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Finite-dimensionality of attractors for dynamical systems with applications: deterministic and random settings

Arthur Cavalcante Cunha

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Arthur Cavalcante Cunha

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Advisor: Prof. Dr. Alexandre Nolasco de Carvalho (ICMC-USP, Brasil)
Advisor: Prof. Dr. José Antonio Langa Rosado (US, Espanha)

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Arthur Cavalcante Cunha

Dimensão fractal de atratores para sistemas dinâmicos com aplicações: problemas determinísticos e aleatórios

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP e à Universidad de Sevilla – US , como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática (ICMC-USP) e PhD (US), de acordo com o convênio acadêmico internacional para dupla titulação de doutorado assinado entre o ICMC-USP e a US. *VERSÃO REVISADA.*

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Orientador: Prof. Dr. Alexandre Nolasco de Carvalho (ICMC-USP, Brasil)
Orientador: Prof. Dr. José Antonio Langa Rosado (US, Espanha)

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To my wife, Anny, with much love.

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"Só há duas opções nesta vida: se resignar ou se indignar.

E eu não vou me resignar nunca."

(Darcy Ribeiro, Antropólogo Brasileiro)

ABSTRACT

CUNHA, A.C. **Finite-dimensionality of attractors for dynamical systems with applications: deterministic and random settings**. 2021. 198 p. Tese (Doutorado em Ciências – Matemática (ICMC-USP) e PhD (US)) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

In this work we obtain estimates on the fractal dimension of attractors in three different settings: global attractors associated to autonomous dynamical systems, uniform attractors associated to non-autonomous dynamical systems and random uniform attractors associated to non-autonomous random dynamical systems. Firstly we give a simple proof of a result due to Mañé (Springer LNM 898, 230–242, 1981) that the global attractor $\mathscr{A}$ (as a subset of a Banach space) for a map S is finite-dimensional if $DS(x) = C(x) + L(x)$, where C is compact and L is a contraction (and both are linear). In particular, we show that if S is compact and differentiable then $\mathscr{A}$ is finite-dimensional. Using a smoothing property for the differential DS we also prove that $\mathscr{A}$ has finite fractal dimension and we make a comparison of this method with Mañé's approach. We give applications to an abstract semilinear parabolic equation and to 2D Navier-Stokes equations. Secondly we prove using a smoothing method that uniform attractors have finite fractal dimension on Banach spaces, with bounds in terms of the dimension of the symbol space and a Kolmogorov entropy number. We also show that the smoothing property is useful to prove the finite-dimensionality of uniform attractors in more regular Banach spaces. In addition, we prove that the finite-dimensionality of the hull of a time-dependent function is fully determined by the tails of the function. We give applications to non-autonomous 2D Navier-Stokes and reaction-diffusion equations. Thirdly we prove using a smoothing and a squeezing method that random uniform attractors have finite fractal dimension. Neither of the two methods implies the other. Estimates on the dimension are given in terms of the dimension of the symbol space plus a term arising from the smoothing/squeezing property; the smoothing is applied also to more regular spaces. In this setting we give applications to a stochastic reaction-diffusion equation with scalar additive noise. In addition, a random absorbing set which absorbs itself after a deterministic period of time is constructed.

Keywords: Fractal dimension, dynamical systems, global attractor, uniform attractor, random uniform attractor.

RESUMO

CUNHA, A.C. **Dimensão fractal de atratores para sistemas dinâmicos com aplicações: problemas determinísticos e aleatórios**. 2021. 198 p. Tese (Doutorado em Ciências – Matemática (ICMC-USP) e PhD (US)) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

Neste trabalho obtemos estimativas para a dimensão fractal de atratores em três contextos: atratores globais associados a sistemas dinâmicos autônomos, atratores uniformes associados a sistemas dinâmicos não-autônomos e atratores uniformes aleatórios associados a sistemas dinâmicos aleatórios não-autônomos. Primeiro, apresentamos uma simples prova de um resultado de Mañé (Springer LNM 898, 230–242, 1981) no qual o atrator global $\mathscr{A}$ (como um subconjunto de um espaço de Banach) para uma função S tem dimensão fractal finita se $DS(x) = C(x) + L(x)$, onde C é compacto e L é uma contração (e ambos são operadores lineares). Em particular, provamos que se S é compacto e diferenciável então $\mathscr{A}$ tem dimensão fractal finita. Supondo uma propriedade de regularização (conhecida como *smoothing*) para a diferencial DS provamos também que $\mathscr{A}$ tem dimensão finita e com isso fazemos uma comparação deste método com o já conhecido método de Mañé. Aplicamos nossos resultados teóricos em um problema parabólico semilinear abstrato e em equações de Navier-Stokes em 2D. Segundo, provamos usando também uma propriedade *smoothing* que atratores uniformes têm dimensão fractal finita em espaços de Banach, com estimativas dadas em termos da dimensão fractal do espaço símbolo mais um número de entropia de Kolmogorov. A propriedade *smoothing* é ainda utilizada para obtermos estimativas na dimensão fractal de atratores uniformes em espaços com maior regularização. Além disso, provamos que a dimensão fractal da envoltória (*hull*) de uma função dependente do tempo é completamente determinada pelo seu comportamento para tempos grandes (positivos e negativos). Aplicações são dadas em equações não-autônomas de reação-difusão e Navier-Stokes em 2D. Terceiro, utilizamos métodos *smoothing* e *squeezing* ("compressão") para obtermos estimativas na dimensão fractal de atratores uniformes aleatórios. Em geral a propriedade *squeezing* pode ser vista como um caso particular da *smoothing*, mas neste caso dos sistemas dinâmicos aleatórios não-autônomos isso não ocorre, e nenhum dos métodos implica no outro. Mais uma vez as estimativas na dimensão fractal são dadas em termos da dimensão do espaço símbolo e dos parâmetros aleatórios da propriedade *smoothing/squeezing*; a propriedade *smoothing* é utilizada ainda para obtermos estimativas na dimensão fractal em espaços mais regulares. Finalmente, consideramos uma perturbação aleatória (a exemplo de um ruído escalar aditivo) da equação de reação-difusão não-autônoma tratada anteriormente. Neste ponto é importante a construção de conjuntos aleatórios que absorvem a si próprios a partir de um período determinístico de tempo, situação a princípio não esperada.

Palavras-chave: Dimensão fractal, sistemas dinâmicos, atrator global, atrator uniforme, atrator uniforme aleatório.

RESUMEN

CUNHA, A.C. **Dimensão fractal de atratores para sistemas dinâmicos com aplicações: problemas determinísticos e aleatórios**. 2021. 198 p. Tese (Doutorado em Ciências – Matemática (ICMC-USP) e PhD (US)) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

En este trabajo obtenemos estimaciones para la dimensión fractal de atractores en tres contextos: atractores globales asociados a los sistemas dinámicos autónomos, atractores uniformes asociados a los sistemas dinámicos no-autónomos y atractores uniformes aleatorios asociados a los sistemas dinámicos aleatorios no-autónomos. En primer lugar, presentamos una prueba simple de un resultado de Mañé (Springer LNM 898, 230-242, 1981) en el cual un atractor global $\mathscr{A}$ (como un subconjunto de un espacio de Banach) para una función S tiene dimensión fractal finita si $DS(x) = C(x) + L(x)$, donde C es compacto y L es una contracción (y ambos son operadores lineales). En particular, probamos que si S es compacto y diferenciable entonces $\mathscr{A}$ tiene dimensión fractal finita. Asumiendo una propriedad de regularización (conocida por *smoothing*) para la diferencial DS probamos también que $\mathscr{A}$ tiene dimensión finita y con eso hacemos una comparación de este método con el ya conocido método de Mañé. Aplicamos nuestros resultados teóricos en un problema parabólico semilineal abstracto y en ecuaciones de Navier-Stokes en 2D. En segundo lugar, probamos utilizando una propriedad *smoothing* que atractores uniformes tienen dimensión fractal finita en espacios de Banach, con estimaciones dadas en términos de la dimensión fractal de los espacios símbolo más un número de entropía de Kolmogorov. La propriedad *smoothing* es utilizada para obtener estimaciones para la dimensión fractal de atractores uniformes en espacios con más regularidad. Además, probamos que la dimensión fractal del *hull* de una función dependiente del tiempo está completamente determinada por su comportamiento para grandes tiempos. Aplicaciones son dadas en ecuaciones no-autónomas de reacción-difusión y Navier-Stokes en 2D. En tercer lugar, utilizamos métodos *smoothing* y *squeezing* ("compresión") para obtener estimaciones en la dimensión fractal de atractores uniformes aleatorios. En general la propriedad *squeezing* puede verse como un caso particular del *smoothing*, pero en el caso de los sistemas dinámicos aleatorios no-autónomos esto no ocurre, y ninguno de los dos métodos implica el otro. Una vez más las estimaciones para la dimensión fractal son dadas en términos de la dimensión del espacio símbolo y parámetros aleatorios de la propriedad *smoothing/squeezing*; la propriedad *smoothing* es utilizada para obtener estimaciones en la dimensión fractal en espacios con más regularidad. Finalmente, consideramos una perturbación aleatoria (a ejemplo de un ruido escalar aditivo) de una ecuación de reacción-difusión no-autónoma. Aqui es importante la construcción de conjuntos aleatorios que absorben a sí mismos a partir de un tiempo determinista, situación en principio no esperada.

Palabras clave: Dimensión fractal, sistemas dinâmicos, atractor global, atractor uniforme, atractor uniforme aleatorio.

CONTENTS

CHAPTER

1

INTRODUCTION

In order to describe the long-time behavior of infinite-dimensional dynamical systems one often studies the attractors associated to them. Depending on the setting a problem is proposed, a number of typical attractors have been introduced and extensively studied: global attractors (see (ROBINSON, 2001) and (TEMAM, 1988)), pullback/cocycle attractors (see (CARVALHO; LANGA; ROBINSON, 2013) and (KLOEDEN; RASMUSSEN, 2011)), uniform attractors (see (CHEPYZHOV; VISHIK, 2002)), random attractors (see (CRAUEL; FLANDOLI, 1994)) and random uniform attractors (see (CUI; LANGA, 2017)), describing in their own way the asymptotic dynamics of the system under consideration.

This object - the attractor - whatever the setting, is more or less described as an invariant compact set which attracts some especial collection of subsets on a phase space, describing in this way its importance in predicting the asymptotic behavior of the system. However, it is still a subset of an infinite-dimensional space (Hilbert or Banach function spaces for instance) and one of the fundamental problems in the study of infinite-dimensional dynamical systems is that of finding a finite-dimensional system, as simple as possible, that exhibits the same asymptotic behavior as the original one. We can call it a finite-dimensional reduction of infinite-dimensional dynamical systems.

As indicated by the pioneering works (MALLET-PARET, 1976) and (MAÑÉ, 1981), who first established the finite-dimensionality of global attractors, estimating the fractal dimension of compact sets provides the information that it can be embedded into an Euclidean space $\mathbb{R}^k$ for some $k \in \mathbb{N}$. This embedding is shown to be linear with a Hölder continuous inverse and one can see it in (ROBINSON, 2011). Such embedding results are applied then to compact attractors which capture the asymptotic behavior of a dynamical system.

In this work we are interested in the study of fractal dimension of attractors in a variety of settings. More specifically, we aim to study the dimensionality of global attractors, uniform attractors and random uniform attractors. The main technique we use in order to obtain estimates on the fractal dimension for each of these objects is mainly known as a *smoothing property*.

A smoothing property can be essentially understood as in the following: for a Banach space $\left(X, \|\cdot\|_X\right)$, a map $S : X \to X$ and a Banach space $\left(Y, \|\cdot\|_Y\right)$ compactly embedded in X, we find $\kappa > 0$ such that $\|S(x) - S(y)\|_Y \leqslant \kappa \|x - y\|_X$, for $x, y \in X$. It means basically a Lipschitz condition satisfied by S, intrinsically describing its regularization aspect. In each case this property is presented accordingly and we split this work into three parts, organized throughout three chapters.

In Chapter 2 we are interested in the study of global attractors associated to autonomous dynamical systems (also known as semigroups) and in how to guarantee that they have finite fractal dimension on Banach spaces. Our motivation question is if the global attractor for a compact semigroup $S : X \to X$ necessarily has finite-dimensionality. The answer for that in general is *no*, but in addition to a differentiation of map S we obtain the result. Actually it is the well-known result in (MAÑÉ, 1981), but in our work we propose a much simpler proof of this fact, being able to present it on a concise and direct proof without any additional theory.

Our theorem, however, is abstract and does not present an explicit bound on the fractal dimension of the global attractor. To remedy this, we assume a quantitative smoothing estimate for the compact operator DS, the differential of S. It gives us a bound which is directly related to the Kolmogov entropy of the compact embedding $Y \hookrightarrow X$. Also in comparison to Mañé's method we could verify that this new procedure with a quantitative smoothing assumption on the derivative provides better estimates on the fractal dimension than Mañé's approach. Finally, as examples we apply the techniques developed in this first part to an abstract semilinear parabolic equation, a 2D Navier-Stokes equation and to an example on a space of sequences.

The Chapter 2 is organized as follows: in Section 2.1 we give a general and brief introduction on the theory of autonomous dynamical systems (semigroups) defining the basic setting and giving conditions for the existence of global attractors. In Section 2.2 we introduce the theory of fractal dimension of compact sets and present our new results on the dimensionality of global attractors as just described above (including the smoothing property, a comparison with Mañé's theory and application to nonlinear partial differential equations). Our contributions are summarized in (CARVALHO *et al.*,).

In Chapter 3 we study uniform attractors associated to non-autonomous dynamical systems. As in the autonomous setting, uniform attractors are intended to capture the forward dynamics of a system, but this time taking into account the time-dependent terms (such as for instance external forces and interaction functions). We prove, under a smoothing condition and the finite dimensionality of the symbol space, that the uniform attractor has finite fractal dimension on Banach phase spaces. Besides that, and dispite the lack of invariance of uniform attractors, we also prove its finite-dimensionality in more regular Banach spaces than the phase space. The smoothing condition has shown to be useful also for this and with the advantage of some relaxations on the smoothing property, considering for instance a Hölder assumption.

We also have constructed new classes of symbol spaces with finite fractal dimension. In general, symbol spaces are considered as the hull of the time-dependent coefficients of the equation under consideration (in a metric space) and it is well-known that hulls of quasiperiodic functions have finite fractal dimension on the space of bounded continuous functions. Now we propose to consider hulls of functions in the more general setting of the space of continuous functions with a Fréchet metric. More precisely, we take hulls of Lipschitz maps eventually exponentially converging to quasiperiodic maps, proving they have finite fractal dimension. Consequently, it implies that in order to prove uniform attractors associated to partial differential equations have finite fractal dimension, we can consider for such equations more general non-autonomous terms which are not almost periodic. As examples we prove the finite-dimensionality of uniform attractors for a reaction-diffusion equation and a 2D Navier-Stokes equation.

The Chapter 3 is organized in the following way. In Section 3.1 we introduce the theory of non-autonomous dynamical systems and uniform attractors providing a basic setting and a decompostion of the uniform attrator in terms of kernel sections. In Section 3.2 we give estimates on the fractal dimension of uniform attractors based on a smoothing condition and in Section 3.3 we construct new symbol spaces with finite-dimensionality. It is also presented in this section an introductory review of almost periodic functions and their hull. Finally, in Section 3.4 we give applications of our theoretical results to non-autonomous partial differential equations, namely a reaction-diffusion problem and a 2D Navier-Stokes equation. The results in this chapter were presented in (CUI *et al.*,).

In Chapter 4 we generalize in great part the results of Chapter 3 for the setting of non-autonomous random dynamical systems and prove that random uniform attractors have finite fractal dimension. At a first glance this generalization seems to be straighforward but it contains particularities which turn the observation interesting. First, we can say that it is much more technical than the deterministic setting and so the calculation must be done carefully. Second, the random nature of the problem brings together a necessity to obtain estimates on the expectation of some random variables, what can imply an extra difficulty in order to apply the results to particular examples.

To get through the second problem we propose then to use a squeezing method instead of the smoothing in order to prove that random uniform attractors have finite fractal dimension. A possible drawback of this approach is that we restrict the applicability of the result to problems on a Hilbert phase space, but fortunately the expectation of random variables in this case is easier to obtain. Finally, we consider a stochastic reaction-diffusion equation with scalar additive noise to apply the smoothing and squeezing methods and prove that its random uniform attractor has finite fractal dimension.

Following the structure of previous chapters we begin Chapter 4 with a revision of non-autonomous random dynamical systems and random uniform attractors in Section 4.1. In Section 4.2 we obtain estimates on the fractal dimension of random uniform attractors based on smoothing and squeezing properties. Finally, in Section 4.3 we apply the theorical results developed in previous section to a stochastic reaction-diffusion problem. The compilation of our contribution in this theme is presented in (CUI; CUNHA; LANGA,).

We organized this work into three different parts which are directly connected to each other and the step forward can be understood as a generalization of the immediately previous step: autonomous dynamical systems, non-autonomous dynamical systems and non-autonomous random dynamical systems. However each part can also be read independently of the others, with only a special attention to Section 2.2.1 where we give the basic setting on the theory of fractal dimension of compact sets which will be essential throughout the complete work. On one hand a considerable part of text brings introductory reviews in which we tried to present as complete as possible in order to give to the reader the big-picture of each setting. On the other hand, our new contributions to the theory of dynamical systems are summarized in Section 2.2 for autonomous dynamical systems; Section 3.2, Section 3.3 (more especifically in Section 3.3.3) and Section 3.4 for non-autonomous dynamical systems; and Section 4.2 and Section 4.3 for non-autonomous random dynamical systems. All the new results can be found in (CARVALHO *et al.*,), (CUI *et al.*,) and (CUI; CUNHA; LANGA,).

