



The Schur degree of additive sets

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ARTICLE INFO

Article history:

Received 21 October 2020

Received in revised form 22 January 2021

Accepted 25 January 2021

Available online 12 February 2021

Keywords:

Sumfree

Schur numbers

Ramsey numbers

Discrete derivative

Minors

ABSTRACT

Let $(G, +)$ be an abelian group. A subset of G is *sumfree* if it contains no elements x, y, z such that $x + y = z$. We extend this concept by introducing the *Schur degree* of a subset of G , where Schur degree 1 corresponds to sumfree. The classical inequality $S(n) \leq R_n(3) - 2$, between the Schur number $S(n)$ and the Ramsey number $R_n(3) = R(3, \dots, 3)$, is shown to remain valid in a wider context, involving the Schur degree of certain subsets of G . Recursive upper bounds are known for $R_n(3)$ but not for $S(n)$ so far. We formulate a conjecture which, if true, would fill this gap. Indeed, our study of the Schur degree leads us to conjecture $S(n) \leq n(S(n-1) + 1)$ for all $n \geq 2$. If true, it would yield substantially better upper bounds on the Schur numbers, e.g. $S(6) \leq 966$ conjecturally, whereas all is known so far is $536 \leq S(6) \leq 1836$.

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1. Introduction

For $a, b \in \mathbb{Z}$, let $[a, b] = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ and $[a, \infty[= \{z \in \mathbb{Z} \mid a \leq z\}$ denote the integer intervals they span. Denote $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$.

A subset of \mathbb{Z} is *sumfree* if it contains no elements x, y, z such that $x + y = z$. The problem of partitioning $[1, N]$ into as few sumfree parts as possible was initiated by Schur [11]. Given $n \in \mathbb{N}_+$, Schur established the existence of a number $S(n)$ such that $[1, N]$ can be partitioned into n sumfree parts if and only if $N \leq S(n)$. The $S(n)$ are called the Schur numbers and, despite more than a century in existence, remain poorly understood at the time of writing. Their only currently known values are

$$(S(1), S(2), S(3), S(4), S(5)) = (1, 4, 13, 44, 160). \quad (1)$$

See Section 5.2 for more details. In his paper, Schur proved the following upper bound and recursive lower bound on the $S(n)$ for $n \geq 2$, namely

$$3S(n-1) + 1 \leq S(n) \leq n!e, \quad (2)$$

leading in particular to $S(n) \geq (3^n - 1)/2$ for all $n \geq 2$.

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For $n \geq 1$, the n -color Ramsey number $R_n(3) = R(3, \dots, 3)$ denotes the smallest N such that, for any n -coloring of the edges of the complete graph K_N on N vertices, there is a monochromatic triangle. See [10] for an extensive dynamic survey on this topic. Only three of the numbers $R_n(3)$ are currently known, namely

$$(R_1(3), R_2(3), R_3(3)) = (3, 6, 17). \tag{3}$$

As for $n = 4$, the presently known bounds are $51 \leq R_4(3) \leq 62$. It is conjectured in [14] that $R_4(3)$ equals 51. Similarly to the upper bound in (2), it was shown in [8] that $R_n(3) \leq n!e + 1$ for all $n \geq 1$. This bound has later been improved to

$$R_n(3) \leq n!(e - 1/6) + 1$$

for all $n \geq 4$ in [15]. See also [4], where the conjecture $R_4(3) = 51$ is shown to imply $R_n(3) \leq n!(e - 5/8) + 1$ for all $n \geq 4$. In fact, there is a well known relationship between the Schur and the Ramsey numbers, namely

$$S(n) \leq R_n(3) - 2. \tag{4}$$

See e.g. [12]. That is, if the set $[1, N]$ admits a partition into n sumfree parts, then $N \leq R_n(3) - 2$.

We shall show here that (4) holds in a more general context. Let $(G, +)$ be an abelian group. As in \mathbb{Z} , a subset of G is *sumfree* if it contains no elements x, y, z such that $x + y = z$. Given a finite sequence $A = (a_1, \dots, a_N)$ in G , let us denote by \hat{A} the set of all *block sums* $a_i + \dots + a_j$ of A , where $1 \leq i \leq j \leq N$. For instance, if $A = (1, \dots, 1)$ of length N in $G = \mathbb{Z}$, then $\hat{A} = [1, N]$.

In this paper, we are concerned with partitioning subsets of G of the form \hat{A} into as few sumfree parts as possible. As just noted, this includes Schur's original problem for the integer intervals $[1, N]$. Our extension of (4) to this more general setting states that if A is a sequence in G of length $|A| = N$ and if \hat{A} can be covered by n sumfree parts, then $N \leq R_n(3) - 2$.

