta^R(\mathbf{s}), \beta \in \mathbb{N}^{(\mathbf{s})}\}$ as a fundamental system of neighborhoods of 0. We will also consider $M[[\mathbf{s}]]$ as a topological $(A[[\mathbf{s}]]; A[[\mathbf{s}]])$-bimodule with $\{\mathfrak{n}_\beta^M(\mathbf{s}), \beta \in \mathbb{N}^{(\mathbf{s})}\}$ as a fundamental system of neighborhoods of 0 for both, a topological left $A[[\mathbf{s}]]$-module structure and a topological right $A[[\mathbf{s}]]$-module structure. If $\mathbf{s}$ is finite, then $\mathfrak{n}_\beta^M(\mathbf{s}) = \sum_{s\in\mathbf{s}} s^{\beta_s+1} M[[\mathbf{s}]] = \sum_{s\in\mathbf{s}} M[[\mathbf{s}]] s^{\beta_s+1}$ and so the above topologies on $R[[\mathbf{s}]]$, and so on $A[[\mathbf{s}]]$, and on $M[[\mathbf{s}]]$ coincide with the $\langle\mathbf{s}\rangle$-adic topologies.

Let us denote by $\mathfrak{n}_\beta^M(\mathbf{s})^{\mathbf{c}} \subset M[\mathbf{s}]$ the intersection of $\mathfrak{n}_\beta^M(\mathbf{s})$ with $M[\mathbf{s}]$, i.e. the subset of $M[\mathbf{s}]$ whose elements are the finite sums $\sum m_\alpha \mathbf{s}^\alpha$ with $m_\alpha = 0$ for all $\alpha \leq \beta$. It is clear that the natural map $R[\mathbf{s}]/\mathfrak{n}_\beta^R(\mathbf{s})^{\mathbf{c}} \longrightarrow R[[\mathbf{s}]]/\mathfrak{n}_\beta^R(\mathbf{s})$ is an isomorphism of rings and the quotient $R[[\mathbf{s}]]/\mathfrak{n}_\beta^R(\mathbf{s})$ is a finitely generated free left (and right) R-module with basis the set of the classes of monomials $\mathbf{s}^\alpha$, $\alpha \leq \beta$.

In the same vein, the $\mathfrak{n}_\beta^M(\mathbf{s})^{\mathbf{c}}$ are sub-$(A[\mathbf{s}]; A[\mathbf{s}])$-bimodules of $M[\mathbf{s}]$ and the natural map $M[\mathbf{s}]/\mathfrak{n}_\beta^M(\mathbf{s})^{\mathbf{c}} \longrightarrow M[[\mathbf{s}]]/\mathfrak{n}_\beta^M(\mathbf{s})$ is an isomorphism of $(A[\mathbf{s}]/\mathfrak{n}_\beta^A(\mathbf{s})^{\mathbf{c}}; A[\mathbf{s}]/\mathfrak{n}_\beta^A(\mathbf{s})^{\mathbf{c}})$-bimodules. Moreover, we have a commutative diagram of natural $\mathbb{Z}$-linear isomorphisms

$$\begin{array}{ccccc}
A[\mathrm{s}]/\mathfrak{n}_\beta^A(\mathrm{s})^{\mathrm{c}} \otimes_A M & \xrightarrow[\simeq]{\varrho} & M[\mathrm{s}]/\mathfrak{n}_\beta^M(\mathrm{s})^{\mathrm{c}} & \xleftarrow[\simeq]{\lambda} & M \otimes_A A[\mathrm{s}]/\mathfrak{n}_\beta^A(\mathrm{s})^{\mathrm{c}} \\
{\scriptstyle \mathrm{nat.}\otimes\mathrm{Id}}\downarrow\simeq & & \downarrow\simeq & & \simeq\downarrow{\scriptstyle \mathrm{Id}\otimes\mathrm{nat.}} \\
A[[\mathrm{s}]]/\mathfrak{n}_\beta^A(\mathrm{s}) \otimes_A M & \xrightarrow[\simeq]{\varrho'} & M[[\mathrm{s}]]/\mathfrak{n}_\beta^M(\mathrm{s}) & \xleftarrow[\simeq]{\lambda'} & M \otimes_A A[[\mathrm{s}]]/\mathfrak{n}_\beta^A(\mathrm{s})
\end{array} \tag{4}$$

where ϱ (resp. ϱ') is an isomorphism of $(A[\mathbf{s}]/\mathfrak{n}_\beta^A(\mathbf{s})^{\mathbf{c}}; A)$-bimodules (resp. of $(A[[\mathbf{s}]]/\mathfrak{n}_\beta^A(\mathbf{s}); A)$-bimodules) and λ (resp. λ') is an isomorphism of bimodules over $(A; A[\mathbf{s}]/\mathfrak{n}_\beta^A(\mathbf{s})^{\mathbf{c}})$(resp. over $(A; A[[\mathbf{s}]]/\mathfrak{n}_\beta^A(\mathbf{s}))$.

It is clear that the natural map

$$R[[\mathbf{s}]] \longrightarrow \varprojlim_{\beta\in\mathbb{N}^{(\mathbf{s})}} R[[\mathbf{s}]]/\mathfrak{n}_\beta^R(\mathbf{s}) \equiv \varprojlim_{\beta\in\mathbb{N}^{(\mathbf{s})}} R[\mathbf{s}]/\mathfrak{n}_\beta^R(\mathbf{s})^{\mathbf{c}}$$

is an isomorphism of rings and so $R[[\mathbf{s}]]$ is complete (hence, separated). Moreover, $R[[\mathbf{s}]]$ appears as the completion of the polynomial ring $R[\mathbf{s}]$ endowed with the topology with $\{\mathfrak{n}_\beta^R(\mathbf{s})^{\mathbf{c}}, \beta \in \mathbb{N}^{(\mathbf{s})}\}$ as a fundamental system of neighborhoods of 0.

Similarly, the natural map

$$M[[\mathbf{s}]] \longrightarrow \varprojlim_{\beta\in\mathbb{N}^{(\mathbf{s})}} M[[\mathbf{s}]]/\mathfrak{n}_\beta^M(\mathbf{s}) \equiv \varprojlim_{\beta\in\mathbb{N}^{(\mathbf{s})}} M[\mathbf{s}]/\mathfrak{n}_\beta^M(\mathbf{s})^{\mathbf{c}}$$

is an isomorphism of $(A[[\mathbf{s}]]; A[[\mathbf{s}]])$-bimodules, and so $M[[\mathbf{s}]]$ is complete (hence, separated). Moreover, $M[[\mathbf{s}]]$ appears as the completion of the bimodule $M[\mathbf{s}]$ over

$(A[\mathbf{s}]; A[\mathbf{s}])$ endowed with the topology with $\{\mathfrak{n}_\beta^M(\mathbf{s})^{\mathbf{c}}, \beta \in \mathbb{N}^{(\mathbf{s})}\}$ as a fundamental system of neighborhoods of 0.

Since the subsets $\{\alpha \in \mathbb{N}^{(\mathbf{s})} \mid \alpha \leq \beta\}$, $\beta \in \mathbb{N}^{(\mathbf{s})}$, are cofinal among the finite subsets of $\mathbb{N}^{(\mathbf{s})}$, the additive isomorphism

$$\sum_{\alpha \in \mathbb{N}^{(\mathbf{s})}} m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]] \mapsto \{m_\alpha\}_{\alpha \in \mathbb{N}^{(\mathbf{s})}} \in M^{\mathbb{N}^{(\mathbf{s})}}$$

is a homeomorphism, where $M^{\mathbb{N}^{(\mathbf{s})}}$ is endowed with the product of discrete topologies on each copy of M. In particular, any formal power series $\sum m_\alpha \mathbf{s}^\alpha$ is the limit of its finite partial sums $\sum_{\alpha \in F} m_\alpha \mathbf{s}^\alpha$, over the filter of finite subsets $F \subset \mathbb{N}^{(\mathbf{s})}$.

Since the quotients $A[[\mathbf{s}]]/\mathfrak{n}_\beta^A(\mathbf{s})$ are free A-modules, we have exact sequences

$$0 \longrightarrow \mathfrak{n}_\beta^A(\mathbf{s}) \otimes_A M \longrightarrow A[[\mathbf{s}]] \otimes_A M \longrightarrow \frac{A[[\mathbf{s}]]}{\mathfrak{n}_\beta^A(\mathbf{s})} \otimes_A M \longrightarrow 0$$

and the tensor product $A[[\mathbf{s}]] \otimes_A M$ is a topological left $A[[\mathbf{s}]]$-module with $\{\mathfrak{n}_\beta^A(\mathbf{s}) \otimes_A M, \beta \in \mathbb{N}^{(\mathbf{s})}\}$ as a fundamental system of neighborhoods of 0. The natural $(A[[\mathbf{s}]]; A)$-linear map

$$A[[\mathbf{s}]] \otimes_A M \longrightarrow M[[\mathbf{s}]]$$

is continuous and, if we denote by $A[[\mathbf{s}]]\widehat{\otimes}_A M$ the completion of $A[[\mathbf{s}]] \otimes_A M$, the induced map $A[[\mathbf{s}]]\widehat{\otimes}_A M \longrightarrow M[[\mathbf{s}]]$ is an isomorphism of $(A[[\mathbf{s}]]; A)$-bimodules, since we have natural $(A[[\mathbf{s}]]; A)$-linear isomorphisms

$$(A[[\mathbf{s}]] \otimes_A M) \,/\, \left(\mathfrak{n}_\beta^A(\mathbf{s}) \otimes_A M\right) \simeq \left(A[[\mathbf{s}]]/\mathfrak{n}_\beta^A(\mathbf{s})\right) \otimes_A M \simeq M[[\mathbf{s}]]/\mathfrak{n}_\beta^M(\mathbf{s})$$

for $\beta \in \mathbb{N}^{(\mathbf{s})}$, and so

$$A[[\mathbf{s}]]\widehat{\otimes}_A M = \varprojlim_{\beta \in \mathbb{N}^{(\mathbf{s})}} \left(\frac{A[[\mathbf{s}]] \otimes_A M}{\mathfrak{n}_\beta^A(\mathbf{s}) \otimes_A M}\right) \simeq \varprojlim_{\beta \in \mathbb{N}^{(\mathbf{s})}} \left(\frac{M[[\mathbf{s}]]}{\mathfrak{n}_\beta^M(\mathbf{s})}\right) \simeq M[[\mathbf{s}]]. \tag{5}$$

Similarly, the natural $(A; A[[\mathbf{s}]])$-linear map $M \otimes_A A[[\mathbf{s}]] \to M[[\mathbf{s}]]$ induces an isomorphism $M\widehat{\otimes}_A A[[\mathbf{s}]] \xrightarrow{\sim} M[[\mathbf{s}]]$ of $(A; A[[\mathbf{s}]])$-bimodules.

If $h : M \to M'$ is an $(A; A)$-linear map between two $(A; A)$-bimodules, the induced map $\overline{h} : M[[\mathbf{s}] \to M'[[\mathbf{s}]$ (see (3)) is clearly continuous and there is a commutative diagram

$$\begin{array}{ccccc}
A[[\mathbf{s}]]\widehat{\otimes}_A M & \xrightarrow{\simeq} & M[[\mathbf{s}]] & \xleftarrow{\simeq} & M\widehat{\otimes}_A A[[\mathbf{s}]] \\
{\scriptstyle \mathrm{Id}\widehat{\otimes} h}\downarrow & & {\scriptstyle \overline{h}}\downarrow & & {\scriptstyle h\widehat{\otimes}\mathrm{Id}}\downarrow \\
A[[\mathbf{s}]]\widehat{\otimes}_A M' & \xrightarrow{\simeq} & M'[[\mathbf{s}]] & \xleftarrow{\simeq} & M'\widehat{\otimes}_A A[[\mathbf{s}]].
\end{array}$$

Similarly, for any ring homomorphism $f : R \to R'$, the induced ring homomorphism $\overline{f} : R[[\mathbf{s}]] \to R'[[\mathbf{s}]]$ is also continuous.

Definition 2 We say that a subset $\Delta \subset \mathbb{N}^{(\mathbf{s})}$ is an *ideal* of $\mathbb{N}^{(\mathbf{s})}$ (resp. a *co-ideal* of $\mathbb{N}^{(\mathbf{s})}$) if whenever $\alpha \in \Delta$ and $\alpha \leq \alpha'$ (resp. $\alpha' \leq \alpha$), then $\alpha' \in \Delta$.

It is clear that Δ is an ideal if and only if its complement Δ^c is a co-ideal, and that the union and the intersection of any family of ideals (resp. of co-ideals) of $\mathbb{N}^{(\mathbf{s})}$ is again an ideal (resp. a co-ideal) of $\mathbb{N}^{(\mathbf{s})}$. Examples of ideals (resp. of co-ideals) of $\mathbb{N}^{(\mathbf{s})}$ are the $\beta + \mathbb{N}^{(\mathbf{s})}$ (resp. the $\mathfrak{n}_\beta(\mathbf{s}) := \{\alpha \in \mathbb{N}^{(\mathbf{s})} \mid \alpha \leq \beta\}$) with $\beta \in \mathbb{N}^{(\mathbf{s})}$. The $\mathfrak{t}_m(\mathbf{s}) := \{\alpha \in \mathbb{N}^{(\mathbf{s})} \mid |\alpha| \leq m\}$ with $m \geq 0$ are also co-ideals. Actually, a subset $\Delta \subset \mathbb{N}^{(\mathbf{s})}$ is an ideal (resp. a co-ideal) if and only if $\Delta = \cup_{\beta\in\Delta} \left(\beta + \mathbb{N}^{(\mathbf{s})}\right) = \Delta + \mathbb{N}^{(\mathbf{s})}$ (resp. $\Delta = \cup_{\beta\in\Delta} \mathfrak{n}_\beta(\mathbf{s})$).

We say that a co-ideal $\Delta \subset \mathbb{N}^{(\mathbf{s})}$ is bounded if there is an integer $m \geq 0$ such that $|\alpha| \leq m$ for all $\alpha \in \Delta$. In other words, a co-ideal $\Delta \subset \mathbb{N}^{(\mathbf{s})}$ is bounded if and only if there is an integer $m \geq 0$ such that $\Delta \subset \mathfrak{t}_m(\mathbf{s})$. Also, a co-ideal $\Delta \subset \mathbb{N}^{(\mathbf{s})}$ is non-empty if and only if $\mathfrak{t}_0(\mathbf{s}) = \mathfrak{n}_0(\mathbf{s}) = \{0\} \subset \Delta$.

For a co-ideal $\Delta \subset \mathbb{N}^{(\mathbf{s})}$ and an integer $m \geq 0$, we denote $\Delta^m := \Delta \cap \mathfrak{t}_m(\mathbf{s})$.

For each co-ideal $\Delta \subset \mathbb{N}^{(\mathbf{s})}$, we denote by Δ_M the sub-$(A[[\mathbf{s}]; A[[\mathbf{s}]])$-bimodule of $M[[\mathbf{s}]]$ whose elements are the formal power series $\sum_{\alpha\in\mathbb{N}^{(\mathbf{s})}} m_\alpha \mathbf{s}^\alpha$ such that $m_\alpha = 0$ whenever $\alpha \in \Delta$. One has

$$\Delta_M = \cdots = \left\{ m \in M[[\mathbf{s}]] \mid \operatorname{supp}(m) \subset \bigcap_{\beta\in\Delta} \mathfrak{n}_\beta(\mathbf{s})^c \right\} =$$

$$\bigcap_{\beta\in\Delta} \left\{ m \in M[[\mathbf{s}]] \mid \operatorname{supp}(m) \subset \mathfrak{n}_\beta(\mathbf{s})^c \right\} = \bigcap_{\beta\in\Delta} \mathfrak{n}_\beta^M(\mathbf{s}),$$

and so Δ_M is closed in $M[[\mathbf{s}]]$. Let $\Delta' \subset \mathbb{N}^{(\mathbf{s})}$ be another co-ideal. We have

$$\Delta_M + \Delta'_M = (\Delta \cap \Delta')_M.$$

If $\Delta \subset \Delta'$, then $\Delta'_M \subset \Delta_M$, and if $a \in \Delta'_A$, $m \in \Delta_M$ we have

$$\operatorname{supp}(am) \subset \operatorname{supp}(a) + \operatorname{supp}(m) \subset \left(\Delta'\right)^c + \Delta^c \subset \left(\Delta'\right)^c \cap \Delta^c = \left(\Delta' \cup \Delta\right)^c,$$

and so $\Delta'_A \Delta_M \subset (\Delta' \cup \Delta)_M$. Is a similar way we obtain $\Delta_M \Delta'_A \subset (\Delta' \cup \Delta)_M$.

Let us denote by $M[[\mathbf{s}]]_\Delta := M[[\mathbf{s}]]/\Delta_M$ endowed with the quotient topology. The elements in $M[[\mathbf{s}]]_\Delta$ are power series of the form

$$\sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha, \quad m_\alpha \in M.$$

It is clear that $M[[\mathbf{s}]]_\Delta$ is a topological $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$-bimodule. A fundamental system of neighborhoods of 0 in $M[[\mathbf{s}]]_\Delta$ consist of

$$\frac{\mathfrak{n}_\beta^M(\mathbf{s}) + \Delta_M}{\Delta_M} = \frac{(\mathfrak{n}_\beta(\mathbf{s}) \cap \Delta)_M}{\Delta_M}, \quad \beta \in \mathbb{N}^{(\mathbf{s})},$$

and since the subsets $\mathfrak{n}_\beta(\mathbf{s}) \cap \Delta$, $\beta \in \mathbb{N}^{(\mathbf{s})}$, are cofinal among the finite subsets of Δ, we conclude that the additive isomorphism

$$\sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]_\Delta \mapsto \{m_\alpha\}_{\alpha\in\Delta} \in M^\Delta$$

is a homeomorphism, where M^Δ is endowed with the product of discrete topologies on each copy of M.

For $\Delta \subset \Delta'$ co-ideals of $\mathbb{N}^{(\mathbf{s})}$, we have natural continuous $(A[[\mathbf{s}]]_{\Delta'}; A[[\mathbf{s}]]_{\Delta'})$-linear projections $\tau_{\Delta'\Delta} : M[[\mathbf{s}]]_{\Delta'} \longrightarrow M[[\mathbf{s}]]_\Delta$, that we also call *truncations*,

$$\tau_{\Delta'\Delta} : \sum_{\alpha\in\Delta'} m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]_{\Delta'} \longmapsto \sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]_\Delta,$$

and continuous $(A; A)$-linear scissions

$$\sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]_\Delta \longmapsto \sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]_{\Delta'}.$$

which are topological immersions.

In particular we have natural continuous $(A; A)$-linear topological embeddings $M[[\mathbf{s}]]_\Delta \hookrightarrow M[[\mathbf{s}]]$ and we define the *support* (resp. the *order*) of any element in $M[[\mathbf{s}]]_\Delta$ as its support (resp. its order) as element of $M[[\mathbf{s}]]$.

We have a bicontinuous isomorphism of $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$-bimodules

$$M[[\mathbf{s}]]_\Delta = \varprojlim_{m\in\mathbb{N}} M[[\mathbf{s}]]_{\Delta^m}.$$

For a ring R, the Δ_R are two-sided closed ideals of $R[[\mathbf{s}]]$, $\Delta_R \Delta'_R \subset (\Delta \cup \Delta')_R$ and we have a bicontinuous ring isomorphism

$$R[[\mathbf{s}]]_\Delta = \varprojlim_{m\in\mathbb{N}} R[[\mathbf{s}]]_{\Delta^m}.$$

When $\mathbf{s}$ is finite, $\mathfrak{t}_m(\mathbf{s})_R$ coincides with the $(m+1)$-power of the two-sided ideal generated by all the variables $s \in \mathbf{s}$.

