ABSOLUTE CONTINUITY THEOREMS FOR ABSTRACT RIEMANN INTEGRATION

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Abstract. Absolute continuity for functionals is studied in the context of proper and abstract Riemann integration examining the relation to absolute continuity for finitely additive measures and giving results in both directions: integrals coming from measures and measures induced by integrals.

To this end, we look for relations between the corresponding integrable functions of absolutely continuous integrals and we deal with the possibility of preserving absolute continuity when extending the elemental integrals.

Keywords: finitely additive integration, abstract Riemann integration, absolute continuity

1. Introduction

It is well known that there are two classical ways of developing an Integration Theory:

On the one hand, there is the set theoretic starting point, which we will denote as (μ/Ω): X is a non empty set, Ω is a σ-algebra of the power set of X and μ is a measure on Ω. In this context, standard and classical methods lead to the $L_1(\Omega,\mu)$ class of the Lebesgue integrable functions (see [11]).

On the other hand, there exists a functional setting which we will denote as (I/B): The starting point here is a Daniell Loomis system, that is a triple (X,B,I) where B is a vector lattice of real functions defined on X and I is a Daniell integral on B ($\{h_n\} \subseteq B$, $h_n \downarrow 0 \Rightarrow I(h_n) \to 0$). In this case we get the corresponding class $L_1(B,I)$ of Daniell integrable functions. For a recent account of the functional extension procedures we refer the reader to [6].

Both contexts have a common hypothesis which plays the central role: continuity. For the (μ/Ω) context it is the σ-additivity of μ and for the (I/B) setting it is the Daniell (or Bourbaki) condition on I.

The interplay between these two schemes, (μ/Ω) and (I/B), is well known: We obtain the corresponding Loomis system (X, B_Ω, I_μ) induced by the measure space (X, Ω, μ) and, when B is stonean (i.e., $1 \wedge B \subseteq B$) the Loomis system (X, B, I) induces the corresponding measure space (X, Ω_B, μ_I). A classical text which clearly shows these facts is the book by Pfeffer [15].

When the continuity of the measure is dropped (and we work without or with weaker continuity conditions) two new paradigms arise: the class $R_1(\mu)$ of the abstract Riemann μ-integrable functions with respect to a finitely additive measure μ (see [13]) versus its functional analogue, the class $R_1(B, I)$ of the abstract Riemann I-integrable functions in [9].

We can trace this back to the works of Loomis [14] and Aumann [4] on integral extension of positive linear functionals. For those attempts there are no Lebesgue convergence theorems. In this functional context, the class $R_1(B, I)$ of the abstract Riemann I-integrable functions was obtained. For this class it is possible to establish results such as Lebesgue convergence type theorems and the usual characterizations of integrability. Moreover, a unified treatment for the Dunford-Schwartz, abstract μ-Riemann, Daniell and Bourbaki integrals is achieved (see [9] and [10]).

The papers [7]–[10] by Díaz-Carrillo and Günzler, and by Díaz-Carrillo and Muñoz-Rivas are the references for this approach. This will be the framework for what follows.

2. Preliminaries

For $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, where $\mathbb{R}$ is the real line, we extend the usual addition in $\mathbb{R}$ to $\overline{\mathbb{R}}$ by the conventions $r + s := 0$ if $r = -s \in \{-\infty, +\infty\}$ and $r - s := r + (-s)$.

We also set $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$, $a^+ := a \vee 0$ and $a^- := -(a \wedge 0)$.

Given an arbitrary nonempty set X, let $\overline{\mathbb{R}}^X$ consist of all functions defined on X with values in $\overline{\mathbb{R}}$. All operations and relations in $\overline{\mathbb{R}}^X$ are defined pointwise, with the convention $\inf \emptyset := +\infty$ and $\sup \emptyset := -\infty$.

A functional $T\colon \overline{\mathbb{R}}^X \longrightarrow \overline{\mathbb{R}}$ will be called *subadditive* if $T(f + g) \leqslant T(f) + T(g)$ for all $f, g \in \overline{\mathbb{R}}^X$ but $T(f) = -T(g) = +\infty$ and $T(f) = -T(g) = -\infty$. The notion of a *superadditive* functional is introduced in a completely dual way.

A triple (X, B, I) is called a *Loomis system* if $B \subseteq \mathbb{R}^X$ is a vector lattice of real functions and $I\colon B \to \mathbb{R}$ is a positive linear functional. We set $+B := \{h \in B\colon h \geqslant 0\}$.

