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Model transform and local parameters. Application to instantaneous attractors

J.A. Galadí [a,b,*], F. Soler-Toscano [c], J.A. Langa [d]

[a] Grupo de Análisis Estocástico de Sistemas Diferenciales, Universidad de Sevilla, Spain
[b] Center for Brain and Cognition, Universitat Pompeu Fabra, Barcelona, Spain
[c] Grupo de Lógica, Lenguaje e Información, Universidad de Sevilla, Spain
[d] Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Spain

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ABSTRACT

The *model transform* fits exactly the parameters of a suitable model to empirical or simulated data in each point in time and/or space. We describe several examples of concrete model transforms and their applications. The model transform allows simple theoretical models to be applied to complex empirical systems in each short interval of time or/and in each local neighborhood. The model can be chosen to identify, for instance, the temporal evolution of the attractor landscape for empirical systems which depict a complex dynamics over time.

1. Introduction

The features characterizing complex systems may be spatially distributed and/or change over time in many examples from Economy, Ecology, sedimentation analysis, study of flexible structures, Physiology, and etcetera. In order to analyze these situations it is crucial to generalize the classical theory to include space-dependent or time-dependent parameter models [7,50]. Thus, time-varying parameters regression models were introduced by Cooley, Prescott and Rosenberg [11,12,37]. Coefficients may vary over time continuously or, alternatively, discontinuously at certain points in time or space which can be either known a priori or estimated from the sample data. In other cases parameters are assumed to evolve in a random way and a stochastic process is modeled, which can be either stationary or non-stationary, with a certain mean and variance. The corresponding dynamical systems can be characterized by differential equations' models which include these time-dependent or random parameters. In last decades a theory of non-autonomous dynamical systems has emerged including time-dependent differential equations and random dynamical systems so that there are now abstract formulations of them [8,27].

Concurrently, an inverse problem arises for the estimation of these parameters using data obtained from empirical observations. Historycally, Quandt was the first one in attempting to estimate the point in time at which one regime switches to other in a linear regression system [35]. These inverse or parameter identification problems take place in the same diverse contexts than the direct problem. For example, the growth of a population depends on the resources, which may be a function of time and space [6]. Furthermore, many important scientific problems require the estimations of distributed parameters in partial differential equations. Thus, time dependent parameters are usually estimated, for instance, for diverse parabolic systems [1]. A general theory for distributed parameters estimation in an abstract setting can be found in [7].

Abbreviations: fMRI, Functional magnetic resonance imaging; EEG, Electroencelophalography; MT, Model transform; KT, Kuramoto transform; SAR, Simultaneous autoregressive; SART, Simultaneous autoregressive transform; MOU, Multivariate Ornstein-Uhlenbeck; RFT, Rate fluctuations transform; DMT, Discrete model transform; DKT, Discrete Kuramoto transform; DSART, Discrete simultaneous autoregressive transform; DRFT, Discrete rate fuctuations transform; DST, Dynamical systems theory; GASS, Globally asymptotically stable solution; GA, Global attractor; DS, Dynamical system; LV, Lotka-Volterra; LCP, Linear complementarity problem; NDD, Negative dominant diagonal; NoEL, Number of energy levels; LVT, Lotka-Volterra transform; CPA, Complementary pivot algorithm; EC, Effective connectivity; SC, Structural connectivity.

* Corresponding author at: Grupo de Análisis Estocástico de Sistemas Diferenciales, Universidad de Sevilla, Spain.
E-mail address: jgaladi@us.es (J.A. Galadí).

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In Estimation Theory, departing from measured empirical data, an estimator attempts to approximate the unknown parameters using the measurements. The usual approach assumes that the measured data are random with probability distribution dependent on the parameters of interest. However, there are some *trivial* situations in which the functional form of the temporal or spatial dependency of the parameters is determined exactly without assuming random components. The simplest example consists of the equations $x_i'(t) = v_i$, $i = 1,2,3$ which could be seen as a "model" of *uniform motion in a straight line* when "parameters" v_i has a fixed value for $i = 1,2,3$. Alternatively, when the values of v_i are allowed to change over time the equations $x_i'(t) = v_i(t)$ become a definition of the *instantaneous velocity* $\vec{v}(t)$ at each time point t for any sufficiently smooth trajectory $\vec{x}(t)$. The *model* could be seen as a tool to *transform* $\vec{x}(t)$ into $\vec{v}(t)$. Thus, $\vec{v}(t)$ can be defined to exactly reproduce "empirical data" $\vec{x}(t)$ integrating the equations $x_i'(t) = v_i(t)$, $i = 1,2,3$ when an initial point $\vec{x}_0 = \vec{x}(t_0)$ is known. Similarly, the *circle* is a trivial "model" which allows to define the *radius of curvature*, a local or spatial-dependent parameter, for any sufficiently smooth curve. The circle equation

$$(x-a)^2 + (y-b)^2 = r^2$$

can be differentiated twice by x. Removing a and b, and after some simplifications we get

$$\left((y')^2 + 1\right)^3 = (ry'')^2 \tag{1}$$

which can be seen as a differential "model" of all circles of radius r when "parameter" r has a fixed value. Alternatively, when r is allowed to change over a (sufficiently smooth) trajectory $y(x)$ the E. (1) becomes a definition of the radius of curvature at the point $P(x,y)$:

$$r(x) \equiv \frac{\sqrt{\left((y')^2 + 1\right)^3}}{y''} \tag{2}$$

The circle model may be imagined as a tool to transform $y(x)$ into $r(x)$ through (2). Thus, $r(x)$ can be defined to exactly reproduce "empirical data" $y(x)$ integrating the Eq. (1) from some initial point of the trajectory.

In *real situations*, the main drawback of the exact fitting of the parameters to the empirical data is the so-called *overfitting*, which is one of the most important concepts in Machine Learning. Understanding how a model fits the data is important to establish the causes of low accuracy in predictions. A model is overfitted when it performs well with the training data, but its accuracy is noticeably lower with the evaluation data. The model has memorized the training data but is unable to generalize the rules for predicting new data in a cross-validation. However, there are situations in which we are not focused on predictions. For example, there are complex systems that have so far resisted real modeling, such as the human brain. The global models developed are capable of reflecting general characteristics such as functional connectivity or metastability of the system, but we are far from being able to predict specific time series of brain activity in the short or long term using, for example, data from fMRI or EEG measurements. Our thesis is that some fundamental characteristics of the different states of complex systems could be reflected through a transformation of the data mediated by the appropriate model. Some of the crucial characteristics could be lost when estimating parameters assuming empirical data with random components. It might be possible that these random factors are a consequence of: a) our lack of knowledge about the true mechanisms governing the dynamics or b) the difficulty of reproducing its actual complexity. Hence, smoothing or averaging the process from real data to estimated parameters would lead to a loss of information that could be essential to characterize the system.

Thus, we propose as an alternative to the classical approach to modeling complex systems the *model transform* (MT), which is a generalization of the transformation performed by trivial models to calculate the instantaneous velocity or the local radius of curvature, and thus find new measurements that could characterize different states of the most complex and difficult systems to model. Therefore, the MT is proposed here not to model but to carry out a transformation of the empirical data of complex systems to time-dependent or local measurements. We call this mathematical tool "transform" because there is not necessarily reduction of information of the dynamics of the system, but rather a transformation to a different phase space. In the conceptual framework of modeling, this would be a clear overfitting. In this sense, the MT would be an overfitting by definition because it is defined so that the differential equations of the model with time-dependent parameters exactly comply with the experimental data. But since we aim not to model the dynamics, it should not be considered overfitting.

The remaining part of the article is arranged as follows: In Section 2, the MT is formally defined and several examples are shown. A discrete version of the MT is defined in Section 3. As an example of the potential applications of the MT, in Section 4 the concept of instantaneous attractors is developed. Other possible applications of the MT are suggested in Section 5. Finally, in Section 6, as a conclusion, we highlight the most important contributions of this work.

2. Model transform

2.1. Definition

For the sake of simplicity, here a model is defined as a system of n first order ordinary differential equations but this definition and the definition of model transform can be extended to second, third, etc., order ordinary differential equations, partial differential equations, or even difference equations.

Definition 1. A model is a system of n first order ordinary differential equations that relate n functions $u_i : \mathbb{R} \to \mathbb{R}$, their derivates $u_i'(x)$, m parameters θ_k and n functions $f_i : \mathbb{R}^{n+m} \to \mathbb{R}$ in the following form:

$$u_i'(x) = f_i(u_1, \ldots, u_n; \theta_1, \ldots, \theta_m), \qquad i = 1, \ldots, n \tag{3}$$

where $m \le n$ and $x \in \mathbb{R}$,

Definition 2. Given a model (3) and a differentiable function $\hat{u}$:

$\hat{u} : \mathbb{R} \to \mathbb{R}^n$
$\hat{u}(x) = (\hat{u}_1(x), \ldots, \hat{u}_i(x), \ldots, \hat{u}_n(x))^T$, for all $x \in \mathbb{R}$
$\hat{u}_i(x) \in \mathbb{R}$, for all $i = 1, \ldots, n$.

we define the model transform of $\hat{u}(x)$ for model (3) as the functions $\widehat{\theta}_k(x)$ such that fulfills:

$$\widehat{u}'_i(x) = f_i\left(\widehat{u}_1, \ldots, \widehat{u}_n; \widehat{\theta}_1(x), \ldots, \widehat{\theta}_m(x)\right), \qquad i = 1, \ldots, n$$

Theorem 1. Given $\hat{u} : \mathbb{R} \to \mathbb{R}^n$, in the particular case of a model where

$$u_i'(x) = f_i(u_1, \ldots, u_n; \theta_i), \qquad i = 1, \ldots, n \tag{4}$$

and the auxiliary functions $\hat{f}_i : \mathbb{R} \to \mathbb{R}$, $\widehat{f}_i(\theta_i) = f_i(\widehat{u}_1, \ldots, \widehat{u}_n; \theta_i)$ *for all* $i = 1, \ldots, n$ are invertible for all $i = 1, \ldots, n$, the model transform of $\widehat{u}(x)$ for model (4) are the functions $\widehat{\theta}_i(x)$ that fulfills:

$$\widehat{\theta}_i(x) = \widehat{f}_i^{-1}\left(\widehat{u}_i'\right), \qquad i = 1, \ldots, n. \tag{5}$$

Proof. We must prove that $\widehat{\theta}_i(x)$ fulfills $\widehat{u}'_i(x) = f_i\left(\widehat{u}_1, \ldots, \widehat{u}_n; \widehat{\theta}_i(x)\right)$. Substituting $u = \widehat{u}(x)$ in $f_i\left(u_1, \ldots, u_n; \widehat{\theta}_i(x)\right)$, for $i = 1, \ldots, n$, is obtained $f_i\left(\widehat{u}_1, \ldots, \widehat{u}_n; \widehat{\theta}_i(x)\right)$ which is $\widehat{u}'_i$ by (4) and by definition of inverse function f^{-1}. Q.E.D.