As in (5) one proves that $A[[\mathbf{s}]]_\Delta \otimes_A M$ (resp. $M \otimes_A A[[\mathbf{s}]]_\Delta$) is endowed with a natural topology in such a way that the natural map $A[[\mathbf{s}]]_\Delta \otimes_A M \to M[[\mathbf{s}]]_\Delta$

(resp. $M \otimes_A A[[\mathbf{s}]]_\Delta \to M[[\mathbf{s}]]_\Delta$) is continuous and gives rise to a $(A[[\mathbf{s}]]_\Delta; A)$-linear (resp. to a $(A; A[[\mathbf{s}]]_\Delta)$-linear) isomorphism

$$A[[\mathbf{s}]]_\Delta \widehat{\otimes}_A M \xrightarrow{\sim} M[[\mathbf{s}]]_\Delta \qquad (\text{resp. } M \widehat{\otimes}_A A[[\mathbf{s}]]_\Delta \xrightarrow{\sim} M[[\mathbf{s}]]_\Delta).$$

If $h : M \to M'$ is an $(A; A)$-linear map between two $(A; A)$-bimodules, the map $\overline{h} : M[[\mathbf{s}]] \to M'[[\mathbf{s}]]$ (see (3)) obviously satisfies $\overline{h}(\Delta_M) \subset \Delta_{M'}$, and so induces another natural $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$-linear continuous map $M[[\mathbf{s}]]_\Delta \to M'[[\mathbf{s}]]_\Delta$, that will be still denoted by $\overline{h}$. We have a commutative diagram

$$\begin{array}{ccccc} A[[\mathbf{s}]]_\Delta \widehat{\otimes}_A M & \xrightarrow{\simeq} & M[[\mathbf{s}]]_\Delta & \xleftarrow{\simeq} & M \widehat{\otimes}_A A[[\mathbf{s}]]_\Delta \\ \downarrow {\scriptstyle \mathrm{Id}\widehat{\otimes} h} & & \downarrow {\scriptstyle \overline{h}} & & \downarrow {\scriptstyle h\widehat{\otimes}\mathrm{Id}} \\ A[[\mathbf{s}]]_\Delta \widehat{\otimes}_A M' & \xrightarrow{\simeq} & M'[[\mathbf{s}]]_\Delta & \xleftarrow{\simeq} & M' \widehat{\otimes}_A A[[\mathbf{s}]]_\Delta. \end{array}$$

Remark 1 In the same way that the correspondences $M \mapsto M[[\mathbf{s}]]$ and $h \mapsto \overline{h}$ define a functor from the category of $(A; A)$-bimodules to the category of $(A[[\mathbf{s}]]; A[[\mathbf{s}]])$-bimodules, we may consider functors $M \mapsto M[[\mathbf{s}]]_\Delta$ and $h \mapsto \overline{h}$ from the category of $(A; A)$-bimodules to the category of $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$-bimodules. We may also consider functors $R \mapsto R[[\mathbf{s}]]_\Delta$ and $f \mapsto \overline{f}$ from the category of rings to itself. Moreover, if R is a k-algebra (over A), then $R[[\mathbf{s}]]_\Delta$ is a $k[[\mathbf{s}]]_\Delta$-algebra (over $A[[\mathbf{s}]]_\Delta$).

Lemma 1 *Under the above hypotheses, Δ_M is the closure of $\Delta_\mathbb{Z} M[[\mathbf{s}]]$.*

Proof Any element in Δ_M is of the form $\sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha$, but $\mathbf{s}^\alpha m_\alpha \in \Delta_\mathbb{Z} M[[\mathbf{s}]]$ whenever $\alpha \in \Delta$ and so it belongs to the closure of $\Delta_\mathbb{Z} M[[\mathbf{s}]]$.

Lemma 2 *Let R be a ring, $\mathbf{s}$ a set and $\Delta \subset \mathbb{N}^{(\mathbf{s})}$ a non-empty co-ideal. The units in $R[[\mathbf{s}]]_\Delta$ are those power series $r = \sum r_\alpha \mathbf{s}^\alpha$ such that r_0 is a unit in R. Moreover, in the special case where $r_0 = 1$, the inverse $r^* = \sum r^*_\alpha \mathbf{s}^\alpha$ of r is given by $r^*_0 = 1$ and*

$$r^*_\alpha = \sum_{d=1}^{|\alpha|} (-1)^d \sum_{\alpha^\bullet \in \mathscr{P}(\alpha,d)} r_{\alpha^1} \cdots r_{\alpha^d} \quad \text{for } \alpha \neq 0,$$

where $\mathscr{P}(\alpha, d)$ is the set of d-uples $\alpha^\bullet = (\alpha^1, \ldots, \alpha^d)$ with $\alpha^i \in \mathbb{N}^{(\mathbf{s})}$, $\alpha^i \neq 0$, and $\alpha^1 + \cdots + \alpha^d = \alpha$.

Proof The proof is standard and it is left to the reader.

Notation 1 *Let R be a ring, $\mathbf{s}$ a set and $\Delta \subset \mathbb{N}^{(\mathbf{s})}$ a non-empty co-ideal. We denote by $\mathscr{U}^\mathbf{s}(R; \Delta)$ the multiplicative sub-group of the units of $R[[\mathbf{s}]]_\Delta$ whose 0-degree coefficient is 1. Clearly, $\mathscr{U}^\mathbf{s}(R; \Delta)^{\mathrm{opp}} = \mathscr{U}^\mathbf{s}(R^{\mathrm{opp}}; \Delta)$. For $\Delta \subset \Delta'$ co-ideals we have $\tau_{\Delta'\Delta}\left(\mathscr{U}^\mathbf{s}(R; \Delta')\right) \subset \mathscr{U}^\mathbf{s}(R; \Delta)$ and the truncation map $\tau_{\Delta'\Delta} : \mathscr{U}^\mathbf{s}(R; \Delta') \to \mathscr{U}^\mathbf{s}(R; \Delta)$ is a group homomorphisms. Clearly, we have*

$$\mathscr{U}^\mathbf{s}(R; \Delta) = \varprojlim_{m\in\mathbb{N}} \mathscr{U}^\mathbf{s}(R; \Delta^m).$$

For any ring homomorphism $f : R \to R'$*, the induced ring homomorphism* $\overline{f} : R[[\mathbf{s}]]_\Delta \to R'[[\mathbf{s}]]_\Delta$ *sends* $\mathscr{U}^{\mathbf{s}}(R; \Delta)$ *into* $\mathscr{U}^{\mathbf{s}}(R'; \Delta)$ *and so it induces natural group homomorphisms* $\mathscr{U}^{\mathbf{s}}(R; \Delta) \to \mathscr{U}^{\mathbf{s}}(R'; \Delta)$.

Definition 3 Let R be a ring, $\mathbf{s}$, $\mathbf{t}$ sets and $\nabla \subset \mathbb{N}^{(\mathbf{s})}$, $\Delta \subset \mathbb{N}^{(\mathbf{t})}$ non-empty co-ideals. For each $r \in R[[\mathbf{s}]]_\nabla$, $r' \in R[[\mathbf{t}]]_\Delta$, the *external product* $r \boxtimes r' \in R[[\mathbf{s} \sqcup \mathbf{t}]]_{\nabla \times \Delta}$ is defined as

$$r \boxtimes r' := \sum_{(\alpha,\beta) \in \nabla \times \Delta} r_\alpha r'_\beta \mathbf{s}^\alpha \mathbf{t}^\beta.$$

Let us notice that the above definition is consistent with the existence of natural isomorphism of $(R; R)$-bimodules $R[[\mathbf{s}]]_\nabla \widehat{\otimes}_R R[[\mathbf{t}]]_\Delta \simeq R[[\mathbf{s} \sqcup \mathbf{t}]]_{\nabla \times \Delta} \simeq R[[\mathbf{t} \sqcup \mathbf{s}]]_{\Delta \times \nabla} \simeq R[[\mathbf{t}]]_\Delta \widehat{\otimes}_R R[[\mathbf{s}]]_\nabla$. Let us also notice that $1 \boxtimes 1 = 1$ and $r \boxtimes r' = (r \boxtimes 1)(1 \boxtimes r')$. Moreover, if $r \in \mathscr{U}^{\mathbf{s}}(R; \nabla)$, $r' \in \mathscr{U}^{\mathbf{t}}(R; \Delta)$, then $r \boxtimes r' \in \mathscr{U}^{\mathbf{s} \sqcup \mathbf{t}}(R; \nabla \times \Delta)$ and $(r \boxtimes r')^* = r'^* \boxtimes r^*$.

Let $k \to A$ be a ring homomorphism between commutative rings, E, F two A-modules, $\mathbf{s}$ a set and $\Delta \subset \mathbb{N}^{(\mathbf{s})}$ a non-empty co-ideal, i.e $\mathfrak{n}_0(\mathbf{s}) = \{0\} \subset \Delta$.

Proposition 1 *Under the above hypotheses, let* $f : E[[\mathbf{s}]]_\Delta \to F[[\mathbf{s}]]_\Delta$ *be a continuous* $k[[\mathbf{s}]]_\Delta$*-linear map. Then, for any co-ideal* $\Delta' \subset \mathbb{N}^{(\mathbf{s})}$ *with* $\Delta' \subset \Delta$ *we have*

$$f\left(\Delta'_E / \Delta_E\right) \subset \Delta'_F / \Delta_F$$

and so there is a unique continuous $k[[\mathbf{s}]]_{\Delta'}$*-linear map* $\overline{f} : E[[\mathbf{s}]]_{\Delta'} \to F[[\mathbf{s}]]_{\Delta'}$ *such that the following diagram is commutative*

$$\begin{array}{ccc} E[[\mathbf{s}]]_\Delta & \xrightarrow{f} & F[[\mathbf{s}]]_\Delta \\ {\scriptstyle nat.}\downarrow & & \downarrow{\scriptstyle nat.} \\ E[[\mathbf{s}]]_{\Delta'} & \xrightarrow{\overline{f}} & F[[\mathbf{s}]]_{\Delta'}. \end{array}$$

Proof It is a straightforward consequence of Lemma 1.

Notation 2 *Under the above hypotheses, the set of all continuous* $k[[\mathbf{s}]]_\Delta$*-linear maps from* $E[[\mathbf{s}]]_\Delta$ *to* $F[[\mathbf{s}]]_\Delta$ *will be denoted by*

$$\operatorname{Hom}^{\text{top}}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta).$$

It is an $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$*-bimodule central over* $k[[\mathbf{s}]]_\Delta$*. For any co-ideals* $\Delta' \subset \Delta \subset \mathbb{N}^{(\mathbf{s})}$*, Proposition 1 provides a natural* $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$*-linear map*

$$\operatorname{Hom}^{\text{top}}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta) \longrightarrow \operatorname{Hom}^{\text{top}}_{k[[\mathbf{s}]]_{\Delta'}}(E[[\mathbf{s}]]_{\Delta'}, F[[\mathbf{s}]]_{\Delta'}).$$

For $E = F$, $\mathrm{End}^{\mathrm{top}}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta)$ *is a* $k[[\mathbf{s}]]_\Delta$*-algebra over* $A[[\mathbf{s}]]_\Delta$.

1. For each $r = \sum_\beta r_\beta \mathbf{s}^\beta \in \mathrm{Hom}_k(E, F)[[\mathbf{s}]]_\Delta$ we define $\widetilde{r} : E[[\mathbf{s}]]_\Delta \to F[[\mathbf{s}]]_\Delta$ by

$$\widetilde{r}\left(\sum_{\alpha\in\Delta} e_\alpha \mathbf{s}^\alpha\right) := \sum_{\alpha\in\Delta}\left(\sum_{\beta+\gamma=\alpha} r_\beta(e_\gamma)\right)\mathbf{s}^\alpha,$$

which is obviously a continuous $k[[\mathbf{s}]]_\Delta$-linear map.

Let us notice that $\widetilde{r} = \sum_\beta \mathbf{s}^\beta \widetilde{r_\beta}$. It is clear that the map

$$r \in \mathrm{Hom}_k(E, F)[[\mathbf{s}]]_\Delta \longmapsto \widetilde{r} \in \mathrm{Hom}^{\mathrm{top}}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta) \tag{6}$$

is $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$-linear.

If $f : E[[\mathbf{s}]]_\Delta \to F[[\mathbf{s}]]_\Delta$ is a continuous $k[[\mathbf{s}]]_\Delta$-linear map, let us denote by $f_\alpha : E \to F$, $\alpha \in \Delta$, the k-linear maps defined by

$$f(e) = \sum_{\alpha\in\Delta} f_\alpha(e)\mathbf{s}^\alpha, \quad \forall e \in E.$$

If $g : E \to F[[\mathbf{s}]]_\Delta$ is a k-linear map, we denote by $g^e : E[[\mathbf{s}]]_\Delta \to F[[\mathbf{s}]]_\Delta$ the unique continuous $k[[\mathbf{s}]]_\Delta$-linear map extending g to $E[[\mathbf{s}]]_\Delta = k[[\mathbf{s}]]_\Delta \widehat{\otimes}_k E$. It is given by

$$g^e\left(\sum_\alpha e_\alpha \mathbf{s}^\alpha\right) := \sum_\alpha g(e_\alpha)\mathbf{s}^\alpha.$$

We have a $k[[\mathbf{s}]]_\Delta$-bilinear and $A[[\mathbf{s}]]_\Delta$-balanced map

$$\langle -, - \rangle : (r, e) \in \mathrm{Hom}_k(E, F)[[\mathbf{s}]]_\Delta \times E[[\mathbf{s}]]_\Delta \longmapsto \langle r, e\rangle := \widetilde{r}(e) \in F[[\mathbf{s}]]_\Delta.$$

Lemma 3 *With the above hypotheses, the following properties hold:*

(1) *The map (6) is an isomorphism of* $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$*-bimodules. When* $E = F$ *it is an isomorphism of* $k[[\mathbf{s}]]_\Delta$*-algebras over* $A[[\mathbf{s}]]_\Delta$.

(2) *The restriction map*

$$f \in \mathrm{Hom}^{\mathrm{top}}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta) \mapsto f|_E \in \mathrm{Hom}_k(E, F[[\mathbf{s}]]_\Delta)$$

is an isomorphism of $(A[[\mathbf{s}]]_\Delta; A)$*-bimodules.*

Proof

(1) One easily sees that the inverse map of $r \mapsto \widetilde{r}$ is $f \mapsto \sum_\alpha f_\alpha \mathbf{s}^\alpha$.
(2) One easily sees that the inverse map of the restriction map $f \mapsto f|_E$ is $g \mapsto g^e$.

Let us call $R = \mathrm{End}_k(E)$. As a consequence of the above lemma, the composition of the maps

$$R[[\mathbf{s}]]_\Delta \xrightarrow{r \mapsto \widetilde{r}} \mathrm{End}^{\mathrm{top}}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) \xrightarrow{f \mapsto f|_E} \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\Delta) \tag{7}$$

is an isomorphism of $(A[[\mathbf{s}]]_\Delta; A)$-bimodules, and so $\mathrm{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$ inherits a natural structure of $k[[\mathbf{s}]]_\Delta$-algebra over $A[[\mathbf{s}]]_\Delta$. Namely, if $g, h \in \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$ with

$$g(e) = \sum_{\alpha \in \Delta} g_\alpha(e)\mathbf{s}^\alpha,\ h(e) = \sum_{\alpha \in \Delta} h_\alpha(e)\mathbf{s}^\alpha, \quad \forall e \in E, \quad g_\alpha, h_\alpha \in \mathrm{Hom}_k(E, E),$$

then the product $hg \in \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$ is given by

$$(hg)(e) = \sum_{\alpha \in \Delta} \left(\sum_{\beta+\gamma=\alpha} (h_\beta \circ g_\gamma)(e) \right) \mathbf{s}^\alpha. \tag{8}$$

Definition 4 Let $\mathbf{s}, \mathbf{t}$ be sets and $\Delta \subset \mathbb{N}^{(\mathbf{s})}$, $\nabla \subset \mathbb{N}^{(\mathbf{t})}$ non-empty co-ideals. For each $f \in \mathrm{End}^{\mathrm{top}}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta)$ and each $g \in \mathrm{End}^{\mathrm{top}}_{k[[\mathbf{t}]]_\nabla}(E[[\mathbf{t}]]_\nabla)$, with

$$f(e) = \sum_{\alpha \in \Delta} f_\alpha(e)\mathbf{s}^\alpha, \quad g(e) = \sum_{\beta \in \nabla} g_\beta(e)\mathbf{t}^\beta \quad \forall e \in E,$$

we define $f \boxtimes g \in \mathrm{End}^{\mathrm{top}}_{k[[\mathbf{s} \sqcup \mathbf{t}]]_{\Delta \times \nabla}}(E[[\mathbf{s} \sqcup \mathbf{t}]]_{\Delta \times \nabla})$ as $f \boxtimes g := h^e$, with:

$$h(x) := \sum_{(\alpha,\beta) \in \Delta \times \nabla} (f_\alpha \circ g_\beta)(x)\mathbf{s}^\alpha \mathbf{t}^\beta \quad \forall x \in E.$$

The proof of the following lemma is clear and it is left to the reader.

Lemma 4 *With the above hypotheses, or each $r \in R[[\mathbf{s}]]_\Delta$, $r' \in R[[\mathbf{t}]]_\nabla$, we have $\widetilde{r \boxtimes r'} = \widetilde{r} \boxtimes \widetilde{r'}$ (see Definition 3).*

Lemma 5 *Let us call $R = \mathrm{End}_k(E)$. For any $r \in R[[\mathbf{s}]]_\Delta$, the following properties are equivalent:*

(a) $r_0 = \mathrm{Id}$.
(b) *The endomorphism $\widetilde{r}$ is compatible with the natural augmentation $E[[\mathbf{s}]]_\Delta \to E$, i.e. $\widetilde{r}(e) \equiv e \mod \mathfrak{n}_0^E(\mathbf{s})/\Delta_E$ for all $e \in E[[\mathbf{s}]]_\Delta$.*

Moreover, if the above properties hold, then $\widetilde{r} : E[[\mathbf{s}]]_\Delta \to E[[\mathbf{s}]]_\Delta$ is a bi-continuous $k[[\mathbf{s}]]_\Delta$-linear automorphism.

Proof The equivalence of (a) and (b) is clear. For the second part, r is invertible since $r_0 = \mathrm{Id}$. So $\widetilde{r}$ is invertible too and $\widetilde{r}^{-1} = \widetilde{r^{-1}}$ is also continuous.

Notation 3 *We denote:*

$$\operatorname{Hom}_k^{\circ}(E, E[[\mathbf{s}]]_\Delta) :=$$
$$\left\{f \in \operatorname{Hom}_k(E, E[[\mathbf{s}]]_\Delta) \mid f(e) \equiv e \bmod \mathfrak{n}_0^E(\mathbf{s})/\Delta_E \quad \forall e \in E\right\},$$

$$\operatorname{Aut}^{\circ}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) :=$$
$$\left\{f \in \operatorname{Aut}^{\mathrm{top}}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) \mid f(e) \equiv e_0 \bmod \mathfrak{n}_0^E(\mathbf{s})/\Delta_E \quad \forall e \in E[[\mathbf{s}]]_\Delta\right\}.$$

Let us notice that a $f \in \operatorname{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$, *given by* $f(e) = \sum_{\alpha\in\Delta} f_\alpha(e)\mathbf{s}^\alpha$, *belongs to* $\operatorname{Hom}_k^{\circ}(E, E[[\mathbf{s}]]_\Delta)$ *if and only if* $f_0 = \operatorname{Id}_E$.

The isomorphism in (7) gives rise to a group isomorphism

$$r \in \mathscr{U}^{\mathbf{s}}(\operatorname{End}_k(E); \Delta) \stackrel{\sim}{\longmapsto} \widetilde{r} \in \operatorname{Aut}^{\circ}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) \tag{9}$$

and to a bijection

$$f \in \operatorname{Aut}^{\circ}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) \stackrel{\sim}{\longmapsto} f|_E \in \operatorname{Hom}_k^{\circ}(E, E[[\mathbf{s}]]_\Delta). \tag{10}$$

So, $\operatorname{Hom}_k^{\circ}(E, E[[\mathbf{s}]]_\Delta)$ is naturally a group with the product described in (8).

3 Substitution Maps

In this section we will assume that k is a commutative ring and A a commutative k-algebra. The following notation will be used extensively.

Notation 4

(i) *For each integer* $r \geq 0$ *let us denote* $[r] := \{1, \ldots, r\}$ *if* $r > 0$ *and* $[0] = \emptyset$.
(ii) *Let* $\mathbf{s}$ *be a set. Maps from a set* Λ *to* $\mathbb{N}^{(\mathbf{s})}$ *will be usually denoted as* $\alpha^\bullet : l \in \Lambda \longmapsto \alpha^l \in \mathbb{N}^{(\mathbf{s})}$, *and its* support *is defined by* $\operatorname{supp}\alpha^\bullet := \{l \in \Lambda \mid \alpha^l \neq 0\}$.
(iii) *For each set* Λ *and for each map* $\alpha^\bullet : \Lambda \to \mathbb{N}^{(\mathbf{s})}$ *with finite support, its* norm *is defined by* $|\alpha^\bullet| := \sum_{l\in\operatorname{supp}\alpha^\bullet} \alpha^l = \sum_{l\in\Lambda} \alpha^l$. *When* $\Lambda = \emptyset$, *the unique map* $\Lambda \to \mathbb{N}^{(\mathbf{s})}$ *is the inclusion* $\emptyset \hookrightarrow \mathbb{N}^{(\mathbf{s})}$ *and its norm is* $0 \in \mathbb{N}^{(\mathbf{s})}$.
(iv) *If* Λ *is a set and* $e \in \mathbb{N}^{(\mathbf{s})}$, *we define*

$$\mathscr{P}^{\circ}(e, \Lambda) := \{\alpha^\bullet : \Lambda \to \mathbb{N}^{(\mathbf{s})} \mid \#\operatorname{supp}\alpha^\bullet < +\infty, |\alpha^\bullet| = e\}.$$

If F *is a finite set and* $e \in \mathbb{N}^{(\mathbf{s})}$, *we define*

$$\mathscr{P}(e, F) := \{\alpha : F \to \mathbb{N}_*^{(\mathbf{s})} \mid |\alpha| = e\} \subset \mathscr{P}^{\circ}(e, F).$$

It is clear that $\mathscr{P}(e, F) = \emptyset$ *whenever* $\#F > |e|$, $\mathscr{P}^\circ(e, \emptyset) = \emptyset$ *if* $e \neq 0$, $\mathscr{P}^\circ(0, \Lambda)$ *consists of only the constant map* 0 *and that* $\mathscr{P}(0, \emptyset) = \mathscr{P}^\circ(0, \emptyset)$ *consists of only the inclusion* $\emptyset \hookrightarrow \mathbb{N}_*^{(\mathbf{s})}$. *If* $\#F = 1$ *and* $e \neq 0$, *then* $\mathscr{P}(e, F)$ *also consists of only one map: the constant map with value e.*

The natural map $\coprod_{F \in \mathfrak{P}_f(\Lambda)} \mathscr{P}(e, F) \longrightarrow \mathscr{P}^\circ(e, \Lambda)$ *is obviously a bijection.*

If $r \geq 0$ *is an integer, we will denote* $\mathscr{P}(e, r) := \mathscr{P}(e, [r])$.