Given (X, Ω, μ) with μ a finite finitely additive measure and Ω a ring, we call (X, B_Ω, I_μ) the *induced Loomis system*, where B_Ω is the vector lattice of μ-simple functions,

$$B_\Omega := \left\{ h \in \mathbb{R}^X : \ h = \sum_{i=1}^{n} a_i \chi_{A_i}, \ a_i \in \mathbb{R}, \ A_i \in \Omega, \ \mu([h \neq 0]) < +\infty \right\},$$

and I_μ is its canonical elemental integral given by

$$I_\mu(h) := \sum_{i=1}^{n} a_i \mu(A_i), \ \forall h \in B_\Omega.$$

3. Proper and abstract Riemann integration

Let (X, B, I) be a Loomis system. For $f \in \overline{\mathbb{R}}^X$, following Loomis in [14] we define by

$$I^-(f) := \inf\{I(h) : \ h \in B, h \geqslant f\},$$
$$I^+(f) := \sup\{I(h) : \ h \in B, h \leqslant f\}$$

the corresponding upper and lower integrals of f, which verify $-\infty \leqslant I^+(f) \leqslant I^-(f) \leqslant +\infty$, $\forall f \in \overline{\mathbb{R}}^X$, I^- is subadditive, I^+ is superadditive, and both are positively homogeneous.

The class of the *properly Riemann integrable functions* is defined by

$$R_{\text{prop}}(B, I) := \{f \in \mathbb{R}^X : \ I^+(f) = I^-(f) \in \mathbb{R}\},$$

or, equivalently, by

$$R_{\text{prop}}(B, I) = \{f \in \mathbb{R}^X : \ \forall \varepsilon > 0, \ \exists h, g \in B, \ h \leqslant f \leqslant g \text{ and } I(g - h) < \varepsilon\}$$

and it is a vector lattice where the functional $I := I^+ = I^-$ is linear and increasing, i.e., it is an integral which extends the original I.

For this class there are no satisfactory Lebesgue convergence type theorems to make a consistent Integration Theory. Therefore, it is necessary to introduce a "local convergence" to ensure this kind of results.

The *local I-convergence* for sequences of functions $\{f_n\}$ in $\overline{\mathbb{R}}^X$ to a function f in $\overline{\mathbb{R}}^X$, denoted by $\{f_n\} \longrightarrow f(I^-)$, means that $\{I^-(|f_n - f| \wedge h)\} \to 0$, $\forall h \in +B$, and it has been used in [9] to define the class $R_1(B,I)$ of the *abstract Riemann integrable functions* as

$$R_1(B,I) := \{f \in \overline{\mathbb{R}}^X:\ \exists\{h_n\} \text{ in } B,\ I\text{-Cauchy};\ \{h_n\} \longrightarrow f(I^-)\}$$

where $\{h_n\}$ I-Cauchy means that $I(|h_n - h_m|) \to 0$, for $n, m \to +\infty$.

Moreover, for $f \in R_1(B,I)$ we set $I(f) := \lim_{n\to+\infty} I(h_n)$ for any sequence $\{h_n\}$ in B I-Cauchy and such that $\{h_n\} \longrightarrow f(I^-)$.

The definition does not depend on the particular sequence $\{h_n\}$ and no confusion arises with this notation since $R_{\text{prop}}(B,I) \subseteq R_1(B,I)$ with coinciding integrals I.

Further relations between the classes $R_{\text{prop}}(B,I)$ and $R_1(B,I)$ are given in [9] by the following characterizations:

$$f \in R_1(B,I) \ \Leftrightarrow\ f^{\pm} \wedge h \in R_{\text{prop}},\ \forall h \in +B \text{ and } I^+(|f|) < +\infty. \tag{1}$$

$$f \in R_{\text{prop}}(B,I) \ \Leftrightarrow\ f \in R_1(B,I) \text{ and } \exists h \in +B\colon\ |f| \leqslant h. \tag{2}$$

In fact, in [9, Th.1.6], it is proved that

$$I(f) = I^+(f),\ \forall f \in +R_1(B,I). \tag{3}$$

We recall that the class of the *null-functions* is introduced in this context by

$$N_1(B,I) := \{f \in R_1(B,I)\colon\ I(|f|) = 0\}$$

or, equivalently, by

$$N_1(B,I) = \{f \in \overline{\mathbb{R}}^X:\ I^-(|f| \wedge h) = 0, \forall h \in +B\}.$$

On the other hand, the *localized functional* I_l^- in the sense of [17] is defined as

$$I_l^-(f) := \sup\{I^-(f \wedge h)\colon\ h \in +B\}.$$

It is easily verified that I_l^- is positively homogeneous, monotone and subadditive. Moreover, $(I_l^-)_l = I_l^-$, $I^+ \leqslant I_l^- \leqslant I^-$ and $I_l^-(f) = I^-(f)$ if $f \leqslant h$ for some $h \in +B$.

Theorem 2 in [10] guarantees that $R_1(B,I)$ is the closure of B in $\overline{\mathbb{R}}^X$ with respect to the integral seminorm $I_l^-(|\cdot|)$ and $I_l^-(f) = I(f),\ \forall f \in R_1(B,I)$ (I is the only I_l^--continuous extension of I from B to $R_1(B,I)$).

Finally, we consider the functional

$$I^*(f) := \inf\{I(g)\colon\ g \in R_1(B,I), g \geqslant f\},$$

which is also positively homogenous, monotone, subadditive and, evidently, extends I from $R_1(B,I)$ to $\overline{\mathbb{R}}^X$.