Currently, the best available theoretical upper bound on $S(n)$ for $n \geq 4$ is the one provided by (4). While the Ramsey numbers $R_n(3)$ satisfy the well known recursive upper bound

$$R_n(3) \leq n(R_{n-1}(3) - 1) + 2$$

for all $n \geq 2$ [8, Theorem 6, p. 6], no similar statement is known yet for the $S(n)$. Here we fill this gap, at least conjecturally, as an outcome of our study of sumfree partitions of sets of the form \hat{A} . Indeed, as we shall see, that study leads us to conjecture the following recursive upper bound, for all $n \geq 2$:

$$S(n) \leq n(S(n - 1) + 1). \tag{5}$$

The contents of this paper are as follows. In Section 2, we introduce the Schur degree and the basic notions and tools needed in the sequel. In Section 3, we prove initial properties of the Schur degree and illustrate them with selected examples in \mathbb{Z} . Our main result, an extension of (4) to sets \hat{A} bounding their Schur degree with the Ramsey numbers $R_n(3)$, is proved in Section 4. The material developed so far leads us in Section 5 to the conjectural recursive upper bound (5), a substantial would-be improvement over (4).

2. Basic notions and tools

Here is the main notion introduced and studied in this paper.

Definition 2.1. Let $(G, +)$ be an abelian group. Let $X \subseteq G$ be a subset. We define the *Schur degree* of X , denoted $\text{sdeg}(X)$, as the smallest $n \geq 1$ such that X can be covered by n sumfree subsets. If no such n exists, we set $\text{sdeg}(X) = \infty$.

For instance, $\text{sdeg}(X) = 1$ if and only if X is sumfree, whereas $\text{sdeg}(X) = \infty$ whenever $0 \in X$, as $\{0\}$ is not sumfree. As another instance, in \mathbb{N} we have

$$\text{sdeg}([1, S(n)]) = n, \quad \text{sdeg}([1, S(n) + 1]) = n + 1 \tag{6}$$

by definition of $S(n)$. Equivalently, $\text{sdeg}([1, N]) \leq n \iff N \leq S(n)$.

Measuring the Schur degree of most subsets is likely to remain an extremely difficult task, even for the integer intervals $[1, N]$ as witnessed by the still highly mysterious Schur numbers $S(n)$. In this paper, we focus on subsets of a certain form \hat{A} , generalizing the intervals $[1, N]$ and introduced below.

2.1. Block sums

Let $(G, +)$ be an abelian group. Let $A = (a_1, \dots, a_N)$ be a finite sequence in G . We denote by $|A| = N$ its length and by $\sigma(A) = \sum_i a_i$ the sum of its elements.

A *block* in A is any nonempty subsequence of consecutive elements of A . That is, any subsequence of the form

$$B = (a_i, \dots, a_j)$$

for some $1 \leq i \leq j \leq N$. A *block sum* in A is a sum $\sigma(B)$ where B is any block in A , i.e. any element in G of the form $a_i + \dots + a_j$ for some $1 \leq i \leq j \leq N$.

Notation 2.2. Let $A = (a_1, \dots, a_N)$ be a sequence in G . We denote by

$$\hat{A} = \{\sigma(B) \mid B \text{ is a block in } A\},$$

the set of block sums in A .

For instance, if $A = (1, \dots, 1)$ of length N in \mathbb{Z} , then $\hat{A} = [1, N]$ as noted above. In this paper, we initiate the study of the Schur degree of subsets of the form \hat{A} for finite sequences A in G , with the hope to shed some light on the basic case $[1, N]$ in \mathbb{Z} . Our main result is [Theorem 4.1](#), an extension of [\(4\)](#) to this context.

2.2. Minors

We show here that the association $A \mapsto \hat{A}$ is monotone with respect to taking minors, as defined below.

Definition 2.3. Let $A = (a_1, \dots, a_N)$ be a sequence in the abelian group G .

- An elementary contraction of A is any sequence \bar{A} obtained by replacing a block B in A by its sum $\sigma(B)$. That is, if $B = (a_i, \dots, a_j)$ for some $1 \leq i \leq j \leq N$, then

$$\bar{A} = (a_1, \dots, a_{i-1}, \sigma(B), a_{j+1}, \dots, a_N).$$

- A contraction of A is any sequence obtained from A by successive elementary contractions.

For instance, let $A = (1, 2, 3, 4)$. Then $(3, 3, 4)$, $(6, 4)$ and $(3, 7)$ are contractions of A , the first two ones being elementary. See also [\[1\]](#).