(v) *Assume that* Λ *is a finite set,* $\mathbf{t}$ *is an arbitrary set and* $\pi : \Lambda \to \mathbf{t}$ *is map. Then, there is a natural bijection*

$$\mathscr{P}^\circ(e, \Lambda) \leftrightarrow \coprod_{e^\bullet \in \mathscr{P}^\circ(e,\mathbf{t})} \prod_{t \in \mathbf{t}} \mathscr{P}^\circ(e^t, \pi^{-1}(t)) = \coprod_{e^\bullet \in \mathscr{P}^\circ(e,\mathbf{t})} \prod_{t \in \operatorname{supp} e^\bullet} \mathscr{P}^\circ(e^t, \pi^{-1}(t)).$$

Namely, to each $\alpha^\bullet \in \mathscr{P}^\circ(e, \Lambda)$ *we associate* $e^\bullet \in \mathscr{P}^\circ(e, \mathbf{t})$ *defined by* $e^t = \sum_{\pi(l)=t} \alpha^l$, *and* $\{\alpha^{t\bullet}\}_{t \in \mathbf{t}} \in \prod_{t \in \mathbf{t}} \mathscr{P}^\circ(e^t, \pi^{-1}(t))$ *with* $\alpha^{t\bullet} = \alpha^\bullet|_{\pi^{-1}(t)}$. *Let us notice that if for some* $t_0 \in \mathbf{t}$ *one has* $\pi^{-1}(t_0) = \emptyset$ *and* $e^{t_0} \neq 0$, *then* $\mathscr{P}^\circ(e^{t_0}, \pi^{-1}(t_0)) = \emptyset$ *and so* $\prod_{t \in \mathbf{t}} \mathscr{P}^\circ(e^t, \pi^{-1}(t)) = \emptyset$. *Hence*

$$\coprod_{e^\bullet \in \mathscr{P}^\circ(e,\mathbf{t})} \prod_{t \in \mathbf{t}} \mathscr{P}^\circ(e^t, \Lambda_t) = \coprod_{e^\bullet \in \mathscr{P}^\circ_\pi(e,\mathbf{t})} \prod_{t \in \mathbf{t}} \mathscr{P}^\circ(e^t, \pi^{-1}(t)) =$$
$$\coprod_{e^\bullet \in \mathscr{P}^\circ_\pi(e,\mathbf{t})} \prod_{t \in \operatorname{supp} e^\bullet} \mathscr{P}^\circ(e^t, \pi^{-1}(t)),$$

where $\mathscr{P}^\circ_\pi(e, \mathbf{t})$ *is the subset of* $\mathscr{P}^\circ(e, \mathbf{t})$ *whose elements are the* $e^\bullet \in \mathscr{P}^\circ(e, \mathbf{t})$ *such that* $e^t = 0$ *whenever* $\pi^{-1}(t) = \emptyset$ *and* $|e^t| \geq \#\pi^{-1}(t)$ *otherwise.*

The preceding bijection induces a bijection

$$\mathscr{P}(e, \Lambda) \longleftrightarrow \coprod_{e^\bullet \in \mathscr{P}^\circ_\pi(e,\mathbf{t})} \prod_{t \in \mathbf{t}} \mathscr{P}(e^t, \pi^{-1}(t)) = \coprod_{e^\bullet \in \mathscr{P}^\circ_\pi(e,\mathbf{t})} \prod_{t \in \operatorname{supp} e^\bullet} \mathscr{P}(e^t, \pi^{-1}(t)). \tag{11}$$

(vi) *If* $\alpha \in \mathbb{N}^{(\mathbf{t})}$, *we denote*

$$[\alpha] := \{(t, r) \in \mathbf{t} \times \mathbb{N}_* \mid 1 \leq r \leq \alpha_t\}$$

endowed with the projection $\pi : [\alpha] \to \mathbf{t}$. *It is clear that* $|\alpha| = \#[\alpha]$, *and so* $\alpha = 0 \Longleftrightarrow [\alpha] = \emptyset$. *We denote* $\mathscr{P}(e, \alpha) := \mathscr{P}(e, [\alpha])$. *Elements in* $\mathscr{P}(e, \alpha)$ *will be written as*

$$\mathscr{E}^{\bullet\bullet} : (t, r) \in [\alpha] \longmapsto \mathscr{E}^{tr} \in \mathbb{N}^{(\mathbf{s})}, \quad \text{with} \sum_{(t,r) \in [\alpha]} \mathscr{E}^{tr} = e.$$

For each $\mathscr{b}^{\bullet\bullet} \in \mathscr{P}(e,\alpha)$ *and each* $t \in \mathbf{t}$*, we denote*

$$\mathscr{b}^{t\bullet} : r \in [\alpha_t] \longmapsto \mathscr{b}^{tr} \in \mathbb{N}^{(\mathbf{s})}, \quad [\mathscr{b}]^{\bullet} : t \in \mathbf{t} \longmapsto [\mathscr{b}]^t := |\mathscr{b}^{t\bullet}| = \sum_{r=1}^{\alpha_t} \mathscr{b}^{tr} \in \mathbb{N}^{(\mathbf{s})}.$$

Notice that $|[\mathscr{b}]^t| \geq \alpha_t$*,* $[\mathscr{b}]^t = 0$ *whenever* $\alpha_t = 0$ *and* $|[\mathscr{b}]^{\bullet}| = e$*. The bijection (11) gives rise to a bijection*

$$\mathscr{P}(e,\alpha) \longleftrightarrow \coprod_{e^{\bullet} \in \mathscr{P}^{\circ}_{\alpha}(e,\mathbf{t})} \prod_{t\in\mathbf{t}} \mathscr{P}(e^t,\alpha_t) = \coprod_{e^{\bullet} \in \mathscr{P}^{\circ}_{\alpha}(e,\mathbf{t})} \prod_{t \in \operatorname{supp} e^{\bullet}} \mathscr{P}(e^t,\alpha_t), \qquad (12)$$

where $\mathscr{P}^{\circ}_{\alpha}(e,\mathbf{t})$ *is the subset of* $\mathscr{P}^{\circ}(e,\mathbf{t})$ *whose elements are the* $e^{\bullet} \in \mathscr{P}^{\circ}(e,\mathbf{t})$ *such that* $e^t = 0$ *if* $\alpha_t = 0$ *and* $|e^t| \geq \alpha_t$ *otherwise.*

2. Let $\mathbf{t}, \mathbf{u}$ be sets and $\Delta \subset \mathbb{N}^{(\mathbf{u})}$ a non-empty co-ideal. Let $\varphi_0 : A[\mathbf{t}] \to A[[\mathbf{u}]]_{\Delta}$ be an A-algebra map given by:

$$\varphi_0(t) =: c^t = \sum_{\substack{\beta\in\Delta \\ 0<|\beta|}} c^t_{\beta} \mathbf{u}^{\beta} \in \mathfrak{n}^A_0(\mathbf{u})/\Delta_A \subset A[[\mathbf{u}]]_{\Delta}, \quad t \in \mathbf{t}.$$

Let us write down the expression of the image $\varphi_0(a)$ of any $a \in A[\mathbf{t}]$ in terms of the coefficients of a and the c^t, $t \in \mathbf{t}$. First, for each $r \geq 0$ and for each $t \in \mathbf{t}$ we have

$$\varphi_0(t^r) = (c^t)^r = \cdots = \sum_{\substack{e\in\Delta \\ |e|\geq r}} \left(\sum_{\beta^{\bullet} \in \mathscr{P}(e,r)} \prod_{k=1}^{r} c^t_{\beta^k} \right) \mathbf{u}^e.$$

Observe that

$$\sum_{\beta^{\bullet} \in \mathscr{P}(e,r)} \prod_{k=1}^{r} c^t_{\beta^k} = \begin{cases} 1 \text{ if } |e| = r = 0 \\ 0 \text{ if } |e| > r = 0. \end{cases} \qquad (13)$$

So, for each $\alpha \in \mathbb{N}^{(\mathbf{t})}$ we have

$$\varphi_0(\mathbf{t}^{\alpha}) = \prod_{t\in\mathbf{t}} (c^t)^{\alpha_t} = \prod_{t\in\operatorname{supp}\alpha} (c^t)^{\alpha_t} = \prod_{t\in\operatorname{supp}\alpha} \left(\sum_{\substack{e\in\Delta \\ |e|\geq\alpha_t}} \left(\sum_{\beta^{\bullet} \in \mathscr{P}(e,\alpha_t)} \prod_{k=1}^{\alpha_t} c^t_{\beta^k} \right) \mathbf{u}^e \right) =$$

$$\sum_{\substack{e^t \in \Delta, t\in\operatorname{supp}\alpha \\ |e^t|\geq\alpha_t}} \prod_{t\in\operatorname{supp}\alpha} \left(\left(\sum_{\beta^{\bullet} \in \mathscr{P}(e^t,\alpha_t)} \prod_{k=1}^{\alpha_t} c^t_{\beta^k} \right) \mathbf{u}^{e^t} \right) =$$

$$\sum_{\substack{e^t\in\Delta, t\in\operatorname{supp}\alpha\\ |e^t|\geq\alpha_t}} \left(\sum_{\substack{\beta^{t\bullet}\in\mathscr{P}(e^t,\alpha_t)\\ t\in\operatorname{supp}\alpha}} \left(\prod_{t\in\operatorname{supp}\alpha} \prod_{k=1}^{\alpha_t} c^t_{\beta^{tk}} \right)\right) \left(\prod_{t\in\operatorname{supp}\alpha} \mathbf{u}^{e^t} \right) =$$

$$\sum_{\substack{e\in\Delta\\ |e|\geq|\alpha|}} \left(\sum_{\substack{e^t\in\Delta, t\in\operatorname{supp}\alpha\\ |e^t|\geq\alpha_t\\ |e^\bullet|=e}} \left(\sum_{\substack{\beta^{t\bullet}\in\mathscr{P}(e^t,\alpha_t)\\ t\in\operatorname{supp}\alpha}} \left(\prod_{t\in\operatorname{supp}\alpha} \prod_{k=1}^{\alpha_t} c^t_{\beta^{tk}} \right)\right)\right) \mathbf{u}^e =$$

$$\sum_{\substack{e\in\Delta\\ |e|\geq|\alpha|}} \left(\sum_{e^\bullet\in\mathscr{P}^\circ_\alpha(e,\mathbf{t})} \left(\sum_{\substack{\beta^{t\bullet}\in\mathscr{P}(e^t,\alpha_t)\\ t\in\operatorname{supp}\alpha}} \left(\prod_{t\in\operatorname{supp}\alpha} \prod_{k=1}^{\alpha_t} c^t_{\beta^{tk}} \right)\right)\right) \mathbf{u}^e = \sum_{\substack{e\in\Delta\\ |e|\geq|\alpha|}} \mathbf{C}_e(\varphi_0,\alpha)\mathbf{u}^e,$$

with (see (12)):

$$\mathbf{C}_e(\varphi_0,\alpha) = \sum_{\beta^{\bullet\bullet}\in\mathscr{P}(e,\alpha)} C_{\beta^{\bullet\bullet}}, \quad C_{\beta^{\bullet\bullet}} = \prod_{t\in\operatorname{supp}\alpha} \prod_{r=1}^{\alpha_t} c^t_{\beta^{tr}}, \quad \text{for } |\alpha|\leq|e|. \tag{14}$$

We have $\mathbf{C}_0(\varphi_0,0) = 1$ and $\mathbf{C}_e(\varphi_0,0) = 0$ for $e\neq 0$. For a fixed $e\in\mathbb{N}^{(\mathbf{u})}$ the support of any $\alpha\in\mathbb{N}^{(\mathbf{t})}$ such that $|\alpha|\leq|e|$ and $\mathbf{C}_e(\varphi_0,\alpha)\neq 0$ is contained in the set

$$\bigcup_{\substack{\beta\in\Delta\\ \beta\leq e}} \{t\in\mathbf{t} \mid c^t_\beta\neq 0\}$$

and so the set of such α's is finite provided that property (17) holds. We conclude that

$$\varphi_0\left(\sum_{\alpha\in\mathbb{N}^{(\mathbf{t})}} a_\alpha\mathbf{t}^\alpha\right) = \sum_{\alpha\in\mathbb{N}^{(\mathbf{t})}} a_\alpha c^\alpha = \sum_{e\in\Delta}\left(\sum_{\substack{\alpha\in\mathbb{N}^{(\mathbf{t})}\\ |\alpha|\leq|e|}} \mathbf{C}_e(\varphi_0,\alpha)a_\alpha\right)\mathbf{u}^e. \tag{15}$$

Observe that for each non-zero $\alpha\in\mathbb{N}^{(\mathbf{t})}$ we have:

$$\operatorname{supp}(\varphi_0(\mathbf{t}^\alpha)) = \operatorname{supp}\left(\prod_{t\in\operatorname{supp}\alpha} \left(c^t\right)^{\alpha_t}\right) \subset \sum_{t\in\operatorname{supp}(\alpha)} \alpha_t\cdot\operatorname{supp}(c^t). \tag{16}$$

Let us notice that if we assign the weight $|\beta|$ to c^t_β, then $\mathbf{C}_e(\varphi_0,\alpha)$ is a quasi-homogeneous polynomial in the variables c^t_β, $t\in\operatorname{supp}\alpha$, $|\beta|\leq|e|$, of weight $|e|$.

The proof of the following lemma is easy and it is left to the reader.

Lemma 6 *For each $e \in \Delta$ and for each $\alpha \in \mathbb{N}^{(\mathbf{t})}$ with $0 < |\alpha| \le |e|$, the following properties hold:*

(1) *If $|\alpha| = 1$, then $\mathbf{C}_e(\varphi_0, \alpha) = c_e^s$, where $\operatorname{supp}\alpha = \{s\}$, i.e. $\alpha = \mathbf{t}^s$ ($\mathbf{t}_t^s = \delta_{st}$).*
(2) *If $|\alpha| = |e|$, then*

$$\mathbf{C}_e(\varphi_0, \alpha) = \sum_{\substack{e^t \in \Delta, t \in \operatorname{supp}\alpha \\ |e^t| = \alpha_t, |e^\bullet| = e}} \left(\prod_{t \in \operatorname{supp}\alpha} \prod_{v \in \operatorname{supp} e^t} \left(c_{\mathbf{u}^v}^t\right)^{e_v^t} \right).$$

Proposition 2 *Let $\mathbf{t}, \mathbf{u}$ be sets and $\Delta \subset \mathbb{N}^{(\mathbf{u})}$ a non-empty co-ideal. For each family*

$$c = \left\{ c^t = \sum_{\substack{\beta \in \Delta \\ \beta \neq 0}} c_\beta^t \mathbf{u}^\beta \in \mathfrak{n}_0^A(\mathbf{u})/\Delta_A \subset A[[\mathbf{u}]]_\Delta,\ t \in \mathbf{t} \right\}$$

(we are assuming that $c_0^t = 0$) satisfying the following property

$$\#\{t \in \mathbf{t} \mid c_\beta^t \neq 0\} < \infty \qquad \text{for all } \beta \in \Delta, \tag{17}$$

there is a unique continuous A-algebra map $\varphi : A[[\mathbf{t}]] \to A[[\mathbf{u}]]_\Delta$ such that $\varphi(t) = c^t$ for all $t \in \mathbf{t}$. Moreover, if $\nabla \subset \mathbb{N}^{(\mathbf{t})}$ is a non-empty co-ideal such that $\varphi(\nabla_A) = 0$, then φ induces a unique continuous A-algebra map $A[[\mathbf{t}]]_\nabla \to A[[\mathbf{u}]]_\Delta$ sending (the class of) each $t \in \mathbf{t}$ to c^t.

Proof Let us consider the unique A-algebra map $\varphi_0 : A[\mathbf{t}] \to A[[\mathbf{u}]]_\Delta$ defined by $\varphi_0(t) = c^t$ for all $t \in \mathbf{t}$. From (14) and (15) in 2, we know that

$$\varphi_0\left(\sum_{\substack{\alpha \in \mathbb{N}^{(\mathbf{t})} \\ \text{finite}}} a_\alpha \mathbf{t}^\alpha \right) = \sum_{e \in \Delta} \left(\sum_{\substack{\alpha \in \mathbb{N}^{(\mathbf{t})} \\ |\alpha| \le |e|}} \mathbf{C}_e(\varphi_0, \alpha) a_\alpha \right) \mathbf{u}^e.$$

Since for a fixed $e \in \mathbb{N}^{(\mathbf{u})}$ the support of the $\alpha \in \mathbb{N}^{(\mathbf{t})}$ such that $|\alpha| \le |e|$ and $\mathbf{C}_e(\varphi_0, \alpha) \neq 0$ is contained in the finite set

$$\bigcup_{\substack{\beta \in \Delta \\ \beta \le e}} \{t \in \mathbf{t} \mid c_\beta^t \neq 0\},$$

the set of such α's is always finite and we deduce that φ_0 is continuous, and so there is a unique continuous extension $\varphi : A[[\mathbf{t}]] \to A[[\mathbf{u}]]_\Delta$ such that $\varphi(t) = c^t$ for all $t \in \mathbf{t}$.

The last part is clear.

Remark 2 Let us notice that, after (16), to get the equality $\varphi(\nabla_A) = 0$ in the above proposition it is enough to have for each $\alpha \in \nabla^c$ (actually, it will be enough to consider the $\alpha \in \nabla^c$ minimal with respect to the ordering $\leq$ in $\mathbb{N}^{(\mathbf{t})}$):

$$\sum_{t \in \operatorname{supp}(\alpha)} \alpha_t \cdot \operatorname{supp}(c^t) \subset \Delta^c.$$

Definition 5 Let $\nabla \subset \mathbb{N}^{(\mathbf{t})}$, $\Delta \subset \mathbb{N}^{(\mathbf{u})}$ be non-empty co-ideals. An A-algebra map $\varphi : A[[\mathbf{t}]]_\nabla \to A[[\mathbf{u}]]_\Delta$ will be called a *substitution map* if the following properties hold:

(1) φ is continuous.
(2) $\varphi(t) \in \mathfrak{n}_0^A(\mathbf{u})/\Delta_A$ for all $t \in \mathbf{t}$.
(3) The family $c = \{\varphi(t), t \in \mathbf{t}\}$ satisfies property (17).

The set of substitution maps $A[[\mathbf{t}]]_\nabla \to A[[\mathbf{u}]]_\Delta$ will be denoted by $\mathcal{S}_A(\mathbf{t}, \mathbf{u}; \nabla, \Delta)$. The *trivial* substitution map $A[[\mathbf{t}]]_\nabla \to A[[\mathbf{u}]]_\Delta$ is the one sending any $t \in \mathbf{t}$ to 0. It will be denoted by $\mathbf{0}$.

Remark 3 In the above definition, a such φ is uniquely determined by the family $c = \{\varphi(t), t \in \mathbf{t}\}$, and will be called the *substitution map associated* with c. Namely, the family c can be lifted to $A[[\mathbf{u}]]$ by means of the natural A-linear scission $A[[\mathbf{u}]]_\Delta \hookrightarrow A[[\mathbf{u}]]$ and we may consider the unique continuous A-algebra map $\psi : A[[\mathbf{t}]] \to A[[\mathbf{u}]]$ such that $\psi(s) = c^s$ for all $s \in \mathbf{s}$. Since φ is continuous, we have a commutative diagram

$$\begin{array}{ccc} A[[\mathbf{t}]] & \xrightarrow{\psi} & A[[\mathbf{u}]] \\ {\scriptstyle \text{proj.}}\downarrow & & \downarrow{\scriptstyle \text{proj.}} \\ A[[\mathbf{t}]]_\nabla & \xrightarrow{\varphi} & A[[\mathbf{u}]]_\Delta, \end{array}$$

and so $\psi(\nabla_A) \subset \Delta_A$. Then, we may identify

$$\mathcal{S}_A(\mathbf{t}, \mathbf{u}; \nabla, \Delta) \equiv \left\{ \overline{\psi} \in \mathcal{S}_A(\mathbf{t}, \mathbf{u}; \mathbb{N}^{(\mathbf{t})}, \Delta) \mid \overline{\psi}(\nabla_A) = 0 \right\}.$$

For $\alpha \in \nabla$ and $e \in \Delta$ with $|\alpha| \leq |e|$ we will write $\mathbf{C}_e(\varphi, \alpha) := \mathbf{C}_e(\varphi_0, \alpha)$, where $\varphi_0 : A[\mathbf{t}] \to A[[\mathbf{u}]]_\Delta$ is the A-algebra map given by $\varphi_0(t) = \varphi(t)$ for all $t \in \mathbf{t}$ (see (14) in 2).