Definition 3.1. A Loomis system (X, B, I) is called $C_{+\infty}$ or *upper continuous* if

$$\lim_{r \to +\infty} I^*(f - f \wedge r) = 0, \ \forall f \in +B.$$

Upper continuity on B is hereditary for the class $R_1(B, I)$; that is:

Lemma 3.2. *If (X, B, I) is $C_{+\infty}$, then so is $R_1(B, I)$.*

In general $R_1(B, I)$ need not be closed under multiplication, but we will use the following two facts which can be easily checked.

Lemma 3.3. *If $BB \subseteq B$, $f \in R_1(B, I)$ and $k \in B$ is bounded, then $fk \in R_1(B, I)$.*

Lemma 3.4. *If (X, B, I) is a $C_{+\infty}$ Loomis system and h and χ_A are in $R_1(B, I)$ then so is $h\chi_A$.*

There are three basic theorems to obtain a good Measure and Integration Theory: Lebesgue, Fubini and Radon-Nikodym type theorems. For the class $R_1(B, I)$, Lebesgue theorems were given by Díaz-Carrillo and Muñoz-Rivas in [9] and Fubini type theorems were found by de Amo and Díaz-Carrillo in [3]. Partial attempts in order to obtain Radon-Nikodym type theorems were done by de Amo, Chiţescu and Díaz-Carrillo (see [1] and [2]). We will now study the notion of absolute continuity in this functional setting of proper and abstract integration and its relations to the notion of absolute continuity for finitely additive measures.

4. Absolute continuity

We recall that, given two finitely additive measures μ and ν on a ring Ω, ν is said to be *absolutely continuous* with respect to μ, and is denoted by $\nu \ll \mu$, if

$$\forall\, \varepsilon > 0, \ \exists\, \delta > 0\colon \ A \in \Omega, \ \mu(A) < \delta \Rightarrow \nu(A) < \varepsilon$$

(see Bochner [5, p. 778], Fefferman [12, p. 35], Dunford-Schwartz [11, p. 131]).

This definition clearly implies the classical one,

$$\mu(A) = 0 \ \Rightarrow \ \nu(A) = 0, \ \forall A \in \Omega,$$

and both are, in fact, equivalent when μ and ν are measures such that $\nu(A) < +\infty$ for all $A \in \Omega$ with $\mu(A) < +\infty$.

The most natural transcription for absolute continuity to the analogous functional context (I/B) is the following one: Let I and J be two positive functionals on B. We say that J is *I-continuous* (continuous with respect to I) if

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0\colon\ h \in +B,\ I(h) < \delta \Rightarrow J(h) < \varepsilon.$$

Unfortunately, this definition fails since I-continuity is, in fact, a kind of boundedness condition.

Proposition 4.1. *Let (X, B, I) be a Loomis system and J a positive functional on B. The following conditions are equivalent:*
(i) *J is I-continuous;*
(ii) $\exists\, M > 0\colon\ J(h) \leqslant MI(h),\ \forall h \in +B$ *($J \leqslant MI$, for abbreviation).*

This equivalence allows us to show that integrals induced by absolute measures need not be continuous in this sense, that is, there exist measures μ and ν such that $\nu \ll \mu$ but I_ν is not I_μ-continuous.

Example 4.2. Let $X = [0, 1]$, let λ be the Lebesgue measure in X and ν the measure given by $\nu(A) = \int_A h \,\mathrm{d}\lambda$, where $h\colon X \longrightarrow \mathbb{R}$ is the λ-integrable function defined by $h(0) = 0$ and $h(t) = 1/\sqrt{t}$, $\forall t \neq 0$.

Evidently, μ is absolutely continuous with respect to λ, but there is no positive M such that $J_\nu \leqslant MI_\lambda$: If we assume that such an M exists then, in particular, we have $\nu([0, 1/n]) \leqslant M\lambda([0, 1/n])$, $\forall n \in \mathbb{N}$, that is

$$\int_0^{1/n} h(t)\,\mathrm{d}t \leqslant M\frac{1}{n} \ \Rightarrow\ n \leqslant \frac{M^2}{4},\ \forall n \in \mathbb{N},$$

which leads to contradiction.

Therefore we have to weaken I-continuity in order to define a satisfactory notion of absolute continuity for functionals. The latter was introduced in [1] and reads as follows:

Definition 4.3. Let (X, B, I) be a Loomis system and J a positive functional on B. J is said to be *absolutely I-continuous* (absolutely continuous with respect to I) and is denoted by $J \ll I$, if

$$\forall \varepsilon > 0, \forall h \in +B,\ \exists \delta > 0\colon\ \forall k \in +B,\ k \leqslant h,\ I(k) < \delta \Rightarrow J(k) < \varepsilon.$$

The next theorem makes evident when absolutely continuous finitely additive measures yield absolutely continuous elementary integrals.