CHAPTER

2

AUTONOMOUS DYNAMICAL SYSTEMS

In this initial chapter we are devoted to the theory of autonomous dynamical systems (also known as semigroups) and to the study of the fractal dimension of their respective global attractors, which are the best mathematical objects in order to describe the asymptotic dynamics of the system. Included in this theme we have the notions of attraction, absorption, invariance, dynamics, global solutions, global attractors, Hausdorff semi-distance, fractal dimension, Kolmogorov numbers among others.

Autonomous dynamical systems describe, in essence, the behavior of solutions of an autonomous evolution equation while the global attractor, a compact invariant set which attracts all bounded sets under the action of the semigroup, works to describe the long-time behavior of such solutions for large times and gives us all the possible dynamics that a given system can produce. Among several important issues on the theory of semigroups, one in especial is of our interest throughout this work, namely the theory of fractal dimension of attractors.

Introduced first in the 70's with (MALLET-PARET, 1976) and (MAÑÉ, 1981) we can say that the theory of dimensionality of global attractors is a well-understood and well-developed topic with a variety of works exclusively dedicated to this. In special some methods, techniques and applications to a large number of differential equations were also given. Despite this great number of contents we are going to present in this chapter a new method to obtain estimates on the fractal dimension of global attractors (more generally it applies to negatively invariant sets). It is based on a smoothing condition for derivatives and surprisingly it allowed us to explicitly compare estimes obtained by different methods. Besides that in our work we could also give a proof of a result in (MAÑÉ, 1981) much simpler than the argument due to Mañé.

This chapter is splitted into two big sections. More specifically, in Section 2.1 we give all the introductory setting related to the theory of semigroups and characterizations of the global attractor, including existence results. Our aim is this section is basically to introduce the reader to the field showing all the background needed to understand great part of the work throughout the text. We decided to give a concise approach proving the essential. The concepts and results in this section are well-known and can be found in classical books such as for example (TEMAM, 1988), (ROBINSON, 2001) and (CHEPYZHOV; VISHIK, 2002).

In Section 2.2, we introduce with a brief review on the theory of fractal dimension of compact sets. In addition, we are interested in a deep discussion on the finite-dimensionality of global attractors. First we are able to give new results on the fractal dimension of negatively invariant sets and then supposing a new version of a smoothing condition we can find an explicitly comparison between methods for the dimension of global attractors. Finally as application of our theoretical analysis we study an abstract semilinear parabolic equation, a 2D Navier-Stokes equation and a problem on a space of sequences in Section 2.2.7. We present these results in (CARVALHO *et al.*,) which are part of our contribution to the field of autonomous dynamical systems and more specifically to the dimensionality of global attractors.

2.1 Autonomous dynamical systems: global attractors

Our aim in this initial section is to present a self-contained material on the theory of autonomous dynamical systems (also known as semigroups), describing the basic notions of attraction, absorption, invariance, dynamics, and ending it with conditions which guarantee the existence of global attractors, the most relevant object in this theory, with the power to describe all the dynamic of the system. Our presentation is done in a general setting of metric spaces, and all the previous knowledge required to the reader is a familiar treatment of this subject. A complete and deep approach on autonomous dynamical systems and its applications is given for instance in the classical works (CHEPYZHOV; VISHIK, 2002), (ROBINSON, 2001) and (TEMAM, 1988).

2.1.1 Semigroups and global attractors

Let (X, d_X) be a metric space with metric $d_X : X \times X \to \mathbb{R}$ and let $\{S(t) : t \geqslant 0\}$ be a family of mappings defined on X, i.e., for each $t \geqslant 0$, $S(t) : X \to X$ is a mapping. Throughout all this work we will denote $\{S(t) : t \geqslant 0\}$ simply by $\{S(t)\}_{t\geqslant 0}$ or even $\{S(t)\}$. An important notion related to a family of mappings as presented before is the notion of *semigroup*.

Definition 2.1.1. *A family $\{S(t)\}_{t\geqslant 0}$ of mappings is called a* semigroup *in X if it satisfies the following conditions:*

i) $S(0) = Id_X$, where Id_X is the identity in X;

ii) $S(t+s) = S(t)S(s)$, for any $t, s \geqslant 0$ (semigroup property);

iii) For each $t \geqslant 0$ the mapping $S(t) : X \to X$ is continuous, i.e., $x \mapsto S(t)x$ is continuous.

Remark 2.1.2. *If a family of mappings is defined for all $t \in \mathbb{R}$, $\{S(t) : t \in \mathbb{R}\}$, and conditions in Definition 2.1.1 hold for all $t, s \in \mathbb{R}$ then we call $\{S(t)\}_{t\in\mathbb{R}}$ a* group*. Throughout this chapter* *we are more concerned with the theory of semigroups, but in later chapters groups will be also necessary.*

Remark 2.1.3. *Semigroups are usually generated by solutions of differential equations, this fact providing a big motivation in putting efforts to the theory in recent decades. We will discuss this point briefly in Section 2.1.4.*

In the following we shall define an important class of subsets of space X which has the purpose of preserving the dynamics of a semigroup $\{S(t)\}_{t\geqslant 0}$, the so called *invariant sets*.

Definition 2.1.4. *We say that a set $A \subseteq X$ is* invariant *under the action of a semigroup $\{S(t)\}_{t\geqslant 0}$ if $S(t)A = A$, for all $t \geqslant 0$.*

Notice that the arbitrarily union of invariant sets under the action of a semigroup $\{S(t)\}_{t\geqslant 0}$ is invariant as well. More precisely, let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of subsets of X which are invariant under $\{S(t)\}$. Then $A := \bigcup_{\lambda\in\Lambda} A_\lambda$ is invariant.

A particular and interesting class of invariant sets is the one composed by orbits of global solutions of semigroups. We make these notions more precise in the sequence.

Definition 2.1.5. *A mapping $\xi : \mathbb{R} \to X$ is said to be a* global solution *for a semigroup $\{S(t)\}_{t\geqslant 0}$ if for all $t \geqslant 0$ and $s \in \mathbb{R}$ we have*

$$S(t)\xi(s) = \xi(t+s).$$

If $\xi(0) = x \in X$ we say that ξ is a global solution through *x. The set of values of ξ, $\gamma(\xi) := \{\xi(t) : t \in \mathbb{R}\} \subseteq X$, is called the* global orbit *of ξ. If $\gamma(\xi) \subseteq X$ is bounded we say that ξ is a* bounded global solution.

In general we will simply call global solutions by *solutions*, global orbits by *orbits* and bounded global solutions by *bounded solutions*. Moreover, global solutions are also known as *complete trajectories* (or simply *trajectories*) and in case it is bounded we call it *bounded complete trajectories* (or simply *bounded trajectories*).

We can easily verify that if $\xi : \mathbb{R} \to X$ is a solution for a semigroup $\{S(t)\}_{t\geqslant 0}$ then its orbit $\gamma(\xi)$ is invariant under the action of $\{S(t)\}_{t\geqslant 0}$. Actually, the relation between invariance and global solutions is deeper than this.

Proposition 2.1.6. *A subset $A \subseteq X$ is invariant for a semigroup $\{S(t)\}_{t\geqslant 0}$ if and only if A is a union of global orbits of $\{S(t)\}_{t\geqslant 0}$.*

Proof. Suppose A is a union of global orbits of $\{S(t)\}$. Since a global orbit is invariant under $\{S(t)\}$ and the union of invariant sets is invariant as well, we conclude that A must be invariant under $\{S(t)\}$.

Now suppose A is invariant and let $x_0 \in A$. Since $S(1)A = A$ there exists $x_{-1} \in A$ such that $x_0 = S(1)x_{-1}$. But $x_{-1} \in A = S(1)A$ and so there is $x_{-2} \in A$ with $x_{-1} = S(1)x_{-2}$, and by the semigroup property we have $x_0 = S(2)x_{-2}$. Analogously for each $n \in \mathbb{N}$ there is $x_{-n} \in A$ such that $x_0 = S(n)x_{-n}$ and $x_{-(m-n)} = S(n)x_{-m}$ for $m \geqslant n$.

Define $\xi : \mathbb{R} \to X$ by

$$\xi(\tau) := \begin{cases} S(\tau)x_0, & \text{if } \tau \geqslant 0 \\ S(\tau+n)x_{-n}, & \text{if } \tau \in [-n, -n+1] \end{cases}.$$

We shall prove that ξ is a global solution for $\{S(t)\}$. Indeed, let $t > 0$ and $\tau \in \mathbb{R}$. If $\tau \geqslant 0$ we have $S(t)\xi(\tau) = S(t)S(\tau)x_0 = S(t+\tau)x_0 = \xi(t+\tau)$.

In case $\tau < 0$ there exists $m \in \mathbb{N}$ such that $\tau \in [-m, -m+1]$ and so $\xi(\tau) = S(\tau+m)x_{-m}$. We have to consider two cases:

Case 1: If $t+\tau \geqslant 0$ we obtain

$$\begin{aligned} S(t)\xi(\tau) &= S(t)S(\tau+m)x_{-m} = S(t+\tau+m)x_{-m} \\ &= S(t+\tau)S(m)x_{-m} = S(t+\tau)x_0 = \xi(t+\tau). \end{aligned}$$

Case 2: If $t+\tau < 0$ let $n \in \mathbb{N}$ be such that $t+\tau \in [-n, -n+1]$. Then $-m \leqslant \tau < t+\tau \leqslant -n+1$ and so $n \leqslant m$. Hence

$$\begin{aligned} S(t)\xi(\tau) &= S(t)S(\tau+m)x_{-m} = S(t+\tau+m)x_{-m} \\ &= S(t+\tau+n)S(m-n)x_{-m} = S(t+\tau+n)x_{-n} \\ &= \xi(t+\tau). \end{aligned}$$

Therefore, $S(t)\xi(\tau) = \xi(t+\tau)$, for all $t > 0$ and $\tau \in \mathbb{R}$. For $t = 0$ there is nothing to do, and so we conclude that ξ is a global solution for $\{S(t)\}$ through x_0, i.e., x_0 is in some orbit of a global solution of $\{S(t)\}$.

Now given $a \in A$ denote by ξ_a the global solution through a constructed as before. On one hand we saw that $A \subseteq \bigcup_{a\in A} \gamma(\xi_a)$. On the other hand by definition of ξ_a we have $\xi_a(t) \in A$ for all $t \in \mathbb{R}$, and therefore $\gamma(\xi_a) \subseteq A$, proving the result. $\square$

The common notion of distance between sets used in the field of dynamical systems is the *Hausdorff semi-distance*. It somehow tells us with some precision when two sets are "close" to each other. We define it in the following and subsequently we prove a triangle inequality.

Definition 2.1.7. *Let $A, B \subseteq X$ be non-empty subsets of X. The* Hausdorff semi-distance *in X between A and B (in that order) is defined as*

$$\operatorname{dist}_X(A,B) := \sup_{a\in A} \inf_{b\in B} d_X(a,b).$$

Notice that if $A = \{a\}$, the Hausdorff semi-distance between A and B coincides with the usual distance between a point and a set

$$\mathrm{dist}_X(a,B) = \inf_{b\in B} d_X(a,b).$$

Because of this we will not distinguish the notation between the usual distance of a point to a set and the Haurdorff semi-distance. It is worth telling that in comparison to the usual distance between sets

$$d_X\big(A,B\big) := \inf_{a\in A}\inf_{b\in B} d_X(a,b)$$

the Hausdorff semi-distance can differ substantially.

We notice that $\mathrm{dist}_X(A,B) = 0$ if and only if $A \subseteq \overline{B}$. In general we have $\mathrm{dist}_X(A,B) \neq \mathrm{dist}_X(B,A)$, and so the mapping $\mathrm{dist}_X : 2^X\backslash\{\emptyset\} \times 2^X\backslash\{\emptyset\} \to \mathbb{R}$ does not define a distance. But fortunately we still have a triangle inequality holding and, as usual, this property carries a great applicability.

Lemma 2.1.8. *Let $A,B,C \subseteq X$ be non-empty subsets of X. Then*

$$\mathrm{dist}_X(A,C) \leqslant \mathrm{dist}_X(A,B) + \mathrm{dist}_X(B,C).$$

Proof. By the triangle inequality for the metric d_X we know that for any $a \in A$, $b \in B$ and $c \in C$ we have $d_X(a,c) \leqslant d_X(a,b) + d_X(b,c)$ and so

$$\mathrm{dist}_X(a,C) \leqslant d_X(a,b) + d_X(b,c).$$

Taking the infimum over $c \in C$ we obtain

$$\mathrm{dist}_X(a,C) \leqslant d_X(a,b) + \mathrm{dist}_X(b,C).$$

But $\mathrm{dist}_X(b,C) \leqslant \mathrm{dist}_X(B,C)$ and hence

$$\mathrm{dist}_X(a,C) \leqslant d_X(a,b) + \mathrm{dist}_X(B,C)$$

and taking the infimum over $b \in B$

$$\mathrm{dist}_X(a,C) \leqslant \mathrm{dist}_X(a,B) + \mathrm{dist}_X(B,C) \leqslant \mathrm{dist}_X(A,B) + \mathrm{dist}_X(B,C).$$

Finally taking the supremum over $a \in A$ we conclude that

$$\mathrm{dist}_X(A,C) \leqslant \mathrm{dist}_X(A,B) + \mathrm{dist}_X(B,C). \qquad \square$$

As mentioned before the Hausdorff semi-distance between sets tells us if these sets are close to each other or not. This is the appropriate notion we need in order to define what an *attracting set* means.

Definition 2.1.9. *Let $B, D \subseteq X$ be non-empty subsets of X and $\{S(t)\}$ be a semigroup in X. We say that D* attracts *B* under the action of $\{S(t)\}$ *if*

$$\operatorname{dist}_X\big(S(t)B, D\big) \to 0, \qquad \text{as } t \to \infty.$$

Definition 2.1.10. *We say that $\mathscr{B} \subseteq X$ is an* attracting set *for the semigroup $\{S(t)\}$ if it attracts every bounded subset B of X, i.e., given $B \subseteq X$ bounded it holds*

$$\operatorname{dist}_X\big(S(t)B, \mathscr{B}\big) \to 0, \qquad \text{as } t \to \infty.$$

Another important definition which plays an essential role in this context is the notion of *absorbing set*.

Definition 2.1.11. *Let $B, D \subseteq X$ be non-empty subsets of X and $\{S(t)\}$ be a semigroup in X. We say that D* absorbs *B under the action of $\{S(t)\}$ if there is a time $t_0 > 0$ such that*

$$S(t)B \subseteq D, \qquad \forall t \geqslant t_0.$$

Definition 2.1.12. *We say that $\mathscr{B} \subseteq X$ is an* absorbing set *for the semigroup $\{S(t)\}$ if it absorbs every bounded subset B of X, i.e., given $B \subseteq X$ bounded there exists a time $t_B > 0$ such that*

$$S(t)B \subseteq \mathscr{B}, \qquad \forall t \geqslant t_B.$$

By the structure of these definitions it is natural to expect some relation between them. Indeed, of course every absorbing set is also an attracting set because if $S(t)B \subseteq \mathscr{B}$ for any $t \geqslant t_B$ then $\operatorname{dist}_X\big(S(t)B, \mathscr{B}\big) = 0$. On the other hand, suppose $\mathscr{B}$ is an attracting set. Then for $B \subseteq X$ bounded, given $\varepsilon > 0$ there exists $t_{B,\varepsilon} > 0$ such that

$$\operatorname{dist}_X\big(S(t)B, \mathscr{B}\big) < \varepsilon, \qquad \forall t \geqslant t_{B,\varepsilon}.$$

Let $x \in B$ be an arbitrary point. So for $t \geqslant t_{B,\varepsilon}$ we have $\operatorname{dist}_X\big(S(t)x, \mathscr{B}\big) < \varepsilon$ and then

$$S(t)B \subseteq B_X\big(\mathscr{B}, \varepsilon\big) := \{y \in X : \operatorname{dist}_X(y, \mathscr{B}) < \varepsilon\}, \qquad \forall t \geqslant t_{B,\varepsilon},$$

and we conclude that the ε-neighborhood of $\mathscr{B}$ is an absorbing set for the semigroup $\{S(t)\}$. More generally, any neighborhood of $\mathscr{B}$ (not necessarily ε-neighborhoods) is an absorbing set for the semigroup.

In the following we define the *global attractor* for a semigroup. This is the most relevant notion related to semigroups and it puts together three important properties: compactness, invariance and attraction. The global attractor has the purpose of describing the dynamics of the system and of providing information based on its structure and dimensionality.

Definition 2.1.13. *A subset $\mathscr{A} \subseteq X$ is a* global attractor *for a semigroup $\{S(t)\}$ if it satisfies the following properties:*

i) $\mathscr{A}$ *is compact;*

ii) $\mathscr{A}$ *is invariant under* $\{S(t)\}$*, i.e.,* $S(t)\mathscr{A} = \mathscr{A}$*, for all* $t \geqslant 0$*;*

iii) $\mathscr{A}$ *is an attracting set for* $\{S(t)\}$*, i.e., if* $B \subseteq X$ *is bounded then*

$$\operatorname{dist}_X\big(S(t)B, \mathscr{A}\big) \to 0, \quad \text{as } t \to \infty.$$

Since we expect the global attractor to be relevant and to give us information about the system, it would be interesting if this set satisfied a uniqueness condition. Fortunately, if there exists a global attractor for a semigroup it must be unique.