Definition 2.4. Let $A = (a_1, \dots, a_N)$ be a sequence in G . A minor of A is either a block B in A or a contraction \bar{A} of A .

Proposition 2.5. Let G be an abelian group. Let A be a finite sequence in G . If B is a minor of A , then $\hat{B} \subseteq \hat{A}$.

Proof. The stated inclusion clearly holds if B is a block in A , since any block sum of B is a block sum of A . If B is an elementary contraction of A then again, any block sum of B is a block sum of A . Therefore, the same holds if B is obtained from A by successive elementary contractions. \square

2.3. The discrete derivative

For subsets X, Y of a group $(G, +)$, their sumset is $X + Y = \{x + y \mid x \in X, y \in Y\}$. Thus, X is sumfree if and only if $(X + X) \cap X = \emptyset$; equivalently, if and only if $(X - X) \cap X = \emptyset$, where $-X = \{-x \mid x \in X\}$.

In this section, for $X \subset \mathbb{Z}$ finite, we relate $X - X$ with a subset of the form \hat{A} for a certain sequence A closely linked to X . This is done with a variant of the discrete derivative, associating to a subset $X \subset \mathbb{Z}$ its sequence of successive jumps. See also [\[1\]](#).

Definition 2.6. Let $X \subset \mathbb{Z}$ be a finite subset. Let the elements of X be $x_0 < x_1 < \dots < x_r$. The discrete derivative of X is the sequence

$$\Delta X = (x_1 - x_0, x_2 - x_1, \dots, x_r - x_{r-1})$$

of successive jumps in X .

The interesting point for our purposes here is that $X - X$ can be read off from the block sums of ΔX .

Proposition 2.7. Let $X \subset \mathbb{Z}$ be a nonempty finite subset, and let $A = \Delta X$. Then

$$\hat{A} = (X - X) \cap \mathbb{N}_+.$$

Proof. Denote by $x_0 < x_1 < \dots < x_r$ the elements of X . Then

$$(X - X) \cap \mathbb{N}_+ = \{x_t - x_s \mid 0 \leq s < t \leq r\}.$$

Let $A = \Delta X = (a_1, \dots, a_r)$, where $a_i = x_i - x_{i-1}$ for $1 \leq i \leq r$. For any indices $0 \leq s < t \leq r$, let $B = (a_{s+1}, \dots, a_t)$ be the corresponding block in A . Then

$$x_t - x_s = \sigma(B). \tag{7}$$

Indeed, $\sigma(B) = \sum_{i=s+1}^t a_i = \sum_{i=s+1}^t (x_i - x_{i-1}) = x_t - x_s$. Hence $x_t - x_s \in \hat{A}$. This concludes the proof of the proposition. \square

The next proposition bounds the Schur degree of certain subsets \hat{A} in \mathbb{Z} . We start with a lemma.

Lemma 2.8. *Let X be a sumfree subset of $[1, N]$ for some $N \in \mathbb{N}_+$. Let $A = \Delta(X)$. Then $\hat{A} \subseteq [1, N - 1] \setminus X$.*

Proof. Denote $X = \{x_0, \dots, x_n\}$ with $1 \leq x_0 < x_1 < \dots < x_n \leq N$. Then $A = (a_1, \dots, a_n)$ where $a_i = x_i - x_{i-1}$ for all $1 \leq i \leq n$. Let $s \in \hat{A}$. Then

$$s = a_i + \dots + a_j = x_j - x_{i-1}$$

for some $1 \leq i \leq j \leq n$. Therefore $1 \leq s \leq N - 1$, and $s \notin X$ since $s + x_{i-1} = x_j$ and X is sumfree. That is, $s \in [1, N - 1] \setminus X$, as desired. \square

Proposition 2.9. *Let $N \geq 1$, and let $X_1 \sqcup \dots \sqcup X_n$ be a sumfree partition of $[1, N]$. Let $A_i = \Delta(X_i)$ for all i . Then $\text{sdeg}(\hat{A}_i) \leq n - 1$.*

Proof. Let $i \in [1, n]$. It follows from Lemma 2.8 that \hat{A}_i is contained in

$$X_1 \sqcup \dots \sqcup X_{i-1} \sqcup X_{i+1} \sqcup \dots \sqcup X_n.$$

This induces a partition of \hat{A}_i into at most $n - 1$ sumfree parts. \square

3. Basic properties of the Schur degree

In this section, we compute the Schur degree in a few examples after giving its first basic properties. Let us start with the monotonicity of the Schur degree with respect to set inclusion.