Theorem 4.4. *Let μ and ν be finitely additive measures such that $\nu(A) < +\infty$ for all $A \in \Omega$ with $\mu(A) < +\infty$. If $\nu \ll \mu$ then $I_\nu \ll I_\mu$.*

Proof. Assume that $\nu \ll \mu$, let $\varepsilon > 0$ and $f \in +B_\Omega$. There are $a_i > 0$ and pairwise disjoint $A_i \in \Omega$ such that $f = \sum_{i=1}^{n} a_i \chi_{A_i}$. Set $A := \bigcup_{i=1}^{n} A_i \in \Omega$ and $\beta := \sup\{a_i\colon\ i = 1, \ldots, n\} > 0$. Note that $\mu(A) < +\infty$, since $\mu([f \neq 0]) < +\infty$.

If $\nu(A) = 0$, then $I_\nu(f) \leqslant \beta\nu(A) = 0$ and therefore $I_\nu(h) \leqslant I_\nu(f) = 0 < \varepsilon$, $\forall h \in +B_\Omega$ with $h \leqslant f$.

Assume that $\nu(A) > 0$ and let $\alpha := \frac{1}{2}\varepsilon/\nu(A) > 0$. Since $\nu \ll \mu$, there exists $\varrho > 0$ such that

$$\forall E \subseteq \Omega,\ \mu(E) < \varrho \Rightarrow \nu(E) < \frac{\varepsilon}{2\beta}.$$

Let $\delta := \alpha\varrho > 0$ and $h \in +B_\Omega$ with $h \leqslant f$ and $I_\mu(h) < \delta$. There are $e_i > 0$ and pairwise disjoint $E_i \in \Omega$ such that $h = \sum_{j=1}^{m} e_j \chi_{E_j}$. Moreover, since $h \leqslant f$ we have $E := \bigcup_{j=1}^{m} E_j \subseteq \bigcup_{i=1}^{n} A_i = A$ and $e_j \leqslant \beta$, $\forall j = 1, \ldots, m$.

Let us now consider sets

$$T := \{t \in \mathbb{N}\colon\ 1 \leqslant t \leqslant m,\ e_t < \alpha\},$$
$$S := \{t \in \mathbb{N}\colon\ 1 \leqslant t \leqslant m,\ e_t \geqslant \alpha\},$$

which are disjoint with $S \cup T = \{1, \ldots, m\}$, and define functions

$$h_1 := \sum_{t \in T} e_t \chi_{E_t} \quad \text{and} \quad h_2 := \sum_{s \in S} e_s \chi_{E_s}.$$

Evidently $h_1, h_2 \in +B_\Omega$ and $h = h_1 + h_2$. Furthermore,

$$I_\nu(h_1) < \alpha \sum_{t \in T} \nu(E_t) \leqslant \alpha\nu(A) = \frac{\varepsilon}{2}$$

and an easy computation shows that $\mu\Big(\bigcup_{s \in S} E_s\Big) < \delta/\alpha = \varrho$. Hence, $\nu\Big(\bigcup_{s \in S} E_s\Big) < \frac{1}{2}\varepsilon/\beta$ and, consequently,

$$I_\nu(h_2) = \sum_{s \in S} e_s \nu(E_s) \leqslant \beta\nu\Big(\bigcup_{s \in S} E_s\Big) < \frac{\varepsilon}{2}.$$

Therefore $I_\nu(h) = I_\nu(h_1) + I_\nu(h_2) < \varepsilon$, which completes the proof. $\square$

5. Absolute continuity and proper Riemann integration

In this section we will study the good behaviour of absolute continuity with respect to proper Riemann integration.

The first result says that absolute continuity of J with respect to I transfers convergence to 0 for B-bounded sequences from the integral I to the integral J:

Lemma 5.1. *Assume that $J \ll I$ and let $\{h_n\}$ be a sequence in $+B$ such that there exists $h \in +B$ with $h_n \leqslant h, \forall n \in \mathbb{N}$ and $I(h_n) \to 0$. Then $J(h_n) \to 0$.*

In particular, we have the following facts:

Corollary 5.2. *If $J \ll I$ and I is Daniell, then so is J.*

Corollary 5.3. *If $J \ll I$ and $\{h_n\}$ is an I-Cauchy sequence in $+B$ such that there exists $h \in +B$ with $|h_n - h_m| \leqslant h$ $\forall n, m \in \mathbb{N}$, then $\{h_n\}$ is J-Cauchy.*

Theorem 5.4. *If $J \ll I$, then*
(i) $R_{\text{prop}}(B, I) \subseteq R_{\text{prop}}(B, J)$,
(ii) $J \ll I$ *on* $R_{\text{prop}}(B, I)$.

Proof. (i) Let $f \in R_{\text{prop}}(B, I)$ and $\varepsilon > 0$. There are $k_\varepsilon, h_\varepsilon \in B$ such that

$$k_\varepsilon \leqslant f \leqslant h_\varepsilon \quad \text{and} \quad I(h_\varepsilon - k_\varepsilon) < \varepsilon.$$

For $\varepsilon > 0$ and $h_\varepsilon - k_\varepsilon \in +B$, since $J \ll I$, there exists $\delta > 0$ such that

$$\forall\, g \in +B,\ g \leqslant h_\varepsilon - k_\varepsilon,\ I(g) < \delta \Rightarrow J(g) < \varepsilon.$$

We can also find $k_\delta, h_\delta \in B$ such that

$$k_\delta \leqslant f \leqslant h_\delta \quad \text{and} \quad I(h_\delta - k_\delta) < \delta.$$

Since δ depends only on ε, we can consider the functions:

$$k'_\varepsilon := k_\varepsilon \vee k_\delta \quad \text{and} \quad h'_\varepsilon := h_\varepsilon \vee h_\delta,$$

which verify that $k'_\varepsilon, h'_\varepsilon \in B$ and $k'_\varepsilon \leqslant f \leqslant h'_\varepsilon$.