Definition 3. If (3) is a model of a dynamical system we use $t \in \mathbb{R}$ (instead of $x \in \mathbb{R}$), and the parameters $\widehat{\theta}_i(t)$ for any given time point $t = t_0$ are called instantaneous parameters in $t = t_0$.

Remark: In practice, (5) makes sense, for instance, when the u_i variables are known (e.g. empirical data), and therefore also are the $\dot{u}_i(t)$. Given θ_i, $u(t)$ is typically the solution of the Eq. (3), but now the situation is the opposite.

2.2. Examples

2.2.1. Trivial examples

In *Differential Calculus*, the most trivial example of MT is the *slope of a curve at a point*. The model would be the differential equation $u'(x) = m$ where m is the parameter of this model. It can be seen as a "*straight line model*" that is trivially invertible and given a sufficiently smooth function $\widehat{u} : \mathrm{R} \to \mathrm{R}$, $\widehat{u}(x)$, *for* $x \in \mathrm{R}$ the transform is $\widehat{m}(x) = \widehat{u}'(x)$ which is the well-known slope $\widehat{m}$ of the curve $\widehat{u}(x)$ at each point x.

Similarly, the linearization of a bivariable function is a model transform. The model is given by the equations

$$\frac{\partial u(x,y)}{\partial x} = q, \quad \frac{\partial u(x,y)}{\partial y} = r$$

where q and r are the parameters of the model. It is a "plane surface model" that is invertible and given a sufficiently smooth function $\widehat{u} : \mathrm{R}^2 \to \mathrm{R}$, $\widehat{u}(x,y)$ for $(x,y) \in \mathbb{R}^2$ the transforms are $\widehat{q}(x,y) = \frac{\partial \widehat{u}(x,y)}{\partial x}$ and $\widehat{r}(x,y) = \frac{\partial \widehat{u}(x,y)}{\partial y}$. In the general case of a linearization of a multivariable function $u(\mathbf{x})$ the model is $\nabla u = \mathbf{q}$, where $\mathbf{q} \in \mathbb{R}^n$ and given a sufficiently smooth function $\widehat{u} : \mathrm{R}^n \to \mathrm{R}$, $\widehat{u}(\mathbf{x})$, *for* $\mathbf{x} \in \mathrm{R}^n$ the transform is $\widehat{\mathbf{q}}(\mathbf{x}) = \nabla \widehat{u}(\mathbf{x})$.

In *Physics*, another trivial example is the *instantaneous velocity*. In this case the model is a *uniform motion in a straight line*, a trivial dynamical system, defined by the three differential equations $x_i'(t) = v_i$, $i = 1,2,3$. Given a sufficiently smooth function $\widehat{x} : \mathrm{R} \to \mathrm{R}^3$, $\widehat{x}_i(t)$, $i = 1,2,3$, for $t \in \mathrm{R}$, the transform is $\widehat{v}(t) = \widehat{x}'(t)$. Thus, in this case the instantaneous parameters are the components of the instantaneous velocity.

In *Differential Geometry*, as a example of simple model transform that generates well-known expressions, let us start from the phase space $\mathbb{R}^3$ and an orbit $\widehat{\gamma}(t)$, to which a *helix* model is now applied in $\mathbb{R}^3$:

$$g(t) = A\cos(\theta(t)) + B\sin(\theta(t)) + Ck\theta(t)t + D$$

where $D \in \mathbb{R}^3$ is a point of the helix axis, $A, B, C \in \mathbb{R}^3$ are three mutually perpendicular vectors, A and B with module ρ and C module 1. Finding the derivative and removing θ, θ' and θ'', it is obtained:

$$g' \cdot \left(g'' \times g'''\right) = \tau \left|g' \times g''\right|^2 \tag{6}$$

where $\tau = \frac{k}{\rho^2 + k^2}$. The expression (6) can be seen as a differential model of helix if the definition of model is generalized to include third order differential equations. Thus, when τ is a constant the helix-shaped paths in $\mathbb{R}^3$ match that equation. However, the model transform can be computed for any sufficiently smooth trajectory $\widehat{\gamma}(t)$ for that differential model and it is obtained:

$$\widehat{\tau}(t) = \frac{\widehat{\gamma}' \cdot \left(\widehat{\gamma}'' \times \widehat{\gamma}\right)}{\left|\widehat{\gamma}' \times \widehat{\gamma}''\right|^2}$$

where the instantaneous parameter of this model transform $\widehat{\tau}(t)$ is the well-known expression of the *torsion* of the curve $\widehat{\gamma}(t)$ in $\mathbb{R}^3$.

Therefore, the slope of a curve in a point, the linearization of a multivariable function, the instantaneous velocity and the torsion are simple particular cases of MT when trivial models are applied. In complex empirical systems the application of non-trivial models can provide relevant information of the system through the corresponding instantaneous and local parameters.

2.2.2. Kuramoto transform

The *Kuramoto* model was motivated by the behavior of systems of *chemical* and *biological* oscillators, and it has found widespread applications in areas such as *neuroscience* and *oscillating flame dynamics*. It describes a set of n coupled oscillators:

$$\theta_i'(t) = \omega_i + g \sum_{j=1}^{n} a_{ij} \sin\left(\theta_j - \theta_i\right), \quad i = 1, \ldots, n. \tag{7}$$

where θ_i are the phases of each oscillator, g is the global coupling, and a_{ij} is the adjacency matrix which express the connectivity among oscillators. Each of the oscillators is considered to have its own *intrinsic frequency* ω_i.

All the n Eqs. (7) are trivially invertible with respect to parameters ω_i, hence a non-trivial example of a model transform can be shown using this model:

Definition 4. Given the values of the parameters $g \in \mathbb{R}^+$ and $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and given a sufficiently smooth function $\widehat{\theta}$ such that:

$$\begin{aligned} &\hat{\theta} : \mathbb{R} \to \mathbb{R}^n \\ &\hat{\theta}(t) = \left(\hat{\theta}_1(t), \ldots, \hat{\theta}_i(t), \ldots, \hat{\theta}_n(t)\right)^T, \quad \text{for all } t \in \mathbb{R} \\ &\hat{\theta}_i(t) \in \mathbb{R}, \text{ for all } i = 1, \ldots, n. \end{aligned}$$

the Kuramoto transform (KT) of $\widehat{\theta}(t)$ for model (7) is defined as the functions $\widehat{\omega}_i(t)$ that fulfills:

$$\widehat{\omega}_i(t) = \widehat{\theta}_i'(t) - g \sum_{j=1}^{n} a_{ij} \sin\left(\widehat{\theta}_j(t) - \widehat{\theta}_i(t)\right), \quad i = 1, \ldots, n.$$

and the numerical values of functions $\widehat{\omega}_i(t)$ for any $t = t_0$ are called instantaneous intrinsic frequencies. In practice, to calculate the KT we may have to previously perform a Hilbert transform [31] that serves to obtain the amplitude and phase of the signal. $\widehat{\theta}_i(t)$ would be an empirical phase in each node i and ω_i the parameter of the model that becomes a function of time and that is the actual result of the KT.

2.2.3. Simultaneous autoregressive transform

Neighboring nodes have more influence on each other than on nodes far away. The same is true of cities, countries, and friends on social media. *Simultaneous autoregressive* (SAR) models can be fit using datasets from *Social Sciences* and *Economy*. The SAR model is composed of a linear combination of the fluctuations within other regions [49]:

$$u_i = g \sum_{j \neq i} \gamma_{ij} u_j + \nu_i, i = 1, \ldots, n. \tag{8}$$

where ν_i stands for *noise*, and u_i, g and γ_{ij} are defined as in the LV model. Usually the ν_i's are modeled as uncorrelated white Gaussian noise.

For SAR Transform (SART) ν_i is the model parameter that is converted to a function of time. Therefore, instead of being modeled as a Gaussian noise, it is calculated exactly from the experimental data:

$$\nu_i(t) = u_i - g\sum_{j\neq i}\gamma_{ij}u_j$$

2.2.4. Rate fluctuations transform

In *Computational Neuroscience*, the *Rate fluctuations model* [19] is a simplification of the Wilson–Cowan model where inhibitory neurons and saturation have been removed. It can be described as a *Multivariate Ornstein-Uhlenbeck* (MOU) process usually applied in disciplines as *Financial Mathematics*, *Physical Sciences*, and *Evolutionary Biology*:

$$\tau\frac{du_i(t)}{dt} = -u_i(t) + g\sum_{j\neq i}\gamma_{ij}u_j(t) + \nu_i. \tag{9}$$

Here τ is the time scale of the excitatory population, and ν_i usually stands for uncorrelated white Gaussian *noise*. u_i, g and γ_{ij} are defined as in the previous models.

In the Rate Fluctuations Transform (RFT) ν_i is again the chosen *instantaneous* parameter:

$$\nu_i(t) = \tau\frac{du_i(t)}{dt} + u_i(t) - g\sum_{j\neq i}\gamma_{ij}u_j(t).$$

3. Discrete model transform (DMT)

3.1. Definition

In practice, both $\widehat{u}_i(t)$ and $\dot{\widehat{u}}_i(t)$ may be empirical or simulated signals that are measured as discrete time series, $t = t_k = k\cdot\Delta t$ where Δt is the time step between measurements and $k = 1, ..., T$ where T is the length of the time series. Hence, we use the notation $u_{i,\,k} = u_i(t = t_k = k\cdot\Delta t)$ and $\theta_{i,\,k} = \theta_i(t = t_k = k\cdot\Delta t)$.