Lemma 3.1. *Let G be an abelian group. If $X \subseteq Y \subseteq G$ then $\text{sdeg}(X) \leq \text{sdeg}(Y)$.*

Proof. Let $n = \text{sdeg}(Y)$. If $n = \infty$, we are done. Otherwise, Y admits a partition into n sumfree parts, inducing a partition of X into at most n sumfree parts. \square

Here is a useful consequence.

Proposition 3.2. *Let A be a finite sequence in the abelian group G . If B is a minor of A , then $\text{sdeg}(\hat{B}) \leq \text{sdeg}(\hat{A})$.*

Proof. We have $\hat{B} \subseteq \hat{A}$ by Proposition 2.5. Now apply Lemma 3.1. \square

Note also that if A' denotes the reverse sequence of A , then $\text{sdeg}(\hat{A}) = \text{sdeg}(\hat{A}')$. Indeed, A and A' have identical block sums, i.e. $\hat{A} = \hat{A}'$.

Our next proposition shows that the Schur degree is also monotone with respect to inverse images under group morphisms. We start with a lemma.

Lemma 3.3. *Let G_1, G_2 be abelian groups and let $f: G_1 \rightarrow G_2$ be a morphism. Let $Y \subseteq G_2$. If Y is sumfree then $f^{-1}(Y)$ also is.*

Proof. Assume that $f^{-1}(Y)$ is not sumfree. Then there exist $x_1, x_2, x_3 \in f^{-1}(Y)$ such that $x_1 + x_2 - x_3 = 0$. Hence $f(x_1) + f(x_2) - f(x_3) = 0$, implying that Y is not sumfree either. \square

Proposition 3.4. *Let G_1, G_2 be abelian groups and let $f: G_1 \rightarrow G_2$ be a morphism. Let $Y \subseteq G_2$. Then $\text{sdeg}(f^{-1}(Y)) \leq \text{sdeg}(Y)$.*

Proof. Let $n = \text{sdeg}(Y)$. Then there exist sumfree subsets $Y_1, \dots, Y_n \subseteq Y$ such that

$$Y = Y_1 \sqcup \dots \sqcup Y_n.$$

Therefore $f^{-1}(Y) = f^{-1}(Y_1) \sqcup \dots \sqcup f^{-1}(Y_n)$, and $f^{-1}(Y_i)$ is sumfree for all i by Lemma 3.3. Hence $\text{sdeg}(f^{-1}(Y)) \leq n$. \square

3.1. Examples

As an illustration, we determine the Schur degree of a few selected subsets of \mathbb{Z} or groups containing \mathbb{Z} . In some cases, the results were obtained using specially written functions in *Mathematica 10* [13].

Example 3.5. Let $B = [1, 2] \cup [m, m + 4]$. We claim that

$$\text{sdeg}(B) = 3$$

for all $m \geq 3$. Indeed, let $A = (1, 1, m, 1, 1)$. Then $\hat{A} = B$, and $\text{sdeg}(\hat{A}) \geq 3$ by Corollary 4.2 in the next section. Equality is obvious here.

Example 3.6. Let $A = (2^i)_{0 \leq i \leq 13}$. Then here also, $\text{sdeg}(\hat{A}) = 3$. But with one more term, i.e. for $B = (2^i)_{0 \leq i \leq 14}$, it is no longer the case as $\text{sdeg}(\hat{B}) = 4$.

Example 3.7. This example is an application of Proposition 3.4. Let x, y be positive integers, and let $A = (x, y, \dots, x, y)$ be the 2-periodic sequence of length 14. Then $\text{sdeg}(\hat{A} \setminus \{7x + 7y\}) = 3$. Indeed, here are three sumfree classes covering that set:

- $C_1 : x, y, 2x + 2y, 5x + 5y, 7x + 6y, 6x + 7y.$
- $C_2 : x + y, 2x + y, x + 2y, 6x + 5y, 5x + 6y, 6x + 6y.$
- $C_3 : 3x + 2y, 2x + 3y, 3x + 3y, 4x + 3y, 3x + 4y, 4x + 4y, 5x + 4y, 4x + 5y.$

Mapping x, y to 1 yields a sumfree 3-partition of $[1, 13]$. In fact, the partition C_1, C_2, C_3 was constructed to do exactly that, using Proposition 3.4.

Example 3.8. For each integer $x \geq 8$, one has

$$\text{sdeg}([1, 6] \cup [x, x + 13]) = 3.$$

Indeed, this is shown by the following sumfree 3-partition of this set:

- $C_1 : 1, 6, x, x + 3, x + 7, x + 10.$
- $C_2 : 2, 5, x + 1, x + 2, x + 8, x + 9.$
- $C_3 : 3, 4, x + 4, x + 5, x + 6, x + 11, x + 12, x + 13.$

However, adjoining 7 to it, one has $\text{sdeg}([1, 7] \cup [x, x + 13]) = 4$.