Proposition 2.1.14. *If there exists a global attractor for a semigroup then it must be unique. More precisely, if* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *are global attractors for a semigroup* $\{S(t)\}$*, then* $\mathscr{A}_1 = \mathscr{A}_2$*.*

Proof. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be global attractors for a semigroup $\{S(t)\}$. Since $\mathscr{A}_1$ is bounded (because it is compact) then it must be attracted by $\mathscr{A}_2$, which means

$$\operatorname{dist}_X\big(S(t)\mathscr{A}_1, \mathscr{A}_2\big) \to 0, \qquad \text{as } t \to \infty.$$

But $\mathscr{A}_1$ is invariant under $\{S(t)\}$ and so $S(t)\mathscr{A}_1 = \mathscr{A}_1$ for all $t \geqslant 0$. Then

$$0 = \lim_{t\to\infty} \operatorname{dist}_X\big(S(t)\mathscr{A}_1, \mathscr{A}_2\big) = \lim_{t\to\infty} \operatorname{dist}_X\big(\mathscr{A}_1, \mathscr{A}_2\big) = \operatorname{dist}_X\big(\mathscr{A}_1, \mathscr{A}_2\big)$$

and we have $\mathscr{A}_1 \subseteq \overline{\mathscr{A}_2} = \mathscr{A}_2$, since $\mathscr{A}_2$ is closed.

Changing the roles of $\mathscr{A}_1$ and $\mathscr{A}_2$ and repeating the procedure as before we obtain $\mathscr{A}_2 \subseteq \mathscr{A}_1$, proving that $\mathscr{A}_1 = \mathscr{A}_2$. □

Next theorem gives us some idea of how the structure of a global attractor looks like, and the subsequently corollary shows the relation between global attractors and bounded global solutions for a semigroup.

Theorem 2.1.15. *Let* $\{S(t)\}$ *be a semigroup with global attractor* $\mathscr{A}$*. Then* $\mathscr{A}$ *is the union of all bounded invariant subsets of* X*.*

Proof. On one hand, since $\mathscr{A}$ is a bounded and invariant subset of X then it must be in the union of all subsets with these properties.

On the other hand, let $B \subseteq X$ be bounded and invariant. So $\mathscr{A}$ attracts B and we have

$$0 = \lim_{t\to\infty} \operatorname{dist}_X\big(S(t)B, \mathscr{A}\big) = \lim_{t\to\infty} \operatorname{dist}_X\big(B, \mathscr{A}\big) = \operatorname{dist}_X\big(B, \mathscr{A}\big),$$

implying $B \subseteq \overline{\mathscr{A}} = \mathscr{A}$. It proves that the union of all bounded invariant subsets of X is in $\mathscr{A}$. □

Corollary 2.1.16. *Let* $\{S(t)\}$ *be a semigroup with global attractor* $\mathscr{A}$*. Then* $\mathscr{A}$ *is the union of all bounded global orbits of* $\{S(t)\}$*.*

Proof. Since $\mathscr{A}$ is invariant, by Proposition 2.1.6 it follows that $\mathscr{A}$ is a union of global orbits of $\{S(t)\}$. But as each of these orbits is in $\mathscr{A}$ and $\mathscr{A}$ is bounded so the orbit must be bounded. Hence $\mathscr{A}$ is in the union of all bounded global orbits of $\{S(t)\}$.

Now if $\xi : \mathbb{R} \to X$ is a bounded global solution of $\{S(t)\}$, we know that its orbit $\gamma(\xi)$ is bounded and invariant under $\{S(t)\}$. By Theorem 2.1.15, $\gamma(\xi) \subseteq \mathscr{A}$, proving that $\mathscr{A}$ is the union of all bounded global orbits of $\{S(t)\}$. $\square$

Remark 2.1.17. *Notice that Theorem 2.1.15 also implies that the global attractor $\mathscr{A}$ is the maximal (with respect to the inclusion of sets) bounded invariant set for $\{S(t)\}$.*

2.1.2 ω-limit sets

In the previous section we have defined global attractors for semigroups and proved that if it exists then it must be unique. We also gave in Theorem 2.1.15 and Corollary 2.1.16 an idea of how is the structure of an attractor. In this section we consider (and define) a special class of semigroups for which we can guarantee the existence of global attractors, the *asymptotically compact* semigroups. Besides that we define ω-limit sets proving how they behave well under the action of this particular class of semigroups. We mention that ω-limit sets will be crucial in order to determine the existence of global attractors.

Definition 2.1.18. *Let $\{S(t)\}$ be a semigroup and $B \subseteq X$ non-empty. The* ω-limit set of B *with respect to $\{S(t)\}$ is defined as*

$$\omega(B) := \bigcap_{t \geqslant 0} \left(\overline{\bigcup_{s \geqslant t} S(s)B}^{X} \right).$$

Clearly $\omega(B)$ is a closed subset of X, since it is the intersection of closed subsets. But this definition of $\omega(B)$ does not allow us to get more significant information and that is why it is much more convenient to consider the following characterization of ω-limit sets. Denote $\mathbb{R}^+ := \{s \in \mathbb{R} : s \geqslant 0\}$.

Lemma 2.1.19. *Let $\{S(t)\}$ be a semigroup and $B \subseteq X$ non-empty. Then*

$$\omega(B) = \Big\{ x \in X : \text{there are sequences } \{x_n\}_{n\in\mathbb{N}} \subseteq B, \{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^+ \text{ with } t_n \to \infty \\ \text{such that } x = \lim_{n\to\infty} S(t_n)x_n \Big\}.$$

Proof. Denote

$$\omega'(B) := \Big\{ x \in X : \text{there are sequences } \{x_n\}_{n\in\mathbb{N}} \subseteq B, \{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^+ \text{ with } t_n \to \infty \\ \text{such that } x = \lim_{n\to\infty} S(t_n)x_n \Big\}.$$

Suppose that $\omega(B) \neq \emptyset$ and let $x \in \omega(B)$. So for each $n \in \mathbb{N}$ we have $x \in \overline{\bigcup_{s \geqslant n} S(s)B}$ and consequently there is $z_n \in \bigcup_{s\geqslant n} S(s)B$ such that $d_X(x, z_n) < \frac{1}{n}$. Moreover, $z_n = S(t_n)x_n$, with

$t_n \geqslant n$ and $x_n \in B$, and we conclude that $x = \lim_{n\to\infty} S(t_n)x_n$, where $\{t_n\} \subseteq \mathbb{R}^+$ with $t_n \to \infty$ and $\{x_n\} \subseteq B$. Hence $x \in \omega'(B)$.

Now let $x \in \omega'(B)$. Then there are sequences $\{x_n\} \subseteq B$ and $\{t_n\} \subseteq \mathbb{R}^+$ with $t_n \to \infty$ and such that $x = \lim_{n\to\infty} S(t_n)x_n$. Given $t \geqslant 0$ let $n_t \in \mathbb{N}$ be such that for any $n \geqslant n_t$ we have $t_n \geqslant t$. Hence for any $n \geqslant n_t$ it holds $S(t_n)x_n \in \bigcup_{s\geqslant t} S(s)B$, and then $x \in \overline{\bigcup_{s\geqslant t} S(s)B}$, for all $t \geqslant 0$. Therefore $x \in \omega(B)$ and we obtain $\omega'(B) = \omega(B)$.

On the other hand, suppose $\omega(B) = \emptyset$. In this case we also have $\omega'(B) = \emptyset$, because otherwise if there is $x \in \omega'(B)$ then repeating the same as before we would have $x \in \omega(B)$. □

For a closed invariant set, its ω-limit satisfies the following.

Lemma 2.1.20. *Let $\{S(t)\}$ be a semigroup and $A \subseteq X$ be a closed invariant set. Then*

$$\omega(A) = A.$$

Proof. On one hand, let $x \in A$. Since $S(n)A = A$, for all $n \in \mathbb{N}$, there is $x_n \in A$ such that $x = S(n)x_n$. Clearly $x \in \omega(A)$ and then $A \subseteq \omega(A)$.

On the other hand, let $x \in \omega(A)$, $\{x_n\} \subseteq A$ and $\{t_n\} \subseteq \mathbb{R}^+$ with $t_n \to \infty$ and $x = \lim_{n\to\infty} S(t_n)x_n$. But since A is invariant we have $S(t_n)x_n \in A$ and so $x \in \overline{A} = A$, proving that $\omega(A) \subseteq A$. □

Remark 2.1.21. *Notice that in Lemma 2.1.20 if A is invariant but it is not closed then we have*

$$A \subseteq \omega(A) \subseteq \overline{A}.$$

In the following we define the *asymptotically compact* semigroups, a class of semigroups for which the ω-limit sets satisfy relevant properties. As mentioned before, for this class there will exist global attractors.

Definition 2.1.22. *A semigroup $\{S(t)\}$ on a metric space X is* asymptotically compact *if there exists a compact attracting set $\mathscr{K}$ for $\{S(t)\}$, i.e., $\mathscr{K} \subseteq X$ is compact and given $B \subseteq X$ bounded it holds*

$$\operatorname{dist}_X\big(S(t)B, \mathscr{K}\big) \to 0, \quad \text{as } t \to \infty.$$

For asymptotically compact semigroups we have:

Lemma 2.1.23. *Let $\{S(t)\}$ be an asymptotically compact semigroup with $\mathscr{K}$ a compact attracting set. If $B \subseteq X$ is non-empty and bounded then $\omega(B)$ satisfies the following properties:*

i) *$\omega(B)$ is non-empty, compact, invariant and $\omega(B) \subseteq \mathscr{K}$;*

ii) *$\omega(B)$ attracts B under the action of $\{S(t)\}$;*

iii) *$\omega(B)$ is the mininal closed subset of X attracting B. More precisely, if $F \subseteq X$ is a closed subset of X attracting B, then $\omega(B) \subseteq F$.*

Proof. First of all, since $\mathscr{K}$ is an attracting set for the semigroup $\{S(t)\}$ then for any bounded subset $D \subset X$ we have

$$\mathrm{dist}_X\big(S(t)D, \mathscr{K}\big) \to 0, \qquad \text{as } t \to \infty. \tag{2.1}$$

Proof of i): Let $x \in B$ be a fixed element and consider $\{t_n\} \subset \mathbb{R}^+$ with $t_n \to \infty$ as $n \to \infty$. Given $\varepsilon > 0$, by (2.1) in particular for $D = \{x\}$ there is $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$ we have

$$\mathrm{dist}_X\big(S(t_n)x, \mathscr{K}\big) < \varepsilon.$$

So there is $y_n \in \mathscr{K}$ satisfying

$$d_X\big(S(t_n)x, y_n\big) < \varepsilon$$

and so

$$d_X\big(S(t_n)x, y_n\big) \to 0, \quad \text{as } n \to \infty.$$

Since $\mathscr{K}$ is compact, $\{y_n\}$ has a subsequence (still denoted by $\{y_n\}$) converging to a point $y \in \mathscr{K}$ and then $\lim_{n\to\infty} S(t_n)x = y$. But from Lemma 2.1.19, $y \in \omega(B)$, which proves $\omega(B) \neq \emptyset$.

Now let us prove that $\omega(B)$ is compact by proving that $\omega(B) \subseteq \mathscr{K}$. Indeed, for any $y \in \omega(B)$ by Lemma 2.1.19 there are $\{x_n\} \subseteq B$ and $\{t_n\} \subseteq \mathbb{R}^+$ with $t_n \to \infty$ such that $\lim_{n\to\infty} S(t_n)x_n = y$. But by the attracting property of $\mathscr{K}$ we obtain

$$\mathrm{dist}_X\big(S(t_n)x_n, \mathscr{K}\big) \to 0, \quad n \to \infty.$$

By the triangle inequality for the Hausdorff semi-distance in Lemma 2.1.8 we conclude that

$$\mathrm{dist}_X(y, \mathscr{K}) \leqslant \mathrm{dist}_X\big(y, S(t_n)x_n\big) + \mathrm{dist}_X\big(S(t_n)x_n, \mathscr{K}\big) \to 0, \quad \text{as } n \to \infty,$$

and we obtain $\mathrm{dist}_X(y, \mathscr{K}) = 0$. But it implies $y \in \overline{\mathscr{K}} = \mathscr{K}$ and therefore $\omega(B) \subseteq \mathscr{K}$. Since $\mathscr{K}$ is compact we conclude that $\omega(B)$ is compact as well ($\omega(B)$ is closed, by definition).

It remains to prove that $\omega(B)$ is invariant under $\{S(t)\}$. Indeed, let $x \in \omega(B)$, $\{x_n\} \subseteq B$ and $\{t_n\} \subseteq \mathbb{R}^+$ with $t_n \to \infty$ and $x = \lim_{n\to\infty} S(t_n)x_n$. For any fixed $t \geqslant 0$ notice that

$$\mathrm{dist}_X\big(S(t+t_n)x_n, \mathscr{K}\big) \leqslant \mathrm{dist}_X\big(S(t+t_n)B, \mathscr{K}\big) \to 0, \quad \text{as } n \to \infty,$$

and just as before there is $y \in \mathscr{K}$ as limit of a subsequence of $\{S(t+t_n)x_n\}$, still denoted by $\{S(t+t_n)x_n\}$, i.e., $y = \lim_{n\to\infty} S(t+t_n)x_n$. By Lemma 2.1.19 we have $y \in \omega(B)$ and by the continuity of $S(t) : X \to X$ and the semigroup property we conclude

$$S(t)x = S(t)\big(\lim_{n\to\infty} S(t_n)x_n\big) = \lim_{n\to\infty} S(t+t_n)x_n = y,$$

proving that $S(t)\omega(B) \subseteq \omega(B)$.

One more time let $x \in \omega(B)$, $\{x_n\} \subseteq B$ and $\{t_n\} \subseteq \mathbb{R}^+$ with $t_n \to \infty$ and $x = \lim_{n\to\infty} S(t_n)x_n$. For $t \geqslant 0$ fixed, let $n_t \in \mathbb{N}$ be sufficiently great such that $t_n \geqslant t$, for all $n \geqslant n_t$. As before we can find $y \in \omega(B)$ as a limit point for a subsequence of $\{S(t_n - t)x_n\}$, still denoted by $\{S(t_n - t)x_n\}$, i.e., $y = \lim_{n\to\infty} S(t_n - t)x_n$. Hence

$$x = \lim_{n\to\infty} S(t_n)x_n = \lim_{n\to\infty} S(t)S(t_n - t)x_n = S(t)\big(\lim_{n\to\infty} S(t_n - t)x_n\big) = S(t)y,$$

proving that $\omega(B) \subseteq S(t)\omega(B)$. Therefore, $S(t)\omega(B) = \omega(B)$, for all $t \geqslant 0$ and $\omega(B)$ is invariant.

Proof of ii): Suppose it is not true. Then there exist $\varepsilon_0 > 0$ and a sequence $\{t_n\} \subseteq \mathbb{R}^+$ with $t_n \to \infty$ and

$$\mathrm{dist}_X\big(S(t_n)B, \omega(B)\big) > \varepsilon_0, \qquad n \in \mathbb{N}.$$

But by the definition of supremum we can find a sequence $\{x_n\} \subseteq B$ in such a way that

$$\mathrm{dist}_X\big(S(t_n)x_n, \omega(B)\big) > \varepsilon_0, \qquad n \in \mathbb{N}. \tag{2.2}$$

Notice that since $\mathscr{K}$ attracts B we have

$$\mathrm{dist}_X\big(S(t_n)x_n, \mathscr{K}\big) \leqslant \mathrm{dist}_X\big(S(t_n)B, \mathscr{K}\big) \to 0, \qquad \text{as } n \to \infty,$$

and as in part *i)* there exists a sequence $\{y_n\} \subseteq \mathscr{K}$ such that

$$d_X\big(S(t_n)x_n, y_n\big) \to 0, \qquad \text{as } n \to \infty.$$

Since $\mathscr{K}$ is compact we may assume $y_n \to y$, where $y \in \mathscr{K}$, and by the last expression we conclude that $\lim_{n\to\infty} S(t_n)x_n = y$, i.e., $y \in \omega(B)$. Moreover, by (2.2) we have

$$0 = \mathrm{dist}_X\big(y, \omega(B)\big) = \lim_{n\to\infty} \mathrm{dist}_X\big(S(t_n)x_n, \omega(B)\big) \geqslant \varepsilon_0 > 0,$$

a contradiction, and in fact $\omega(B)$ attracts B.

Proof of iii): Let F be a closed set attracting B under the action of $\{S(t)\}$. By Lemma 2.1.19, for any $y \in \omega(B)$ there exist sequences $\{x_n\} \subseteq B$ and $\{t_n\} \subseteq \mathbb{R}^+$ with $t_n \to \infty$ such that $\lim_{n\to\infty} S(t_n)x_n = y$. Since F attracts B we have

$$\mathrm{dist}_X\big(S(t_n)x_n, F\big) \leqslant \mathrm{dist}_X\big(S(t_n)B, F\big) \to 0, \qquad \text{as } n \to \infty,$$

and then $\mathrm{dist}_X(y, F) = 0$. But as we know it means $y \in \overline{F} = F$ and finally we conclude that $\omega(B) \subseteq F$. $\square$

2.1.3 Existence of global attractors

With the concepts presented until this point we are ready to prove the existence of global attractors. A sufficient and necessary condition for this is the semigroup $\{S(t)\}$ to be asymptotically compact. We notice that for the applications we are going to consider in this work we usually apply next theorem to guarantee the existence of attractors.