The simplest discrete version of the Eq. (3) for the particular case (4) in a dynamical system model where $x = t$ is:

$$\frac{u_{i,2}-u_{i,1}}{\Delta t} = f_i(u_{1,1},\ldots,u_{n,1};\theta_{i,1}),\ \ i=1,\ldots,n$$
$$\frac{u_{i,k+1}-u_{i,k-1}}{2\cdot\Delta t} = f_i(u_{1,k},\ldots,u_{n,k};\theta_{i,k}),\ \ i=1,\ldots,n;\ \ k=2,\ldots,T-1$$
$$\frac{u_{i,T}-u_{i,T-1}}{\Delta t} = f_i(u_{1,T},\ldots,u_{n,T};\theta_{i,T}),\ \ i=1,\ldots,n$$

where n is the number of nodes in the model and each different i from 1 to n corresponds to a different node. Solving for θ, the corresponding discrete model transform (DMT) and substituting data $\widehat{u}_{i,k}$:

$$\widehat{\theta}_{i,1} = \widehat{f}_i^{\,-1}\left(\widehat{u}_{1,1},\ldots,\widehat{u}_{n,1};\frac{\widehat{u}_{i,2}-\widehat{u}_{i,1}}{\Delta t}\right),\quad i=1,\ldots,n$$
$$\widehat{\theta}_{i,k} = \widehat{f}_i^{\,-1}\left(\widehat{u}_{1,k},\ldots,\widehat{u}_{n,k};\frac{\widehat{u}_{i,k+1}-\widehat{u}_{i,k-1}}{2\cdot\Delta t}\right),\quad i=1,\ldots,n\quad k=2,\ldots,T-1$$
$$\widehat{\theta}_{i,T} = \widehat{f}_i^{\,-1}\left(\widehat{u}_{1,T},\ldots,\widehat{u}_{n,T};\frac{\widehat{u}_{i,T}-\widehat{u}_{i,T-1}}{\Delta t}\right),\quad i=1,\ldots,n$$

where the discrete version (3) and the $\widehat{\theta}_{i,k}$ are defined to reproduce the empirical or simulated signal $\widehat{u}_{i,k}$ following the Euler method:

$$u_{i,2} = u_{i,1} + \Delta t\cdot f_i\left(u_{1,1},\ldots,u_{n,1};\widehat{\theta}_{i,1}\right),\quad i=1,\ldots,n$$
$$u_{i,k} = u_{i,k-2} + 2\cdot\Delta t\cdot f_i\left(u_{1,k},\ldots,u_{n,k};\widehat{\theta}_{i,k}\right),\quad i=1,\ldots,n\quad k=3,\ldots,T$$

with initial conditions $u_{i,1} = \widehat{u}_{i,1}$ for $i = 1, ..., n$.

Thus, at each time step k the DMT provides a $\theta_{i,\,k}$ column with n components, and a temporal series of T different $1\times n$ columns is obtained.

3.2. Examples

Here we show examples of DMTs such as the discrete KT, the discrete SART, and the discrete RFT. See another example of discrete MT in 4.7.

3.2.1. Discrete Kuramoto transform

The simplest discrete version of the Eq. (7) is:

$$\frac{\theta_{i,2}-\theta_{i,1}}{\Delta t} = \omega_{i,1} + g\sum_{j=1}^{n}a_{ij}\ sin\ (\theta_{j,1}-\theta_{i,1}),\ \ i=1,\ldots,n$$
$$\frac{\theta_{i,k+1}-\theta_{i,k-1}}{2\cdot\Delta t} = \omega_{i,k} + g\sum_{j=1}^{n}a_{ij}\ sin\ (\theta_{j,k}-\theta_{i,k}),\ \ i=1,\ldots,n;\ \ k=2,\ldots,T-1$$
$$\frac{\theta_{i,T}-\theta_{i,T-1}}{\Delta t} = \omega_{i,T} + g\sum_{j=1}^{n}a_{ij}\ sin\ (\theta_{j,T}-\theta_{i,T}),\ \ i=1,\ldots,n$$

where each different k from 1 to T corresponds to a different time step. Solving for $\omega_{i,\,k}$ and substituting the discrete data $\widehat{\theta}_{i,k}$ the corresponding discrete Kuramoto transform (DKT) $\widehat{\omega}_{i,k}$ is:

$$\widehat{\omega}_{i,1}(t) = \frac{\widehat{\theta}_{i,2}-\widehat{\theta}_{i,1}}{\Delta t} - g\sum_{j=1}^{n}a_{ij}\ sin\left(\widehat{\theta}_{j,1}-\widehat{\theta}_{i,1}\right),\quad i=1,\ldots,n$$
$$\widehat{\omega}_{i,k}(t) = \frac{\widehat{\theta}_{i,k+1}-\widehat{\theta}_{i,k-1}}{\Delta t} - g\sum_{j=1}^{n}a_{ij}\ sin\left(\widehat{\theta}_{j,k}-\widehat{\theta}_{i,k}\right),\quad i=1,\ldots,n\quad k=2,\ldots,T-1$$
$$\widehat{\omega}_{i,T}(t) = \frac{\widehat{\theta}_{i,T}-\widehat{\theta}_{i,T-1}}{\Delta t} - g\sum_{j=1}^{n}a_{ij}\ sin\left(\widehat{\theta}_{j,T}-\widehat{\theta}_{i,T}\right),\quad i=1,\ldots,n$$

3.2.2. Discrete simultaneous autoregressive transform

The discrete version of the Eq. (8) is:

$$u_{i,k} = g\sum_{j\neq i}\gamma_{ij}u_{j,k} + \nu_{i,k},\ \ i=1,\ldots,n;\ \ k=1,\ldots,T$$

for the time step k from 1 to T. Solving for $\nu_{i,\,k}$ and substituting the discrete data $\widehat{u}_{i,k}$ the corresponding discrete simultaneous autoregressive transform (DSART) $\widehat{\nu}_{i,k}$ is:

$$\widehat{\nu}_{i,k} = \widehat{u}_{i,k} - g\sum_{j\neq i}\gamma_{ij}\widehat{u}_{j,k},\quad i=1,\ldots,n\quad k=1,\ldots,T$$

3.2.3. Discrete rate fluctuations transform

The simplest discrete version of the Eq. (9) is:

$$\tau\frac{u_{i,2}-u_{i,1}}{\Delta t} = -u_{i,1} + g\sum_{j\neq i}\gamma_{ij}u_{j,1} + \nu_{i,1},\ \ i=1,\ldots,n$$
$$\tau\frac{u_{i,k+1}-u_{i,k-1}}{2\cdot\Delta t} = -u_{i,k} + g\sum_{j\neq i}\gamma_{ij}u_{j,k} + \nu_{i,k},\ \ i=1,\ldots,n;\ \ k=2,\ldots,T-1$$
$$\tau\frac{u_{i,T}-u_{i,T-1}}{\Delta t} = -u_{i,T} + g\sum_{j\neq i}\gamma_{ij}u_{j,T} + \nu_{i,T},\ \ i=1,\ldots,n$$

for time step $1\le k\le T$. Solving for $\nu_{i,\,k}$ and substituting the discrete data $\widehat{u}_{i,k}$ the corresponding discrete rate fluctuations transform (DRFT) $\widehat{\nu}_{i,k}$ is:

$$\widehat{\nu}_{i,1} = \tau\frac{\widehat{u}_{i,2}-\widehat{u}_{i,1}}{\Delta t} + \widehat{u}_{i,1} - g\sum_{j\neq i}\gamma_{ij}\widehat{u}_{j,1},\quad i=1,\ldots,n$$
$$\widehat{\nu}_{i,k} = \tau\frac{\widehat{u}_{i,k+1}-\widehat{u}_{i,k-1}}{\Delta t} + \widehat{u}_{i,k} - g\sum_{j\neq i}\gamma_{ij}\widehat{u}_{j,k},\quad i=1,\ldots,n\quad k=2,\ldots,T-1$$
$$\widehat{\nu}_{i,T} = \tau\frac{\widehat{u}_{i,T}-\widehat{u}_{i,T-1}}{\Delta t} + \widehat{u}_{i,T} - g\sum_{j\neq i}\gamma_{ij}\widehat{u}_{j,T},\quad i=1,\ldots,n$$

4. Instantaneous attractors

In this section, as an example of application of the concept of MT we search for a model that allows us to find the stationary states towards which a dynamical system is attracted at each time point and how the corresponding attractor landscape is characterized. In different Subsections we a) justify the method, b) define the concept of minimal model, c) introduce some ideas and results from the Dynamical Systems Theory (DST), d) find the simplest model that meets certain conditions for our objective, e) apply the DST to the particular case of the suitable minimal model, f) define the corresponding MT to this minimal model, g) sumarize the formalism, h) apply it in several examples, i) show the usefulness of the method, and j) explore its application in high-dimensional systems.

4.1. Justification, benefits and advantages of the method

Especially in very complex systems such as the human brain, it is not realistic to imagine a stationary attractor landscape in time that would allow explaining its dynamics. On the other hand, if we assume that there is a non-stationary attractor landscape that explains brain dynamics, we would have to develop a method to approximate it, since this has not yet been done.

To develop it, we take inspiration from trivial examples such as the concept of velocity, which has a fixed value in uniform rectilinear motion but can be extended by difference calculus to variable values in other motions. The same differential equation $d\vec{r}/dt = \vec{v}$ that can be used as a "differential model" of uniform rectilinear motion when the parameter $\vec{v}$ is fixed can, alternatively, be taken as the definition of instantaneous velocity when its value is allowed to vary with time. In this second case, it is no longer a differential equation but the definition of $\vec{v}$, which could be calculated knowing, for example, the empirical values of $\vec{r}$.

Starting from that trivial example we try to transfer this idea what we formalize as Model Transform to the concept of global attractor. Our goal is to find a simple but non-trivial model for which, when the corresponding parameters are fixed, this global attractor is well known. Interestingly, this global attractor varies when the parameters are allowed to vary over time. In analogy to the trivial example of velocity, the same equation that defines the model serves for the empirical calculation of the parameters as functions of time, substituting in it the empirical values, typically, empirical time series.

The ideal model for this purpose should be the simplest one that meets the following requirements: a nontrivial model with a finite number of stationary points which includes at least the trivial solution $(0,0,\ldots,0)$ and a globally asymptotically stable solution (GASS). This GASS is the state the system is attracted to, since from any initial point the system will converge to the GASS. Here, "nontrivial" means that the model includes more than one stationary point. Finally, in order to have a manageable and interesting structure to study, a finite number (larger than one) of stationary points is required.

It will be shown that the model proposed has multiple advantages, such as: i) it includes a rich attractor landscape with a large number of stationary points, each being a unique combination of active and inactive nodes of the dynamical system, ii) it includes empirical information regarding the interaction matrix of the dynamical system, iii) it has a well-known condition for existence and uniqueness of the GASS, iv) it supports a Lyapunov function so that its isolated invariants could be ordered according to its energy levels, v) it supports an algorithm of low computational complexity to find the GASS (see Section 4.10), and vi) it has a well-known structure of the global attractor.