Example 1 For any family of integers $\nu = \{\nu_t \geq 1, t \in \mathbf{t}\}$, we will denote $[\nu] : A[[\mathbf{t}]]_\nabla \to A[[\mathbf{t}]]_{\nu\nabla}$ the substitution map determined by $[\nu](t) = t^{\nu_t}$ for all $t \in \mathbf{t}$, where

$$\nu\nabla := \{\gamma \in \mathbb{N}^{(\mathbf{t})} \mid \exists \alpha \in \nabla, \gamma \leq \nu\alpha\}.$$

We obviously have $[\nu\nu'] = [\nu]\circ[\nu']$.

Lemma 7 *The composition of two substitution maps* $A[[\mathbf{t}]]_\nabla \xrightarrow{\varphi} A[[\mathbf{u}]]_\Delta \xrightarrow{\psi} A[[\mathbf{s}]]_\Omega$ *is a substitution map and we have*

$$\mathbf{C}_f(\psi\circ\varphi,\alpha) = \sum_{\substack{e\in\Delta\\ |f|\geq|e|\geq|\alpha|}} \mathbf{C}_e(\varphi,\alpha)\mathbf{C}_f(\psi,e), \quad \forall f\in\Omega, \forall\alpha\in\nabla, |\alpha|\leq|f|.$$

Moreover, if one of the substitution maps is trivial, then the composition is trivial too.

Proof Properties (1) and (2) in Definition 5 are clear. Let us see property (3). For each $t\in\mathbf{t}$ let us write:

$$\varphi(t) =: c^t = \sum_{\substack{\beta\in\Delta\\ 0<|\beta|}} c^t_\beta \mathbf{u}^\beta \in \mathfrak{n}^A_0(\mathbf{u})/\Delta_A \subset A[[\mathbf{u}]]_\Delta,$$

and so

$$(\psi\circ\varphi)(t) = \psi\left(\sum_{\substack{\beta\in\Delta\\ 0<|\beta|}} c^t_\beta\mathbf{u}^\beta\right) = \sum_{\substack{\beta\in\Delta\\ 0<|\beta|}} c^t_\beta\left(\sum_{\substack{f\in\Omega\\ |f|\geq|\beta|}}\mathbf{C}_f(\psi,\beta)\mathbf{s}^f\right) = \sum_{\substack{f\in\Omega\\ |f|>0}} d^t_f\mathbf{s}^f$$

with

$$d^t_f = \sum_{\substack{\beta\in\Delta\\ 0<|\beta|\leq|f|}} c^t_\beta\mathbf{C}_f(\psi,\beta)$$

and for a fixed $f\in\Omega$ the set

$$\{t\in\mathbf{t}\mid d^t_f\neq 0\}\subset \bigcup_{\substack{\beta\in\nabla,|\beta|\leq|f|\\ \mathbf{C}_f(\psi,\beta)\neq 0}} \{t\in\mathbf{t}\mid c^t_\beta\neq 0\}$$

is finite. On the other hand

$$(\psi\circ\varphi)(\mathbf{t}^\alpha) = \psi\left(\sum_{\substack{e\in\Delta\\ |e|\geq|\alpha|}}\mathbf{C}_e(\varphi,\alpha)\mathbf{u}^e\right) = \sum_{\substack{e\in\Delta\\ |e|\geq|\alpha|}}\mathbf{C}_e(\varphi,\alpha)\left(\sum_{\substack{f\in\Omega\\ |f|\geq|e|}}\mathbf{C}_f(\psi,e)\mathbf{s}^f\right) =$$

$$\sum_{\substack{f\in\Omega\\ |f|\geq|\alpha|}}\left(\sum_{\substack{e\in\Delta\\ |f|\geq|e|\geq|\alpha|}}\mathbf{C}_e(\varphi,\alpha)\mathbf{C}_f(\psi,e)\right)\mathbf{u}^f$$

and so

$$\mathbf{C}_f(\psi \circ \varphi, \alpha) = \sum_{\substack{e \in \Delta \\ |f| \geq |e| \geq |\alpha|}} \mathbf{C}_e(\varphi, \alpha) \mathbf{C}_f(\psi, e), \quad \forall f \in \Omega, \forall \alpha \in \nabla, |\alpha| \leq |f|.$$

If B is a commutative A-algebra, then any substitution map $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ induces a natural substitution map $\varphi_B : B[[\mathbf{s}]]_\nabla \to B[[\mathbf{t}]]_\Delta$ making the following diagram commutative

$$\begin{array}{ccc} B\widehat{\otimes}_A A[[\mathbf{s}]]_\nabla & \xrightarrow{\mathrm{Id}\widehat{\otimes}\varphi} & B\widehat{\otimes}_A A[[\mathbf{t}]]_\Delta \\ \text{nat.}\downarrow \simeq & & \simeq \downarrow \text{nat.} \\ B[[\mathbf{s}]]_\nabla & \xrightarrow{\varphi_B} & B[[\mathbf{t}]]_\Delta. \end{array}$$

3. For any substitution map $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ and for any integer $n \geq 0$ we have $\varphi(\nabla^n_A / \nabla_A) \subset \Delta^n_A / \Delta_A$ and so there are induced substitution maps $\tau_n(\varphi) : A[[\mathbf{s}]]_{\nabla^n} \to A[[\mathbf{t}]]_{\Delta^n}$ making commutative the following diagram

$$\begin{array}{ccc} A[[\mathbf{s}]]_\nabla & \xrightarrow{\varphi} & A[[\mathbf{t}]]_\Delta \\ \text{nat.}\downarrow & & \downarrow \text{nat.} \\ A[[\mathbf{s}]]_{\nabla^n} & \xrightarrow{\tau_n(\varphi)} & A[[\mathbf{t}]]_{\Delta^n}. \end{array}$$

Moreover, if φ is the substitution map associated with a family $c = \{c^s, s \in \mathbf{s}\}$,

$$c^s = \sum_{\beta \in \Delta} c^s_\beta \mathbf{t}^\beta \in \mathfrak{n}^A_0(\mathbf{t})/\Delta_A \subset A[[\mathbf{t}]]_\Delta,$$

then $\tau_n(\varphi)$ is the substitution map associated with the family $\tau_n(c) = \{\tau_n(c)^s, s \in \mathbf{s}\}$, with

$$\tau_n(c)^s := \sum_{\substack{\beta \in \Delta \\ |\beta| \leq n}} c^s_\beta \mathbf{t}^\beta \in \mathfrak{n}^A_0(\mathbf{t})/\Delta^n_A \subset A[[\mathbf{t}]]_{\Delta^n}.$$

So, we have truncations $\tau_n : \mathcal{S}_A(\mathbf{s}, \mathbf{t}; \nabla, \Delta) \longrightarrow \mathcal{S}_A(\mathbf{s}, \mathbf{t}; \nabla^n, \Delta^n)$, for $n \geq 0$.

We may also add two substitution maps $\varphi, \varphi' : A[[\mathbf{s}]] \to A[[\mathbf{t}]]_\Delta$ to obtain a new substitution map $\varphi + \varphi' : A[[\mathbf{s}]] \to A[[\mathbf{t}]]_\Delta$ determined by[1]:

$$(\varphi + \varphi')(s) = \varphi(s) + \varphi'(s), \quad \text{for all } s \in \mathbf{s}.$$

[1]Pay attention that $(\varphi + \varphi')(r) \neq \varphi(r) + \varphi'(r)$ for arbitrary $r \in A[[\mathbf{s}]]_\nabla$.

It is clear that $\mathscr{S}_A(\mathbf{s}, \mathbf{t}; \mathbb{N}^{(\mathbf{s})}, \Delta)$ becomes an abelian group with the addition, the zero element being the trivial substitution map $\mathbf{0}$.

If $\psi : A[[\mathbf{t}]]_\Delta \to A[[\mathbf{u}]]_\Omega$ is another substitution map, we clearly have

$$\psi \circ (\varphi + \varphi') = \psi \circ \varphi + \psi \circ \varphi'.$$

However, if $\psi : A[[\mathbf{u}]] \to A[[\mathbf{s}]]$ is a substitution map, we have in general

$$(\varphi + \varphi') \circ \psi \neq \varphi \circ \psi + \varphi' \circ \psi.$$

Definition 6 We say that a substitution map $\varphi : A[[\mathbf{t}]]_\nabla \to A[[\mathbf{u}]]_\Delta$ has *constant coefficients* if $c^t_\beta \in k$ for all $t \in \mathbf{t}$ and all $\beta \in \Delta$, where

$$\varphi(t) = c^t = \sum_{\substack{\beta \in \Delta \\ 0 < |\beta|}} c^t_\beta \mathbf{u}^\beta \in \mathfrak{n}^A_0(\mathbf{u})/\Delta_A \subset A[[\mathbf{u}]]_\Delta.$$

This is equivalent to saying that $\mathbf{C}_e(\varphi, \alpha) \in k$ for all $e \in \Delta$ and for all $\alpha \in \nabla$ with $0 < |\alpha| \leq |e|$. Substitution maps which constant coefficients are induced by substitution maps $k[[\mathbf{t}]]_\nabla \to k[[\mathbf{u}]]_\Delta$.

We say that a substitution map $\varphi : A[[\mathbf{t}]]_\nabla \to A[[\mathbf{u}]]_\Delta$ is *combinatorial* if $\varphi(t) \in \mathbf{u}$ for all $t \in \mathbf{t}$. A combinatorial substitution map has constant coefficients and is determined by (and determines) a map $\mathbf{t} \to \mathbf{u}$, necessarily with finite fibers. If $\iota : \mathbf{t} \to \mathbf{u}$ is such a map, we will also denote by $\iota : A[[\mathbf{t}]]_\nabla \to A[[\mathbf{u}]]_{\iota_*(\nabla)}$ the corresponding substitution map, with

$$\iota_*(\nabla) := \{\beta \in \mathbb{N}^{(\mathbf{u})} \mid \beta \circ \iota \in \nabla\}.$$

4. Let $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ be a continuous A-linear map. It is determined by the family $K = \{K_{e,\alpha}, e \in \Delta, \alpha \in \nabla\} \subset A$, with $\varphi(\mathbf{s}^\alpha) = \sum_{e \in \Delta} K_{e,\alpha} \mathbf{t}^e$. We will assume that

- φ is compatible with the order filtration, i.e. $\varphi(\nabla^n_A/\nabla_A) \subset \Delta^n_A/\Delta_A$ for all $n \geq 0$.
- φ is compatible with the natural augmentations $A[[\mathbf{s}]]_\nabla \to A$ and $A[[\mathbf{t}]]_\Delta \to A$.

These properties are equivalent to the fact that $K_{e,\alpha} = 0$ whenever $|\alpha| > |e|$ and $K_{0,0} = 1$.

Let $K = \{K_{e,\alpha}, e \in \Delta, \alpha \in \nabla, |\alpha| \leq |e|\}$ be a family of elements of A with

$$\#\{\alpha \in \nabla \mid |\alpha| \leq |e|, K_{e,\alpha} \neq 0\} < +\infty, \quad \forall e \in \Delta,$$

and $K_{0,0} = 1$, and let $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ be the A-linear map given by

$$\varphi\left(\sum_{\alpha\in\nabla} a_\alpha \mathbf{s}^\alpha\right) = \sum_{e\in\Delta}\left(\sum_{\substack{\alpha\in\nabla\\|\alpha|\leq|e|}} K_{e,\alpha} a_\alpha\right)\mathbf{t}^e.$$

It is clearly continuous and since $\varphi(\mathbf{s}^\alpha) = \sum_{\substack{e\in\Delta\\|\alpha|\leq|e|}} K_{e,\alpha}\mathbf{t}^e$, it determines the family K.

Proposition 3 *With the above notations, the following properties are equivalent:*

(a) *φ is a substitution map.*
(b) *For each $\mu, \nu \in \nabla$ and for each $e \in \Delta$ with $|\mu+\nu| \leq |e|$, the following equality holds:*

$$K_{e,\mu+\nu} = \sum_{\substack{\beta+\gamma=e\\|\mu|\leq|\beta|,|\nu|\leq|\gamma|}} K_{\beta,\mu} K_{\gamma,\nu}.$$

Moreover, if the above equality holds, then $K_{e,0} = 0$ whenever $|e| > 0$ and φ is the substitution map determined by

$$\varphi(u) = \sum_{\substack{e\in\Delta\\0<|e|}} K_{e,\mathbf{s}^u}\mathbf{t}^e, \quad u \in \mathbf{s}.$$

Proof (a) $\Rightarrow$ (b) If φ is a substitution map, there is a family

$$c^s = \sum_{\beta\in\Delta} c^s_\beta \mathbf{t}^\beta \in A[[\mathbf{t}]]_\Delta, \quad s \in \mathbf{s},$$

such that $\varphi(s) = c^s$. So, from (15), we deduce

$$K_{e,\alpha} = \mathbf{C}_e(\varphi, \alpha) = \sum_{\ell^{\bullet\bullet}\in\mathscr{P}(e,\alpha)} C_{\ell^{\bullet\bullet}} \quad \text{for } |\alpha| \leq |e|,$$

with $C_{\ell^{\bullet\bullet}} = \prod_{s\in\operatorname{supp}\alpha}\prod_{r=1}^{\alpha_s} c^s_{\ell^{sr}}$.

For each ordered pair (r, s) of non-negative integers there are natural injective maps

$$i \in [r] \mapsto i \in [r+s], \quad i \in [s] \mapsto r+i \in [r+s]$$

inducing a natural bijection $[r] \sqcup [s] \longleftrightarrow [r+s]$. Consequently, for $(\mu, \nu) \in \mathbb{N}^{(\mathbf{s})} \times \mathbb{N}^{(\mathbf{s})}$ there are natural injective maps $[\mu] \hookrightarrow [\mu+\nu] \hookleftarrow [\nu]$ inducing a natural bijection $[\mu] \sqcup [\nu] \longleftrightarrow [\mu+\nu]$. So, for each $e \in \mathbb{N}^{(\mathbf{t})}$ and each $f^{\bullet\bullet} \in \mathscr{P}(e, \mu+\nu)$, we can consider the restrictions $g^{\bullet\bullet} = f^{\bullet\bullet}|_{[\mu]} \in \mathscr{P}(\beta, \mu)$, $h^{\bullet\bullet} = f^{\bullet\bullet}|_{[\nu]} \in \mathscr{P}(\gamma, \nu)$, with $\beta = |g^{\bullet\bullet}|$ and $\gamma = |h^{\bullet\bullet}|$, $\beta+\gamma = e$. The correspondence $f^{\bullet\bullet} \longmapsto (\beta, \gamma, g^{\bullet\bullet}, h^{\bullet\bullet})$ establishes a bijection between $\mathscr{P}(e, \mu+\nu)$ and the set of $(\beta, \gamma, g^{\bullet\bullet}, h^{\bullet\bullet})$ with $\beta, \gamma \in \mathbb{N}^{(\mathbf{t})}$, $g^{\bullet\bullet} \in \mathscr{P}(\beta, \mu)$, $h^{\bullet\bullet} \in \mathscr{P}(\gamma, \nu)$ and $|\beta| \geq |\mu|$, $|\gamma| \geq |\nu|$, $\beta+\gamma = e$. Moreover, under this bijection we have $C_{f^{\bullet\bullet}} = C_{g^{\bullet\bullet}} C_{h^{\bullet\bullet}}$ and we deduce

$$K_{e,\mu+\nu} = \mathbf{C}_e(\varphi, \mu+\nu) = \sum_{f^{\bullet\bullet}} C_{f^{\bullet\bullet}} = \sum_{\substack{\beta+\gamma=e \\ |\mu|\leq|\beta| \\ |\nu|\leq|\gamma|}} \sum_{g^{\bullet\bullet}, h^{\bullet\bullet}} C_{g^{\bullet\bullet}} C_{h^{\bullet\bullet}} =$$

$$\sum_{\substack{\beta+\gamma=e \\ |\mu|\leq|\beta| \\ |\nu|\leq|\gamma|}} \left(\sum_{g^{\bullet\bullet}} C_{g^{\bullet\bullet}} \right) \left(\sum_{h^{\bullet\bullet}} C_{h^{\bullet\bullet}} \right) = \sum_{\substack{\beta+\gamma=e \\ |\mu|\leq|\beta| \\ |\nu|\leq|\gamma|}} \mathbf{C}_\beta(\varphi, \mu) \mathbf{C}_\gamma(\varphi, \nu) = \sum_{\substack{\beta+\gamma=e \\ |\mu|\leq|\beta| \\ |\nu|\leq|\gamma|}} K_{\beta,\mu} K_{\gamma,\nu}.$$

where $f^{\bullet\bullet} \in \mathscr{P}(e, \mu+\nu)$, $g^{\bullet\bullet} \in \mathscr{P}(\beta, \mu)$ and $h^{\bullet\bullet} \in \mathscr{P}(\gamma, \nu)$.

(b) $\Rightarrow$ (a) First, one easily proves by induction on $|e|$ that $K_{e,0} = 0$ whenever $|e| > 0$, and so $\varphi(1) = \varphi(\mathbf{s}^0) = K_{0,0} = 1$. Let $a = \sum_\alpha a_\alpha \mathbf{s}^\alpha$, $b = \sum_\alpha b_\alpha \mathbf{s}^\alpha$ be elements in $A[[t]]_\Delta$, and $c = ab = \sum_\alpha c_\alpha \mathbf{s}^\alpha$ with $c_\alpha = \sum_{\mu+\nu=\alpha} a_\mu b_\nu$. We have:

$$\varphi(ab) = \varphi(c) = \sum_{e\in\Delta} \left(\sum_{\substack{\alpha\in\nabla \\ |\alpha|\leq|e|}} K_{e,\alpha} c_\alpha \right) \mathbf{t}^e = \sum_{e\in\Delta} \left(\sum_{\substack{\mu,\nu\in\nabla \\ |\mu+\nu|\leq|e|}} K_{e,\mu+\nu} a_\mu b_\nu \right) \mathbf{t}^e =$$

$$\sum_e \left(\sum_{|\mu+\nu|\leq|e|} \sum_{\substack{\beta+\gamma=e \\ |\mu|\leq|\beta|, |\nu|\leq|\gamma|}} K_{\beta,\mu} K_{\gamma,\nu} a_\mu b_\nu \right) \mathbf{t}^e = \cdots = \varphi(a)\varphi(b).$$

We conclude that φ is a (continuous) A-algebra map determined by the images

$$\varphi(u) = \varphi\left(\mathbf{s}^{\mathbf{s}^u}\right) = \sum_{\substack{e\in\Delta \\ 0<|e|}} K_{e,\mathbf{s}^u} \mathbf{t}^e, \quad u \in \mathbf{s},$$

(remember that $\{\mathbf{s}^u\}_{u\in\mathbf{s}}$ is the canonical basis of $\mathbb{N}^{(\mathbf{s})}$) and so it is a substitution map.

Definition 7 The *tensor product* of two substitution maps $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$, $\psi : A[[\mathbf{u}]]_{\nabla'} \to A[[\mathbf{v}]]_{\Delta'}$ is the unique substitution map

$$\varphi \otimes \psi : A[[\mathbf{s} \sqcup \mathbf{u}]]_{\nabla\times\nabla'} \longrightarrow A[[\mathbf{t} \sqcup \mathbf{v}]]_{\Delta\times\Delta'}$$

making commutative the following diagram

$$\begin{array}{ccccc} A[[\mathbf{s}]]_\nabla & \longrightarrow & A[[\mathbf{s}\sqcup\mathbf{u}]]_{\nabla\times\nabla'} & \longleftarrow & A[[\mathbf{u}]]_{\nabla'} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi\otimes\psi} & & \downarrow{\scriptstyle\psi} \\ A[[\mathbf{t}]]_\Delta & \longrightarrow & A[[\mathbf{t}\sqcup\mathbf{v}]]_{\Delta\times\Delta'} & \longleftarrow & A[[\mathbf{v}]]_{\Delta'}, \end{array}$$

where the horizontal arrows are the combinatorial substitution maps induced by the inclusions $\mathbf{s},\mathbf{u}\hookrightarrow \mathbf{s}\sqcup\mathbf{u}$, $\mathbf{t},\mathbf{v}\hookrightarrow \mathbf{t}\sqcup\mathbf{v}$[2].