Theorem 2.1.24. *Let $\{S(t)\}$ be a semigroup in a metric space X. The following conditions are equivalent:*

1. *$\{S(t)\}$ is asymptotically compact (with $\mathscr{K} \subseteq X$ a compact attracting set);*
2. *$\{S(t)\}$ has a global attractor $\mathscr{A}$.*

Moreover, in this case we have

$$\mathscr{A} = \bigcup_{B \in \mathscr{B}(X)} \omega(B) = \omega(\mathscr{K}), \tag{2.3}$$

where $\mathscr{B}(X)$ denotes the collection of all bounded subsets of X.

Proof. (1) $\implies$ (2): Let

$$\mathscr{A} := \bigcup_{B \in \mathscr{B}(X)} \omega(B).$$

We have to prove that $\mathscr{A}$ is compact, invariant and attracts all bounded subsets of X under the action of $\{S(t)\}$. Indeed, since $\{S(t)\}$ is asymptotically compact we know by Lemma 2.1.23, *i)*, that for any bounded $B \subseteq X$, $\omega(B)$ is invariant. So as $\mathscr{A}$ is a union of invariant sets it follows that $\mathscr{A}$ must be invariant.

Now given $B \in \mathscr{B}(X)$, we have by Lemma 2.1.23, *ii)*, that $\omega(B)$ attracts B under $\{S(t)\}$ and so

$$\operatorname{dist}_X\big(S(t)B, \mathscr{A}\big) \leqslant \operatorname{dist}_X\big(S(t)B, \omega(B)\big) \to 0, \qquad \text{as } t \to \infty,$$

proving that $\mathscr{A}$ is an attracting set for $\{S(t)\}$.

Let $\mathscr{K}$ be a compact attracting set for the semigroup $\{S(t)\}$. By Lemma 2.1.23, *i)*, for any bounded subset $B \subset X$ we already know that $\omega(B) \subseteq \mathscr{K}$, and this implies $\mathscr{A} \subseteq \mathscr{K}$. As a consequence, $\omega(B) \subseteq \omega(\mathscr{K})$. Since $\mathscr{K} \in \mathscr{B}(X)$, we also have $\omega(\mathscr{K}) \subseteq \mathscr{A}$, and then

$$\mathscr{A} \subseteq \omega(\mathscr{A}) \subseteq \omega(\mathscr{K}) \subseteq \mathscr{A}.$$

But it means $\mathscr{A} = \omega(\mathscr{K})$, and by Lemma 2.1.23, *i)*, we conclude that $\mathscr{A}$ is compact, proving that $\mathscr{A}$ is the global attractor for $\{S(t)\}$ and that (2.3) holds.

(2) $\implies$ (1): If there exists the global attractor $\mathscr{A}$ for $\{S(t)\}$, then $\mathscr{A}$ itself is a compact attracting set for $\{S(t)\}$. □

2.1.4 Semigroups generated by autonomous evolution equations

In this section we present a basic setting in which semigroups are generated by evolution equations in Banach spaces. For this let X be a Banach space. An autonomous evolution equation is represented as

$$\frac{\partial u}{\partial t} = f(u), \tag{2.4}$$

where $f : X \to X$ is a non-linear function. We supplement equation (2.4) with a initial data $u_0 \in X$ at $t = 0$,

$$\frac{\partial u}{\partial t} = f(u), \qquad t > 0, \tag{2.5}$$

$$u|_{t=0} = u_0, \qquad u_0 \in X. \tag{2.6}$$

Let us suppose that problem (2.5)-(2.6) has a unique solution $v(\cdot) = v(\cdot, 0; u_0)$ for arbitrary $u_0 \in X$, i.e., $v(\cdot)$ satisfies (2.5) and $v(0) = v(0, 0; u_0) = u_0$. Suppose in addition that for each $t \geqslant 0$, the mapping $v(t, 0; \cdot) : X \to X$ is continuous. Under these conditions, for any $t \geqslant 0$ and any $u_0 \in X$ it is well-defined the mapping $S(t) : X \to X$ given as

$$S(t)u_0 := v(t, 0; u_0), \qquad u_0 \in X.$$

We note that the family of mappings $\{S(t) : t \geqslant 0\}$ satisfies the properties

i) $S(0) = Id_X$,

ii) $S(t)S(s) = S(t+s)$, for any $t, s \geqslant 0$,

iii) For each $t \geqslant 0$, the mapping $S(t) : X \to X$ is continuous.

Indeed, it is immediate that $S(0)u_0 = u_0$, for any $u_0 \in X$, and so $S(0) = Id_X$. Now given $u_0 \in X$ notice that $v\big(t, 0; v(s, 0; u_0)\big)$ and $v(t+s, 0; u_0)$ are both solutions of the initial value problem

$$\frac{\partial u}{\partial t} = f(u),$$
$$u|_{t=0} = v(s, 0; u_0),$$

and it follows by the uniqueness of solutions that they are equal, i.e.,

$$S(t)S(s)u_0 = v\big(t, 0; v(s, 0; u_0)\big) = v(t+s, 0; u_0) = S(t+s)u_0$$

for arbitrary $u_0 \in X$. Finally since for each $t \geqslant 0$ the mapping $v(t, 0; \cdot) : X \to X$ is continuous then $S(t)$ is continuous as well.

Therefore with this simple analysis we verify that the family of mappings $\{S(t)\}_{t \geqslant 0}$ is a semigroup, and we call it a semigroup generated by the evolution equation (2.4). This gives us a good motivation to put attention in studying the theory of semigroups since it is directly related to the study of differential equations modeling a significantly number of natural phenomena.

Another important point and which we can not pass without mentioning it, is that, as indicated by Corollary 2.1.16, the global attractor of a semigroup gathers all the bounded solutions of the problem. Since these solutions are the ones with more relevance in practical analysis and applications, getting information from the attractor gives us information for the solutions of the problem, and consequently for the problem itself.

2.2 Finite dimension of negatively invariant subsets of Banach spaces

We want to estimate the fractal dimension of attractors associated to dynamical systems on Banach spaces. As we know in the continuous-time case, the standard abstract setting is the following. We say that a family of continuous maps $\{S(t)\}_{t\geq 0}$ from a Banach space $\big(X, \|\cdot\|_X\big)$ into itself is a *semigroup* if

(i) $S(0) = Id_X$,

(ii) $S(t+s) = S(t)S(s)$, for all $t,s \geqslant 0$,

(iii) for each $t \geqslant 0$ the mapping $S(t) : X \to X$ is continuous, i.e., $x \mapsto S(t)x$ is continuous.

We say that a subset $\mathscr{A} \subset X$ is *invariant* under the action of the semigroup $\{S(t)\}$ if $S(t)\mathscr{A} = \mathscr{A}$ for all $t \geqslant 0$, and we say that $\mathscr{A}$ *attracts* a subset D of X under the action of the semigroup if $\mathrm{dist}_X\big(S(t)D, \mathscr{A}\big) \to 0$ as $t \to \infty$, where $\mathrm{dist}_X(A,B) = \sup\limits_{a\in A} \inf\limits_{b\in B} \|a-b\|_X$.

A subset $\mathscr{A}$ of X is said to be the *global attractor* for $\{S(t)\}$ if it is compact, invariant and attracts all bounded subsets B of X under the action of $\{S(t)\}$. The global attractor for the semigroup $\{S(t)\}$ is the same as the global attractor for the discrete semigroup $\{S^n : n = 0,1,2,\cdots\}$, where we can take $S = S(t_0)$ for any $t_0 > 0$; so throughout this section we consider only the discrete case. In fact our results are valid for compact sets $\mathscr{A} \subset X$ such that $\mathscr{A} \subseteq S(\mathscr{A})$, i.e. that are *negatively invariant* under S. The results in this section were presented in (CARVALHO *et al.*,) and are part of our contribution to the field of autonomous dynamical systems and more specifically to the dimensionality of global attractors.

The earliest result on finite dimensionality of attractors for dynamical systems is due to Mallet-Paret in 1976 (see (MALLET-PARET, 1976)), who considered separable Hilbert spaces. Mañé generalised this result to Banach spaces in 1981 (see (MAÑÉ, 1981)); his proof is taken up and somewhat improved in (CARVALHO; LANGA; ROBINSON, 2010). All these three papers treat a map S whose derivative is everywhere equal to the sum of a compact map and a contraction, and the proofs all rely on using the compactness assumption to find a finite-dimensional subspace U such that the image under DS of the unit ball in U provides a good approximation of the image of the unit ball in X. The resulting dimension estimate involves the dimension of U, which means that, in practice, it is hard to use the results to give explicit bounds on the dimension of $\mathscr{A}$.

For explicit bounds the standard technique relies on setting the problem in a Hilbert space, and then one can obtain estimates using the theory of Lyapunov exponents, as developed in (CONSTANTIN; FOIAS, 1985) (see also (CHEPYZHOV; VISHIK, 2002) or (CARVALHO; LANGA; ROBINSON, 2013)).

Our starting point for this presentation was the following question: if $\mathscr{A}$ is the attractor of a compact map, is it a finite-dimensional set? A relatively simple example shows that the answer to the question is generically *no*. Indeed, consider the map $S : \ell^2 \to \ell^2$, where $\ell^2 := \Big\{ \mathbf{x} = (x_1, x_2, \cdots) : \|\mathbf{x}\|_2^2 = \sum_{j=1}^{\infty} |x_j|^2 < \infty \Big\}$, given by

$$(S\mathbf{x})_j := \begin{cases} j^{-1} \frac{x_j}{|x_j|} & , |x_j| > j^{-1} \\ x_j & , |x_j| \leqslant j^{-1} \end{cases}.$$

This map is compact, but its attractor is the set

$$\mathscr{A} = \{ \mathbf{x} \in \ell^2 : \ |x_j| \leqslant 1/j \}$$

which is an infinite-dimensional subset of ℓ^2. However, it turns out that the answer to this question is *yes* if as well as being compact S is differentiable: in this case it follows that DS is compact, and the finite-dimensionality can then be obtained from Mañé's result (MAÑÉ, 1981). In fact this holds whenever DS is the sum of a compact map and a contraction (in an appropriately uniform way over the attractor); here (see Theorem 2.2.8) we give a proof of this fact that is much simpler than the argument due to Mañé.

However, our argument gives no explicit bound on the attractor dimension. In order to remedy this, we use ideas due to (MÁLEK; RUZICKA; THÄTER, 1994), and assume a quantitative smoothing estimate for the compact part of DS: we suppose that for $x \in \mathscr{A}$ we have $DS(x) = C(x) + L(x)$, where $C(x)$ and $L(x)$ are both linear, that $\|L(x)\|_{\mathscr{L}(X)} < 1/4$, and that $C(x)$ satisfies

$$\|C(x)u\|_Z \leqslant \kappa \|u\|_X, \qquad u \in X,$$

for some space Z that is compactly embedded in X. This enables us to give an explicit bound on the dimension of $\mathscr{A}$, which (see (ZELIK, 2000)) involves the Kolmogorov ε-entropy of the compact embedding of Z into X. Some of the arguments here are also inspired by those in the paper (CARVALHO; SONNER, 2013), which uses such a quantitative smoothing property and a similar splitting to bound the dimension of pullback exponential attractors.

In Section 2.2.1 we present the definition of the fractal dimension, and give two simple lemmas that enable us to bound the fractal dimension based on iterated coverings of $\mathscr{A}$. We then prove two results based on DS in Section 2.2.2. In the next section we relate properties of S to properties of DS, showing in particular in Corollary 2.2.11 that a differentiable compact map has a finite-dimensional attractor. In Section 2.2.4 we use the smoothing property from (MÁLEK; RUZICKA; THÄTER, 1994) and (ZELIK, 2000) to work directly with assumptions on the map S itself.

We then discuss in Section 2.2.5 the Kolomogorov ε-entropy which enters the dimension bounds, giving a simple argument to bound this in the case of L^2-based Sobolev spaces. In Section 2.2.6 we present Mañé's method on a new setting being able to make an explicitly comparison between estimates. In the next section we apply the theory to two classical examples: abstract semilinear parabolic equations, and the two-dimensional Navier–Stokes equations. Finally, we give an example of a map defined on space of sequences showing how to apply our results also in this case.

2.2.1 Bounding the fractal dimension of negatively invariant sets via simple covering lemmas

For a compact subset $\mathscr{A}$ of a normed space X (or more generally a complete metric space (X,d)), let $N_X[\mathscr{A};\varepsilon]$ denote the minimum number of open ε-balls in X centred at points of $\mathscr{A}$ that are necessary to cover $\mathscr{A}$.

Definition 2.2.1. *The (upper) fractal dimension of $\mathscr{A}$ in X, denoted by $\dim_F(\mathscr{A};X)$, is defined as*

$$\dim_F(\mathscr{A};X) := \limsup_{\varepsilon\to 0^+} \frac{\ln N_X[\mathscr{A};\varepsilon]}{-\ln\varepsilon}. \tag{2.7}$$

Essentially, the last expression extracts the exponent in

$$N_X[\mathscr{A};\varepsilon] \sim \varepsilon^{-\dim_F(\mathscr{A};X)}.$$

A detailed treatment of the fractal dimension of compact sets can be found in references (FALCONER, 2003) and (ROBINSON, 2011). According to Definition 2.2.1 it can happen $\dim_F(\mathscr{A};X)=\infty$. Notice that the fractal dimension of a compact set does not increase under the action of Lipschitz maps. More precisely:

Lemma 2.2.2 ((CARVALHO; LANGA; ROBINSON, 2013), Lemma 4.2)**.** *Let $f: X\to Y$ be a Lipschitz map with X and Y Banach spaces, and $\mathscr{A}\subset X$ be a compact set. Then*

$$\dim_F\big(f(\mathscr{A});Y\big) \leqslant \dim_F\big(\mathscr{A};X\big).$$

Proof. Since $\mathscr{A}$ is compact in X then given $\varepsilon>0$ we can cover $\mathscr{A}$ with $N_X[\mathscr{A};\varepsilon]$ balls $B_X(x_i,\varepsilon)$ in X with centres $x_i\in\mathscr{A}$, and so

$$f(\mathscr{A}) \subseteq \bigcup_{i=1}^{N_X[\mathscr{A};\varepsilon]} f\big(B_X(x_i,\varepsilon)\big) \subseteq \bigcup_{i=1}^{N_X[\mathscr{A};\varepsilon]} B_Y\big(f(x_i),L\varepsilon\big),$$

where $L>0$ is a Lipschitz constant for f. Hence $N_Y\big[f(\mathscr{A});L\varepsilon\big] \leqslant N_X[\mathscr{A};\varepsilon]$, which implies

$$\dim_F\big(f(\mathscr{A});Y\big) \leqslant \dim_F\big(\mathscr{A};X\big). \qquad \square$$

If clear enough, we will often drop the X from expression (2.7). The lim sup in (2.7) can also be taken over a geometrically decreasing sequence.

Lemma 2.2.3 ((CARVALHO; LANGA; ROBINSON, 2013), Lemma 4.1)**.** *Given a compact subset $\mathscr{A}$ of X, $r>0$, and any $\eta\in(0,1)$*

$$\dim_F(\mathscr{A}) = \limsup_{k\to\infty} \frac{\ln N[\mathscr{A};\eta^k r]}{-k\ln\eta}. \tag{2.8}$$

Proof. Denote by d' the right-hand side of (2.8) and notice that by (2.7) we have $d' \leqslant \dim_F(\mathscr{A})$. It remains to prove the reverse inequality. Indeed, given $\varepsilon \leqslant r$ let $k_\varepsilon \in \mathbb{N}$ be such that

$$\eta^{k_\varepsilon} r < \varepsilon \leqslant \eta^{k_\varepsilon - 1} r,$$

where the numbers k_ε can be chosen with $k_\varepsilon \to \infty$ as $\varepsilon \to 0^+$. Without loss of generality suppose $\eta^{k_\varepsilon - 1} r < 1$ (otherwise we take ε smaller and smaller). Then

$$\frac{\ln N\big[\mathscr{A};\varepsilon\big]}{-\ln\varepsilon} \leqslant \frac{\ln N\big[\mathscr{A};\eta^{k_\varepsilon} r\big]}{-\ln(\eta^{k_\varepsilon - 1} r)} = \frac{\ln N\big[\mathscr{A};\eta^{k_\varepsilon} r\big]}{-\ln(\eta^{k_\varepsilon} r) + \ln\eta},$$

and taking the superior limit as $\varepsilon \to 0^+$ we obtain

$$\dim_F(\mathscr{A}) \leqslant d'. \qquad \square$$

We now prove two simple lemmas that allow us to obtain bounds on the fractal dimension of $\mathscr{A}$ by applying S or DS to given coverings of $\mathscr{A}$. As before we write $B_X(x,r)$ for the open ball in X centred at x of radius r, dropping the X subscript when clear from the context.

Lemma 2.2.4. *Let $\mathscr{A}$ be a compact subset of a Banach space X that is negatively invariant for $S: X \to X$, i.e. $\mathscr{A} \subseteq S(\mathscr{A})$. If there exist $M \geqslant 1$, $0 < \beta < 1/2$, and $r_0 > 0$ such that for all $x \in \mathscr{A}$ and all $0 < r \leqslant r_0$*

$$N\big[S(B(x,r));\beta r\big] \leqslant M, \tag{2.9}$$

then

$$\dim_F(\mathscr{A}) \leqslant \frac{\ln M}{-\ln 2\beta}. \tag{2.10}$$

Proof. Cover $\mathscr{A}$ with $N = N[\mathscr{A};r_0]$ balls of radius r_0, $\{B(x_i,r_0)\}_{i=1}^N$, with centres $x_i \in \mathscr{A}$. Apply S to every element of this cover. Since $\mathscr{A} \subseteq S(\mathscr{A})$, this provides a new cover of $\mathscr{A}$, $\big\{S(B(x_i,r_0))\big\}_{i=1}^N$. Using (2.9) each of these images can be covered by M balls of radius βr_0, with centres $y_{ij} \in X$; by enlarging these to balls of twice the radius we can take new centres x_{ij} to be in $\mathscr{A}$ once again, and so we obtain

$$N[\mathscr{A};2\beta r_0] \leqslant MN[\mathscr{A};r_0].$$

Since the centres of the balls in this new cover lie in $\mathscr{A}$ and $2\beta < 1$ we can apply the same argument n times to obtain

$$N[\mathscr{A};(2\beta)^n r_0] \leqslant M^n N[\mathscr{A};r_0],$$

and Lemma 2.2.3 yields

$$\dim_F(\mathscr{A}) \leqslant \frac{\ln M}{-\ln 2\beta}. \qquad \square$$

One way to obtain the bound in (2.9) is to have a similar bound on covers for the unit ball under the derivative $DS(x)$ on the attractor. We need some uniformity in what it means for S to be differentiable 'on $\mathscr{A}$'.