Example 3.9. Let G be an abelian group containing \mathbb{Z} and let $x \in G \setminus \mathbb{Z}$. Then

$$\begin{aligned} \text{sdeg}(\{1, 2\} \cup [x, x + 3]) &= 2, \\ \text{sdeg}(\{1, 2\} \cup [x, x + 4]) &= 3. \end{aligned}$$

Indeed, as easily seen, the *only* sumfree 2-coloring of $\{1, 2\} \cup [x, x + 3]$ is given by the two color classes $\{1, x, x + 3\}$ and $\{2, x + 1, x + 2\}$. Hence, it is impossible to add $x + 4$ to either class while maintaining the sumfree property.

Example 3.10. Let G be an abelian group containing \mathbb{Z} . Let $x \in G \setminus \mathbb{Z}$ be such that $\{1, x\}$ is \mathbb{Z} -free, i.e. spans a free-abelian subgroup of rank 2 of G . Then

$$\text{sdeg}([1, 6] \cup (x + \mathbb{N})) = 3.$$

Indeed, consider the 3-partition of Example 3.8 and extend it periodically as follows:

- $C_1 : 1, 6, x, x + 3, x + 7, x + 10, x + 14, x + 17, \dots$
- $C_2 : 2, 5, x + 1, x + 2, x + 8, x + 9, x + 15, x + 16, \dots$
- $C_3 : 3, 4, x + 4, x + 5, x + 6, x + 11, x + 12, x + 13, x + 18, x + 19, x + 20, \dots$

One can also extend it towards the left. Thus in fact, $\text{sdeg}([1, 6] \cup (x + \mathbb{Z})) = 3$. But here again, adjoining 7 to it, one has $\text{sdeg}([1, 7] \cup (x + \mathbb{Z})) = 4$.

4. Comparison with $R_n(3)$

Recall that, for $n \geq 1$, the Ramsey number $R_n(3)$ denotes the smallest N such that, for any n -coloring of the edges of the complete graph K_N , there is a monochromatic triangle. There is a well known relationship between the Schur and the Ramsey numbers, namely

$$S(n) \leq R_n(3) - 2. \tag{8}$$

Using the Schur degree of $[1, N]$, this may be expressed as follows:

$$N \geq R_n(3) - 1 \implies \text{sdeg}([1, N]) \geq n + 1.$$

Theorem 4.1 below extends this relationship to the Schur degree of \hat{A} for any finite sequence A in an abelian group.

Theorem 4.1. Let G be an abelian group. Let A be a finite sequence in G . If $|A| \geq R_n(3) - 1$ then $\text{sdeg}(\hat{A}) \geq n + 1$.

Proof. Let $N = |A| \geq R_n(3) - 1$. Denote $b(i, j) = x_i + \dots + x_{j-1}$ for all $1 \leq i < j \leq N + 1$. Then

$$\hat{A} = \{b(i, j) \mid 1 \leq i < j \leq N + 1\}.$$

Let $\chi: \hat{A} \rightarrow [1, n]$ be an arbitrary n -coloring of \hat{A} . Consider the complete graph $K_{N+1} = (V, E)$ on the vertex set $V = [1, N + 1]$. Then χ induces an n -coloring $\chi': E \rightarrow [1, n]$ on E defined by

$$\chi'(\{i, j\}) = \chi(b(i, j))$$

for all $1 \leq i < j \leq N + 1$. Since $N + 1 \geq R_n(3)$, there is a monochromatic triangle under χ' in K_{N+1} , say with vertices i, j, h for some $1 \leq i < j < h \leq N + 1$. This yields, under χ , the monochromatic subset

$$\{b(i, j), b(j, h), b(i, h)\} \subset \hat{A}.$$

Since $b(i, j) + b(j, h) = b(i, h)$, the corresponding color class in \hat{A} is not sumfree. Since χ was an arbitrary n -coloring of \hat{A} , we conclude that $\text{sdeg}(\hat{A}) \geq n + 1$. \square

In particular, for $n = 2, 3$ and 4 , one has the following consequences.