Therefore, it is about extending the concept of global attractor taken from the dynamical systems theory (DST) to non-stationary attractor landscapes, such that the state to which a system is attracted at each instant of time can be approximated.

4.2. Minimal model

The specific choice of the model and parameters for the model transform (MT) depends on the question we want to answer. In trivial models, if we want to quantify the "departure from rest" or "distance" between the rest and a given motion we must look for the simplest possible motion that allows us to measure that distance. The simplest motion in this case is the uniform motion in a straight line and, fitting it to each small neighborhood of $\vec{x}(t)$, the variation per unit of time of $\vec{x}(t)$ is a measure of "remoteness from rest". Therefore, we say that the uniform motion in a straight line is the minimal model that transforms a motion $\vec{x}(t)$ into $\vec{v}(t)$, or conversely that $\vec{v}(t)$ is the MT of $\vec{x}(t)$ when the minimal model is the uniform motion in a straight line. Note that first, we have an intuitive idea of what we want to measure and then, after looking for a minimal model, the intuitive idea is specified and formalized. While the uniform motion in a straight line is the simplest model that allows us to know "how far an object moves from rest at each time instant", i.e. how a motion $\vec{x}(t)$ "moves away from rest", in Table 1 other examples of minimal models are shown. Note that the *curvature* or "How far a curve moves away from a straight line in any small segment" can be measured using, for example, a parabola as model, but that the simplest model that allows us to measure the curvature through a MT is the circle.

Hereafter, we search for the minimal model that allows us to answer the questions "towards which state is the dynamical system attracted to at each time point?" and "what does the corresponding instantaneous attractor look like?" These questions will be faced with help of the Dynamical Systems Theory (DST).

4.3. Introduction to Dynamical Systems Theory (DST)

The *Global Attractor* (GA) is an important concept in DST which describes all the future scenarios of a Dynamical System (DS). The GA is defined as follows [5,22,24,28,36,47]:

A set $\mathcal{A}$ in the phase space X, $\mathcal{A}\subseteq X$ is a GA for the semigroup $\{S(t) : t \geq 0\}$ (which characterizes the DS) if it is

(i) compact,
(ii) invariant under $\{S(t) : t \geq 0\}$, i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and.
(iii) attracts bounded subsets of X under $\{S(t) : t \geq 0\}$ for the Hausdorff semidistance; i.e., for all bounded $B \subset X$

$$\lim_{t\to+\infty} \mathrm{dist}_H(S(t)B, \mathcal{A}) := \lim_{t\to+\infty} \sup_{b\in B} \inf_{a\in\mathcal{A}} d(S(t)b, a) = 0.$$

where dist_H is the Hausdorff distance.

A global solution is a function $\xi : \mathbb{R} \to X$ such that $\xi(t+s) = S(t)\xi(s)$ for all $s \in \mathbb{R}$, $t \in \mathbb{R}^+$. A *orbit* initiating at u_0 is the set $\{S(t)u_0 : t \in \mathbb{R}^+\} \subset X$. $u^* \in X$ is a stationary point if $S(t)u^* = u^*$, for all $t \geq 0$. A stationary point is the simplest instance of global solution associated with $S(t)$. A stationary point $u^* \in X$ is said to be *stable* if for any neighborhood U of u^*, there

Table 1
The *model transform* (MT) allows simple theoretical models to be applied to complex empirical systems and the specific choice of model and parameter depends on the question we want to answer.

Minimal model	Variable parameter	Question we want to answer
Uniform motion in a straight line	Instantaneous velocity	How far an object moves from rest at each time point
Circle	Curvature or reciprocal of the radius of curvature	How far a curve moves away from a straight line in any small segment
Helix	Torsion	Distance between any three-dimensional curve and a planar curve

exists a neighborhood W of u^* such that any orbit initiating in W at time $t = 0$ remains in U for all $t \geq 0$. It is said to be *asymptotically stable* if it is stable and the orbit converges to u^*. If u^* is not stable, it is said to be *unstable*. The *basin of attraction* of u^* is defined by the set of points u_0 satisfying $S(t)u_0 \to u^*$ as $t \to +\infty$. When the basin of attraction of u^* is the whole phase space or at least its interior and u^* is stable, u^* is said to be *globally stable*. When a stationary point is asymptotically and globally stable it is called *globally asymptotically stable solution* (GASS) [46].

Generically, the *GA* structure can be described by isolated invariant sets (typically stationary points or periodic orbits [10,23,25,34], see definition of isolated invariant set in Supplementary Materials) and connecting global solutions among them ([2,10], see definition of connecting global solution in Supplementary Materials). Those connections among invariant sets describe its structure [4,8].

We will say that a semigroup $\{S(t) : t \geqslant 0\}$ with a GA $\mathcal{A}$ and a disjoint family of isolated invariant sets $\Omega = \{\Xi_1, \ldots, \Xi_N\}$ is a gradient semigroup with respect to Ω, if there exists a continuous function $V : X \to \mathbb{R}$ such that $[0,\infty) \ni t \mapsto V(S(t)u) \in \mathbb{R}$ is decreasing for each $u \in X$, V is constant in Ξ_i for each $1 \leqslant i \leqslant N$, and $V(S(t)u) = V(u)$ for all $t \geq 0$ if and only if $u \in \bigcup_{j=1}^{N} \Xi_j$. V is called Lyapunov function respect to Ω (see more about gradient semigroups in Supplementary Materials).

So, a DS is called gradient if there is some continuous real-valued function which is strictly decreasing on nonconstant solutions. This description shows a geometrical picture of the GA of a dynamically gradient system, in which all the stationary points or isolated invariant sets are ordered by connections related to its *level of attraction* [3] or stability. Thus a consequence of it is that the GA is isomorphic to a directed graph: Each of the Ξ_i is associated with a vertex or node from the graph, and there is a directed edge from the vertex associated to Ξ_i towards the vertex associated to Ξ_j if and only if there is a global solution that connects Ξ_i to Ξ_j. That edge is directed because there is a Lyapunov function which is decreasing across the global solution that connects Ξ_i to Ξ_j. The resulting directed graph is called *Informational Structure* (IS) [16,26].

We can define an order between two isolated invariant sets Ξ_i and Ξ_j saying that Ξ_i preceds Ξ_j ($\Xi_i \prec \Xi_j$) if there is a chain of global solutions

$$\{\xi_\ell,\ \ i \leq \ell \leq j-1\} \tag{10}$$

with

$$\lim_{t \to -\infty} dist(\xi_\ell(t), \Xi_\ell) = 0$$

and

$$\lim_{t \to \infty} dist(\xi_\ell(t), \Xi_{\ell+1}) = 0.$$

This implies that, given any gradient semigroup with respect to the disjoint family of isolated invariant sets $\Omega = \{\Xi_1, \ldots, \Xi_N\}$, there exists a partial order in Ω.

There exists a dynamical description of a gradient semigroup by reordering and regrouping the corresponding isolated invariant subsets to obtain a totally ordered family of isolated invariant sets that we refer to as energy levels.

Let us consider

$$\mathrm{M}_1 := \{\Xi_l \in \Omega : \text{there is no element } \Xi \in \Omega \text{ that preceeds } \Xi_l\}$$

and, for any integer $k \geq 2$,

$$\mathrm{M}_k := \{\Xi_1 \in \Omega : \text{if } \Xi \in \Omega \text{ and } \Xi \prec \Xi_l \text{ then } \Xi \in \mathrm{M}_{k-1}\}.$$

Note that, by definition, $\mathscr{M}_k \subset \mathscr{M}_{k+1}$.

We now define the sets

$$\mathcal{N}_1 := \bigcup_{\Xi \in \mathrm{M}_1} \Xi$$

and

$$\mathcal{N}_k := \bigcup_{\Xi \in \mathrm{M}_k \setminus \mathrm{M}_{k-1}} \Xi\ ,$$

for all $k \geq 2$.

Each of the levels $\mathcal{N}_i$, $1 \leq i \leq q$ is made of a finite union of the isolated invariant sets in Ω and $\mathcal{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_q\}$ is totally ordered by the dynamics defined by (10). Indeed, the associated Lyapunov function has strictly decreasing values in any global solution linking two different level-sets of $\mathcal{N}$ and any two elements of Ω which are contained in the same element of $\mathcal{N}$ (same energy level) are not connected (see more about energy levels in Supplementary Materials).

4.4. Minimal model to find the instantaneous global attractor

Mathematically speaking, we aim to find the simplest nontrivial model with a finite number of stationary points which includes at least the trivial solution (0,0,…,0) and a globally asymptotically stable solution (GASS). This GASS is the state the system is attracted to, since from any initial point (the basin of attraction of is the whole phase space or at least its interior) the system will converge to the GASS. Here, "nontrivial" means that the model includes more than one stationary point. Finally, in order to have a manageable and interesting structure to study, a finite number (larger than one) of stationary points is required.

We can choose a system of higher order differential equations but for the sake of simplicity, we choose first order differential equations: the lower the order the simpler the model. The most general form of a model of n first order differential equations is:

$$\frac{du_i}{dt} = F_i(u_1, \ldots, u_j, \ldots, u_n),\ \ i \in \{1, \ldots, n\}, \tag{11}$$

where each i corresponds to a different node. We restrict $F_i(u_1,\ldots,u_j,\ldots,u_n)$ to analytical functions for all $i \in \{1,\ldots,n\}$, again for simplicity. Thus, it has a Taylor series at the origin (Maclaurin series):

$$F_i(u_1, \ldots, u_j, \ldots, u_n) = F_i(0, \ldots, 0) + \sum_{j=1}^{n} \frac{\partial F_i(0, \ldots, 0)}{\partial u_j} u_j +$$
$$+ \frac{1}{2!} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 F_i(0, \ldots, 0)}{\partial u_j \partial u_k} u_j u_k + \frac{1}{3!} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial^3 F_i(0, \ldots, 0)}{\partial u_j \partial u_k \partial u_l} u_j u_k u_l + \ldots$$

The more terms we consider the more complex our model is. But if we preserve only the terms up to the first order, the model is trivial since it will include at most a single stationary point. Therefore, we keep the terms up to the second order

$$\frac{du_i}{dt} = F_i(0, \ldots, 0) + \sum_{j=1}^{n} \frac{\partial F_i(0, \ldots, 0)}{\partial u_j} u_j + \frac{1}{2!} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 F_i(0, \ldots, 0)}{\partial u_j \partial u_k} u_j u_k,\ \ i \in \{1, \ldots, n\}. \tag{12}$$

This new $F_i(u_1,\ldots,u_j,\ldots,u_n)$ is a non-homogeneous multivariate quadratic form that must include at least two zeros for the model to be nontrivial. To avoid complex cases with chaotic solutions (see, for instance, the Lorenz [29], Rössler [38], Chen [9], and Lü [30] systems) we require that F_i can be factored into non-homogeneous multivariate linear forms:

$$F_i = F_i(u_1, \ldots, u_j, \ldots, u_n) = \left(\alpha_i + \sum_{j=1}^{n} a_{ij} u_j\right)\left(\beta_i + \sum_{j=1}^{n} b_{ij} u_j\right).$$

This also simplifies the search for stationary points: $\frac{du_i^*}{dt} = 0 \Leftrightarrow \gamma_i + \sum_{j=1}^n c_{ij} u_j^* = 0$ where γ_i and c_{ij} can be α_i and a_{ij}, or β_i and b_{ij} generating up to 2^n different systems of equations that, if they are determinate compatible, would involve up to 2^n stationary points.