Definition 2.2.5. *We say that $S : X \to X$ is* uniformly differentiable for $x \in \mathscr{A}$ *if for every $x \in \mathscr{A}$ there exists a bounded linear map $DS(x) : X \to X$ such that for every $\eta > 0$ there exists a positive constant $r_0(\eta) > 0$ such that*

$$\|S(x+h) - S(x) - DS(x)h\| < \eta \|h\|, \qquad \forall 0 < r \leqslant r_0(\eta),\ x \in \mathscr{A},\ h \in X \text{ with } \|h\| < r.$$

Such uniform differentiability follows whenever S is continuously differentiable on an open neighborhood of $\mathscr{A}$, as follows.

Proposition 2.2.6. *If a map $S : X \to X$ is continuously differentiable on a neighborhood $\mathscr{U}$ of a compact set $\mathscr{A}$ then it is uniformly differentiable for $x \in \mathscr{A}$. Conversely, if $S : X \to X$ is uniformly differentiable on $\mathscr{A}$ then it is continuously differentiable.*

Proof. Without loss of generality (since $\mathscr{A}$ is compact) let $\mathscr{U} = B_X(\mathscr{A}, \delta) := \bigcup_{x \in \mathscr{A}} B_X(x, \delta)$, for some $\delta > 0$. First note that given $\eta > 0$ there exists $r_0 = r_0(\eta) > 0$ and some neighborhood $\mathscr{V}$ of $\mathscr{A}$ such that

$$\|DS(w) - DS(z)\| < \eta, \qquad \forall 0 < r \leqslant r_0,\ w \in \mathscr{V},\ z \in \mathscr{A}, \|w - z\| < r. \tag{2.11}$$

Indeed, let $x \in \mathscr{A}$. Since $DS(\cdot)$ is continuous then given $\eta > 0$ there is $\delta_x = \delta_x(\eta) > 0$ such that

$$\|DS(w) - DS(x)\| < \frac{\eta}{2}, \qquad \forall w \in \mathscr{U},\ \|w - x\| < \delta_x. \tag{2.12}$$

Since $\mathscr{A}$ is a compact subset we have $\mathscr{A} \subseteq \bigcup_{i=1}^{m} B_X(x_i, \delta_{x_i}/2)$, with $x_i \in \mathscr{A}$ for each $i = 1, \cdots, m$. Let $r_0 := \min\{\delta, \delta_{x_1}/2, \cdots, \delta_{x_m}/2\}$ and define

$$\mathscr{V} := B_X(\mathscr{A}, r_0).$$

Clearly $\mathscr{V} \subseteq \mathscr{U}$. Now take $0 < r \leqslant r_0$ and any $w \in \mathscr{V}$ and $z \in \mathscr{A}$ with $\|w - z\| < r$. Hence $z \in B_X(x_i, \delta_{x_i}/2)$ for some $i = 1, \cdots, m$ and then $\|w - x_i\| \leqslant \|w - z\| + \|z - x_i\| < r + \delta_{x_i}/2 \leqslant \delta_{x_i}/2 + \delta_{x_i}/2 = \delta_{x_i}$, i.e., $w \in B_X(x_i, \delta_{x_i})$. Finally, from (2.12)

$$\|DS(w) - DS(z)\| \leqslant \|DS(w) - DS(x_i)\| + \|DS(x_i) - DS(z)\| < \frac{\eta}{2} + \frac{\eta}{2} = \eta,$$

proving (2.11).

Let $0 < r \leqslant r_0$, $x \in \mathscr{A}$ and $z \in \mathscr{V}$ with $\|x - z\| < r$. Note that for each $t \in [0,1]$ the line $x_t := tz + (1-t)x \in \mathscr{V}$ ($x_t \in B_X(x, r) \subseteq \mathscr{V}$). So from the fundamental theorem of calculus and (2.11) we have

$$\begin{aligned} \|S(z) - S(x) - DS(x)(z-x)\| &= \left\| \int_0^1 DS(x_t)(z-x)dt - DS(x)(z-x) \right\| \\ &= \left\| \int_0^1 \big[DS(x_t)(z-x) - DS(x)(z-x)\big] dt \right\| \\ &\leqslant \int_0^1 \|DS(x_t) - DS(x)\| \|z-x\| \, dt \\ &< \int_0^1 \eta \|z-x\| \, dt \\ &= \eta \|z-x\|, \end{aligned}$$

i.e.,

$$\|S(z)-S(x)-DS(x)(z-x)\| < \eta\|z-x\|, \qquad \forall x\in\mathscr{A},\ z\in\mathscr{V},\ \|z-x\|<r,\ 0<r\leqslant r_0.$$

Note that this is equivalent to the expression in Definition 2.2.5.

Now suppose that S is uniformly differentiable for $x\in\mathscr{A}$. Then given $\eta>0$ there is $r_0=r_0(\eta)>0$ such that

$$\|S(x+h)-S(x)-DS(x)h\| \leqslant \eta\|h\|, \qquad \forall x\in\mathscr{A},\ h\in X,\ \|h\|\leqslant r_0. \tag{2.13}$$

Given $x,y\in\mathscr{A}$ note that

$$\begin{aligned}\|DS(x)-DS(y)\| &= \sup_{\|w\|=1}\|DS(x)w-DS(y)w\|\\ &= \frac{2}{r_0}\sup_{\|w\|=1}\Big\|DS(x)(wr_0/2)-DS(y)(wr_0/2)\Big\|\end{aligned}$$

and so

$$\begin{aligned}&\Big\|DS(x)(wr_0/2)-DS(y)(wr_0/2)\Big\| \leqslant\\ &\qquad\leqslant 2\eta\|wr_0/2\|+\|S(x+wr_0/2)-S(x)+S(y)-S(y+wr_0/2)\|\\ &\qquad= r_0\eta+\|S(x+wr_0/2)-S(x)+S(x+z)-S(x+z+wr_0/2)\|,\end{aligned}$$

where $z=y-x$. Suppose that $\|z\|<r_0/2$. Then $\|z+wr_0/2\|<r_0$ and using the differentiability (2.13) three times (in directions $wr_0/2$, z and $z+wr_0/2$) we obtain

$$\begin{aligned}\|S(x+wr_0/2)-S(x)+S(x+z)-S(x+z+wr_0/2)\| &\leqslant 2\eta\big(\|z\|+\|wr_0/2\|\big)\\ &< 2r_0\eta.\end{aligned}$$

Finally for any $x,y\in\mathscr{A}$ with $\|x-y\|<r_0/2$ we have

$$\begin{aligned}\|DS(x)-DS(y)\| &< \frac{2}{r_0}\big(r_0\eta+2r_0\eta\big)\\ &= 6\eta,\end{aligned}$$

and then $DS(x)$ is uniformly continuous (over x) on $\mathscr{A}$. □

Using Definition 2.2.5 we can now prove a result based on assumptions on DS.

Lemma 2.2.7. *Let $\mathscr{A}$ be a compact subset of a Banach space X that is negatively invariant for a map $S: X\to X$ that is uniformly differentiable for $x\in\mathscr{A}$. If there exist $\alpha\in\big(0,1/2\big)$ and $M\geqslant 1$ such that*

$$N\big[DS(x)B(0,1);\alpha\big]\leqslant M, \qquad x\in\mathscr{A}, \tag{2.14}$$

then

$$\dim_F(\mathscr{A}) \leqslant \frac{\ln M}{-\ln 2\alpha}. \tag{2.15}$$

Proof. The uniform differentiability assumption implies for $\eta \in \big(0, 1/2 - \alpha\big)$ that

$$S(B(x,r)) \subseteq S(x) + DS(x)B(0,r) + B(0,\eta r), \qquad x \in \mathscr{A},\ r \leqslant r_0(\eta);$$

combining this with the covering assumption in (2.14) it follows that

$$N\big[S(B(x,r)); (\alpha+\eta)r\big] \leqslant M,$$

for all $x \in \mathscr{A}$ and for all $0 < r \leqslant r_0(\eta)$. Lemma 2.2.4 now guarantees that

$$\dim_F(\mathscr{A}) \leqslant \frac{\ln M}{-\ln 2(\alpha+\eta)},$$

and since this holds for all $0 < \eta < 1/2 - \alpha$ the estimate in (2.15) follows. □

2.2.2 Finite-dimensional attractors from assumptions on DS

For our first results we make assumptions on the derivative of S; these results parallel the original ones in (MALLET-PARET, 1976) and (MAÑÉ, 1981).

2.2.2.1 The dimension is finite

Our first theorem revisits the classical result of (MAÑÉ, 1981) (see also (CARVALHO; LANGA; ROBINSON, 2010)) for a map whose derivative is the sum of a compact map and a contraction, but the proof is considerably simpler. However, our assumptions do not yield an explicit bound on the dimension.

We say that a map $L : X \to X$ is a λ-contraction if

$$\|Lx - Ly\| \leqslant \lambda \|x - y\|, \qquad x, y \in X.$$

If L is linear then it is a λ-contraction if $\|L\|_{\mathscr{L}(X)} \leqslant \lambda$.

Theorem 2.2.8. *Let X be a Banach space and $\mathscr{A}$ a compact subset of X that is negatively invariant for a map $S : X \to X$ that is uniformly differentiable for $x \in \mathscr{A}$. Suppose that for each $x \in \mathscr{A}$ we can write*

$$DS(x) = C_x + L_x,$$

where

- *C_x and L_x are both linear;*
- *$C_x : X \to X$ is compact for each $x \in \mathscr{A}$;*
- *C_x is continuous in x (on $\mathscr{A}$); and*
- *there exists $0 < \lambda < 1/4$ such that $\|L_x\|_{\mathscr{L}(X)} \leqslant \lambda$ for every $x \in \mathscr{A}$ (L_x is a λ-contraction).*

Then

$$\dim_F(\mathscr{A}) < \infty.$$

Proof. Fix $\theta > 0$ such that $0 < \theta + \lambda < 1/4$. Since C_x is compact, for each $x \in \mathscr{A}$ there exists $M(x,\theta) > 0$ such that

$$N\big[C_x B(0,1); \theta/2\big] \leqslant M(x,\theta).$$

Since C_x is continuous on the compact set $\mathscr{A}$ it is uniformly continuous, so there exists $\delta = \delta(\theta) > 0$ such that

$$\|x - x'\|_X < \delta, \;\; x, x' \in \mathscr{A} \qquad \Rightarrow \qquad \|C_x - C_{x'}\|_{\mathscr{L}(X)} < \theta/2.$$

It follows that if $\|x' - x\|_X < \delta$ with $x, x' \in \mathscr{A}$ then

$$\begin{aligned} C_{x'}B(0,1) &\subseteq C_x B(0,1) + [C_{x'} - C_x]B(0,1) \\ &\subseteq C_x B(0,1) + B(0,\theta/2), \end{aligned}$$

and so

$$N\big[C_{x'}B(0,1); \theta\big] \leqslant M(x,\theta)$$

for every $x' \in B(x,\delta) \cap \mathscr{A}$.

Since

$$\mathscr{A} = \bigcup_{x \in \mathscr{A}} B(x,\delta) \cap \mathscr{A}$$

and $\mathscr{A}$ is compact we can find $x_1, \ldots, x_k \in \mathscr{A}$ such that

$$\mathscr{A} = \bigcup_{i=1}^{k} B(x_i,\delta) \cap \mathscr{A}.$$

It now follows that by taking

$$M^*(\theta) := \max_{i=1,\ldots,k} M(x_i,\theta)$$

we have

$$\sup_{x \in \mathscr{A}} N\big[C_x B(0,1); \theta\big] \leqslant M^*(\theta),$$

with M^* independent of x.

Now for any $x \in \mathscr{A}$

$$
\begin{aligned}
DS(x)B(0,1) &= [C_x + L_x]B(0,1) \\
&\subseteq C_x B(0,1) + L_x B(0,1) \\
&\subseteq \bigcup_{i=1}^{M^*(\theta)} B(y_i,\theta) + B(0,\lambda) \\
&\subseteq \bigcup_{i=1}^{M^*(\theta)} B(y_i,\theta+\lambda) \\
&\subseteq \bigcup_{i=1}^{M^*(\theta)} B(\tilde{y}_i, 2(\theta+\lambda)),
\end{aligned}
$$

for some $\tilde{y}_i \in DS(x)B(0,1)$ and then

$$\sup_{x\in\mathscr{A}} N\big[DS(x)B(0,1); 2(\theta+\lambda)\big] \leqslant M^*(\theta).$$

From Lemma 2.2.7 we conclude that

$$\dim_F(\mathscr{A}) \leqslant \frac{\ln M^*(\theta)}{-\ln 4(\theta+\lambda)} < \infty. \qquad \square$$

We now discuss, briefly, how this method relates to that of Mañé in (MAÑÉ, 1981) (we will discuss Mañé's in more details in Section 2.2.6). Certain particular examples, such as the semilinear parabolic equation we shaw treat here in Section 2.2.7.1 (see also (CARVALHO; LANGA; ROBINSON, 2010)), generate a semigroup S that is continuously differentiable and for which it is possible to obtain a sequence of finite rank projections $\{P_n\}_{n\in\mathbb{N}}$ such that for points $x \in \mathscr{A}$ we have

$$DS(x) = P_n DS(x) + (I-P_n)DS(x) = C_x + L_x,$$

where $C_x = P_n DS(x)$ is compact and for sufficiently large n the operator $L_x = (I-P_n)DS(x)$ is a contraction on X with contraction constant less than $1/4$. Therefore, in this case S satisfies hypotheses in Theorem 2.2.8 and its attractor $\mathscr{A}$ has finite fractal dimension. We note that for each $x \in \mathscr{A}$ we have $C_x B(0,1) \subseteq U_x$, for some linear subspace $U_x \subseteq X$ with finite algebraic dimension $\dim(U_x) < \infty$; since $\mathscr{A}$ is compact, given $\varepsilon > 0$ there exists a linear subspace $U_\varepsilon \subseteq X$ with finite algebraic dimension such that

$$\operatorname{dist}_X\big(C_x B(0,1), U_\varepsilon\big) < \varepsilon, \qquad x \in \mathscr{A}.$$

Indeed, just as in the proof of Theorem 2.2.8, given $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$\|x - x'\|_X < \delta, \;\; x, x' \in \mathscr{A} \qquad \Rightarrow \qquad \|C_x - C_{x'}\|_{\mathscr{L}(X)} < \varepsilon/2.$$

Now

$$
\begin{aligned}
C_{x'}B(0,1) &\subseteq C_x B(0,1) + [C_{x'} - C_x]B(0,1) \\
&\subseteq U_x + B(0,\varepsilon/2),
\end{aligned}
$$

and so

$$\mathrm{dist}_X\big(C_{x'}B(0,1),U_x\big) < \varepsilon, \qquad x' \in B(x,\delta)\cap\mathscr{A}.$$

Since $\mathscr{A} = \bigcup_{i=1}^k B(x_i,\delta)\cap\mathscr{A}$, if we set $U_\varepsilon := \bigcup_{i=1}^k U_{x_i}$, then we have

$$\mathrm{dist}_X\big(C_xB(0,1),U_\varepsilon\big) < \varepsilon, \qquad x\in\mathscr{A}. \tag{2.16}$$

Expression (2.16) essentially portrays the fundamental property needed to follow Mañé's approach: it provides a finite-dimensional subspace U that is a good approximation of the image under C_x of the unit ball in X. For more general situations (in which C_x is not necessarily a finite rank operator) (2.16) is restated and achieved using the compactness of operators C_x (see (CARVALHO; LANGA; ROBINSON, 2010), Lemma 2.4) or alternatively by Lemma 2.2.21 in Section 2.2.6 using a smoothing condition.

2.2.2.2 Bounds on the dimension: the smoothing method

Theorem 2.2.8 in the previous section ensures that the fractal dimension of $\mathscr{A}$ is finite, but it does not provide any explicit estimate on this dimension.

In order to give a bound on the fractal dimension we now consider an auxiliary Banach space Z that is compactly embedded in X, and impose a Lipschitz continuity property between these spaces for the derivative DS of S, which here we refer to as the *smoothing property*. The bound we will obtain will involve the quantities

$$N_\varepsilon := N_X[B_Z(0,1);\varepsilon],$$

i.e. the minimum number of ε-balls in X that are needed to cover the unit ball $B_Z(0,1)$ in the subspace Z. These are related to the *Kolmogorov entropy numbers* for the compact embedding of Z into X; we discuss this further in Section 2.2.5. This method is based on the techniques developed by (MÁLEK; RUZICKA; THÄTER, 1994) (see also (ZELIK, 2000)).