Corollary 4.2. *Let A be a sequence in an abelian group G . If $|A| \geq 5$, then $\text{sdeg}(\hat{A}) \geq 3$. If $|A| \geq 16$, then $\text{sdeg}(\hat{A}) \geq 4$. If $|A| \geq 61$, then $\text{sdeg}(\hat{A}) \geq 5$.*

Proof. Follows from [Theorem 4.1](#) and the well-known values $R_2(3) = 6, R_3(3) = 17$ and current upper bound $R_4(3) \leq 62$. \square

The converse of [Theorem 4.1](#) does not hold in general. For instance, for $n = 3$ and $A = (1, \dots, 1)$ of length 14 in \mathbb{Z} , by [\(6\)](#) we have $\text{sdeg}(\hat{A}) \geq 4$ since $\hat{A} = [1, 14]$ and $S(3) = 13$, yet $|A| \leq R_3(3) - 2 = 15$. However, here is a partial converse showing that [Theorem 4.1](#) is best possible. First observe that if $|A| = N$, then

$$|\hat{A}| \leq 1 + 2 + \dots + N = \binom{N + 1}{2}.$$

The case of equality, where all block sums in A are pairwise distinct, is of interest. It occurs for instance if A is \mathbb{Z} -free, i.e. generates a subgroup isomorphic to \mathbb{Z}^N .

Theorem 4.3. *Let A be a finite sequence in an abelian group G . If $|A| \leq R_n(3) - 2$ and A is \mathbb{Z} -free, then $\text{sdeg}(\hat{A}) \leq n$.*

Proof. Denote $A = \{x_1, \dots, x_N\}$. Reusing the notation introduced in the proof of [Theorem 4.1](#), we have

$$\hat{A} = \{b(i, j) \mid 1 \leq i < j \leq N + 1\}.$$

Again, let $K_{N+1} = (V, E)$ be the complete graph on the vertex set $V = [1, N + 1]$. Consider the map $f: E \rightarrow \hat{A}$ defined by

$$f(\{i, j\}) = b(i, j) \tag{9}$$

for all $1 \leq i < j \leq N + 1$. Since $|E| = |\hat{A}|$ and the $b(i, j)$ are pairwise distinct by assumption, the map f is a bijection. Since $N + 1 \leq R_n(3) - 1$, there is an n -coloring $\chi: E \rightarrow [1, n]$ without any monochromatic triangle. Consider the composed map

$$\chi \circ f^{-1}: \hat{A} \rightarrow [1, n].$$

We claim that under this n -coloring of \hat{A} , every color class is sumfree. Indeed, let u_1, u_2, u_3 be any triple in \hat{A} satisfying $u_1 + u_2 = u_3$. We claim that it cannot be monochromatic under $\chi \circ f^{-1}$. We have $u_1 = b(i_1, j_1), u_2 = b(i_2, j_2), u_3 = b(i_3, j_3)$ for some indices $i_1 < j_1, i_2 < j_2, i_3 < j_3$ in $[1, N + 1]$. The relation $u_1 + u_2 = u_3$ then becomes

$$(x_{i_1} + \dots + x_{j_1-1}) + (x_{i_2} + \dots + x_{j_2-1}) = (x_{i_3} + \dots + x_{j_3-1}).$$

We may freely assume $i_1 \leq i_2$. Since the sequence x_1, \dots, x_N is \mathbb{Z} -free by hypothesis, the above equality is only possible if $i_1 = i_3, j_1 = i_2$ and $j_2 = j_3$. That is, if the three edges $\{i_1, j_1\}, \{i_2, j_2\}, \{i_3, j_3\}$ form a triangle in K_{N+1} . Since that triangle is not monochromatic under χ , the triple $u_1, u_2, u_3 = u_1 + u_2$ in \hat{A} is not monochromatic under $\chi \circ f^{-1}$ either, since $f^{-1}(u_k) = \{i_k, j_k\}$ for $k = 1, 2, 3$ by [\(9\)](#). Hence $\text{sdeg}(\hat{A}) \leq n$, as claimed. \square

Remark 4.4. The hypothesis that A be \mathbb{Z} -free is not strictly needed in [Theorem 4.3](#). For instance, let $A = (1, 3, 3^2, \dots, 3^{N-1})$. Even though A is not \mathbb{Z} -free, it is still true that if $N \leq R_n(3) - 2$ then $\text{sdeg}(\hat{A}) \leq n$. This derives from the above proof and the fact that the only triples $u, v, u + v$ in \hat{A} are those of the form $b(i, j), b(j, h), b(i, h)$.

5. A recursive upper bound on $S(n)$?

The Ramsey numbers admit well-known recursive upper bounds, including

$$R_n(3) \leq n(R_{n-1}(3) - 1) + 2 \tag{10}$$

[8, Theorem 6, p. 6]. To the best of our knowledge, no recursive upper bounds are known yet for the Schur numbers. We propose here a conjecture which, if true, would fill this gap. Let us start with an upper bound on $S(n)$ involving the number $L(n)$ defined below.