For all $(\alpha,\beta)\in\nabla\times\nabla'\subset\mathbb{N}^{(\mathbf{s})}\times\mathbb{N}^{(\mathbf{u})}\equiv\mathbb{N}^{(\mathbf{s}\sqcup\mathbf{u})}$ we have

$$(\varphi\otimes\psi)(\mathbf{s}^\alpha\mathbf{u}^\beta)=\varphi(\mathbf{s}^\alpha)\psi(\mathbf{u}^\beta)=\cdots=\sum_{\substack{e\in\Delta,f\in\Delta'\\ |e|\geq|\alpha|\\ |f|\geq|\beta|}}\mathbf{C}_e(\varphi,\alpha)\mathbf{C}_f(\psi,\beta)\mathbf{t}^e\mathbf{v}^f$$

and so, for all $(e,f)\in\Delta\times\Delta'$ and all $(\alpha,\beta)\in\nabla\times\nabla'$ with $|e|+|f|=|(e,f)|\geq|(\alpha,\beta)|=|\alpha|+|\beta|$ we have

$$\mathbf{C}_{(e,f)}(\varphi\otimes\psi,(\alpha,\beta))=\begin{cases}\mathbf{C}_e(\varphi,\alpha)\mathbf{C}_f(\psi,\beta) & \text{if } |\alpha|\leq|e| \text{ and } |\beta|\leq|f|,\\ 0 & \text{otherwise.}\end{cases}$$

4 The Action of Substitution Maps

In this section k will be a commutative ring, A a commutative k-algebra, M an $(A;A)$-bimodule, $\mathbf{s}$ and $\mathbf{t}$ sets and $\nabla\subset\mathbb{N}^{(\mathbf{s})}$, $\Delta\subset\mathbb{N}^{(\mathbf{t})}$ non-empty co-ideals.

Any A-linear continuous map $\varphi:A[[\mathbf{s}]]_\nabla\to A[[\mathbf{t}]]_\Delta$ satisfying the assumptions in 4 induces $(A;A)$-linear maps

$$\varphi_M:=\varphi\widehat{\otimes}\mathrm{Id}_M:M[[\mathbf{s}]]_\nabla\equiv A[[\mathbf{s}]]_\Delta\widehat{\otimes}_A M\longrightarrow M[[\mathbf{t}]]_\Delta\equiv A[[\mathbf{t}]]_\Delta\widehat{\otimes}_A M$$

and

$$_M\varphi:=\mathrm{Id}_M\widehat{\otimes}\varphi:M[[\mathbf{s}]]_\nabla\equiv M\widehat{\otimes}_A A[[\mathbf{s}]]_\nabla\longrightarrow M[[\mathbf{t}]]_\Delta\equiv M\widehat{\otimes}_A A[[\mathbf{t}]]_\Delta.$$

[2]Let us notice that there are canonical continuous isomorphisms of A-algebras $A[[\mathbf{s}\sqcup\mathbf{u}]]_{\nabla\times\nabla'}\simeq A[[\mathbf{s}]]_\nabla\widehat{\otimes}_A A[[\mathbf{u}]]_{\nabla'}$, $A[[\mathbf{s}\sqcup\mathbf{u}]]_{\Delta\times\Delta'}\simeq A[[\mathbf{s}]]_\Delta\widehat{\otimes}_A A[[\mathbf{u}]]_{\Delta'}$.

If φ is determined by the family $K = \{K_{e,\alpha}, e \in \nabla, \alpha \in \Delta, |\alpha| \le |e|\} \subset A$, with $\varphi(\mathbf{s}^\alpha) = \sum_{\substack{e\in\Delta\\|e|\ge|\alpha|}} K_{e,\alpha}\mathbf{t}^e$, then

$$\varphi_M\left(\sum_{\alpha\in\nabla} m_\alpha \mathbf{s}^\alpha\right) = \sum_{\alpha\in\nabla} \varphi(\mathbf{s}^\alpha) m_\alpha = \sum_{e\in\Delta}\left(\sum_{\substack{\alpha\in\nabla\\|\alpha|\le|e|}} K_{e,\alpha} m_\alpha\right)\mathbf{t}^e, \quad m \in M[[\mathbf{s}]]_\nabla,$$

$$_M\varphi\left(\sum_{\alpha\in\nabla} m_\alpha \mathbf{s}^\alpha\right) = \sum_{\alpha\in\nabla} m_\alpha \varphi(\mathbf{s}^\alpha) = \sum_{e\in\Delta}\left(\sum_{\substack{\alpha\in\nabla\\|\alpha|\le|e|}} m_\alpha K_{e,\alpha}\right)\mathbf{t}^e, \quad m \in M[[\mathbf{s}]]_\nabla.$$

If $\varphi' : A[[\mathbf{t}]]_\Delta \to A[[\mathbf{u}]]_\Omega$ is another A-linear continuous map satisfying the assumptions in 4 and $\varphi'' = \varphi \circ \varphi'$, we have $\varphi''_M = \varphi_M \circ \varphi'_M$, $_M\varphi'' = {}_M\varphi \circ {}_M\varphi'$.

If $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ is a substitution map and $m \in M[[\mathbf{s}]]_\nabla$, $a \in A[[\mathbf{s}]]_\nabla$, we have

$$\varphi_M(am) = \varphi(a)\varphi_M(m), \ \ _M\varphi(ma) = {}_M\varphi(m)\varphi(a),$$

i.e. φ_M is $(\varphi; A)$-linear and $_M\varphi$ is $(A; \varphi)$-linear. Moreover, φ_M and $_M\varphi$ are compatible with the augmentations, i.e.

$$\varphi_M(m) \equiv m_0, \ {}_M\varphi(m) \equiv m_0 \bmod \mathfrak{n}_0^M(\mathbf{t})/\Delta_M, \quad m \in M[[\mathbf{s}]]_\nabla. \tag{18}$$

If φ is the trivial substitution map (i.e. $\varphi(s) = 0$ for all $s \in \mathbf{s}$), then $\varphi_M : M[[\mathbf{s}]]_\nabla \to M[[\mathbf{t}]]_\Delta$ and $_M\varphi : M[[\mathbf{s}]]_\nabla \to M[[\mathbf{t}]]_\Delta$ are also trivial, i.e.

$$\varphi_M(m) = {}_M\varphi(m) = m_0, \ m \in M[[\mathbf{s}]]_\nabla.$$

5. The above constructions apply in particular to the case of any k-algebra R over A, for which we have two induced continuous maps, $\varphi_R = \varphi\widehat{\otimes}\mathrm{Id}_R : R[[\mathbf{s}]]_\nabla \to R[[\mathbf{t}]]_\Delta$, which is $(A; R)$-linear, and $_R\varphi = \mathrm{Id}_R\widehat{\otimes}\varphi : R[[\mathbf{s}]]_\nabla \to R[[\mathbf{t}]]_\Delta$, which is $(R; A)$-linear.

For $r \in R[[\mathbf{s}]]_\nabla$ we will denote

$$\varphi \bullet r := \varphi_R(r), \quad r \bullet \varphi := {}_R\varphi(r).$$

Explicitly, if $r = \sum_\alpha r_\alpha \mathbf{s}^\alpha$ with $\alpha \in \nabla$, then

$$\varphi \bullet r = \sum_{e\in\Delta}\left(\sum_{\substack{\alpha\in\nabla\\|\alpha|\le|e|}} \mathbf{C}_e(\varphi,\alpha) r_\alpha\right)\mathbf{t}^e, \quad r \bullet \varphi = \sum_{e\in\Delta}\left(\sum_{\substack{\alpha\in\nabla\\|\alpha|\le|e|}} r_\alpha \mathbf{C}_e(\varphi,\alpha)\right)\mathbf{t}^e. \tag{19}$$

From (18), we deduce that $\varphi_R(\mathscr{U}^{\mathbf{s}}(R;\nabla)) \subset \mathscr{U}^{\mathbf{t}}(R;\Delta)$ and ${}_R\varphi(\mathscr{U}^{\mathbf{s}}(R;\nabla)) \subset \mathscr{U}^{\mathbf{t}}(R;\Delta)$. We also have $\varphi \bullet 1 = 1 \bullet \varphi = 1$.

If φ is a substitution map with constant coefficients, then $\varphi_R = {}_R\varphi$ is a ring homomorphism over φ. In particular, $\varphi \bullet r = r \bullet \varphi$ and $\varphi \bullet (rr') = (\varphi \bullet r)(\varphi \bullet r')$.

If $\varphi = \mathbf{0} : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ is the trivial substitution map, then $\mathbf{0} \bullet r = r \bullet \mathbf{0} = r_0$ for all $r \in R[[\mathbf{s}]]_\nabla$. In particular, $\mathbf{0} \bullet r = r \bullet \mathbf{0} = 1$ for all $r \in \mathscr{U}^{\mathbf{s}}(R;\nabla)$.

If $\psi : R[[\mathbf{t}]]_\Delta \to R[[\mathbf{u}]]_\Omega$ is another substitution map, one has

$$\psi \bullet (\varphi \bullet r) = (\psi \circ \varphi) \bullet r, \quad (r \bullet \varphi) \bullet \psi = r \bullet (\psi \circ \varphi).$$

Since $(R[[\mathbf{s}]]_\nabla)^{\text{opp}} = R^{\text{opp}}[[\mathbf{s}]]_\nabla$, for any substitution map $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ we have $(\varphi_R)^{\text{opp}} = {}_{R^{\text{opp}}}\varphi$ and $({}_R\varphi)^{\text{opp}} = \varphi_{R^{\text{opp}}}$.

The proof of the following lemma is straightforward and it is left to the reader.

Lemma 8 *If $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ is a substitution map, then:*

(i) *φ_R is left φ-linear, i.e. $\varphi_R(ar) = \varphi(a)\varphi_R(r)$ for all $a \in A[[\mathbf{s}]]_\nabla$ and for all $r \in R[[\mathbf{s}]]_\nabla$.*
(ii) *${}_R\varphi$ is right φ-linear, i.e. ${}_R\varphi(ra) = {}_R\varphi(r)\varphi(a)$ for all $a \in A[[\mathbf{s}]]_\nabla$ and for all $r \in R[[\mathbf{s}]]_\nabla$.*

Let us assume again that $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ is an A-linear continuous map satisfying the assumptions in 4. We define the $(A;A)$-linear map

$$\varphi_* : f \in \operatorname{Hom}_k(A, A[[\mathbf{s}]]_\nabla) \longmapsto \varphi_*(f) = \varphi \circ f \in \operatorname{Hom}_k(A, A[[\mathbf{t}]]_\Delta)$$

which induces another one $\overline{\varphi_*} : \operatorname{End}^{\text{top}}_{k[[\mathbf{s}]]_\nabla}(A[[\mathbf{s}]]_\nabla) \longrightarrow \operatorname{End}^{\text{top}}_{k[[\mathbf{t}]]_\Delta}(A[[\mathbf{t}]]_\Delta)$ defined by

$$\overline{\varphi_*}(f) := (\varphi_*(f|_A))^e = (\varphi \circ f|_A)^e, \quad f \in \operatorname{End}^{\text{top}}_{k[[\mathbf{s}]]_\nabla}(A[[\mathbf{s}]]_\nabla).$$

More generally, for a given left A-module E (which will be considered as a trivial $(A;A)$-bimodule) we have $(A;A)$-linear maps

$$(\varphi_E)_* : f \in \operatorname{Hom}_k(E, E[[\mathbf{s}]]_\nabla) \mapsto (\varphi_E)_*(f) = \varphi_E \circ f \in \operatorname{Hom}_k(E, E[[\mathbf{t}]]_\Delta),$$
$$\overline{(\varphi_E)_*} : \operatorname{End}^{\text{top}}_{k[[\mathbf{s}]]_\nabla}(E[[\mathbf{s}]]_\nabla) \to \operatorname{End}^{\text{top}}_{k[[\mathbf{t}]]_\Delta}(E[[\mathbf{t}]]_\Delta), \quad \overline{(\varphi_E)_*}(f) := (\varphi_E \circ f|_A)^e.$$

Let us denote $R = \operatorname{End}_k(E)$. For each $r \in R[[\mathbf{s}]]_\nabla$ and for each $e \in E$ we have

$$\widetilde{\varphi_R(r)}(e) = \varphi_E\left(\widetilde{r}(e)\right),$$

or more graphically, the following diagram is commutative (see (7)):

$$\begin{array}{ccccc}
R[[\mathbf{s}]]_\nabla & \xrightarrow[r\mapsto\widetilde{r}]{\sim} & \mathrm{End}^{\mathrm{top}}_{k[[\mathbf{s}]]_\nabla}(E[[\mathbf{s}]]_\nabla) & \xrightarrow[\text{rest.}]{\sim} & \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\nabla) \\
\varphi_R \big\downarrow & & \big\downarrow \overline{(\varphi_E)_*} & & (\varphi_E)_* \big\downarrow \\
R[[\mathbf{t}]]_\Delta & \xrightarrow[r\mapsto\widetilde{r}]{\sim} & \mathrm{End}^{\mathrm{top}}_{k[[\mathbf{t}]]_\Delta}(E[[\mathbf{t}]]_\Delta) & \xrightarrow[\text{rest.}]{\sim} & \mathrm{Hom}_k(E, E[[\mathbf{t}]]_\Delta).
\end{array}$$

In order to simplify notations, we will also write

$$\varphi\bullet f := \overline{(\varphi_E)_*}(f) \quad \forall f \in \mathrm{End}^{\mathrm{top}}_{k[[\mathbf{s}]]_\nabla}(E[[\mathbf{s}]]_\nabla)$$

and so have $\widetilde{\varphi\bullet r} = \varphi\bullet\widetilde{r}$ for all $r \in R[[\mathbf{s}]]_\nabla$. Let us notice that $(\varphi\bullet f)(e) = (\varphi_E \circ f)(e)$ for all $e \in E$, i.e.

$$\boxed{(\varphi\bullet f)|_E = (\varphi_E \circ f)|_E, \ \text{but in general} \varphi\bullet f \neq \varphi_E \circ f.} \tag{20}$$

If φ is the trivial substitution map, then $(\varphi_E)_*$ (resp. $\overline{(\varphi_E)_*}$) is also trivial in the sense that if $f = \sum_\alpha f_\alpha \mathbf{s}^\alpha \in \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\nabla)$ (resp. $f = \sum_\alpha f_\alpha \mathbf{s}^\alpha \in \mathrm{End}_k(E)[[\mathbf{s}]]_\nabla \equiv \mathrm{End}^{\mathrm{top}}_{k[[\mathbf{s}]]_\nabla}(E[[\mathbf{s}]]_\nabla)$), then $(\varphi_E)_*(f) = f_0 \in \mathrm{End}_k(E) \subset \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\nabla)$ (resp. $\overline{(\varphi_E)_*}(f) = f_0^e \in \mathrm{End}^{\mathrm{top}}_{k[[\mathbf{s}]]_\nabla}(E[[\mathbf{s}]]_\nabla)$, with $f_0^e(\sum_\alpha e_\alpha \mathbf{s}^\alpha) = \sum_\alpha f_0(e_\alpha)\mathbf{s}^\alpha$).

If $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\nabla$ is a substitution map, we have

$$(\varphi_E)_*(af) = \varphi(a)(\varphi_E)_*(f) \quad \forall a \in A[[\mathbf{s}]]_\nabla, \forall f \in \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\nabla)$$

and so

$$\overline{(\varphi_E)_*}(af) = \varphi(a)\overline{(\varphi_E)_*}(f) \quad \forall a \in A[[\mathbf{s}]]_\nabla, \forall f \in \mathrm{End}^{\mathrm{top}}_{k[[\mathbf{s}]]_\nabla}(E[[\mathbf{s}]]_\nabla).$$

Moreover, the following inclusions hold

$$(\varphi_E)_*(\mathrm{Hom}^\circ_k(E, M[[\mathbf{s}]]_\nabla)) \subset \mathrm{Hom}^\circ_k(E, E[[\mathbf{t}]]_\Delta),$$
$$\overline{(\varphi_E)_*}\left(\mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\nabla}(E[[\mathbf{s}]]_\nabla)\right) \subset \mathrm{Aut}^\circ_{k[[\mathbf{t}]]_\Delta}(E[[\mathbf{t}]]_\Delta),$$

and so we have a commutative diagram:

$$\begin{array}{ccccc}
\mathscr{U}^{\mathbf{s}}(R;\nabla) & \xrightarrow[r\mapsto\widetilde{r}]{\sim} & \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\nabla}(E[[\mathbf{s}]]_\nabla) & \xrightarrow[\text{rest.}]{\sim} & \mathrm{Hom}^\circ_k(E, E[[\mathbf{s}]]_\nabla) \\
\varphi_R \big\downarrow & & \big\downarrow \overline{(\varphi_E)_*} & & (\varphi_E)_* \big\downarrow \\
\mathscr{U}^{\mathbf{t}}(R;\Delta) & \xrightarrow[r\mapsto\widetilde{r}]{\sim} & \mathrm{Aut}^\circ_{k[[\mathbf{t}]]_\Delta}(E[[\mathbf{t}]]_\Delta) & \xrightarrow[\text{rest.}]{\sim} & \mathrm{Hom}^\circ_k(E, E[[\mathbf{t}]]_\Delta).
\end{array} \tag{21}$$

Lemma 9 *With the notations above, if* $\varphi : k[[\mathbf{s}]]_\nabla \to k[[\mathbf{t}]]_\Delta$ *is a substitution map with constant coefficients, then*

$$\langle \varphi \bullet r, \varphi_E(e)\rangle = \varphi_E(\langle r, e\rangle), \quad \forall r \in R[[\mathbf{s}]]_\nabla, \forall e \in E[[\mathbf{s}]]_\nabla.$$

Proof Let us write $r = \sum_\alpha r_\alpha \mathbf{s}^\alpha$, $r_\alpha \in R = \operatorname{End}_k(E)$ and $e = \sum_\alpha e_\alpha \mathbf{s}^\alpha$, $e_\alpha \in E$. We have

$$\langle \varphi \bullet r, \varphi_E(e)\rangle = (\widetilde{\varphi \bullet r})(\varphi_E(e)) = \left(\sum_\alpha \varphi(\mathbf{s}^\alpha)\widetilde{r_\alpha}\right)\left(\sum_\alpha \varphi(\mathbf{s}^\alpha) e_\alpha\right) =$$

$$\sum_{\alpha,\beta} \varphi(\mathbf{s}^\alpha)\widetilde{r_\alpha}\left(\varphi(\mathbf{s}^\beta)e_\beta\right) = \sum_{\alpha,\beta} \varphi(\mathbf{s}^\alpha)\varphi(\mathbf{s}^\beta)\widetilde{r_\alpha}\left(e_\beta\right) = \sum_{\alpha,\beta} \varphi(\mathbf{s}^{\alpha+\beta})\widetilde{r_\alpha}(e_\beta) =$$

$$\sum_\gamma \varphi(\mathbf{s}^\gamma)\left(\sum_{\alpha+\beta=\gamma} \widetilde{r_\alpha}(e_\beta)\right) = \varphi_E\left(\sum_\gamma \left(\sum_{\alpha+\beta=\gamma} \widetilde{r_\alpha}(e_\beta)\right)\mathbf{s}^\gamma\right)$$

$$= \varphi_E\left(\widetilde{r}(e)\right) = \varphi_E(\langle r, e\rangle).$$

Notice that if $\varphi : k[[\mathbf{s}]]_\nabla \to k[[\mathbf{t}]]_\Delta$ is a substitution map with constant coefficients, we already pointed out that ${}_R\varphi = \varphi_R$, and indeed, $\varphi \bullet r = r \bullet \varphi$ for all $r \in R[[\mathbf{s}]]_\nabla$.

6. Let us denote $\iota : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{s} \sqcup \mathbf{t}]]_{\nabla\times\Delta}$, $\kappa : A[[\mathbf{t}]]_\Delta \to A[[\mathbf{s} \sqcup \mathbf{t}]]_{\nabla\times\Delta}$ the combinatorial substitution maps given by the inclusions $\mathbf{s} \hookrightarrow \mathbf{s} \sqcup \mathbf{t}$, $\mathbf{t} \hookrightarrow \mathbf{s} \sqcup \mathbf{t}$.

Let us notice that for $r \in R[[\mathbf{s}]]_\nabla$ and $r' \in R[[\mathbf{t}]]_\Delta$, we have (see Definition 3) $r \boxtimes r' = (\iota \bullet r)(\kappa \bullet r') \in R[[\mathbf{s} \sqcup \mathbf{t}]]_{\nabla\times\Delta}$.