Theorem 2.2.9. *Let Z and X be two Banach spaces such that Z is compactly embedded in X, and let $\mathscr{A}\subset Z$ be a compact subset of X that is negatively invariant for a map $S: X\to X$ that is uniformly differentiable for $x\in\mathscr{A}$. Suppose in addition that for each $x\in\mathscr{A}$ we have*

$$DS(x) = C_x + L_x,$$

where

- *$C_x\in\mathscr{L}(X,Z)$ and $L_x\in\mathscr{L}(X)$;*
- *$C_x : X\to Z$ satisfies a smoothing property, i.e. there exists $\kappa>0$ such that*

$$\|C_x(u)\|_Z \leqslant \kappa\|u\|_X, \qquad u\in X; \tag{2.17}$$

- *there exists $0<\lambda<1/4$ such that $\|L_x\|_{\mathscr{L}(X)}\leqslant\lambda$ (L_x is a λ-contraction).*

Then

$$\dim_F(\mathscr{A};X) \leqslant \frac{\ln N_{(\nu+\lambda)/2\kappa}}{-\ln 4(\nu+\lambda)}, \qquad \textit{for } \nu \in \left(0, \frac{1}{4} - \lambda\right).$$

Proof. For each $x \in \mathscr{A}$, by (2.17) and the contraction of L_x we have

$$\begin{aligned} DS(x)B_X(0,1) &\subseteq C_x B_X(0,1) + L_x B_X(0,1) \\ &\subseteq B_Z(0,\kappa) + B_X(0,\lambda). \end{aligned}$$

But given $0 < \nu < 1/4 - \lambda$, if

$$B_Z(0,1) \subseteq \bigcup_{i=1}^{N_{(\nu+\lambda)/2\kappa}} B_X\big(x_i, (\nu+\lambda)/2\kappa\big),$$

where $x_i \in B_Z(0,1)$, then

$$\begin{aligned} DS(x)B_X(0,1) &\subseteq \left[\bigcup_{i=1}^{N_{(\nu+\lambda)/2\kappa}} B_X\big(\kappa x_i, (\nu+\lambda)/2\big)\right] + B_X(0,\lambda) \\ &\subseteq \bigcup_{i=1}^{N_{(\nu+\lambda)/2\kappa}} B_X\big(y_i, 2(\nu+\lambda)\big), \end{aligned}$$

with $y_i \in DS(x)B_X(0,1)$. So for any $x \in \mathscr{A}$ we obtain

$$N_X\big[DS(x)B_X(0,1); 2(\nu+\lambda)\big] \leqslant N_{(\nu+\lambda)/2\kappa}$$

and from Lemma 2.2.7 we conclude that

$$\dim_F(\mathscr{A};X) \leqslant \frac{\ln N_{(\nu+\lambda)/2\kappa}}{-\ln 4(\nu+\lambda)}, \qquad \nu \in \big(0, 1/4 - \lambda\big). \qquad \square$$

In the next section we prove some results on the relationship between maps and their derivatives, which will allow us to deduce results on the dimension of $\mathscr{A}$ by imposing conditions on S rather than DS.

2.2.3 Maps and their derivatives

In this section we relate properties of the original map S to properties of its derivative, and vice versa.

First we show that if S is compact and differentiable then DS is also compact.

Lemma 2.2.10. *Let X be a Banach space and suppose that $S : X \to X$ is a compact map. If S is Fréchet differentiable at $x \in X$ with derivative $DS(x)$, then $DS(x) : X \to X$ is compact.*

Proof. Suppose that the operator $DS(x)$ is not compact. Then there exist $\varepsilon_0 > 0$ and a sequence $\{y_n\}_n \subset X$ such that $\|y_n\| \leqslant 1$ and

$$\|DS(x)y_n - DS(x)y_m\| \geqslant \varepsilon_0, \qquad n \neq m.$$

By the definition of the derivative there exists $\delta > 0$ such that for all $\|h\| \leqslant \delta$ we have

$$\|S(x+h) - S(x) - DS(x)h\| \leqslant \frac{\varepsilon_0}{4}\|h\|.$$

Choosing $\tau > 0$ such that $\|\tau y_n\| < \delta$ for all $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
\|S(x+\tau y_n) - S(x+\tau y_m)\| &= \|S(x+\tau y_n) - S(x) - DS(x)(\tau y_n) \\
&\qquad + DS(x)(\tau y_n) - DS(x)(\tau y_m) \\
&\qquad + S(x) - S(x+\tau y_m) + DS(x)(\tau y_m)\| \\
&\geqslant \tau\|DS(x)y_n - DS(x)y_m\| \\
&\qquad - \|S(x+\tau y_n) - S(x) - DS(x)(\tau y_n)\| \\
&\qquad - \|S(x+\tau y_m) - S(x) - DS(x)(\tau y_m)\| \\
&\geqslant \tau\varepsilon_0 - 2\frac{\tau}{4}\varepsilon_0 = \frac{\tau}{2}\varepsilon_0.
\end{aligned}$$

Since $\{x+\tau y_n\}_n$ is a bounded sequence, this shows that $\{S(x+\tau y_n)\}_n$ can have no Cauchy subsequence, and so S is not compact, a contradiction. □

This result enables us to answer the question posed in the introduction in the affirmative if S is compact and differentiable.

Corollary 2.2.11. *If $\mathscr{A}$ is a compact subset of X that is negatively invariant under a map $S : X \to X$, and S is compact and uniformly differentiable for $x \in \mathscr{A}$ with DS continuous on $\mathscr{A}$, then* $\dim_F(\mathscr{A}) < \infty$.

In fact we can prove a 'compact map plus contraction' result in this setting too, although this is perhaps a little less natural, since we require a splitting $S = C + L$ in which both C and L are differentiable. First we show that if $L : X \to X$ is a λ-contraction (i.e. $\|Lx - Ly\| \leqslant \lambda\|x-y\|$ for every $x, y \in X$) then so is DL.

Lemma 2.2.12. *Suppose that $L : X \to X$ is a λ-contraction. If L is differentiable at $x \in X$ then $DL(x) \in \mathscr{L}(X)$ is also a λ-contraction.*

Proof. Since

$$L(x+h) = L(x) + DL(x)h + o(\|h\|)$$

then

$$\|DL(x)h\| \leqslant \lambda\|h\| + o(\|h\|).$$

Given $\varepsilon > 0$ there is $\delta > 0$ such that for $\|h\| < \delta$ we have

$$\|DL(x)h\| \leqslant (\lambda + \varepsilon)\|h\|.$$

Now given $h \in X$ let $h_1 := \frac{\delta}{2\|h\|}h$ and then $\|h_1\| < \delta$. Since $DL(x)$ is linear we conclude

$$\|DL(x)h\| \leqslant (\lambda + \varepsilon)\|h\|, \qquad \text{for all } \varepsilon > 0,$$

and so $DL(x)$ is a λ-contraction. □

Corollary 2.2.13. *Suppose that $\mathscr{A}$ is a compact subset of X that is negatively invariant under a map $S : X \to X$, with $S = C + L$, where C is compact and L is a λ-contraction with $\lambda < 1/4$. If C and L are uniformly differentiable for $x \in \mathscr{A}$, and DC is continuous on $\mathscr{A}$ then* $\dim_F(\mathscr{A}) < \infty$.

We can also transfer a smoothing property for C to one for DC; but rather than argue this way in the next section we instead prove directly that a smoothing property for C can be used to bound the dimension of $\mathscr{A}$. Indeed, this was the original way that this property was used by (MÁLEK; RUZICKA; THÄTER, 1994) and (ZELIK, 2000).

2.2.4 Results using assumptions on S

We now prove a result in which we assume that S can be written in the form $S = C + L$, where C and L satisfy (2.18) and (2.19) below.

This method is a powerful tool in constructing exponential attractors in various settings (besides the autonomous case in (MÁLEK; RUZICKA; THÄTER, 1994) and (ZELIK, 2000), see (CARVALHO; SONNER, 2013) for the non-autonomous case and (CARABALLO; SONNER, 2017) for the random case) and then determining the dimension of attractors. As in Theorem 2.2.9 the estimates for the fractal dimension are once again given in terms of the numbers N_ε related to the compact embedding of Z into X. Next theorem uses a similar assumption to that in (CARVALHO; SONNER, 2013) (which generalises (MÁLEK; RUZICKA; THÄTER, 1994) and (ZELIK, 2000)) by allowing the additional contraction term L.

Theorem 2.2.14. *Let Z and X be two Banach spaces with Z compactly embedded in X, and suppose that $\mathscr{A} \subset Z$ is a compact subset of X that is negatively invariant for a map $S : X \to X$. Suppose in addition that*

$$S = C + L,$$

where

- *$C : X \to Z$ and $L : X \to X$ are continuous maps;*
- *$C : X \to Z$ satisfies a smoothing property on $\mathscr{A}$, i.e. there exists $\kappa > 0$ such that*

$$\|C(x) - C(y)\|_Z \leqslant \kappa \|x - y\|_X, \qquad x, y \in \mathscr{A}; \tag{2.18}$$

- *$L : X \to X$ is a λ-contraction on $\mathscr{A}$, with $0 < \lambda < 1/4$, i.e.*

$$\|L(x) - L(y)\|_X \leqslant \lambda \|x - y\|_X, \qquad x, y \in \mathscr{A}. \tag{2.19}$$

Then

$$\dim_F(\mathscr{A}; X) \leqslant \frac{\ln N_{(\nu+\lambda)/2\kappa}}{-\ln 4(\nu+\lambda)}, \qquad \textit{for } \nu \in \left(0\,, \frac{1}{4} - \lambda\right).$$

Proof. Given $0 < \nu < 1/4 - \lambda$, let $x_0 \in \mathscr{A}$ and $R > 0$ be such that

$$\mathscr{A} = B_X(x_0, R) \cap \mathscr{A}. \tag{2.20}$$

By the smoothing property for C in (2.18) and the definition of $N_{(\nu+\lambda)/2\kappa}$ we have

$$\begin{aligned} C\big(B_X(x_0,R)\cap\mathscr{A}\big) &\subseteq B_Z\big(C(x_0),R\kappa\big) \\ &\subseteq \bigcup_{j=1}^{N_{(\nu+\lambda)/2\kappa}} B_X\big(C(x_0)+R\kappa x_j, R(\nu+\lambda)\big) \\ &\subseteq \bigcup_{j=1}^{N_{(\nu+\lambda)/2\kappa}} B_X\big(y_j, 2R(\nu+\lambda)\big), \end{aligned}$$

for some $y_j \in X$. Now, by the contraction property (2.19) for L we obtain

$$L\big(B_X(x_0,R)\cap\mathscr{A}\big) \subseteq B_X\big(L(x_0),R\lambda\big)$$

and then applying S in (2.20) we have

$$\begin{aligned} \mathscr{A} &= S(\mathscr{A})\cap\mathscr{A} \\ &= \big[\,C\big(B_X(x_0,R)\cap\mathscr{A}\big)+L\big(B_X(x_0,R)\cap\mathscr{A}\big)\,\big]\cap\mathscr{A} \\ &= \left[\left(\bigcup_{j=1}^{N_{(\nu+\lambda)/2\kappa}} B_X\big(y_j,2R(\nu+\lambda)\big)\right)+B_X\big(L(x_0),R\lambda\big)\right]\cap\mathscr{A} \\ &= \bigcup_{j=1}^{N_{(\nu+\lambda)/2\kappa}} B_X\big(y_j+L(x_0),2R2(\nu+\lambda)\big)\cap\mathscr{A} \\ &= \bigcup_{j=1}^{N_{(\nu+\lambda)/2\kappa}} B_X\big(z_j,2R[4(\nu+\lambda)]\big)\cap\mathscr{A}, \end{aligned}$$

for some $z_j \in \mathscr{A}$, i.e.,

$$\mathscr{A} = \bigcup_{j=1}^{N_{(\nu+\lambda)/2\kappa}} B_X\big(z_j,2R[4(\nu+\lambda)]\big)\cap\mathscr{A}.$$

Analogously, for each $n \in \mathbb{N}$, there exists a subset $V_n \subset \mathscr{A}$ with $\sharp V_n \leqslant N^n_{(\nu+\lambda)/2\kappa}$ such that

$$\mathscr{A} = \bigcup_{z\in V_n} B_X\big(z,2R[4(\nu+\lambda)]^n\big)\cap\mathscr{A},$$

and so

$$N_X\big[\mathscr{A};2R[4(\nu+\lambda)]^n\big] \leqslant N^n_{(\nu+\lambda)/2\kappa}.$$

Finally

$$\frac{\ln N_X\big[\mathscr{A};2R[4(\nu+\lambda)]^n\big]}{-n\ln 4(\nu+\lambda)} \leqslant \frac{n\ln N_{(\nu+\lambda)/2\kappa}}{-n\ln 4(\nu+\lambda)} = \frac{\ln N_{(\nu+\lambda)/2\kappa}}{-\ln 4(\nu+\lambda)}$$

and from Lemma 2.2.3 we obtain

$$\dim_F\big(\mathscr{A};X\big) \leqslant \frac{\ln N_{(\nu+\lambda)/2\kappa}}{-\ln 4(\nu+\lambda)}, \qquad \nu \in \Big(0,\frac{1}{4}-\lambda\Big). \qquad \square$$

Note that in the above result we have estimated the fractal dimension of $\mathscr{A}$ in the space X. However, when the contraction term L is absent (i.e. when $\lambda = 0$) then we have

$$\dim_F(\mathscr{A};Z) = \dim_F(\mathscr{A};X). \tag{2.21}$$

Indeed, if

$$\|S(x) - S(y)\|_Z \leqslant \kappa \|x-y\|_X, \qquad x,y \in \mathscr{A},$$

then the map $S : \mathscr{A} \to Z$ is Lipschitz and

$$\dim_F(\mathscr{A};Z) \leqslant \dim_F\big(S(\mathscr{A});Z\big) \leqslant \dim_F(\mathscr{A};X) \leqslant \dim_F(\mathscr{A};Z),$$

where for the second inequality we use Lemma 2.2.2 and for the last inequality the fact that Z is continuously embedded in X.

Remark 2.2.15. *There is another smoothing property that we could use in the above theorem and obtain the same bounds on the dimension. We assume that $\mathscr{A}$ is a compact subset of X that is negatively invariant for a map $S : X \to X$, but now we suppose that $S = C + L$ and for all $x,y \in \mathscr{A}$*

$$\|C(x) - C(y)\|_X \leqslant \kappa \|x-y\|_Y,$$

where X is compactly embedded in Y; once again we take L to be a λ-contraction on $\mathscr{A}$ for some $\lambda \in (0,1/4)$. Then we obtain

$$\dim_F(\mathscr{A};X) \leqslant \frac{\ln \tilde{N}_{(\nu+\lambda)/2\kappa}}{-\ln 4(\nu+\lambda)}, \qquad \textit{for } \nu \in \big(0, 1/4-\lambda\big),$$

where $\tilde{N}_\varepsilon := N_Y\big[B_X(0,1);\varepsilon\big]$, $\varepsilon > 0$. The proof of this result is almost identical and uses the same argumentation as before.

2.2.5 Kolmogorov entropy numbers

In this section we recall the notion of the Kolmogorov entropy numbers for the compact embedding between two Banach spaces Z and X and discuss how it relates to the estimates for the fractal dimension presented above (see (KOLMOGOROV; TIKHOMIROV, 1993) for a detailed treatment of this subject). We shall also see that it is possible to obtain in Theorem 2.2.9 (and Theorem 2.2.14) the best estimate among all the family (in ν) of estimates.

The quantities $N_\varepsilon = N_X\big[B_Z(0,1);\varepsilon\big]$ are related with the entropy numbers of the compact embedding between the spaces Z and X: the *Kolmogorov ε-entropy* of that embedding, denoted by $\mathbf{H}_\varepsilon(Z,X)$, is

$$\mathbf{H}_\varepsilon(Z,X) := \log_2 N_\varepsilon.$$

For many function spaces it is possible to determine estimates for these numbers. In particular, for Sobolev spaces we generically have a polynomial growth in ε (as $\varepsilon \to 0^+$): if $\mathscr{O} \subset \mathbb{R}^d$ is a smooth bounded domain, then the embedding

$$W^{l_1,p_1}(\mathscr{O}) \hookrightarrow W^{l_2,p_2}(\mathscr{O})$$

is compact if $l_1, l_2 \in \mathbb{R}$, $p_1, p_2 \in (1,\infty)$ with $l_1 > l_2$ and $l_1 - \frac{d}{p_1} > l_2 - \frac{d}{p_2}$. It is shown in (TRIEBEL, 1978) (Section 4.10.3, Remark 3) that in this case there exist constants $c_1, c_2 > 0$ such that

$$c_1 \varepsilon^{-\frac{d}{l_1-l_2}} \leqslant \mathbf{H}_\varepsilon\big(W^{l_1,p_1}(\mathscr{O}), W^{l_2,p_2}(\mathscr{O})\big) \leqslant c_2 \varepsilon^{-\frac{d}{l_1-l_2}}, \quad \text{for all } \varepsilon > 0. \tag{2.22}$$

We therefore suppose that Z and X are spaces such that

$$c_1 \varepsilon^{-\gamma} \leqslant \mathbf{H}_\varepsilon(Z,X) \leqslant c_2 \varepsilon^{-\gamma}, \quad \text{for all } \varepsilon > 0 \tag{2.23}$$

for some $c_1, c_2 > 0$ and $\gamma > 0$, so that

$$\ln N_\varepsilon \leqslant (\ln 2^{c_2})\varepsilon^{-\gamma}. \tag{2.24}$$

Using the estimate in (2.24) and from Theorem 2.2.9, it follows that for each choice of $\nu \in \big(0, 1/4-\lambda\big)$ we have

$$\begin{aligned}
\dim_F(\mathscr{A};X) &\leqslant \frac{\ln N_{(\nu+\lambda)/2\kappa}}{-\ln 4(\nu+\lambda)} \\
&\leqslant \frac{(\ln 2^{c_2})\left(\frac{\nu+\lambda}{2\kappa}\right)^{-\gamma}}{-\ln 4(\nu+\lambda)} \\
&= (\ln 2^{c_2})(2\kappa)^{\gamma} \frac{\left(\frac{1}{\nu+\lambda}\right)^{\gamma}}{-\ln 4(\nu+\lambda)}.
\end{aligned}$$

This expression attains its minimun at $\big(0, 1/4-\lambda\big)$ for $\nu_0 := \dfrac{1}{4e^{\frac{1}{\gamma}}} - \lambda$ and so

$$\dim_F(\mathscr{A};X) \leqslant c(8\kappa)^{\gamma}\gamma, \tag{2.25}$$

for some positive constant $c > 0$. The estimate in (2.25) is the best among all in Theorem 2.2.9.