The trivial solution (0,0,...,0) is included in

$$\frac{du_i}{dt} = \left(\alpha_i + \sum_{j=1}^n a_{ij} u_j\right)\left(\beta_i + \sum_{j=1}^n b_{ij} u_j\right), \quad i \in \{1, \ldots, n\}$$

if either α_i or β_i are null for all $i \in \{1,...,n\}$. We choose, without loss of generality, $\beta_i = 0$ for all $i \in \{1,...,n\}$:

$$\frac{du_i}{dt} = \left(\alpha_i + \sum_{j=1}^n a_{ij} u_j\right)\left(\sum_{j=1}^n b_{ij} u_j\right), \quad i \in \{1, \ldots, n\}. \tag{13}$$

But $\sum_{j=1}^n b_{ij} u_j = 0$ could include infinite solutions if $B = (b_{ij})$ is singular, i.e., it has determinant $|B| = 0$. The simplest regular matrix is the identity matrix $I = (\delta_{ij})$, so choosing $B = I$ the model reaches the form:

$$\dot{u}_i = u_i\left(\alpha_i + \sum_{j=1}^n a_{ij} u_j\right), \qquad i = 1, \ldots, n. \tag{"NewLabel"}$$

These equations are called Lotka-Volterra (LV) model. It was initially proposed by Alfred J. Lotka in the theory of autocatalytic *Chemical Reactions*. It is frequently used to describe the *Dynamics of Biological Systems*. The LV equations have also a long history of use in *Economic Theory*. In practice, it is accepted that $u_i \geq 0$ for $i = 1, ..., n$ as the positive cone R_+^n is an invariant set [45]. In the next subsection we show that this model:

i) includes a rich attractor landscape with a large number of stationary points, each being a unique combination of active ($u_i^* \neq 0$) and inactive ($u_i^* = 0$) nodes of the dynamical system,
ii) includes empirical information regarding the interaction matrix a_{ij} of the dynamical system,
iii) has a well-known condition for existence and uniqueness of the state towards which the system is attracted to, i.e. the GASS [45],
iv) supports a Lyapunov function so that its isolated invariants could be ordered according to its energy levels,
v) supports an algorithm of low computational complexity to find the GASS,
vi) has a well-known structure of the global attractor.

4.5. DST applied to Lotka-Volterra systems

Here they are shown the concepts of stationary solutions, GASS, GA, and energy levels for Lotka-Volterra (LV) systems.

Originally LV model comes from populations dynamics and its solutions are restricted to positive values, so the phase space for Eq. ("NewLabel") is the positive orthant, which is an invariant set,

$$\mathrm{R}_+^n = \{u = (u_1, \ldots, u_n) \in \mathrm{R}^n, u_i \geq 0, \quad i = 1, \ldots, n\}. \tag{14}$$

Given an initial data for Eq. ("NewLabel"), sufficient conditions for the existence and uniqueness of global solutions are well-known (see, for instance, [32,45]).

LV systems include empirical information regarding the interaction matrix simply choosing a_{ij} as the empirical connectivity of the system for nodes i and j which is usually called *structural connectivity*.

4.5.1. Stationary solutions for the Lotka-Volterra model

The next theorem shows that model assumes up to 2^n stationary points:

Theorem 2. Each stationary point $u^* = (u_1^*, u_2^*, ..., u_m^*)$ of Eq. ("NewLabel") consists of a unique combination of null and non-zero variables.

Proof: We have to prove that given I a subset of $M = \{1,...,m\}$ such that $u_l^* = 0$ if and only if $l \in I$ the solution u^* is unequivocally determined. Each stationary solution holds $u_i^*(\alpha_i + \sum_{j=1}^n a_{ij} u_j^*) = 0$ for $i = 1, ..., n$, so for each i it can either hold $u_i^* = 0$ or

$$\alpha_i + \sum_{j=1}^n a_{ij} u_j^* = 0. \tag{15}$$

For any $k \notin I$ and due to (15) we have

$$u_k^* = \frac{-1}{a_{kk}}\left(\alpha_k + \sum_{j \neq k, j \notin I}^n a_{kj} u_j^*\right)$$

which univocally defines a positive number since it can not be zero by definition of I and can not be negative because $u^* \in \mathrm{R}_+^n$. Therefore the solution u^* is univocally defined by I. Q.E.D.

4.5.2. Globally asymptotically stable solution in the LV model

Hereafter, it is shown that the linear complementarity problem (LCP) in the theory of mathematical programming are closely connected with the problem to find out a globally asymptotically stable solution (GASS) in LV systems.

The Linear Complementarity Problem (LCP) (see [13,33]) states that, given $r \in \mathbb{R}^n$ and a matrix M of order n, we try to find $(w,z) \in \mathbb{R}^{2n}$, $w = (w_1, w_2, ..., w_n)^T$, $z = (z_1, ..., z_n)^T$, such that

$$\begin{aligned} &w = r + Mz \\ &w \geq 0, z \geq 0 \ \text{ and } w_i z_i = 0 \text{ for all } i = 1, \ldots, n. \end{aligned} \tag{16}$$

(See more about the LCP and its solutions in Supplementary Materials). Existence and uniqueness of solution to the LCP depends on the stability of the matrix M. Hereafter we show some definitions and results about stability of matrices:

$A \in \mathbb{R}^{n \times n}$ is said to be stable if all associated eigenvalues has negative real part.

A is positive semi-definite (negative semi-definite), if $u^T A u \geq 0$ ($u^T A u \leq 0$) for all $u \in \mathbb{R}^n$. It is positive definite (negative definite) if $u^T A u > 0$ ($u^T A u < 0$) for all $u \in \mathbb{R}^n \backslash \{0\}$..

A matrix A belongs to class S_w or is Lyapunov-stable (see [14]), $A \in S_w$, if there exists a diagonal positive matrix W such that $WA + A^T W$ is negative definite.

A is called negative dominant diagonal, $A \in NDD$ if there exist n positive numbers $v_i > 0$ such that

$$-v_i a_{ii} > \sum_{i \neq j}^n |a_{ij}| v_j, \quad i = 1, \ldots, n.$$

Recall that a minor of a matrix A is the determinant of some smaller square matrix, cut down from A by removing one or more of its rows or columns. If I and J are subsets of $\{1,...,n\}$ with k elements, then we write $[A]_{I,J}$ for the $k \times k$ minor of A that corresponds to the rows with index in I and the columns with index in J. If $I = J$, then $[A]_{I,J}$ is called a principal minor. A is said to be a P-matrix, $A \in P$, if all principal minors of A are positive.

A matrix A in S_w implies $-A$ to be a P-matrix [45].

A belongs to S_w if any of the following conditions hold:

1. A is negative diagonal dominant;
2. A es negative definite.

The $LCP(r,M)$ possesses a unique solution $r \in \mathbb{R}^n$ if and only if M is a P-matrix.

Now we show the relationship between the LV systems and the LCP:

Consider the stationary point $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ of the LV equations expressed in the general way:

$$\frac{du_i}{dt} = u_i\left(\alpha_i + \sum_{j=1}^{n} a_{ij}u_j\right), \quad i = 1, \ldots, n, \tag{17}$$

In a stationary point $\frac{du_i}{dt} = 0$ for $i = 1, \ldots, n$ and by (14) $u_i^* \geq 0$. So any stationary point u^* holds that:

$$\begin{cases} u_i^* & \geq & 0, \\ u_i^*\left(\alpha_i + \sum_{j=1}^{n} a_{ij}u_j^*\right) & = & 0, \quad i = 1, \ldots, n. \end{cases} \tag{18}$$

There exists and equivalence between looking for the GASS of the LV system and the solution of a *LCP*, as shown by the following result:

Lemma 1. *The LCP*$(-\alpha, -A)$ *where* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ *and* $A = [a_{ij}]$ *is equivalent to find a nonnegative* u^* *stationary point of* (17) *satisfying*:

$$\alpha_i + \sum_{j=1}^{n} a_{ij}u_j^* \leq 0 \quad \text{for } i = 1, \ldots, n. \tag{19}$$

Proof: It is enough to take

$$z = u^* \quad y \quad w = -b - Au^*,$$

so that $M = -A$ and $r = -\alpha$ and thus it holds $w = Mz + r$, $w \geq 0, z \geq 0$ and $w_i z_i = 0$, for all $i = 1, \ldots, n$. Q.E.D.

u^* satisfying (19) is called saturated.

In this way, LV Eq. (17) has a unique saturated equilibrium point for each $\alpha \in \mathbb{R}^n$ if and only if the LCP$(-\alpha, -A)$ possesses a unique solution.

The following important result (see [46]) now gives us the global stability of saturated equilibria for LV systems:

Theorem 3. Suppose $A \in S_w$. Then the LV system (17) possesses a saturated stationary point u^* for each $\alpha \in \mathbb{R}^n$ which is globally stable in $\mathbb{R}_I^n$.

It assures existence of the GASS, i.e., the stationary point for which (19) is satisfied (see [45]).

Note that if $A \in S_w$, every principal submatrix of A also belong to S_w.

We finally have (see [45]):

Corollary 1. If $A \in S_w$, then the LV system and all its associated subsystems possess a unique globally asymptotically stable solution (GASS) for each $\alpha \in \mathbb{R}^n$.