If $\nabla' \subset \nabla \subset \mathbb{N}^{(\mathbf{s})}$, $\Delta' \subset \Delta \subset \mathbb{N}^{(\mathbf{t})}$ are non-empty co-ideals, we have

$$\tau_{\nabla\times\Delta,\nabla'\times\Delta'}(r \boxtimes r') = \tau_{\nabla,\nabla'}(r) \boxtimes \tau_{\Delta,\Delta'}(r').$$

If we denote by $\Sigma : R[[\mathbf{s} \sqcup \mathbf{s}]]_{\nabla\times\nabla} \to R[[\mathbf{s}]]_\nabla$ the combinatorial substitution map given by the co-diagonal map $\mathbf{s} \sqcup \mathbf{s} \to \mathbf{s}$, it is clear that for each $r, r' \in R[[\mathbf{s}]]_\nabla$ we have

$$rr' = \Sigma \bullet (r \boxtimes r'). \tag{22}$$

If $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{u}]]_\Omega$ and $\psi : A[[\mathbf{t}]]_\Delta \to A[[\mathbf{v}]]_{\Omega'}$ are substitution maps, we have new substitution maps $\varphi \otimes \operatorname{Id} : A[[\mathbf{s} \sqcup \mathbf{t}]]_{\nabla\times\Delta} \to A[[\mathbf{u} \sqcup \mathbf{t}]]_{\Omega\times\Delta}$ and

$\mathrm{Id} \otimes \psi : A[[\mathbf{s} \sqcup \mathbf{t}]]_{\nabla \times \Delta} \to A[[\mathbf{s} \sqcup \mathbf{v}]]_{\nabla \times \Omega'}$ (see Definition 7) taking part in the following commutative diagrams of $(A; A)$-bimodules

$$\begin{array}{ccc} R[[\mathbf{s}]]_{\nabla} \otimes_R R[[\mathbf{t}]]_{\Delta} & \xrightarrow{\varphi_R \otimes \mathrm{Id}} & R[[\mathbf{u}]]_{\Omega} \otimes_R R[[\mathbf{t}]]_{\Delta} \\ \text{can.} \downarrow & & \downarrow \text{can.} \\ R[[\mathbf{s} \sqcup \mathbf{t}]]_{\nabla \times \Delta} & \xrightarrow{(\varphi \otimes \mathrm{Id})_R} & R[[\mathbf{u} \sqcup \mathbf{t}]]_{\Omega \times \Delta} \end{array}$$

and

$$\begin{array}{ccc} R[[\mathbf{s}]]_{\nabla} \otimes_R R[[\mathbf{t}]]_{\Delta} & \xrightarrow{\mathrm{Id} \otimes \psi} & R[[\mathbf{s}]]_{\nabla} \otimes_R R[[\mathbf{v}]]_{\Omega'} \\ \text{can.} \downarrow & & \downarrow \text{can.} \\ R[[\mathbf{s} \sqcup \mathbf{t}]]_{\nabla \times \Delta} & \xrightarrow{(\mathrm{Id} \otimes \varphi)_R} & R[[\mathbf{s} \sqcup \mathbf{v}]]_{\nabla \times \Omega'}. \end{array}$$

So $(\varphi \bullet r) \boxtimes r' = (\varphi \otimes \mathrm{Id}) \bullet (r \boxtimes r')$ and $r \boxtimes (r' \bullet \psi) = (r \boxtimes r') \bullet (\mathrm{Id} \otimes \psi)$.

5 Multivariate Hasse-Schmidt Derivations

In this section we study multivariate (possibly ∞-variate) Hasse–Schmidt derivations. The original reference for 1-variate Hasse–Schmidt derivations is [4]. This notion has been studied and developed in [8, §27] (see also [13] and [10]). In [6] the authors study "finite dimensional" Hasse–Schmidt derivations, which correspond in our terminology to p-variate Hasse–Schmidt derivations.

From now on k will be a commutative ring, A a commutative k-algebra, $\mathbf{s}$ a set and $\Delta \subset \mathbb{N}^{(\mathbf{s})}$ a non-empty co-ideal.

Definition 8 A $(\mathbf{s}, \Delta)$-*variate Hasse-Schmidt derivation*, or a $(\mathbf{s}, \Delta)$-*variate HS-derivation* for short, of A over k is a family $D = (D_\alpha)_{\alpha \in \Delta}$ of k-linear maps $D_\alpha : A \longrightarrow A$, satisfying the following Leibniz type identities:

$$D_0 = \mathrm{Id}_A, \quad D_\alpha(xy) = \sum_{\beta+\gamma=\alpha} D_\beta(x) D_\gamma(y)$$

for all $x, y \in A$ and for all $\alpha \in \Delta$. We denote by $\mathrm{HS}_k^{\mathbf{s}}(A; \Delta)$ the set of all $(\mathbf{s}, \Delta)$-variate HS-derivations of A over k and $\mathrm{HS}_k^{\mathbf{s}}(A) =$ for $\Delta = \mathbb{N}^{(\mathbf{s})}$. In the case where $\mathbf{s} = \{1, \ldots, p\}$, a $(\mathbf{s}, \Delta)$-variate HS-derivation will be simply called a (p, Δ)-*variate HS-derivation* and we denote $\mathrm{HS}_k^p(A; \Delta) := \mathrm{HS}_k^{\mathbf{s}}(A; \Delta)$ and $\mathrm{HS}_k^p(A) := \mathrm{HS}_k^{\mathbf{s}}(A)$. For $p = 1$, a 1-variate HS-derivation will be simply called a *Hasse–Schmidt derivation* (a HS-derivation for short), or a *higher derivation*[3], and we will simply write $\mathrm{HS}_k(A;m) := \mathrm{HS}^1_k(A;\Delta)$ for $\Delta = \{q \in \mathbb{N} \mid q \leq m\}$[4] and $\mathrm{HS}_k(A) := \mathrm{HS}^1_k(A)$.

7. The above Leibniz identities for $D \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$ can be written as

$$D_\alpha x = \sum_{\beta+\gamma=\alpha} D_\beta(x) D_\gamma, \quad \forall x \in A, \forall \alpha \in \Delta. \tag{23}$$

Any $(\mathbf{s},\Delta)$-variate HS-derivation D of A over k can be understood as a power series

$$\sum_{\alpha\in\Delta} D_\alpha \mathbf{s}^\alpha \in \mathrm{End}_k(A)[[\mathbf{s}]]_\Delta$$

and so we consider $\mathrm{HS}^{\mathbf{s}}_k(A;\Delta) \subset \mathrm{End}_k(A)[[\mathbf{s}]]_\Delta$.

Proposition 4 *Let $D \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$ be a HS-derivation. Then, for each $\alpha \in \Delta$, the component $D_\alpha : A \to A$ is a k-linear differential operator or order $\leq |\alpha|$ vanishing on k. In particular, if $|\alpha| = 1$ then $D_\alpha : A \to A$ is a k-derivation.*

Proof The proof follows by induction on $|\alpha|$ from (23).

The map

$$D \in \mathrm{HS}^{\mathbf{s}}_k(A;\mathfrak{t}_1(\mathbf{s})) \mapsto \{D_\alpha\}_{|\alpha|=1} \in \mathrm{Der}_k(A)^{\mathbf{s}} \tag{24}$$

is clearly a bijection.

The proof of the following proposition is straightforward and it is left to the reader (see Notation 1 and 2).

Proposition 5 *Let us denote $R = \mathrm{End}_k(A)$ and let $D = \sum_\alpha D_\alpha \mathbf{s}^\alpha \in R[[\mathbf{s}]]_\Delta$ be a power series. The following properties are equivalent:*

(a) *D is a $(\mathbf{s},\Delta)$-variate HS-derivation of A over k.*
(b) *The map $\widetilde{D} : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{s}]]_\Delta$ is a (continuous) $k[[\mathbf{s}]]_\Delta$-algebra homomorphism compatible with the natural augmentation $A[[\mathbf{s}]]_\Delta \to A$.*
(c) *$D \in \mathscr{U}^{\mathbf{s}}(R;\Delta)$ and for all $a \in A[[\mathbf{s}]]_\Delta$ we have $Da = \widetilde{D}(a)D$.*
(d) *$D \in \mathscr{U}^{\mathbf{s}}(R;\Delta)$ and for all $a \in A$ we have $Da = \widetilde{D}(a)D$.*

Moreover, in such a case $\widetilde{D}$ is a bi-continuous $k[[\mathbf{s}]]_\Delta$-algebra automorphism of $A[[\mathbf{s}]]_\Delta$.

Corollary 1 *Under the above hypotheses, $\mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$ is a (multiplicative) subgroup of $\mathscr{U}^{\mathbf{s}}(R;\Delta)$.*

[3] This terminology is used for instance in [8].

[4] These HS-derivations are called of length m in [10].

If $\Delta' \subset \Delta \subset \mathbb{N}^{(\mathbf{s})}$ are non-empty co-ideals, we obviously have group homomorphisms $\tau_{\Delta\Delta'} : \mathrm{HS}^{\mathbf{s}}_k(A;\Delta) \longrightarrow \mathrm{HS}^{\mathbf{s}}_k(A;\Delta')$. Since any $D \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$ is determined by its finite truncations, we have a natural group isomorphism

$$\mathrm{HS}^{\mathbf{s}}_k(A) = \varprojlim_{\substack{\Delta' \subset \Delta \\ \sharp\Delta' < \infty}} \mathrm{HS}^{\mathbf{s}}_k(A;\Delta').$$

In the case $\Delta' = \Delta^1 = \Delta \cap \mathfrak{t}_1(\mathbf{s})$, since $\mathrm{HS}^{\mathbf{s}}_k(A;\Delta^1) \simeq \mathrm{Der}_k(A)^{\Delta^1}$, we can think on $\tau_{\Delta\Delta^1}$ as a group homomorphism $\tau_{\Delta\Delta^1} : \mathrm{HS}^{\mathbf{s}}_k(A;\Delta) \to \mathrm{Der}_k(A)^{\Delta^1}$ whose kernel is the normal subgroup of $\mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$ consisting of HS-derivations D with $D_\alpha = 0$ whenever $|\alpha| = 1$.

In the case $\Delta' = \Delta^n = \Delta \cap \mathfrak{t}_n(\mathbf{s})$, for $n \geq 1$, we will simply write $\tau_n = \tau_{\Delta,\Delta^n} : \mathrm{HS}^{\mathbf{s}}_k(A;\Delta) \longrightarrow \mathrm{HS}^{\mathbf{s}}_k(A;\Delta^n)$.

Remark 4 Since for any $D \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$ we have $D_\alpha \in \mathscr{D}\mathrm{iff}^{|\alpha|}_{A/k}(A)$, we may also think on D as an element in a generalized Rees ring of the ring of differential operators:

$$\widehat{\mathscr{R}}^{\mathbf{s}}\left(\mathscr{D}_{A/k}(A);\Delta\right) := \left\{ \sum_{\alpha\in\Delta} r_\alpha \mathbf{s}^\alpha \in \mathscr{D}_{A/k}(A)[[\mathbf{s}]]_\Delta \mid r_\alpha \in \mathscr{D}\mathrm{iff}^{|\alpha|}_{A/k}(A) \right\}.$$

The group operation in $\mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$ is explicitly given by

$$(D,E) \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta) \times \mathrm{HS}^{\mathbf{s}}_k(A;\Delta) \longmapsto D \circ E \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$$

with

$$(D \circ E)_\alpha = \sum_{\beta+\gamma=\alpha} D_\beta \circ E_\gamma,$$

and the identity element of $\mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$ is $\mathbb{I}$ with $\mathbb{I}_0 = \mathrm{Id}$ and $\mathbb{I}_\alpha = 0$ for all $\alpha \neq 0$. The inverse of a $D \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$ will be denoted by D^*.

Proposition 6 *Let $D \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$, $E \in \mathrm{HS}^{\mathbf{t}}_k(A;\nabla)$ be HS-derivations. Then their external product $D \boxtimes E$ (see Definition 3) is a $(\mathbf{s} \sqcup \mathbf{t}, \nabla \times \Delta)$-variate HS-derivation.*

Proof From Lemma 4 we know that $\widetilde{D \boxtimes E} = \widetilde{D} \boxtimes \widetilde{E}$ and we conclude by Proposition 5.

Definition 9 For each $a \in A^{\mathbf{s}}$ and for each $D \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$, we define $a \bullet D$ as

$$(a \bullet D)_\alpha := a^\alpha D_\alpha, \quad \forall \alpha \in \Delta.$$

It is clear that $a \bullet D \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$, $a' \bullet (a \bullet D) = (a'a) \bullet D$, $1 \bullet D = D$ and $0 \bullet D = \mathbb{I}$.

If $\Delta' \subset \Delta \subset \mathbb{N}^{(\mathbf{s})}$ are non-empty co-ideals, we have $\tau_{\Delta\Delta'}(a \bullet D) = a \bullet \tau_{\Delta\Delta'}(D)$. Hence, in the case $\Delta' = \Delta^1 = \Delta \cap \mathfrak{t}_1(\mathbf{s})$, since $\operatorname{HS}^{\mathbf{s}}_k(A;\Delta^1) \simeq \operatorname{Der}_k(A)^{\Delta^1}$, the image of $\tau_{\Delta\Delta^1} : \operatorname{HS}^{\mathbf{s}}_k(A;\Delta) \to \operatorname{Der}_k(A)^{\Delta^1}$ is an A-submodule.

The following lemma provides a dual way to express the Leibniz identity (23), 7.

Lemma 10 *For each $D \in \operatorname{HS}^{\mathbf{s}}_k(A;\Delta)$ and for each $\alpha \in \Delta$, we have*

$$x D_\alpha = \sum_{\beta+\gamma=\alpha} D_\beta\, D^*_\gamma(x), \quad \forall x \in A.$$

Proof We have

$$\sum_{\beta+\gamma=\alpha} D_\beta\, D^*_\gamma(x) = \sum_{\beta+\gamma=\alpha}\sum_{\mu+\nu=\beta} D_\mu(D^*_\gamma(x)) D_\nu =$$

$$\sum_{e+\nu=\alpha}\left(\sum_{\mu+\gamma=e} D_\mu(D^*_\gamma(x))\right) D_\nu = x D_\alpha.$$

It is clear that the map (24) is an isomorphism of groups (with the addition on $\operatorname{Der}_k(A)$ as internal operation) and so $\operatorname{HS}^{\mathbf{s}}_k(A;\mathfrak{t}_1(\mathbf{s}))$ is abelian.

Notation 5 *Let us denote*

$$\operatorname{Hom}^\circ_{k-\text{alg}}(A, A[[\mathbf{s}]]_\Delta) :=$$

$$\left\{ f \in \operatorname{Hom}_{k-\text{alg}}(A, A[[\mathbf{s}]]_\Delta) \mid f(a) \equiv a \bmod \mathfrak{n}^A_0(\mathbf{s})/\Delta_A\ \forall a \in A \right\},$$

$$\operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta-\text{alg}}(A[[\mathbf{s}]]_\Delta) :=$$

$$\left\{ f \in \operatorname{Aut}^{\text{top}}_{k[[\mathbf{s}]]_\Delta-\text{alg}}(A[[\mathbf{s}]]_\Delta) \mid f(a) \equiv a_0 \bmod \mathfrak{n}^A_0(\mathbf{s})/\Delta_A\ \forall a \in A[[\mathbf{s}]]_\Delta \right\}.$$

It is clear that (see Notation 3) $\operatorname{Hom}^\circ_{k-\text{alg}}(A, A[[\mathbf{s}]]_\Delta) \subset \operatorname{Hom}^\circ_k(A, A[[\mathbf{s}]]_\Delta)$ and $\operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta-\text{alg}}(A[[\mathbf{s}]]_\Delta) \subset \operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(A[[\mathbf{s}]]_\Delta)$ are subgroups and we have group isomorphisms (see (10) and (9)):

$$\operatorname{HS}^{\mathbf{s}}_k(A;\Delta) \xrightarrow[\simeq]{D \mapsto \widetilde{D}} \operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta-\text{alg}}(A[[\mathbf{s}]]_\Delta) \xrightarrow[\simeq]{\text{restriction}} \operatorname{Hom}^\circ_{k-\text{alg}}(A, A[[\mathbf{s}]]_\Delta). \tag{25}$$

The composition of the above isomorphisms is given by

$$D \in \operatorname{HS}^{\mathbf{s}}_k(A;\Delta) \stackrel{\sim}{\longmapsto} \Phi_D := \left[a \in A \mapsto \sum_{\alpha\in\Delta} D_\alpha(a)\mathbf{s}^\alpha \right] \in \operatorname{Hom}^\circ_{k-\text{alg}}(A, A[[\mathbf{s}]]_\Delta). \tag{26}$$

For each HS-derivation $D \in \operatorname{HS}^{\mathbf{s}}_k(A;\Delta)$ we have

$$\widetilde{D}\left(\sum_{\alpha\in\Delta} a_\alpha \mathbf{s}^\alpha\right) = \sum_{\alpha\in\Delta} \Phi_D(a_\alpha)\mathbf{s}^\alpha,$$

for all $\sum_\alpha a_\alpha \mathbf{s}^\alpha \in A[[\mathbf{s}]]_\Delta$, and for any $E \in \operatorname{HS}^{\mathbf{s}}_k(A;\Delta)$ we have $\Phi_{D\circ E} = \widetilde{D}\circ\Phi_E$. If $\Delta' \subset \Delta$ is another non-empty co-ideal and we denote by $\pi_{\Delta\Delta'} : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{s}]]_{\Delta'}$ the projection, one has $\Phi_{\tau_{\Delta\Delta'}(D)} = \pi_{\Delta\Delta'}\circ\Phi_D$.

Definition 10 For each HS-derivation $E \in \operatorname{HS}^{\mathbf{s}}_k(A;\Delta)$, we denote

$$\ell(E) := \min\{r \geq 1 \mid \exists\alpha \in \Delta, |\alpha| = r, E_\alpha \neq 0\} \geq 1$$

if $E \neq \mathbb{I}$ and $\ell(E) = \infty$ if $E = \mathbb{I}$. In other words, $\ell(E) = \operatorname{ord}(E - \mathbb{I})$. Clearly, if Δ is bounded, then $\ell(E) > \max\{|\alpha| \mid \alpha \in \Delta\} \iff \ell(E) = \infty \iff E = \mathbb{I}$.

We obviously have $\ell(E\circ E') \geq \min\{\ell(E), \ell(E')\}$ and $\ell(E^*) = \ell(E)$. Moreover, if $\ell(E') > \ell(E)$, then $\ell(E\circ E') = \ell(E)$:

$$\ell(E\circ E') = \operatorname{ord}(E\circ E' - \mathbb{I}) = \operatorname{ord}(E\circ(E' - \mathbb{I}) + (E - \mathbb{I}))$$

and since $\operatorname{ord}(E\circ(E'-\mathbb{I})) \geq$[5] $\operatorname{ord}(E'-\mathbb{I}) = \ell(E') > \ell(E) = \operatorname{ord}(E-\mathbb{I})$ we obtain

$$\ell(E\circ E') = \cdots = \operatorname{ord}(E\circ(E'-\mathbb{I}) + (E-\mathbb{I})) = \operatorname{ord}(E-\mathbb{I}) = \ell(E).$$

Proposition 7 *For each $D \in \operatorname{HS}^{\mathbf{s}}_k(A;\Delta)$ we have that D_α is a k-linear differential operator or order $\leq \lfloor \frac{|\alpha|}{\ell(D)} \rfloor$ for all $\alpha \in \Delta$. In particular, D_α is a k-derivation if $|\alpha| = \ell(D)$, whenever $\ell(D) < \infty$ ($\Leftrightarrow D \neq \mathbb{I}$).*

Proof We may assume $D \neq \mathbb{I}$. Let us call $n := \ell(D) < \infty$ and, for each $\alpha \in \Delta$, $q_\alpha := \lfloor \frac{|\alpha|}{n} \rfloor$ and $r_\alpha := |\alpha| - q_\alpha n$, $0 \leq r_\alpha < n$. We proceed by induction on q_α. If $q_\alpha = 0$, then $|\alpha| < n$, $D_\alpha = 0$ and the result is clear. Assume that the order of D_β is less or equal than q_β whenever $0 \leq q_\beta \leq q$. Now take $\alpha \in \Delta$ with $q_\alpha = q + 1$. For any $a \in A$ we have

$$[D_\alpha, a] = \sum_{\substack{\gamma+\beta=\alpha\\ |\gamma|>0}} D_\gamma(a) D_\beta = \sum_{\substack{\gamma+\beta=\alpha\\ |\gamma|\geq n}} D_\gamma(a) D_\beta,$$

but any β in the index set of the above sum must have norm $\leq |\alpha| - n$ and so $q_\beta < q_\alpha = q + 1$ and D_β has order $\leq q_\beta$. Hence $[D_\alpha, a]$ has order $\leq q$ for any $a \in A$ and D_α has order $\leq q + 1 = q_\alpha$.

[5] Actually, here an equality holds since the 0-term of E (as a series) is 1.

The following example shows that the group structure on HS-derivations takes into account the Lie bracket on usual derivations.

Example 2 If $D, E \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$, then we may apply the above proposition to $[D,E] = D\circ E\circ D^*\circ E^*$ to deduce that $[D,E]_\alpha \in \mathrm{Der}_k(A)$ whenever $|\alpha| = 2$. Actually, for $|\alpha| = 2$ we have:

$$[D,E]_\alpha = \begin{cases} [D_{\mathbf{s}^t}, E_{\mathbf{s}^t}] & \text{if } \alpha = 2\mathbf{s}^t \\ [D_{\mathbf{s}^t}, E_{\mathbf{s}^u}] + [D_{\mathbf{s}^u}, E_{\mathbf{s}^t}] & \text{if } \alpha = \mathbf{s}^t + \mathbf{s}^u, \text{ with } t \neq u. \end{cases}$$

Proposition 8 *For any $D, E \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$ we have $\ell([D,E]) \geq \ell(D) + \ell(E)$.*

Proof We may assume $D, E \neq \mathbb{I}$. Let us write $m = \ell(D) = \ell(D^*)$, $n = \ell(E) = \ell(E^*)$. We have $D_\beta = D^*_\beta = 0$ whenever $0 < |\beta| < m$ and $E_\gamma = E^*_\gamma = 0$ whenever $0 < |\gamma| < n$.