It seems useful to sketch an elementary proof of the estimate in (2.23) in the case that X is a separable Hilbert space and Z is the fractional power space $X^\alpha := D(A^\alpha)$ of a linear operator A (e.g. $X = L^2(\mathscr{O})$, with $\mathscr{O}$ a smooth bounded domain in $\mathbb{R}^d$ and $A := -\Delta$ the Dirichlet Laplacian). For a review on fractional power spaces see (CARVALHO; LANGA; ROBINSON, 2013), Chapter 6. Let A have eigenvalues $(\lambda_j)_{j=1}^\infty$ with $\lambda_{j+1} \geqslant \lambda_j$ and $\lambda_j \to \infty$ as $j \to \infty$, with orthonormal eigenfunctions $(e_j)_{j=1}^\infty$ that form a basis for X. We will assume that

$$\lambda_j \sim j^\theta, \tag{2.26}$$

i.e. there exist constants $c_1, c_2 > 0$ such that $c_1 j^\theta \leqslant \lambda_j \leqslant c_2 j^\theta$. If $A = -\Delta$ on a bounded domain in $\mathbb{R}^d$, then $\theta = 2/d$ (see (DAVIES, 1995)).

We want to cover the unit ball in X^α,

$$B_\alpha := \left\{ x = \sum_{j=1}^\infty x_j e_j : \sum_{j=1}^\infty \lambda_j^{2\alpha}|x_j|^2 \leqslant 1 \right\},$$

with 2ε-balls in X, estimating then $N_{2\varepsilon} = N_X\big[B_\alpha; 2\varepsilon\big]$.

Given $\varepsilon > 0$ let n be the smallest integer such that $\lambda_{n+1}^{-\alpha} \leqslant \varepsilon$. Then for every $x \in B_\alpha$ we have

$$\Bigg\| \sum_{j=n+1}^{\infty} x_j e_j \Bigg\|^2 = \sum_{j=n+1}^{\infty} |x_j|^2 \leqslant \lambda_{n+1}^{-2\alpha} \left[\sum_{j=n+1}^{\infty} \lambda_j^{2\alpha} |x_j|^2 \right] \leqslant \varepsilon^2;$$

so if we can cover the finite-dimensional set

$$E := \left\{ x = \sum_{j=1}^{n} x_j e_j : \ \sum_{j=1}^{n} \lambda_j^{2\alpha} |x_j|^2 \leqslant 1 \right\}$$

with ε-balls in X we can cover B_α with the same number of 2ε-balls.

Lemma III.2.1 in (CHEPYZHOV; VISHIK, 2002) shows that we can cover the ellipse E, whose semiaxes are $\{\lambda_j^{-\alpha}\}_{j=1}^n$, using no more than

$$4^n \frac{\prod_{j=1}^n \lambda_j^{-\alpha}}{\varepsilon^n} \leqslant 4^n \frac{\prod_{j=1}^n \lambda_j^{-\alpha}}{\lambda_{n+1}^{-\alpha n}} = 4^n \frac{\lambda_{n+1}^{\alpha n}}{\prod_{j=1}^n \lambda_j^{\alpha}}$$

ε-balls. The assumption that $\lambda_j \sim j^\theta$ yields, using the lower bound[1] $n! \geqslant n^n e^{-n}$, that

$$\begin{aligned} N_{2\varepsilon} &\leqslant 4^n \frac{\lambda_{n+1}^{\alpha n}}{\prod_{j=1}^n \lambda_j^{\alpha}} \leqslant (c_2/c_1)^{\alpha n} 4^n \frac{(n+1)^{n\theta\alpha}}{(n!)^{\theta\alpha}} \leqslant (c_2/c_1)^{\alpha n} 4^n \left(\frac{n+1}{n}\right)^{n\theta\alpha} e^{n\theta\alpha} \\ &\leqslant (c_2/c_1)^{\alpha n} 4^n 2^{n\theta\alpha} e^{n\theta\alpha} = \beta^n, \end{aligned}$$

where $\beta = 4(c_2/c_1)^\alpha 2^{\theta\alpha} e^{\theta\alpha}$.

The ε-entropy is therefore bounded by

$$\mathbf{H}_{2\varepsilon}\big(X^\alpha, X\big) = \log_2 N_{2\varepsilon} \leqslant n \log_2 \beta;$$

since n is the smallest integer such that $\lambda_{n+1}^{-\alpha} \leqslant \varepsilon$ and $\lambda_n \geqslant c_1 n^\theta$ it follows that $n \leqslant c_1^{-1/\theta} \varepsilon^{-1/\alpha\theta}$, and

$$\mathbf{H}_{2\varepsilon}\big(X^\alpha, X\big) \leqslant C\varepsilon^{-1/\alpha\theta}.$$

In the case of Laplacian on a bounded domain in $\mathbb{R}^d$ we have $\theta = 2/d$ and then

$$\mathbf{H}_{\varepsilon}\big(X^\alpha, X\big) \leqslant C\varepsilon^{-d/2\alpha}.$$

It follows easily using essentially the same argument that if $0 < \alpha < \beta$ we have

$$\mathbf{H}_{\varepsilon}\big(X^\beta, X^\alpha\big) \leqslant C\varepsilon^{-d/2(\beta-\alpha)},$$

which in particular agrees with the upper bound in (2.22) in the case $p_1 = p_2 = 2$, $l_1 = 2\beta$, $l_2 = 2\alpha$.

[1] Notice that $\ln n! = \sum_{k=1}^n \ln k \geqslant \int_1^n \ln x \, dx = n\ln n - n + 1 \geqslant n \ln n - n$, which implies $n! \geqslant n^n e^{-n}$.

2.2.6 A comparison with Mañé's method

For this section let us consider the setting proposed in Remark 2.2.15, i.e., X and Y are Banach spaces with X compactly embedded in Y, and $S : X \to X$, $S = C + L$. Denote $\tilde{N}_\varepsilon := N_Y\left[B_X(0,1);\varepsilon\right]$, for $\varepsilon > 0$. So we can restate Theorem 2.2.9 with a very slightly modification on the proof and guarantee the following result on fractal dimensionality of negatively invariant sets.

Theorem 2.2.16. *Let X and Y be two Banach spaces such that X is compactly embedded in Y, and let $\mathscr{A} \subset X$ be a compact subset of X that is negatively invariant for a map $S : X \to X$ that is uniformly differentiable for $x \in \mathscr{A}$. Suppose in addition that for all $x \in \mathscr{A}$ we can write*

$$DS(x) = C_x + L_x,$$

where

- *$C_x, L_x \in \mathscr{L}(X)$;*
- *$C_x : X \to X$ satisfies a smoothing property on $\mathscr{A}$, i.e. there exists $\kappa > 0$ such that*

$$\|C_x(u)\|_X \leqslant \kappa \|u\|_Y, \qquad u \in X; \tag{2.27}$$

- *$L_x : X \to X$ is a λ-contraction with $0 < \lambda < 1/4$, i.e.*

$$\|L_x\|_{\mathscr{L}(X)} < \lambda. \tag{2.28}$$

Then

$$\dim_F(\mathscr{A};X) \leqslant \frac{\ln \tilde{N}_{(\nu+\lambda)/2\kappa}}{-\ln 4(\nu+\lambda)}, \qquad \textit{for } \nu \in \left(0, \frac{1}{4} - \lambda\right).$$

In order to provide a comparison between the smoothing method presented here (Theorem 2.2.16 above) and Mañé's method (as developed in (CARVALHO; LANGA; ROBINSON, 2010)) we need first to discuss techniques which describe how to cover balls in an n-dimensional normed space by balls with smaller radius. Afterwards we shall present a new version of Lemma 2.4 in (CARVALHO; LANGA; ROBINSON, 2010) and so following Mañé's approach we will be able to estimate the fractal dimension of $\mathscr{A}$ (see Theorem 2.2.22) also in terms of the Kolmogorov numbers $\tilde{N}_\varepsilon$, for some particular $\varepsilon > 0$, and it will allows us to compare explicitly estimates achieved in Theorem 2.2.16 and Theorem 2.2.22.

Let $\mathbb{R}^n_\infty$ denote the space $\mathbb{R}^n$ endowed with the norm $\|z\|_\infty := \max\limits_{1 \leqslant j \leqslant n} |z_j|$ for $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$. If U is an n-dimensional subspace of a normed space we shall construct in the following an *Auerbach basis* for U.

Lemma 2.2.17 ((CARVALHO; LANGA; ROBINSON, 2013), Lemma 4.7)**.** *Let U be an (algebraic) n-dimensional real normed space. Then there exist a basis $\{x_1, \cdots, x_n\}$ of U and a basis $\{f_1^*, \cdots, f_n^*\}$ of U^* such that $\|x_j\|_U = \|f_j^*\|_{U^*} = 1$ and $f_i^*(x_j) = \delta_{ij}$, $1 \leqslant i, j \leqslant n$, where δ_{ij} is the Kronecker delta. In this case, $\{x_1, \cdots, x_n\}$ is called an Auerbach basis for U.*

Proof. Let $E = \{v_1, \cdots, v_n\}$ be an arbitrary basis for U. Given $u \in U$ we have $u = a_1 v_1 + \cdots + a_n v_n$, for some $a_j \in \mathbb{R}$. Let us denote by $\hat{u}$ the column vector $\hat{u} := (a_1 \cdots a_n)^T$ and consider the map $U^n \ni (u_1, \cdots, u_n) \mapsto \det[\hat{u}_1, \cdots, \hat{u}_n] \in \mathbb{R}$.

Let $B := \overline{B_U(0,1)}$ be the closed unit ball in U and notice that if $\{x_1, \cdots, x_n\}$ is a point where the function $\left|\det[\cdot, \cdots, \cdot]\right|$ attains a maximum in B^n then $\{x_1, \cdots, x_n\}$ must be a basis for U, and hence $\det[\hat{x}_1, \cdots, \hat{x}_n] \neq 0$. Moreover $\|x_j\|_U = 1$, for all $1 \leqslant j \leqslant n$, because otherwise if $\|x_j\|_U < 1$ we have for some $|a| > 1$ that $\|a x_j\|_U = 1$ and $\left|\det[\hat{x}_1, \cdots, \hat{x}_{j-1}, a\hat{x}_j, \hat{x}_{j+1}, \cdots, \hat{x}_n]\right| = |a| \left|\det[\hat{x}_1, \cdots, \hat{x}_n]\right| > \left|\det[\hat{x}_1, \cdots, \hat{x}_n]\right|$ with $(x_1, \cdots, x_{j-1}, a x_j, x_{j+1}, \cdots, x_n) \in B^n$, an absurd.

Finally we define the linear functionals $f_i^* : U \to \mathbb{R}$ for each $1 \leqslant i \leqslant n$ as

$$f_i^*(u) := \frac{\det[\hat{x}_1, \cdots, \hat{x}_{i-1}, \hat{u}, \hat{x}_{i+1}, \cdots, \hat{x}_n]}{\det[\hat{x}_1, \cdots, \hat{x}_n]}, \qquad u \in U.$$

Notice that $\|x_j\|_U = \|f_j^*\|_{U^*} = 1$ and $f_i^*(x_j) = \delta_{ij}$, for all $1 \leqslant i, j \leqslant n$. Moreover, since $\{f_1^*, \cdots, f_n^*\}$ is linearly independent it is a basis for U^*. In this case $\{x_1, \cdots, x_n\}$ is an Auerbach basis for X. $\square$

Corollary 2.2.18 ((CARVALHO; LANGA; ROBINSON, 2010), Proposition 2.2)**.** *Let U be an n-dimensional real normed space. Then there is an isomorphism $T : \mathbb{R}^n_\infty \to U$ such that*

$$\|T\|_{\mathscr{L}(\mathbb{R}^n_\infty, U)} \|T^{-1}\|_{\mathscr{L}(U, \mathbb{R}^n_\infty)} \leqslant n.$$

Proof. By Lemma 2.2.17 let $\{x_1, \cdots, x_n\}$ be an Auerbach basis for U and define the mapping $T : \mathbb{R}^n_\infty \to U$ by setting

$$T(z) := \sum_{j=1}^n z_j x_j, \qquad z = (z_1, \cdots, z_n) \in \mathbb{R}^n_\infty.$$

Then

$$\|T(z)\|_U = \Big\| \sum_{j=1}^n z_j x_j \Big\|_U \leqslant \sum_{j=1}^n |z_j| \leqslant n \|z\|_\infty$$

and so

$$\|T\|_{\mathscr{L}(\mathbb{R}^n_\infty, U)} \leqslant n.$$

Now for any $u \in B_U(0,1)$, $u = a_1 x_1 + \cdots + a_n x_n$, we have $a_j = f_j^*(u)$, and

$$\|T^{-1}(u)\|_\infty = \|(a_1, \cdots, a_n)\|_\infty = \max_{1 \leqslant j \leqslant n} |a_j| = \max_{1 \leqslant j \leqslant n} |f_j^*(u)| \leqslant \max_{1 \leqslant j \leqslant n} \|f_j^*\|_{U^*} \|u\|_U = \|u\|_U,$$

proving that

$$\|T^{-1}\|_{\mathscr{L}(U, \mathbb{R}^n_\infty)} \leqslant 1.$$

Finally,

$$\|T\|_{\mathscr{L}(\mathbb{R}^n_\infty, U)} \|T^{-1}\|_{\mathscr{L}(U, \mathbb{R}^n_\infty)} \leqslant n.$$ $\square$

As a consequence of the existence of isomorphism T we are able to estimate the number of balls which are necessary to cover a ball in an n-dimensional normed space.

Lemma 2.2.19 ((CARVALHO; LANGA; ROBINSON, 2013), Lemma 4.8)**.** *If U is an n-dimensional real normed space then*

$$N\big[B_U(0,R);r\big] \leqslant (n+1)^n\Big(\frac{R}{r}\Big)^n, \qquad 0<r\leqslant R.$$

Proof. From Corollary 2.2.18 there is an isomorphism $T:\mathbb{R}^n_\infty \to U$ such that

$$\|T\|_{\mathscr{L}(\mathbb{R}^n_\infty,U)}\|T^{-1}\|_{\mathscr{L}(U,\mathbb{R}^n_\infty)} \leqslant n.$$

Notice that

$$B_U(0,R) = TT^{-1}B_U(0,R) \subseteq TB_{\mathbb{R}^n_\infty}\big(0,\|T^{-1}\|R\big),$$

and since we can cover $B_{\mathbb{R}^n_\infty}\big(0,\|T^{-1}\|R\big)$ in $\mathbb{R}^n_\infty$ with no more than

$$\Big(1+\frac{\|T^{-1}\|R}{r/\|T\|}\Big)^n = \Big(1+\|T^{-1}\|\|T\|\frac{R}{r}\Big)^n \leqslant \Big(1+n\frac{R}{r}\Big)^n \leqslant (n+1)^n\Big(\frac{R}{r}\Big)^n$$

balls of radius $r/\|T\|$ we conclude that

$$N\big[B_U(0,R);r\big] \leqslant (n+1)^n\Big(\frac{R}{r}\Big)^n.$$

□

Remark 2.2.20. *In a case of an n-dimensional complex normed space U we have the estimate*

$$N\big[B_U(0,R);r\big] \leqslant (n+1)^{2n}\Big(\frac{R}{r}\Big)^{2n}, \qquad 0<r\leqslant R,$$

since we need no more than $\big(1+M/m\big)^{2n}$ balls of radius m to cover the ball $B_{\mathbb{C}^n_\infty}(0,M)$, with $m\leqslant M$. Notice that we have to do the necessary changes on Lemma 2.2.17 and Corollary 2.2.18.

In the following we shall present a new version of Lemma 2.4 in (CARVALHO; LANGA; ROBINSON, 2010), in which we obtain good approximations of the image under a linear map of the unit ball on a Banach space X by the image of the unit ball on a finite dimensional subspace of X. The (algebraic) dimension of this subspace will be estimated in terms of $\tilde{N}_\varepsilon$ and it will be useful to follow Mañé's method in Theorem 2.2.22. Recall that $\tilde{N}_\varepsilon = N_Y\big[B_X(0,1);\varepsilon\big]$.