Note: If $A \in S_w$, then $-A$ is a *P*-matrix and $a_{ii} < 0 \quad \forall\, i$. Thus, without loss of generality they can be chosen $a_{ii} = -1\ \forall i$ by re-scaling each u_i. In addition, it can be introduced a global coupling strength parameter g so that $a_{ij} = g\gamma_{ij}$, $\forall j \neq i$:

$$\dot{u}_i = u_i\left(\alpha_i - u_i + g\sum_{j\neq i}^{n} \gamma_{ij}u_j\right), \quad i = 1, \ldots, n, \tag{20}$$

or, expressed in matrix form, $\dot{u} = u(\alpha - u + g\Gamma u)$.

Theorem 4. The system (20) has an asymptotically globally stable solution if $g < 1/\rho(\Gamma)$, where ρ is the spectral radius of a matrix Γ, i.e. $\rho(\Gamma) = max_\lambda\, |\lambda(\Gamma)|$, with $\lambda(\Gamma)$ denoting an eigenvalue of Γ.

4.5.3. Structure of the global attractor and energy levels in a gradient LV system

They can be shown that if A in $\frac{du}{dt} = u(\alpha + Au)$ is Lyapunov-stable:

1) There is a GASS in the positive cone, and a finite set of all stationary points $U^* := \{u_1^*, u_2^*, \ldots, u_m^*\}$.
2) The semigroup associated with the equations $u' = u(\alpha + Au)$, is a *gradient semigroup* with respect to U^*, so the strucure of the GA is a directed graph in such a way that just one of the stationary points in the positive cone, R_+^n, is a globally asymptotically stable solution (GASS).
3) This GASS can be interpreted as the lower end of the structure of the GA if it is represented with the directed links going from up to down (see Fig. 1).
4) Each stationary point in U^* has a unique combination of null and non-zero variables. So the structure of the global attractor is isomorphic to a subgraph of the directed hypercube $[0,1]^n$. This hypercube has 2^n vertices (or nodes) and $2^{n-1}n$ edges (or links) where each directed link goes to the vertex nearest $(0,0,\ldots,0)$ to the vertex nearest $(1,1,\ldots1)$.
5) Recall that as the DS is defined on a graph, two different graphs are defined: A the proper graph on which the DS is defined (structural network, Fig. 1A Left) and B the structure of the global attractor (Fig. 1A Right). In the particular case of the LV model there is also a unique subgraph of the structural graph associated to each stationary point (nodes of B) since each stationary point has a unique combination of null and non-zero variables and the corresponding subgraph would be the one that only includes the nodes corresponding to the non-null variables. Thus, the GA and its corresponding directed graph can be understood as *a new dynamical network describing all the possible feasible future networks* [20,21].
6) Each energy level is formed by stationary points with the same number of non-zero components. The number of energy levels (NoEL) equals the number of non-zero entries in the GASS plus one. The first level of energy (associated with a source, a node or vertex with only outgoing edges) is always comprised of the trivial solution (all variables equal zero), while the last energy level includes only the GASS (associated with a sink, a node or vertex with only incoming edges). Depending on α and A all the components of the GASS could be non-zero and the number of energy levels would be $n + 1$. In some cases, the structure may even be complete, i.e., including 2^n stationary points (as in Fig. 1A Right).

4.6. Lotka-Volterra transform

Definition 5. Given the values of the parameters $g \in \mathbb{R}^+$ and $\Gamma = (\gamma_{ij}) \in \mathbb{R}^{n\times n}$ in a LV model, and given a sufficiently smooth function of time $\hat{u}$:

$$\begin{aligned} &\hat{u} : \mathbb{R} \to \mathbb{R}_+^n \\ &\hat{u}(t) = (\hat{u}_1(t), \ldots, \hat{u}_i(t), \ldots, \hat{u}_n(t))^T, \quad \text{for all } t \in \mathbb{R} \\ &\hat{u}_i(t) \in \mathbb{R}^+, \text{for all } i = 1, \ldots, n. \end{aligned}$$

the Lotka-Volterra transform (LVT) of $\hat{u}(t)$ is defined as the function $\hat{\alpha}(t)$ that fulfills:

$$\begin{aligned} &\hat{\alpha} : \mathbb{R} \to \mathbb{R}^n \\ &\hat{\alpha}(t) = (\hat{\alpha}_1(t), \ldots, \hat{\alpha}_i(t), \ldots, \hat{\alpha}_n(t))^T, \quad \text{for all } t \in \mathbb{R} \\ &\hat{\alpha}_i(t) = \frac{\dot{\hat{u}}_i(t)}{\hat{u}_i(t)} + \hat{u}_i(t) - g\sum_{j=1}^{n} \gamma_{ij}\hat{u}_j(t), i = 1, \ldots, n. \end{aligned}$$

and the numerical values of functions $\hat{\alpha}_i(t)$ for any $t = t_0$ are called instantaneous growth rates.

Remark: Again, notice that the previous expression have been solved for α_i which makes sense, for example, when data $\hat{u}_i(t)$ are known, and, therefore, also $\dot{\hat{u}}_i(t)$. Usually $u(t)$ is the solution of the equation given $\alpha(t)$ but now the situation is the opposite. LVT is defined such that given the initial values of $\hat{u}(t)$, the LV equations' solutions fit exactly the empirical or simulated data $\hat{u}(t)$. Nevertheless, in the context of the LVT the LV equations are not assessed as a model, since choosing the suitable $\alpha(t)$ the fitting is always perfect for any sufficiently smooth $\hat{u}(t)$.

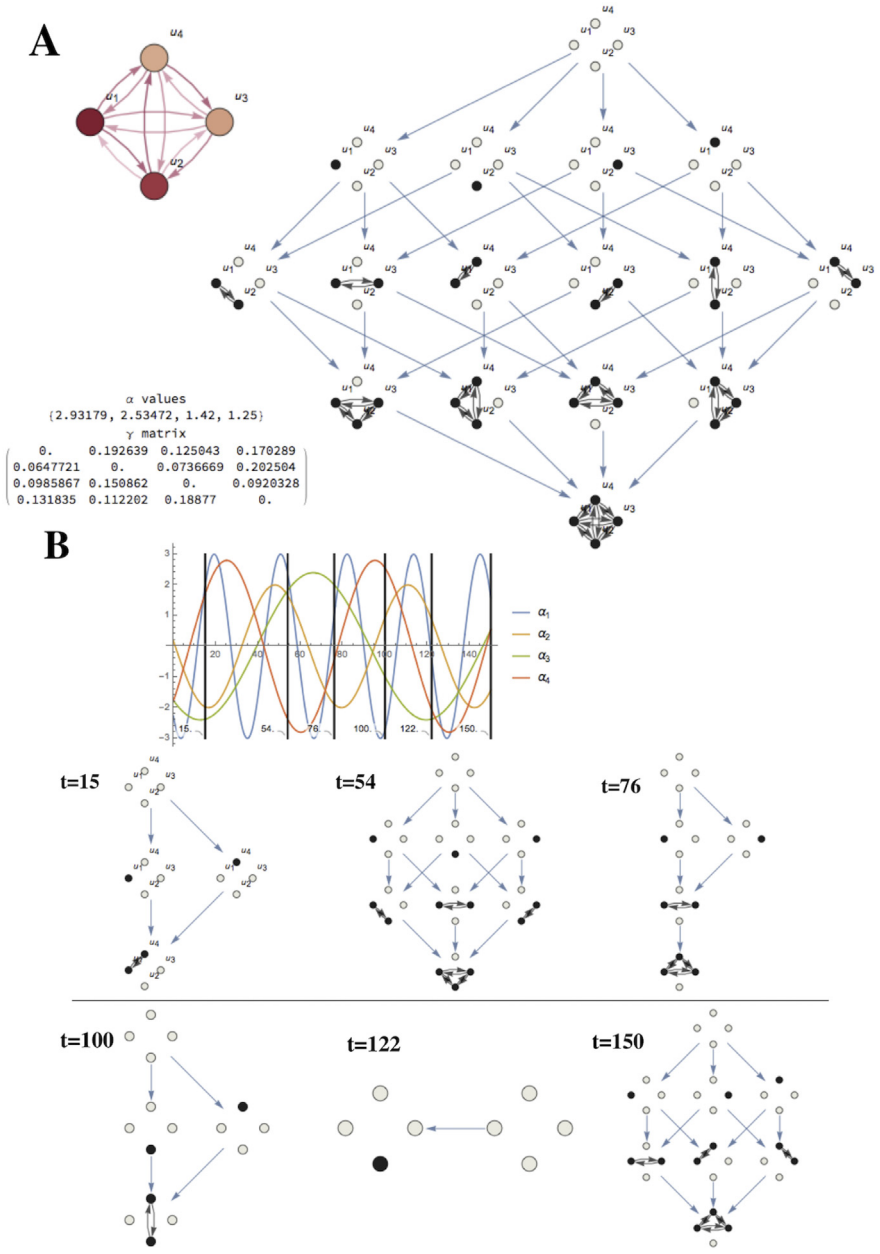

Fig. 1. A, (Left) Structural network of a cooperative ($\gamma_{i\neq j} \geq 0$) 4-dimensional ($n = 4$) Lotka-Volterra (LV) system given by a γ matrix for a equation like Eq. (20). (Right) Structure of the global attractor. Each node is a stationary point which can be represented as a subgraph of the structural network where non-null components are shown in black and null in grey. B, (Top) Evolution in time of α_i parameters of a four-node system. In this example α_i are periodic functions of time. Changes in the parameters governing the dynamics produce changes in the corresponding structures of the Global Attractor. (Bottom) They are shown the corresponding different structures for the time steps shown above. For these LV systems each energy level is formed by stationary points with the same number of non-zero components. The number of energy levels (NoEL) changes over time also, but equals the number of non-zero entries in the GASS plus one.

4.7. Recapitulation and procedure

The LV model is ideal for the purpose of answering the questions "towards which GASS is the dynamical system attracted to at each time point?" and "how is the structure of the corresponding global attractor?", because LV is the simplest (minimal) model with a finite number of stationary points (but more than one) which includes at least the trivial solution (0,0,…,0) and a GASS. Furthermore, this model includes a rich attractor landscape with a large number of stationary points, each being a unique combination of active ($u_i^* \neq 0$) and inactive ($u_i^* = 0$) nodes of the dynamical system (as a consequence of including the trivial solution) which makes it easy to interpret the meaning of the stationary points in practical applications. The model also includes the empirical structural connectivity of the dynamical system, and the conditions for existence and uniqueness of the GASS are well-known. Cooperative LV systems have a well-known structure of the global attractor, their invariants can be ordered according to its energy levels, and supports an algorithm of low computational complexity to find the GASS and the NoEL [3,18,21].