Definition 5.1. Let $n \geq 2$. We define $L(n)$ to be the smallest positive integer with the following property: for every sequence A in \mathbb{N}_+ of length $|A| = L(n)$ and average $\mu(A) \leq n$, one has $\text{sdeg}(\hat{A}) \geq n$.

Example 5.2. Let $n = 2$. Then $L(2) = 2$. Indeed, up to symmetry, the only sequences A to consider are $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 2)$. This yields $\hat{A} = \{1, 2\}$, $\{1, 2, 3\}$, $\{1, 3, 4\}$, $\{2, 4\}$, respectively. As none is sumfree, we have $\text{sdeg}(\hat{A}) \geq 2$ in all cases, as required.

Let us now establish the existence of $L(n)$ in full generality.

Proposition 5.3. For all $n \geq 2$, the number $L(n)$ exists and satisfies

$$S(n - 1) + 1 \leq L(n) \leq R_{n-1}(3) - 1. \tag{11}$$

Proof. If A is any sequence in \mathbb{N}_+ of length $|A| = R_{n-1}(3) - 1$, then irrespective of its average $\mu(A)$, we have $\text{sdeg}(\hat{A}) \geq n$ by Theorem 4.1, as desired. Thus $L(n)$ exists and is bounded above by $R_{n-1}(3) - 1$. On the other hand, let $A = (1, \dots, 1)$ of length $L(n)$ and average $\mu(A) = 1$. Then $\hat{A} = [1, L(n)]$, whence $\text{sdeg}([1, L(n)]) \geq n$ by hypothesis. Hence $L(n) \geq S(n - 1) + 1$, by definition of $S(n - 1)$. \square

Here is our upper bound on $S(n)$ involving $L(n)$.

Theorem 5.4. We have $S(n) \leq nL(n)$ for all $n \geq 2$.

Proof. We claim that $[1, nL(n) + 1]$ has Schur degree at least $n + 1$. This will imply $nL(n) + 1 \geq S(n) + 1$, the desired conclusion. Assume for a contradiction that $nL(n) + 1 \leq S(n)$. Let then

$$[1, nL(n) + 1] = X_1 \sqcup \dots \sqcup X_n \tag{12}$$

be a sumfree partition. By the pigeonhole principle, one of the X_i 's has cardinality at least $L(n) + 1$, say $|X_1| \geq L(n) + 1$. Let $A = \Delta(X_1)$. Then $|A| \geq L(n)$, and $\text{sdeg}(\hat{A}) \leq n - 1$ by Proposition 2.9. Let B be a block of A of length $|B| = L(n)$. Since B is a minor of A , Proposition 2.5 implies

$$\text{sdeg}(\hat{B}) \leq \text{sdeg}(\hat{A}) \leq n - 1. \tag{13}$$

Let $s = \min(X_1)$, $t = \max(X_1)$. Then $\sigma(A) = t - s$ by (7), and $t - s \leq nL(n)$ since $X_1 \subseteq [1, nL(n) + 1]$ by (12). Hence $\sigma(B) \leq nL(n)$ and so $\mu(B) = \sigma(B)/L(n) \leq n$. Since $|B| = L(n)$, the defining property of $L(n)$ implies $\text{sdeg}(\hat{B}) \geq n$, contradicting (13). This concludes the proof of the theorem. \square

Remark 5.5. Proposition 5.3 and Theorem 5.4 imply the upper bound

$$S(n) \leq n(R_{n-1}(3) - 1)$$

for all $n \geq 2$. However, this also follows by combining (8) and (10), namely $S(n) \leq R_n(3) - 2$ and $R_n(3) \leq n(R_{n-1}(3) - 1) + 2$.

5.1. Conjectures

Given $n \geq 2$, what is the exact value of $L(n)$? It follows from Proposition 5.3 that

$$\text{if } S(n - 1) + 1 = R_{n-1}(3) - 1 \text{ then } L(n) = S(n - 1) + 1. \tag{14}$$

This occurs for $n = 2$ and 3 , since by (1) and (3), we have $(S(1), R_1(3)) = (1, 3)$ and $(S(2), R_2(3)) = (4, 6)$. Thus $L(2) = 2$ as already seen, and $L(3) = 5$. As for $n = 4$, we have

$$(S(3), R_3(3)) = (13, 17).$$

Proposition 5.3 then implies $14 \leq L(4) \leq 16$. We conjecture that $L(4) = 14$ and, more generally, that the lower bound on $L(n)$ in (11) is optimal.