Let $\alpha \in \Delta$ be with $0 < |\alpha| < m + n$. If $|\alpha| < m$ or $|\alpha| < n$ it is clear that $[D,E]_\alpha = 0$. Assume that $m, n \leq |\alpha| < m + n$:

$$[D,E]_\alpha = \sum_{\beta+\gamma+\lambda+\mu=\alpha} D_\beta \circ E_\gamma\, D^*_\lambda\, E^*_\mu = \sum_{\gamma+\mu=\alpha} E_\gamma\, E^*_\mu +$$

$$\sum_{\substack{\beta+\gamma+\lambda+\mu=\alpha\\ |\beta+\lambda|>0}} D_\beta\, E_\gamma\, D^*_\lambda\, E^*_\mu = 0 + \sum_{\substack{\gamma+\lambda+\mu=\alpha\\ |\lambda|>0}} E_\gamma\, D^*_\lambda\, E^*_\mu + \sum_{\substack{\beta+\gamma+\mu=\alpha\\ |\beta|>0}} D_\beta\, E_\gamma\, E^*_\mu +$$

$$\sum_{\substack{\beta+\gamma+\lambda+\mu=\alpha\\ |\beta|,|\lambda|>0}} D_\beta\, E_\gamma\, D^*_\lambda\, E^*_\mu = \sum_{\substack{\gamma+\lambda+\mu=\alpha\\ |\lambda|\geq m}} E_\gamma\, D^*_\lambda\, E^*_\mu + \sum_{\substack{\beta+\gamma+\mu=\alpha\\ |\beta|\geq m}} D_\beta\, E_\gamma\, E^*_\mu +$$

$$\sum_{\substack{\beta+\gamma+\lambda+\mu=\alpha\\ |\beta|,|\lambda|\geq m}} D_\beta\, E_\gamma\, D^*_\lambda\, E^*_\mu = D^*_\alpha + \sum_{\substack{\gamma+\lambda+\mu=\alpha\\ |\lambda|\geq m, |\gamma+\mu|>0}} E_\gamma\, D^*_\lambda\, E^*_\mu + D_\alpha +$$

$$\sum_{\substack{\beta+\mu=\alpha\\ |\beta|\geq m\\ |\gamma+\mu|>0}} D_\beta\, E_\gamma\, E^*_\mu + \sum_{\substack{\beta+\lambda=\alpha\\ |\beta|,|\lambda|\geq m}} D_\beta\, D^*_\lambda + \sum_{\substack{\beta+\gamma+\lambda+\mu=\alpha\\ |\beta|,|\lambda|\geq m\\ |\gamma+\mu|>0}} D_\beta\, E_\gamma\, D^*_\lambda\, E^*_\mu =$$

$$D^*_\alpha + 0 + D_\alpha + 0 + \sum_{\substack{\beta+\lambda=\alpha\\ |\beta|,|\lambda|>0}} D_\beta\, D^*_\lambda + 0 = \sum_{\beta+\lambda=\alpha} D_\beta\, D^*_\lambda = 0.$$

So, $\ell([D,E]) \geq \ell(D) + \ell(E)$.

Corollary 2 *Assume that Δ is bounded and let m be the* max *of $|\alpha|$ with $\alpha \in \Delta$. Then, the group $\mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$ is nilpotent of nilpotent class $\leq m$, where a central series is*[6]

$$\{\mathbb{I}\} = \{E \mid \ell(E) > m\} \lhd \{E \mid \ell(E) \geq m\} \lhd \cdots \lhd \{E \mid \ell(E) \geq 1\} = \mathrm{HS}^{\mathbf{s}}_k(A;\Delta).$$

[6] Let us notice that $\{E \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta) \mid \ell(E) > r\} = \ker \tau_{\Delta,\Delta_r}$.

Proposition 9 *For each* $D \in \mathrm{HS}_k^{\mathbf{s}}(A; \Delta)$*, its inverse* D^* *is given by* $D_0^* = \mathrm{Id}$ *and*

$$D_\alpha^* = \sum_{d=1}^{|\alpha|} (-1)^d \sum_{\alpha^\bullet \in \mathscr{P}(\alpha,d)} D_{\alpha^1} \circ \cdots \circ D_{\alpha^d}, \quad \alpha \in \Delta.$$

Moreover, $\sigma_{|\alpha|}(D_\alpha^*) = (-1)^{|\alpha|} \sigma_{|\alpha|}(D_\alpha)$.

Proof The first assertion is a straightforward consequence of Lemma 2. For the second assertion, first we have $D_\alpha^* = -D_\alpha$ for all α with $|\alpha| = 1$, and if we denote by $-\mathbf{1} \in A^{\mathbf{s}}$ the constant family -1 and $E = D \circ ((-\mathbf{1}) \bullet D)$, we have $\ell(E) > 1$. So, $D^* = ((-\mathbf{1}) \bullet D) \circ E^*$ and

$$D_\alpha^* = \sum_{\beta+\gamma=\alpha} (-1)^{|\beta|} D_\beta E_\gamma^* = (-1)^{|\alpha|} D_\alpha + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|>0}} (-1)^{|\beta|} D_\beta E_\gamma^*.$$

From Proposition 7, we know that E_γ^* is a differential operator of order strictly less than $|\gamma|$ and so $\sigma_{|\alpha|}(D_\alpha^*) = (-1)^{|\alpha|} \sigma_{|\alpha|}(D_\alpha)$.

6 The Action of Substitution Maps on HS-Derivations

In this section, k will be a commutative ring, A a commutative k-algebra, $R = \mathrm{End}_k(A)$, $\mathbf{s}, \mathbf{t}$ sets and $\Delta \subset \mathbb{N}^{(\mathbf{s})}$, $\nabla \subset \mathbb{N}^{(\mathbf{t})}$ non-empty co-ideals.

We are going to extend the operation $(a, D) \in A^{\mathbf{s}} \times \mathrm{HS}_k^{\mathbf{s}}(A; \Delta) \mapsto a \bullet D \in \mathrm{HS}_k^{\mathbf{s}}(A; \Delta)$ (see Definition 9) by means of the constructions in section 4.

Proposition 10 *For any substitution map* $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$*, we have:*

(1) $\varphi_* \left(\mathrm{Hom}_{k-\mathrm{alg}}^\circ(A, A[[\mathbf{s}]]_\Delta)\right) \subset \mathrm{Hom}_{k-\mathrm{alg}}^\circ(A, A[[\mathbf{t}]]_\nabla)$,

(2) $\varphi_R \left(\mathrm{HS}_k^{\mathbf{s}}(A; \Delta)\right) \subset \mathrm{HS}_k^{\mathbf{t}}(A; \nabla)$,

(3) $\overline{\varphi_*} \left(\mathrm{Aut}_{k[[\mathbf{s}]]_\Delta-\mathrm{alg}}^\circ(A[[\mathbf{s}]]_\Delta)\right) \subset \mathrm{Aut}_{k[[\mathbf{t}]]_\nabla-\mathrm{alg}}^\circ(A[[\mathbf{t}]]_\nabla)$.

Proof By using diagram (21) and (25), it is enough to prove the first inclusion, but if $f \in \mathrm{Hom}_{k-\mathrm{alg}}^\circ(A, A[[\mathbf{s}]]_\Delta)$, it is clear that $\varphi_*(f) = \varphi \circ f : A \to A[[\mathbf{t}]]_\nabla$ is a k-algebra map. Moreover, since $\varphi(\mathrm{t}_0^A(\mathbf{s})/\Delta_A) \subset \mathrm{t}_0^A(\mathbf{t})/\nabla_A$ (see 3) and $f(a) \equiv a \mod \mathrm{t}_0^A(\mathbf{s})/\Delta_A$ for all $a \in A$, we deduce that $\varphi(f(a)) \equiv \varphi(a) \mod \mathrm{t}_0^A(\mathbf{t})/\nabla_A$ for all $a \in A$, but φ is an A-algebra map and $\varphi(a) = a$. So $\varphi_*(f) \in \mathrm{Hom}_{k-\mathrm{alg}}^\circ(A, A[[\mathbf{t}]]_\nabla)$.

As a consequence of the above proposition and diagram (21) we have a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}^{\circ}_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_{\Delta}) & \xleftarrow[\Phi_D \leftarrow\!\shortmid D]{\sim} & \mathrm{HS}^{\mathbf{s}}_k(A;\Delta) & \xrightarrow{\sim} & \mathrm{Aut}^{\circ}_{k[[\mathbf{s}]]_{\Delta}-\mathrm{alg}}(A[[\mathbf{s}]]_{\Delta}) \\
\downarrow \varphi_* & & \downarrow \varphi_R & & \downarrow \overline{\varphi_*} \\
\mathrm{Hom}^{\circ}_{k-\mathrm{alg}}(A, A[[\mathbf{t}]]_{\nabla}) & \xleftarrow[\Phi_D \leftarrow\!\shortmid D]{\sim} & \mathrm{HS}^{\mathbf{t}}_k(A;\nabla) & \xrightarrow{\sim} & \mathrm{Aut}^{\circ}_{k[[\mathbf{t}]]_{\nabla}-\mathrm{alg}}(A[[\mathbf{t}]]_{\nabla}).
\end{array} \tag{27}$$

The inclusion (2) in Proposition 10 can be rephrased by saying that for any substitution map $\varphi : A[[\mathbf{s}]]_{\Delta} \to A[[\mathbf{t}]]_{\nabla}$ and for any HS-derivation $D \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$ we have $\varphi \bullet D \in \mathrm{HS}^{\mathbf{t}}_k(A;\nabla)$ (see 5). Moreover $\Phi_{\varphi \bullet D} = \varphi \circ \Phi_D$.

It is clear that for any co-ideals $\Delta' \subset \Delta$ and $\nabla' \subset \nabla$ with $\varphi\left(\Delta'_A/\Delta_A\right) \subset \nabla'_A/\nabla_A$ we have

$$\tau_{\nabla\nabla'}(\varphi \bullet D) = \varphi' \bullet \tau_{\Delta\Delta'}(D), \tag{28}$$

where $\varphi' : A[[\mathbf{s}]]_{\Delta'} \to A[[\mathbf{t}]]_{\nabla'}$ is the substitution map induced by φ.

Let us notice that any $a \in A^{\mathbf{s}}$ gives rise to a substitution map $\varphi : A[[\mathbf{s}]]_{\Delta} \to A[[\mathbf{s}]]_{\Delta}$ given by $\varphi(s) = a_s s$ for all $s \in \mathbf{s}$, and one has $a \bullet D = \varphi \bullet D$.

8. Let $\varphi \in \mathcal{S}_A(\mathbf{s}, \mathbf{t}; \nabla, \Delta)$, $\psi \in \mathcal{S}_A(\mathbf{t}, \mathbf{u}; \Delta, \Omega)$ be substitution maps and $D, D' \in \mathrm{HS}^{\mathbf{s}}_k(A;\nabla)$ HS-derivations. From 5 we deduce the following properties:

- If we denote $E := \varphi \bullet D \in \mathrm{HS}^{\mathbf{t}}_k(A;\Delta)$, we have

$$E_0 = \mathrm{Id}, \quad E_e = \sum_{\substack{\alpha\in\nabla \\ |\alpha|\leq|e|}} \mathbf{C}_e(\varphi,\alpha) D_\alpha, \quad \forall e \in \Delta. \tag{29}$$

- If φ has <u>constant coefficients</u>, then $\varphi \bullet (D \circ D') = (\varphi \bullet D) \circ (\varphi \bullet D')$. The general case will be treated in Proposition 11.
- If $\varphi = \mathbf{0}$ is the trivial substitution map or if $D = \mathbb{I}$, then $\varphi \bullet D = \mathbb{I}$.
- $\psi \bullet (\varphi \bullet D) = (\psi \circ \varphi) \bullet D$.

Remark 5 We recall that a HS-derivation $D \in \mathrm{HS}_k(A)$ is called *iterative* (see [8, pg. 209]) if

$$D_i \circ D_j = \binom{i+j}{i} D_{i+j} \quad \forall i, j \geq 0.$$

This notion makes sense for $\mathbf{s}$-variate HS-derivations of any length. Actually, iterativity may be understood through the action of substitution maps. Namely, if we denote by $\iota, \iota' : s \hookrightarrow s \sqcup s$ the two canonical inclusions and $\iota + \iota' : A[[\mathbf{s}]] \to A[[\mathbf{s} \sqcup \mathbf{s}]]$ is the substitution map determined by

$$(\iota + \iota')(s) = \iota(s) + \iota'(s), \quad \forall s \in \mathbf{s},$$

then a HS-derivation $D \in \mathrm{HS}^{\mathbf{s}}_k(A)$ is iterative if and only if

$$(\iota + \iota') \bullet D = (\iota \bullet D) \circ (\iota' \bullet D).$$

A similar remark applies for any formal group law instead of $\iota + \iota'$ (cf. [5]).

Proposition 11 *Let $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ be a substitution map. Then, the following assertions hold:*

(i) *For each $D \in \mathrm{HS}^{\mathbf{s}}_k(A; \nabla)$ there is a unique substitution map $\varphi^D : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ such that $\left(\widetilde{\varphi \bullet D}\right) \circ \varphi^D = \varphi \circ \widetilde{D}$. Moreover, $(\varphi \bullet D)^* = \varphi^D \bullet D^*$ and $\varphi^{\mathbb{I}} = \varphi$.*

(ii) *For each $D, E \in \mathrm{HS}^{\mathbf{s}}_k(A; \nabla)$, we have $\varphi \bullet (D \circ E) = (\varphi \bullet D) \circ (\varphi^D \bullet E)$ and $\left(\varphi^D\right)^E = \varphi^{D \circ E}$. In particular, $\left(\varphi^D\right)^{D^*} = \varphi$.*

(iii) *If ψ is another composable substitution map, then $(\varphi \circ \psi)^D = \varphi^{\psi \bullet D} \circ \psi^D$.*

(iv) *$\tau_n(\varphi^D) = \tau_n(\varphi)^{\tau_n(D)}$, for all $n \geq 1$.*

(v) *If φ has constant coefficients then $\varphi^D = \varphi$.*

Proof

(i) We know that

$$\widetilde{D} \in \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\nabla - \mathrm{alg}}(A[[\mathbf{s}]]_\nabla) \quad \text{and} \quad \widetilde{\varphi \bullet D} \in \mathrm{Aut}^\circ_{k[[\mathbf{t}]]_\Delta - \mathrm{alg}}(A[[\mathbf{t}]]_\Delta).$$

The only thing to prove is that

$$\varphi^D := \left(\widetilde{\varphi \bullet D}\right)^{-1} \circ \varphi \circ \widetilde{D}$$

is a substitution map $A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ (see Definition 5). Let start by proving that φ^D is an A-algebra map. Let us write $E = \varphi \bullet D$. For each $a \in A$ we have

$$\varphi^D(a) = \widetilde{E}^{-1}\left(\varphi\left(\widetilde{D}(a)\right)\right) = \widetilde{E}^{-1}\left(\varphi\left(\Phi_D(a)\right)\right) =$$
$$\widetilde{E}^{-1}\left((\varphi \circ \Phi_D)(a)\right)\right) = \widetilde{E}^{-1}\left(\Phi_{\varphi \bullet D}(a)\right) = \widetilde{E}^{-1}\left(\left(\widetilde{\varphi \bullet D}\right)(a)\right) = a,$$

and so φ^D is A-linear. The continuity of φ^D is clear, since it is the composition of continuous maps. For each $s \in \mathbf{s}$, let us write

$$\varphi(s) = \sum_{\substack{\beta \in \Delta \\ |\beta| > 0}} c^s_\beta \mathbf{t}^\beta.$$

Since φ is a substitution map, property (17) holds:

$$\#\{s \in \mathbf{s} \mid c^s_\beta \neq 0\} < \infty \qquad \text{for all } \ \beta \in \Delta.$$

We have

$$\varphi^D(s) = \widetilde{E^*}\left(\varphi(\widetilde{D}(s))\right) = \widetilde{E^*}\left(\varphi(s)\right) = \sum_{\beta\in\Delta}\left(\sum_{\alpha+\gamma=\beta} E^*_\alpha(c^s_\gamma)\right)\mathbf{t}^\beta = \sum_{\beta\in\Delta} d^s_\beta \mathbf{t}^\beta$$

with $d^s_\beta = \sum_{\alpha+\gamma=\beta} E^*_\alpha(c^s_\gamma)$. So, for each $\beta \in \Delta$ we have

$$\{s \in \mathbf{s} \mid c^s_\beta \neq 0\} \subset \bigcup_{\gamma\leq\beta} \{s \in \mathbf{s} \mid c^s_\gamma \neq 0\}$$

and φ^D satisfies property (17) too. We conclude that φ^D is a substitution map, and obviously it is the only one such that $\left(\widetilde{\varphi\bullet D}\right)\circ\varphi^D = \varphi\circ\widetilde{D}$. From there, we have

$$\varphi^D \circ \widetilde{D^*} = \varphi^D \circ \widetilde{D}^{-1} = \left(\widetilde{\varphi\bullet D}\right)^{-1} \circ \varphi = \widetilde{(\varphi\bullet D)^*}\circ\varphi,$$

and taking restrictions to A we obtain $\varphi^D \circ \Phi_{D^*} = \Phi_{(\varphi\bullet D)^*}$ and so $\varphi^D \bullet D^* = (\varphi\bullet D)^*$.

On the other hand, it is clear that if $D = \mathbb{I}$, then $\varphi^{\mathbb{I}} = \varphi$ and if $\varphi = \mathbf{0}$, $\mathbf{0}^D = \mathbf{0}$.

(ii) In order to prove the first equality, we need to prove the equality $\widetilde{\varphi\bullet(D\circ E)} = \left(\widetilde{\varphi\bullet D}\right)\circ\left(\widetilde{\varphi^D\bullet E}\right)$. For this it is enough to prove the equality after restriction to A, but

$$\left(\widetilde{\varphi\bullet(D\circ E)}\right)|_A = \Phi_{\varphi\bullet(D\circ E)} = \varphi\circ\Phi_{D\circ E} = \varphi\circ\widetilde{D}\circ\Phi_E,$$

$$\left(\left(\widetilde{\varphi\bullet D}\right)\circ\left(\widetilde{\varphi^D\bullet E}\right)\right)|_A = \left(\widetilde{\varphi\bullet D}\right)\circ\Phi_{\varphi^D\bullet E} = \left(\widetilde{\varphi\bullet D}\right)\circ\varphi^D\circ\Phi_E$$

and both are equal by (i). For the second equality, we have $\left(\varphi^D\right)^{D^*} = \varphi^{\mathbb{I}} = \varphi$.

(iii) Since

$$((\widetilde{\varphi\circ\psi)\bullet} D)\circ\left(\varphi^{\psi\bullet D}\circ\psi^D\right) = (\varphi\widetilde{\bullet(\psi\bullet} D))\circ\varphi^{\psi\bullet D}\circ\psi^D =$$

$$\varphi\circ\left(\widetilde{\psi\bullet D}\right)\circ\psi^D = \varphi\circ\psi\circ\widetilde{D},$$

we deduce that $(\varphi\circ\psi)^D = \varphi^{\psi\bullet D}\circ\psi^D$ from the uniqueness in (i).

Part (iv) is also a consequence of the uniqueness property in (i).

(v) Let us assume that φ has constant coefficients. We know from Lemma 9 that $\langle\varphi\bullet D, \varphi(a)\rangle = \varphi\left(\langle D, a\rangle\right)$ for all $a \in A[[\mathbf{s}]]_\nabla$, and so $\left(\widetilde{\varphi\bullet D}\right)\circ\varphi = \varphi\circ\widetilde{D}$. Hence, by the uniqueness property in (i) we deduce that $\varphi^D = \varphi$.

The following proposition gives a recursive formula to obtain φ^D from φ.

Proposition 12 *With the notations of Proposition 11, we have*

$$\mathbf{C}_e(\varphi, f+\nu) = \sum_{\substack{\beta+\gamma=e \\ |f+g|\leq|\beta|, |\nu|\leq|\gamma|}} \mathbf{C}_\beta(\varphi, f+g) D_g(\mathbf{C}_\gamma(\varphi^D, \nu))$$

for all $e \in \Delta$ and for all $f, \nu \in \nabla$ with $|f+\nu| \leq |e|$. In particular, we have the following recursive formula

$$\mathbf{C}_e(\varphi^D, \nu) := \mathbf{C}_e(\varphi, \nu) - \sum_{\substack{\beta+\gamma=e \\ |g|\leq|\beta|, |\nu|\leq|\gamma|<|e|}} \mathbf{C}_\beta(\varphi, g) D_g(\mathbf{C}_\gamma(\varphi^D, \nu)).$$

for $e \in \Delta$, $\nu \in \nabla$ with $|e| \geq 1$ and $|\nu| \leq |e|$, starting with $\mathbf{C}_0(\varphi^D, 0) = 1$.

Proof First, the case $f = 0$ easily comes from the equality

$$\sum_{\substack{e\in\Delta \\ |\nu|\leq|e|}} \mathbf{C}_e(\varphi, \nu)\mathbf{t}^e = \varphi(\mathbf{s}^\nu) = (\varphi \circ \widetilde{D})(\mathbf{s}^\nu) = \left(\left(\widetilde{\varphi \bullet D}\right) \circ \varphi^D\right)(\mathbf{s}^\nu) \quad \forall \nu \in \nabla.$$

For arbitrary f one has to use Proposition 3. Details are left to the reader.