Hence, we consider the Lotka-Volterra transform (LVT), given by:

$$\alpha_i(t) = \frac{\dot{u}_i(t)}{u_i(t)} + u_i(t) - g\sum_{j=1}^{n} \gamma_{ij} u_j(t), \qquad i = 1, \ldots, n. \tag{21}$$

In practice, $u(t)$, $\dot{u}(t)$, g, and γ_{ij} are empirical values where $u(t)$, $\dot{u}(t)$ are obtained in the form of discrete time series. The discrete LVT is a particular case of DMT (see Section 3). The simplest discrete version of the LV equations are:

$$\frac{u_{i,2}-u_{i,1}}{\Delta t} = u_{i,1}\left(\alpha_{i,1}-u_{i,1}+g\sum_{j=1}^{n}\gamma_{ij}u_{j,1}\right),\quad i=1,\ldots,n$$

$$\frac{u_{i,k+1}-u_{i,k-1}}{2\cdot\Delta t} = u_{i,k}\left(\alpha_{i,k}-u_{i,k}+g\sum_{j=1}^{n}\gamma_{ij}u_{j,k}\right),\quad i=1,\ldots,n;\quad k=2,\ldots,T-1$$

$$\frac{u_{i,T}-u_{i,T-1}}{\Delta t} = u_{i,T}\left(\alpha_{i,T}-u_{i,T}+g\sum_{j=1}^{n}\gamma_{ij}u_{j,T}\right),\quad i=1,\ldots,n$$

and, solving for α, the corresponding discrete LVT:

$$\alpha_{i,1} = \frac{u_{i,2}-u_{i,1}}{\Delta t\cdot u_{i,1}}+u_{i,1}-g\sum_{j=1}^{n}\gamma_{ij}u_{j,1},\quad i=1,\ldots,n$$

$$\alpha_{i,k} = \frac{u_{i,k+1}-u_{i,k-1}}{2\Delta t\cdot u_{i,k}}+u_{i,k}-g\sum_{j=1}^{n}\gamma_{ij}u_{j,k},\quad i=1,\ldots,n;\quad k=2,\ldots,T-1$$

$$\alpha_{i,T} = \frac{u_{i,T}-u_{i,T-1}}{\Delta t\cdot u_{i,T}}+u_{i,T}-g\sum_{j=1}^{n}\gamma_{ij}u_{j,T},\quad i=1,\ldots,n$$

We call this new mathematical tool "*transform*" because it starts from a $n\times T$ data matrix $u_{i,k}$ (n nodes, T time steps) while the result of the discrete LVT is another $n\times T$ matrix called *alpha*. Thus, at each time step k the output of the LVT is a n-dimensional column vector $\alpha_{i,k}$ ($i=1,\ldots,n$), such that we obtain a temporal series of T different columns vectors. Then, for each time step k we calculate the global attractor of a LV system using the column vector $\alpha_{i,k}$. Thus, we obtain a global attractors time series (see Fig. 1A) since for each time step k the column vector $\alpha_{i,k}$ defines a LV system for which the GA, its structure of stationary points and the GASS can be calculated. In practice, if we are interested only in the GASS and the NoEL and not in the whole structure of the GA, the linear complementarity problem (LCP) can be used. Let us remember that exists an equivalence between obtaining the GASS of collaborative LV systems and solving a LCP (see Section 4.5.2). Furthermore, the NoEL equals the number of non-zero entries in the GASS plus one (see Section 4.5.3). The LCP is solved using, for instance, the Complementary Pivot Algorithm (CPA, [13,33]) an algorithm of low computational complexity which allow us to calculate large time series of GASSs in a short period of time (See Section 4.10 for more information about the low complexity of the CPA).

4.8. Examples of application of LVT

In a first example (see Fig. 2AB), the starting data consists of a time series generated by a specific model and the MT associated with that same model is performed on these data. In such cases the result will be constant over time. Specifically, we have generated data from a Lotka-Volterra model with $n=7$ and a given alpha column vector. We also need a certain connectivity matrix and a certain value of g to generate the data. When applying the LVT to these data, we obtain, as expected, a constant alpha column vector in time, precisely the one from which we started, except for the variations produced by the noise added in the data generation.

In a second example (Fig. 2CD), the generating model and the transforming model are again the same (LV), but the parameters (alpha) are time-varying. Specifically, the components of alpha are piecewise functions. Again, when applying the LVT we obtain the components of the alpha function from which we started, except for the variations produced by the added noise.

But the information that MT can extract from a empirical or simulated system is not reduced to these trivial cases. More interestingly, in a third example (Fig. 2EF), the data has been generated by a different model than the one used for the MT. Specifically, we use a *Multivariable Ornstein–Uhlenbeck* (MOU, a model with applications in *Financial Mathematics*, *Physical Sciences*, *Evolutionary Biology*, and *Computational Neuroscience*) process ($n=4$) to generate the data:

$$\frac{du_i(t)}{dt} = -\frac{u_i(t)}{\tau}+\sum C_{ij}u_j(t)+\nu_i.$$

where τ is the leakage time constant (common to all nodes) and ν_i the input noise. In our simulations, the connectivity matrix

$$C=\begin{pmatrix}0&0&\beta&0\\ \beta&0&0&0\\ 0&\beta&0&\beta/2\\ 0&0&\beta/2&0\end{pmatrix}=\beta\begin{pmatrix}0&0&1&0\\ 1&0&0&0\\ 0&1&0&1/2\\ 0&0&1/2&0\end{pmatrix}=\beta D$$

depends on β. β can be considered as a measure of the effective connectivity (EC) of a system whose structural connectivity (SC) represented by D remains constant. For each β value, time series of data are generated through the MOU to which the LVT is applied. For each β the $n\times T$ data matrix (n nodes, T time steps) is transformed into another $n\times T$ alpha matrix. For each β value and for each time step we calculate the number of energy levels (NoEL) of the corresponding global attractor (as explained in Section 4.7), which will be between 1 and $n+1$. Therefore, for each β we obtain a time series of NoEL values and we can calculate its mean and standard deviation.

The results are shown in Fig. 2EF. It is observed that both the average NoEL and its standard deviation are linearly correlated with the effective connectivity of the system. From this we can infer that the LVT is a useful tool to estimate the EC in complex systems in which, although the SC remains constant, the EC varies depending on the state of the complex system. In fact, this is compatible with the results obtained in [18,44], considering that low-consciousness brain states such as *sleep* and *coma* are generally considered to be associated with lower levels of EC. Note that it is not obvious a priori what information each specific MT will extract from each empirical system, and it will often be inferred a posteriori from applications of each specific MT to both simulated and empirical datasets.

4.9. Usefulness of the LVT

In Section 4.8 it has been shown that the LVT could be a useful tool to estimate the EC for complex systems in which, while the SC remains constant, the EC varies depending on the state of the complex system. Nevertheless, the general usefulness of the LVT is to calculate where an empirical system is attracted to at each time point: that is, the instantaneous attractor. The same method also allows to approximate the surrounding attractor landscape including its stationary points (saddle and unstable), the structure of the corresponding global attractor and its number of energy levels. All the stationary points, which are mostly saddle points of the AL, repel or attract the state of the system. This can be visualized by imagining these points curving the phase space, and, indirectly, affecting the dynamics of the system (see Fig. 3 in [16]).

The novelty of this approach is essentially a first transformation of empirical data into a non-stationary description of the attractor landscape by a dynamical system approach. On the side of applied mathematics and DST, this way to interpret attractors as an instantaneous object is also new.

Another motivation for this appliaction of the MT is based on the thesis that the fundamental information of a dynamical system is expressed in the structure of its global attractor. Thus, in the case of the human brain as a complex dynamical system, the structure of the attractor could be related to the corresponding states of consciousness. A validation of this is that our method has been used [18] to define new measures of consciousness.

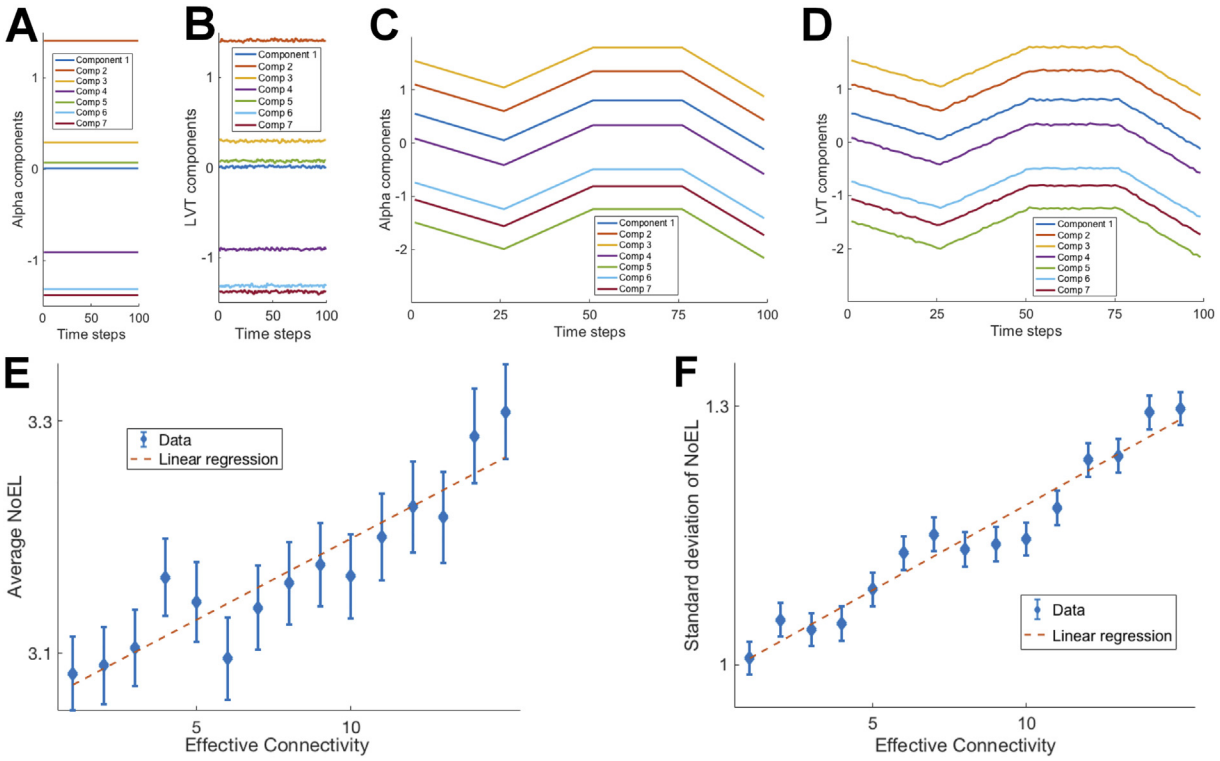

Fig. 2. A, In a cooperative ($\gamma_{i\neq j} \geq 0$) 7-dimensional ($n = 7$) Lotka-Volterra (LV) system (Eq. (20)), a time series dataset is generated by specific constant alpha components α_i. B, The LVT is performed on these data. and as expected, the same alpha column vector is obtained except for the noise added in the generation process. C, Again a cooperative 7-dimensional LV system, but now the components of alpha α_i are piecewise functions. D, When the LVT is applied on these data the original piecewise α_i are obtained except for the added noise. E, In this example a four-node system evolves in time according to a multivariable Ornstein–Uhlenbeck process. α_i are calculated from the generated data through the LVT. Changes in the β parameter governing the effective connectivity (EC) produce changes in the corresponding distribution of the global attractors and their number of energy levels (NoEL). The average NoEL is linearly correlated with the EC of the system. A linear regression is shown with $R^2 = 0.85$ (it indicates the amount of variance in the average NoEL that is explained by the EC). Similar results are obtained in different realizations of the simulation. F, The standard deviation of the NoEL is also linearly correlated with the EC of the system. In this realization the linear regression performs with $R^2 = 0.95$. Thus, the variability of the NoEL is even a better predictor of the EC. Again, similar results are obtained in different realizations.