Therefore, we have found that

$$h'_\varepsilon - k'_\varepsilon \in +B,\ h'_\varepsilon - k'_\varepsilon \leqslant h_\varepsilon - k_\varepsilon \quad \text{with} \quad I(h'_\varepsilon - k'_\varepsilon) \leqslant I(h_\delta - k_\delta) < \delta$$

and, consequently, $J(h'_\varepsilon - k'_\varepsilon) < \varepsilon$. Thus $f \in R_{\text{prop}}(B, J)$.

(ii) Assume that $J \ll I$ and let $\varepsilon > 0$. Given $f \in +R_{\text{prop}}(B, I)$ we can take $k_\varepsilon = k_\varepsilon(f) \in +B$ such that $f \leqslant k_\varepsilon$. I-continuity of J gives $\delta = \delta(\varepsilon, f) > 0$ such that

$$\forall\, h \in +B,\ h \leqslant k_\varepsilon,\ I(h) < \delta \Rightarrow J(h) < \varepsilon.$$

Let $\varrho := \frac{1}{2}\delta$. Given $g \in +R_{\text{prop}}(B, I)$, $g \leqslant f$ with $I(g) < \varrho$ there are $h_\delta, k_\delta \in +B$ with $h_\delta \leqslant g \leqslant k_\delta$ and $I(k_\delta - h_\delta) < \varrho$.

Taking $h_\varepsilon := k_\varepsilon \wedge k_\delta \in +B$, we have $g \leqslant k_\varepsilon \wedge k_\delta = h_\varepsilon$ and

$$I(h_\varepsilon) \leqslant I(k_\delta) \leqslant I(k_\delta - h_\delta) + I(h_\delta) < \varrho + I(g) < \varrho + \varrho = \delta.$$

Therefore, we deduce that $J(h_\varepsilon) < \varepsilon$ and hence

$$J(g) = J^-(g) = \inf\{J(h)\colon\ h \in +B,\ g \leqslant h\} \leqslant J(h_\varepsilon) < \varepsilon,$$

that is, $J \ll I$ on $R_{\text{prop}}(B, I)$. $\square$

We are able to give a first sufficient condition for finitely additive measures induced by absolutely continuous integrals to be absolutely continuous.

Given two positive functionals I and J, let $(X, \Omega(I), \mu_I)$ and $(X, \Omega(J), \nu_J)$ be their respective finitely additive measure induced spaces, that is,

$$\Omega(I) = \{A \subseteq X\colon\ \chi_A \in R_1(B, I)\},\quad \mu_I(A) = I(\chi_A),\ \forall A \in \Omega,$$
$$\Omega(J) = \{A \subseteq X\colon\ \chi_A \in R_1(B, J)\},\quad \nu_J(A) = J(\chi_A),\ \forall A \in \Omega.$$

Proposition 5.5. *If $1 \in R_{\text{prop}}(B, I)$ and $J \ll I$, then $\nu_J \ll \mu_I$ (on $\Omega(I) \cap \Omega(J)$).*

P r o o f. Since $J \ll I$, Theorem 5.4 says that $J \ll I$ on $R_{\text{prop}}(B, I) \subseteq R_{\text{prop}}(B, J)$. Thus, for $\varepsilon > 0$ and $1 \in R_{\text{prop}}(B, I)$ there exists $\delta > 0$ such that

$$\forall h \in R_{\text{prop}}(B, I) \text{ with } h \leqslant 1 \text{ and } I(h) < \delta \Rightarrow J(h) < \varepsilon.$$

Given $A \in \Omega(I) \cap \Omega(J)$ with $\mu_I(A) < \delta$ we have $\chi_A \wedge h \in R_{\text{prop}}(B, I), \forall h \in +B$, $\chi_A \wedge h \leqslant 1$ and $I(\chi_A \wedge h) \leqslant I(\chi_A) = \mu_I(A) < \delta$.

Therefore, it follows that $J(\chi_A \wedge h) < \varepsilon$, $\forall h \in +B$ and, keeping in mind that $\chi_A \in +R_1(B, J)$, we conclude that

$$\nu_J(A) = J(\chi_A) = J_l^-(\chi_A) = \sup\{J^-(\chi_A \wedge h)\colon\ h \in +B\} < \varepsilon.$$

$\square$

In the following section, the condition $1 \in R_{\text{prop}}(B, I)$ will be relaxed to $1 \in R_1(B, I)$ and $\Omega(I) \cap \Omega(J)$ will be, in fact, $\Omega(I)$ (see Corollary 6.9).

From now on I and J will be two positive linear functionals defined on the same vector lattice B.