The proof of the following corollary is a consequence of Lemma 10.

Corollary 3 *Under the hypotheses of Proposition 11, the following identity holds for each $e \in \Delta$*

$$(\varphi \bullet D)^*_e = \sum_{|\mu+\nu|\leq|e|} D^*_\mu \cdot D_\nu \left(\mathbf{C}_e(\varphi^D, \mu+\nu)\right).$$

Proposition 13 *Let $D \in \mathrm{HS}^{\mathbf{t}}_k(A; \Delta)$ be a HS-derivation and $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ a substitution map. Then, the following identity holds:*

$$\widetilde{D} \circ \varphi = (D(\varphi) \otimes \pi) \circ \left(\widetilde{\kappa \bullet D}\right) \circ \iota,$$

where:

- *$D(\varphi) : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$ is the substitution map determined by $D(\varphi)(s) = \widetilde{D}(\varphi(s))$ for all $s \in \mathbf{s}$.*
- *$\pi : A[[\mathbf{t}]]_\Delta \to A$ is the augmentation, or equivalently, the substitution map[7] given by $\pi(t) = 0$ for all $t \in \mathbf{t}$.*

[7]The map π can be also understood as the truncation $\tau_{\Delta,\{0\}} : A[[\mathbf{t}]]_\Delta \to A[[\mathbf{t}]]_{\{0\}} = A$.

- $\iota : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{s} \sqcup \mathbf{t}]]_{\nabla\times\Delta}$ *and* $\kappa : A[[\mathbf{t}]]_\Delta \to A[[\mathbf{s} \sqcup \mathbf{t}]]_{\nabla\times\Delta}$ *are the combinatorial substitution maps determined by the inclusions* $\mathbf{s} \hookrightarrow \mathbf{s} \sqcup \mathbf{t}$ *and* $\mathbf{t} \hookrightarrow \mathbf{s} \sqcup \mathbf{t}$*, respectively.*

Proof It is enough to check that both maps coincide on any $a \in A$ and on any $s \in \mathbf{s}$. Details are left to the reader.

Remark 6 Let us notice that with the notations of Propositions 11 and 13, we have $\varphi^D = (\varphi\bullet D)^*(\varphi)$.

The following proposition will not be used in this paper and will be stated without proof.

Proposition 14 *For any HS-derivation* $D \in \mathrm{HS}^{\mathbf{s}}_k(A;\nabla)$ *and any substitution map* $\varphi \in \mathcal{S}(\mathbf{t},\mathbf{u};\Delta,\Omega)$*, there exists a substitution map* $D\star\varphi \in \mathcal{S}(\mathbf{s}\sqcup\mathbf{t},\mathbf{s}\sqcup\mathbf{u};\nabla\times\Delta,\nabla\times\Omega)$ *such that for each HS-derivation* $E \in \mathrm{HS}^{\mathbf{t}}_k(A;\Delta)$ *we have:*

$$D\boxtimes(\varphi\bullet E) = (D\star\varphi)\bullet(D\boxtimes E).$$

7 Generating HS-Derivations

In this section we show how the action of substitution maps allows us to express any HS-derivation in terms of a fixed one under some natural hypotheses. We will be concerned with $(\mathbf{s},\mathfrak{t}_m(\mathbf{s}))$-variate HS-derivations, where $\mathfrak{t}_m(\mathbf{s}) = \{\alpha \in \mathbb{N}^{(\mathbf{s})} \mid |\alpha| \leq m\}$. To simplify we will write $A[[\mathbf{s}]]_m := A[[\mathbf{s}]]_{\mathfrak{t}_m(\mathbf{s})}$ and $\mathrm{HS}^{\mathbf{s}}_k(A;m) := \mathrm{HS}^{\mathbf{s}}_k(A;\mathfrak{t}_m(\mathbf{s}))$ for any integer $m \geq 1$, and $\mathrm{HS}^{\mathbf{s}}_k(A;\infty) := \mathrm{HS}^{\mathbf{s}}_k(A)$. For $m \geq n \geq 1$ we will denote $\tau_{mn} : \mathrm{HS}^{\mathbf{s}}_k(A;m) \to \mathrm{HS}^{\mathbf{s}}_k(A;n)$ the truncation map.

Assume that $m \geq 1$ is an integer and let $\varphi : A[[\mathbf{s}]]_m \to A[[\mathbf{t}]]_m$ be a substitution map. Let us write

$$\varphi(s) = c^s = \sum_{\substack{\beta\in\mathbb{N}^{(\mathbf{t})}\\ 0<|\beta|\leq m}} c^s_\beta \mathbf{t}^\beta \in \mathfrak{n}_0(\mathbf{t})/\mathfrak{t}_m(\mathbf{t}) \subset A[[\mathbf{t}]]_m, \quad s \in \mathbf{s}$$

and let us denote by $\varphi_m, \varphi_{<m} : A[[\mathbf{s}]]_m \to A[[\mathbf{t}]]_m$ the substitution maps determined by

$$\varphi_m(s) = c^s_m := \sum_{\substack{\beta\in\mathbb{N}^{(\mathbf{t})}\\ |\beta|=m}} c^s_\beta \mathbf{t}^\beta \in \mathfrak{n}_0(\mathbf{t})/\mathfrak{t}_m(\mathbf{t}) \in A[[\mathbf{t}]]_m, \quad s \in \mathbf{s},$$

$$\varphi_{<m}(s) = c^s_{<m} := \sum_{\substack{\beta\in\mathbb{N}^{(\mathbf{t})}\\ 0<|\beta|<m}} c^s_\beta \mathbf{t}^\beta \in \mathfrak{n}_0(\mathbf{t})/\mathfrak{t}_m(\mathbf{t}) \in A[[\mathbf{t}]]_m, \quad s \in \mathbf{s}.$$

We have $c^s = c^s_m + c^s_{<m}$ and so $\varphi = \varphi_m + \varphi_{<m}$ (see 3).

Proposition 15 *With the above notations, for any HS-derivation $D \in \mathrm{HS}^{\mathbf{s}}_k(A; m)$ the following properties hold:*

(1) $(\varphi_m \bullet D)_e = 0$ *for* $0 < |e| < m$ *and* $(\varphi_m \bullet D)_e = \sum_{t\in\mathbf{s}} c^t_e D_{\mathbf{s}^t}$ *for* $|e| = m$*, where the* $\mathbf{s}^t$ *are the elements of the canonical basis of* $\mathbb{N}^{(\mathbf{s})}$*.*

(2) $\varphi \bullet D = (\varphi_m \bullet D) \circ (\varphi_{<m} \bullet D) = (\varphi_{<m} \bullet D) \circ (\varphi_m \bullet D)$.

Proof

(1) Let us denote $E' = \varphi_m \bullet D$. Since $\tau_{m,m-1}(E')$ coincides with $\tau_{m,m-1}(\varphi_m) \bullet \tau_{m,m-1}(D)$ (see (28)) and $\tau_{m,m-1}(\varphi_m)$ is the trivial substitution map, we deduce that $\tau_{m,m-1}(E') = \mathbb{I}$, i.e. $E_e = 0$ whenever $0 < |e| < m$.

From (29) and (14), for $|e| > 0$ we have $E'_e = \sum_{0<|\alpha|\leq|e|} \mathbf{C}_e(\varphi_m, \alpha) D_\alpha$, with

$$\mathbf{C}_e(\varphi_m, \alpha) = \sum_{f^{\bullet\bullet}\in\mathscr{P}(e,\alpha)} C_{f^{\bullet\bullet}} \quad \text{for } |\alpha| \leq |e|, \quad C_{f^{\bullet\bullet}} = \prod_{s\in\operatorname{supp}\alpha} \prod_{r=1}^{\alpha_s} (c^s_m)_{f^{sr}}.$$

Assume now that $|e| = m$, $1 < |\alpha| \leq m$ and let $f^{\bullet\bullet} \in \mathscr{P}(e, \alpha)$. Since

$$\sum_{s\in\operatorname{supp}\alpha} \sum_{r=1}^{\alpha_s} f^{sr} = e,$$

we deduce that $|f^{sr}| < |e| = m$ for all s, r and so $(c^s_m)_{f^{sr}} = 0$ and $C_{f^{\bullet\bullet}} = 0$. Consequently, $\mathbf{C}_e(\varphi_m, \alpha) = 0$.

If $|\alpha| = 1$, then α must be an element $\mathbf{s}^t$ of the canonical basis of $\mathbb{N}^{(\mathbf{s})}$ and from Lemma 6, (1), we know that $\mathbf{C}_e(\varphi_m, \mathbf{s}^t) = (c^t_m)_e$. We conclude that

$$E'_e = \cdots = \sum_{t\in\mathbf{s}} \mathbf{C}_e(\varphi_m, \mathbf{s}^t) D_{\mathbf{s}^t} = \sum_{t\in\mathbf{s}} (c^t_m)_e D_{\mathbf{s}^t} = \sum_{t\in\mathbf{s}} c^t_e D_{\mathbf{s}^t}.$$

(2) Let us write $E = \varphi \bullet D$, $E' = \varphi_m \bullet D$ and $E'' = \varphi_{<m} \bullet D$. We have

$$\tau_{m,m-1}(E) = \tau_{m,m-1}(\varphi) \bullet \tau_{m,m-1}(D) =$$
$$\tau_{m,m-1}(\varphi_{<m}) \bullet \tau_{m,m-1}(D) = \tau_{m,m-1}(E'').$$

By property (1), we know that $\tau_{m,m-1}(E')$ is the identity and we deduce that $\tau_{m,m-1}(E) = \tau_{m,m-1}(E' \circ E'') = \tau_{m,m-1}(E'' \circ E')$. So $E_e = (E' \circ E'')_e = (E'' \circ E')_e$ for $|e| < m$.

Now, let $e \in \mathbb{N}^{(\mathbf{t})}$ be with $|e| = m$. By using again that $\tau_{m,m-1}(E')$ is the identity, we have $(E' \circ E'')_e = \cdots = E'_e + E''_e = \cdots = (E'' \circ E')_e$, and we conclude that $E' \circ E'' = E'' \circ E'$.

On the other hand, from Lemma 6, (1), we have that $\mathbf{C}_e(\varphi_{<m},\alpha) = 0$ whenever $|\alpha| = 1$, and one can see that $\mathbf{C}_e(\varphi,\alpha) = \mathbf{C}_e(\varphi_{<m},\alpha)$ whenever that $2 \leq |\alpha| \leq |e|$. So:

$$E_e = \sum_{1\leq|\alpha|\leq m} \mathbf{C}_e(\varphi,\alpha)D_\alpha = \sum_{|\alpha|=1} \mathbf{C}_e(\varphi,\alpha)D_\alpha + \sum_{2\leq|\alpha|\leq m} \mathbf{C}_e(\varphi,\alpha)D_\alpha =$$

$$\sum_{t\in\mathbf{s}} c_e^t D_{\mathbf{s}^t} + \sum_{2\leq|\alpha|\leq m} \mathbf{C}_e(\varphi_{<m},\alpha)D_\alpha = E'_e + \sum_{1\leq|\alpha|\leq m} \mathbf{C}_e(\varphi_{<m},\alpha)D_\alpha = E'_e + E''_e$$

and $E = E'\circ E'' = E''\circ E'$.

The following theorem generalizes Theorem 2.8 in [3] to the case where $\operatorname{Der}_k(A)$ is not necessarily a finitely generated A-module. The use of substitution maps makes its proof more conceptual.

Theorem 1 *Let $m \geq 1$ be an integer, or $m = \infty$, and $D \in \operatorname{HS}_k^{\mathbf{s}}(A;m)$ a $\mathbf{s}$-variate HS-derivation of length m such that $\{D_\alpha, |\alpha| = 1\}$ is a system of generators of the A-module $\operatorname{Der}_k(A)$. Then, for each set $\mathbf{t}$ and each HS-derivation $G \in \operatorname{HS}_k^{\mathbf{t}}(A;m)$ there is a substitution map $\varphi : A[[\mathbf{s}]]_m \to A[[\mathbf{t}]]_m$ such that $G = \varphi\bullet D$. Moreover, if $\{D_\alpha, |\alpha| = 1\}$ is a basis of $\operatorname{Der}_k(A)$, φ is uniquely determined.*

Proof For m finite, we will proceed by induction on m. For $m = 1$ the result is clear. Assume that the result is true for HS-derivations of length $m-1$ and consider a $D \in \operatorname{HS}_k^{\mathbf{s}}(A;m)$ such that $\{D_\alpha, |\alpha| = 1\}$ is a system of generators of the A-module $\operatorname{Der}_k(A)$ and a $G \in \operatorname{HS}_k^{\mathbf{t}}(A;m)$. By the induction hypothesis, there is a substitution map $\varphi' : A[[\mathbf{s}]]_{m-1} \to A[[\mathbf{t}]]_{m-1}$, given by $\varphi'(s) = \sum_{|\beta|\leq m-1} c_\beta^s \mathbf{t}^\beta$, $s \in \mathbf{s}$, and such that $\tau_{m,m-1}(G) = \varphi'\bullet\tau_{m,m-1}(D)$. Let $\varphi'' : A[[\mathbf{s}]]_m \to A[[\mathbf{u}]]_m$ be the substitution map lifting φ' (i.e. $\tau_{m,m-1}(\varphi'') = \varphi'$) given by $\varphi''(s) = \sum_{|\beta|\leq m-1} c_\beta^s \mathbf{t}^\beta \in A[[\mathbf{t}]]_m$, $s \in \mathbf{s}$, and consider $F = \varphi''\bullet D$. We obviously have $\tau_{m,m-1}(F) = \tau_{m,m-1}(G)$ and so, for $H = G\circ F^*$, the truncation $\tau_{m,m-1}(H)$ is the identity and $H_e = 0$ for $0 < |e| < m$. We deduce that each component of H of highest order, H_e with $|e| = m$, must be a k-derivation of A and so there is a family $\{c_e^s, s \in \mathbf{s}\}$ of elements of A such that $c_e^s = 0$ for all s except a finite number of indices and $H_e = \sum_{s\in\mathbf{s}} c_e^{\mathbf{s}} D_{\mathbf{s}^s}$, where $\{\mathbf{s}^s, s \in \mathbf{s}\}$ is the canonical basis of $\mathbb{N}^{(\mathbf{s})}$. To finish, let us consider the substitution map $\varphi : A[[\mathbf{s}]]_m \to A[[\mathbf{t}]]_m$ given by $\varphi(s) = \sum_{|\beta|\leq m} c_\beta^s \mathbf{t}^\beta$, $s \in \mathbf{s}$. From Proposition 15 we have

$$\varphi\bullet D = (\varphi_m\bullet D)\circ(\varphi_{<m}\bullet D) = H\circ(\varphi''\bullet D) = H\circ F = G.$$

For HS-derivations of infinite length, following the above procedure we can construct φ as a projective limit of substitution maps $A[[\mathbf{s}]]_m \to A[[\mathbf{t}]]_m$, $m \geq 1$.

Now assume that the set $\{D_\alpha, |\alpha| = 1\}$ is linearly independent over A and let us prove that

$$\varphi\bullet D = \psi\bullet D \quad\Longrightarrow\quad \varphi = \psi. \tag{30}$$

The infinite length case can be reduced to the finite case since $\varphi = \psi$ if and only if all their finite truncations are equal. For the finite length case, we proceed by induction on the length m. Assume that the substitution maps are given by

$$\varphi(s) = c^s := \sum_{\substack{\beta \in \mathbb{N}^{(\mathbf{t})} \\ 0<|\beta|\leq m}} c^s_\beta \mathbf{t}^\beta \in \mathfrak{n}_0(\mathbf{t})/\mathfrak{t}_m(\mathbf{t}) \subset A[[\mathbf{t}]]_m, \quad s \in \mathbf{s}$$

$$\psi(s) = d^s := \sum_{\substack{\beta \in \mathbb{N}^{(\mathbf{t})} \\ 0<|\beta|\leq m}} d^s_\beta \mathbf{t}^\beta \in \mathfrak{n}_0(\mathbf{t})/\mathfrak{t}_m(\mathbf{t}) \subset A[[\mathbf{t}]]_m, \quad s \in \mathbf{s}.$$

If $m = 1$, then $\varphi = \varphi_1$ and $\psi = \psi_1$ and for each $e \in \mathbb{N}^{(\mathbf{t})}$ with $|e| = 1$ we have from Proposition 15

$$\sum_{s\in\mathbf{s}} c^s_e D_{\mathbf{s}^s} = (\varphi_1 \bullet D)_e = (\varphi \bullet D)_e = (\psi \bullet D)_e = (\psi_1 \bullet D)_e = \sum_{s\in\mathbf{s}} d^s_e D_{\mathbf{s}^s}$$

and we deduce that $c^s_e = d^s_e$ for all $s \in \mathbf{s}$ and so $\varphi = \psi$.

Now assume that (30) is true whenever the length is $m - 1$ and take D, φ and ψ as before of length m with $\varphi \bullet D = \psi \bullet D$. By considering $(m - 1)$-truncations and using the induction hypothesis we deduce that $\tau_{m,m-1}(\varphi) = \tau_{m,m-1}(\psi)$, or equivalently $\varphi_{<m} = \psi_{<m}$.

From Proposition 15 we obtain first that $\varphi_m \bullet D = \psi_m \bullet D$ and second that for each $e \in \mathbb{N}^{(\mathbf{t})}$ with $|e| = m$

$$\sum_{s\in\mathbf{s}} c^s_e D_{\mathbf{s}^s} = \sum_{s\in\mathbf{s}} d^s_e D_{\mathbf{s}^s}.$$

We conclude that $\varphi_m = \psi_m$ and so $\varphi = \psi$.

Now we recall the definition of integrability.

Definition 11 (Cf. [1, 7]) Let $m \geq 1$ be an integer or $m = \infty$ and $\mathbf{s}$ a set.

(i) We say that a k-derivation $\delta : A \to A$ is *m-integrable* (over k) if there is a Hasse–Schmidt derivation $D \in \mathrm{HS}_k(A; m)$ such that $D_1 = \delta$. Any such D will be called an *m-integral* of δ. The set of m-integrable k-derivations of A is denoted by $\mathrm{Ider}_k(A; m)$. We simply say that δ is *integrable* if it is ∞-integrable and we denote $\mathrm{Ider}_k(A) := \mathrm{Ider}_k(A; \infty)$.

(ii) We say that a $\mathbf{s}$-variate HS-derivation $D' \in \mathrm{HS}^{\mathbf{s}}_k(A; n)$, with $1 \leq n < m$, is *m-integrable* (over k) if there is a $\mathbf{s}$-variate HS-derivation $D \in \mathrm{HS}^{\mathbf{s}}_k(A; m)$ such that $\tau_{mn} D = D'$. Any such D will be called an *m-integral* of D'. The set of m-integrable $\mathbf{s}$-variate HS-derivations of A over k of length n is denoted by $\mathrm{IHS}^{\mathbf{s}}_k(A; n; m)$. We simply say that D' is *integrable* if it is ∞-integrable and we denote $\mathrm{IHS}^{\mathbf{s}}_k(A; n) := \mathrm{IHS}^{\mathbf{s}}_k(A; n; \infty)$.

Corollary 4 *Let $m \geq 1$ be an integer or $m = \infty$. The following properties are equivalent:*

(1) $\mathrm{Ider}_k(A; m) = \mathrm{Der}_k(A)$.
(2) $\mathrm{IHS}_k^{\mathbf{s}}(A; n; m) = \mathrm{HS}_k^{\mathbf{s}}(A; n)$ *for all n with $1 \leq n < m$ and all sets* $\mathbf{s}$.

Proof We only have to prove (1) $\Longrightarrow$ (2). Let $\{\delta_t, t \in \mathbf{t}\}$ be a system of generators of the A-module $\mathrm{Der}_k(A)$, and for each $t \in \mathbf{t}$ let $D^t \in \mathrm{HS}_k(A; m)$ be an m-integral of δ_t. By considering some total ordering $<$ on $\mathbf{t}$, we can define $D \in \mathrm{HS}_k^{\mathbf{t}}(A; m)$ as the external product (see Definition 3) of the ordered family $\{D^t, t \in \mathbf{t}\}$, i.e. $D_0 = \mathrm{Id}$ and for each $\alpha \in \mathbb{N}^{(\mathbf{t})}$, $\alpha \neq 0$,

$$D_\alpha = D^{t_1}_{\alpha_{t_1}} \circ \cdots \circ D^{t_e}_{\alpha_{t_e}} \quad \text{with} \quad \operatorname{supp}\alpha = \{t_1 < \cdots < t_e\}.$$

Let n be an integer with $1 \leq n < m$, $\mathbf{s}$ a set and $E \in \mathrm{HS}_k^{\mathbf{s}}(A; n)$. After Theorem 1, there exists a substitution map $\varphi : A[[\mathbf{t}]]_n \to A[[\mathbf{s}]]_n$ such that $E = \varphi \bullet \tau_{mn}(D)$. By considering any substitution map $\varphi' : A[[\mathbf{t}]]_m \to A[[\mathbf{s}]]_m$ lifting φ we find that $\varphi' \bullet D$ is an m-integral of E and so $E \in \mathrm{IHS}_k^{\mathbf{s}}(A; n; m)$.

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