Table 1
 $S(n) \leq n(S(n-1) + 1)$ for $2 \leq n \leq 5$.

n	$S(n)$	$n(S(n-1) + 1)$
1	1	
2	4	4
3	13	15
4	44	56
5	160	225

Conjecture 5.6. *Let $n \geq 2$. Then $L(n) = S(n-1) + 1$. That is, every sequence A in \mathbb{N}_+ of length $|A| = S(n-1) + 1$ and average $\mu(A) \leq n$ satisfies $\text{sdeg}(\hat{A}) \geq n$.*

As shown below, this has very interesting consequences for the Schur numbers themselves.

We have seen above that [Conjecture 5.6](#) holds for $n = 2$ and 3 . Does it hold for $n = 4$? That is, is it true that for any sequence A in \mathbb{N}_+ of length 14 and average $\mu(A) \leq 4$, one has $\text{sdeg}(\hat{A}) \geq 4$? We do not know yet. In any case, some hypothesis bounding $\mu(A)$ from above cannot be completely dispensed of. For instance, consider the sequence

$$A = (23, 375, 23, 209, 209, 60, 60, 60, 23, 1, 60, 261, 209, 23)$$

of length 14 . Then $|\hat{A}| = 83$, and $\text{sdeg}(\hat{A}) = 3$ as can be verified. But this does not contradict [Conjecture 5.6](#) for $n = 4$, since $\mu(A) = 114$ here. Such exotic examples in length 14 are hard to come by. This one was found with a semi-random search by computer. See also [Example 3.6](#) with the powers of 2 , also of length 14 but with a still higher average.

Here is a worthwhile consequence of [Conjecture 5.6](#) for the Schur numbers, potentially the first known recursive upper bound for them.

Conjecture 5.7. $S(n) \leq n(S(n-1) + 1)$ for all $n \geq 2$.

This directly follows from [Theorem 5.4](#) and [Conjecture 5.6](#). [Table 1](#) shows that [Conjecture 5.7](#) actually holds for $2 \leq n \leq 5$.

5.2. Comparisons

Let us now compare this conjectural upper bound on $S(n)$ with the general currently known ones given by [\(8\)](#) and [\(10\)](#), namely

$$S(n) \leq R_n(3) - 2, \quad R_n(3) \leq n(R_{n-1}(3) - 1) + 2. \tag{15}$$

The currently known bounds on $R_4(3)$ are $51 \leq R_4(3) \leq 62$, established in [\[3\]](#) and [\[6\]](#), respectively. Starting with $R_4(3) \leq 62$, the bounds [\(15\)](#) yield

$$S(5) \leq R_5(3) - 2 \leq 305, \quad S(6) \leq R_6(3) - 2 \leq 1836.$$

- For $n = 4$, the equality $S(4) = 44$ was established by computer [\[2\]](#). But, as far as theory is concerned, nothing better than $S(4) \leq R_4(3) - 2 \leq 60$ is currently known. A proof of [Conjecture 5.6](#) for $n = 4$ would yield $S(4) \leq 56$, still far away from the true value 44 , yet a little closer to it.

- For $n = 5$, the bound $S(5) \geq 160$ was first established in [\[5\]](#), with equality later conjectured to hold in [\[7\]](#). Indeed, the exact value $S(5) = 160$ has recently been established by massive computer calculations with a certified SAT solver [\[9\]](#). A proof of [Conjecture 5.6](#) for $n = 5$, namely that every sequence A in \mathbb{N}_+ such that $|A| = 45$ and $\mu(A) \leq 5$ satisfies $\text{sdeg}(\hat{A}) \geq 5$, would imply $S(5) \leq 225$. Here again, it would still be far away from the true value, yet it would provide a marked improvement over the currently best known theoretical upper bound $S(5) \leq 305$.

- For $n = 6$, on the one hand we have $S(6) \geq 536$ by [\[7\]](#), while at the time of writing, the best known upper bound is again the one given above, namely

$$S(6) \leq R_6(3) - 2 \leq 1836.$$

By sharp contrast, using the true value $S(5) = 160$, [Conjecture 5.7](#) implies the following substantial improvement.

Conjecture 5.8. $S(6) \leq 966$.

- As for $n = 7$, [Conjectures 5.7](#) and [5.8](#) yield the conjectural upper bound

$$S(7) \leq 6769,$$

to be compared with the known ones given by [\(15\)](#), namely $S(7) \leq R_7(3) - 2 \leq 12859$. For a lower bound, the best we currently have is $S(7) \geq 1680$, by [\[7\]](#) again.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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