The definition of absolute continuity given in 4.3 allows us to prove an analogous with Lemma 5.1 where I and J are replaced by I^- and J^-, respectively.

Proposition 6.1. *Assume that $J \ll I$ and let $\{g_n\}$ be a sequence in $\overline{\mathbb{R}}_+^X$ such that there is $h \in +B$ with $g_n \leqslant h, \forall n \in \mathbb{N}$ and $I^-(g_n) \to 0$. Then $J^-(g_n) \to 0$.*

Proof. Let $\varepsilon > 0$. Since $J \ll I$, for this ε and for h from the proposition, there exists $\delta > 0$ such that

$$\forall\, k \in +B,\ k \leqslant h,\ I(k) < \delta \Rightarrow J(k) < \varepsilon.$$

Since $I^-(g_n) \to 0$, there is $n_0 \in \mathbb{N}$ such that $\forall n \geqslant n_0$, $I^-(g_n) < \delta$. Therefore, for each $n \geqslant n_0$ we can find $k_n \in B$, $g_n \leqslant k_n$ with $I(k_n) < \delta$.

Set $k'_n := k_n \wedge h \in +B$, $\forall n \geqslant n_0$. Obviously, $k'_n \leqslant h$ and $I(k'_n) \leqslant I(k_n) < \delta$. Therefore, it follows that $J(k'_n) < \varepsilon$, $\forall n \geqslant n_0$. Since $g_n \leqslant k'_n$, we deduce that $J^-(g_n) \leqslant J(k'_n) < \varepsilon$, $\forall n \geqslant n_0$. Thus, we have proved that $J^-(g_n) \to 0$, as we wanted. □

As an immediate consequence of Proposition 6.1, local I-convergence implies local J-convergence whenever J is absolutely continuous with respect to I.

Theorem 6.2. *If $J \ll I$ and $f_n \longrightarrow f(I^-)$, then $f_n \longrightarrow f(J^-)$, $\forall f_n, f \in \overline{\mathbb{R}}^X$, $\forall n \in \mathbb{N}$.*

Proof. Assume that $f_n \longrightarrow f(I^-)$ and let $h \in +B$. We have $I^-(|f_n - f| \wedge h) \to 0$, that is, $I^-(g_n) \to 0$ where $g_n := |f_n - f| \wedge h$ verifies $0 \leqslant g_n \leqslant h$ $\forall n \in \mathbb{N}$. Proposition 6.1 says that $J^-(g_n) \to 0$. Thus, $J^-(|f_n - f| \wedge h) \to 0$, $\forall h \in +B$, that is, $f_n \longrightarrow f(J^-)$. □

Theorem 6.2 allows us to see that the classical definition of absolute continuity by null sets is weaker than this one (Definition 4.3), in the same way as in the finitely additive measure context.

Corollary 6.3. *If $J \ll I$, then $N_1(B, I) \subseteq N_1(B, J)$.*

Proof. Let $f \in N_1(B, I)$. Since $I^-(|f| \wedge h) = 0$ $\forall h \in +B$, the sequence $h_n := 0 \in B$ verifies that $h_n \longrightarrow f(I^-)$ and $I(h_n) \to 0$. From Lemma 5.1 and Theorem 6.2 we deduce that $h_n \longrightarrow f(J^-)$ and $J(h_n) \to 0$. Thus, $f \in R_1(B, J)$ with $J(|f|) = \lim J(|h_n|) = 0$, that is, $f \in N_1(B, J)$. □

At this point, since absolute continuity has a good behaviour with respect to local convergence, one can expect that if J is absolutely I-continuous then $R_1(B, I) \subseteq R_1(B, J)$, but this is not, in general, true.

Example 6.4. Let $X :=]0, 1]$, let Ω be the ring generated by the semi-ring $\{]a, b]\colon\ 0 < a < b < 1\}$, let $B := B_\Omega$ be the vector lattice of all Ω-simple functions and I its canonical elemental integral.

Consider the function f defined by

$$f(x) := \sum_{n=1}^{+\infty} n\chi_{]1/(n+1)^2, 1/n^2]}, \forall x \in]0, 1],$$

and the linear functional $J\colon\ B \longrightarrow \mathbb{R}$ given by $J(h) := I(fh)$, $\forall h \in B$.

Let us see that $f \in R_1(B, I)$. Since $f \wedge h \in B \subseteq R_{\text{prop}}(B, I)$ for all $h \in B$, we only have to check that $I^+(f) < +\infty$. To see this, let $I_k =]1/(k+1)^2, 1/k^2]$ for each $k \in \mathbb{N}$ and consider the functions $h_n := \sum\limits_{k=1}^{n} k\chi_{I_k}$. It is easy to check that for each $h \in +B$ with $h \leqslant f$, there exists $m \in \mathbb{N}$ such that $h \leqslant h_m + m\chi_{]0, 1/(m+1)^2]}$.