Furthermore, the method proposes here could help to calculate the empirical "ghost" attractors of complex systems as the human brain. These attractors, suggested in [15,51], would correspond to distinct foci of high activity in particular nodes. According to [15], at the edge of the transition from resting state to task context in the human brain the local attractors would not exist as stable fixed points yet, since they are either saddle points, or regimes with close to zero flow in the phase space. However, as a possible neurobiological explanation of the resting state dynamics, these states could be easily stabilized when needed in a given task context or for a given function. We believe that our new formalism could be used to detect empirical ghost attractors since, in a non-stationary AL, stable attractors are never reached.

4.10. LVT for high-dimensional systems

Here we discuss about the validity of our approach when the dimension n of the underlying dynamical system becomes large. Even a relatively simple model, as LV is, could include a complex attractor landscape with up to 2^n stationary points (for instance 1.238×10^{27} when $n = 90$). It is known that many data analysis method fail for high-dimensional dynamics. On the contrary, one of the advantages of our approach is that it is valid when the dimension of the underlying dynamical system becomes large. The problem of calculating the GASS (and consequently the number of energy levels of the GA) in a cooperative LV system is equivalent to solving a LCP. Algorithms to solve the LCP such as the Complementary Pivot Algorithm (CPA) have shown low computational complexity [33].

The computational complexity of an algorithm measures the growth of the computational effort involved in executing the algorithm as a function of the size of the problem. In the CPA, the computational effort is assessed by the number of pivot steps carried out before the algorithm terminates.

The Probabilistic Average Computational Complexity is a measure for studying the computational complexity of an algorithm where the data in the problem are assumed randomly generated according to some probability distribution. The average computational complexity of the algorithm under this model is then defined to be the statistical expectation of the number of steps needed by the algorithm before termination on problem instances with this data. The average analysis has been performed [48] under certain assumptions on the distribution of the data and the expected number of pivot steps taken by the CPA before termination when applied on the LCP was at most $n(n + 1)/4$.

The Empirical Average Computation Complexity is another measure for computational complexity used more in the spirit of simulation. The computational experiment is performed by applying the algorithm on a large number of problem instances of various sizes, and summary statistics are then prepared on how the algorithm performed on them. Computational experiments indicate that on problems on which it did terminate, the average number of simplices that the algorithm walked through before termination is $\mathcal{O}(n^2)$, as a function of the dimension of the problem (see [39–42]).

As a conclusion, the CPA can be applied in systems of n dimensions, being relatively fast as n grows.

5. Other applications

Metastability is usually measured in empirical *complex systems* with the standard deviation σ_R of the Kuramoto order parameter R, where R is the module of the complex number $\frac{1}{n}\sum_{j=1}^{n} e^{i\theta_j}$ [43]. However,

what really expresses σ_R is a variability in the overall synchronization of the system. In the framework of the previous section, an alternative is the standard deviation of the NoEL which assesses the tendency of the system to change the local attractors (the GASSs) and could be called *structural metastability* [17].

In this LVT context, *cooperation and integration measures* can be developed after the structure of the global attractor. Structures with NoEL equal to $q + 1$ can have up to 2^q nodes, where $q \leq n$ but not all nodes are always present. The number of nodes divided by 2^q is a measure of the lack of integration in the system. This is because when this ratio is small, it is reached a stable local attractor (GASS) with certain active nodes while stationary points corresponding to combinations of some of those active nodes do not appear in the GA. The explanation, therefore, is that the integrative interaction between different nodes facilitates the existence of that stable attractor. Furthermore, in cooperative LV systems ($\gamma_{ij} \geq 0 \quad \forall\, i \neq j$) once a node n_i appears as active in a stationary point at certain energy level, it will also appear in the more stable levels and also in the GASS. But we can focus on the stationary points which contain some new apparition of a certain node, and call them *cooperative points*. Several measures can be defined by looking at the cooperative nodes of a GA. Thus, the *highest energy level* in which cooperation appears is the *highest cooperative level*.

Again in the framework of the previous section *criticality and synchronicity measures* can be defined. In the context of the LCP solved to calculate the GASS 16 joining w and z there are, at most, only 7 non-zero components different that are also positive which will be called r_i. Recall that each GASS of a LV system is a unique combination of null and non-null components. At the points where the transition between GASS occurs there are more than 7 null components of r. Thus, one way to measure the proximity of the GASS transition is to calculate the minimum of the r_i. Hence, the minimum of the $\{r_i\}$ is a measure of *criticality* as it indicates the proximity of a phase transition. Additionally, the GASS divides the nodes of the system into two subsets: inactives and actives ones. But there are two special kinds of GASSs: when all nodes are actives and when all nodes are inactives. *Synchronicity* is defined as the ratio of these extreme GASSs in the GASS time series.

In a general framework beyond the LVT, *empirical noise distributions* can be calculated using different MTs. In *Physics, Acoustical Engineering, Telecommunications, and Statistical Forecasting noise* refers to a statistical model for signals and signal sources. *Noise time series* can be calculated using, for example, the SAR Transform (SART) and the Rate Fluctuations Transform (RFT). In both cases and whatever cases in which the noise is the model parameter converted to a function of time, instead of being modeled as a Gaussian noise, noise is calculated exactly from the experimental data.

In *Machine Learning, classification* is the problem of identifying which of a set of categories an observation belongs to. An algorithm that implements classification in a concrete implementation is known as a classifier. The observations are termed instances, the explanatory variables are known as features, and the categories to be predicted are called classes. Classifiers can be trained for using the instantaneous parameters values from the MT. Indeed, a parameter generally, is any characteristic that can help in defining or classifying a particular system. Thus, our proposal is that the precision of the classification can be increased by using the appropriate MT for each case, because the instantaneous parameters can reveal characteristics of the system not shown in the raw data. In [18], it has been proved that the LVT is useful for classification of different human brain states.

When the parameters are fixed, a model can be used to generate data through simulations given initial values or boundary conditions. When the MT associated with the model that generated the data is applied to these simulated data, the fixed parameters from which it was started are obtained. The variability of the MT result in this case would therefore be zero. Said variability can be measured in different ways as, for example, by standard deviation. Any of these measurements can be used as a *model distance* that can be defined as the distance between the empirical model and the model used in the MT.

In Structural Engineering, Electrodynamics, and Mechanics the Modal analysis is the study of the dynamic properties of systems in the frequency domain. The analysis of the signals typically relies on Fourier analysis. The resulting transfer function will show one or more resonances, whose characteristic mass, frequency and damping ratio can be estimated from the measurements. An alternative in the framework of the model transform is the Kuramoto transform which can provide instantaneous intrinsic frequencies from empirical data.

6. Conclusion

A model was defined as a system of n first order ordinary differential equations but this definition can be extended to second, third, etc., order ordinary differential equations, difference equations, or other formulations of a theoretical dynamical system. In the model transform (MT) the model is solved for parameters which makes sense, for instance, when empirical time series of the model variables are known. Typically the solution of a model is assessed given fixed parameters, but, contrary to this, in the MT the situation is just the opposite. The MT is defined as the time dependent parameters with which the model equations fit exactly the empirical data as a solution. Nevertheless, it is very important noticing that *the model equations are not being assessed as a model*. In other words, the model could be too simple compared to complex empirical time series.

The Lotka-Volterra transform (LVT) is a particular case of the general model transform. It was desirable to take advantage of the well-known structure of the the global attractor (GA) of the Lotka-Volterra (LV) model. Using LV equations the globally asymptotically stable solution (GASS) and the number of energy levels (NoEL) are easily computed by means of the efficient Complementary Pivot Algorithm. No similar algorithm is known to calculate the GASS in more complex models. A GA is obtained for that LV model for each time point resulting a continuous movement of attractor landscapes.

Thus, in complex empirical systems the application of the LVT can provide relevant system information through the corresponding instantaneous parameters. The LVT can be analytically computed for dynamical activity of complex systems with local node dynamics $u(t)$ defined on networks where the connectivity γ_{ij} is the underlying structure of a system. Changes in the parameters characterize the dynamics of the system and produce changes in the corresponding GA.

In practice, the empirical signals are measured as time series, that is, their values are known for discrete values of time. At each time step the data define a column matrix, and, finally, it is obtained a temporal series of different attractors. It opens the door to describe landscapes of attractors that change over time and to calculate the associated structures for any empirical system on a network.

In *Neuroscience*, it has been proved [18] that the LVT is useful for *studies of consciousness*. This approach also opens the door to use the MT for different models, for instance, the Kuramoto model for computing instantaneous intrinsic frequencies. In this and other *research fields* the *applications* of the MT may be countless.

CRediT authorship contribution statement

J.A. Galadí: Conceptualization, Investigation, Methodology, Software, Formal analysis, Writing – original draft, Visualization, Writing – review & editing. **F. Soler-Toscano:** Investigation, Visualization, Writing – review & editing. **J.A. Langa:** Funding acquisition, Investigation, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

The paper has never been published before, and it is not under review by another journal. We will not re-submit it to another journal during the reviewing procedure.

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Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.chaos.2022.112094.

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