Therefore, $I^+(f) = \sup\{I(h)\colon\ h \leqslant f, h \in +B\}$ can be bounded in the following way:

$$I^+(f) \leqslant \lim_{m \to +\infty} I(h_m) + \lim_{m \to +\infty} m\ I(\chi_{]0, 1/(m+1)^2]}) \leqslant \sum_{k=1}^{+\infty} \frac{2k^2 + k}{k^2(k+1)^2} < +\infty.$$

Since $f \in R_1(B, I)$, the functions in B are bounded and $BB \subseteq B$, Lemma 3.3 guaranties that J is well-defined.

Moreover, if λ is the Lebesgue measure on X and ν is the measure given by $\nu(A) := \int_A f\, \mathrm{d}\lambda$, both defined on the σ-algebra $\sigma(\Omega)$ generated by Ω, then it is clear that $\nu \ll \lambda$ on $\sigma(\Omega)$ and so, in particular, on Ω. Thus, Theorem 4.4 says that $J \ll I$ on B (since I and J on B are induced by λ and ν on Ω, respectively).

However, $f \notin R_1(B, J)$, since $J^+(f) = I^+(f^2) = +\infty$.

To find the condition under which $R_1(B, I) \subseteq R_1(B, J)$ holds, we have to consider the measurable functions. The characterization (1) of $R_1(B, I)$, given in [9], suggests the following definition of measurability (in the sense of Stone, [16]).

Definition 6.5. The class of *measurable functions* with respect to a Loomis system (X, B, I) (I-measurable functions) is defined by

$$M_1(B, I) := \{f \in \overline{\mathbb{R}}^X\colon\ f^\pm \wedge h \in R_{\text{prop}}(B, I),\ \forall h \in +B\}.$$

Thus, we have that every integrable function is measurable and that every measurable function with $I^+(|f|) < +\infty$ is, in fact, integrable. Moreover, note that we can use either $R_{\text{prop}}(B,I)$ or $R_1(B,I)$ in the second member of the previous definition of $M_1(B,I)$.

By Theorem 6.2 we deduce that every I-measurable function is J-measurable whenever $J \ll I$.

Corollary 6.6. *If $J \ll I$, then $M_1(B,I) \subseteq M_1(B,J)$.*

Proof. Let $f \in M_1(B,I)$ and $h \in +B$. We can assume that $f \geqslant 0$ (otherwise, use f^+ and f^- instead of f). Since $f \wedge h \in R_1(B,I)$ there exists $\{h_n\} \in B$, I-Cauchy, such that

$$h_n \leqslant h_{n+1} \leqslant f \wedge h,\ \forall n \in \mathbb{N} \quad \text{and} \quad h_n \longrightarrow f \wedge h(I^-).$$

Moreover, we have

$$|h_n - h_m| \leqslant h_n + h_m \leqslant 2h,\ \forall n, m \in \mathbb{N}.$$

Therefore Corollary 5.3 and Theorem 6.2 say that $\{h_n\}$ is a J-Cauchy sequence and $h_n \longrightarrow f \wedge h(J^-)$, that is, $f \in R_1(B,J)$, $\forall h \in +B$. Thus $f \in M_1(B,J)$. $\square$

It is now easy to find the condition that we have to add to $J \ll I$ in order to obtain $R_1(B,I) \subseteq R_1(B,J)$.

Corollary 6.7. *If $J \ll I$ and $J^+(|f|) < +\infty$ for all f with $I^+(|f|) < +\infty$, then $R_1(B,I) \subseteq R_1(B,J)$.*

Proof. Let $f \in R_1(B,I)$. In particular, $f \in M_1(B,I)$ and, since $J \ll I$, Corollary 6.6 yields $f \in M_1(B,J)$. Moreover, we have $I^+(|f|) < +\infty$ and hence $J^+(|f|) < +\infty$ by hypothesis, which gives $f \in R_1(B,J)$. $\square$

As we did in the preceding section for Riemann proper integration, once we know when $R_1(B,I) \subseteq R_1(B,J)$ for $J \ll I$ we can now prove that the corresponding extensions of I and J to $R_1(B,I)$, which for positive values coincide, respectively, with J^+ and I^+ (see (3)), satisfy $J \ll I$.

Theorem 6.8. *If $J \ll I$ and $J^+(f) < +\infty$ for all $f \in +\overline{\mathbb{R}}^X$ with $I^+(f) < +\infty$, then $J \ll I$ on $R_1(B,I)$.*

Proof. Assume that $J \ll I$ on B and let $\varepsilon > 0$ and $f \in +R_1(B,I)$. Since $I^+(f) < +\infty$, it follows that $J^+(f) < +\infty$ and, therefore, there exists $h \in +B$ with $J^+(f) - \frac{1}{4}\varepsilon < J(h)$, that is, $J^+(f) - J(h) < \frac{1}{4}\varepsilon$.

By absolute continuity, we find $\delta > 0$ such that

$$\forall\, k \in +B,\ k \leqslant h,\ I(k) < \delta \Rightarrow J(k) < \frac{\varepsilon}{4}.$$

Let $g \in +R_1(B,I)$ with $g \leqslant f$ and $I^+(g) < \delta$, and let us prove that $J^+(g) < \varepsilon$. Given $u \in +B$ with $u \leqslant g$, it is clear that $u \wedge h \in +B$, $u \wedge h \leqslant h$ and $I(u \wedge h) \leqslant I(u) \leqslant I^+(g) < \delta$ and hence $J(u \wedge h) < \frac{1}{4}\varepsilon$.

Thus, $J(u)+J(h) = J(u\wedge h)+J(u\vee h) \leqslant J(u\wedge h)+J^+(f)$ implies that $J(u) < \frac{1}{2}\varepsilon$.

Therefore, $J(u) < \frac{1}{2}\varepsilon$ for all $u \in +B$, $u \leqslant g$; that is, $J^+(f) \leqslant \frac{1}{2}\varepsilon < \varepsilon$, which gives $J \ll I$ on $R_1(B,I)$. □

We are now in a position to give the announced sufficient conditions for finitely additive measures induced by absolute continuous functionals to be absolutely continuous.

Consider the finitely additive measure space induced by I, that is, (X,Ω,μ_I) with $\Omega = \{A \subseteq X\colon\ \chi_A \in R_1(B,I)\}$ and $\mu_I(A) = I(\chi_A)\ \forall A \in \Omega$. Under the assumptions of Corollary 6.7, $R_1(B,I) \subseteq R_1(B,J)$ and, therefore, we can also define the finitely additive measure μ_J on Ω as $\mu_J(A) := J(\chi_A),\ \forall A \in \Omega$.

Corollary 6.9. *If $1 \in R_1(B,I)$, $J \ll I$ and $J^+(f) < +\infty$ for all $f \in +\overline{\mathbb{R}}^X$ with $I^+(f) < +\infty$, then $\nu_J \ll \mu_I$.*

P r o o f. Theorem 6.8 and Corollary 6.7 say that $J \ll I$ on $R_1(B,I) \subseteq R_1(B,J)$. Thus, for $\varepsilon > 0$ and $1 \in R_1(B,I)$ there exists $\delta > 0$ such that

$$\forall g \in R_1(B,I) \text{ with } g \leqslant 1 \text{ and } I(g) < \delta \Rightarrow J(g) < \varepsilon.$$

Given $A \in \Omega(I) \cap \Omega(J)$ with $\mu_I(A) < \delta$ we have $\chi_A \wedge h \in R_{\text{prop}}(B,I), \forall h \in +B$, $\chi_A \wedge h \leqslant 1$ and $I(\chi_A \wedge h) \leqslant I(\chi_A) = \mu_I(A) < \delta$.

Therefore, it follows that $J(\chi_A \wedge h) < \varepsilon,\ \forall h \in +B$ and, since $\chi_A \in R_1(B,I) \subseteq R_1(B,J)$, we conclude that

$$\nu_J(A) = J(\chi_A) = J_l^-(\chi_A) = \sup\{J^-(\chi_A \wedge h)\colon\ h \in +B\} < \varepsilon.$$

□

Furthermore, assuming that the Loomis system (X,B,I) is $C_{+\infty}$, we are able to obtain the absolute continuity of certain induced finitely additive measures μ_f with respect to the finitely additive measure μ induced by I.

To be more specific, given $f \in +R_1(B,I)$, Lemma 3.4 allows us to define the finitely additive measure μ_f by $\mu_f(A) := I(f\chi_A), \forall A \in \Omega$, where $\Omega = \{A \subseteq X\colon\ \chi_A \in R_1(B,I)\}$. Setting $\mu(A) := I(\chi_A), \forall A \in \Omega$, we can prove that $\mu_f \ll \mu$.

Theorem 6.10. *If (X, B, I) is a $C_{+\infty}$ Loomis system and $f \in R_1(B, I)$, then*

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0\colon\ A \subseteq X,\ I^*(\chi_A) < \delta \Rightarrow I^*(|f|\chi_A) < \varepsilon.$$

Proof. There is no loss of generality in assuming that $f \in +R_1(B, I)$.

Given $A \subseteq X$, $r > 0$ and $\varepsilon > 0$, we have the inequality

$$f\chi_A = (f - f \wedge r + f \wedge r)\chi_A = (f - f \wedge r)\chi_A + (f \wedge r)\chi_A \leqslant (f - f \wedge r) + r\chi_A.$$

Since B is $C_{+\infty}$ and so is $R_1(B, I)$ (see Lemma 3.2), there exists $s > 0$ such that $I^*(f - f \wedge s) < \frac{1}{2}\varepsilon$.

Let $\delta := \frac{1}{2}\varepsilon/s$. Given $A \subseteq X$ with $I^*(\chi_A) < \delta$, we have

$$I^*(f\chi_A) \leqslant I^*(f - f \wedge s) + I^*(s\chi_A) < \frac{\varepsilon}{2} + s\frac{\varepsilon}{2s} = \varepsilon.$$

□

Corollary 6.11. *If (X, B, I) is a $C_{+\infty}$ Loomis system and $0\chi f \in R_1(B, I)$, then*

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0\colon\ A \in \Omega,\ \mu(A) < \delta \Rightarrow \mu_f(A) < \varepsilon.$$

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