Lemma 2.2.21. *Let X and Y be two Banach spaces with X compactly embedded in Y and $C\in\mathscr{L}(X)$ be a bounded linear operator satisfying the smoothing property, i.e. there is $\kappa>0$ such that*

$$\|C(u)\|_X \leqslant \kappa\|u\|_Y, \qquad \forall u\in X. \tag{2.29}$$

Then given $\varepsilon>0$ there exists a subspace $U_\varepsilon\subset X$ with $\dim(U_\varepsilon)\leqslant\tilde{N}_\varepsilon$ such that

$$\operatorname{dist}_X\big(CB_X(0,1),CB_{U_\varepsilon}(0,1)\big)\leqslant\kappa\varepsilon.$$

Proof. Given $\varepsilon>0$,

$$B_X(0,1)\subseteq\bigcup_{i=1}^{\tilde{N}_\varepsilon}B_Y(x_i,\varepsilon), \tag{2.30}$$

where $x_i \in B_X(0,1)$ for each $i = 1,2,\cdots,\tilde{N}_\varepsilon$. Let $U_\varepsilon := \operatorname{span}\{x_i : 1 \leqslant i \leqslant \tilde{N}_\varepsilon\} \subset X$ and $x \in B_X(0,1)$ be an arbitrary point in the unit ball of X. Then $\|x - x_{i_0}\|_Y < \varepsilon$, for some $1 \leqslant i_0 \leqslant \tilde{N}_\varepsilon$, and since $B_{U_\varepsilon}(0,1) = U_\varepsilon \cap B_X(0,1)$ then we have by hypothesis (2.29) that

$$\begin{aligned}
\operatorname{dist}_X\big(C(x), CB_{U_\varepsilon}(0,1)\big) &\leqslant \|C(x) - C(x_{i_0})\|_X \\
&\leqslant \kappa\|x - x_{i_0}\|_Y \\
&< \kappa\varepsilon.
\end{aligned}$$

Therefore, taking the supremum over $x \in B_X(0,1)$ we obtain

$$\operatorname{dist}_X\big(CB_X(0,1), CB_{U_\varepsilon}(0,1)\big) \leqslant \kappa\varepsilon,$$

and the result follows. □

In particular, given $\nu > 0$, we can choose $\varepsilon = \frac{\nu}{2\kappa}$ in last lemma and take $U \subset X$ with $\dim(U) \leqslant \tilde{N}_{\nu/2\kappa} = N_Y\big[B_X(0,1); \frac{\nu}{2\kappa}\big]$ and such that

$$\operatorname{dist}_X\big(CB_X(0,1), CB_U(0,1)\big) < \nu. \tag{2.31}$$

In this new setting we can estimate the fractal dimension of negatively invariant sets as follows.

Theorem 2.2.22 (Based on (CARVALHO; LANGA; ROBINSON, 2010), Theorem 2.5)**.** *Suppose the same conditions as in Theorem 2.2.16. Then $D := \sup_{x\in\mathscr{A}} \|DS(x)\|_{\mathscr{L}(X)} < \infty$ and we can estimate the fractal dimension of $\mathscr{A}$ on X as*

$$\dim_F\big(\mathscr{A}; X\big) \leqslant 2\tilde{N}_{\nu/2\kappa}\left[\frac{\ln\Big((\tilde{N}_{\nu/2\kappa}+1)\frac{2D}{\nu}\Big)}{-\ln 4(\nu+\lambda)}\right], \qquad \text{for } \nu \in \Big(0, \min\{1/4-\lambda, D\}\Big).$$

Remark 2.2.23. *Since X is in particular continuously embedded in Y then we have*

$$\dim_F\big(\mathscr{A}; Y\big) \leqslant \dim_F\big(\mathscr{A}; X\big) < \infty.$$

Proof of Theorem 2.2.22. From Proposition 2.2.6 we know that S is continuously differentiable on $\mathscr{A}$ and then it follows immediately that $D = \sup_{x\in\mathscr{A}} \|DS(x)\|_{\mathscr{L}(X)} < \infty$.

Besides that, let $\nu \in \big(0, \min\{1/4-\lambda, D\}\big)$ and choose $U \subset X$ as in (2.31), i.e., $\dim(U) \leqslant \tilde{N}_{\nu/2\kappa}$ and

$$\operatorname{dist}_X\big(C_xB_X(0,1), C_xB_U(0,1)\big) < \nu, \qquad x \in \mathscr{A}.$$

Notice that U is independent of $x \in \mathscr{A}$ provided that $\kappa > 0$ is independent of x over $\mathscr{A}$.

For all $u \in B_X(0,1)$ and $z \in B_U(0,1)$ we obtain

$$\begin{aligned}
\|DS(x)u - DS(x)z\|_X &\leqslant \|C_x(u) - C_x(z)\|_X + \|L_x(u) - L_x(z)\|_X \\
&\leqslant \|C_x(u) - C_x(z)\|_X + 2\|L_x\|_{\mathscr{L}(X)} \\
&\leqslant \|C_x(u) - C_x(z)\|_X + 2\lambda
\end{aligned}$$

and therefore

$$\begin{aligned}
&\operatorname{dist}_X\big(DS(x)u, DS(x)B_U(0,1)\big) \leqslant \|C_x(u)-C_x(z)\|_X + 2\lambda \implies \\
&\operatorname{dist}_X\big(DS(x)u, DS(x)B_U(0,1)\big) \leqslant \operatorname{dist}_X\big(C_x(u), C_xB_U(0,1)\big) + 2\lambda \implies \\
&\operatorname{dist}_X\big(DS(x)u, DS(x)B_U(0,1)\big) \leqslant \operatorname{dist}_X\big(C_xB_X(0,1), C_xB_U(0,1)\big) + 2\lambda \implies \\
&\operatorname{dist}_X\big(DS(x)B_X(0,1), DS(x)B_U(0,1)\big) \leqslant \operatorname{dist}_X\big(C_xB_X(0,1), C_xB_U(0,1)\big) + 2\lambda,
\end{aligned}$$

implying

$$\operatorname{dist}_X\big(DS(x)B_X(0,1), DS(x)B_U(0,1)\big) < \nu + 2\lambda. \tag{2.32}$$

Since the space $DS(x)U$ is also finite-dimensional with $\dim\big(DS(x)U\big) \leqslant \tilde{N}_{\nu/2\kappa}$ we can apply Lemma 2.2.19 (Remark 2.2.20) and cover the ball $B_{DS(x)U}(0,D)$ with balls $B_X\big(y_i, \nu/2\big)$, for $1 \leqslant i \leqslant p$, such that $y_i \in B_{DS(x)U}(0,D)$ and

$$p \leqslant \left[(\tilde{N}_{\nu/2\kappa}+1)\frac{2D}{\nu}\right]^{2\tilde{N}_{\nu/2\kappa}}.$$

Hence

$$\begin{aligned}
DS(x)B_U(0,1) &\subseteq B_{DS(x)U}\big(0, \|DS(x)\|_{\mathscr{L}(X)}\big) \\
&\subseteq B_{DS(x)U}\big(0,D\big) \\
&\subseteq \bigcup_{i=1}^{p} B_X\big(y_i, \nu/2\big),
\end{aligned}$$

and we conclude that

$$DS(x)B_U(0,1) \subseteq \bigcup_{i=1}^{p} B_X\big(\tilde{y}_i, \nu\big), \tag{2.33}$$

where we choose $\tilde{y}_i \in DS(x)B_U(0,1)$. Notice that if $u \in B_X(0,1)$, from (2.32) there exists $w \in DS(x)B_U(0,1)$ such that $\|DS(x)u - w\|_X < \nu + 2\lambda$. But by (2.33), $\|w-\tilde{y}_i\|_X < \nu$ for some $i = 1, \cdots, p$ and it follows that

$$\|DS(x)u - \tilde{y}_i\|_X \leqslant \|DS(x)u - w\|_X + \|w-\tilde{y}_i\|_X < 2(\nu+\lambda),$$

concluding that

$$DS(x)B_X(0,1) \subseteq \bigcup_{i=1}^{p} B_X\big(\tilde{y}_i, 2(\nu+\lambda)\big).$$

Finally,

$$N_X\big[DS(x)B_X(0,1); 2(\nu+\lambda)\big] \leqslant \left[(\tilde{N}_{\nu/2\kappa}+1)\frac{2D}{\nu}\right]^{2\tilde{N}_{\nu/2\kappa}}, \qquad \forall x \in \mathscr{A},$$

and we have from Lemma 2.2.7 that

$$\dim_F\big(\mathscr{A}; X\big) \leqslant 2\tilde{N}_{\nu/2\kappa}\left[\frac{\ln\Big((\tilde{N}_{\nu/2\kappa}+1)\frac{2D}{\nu}\Big)}{-\ln 4(\nu+\lambda)}\right], \qquad \text{for } \nu \in \Big(0, \min\{1/4-\lambda, D\}\Big). \qquad \square$$

We note that by Theorem 2.2.16 the estimates on the fractal dimension of $\mathscr{A}$ is of order $\sim \ln N$ while by Theorem 2.2.22 we have estimates of order $\sim N \log N$. This clearly shows that a quantitative compactness (presented here as a smoothing property) instead of a qualitative compactness (and following Mañé's approach) provides better estimates on the fractal dimension of negatively invariant subsets of Banach spaces. Particularly, all these results are applied to global attractors.

2.2.7 Applications

In this section we intend to present some applications in which we can apply the theorems studied in the previous sections. We shall see an abstract semilinear parabolic problem, a 2D Navier-Stokes equation and finally a problem on a space of sequences.

2.2.7.1 Application 1: An abstract semilinear parabolic problem

For the general abstract model we treat in this section we can apply either Theorem 2.2.8 or Theorem 2.2.9 to determine that the fractal dimension of the associated attractor $\mathscr{A}$ is finite.

For the notation and terms in the following you can see (CARVALHO; LANGA; ROBINSON, 2013), Chapter 6. Let X be a Banach space, $A : D(A) \subset X \to X$ be a sectorial operator with $\operatorname{Re} \sigma(A) > a > 0$ and such that A has compact resolvent. By X^γ, with $\gamma \geqslant 0$, we represent the associated fractional power spaces of X. Now, for a fixed $\alpha \in (0,1)$, suppose $F : X^\alpha \to X$ is continuously differentiable, Lipschitz continuous in bounded subsets of X^α (with Lipschitz constant $L_{\alpha,B}$, for B a bounded subset of X^α). For $\beta \in (\alpha,1)$, note that $F : X^\beta \to X$ satisfies the same hypotheses as before with Lipschitz constant replaced by $L_{\beta,D}$, for D any bounded subset of X^β. Suppose that the semigroup $\{S(t) : X^\alpha \to X^\alpha \;\; : \;\; t \geqslant 0\}$ associated to the abstract parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = F(u), \;\; t > 0 \\ u(0) = u_0 \in X^\alpha \end{cases}$$

has a global attractor $\mathscr{A} \subset X^\beta$. By hypothesis we know that

$$\|e^{-At}\|_{\mathscr{L}(X^\alpha, X^\theta)} \leqslant c_{\theta-\alpha}\, t^{-(\theta-\alpha)} e^{-at}, \quad t > 0 \;\; , \;\; 0 \leqslant \alpha \leqslant \theta \leqslant \beta,$$

for some positive constants $c_\rho > 0$, $\rho \geqslant 0$ (for details see (HENRY, 1981), Theorem 1.4.3).

For $u \in \mathscr{A}$ we have

$$S(t)u = e^{-At}u + \int_0^t e^{-A(t-s)} F\big(S(s)u\big)\, ds \tag{2.34}$$

and differentiating this expression with respect to u, denoting it by $S_u(t)$, we obtain

$$S_u(t) = e^{-At} + \int_0^t e^{-A(t-s)} DF\big(S(s)u\big) S_u(s)\, ds. \tag{2.35}$$

So for any $t > 0$ and for any $v \in X^\alpha$ we have

$$\|S_u(t)v\|_{X^\beta} \leqslant \frac{c_{\beta-\alpha}}{t^{\beta-\alpha}} \|v\|_{X^\alpha} + \int_0^t \frac{c_\beta N}{(t-s)^\beta} \|S_u(s)v\|_{X^\beta}\, ds,$$

where $N := \sup_{u\in\mathscr{A}} \left\{ \|DF(u)\|_{\mathscr{L}(X^\beta,X)} \right\}$.

By a Volterra's inequality (see (CHOLEWA; DLOTKO, 2000), formulas (1.2.21) and (1.2.30) in Lemma 1.2.9) we obtain for t_0 satisfying

$$\frac{c_\beta N t_0^{1-\beta}}{2^{\alpha-2\beta}}\left(\frac{1}{1-\beta+\alpha}+\frac{1}{1-\beta}\right)=1 \tag{2.36}$$

that

$$\|S_u(t_0)v\|_{X^\beta} \leqslant 2c_{\beta-\alpha}t_0^{\alpha-\beta}\|v\|_{X^\alpha}, \qquad \text{for all } v\in X^\alpha, \tag{2.37}$$

and this is precisely the smoothing property corresponding to the compact embedding (since the operator A has compact resolvent) of X^β into X^α with

$$\kappa := 2c_{\beta-\alpha}t_0^{\alpha-\beta}.$$

Note that $\kappa>0$ is uniform with respect to $\mathscr{A}$. Then, applying Theorem 2.2.9 to $S := S(t_0)$ it follows that

$$\dim_F(\mathscr{A};X^\alpha) \leqslant \frac{\ln N_{\nu/2\kappa}}{-\ln 4\nu}, \qquad \text{for all } \nu\in\left(0,\frac{1}{4}\right).$$

Besides that, condition (2.37) implies S is Lipschitz from $\mathscr{A}$ to X^β and arguing as in (2.21) we have

$$\dim_F(\mathscr{A};X^\beta)=\dim_F(\mathscr{A};X^\alpha).$$

Moreover, following steps in Corollary 3.4 in (CARVALHO; LANGA; ROBINSON, 2010) we say that operator A is an *admissible sectorial operator* if it is a sectorial operator and there are a sequence of finite rank projections $\{P_n\}_{n\in\mathbb{N}}$, a sequence of positive real numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_{n+1}\geqslant\lambda_n$ and $\lambda_n\to\infty$ as $n\to\infty$, and $M>0$ such that

$$\|e^{-At}(I-P_n)\|_{\mathscr{L}(X^\alpha,X^\theta)} \leqslant Mt^{-(\theta-\alpha)}e^{-\lambda_n t}, \qquad t>0,\ \ 0\leqslant\alpha\leqslant\theta\leqslant\beta.$$

Then from (2.35) we have (choose $M\geqslant c_0$)

$$\|S_u(t)\|_{\mathscr{L}(X^\alpha)} \leqslant M + c_\alpha N_\alpha \int_0^t (t-s)^{-\alpha}\|S_u(s)\|_{\mathscr{L}(X^\alpha)}\,ds,$$

where $N_\alpha := \sup_{u\in\mathscr{A}}\left\{\|DF(u)\|_{\mathscr{L}(X^\alpha,X)}\right\}$, and by a Gronwall's inequality [2] we obtain

$$\|S_u(t)\|_{\mathscr{L}(X^\alpha)} \leqslant 2Me^{(2c_\alpha N_\alpha\Gamma(1-\alpha))^{1/(1-\alpha)}t},$$

[2] [Singular Gronwall Lemma, see (CARVALHO; LANGA; ROBINSON, 2013), Lemma 6.24] Suppose that $f\in L^1\big((0,\infty);\mathbb{R}^+\big)$ satisfies

$$f(t)\leqslant a+b\int_0^t (t-s)^{-(1-\alpha)}f(s)\,ds, \qquad t\in(0,\infty),$$

for some $\alpha\in(0,1]$ and $a,b\in(0,\infty)$. Then

$$f(t)\leqslant 2ae^{(2b\Gamma(\alpha))^{1/\alpha}t},$$

where $\Gamma(x) := \displaystyle\int_0^\infty e^{-s}s^{x-1}\,ds$ is the gamma function.

where

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1}\, dt \tag{2.38}$$

is the gamma function.

Now if $Q_n := I - P_n$ then by (2.35) and the fact that (use the change of variables $t = as$ in (2.38))

$$\int_0^\infty r^b e^{-ar}\, dr = \frac{\Gamma(1+b)}{a^{1+b}}, \qquad a > 0,\ b \in \mathbb{R},$$

we have

$$\begin{aligned}
&\|Q_n S_u(t)\|_{\mathscr{L}(X^\alpha)} \leqslant \\
&\quad \leqslant M e^{-\lambda_n t} + N_\alpha M \int_0^t (t-s)^{-\alpha} e^{-\lambda_n (t-s)} \|S_u(s)\|_{\mathscr{L}(X^\alpha)}\, ds \\
&\quad \leqslant M e^{-\lambda_n t} + 2N_\alpha M^2 \int_0^t (t-s)^{-\alpha} e^{-\lambda_n (t-s)} e^{(2c_\alpha N_\alpha \Gamma(1-\alpha))^{1/(1-\alpha)} s}\, ds \\
&\quad = M e^{-\lambda_n t} + 2N_\alpha M^2 e^{(2c_\alpha N_\alpha \Gamma(1-\alpha))^{1/(1-\alpha)} t} \int_0^t (t-s)^{-\alpha} e^{-(\lambda_n + (2c_\alpha N_\alpha \Gamma(1-\alpha))^{1/(1-\alpha)})(t-s)}\, ds \\
&\quad = M e^{-\lambda_n t} + 2N_\alpha M^2 e^{(2c_\alpha N_\alpha \Gamma(1-\alpha))^{1/(1-\alpha)} t} \int_0^t r^{-\alpha} e^{-(\lambda_n + (2c_\alpha N_\alpha \Gamma(1-\alpha))^{1/(1-\alpha)}) r}\, dr \\
&\quad = M e^{-\lambda_n t} + 2N_\alpha M^2 e^{(2c_\alpha N_\alpha \Gamma(1-\alpha))^{1/(1-\alpha)} t} \frac{\Gamma(1-\alpha)}{\left[\lambda_n + (2c_\alpha N_\alpha \Gamma(1-\alpha))^{1/(1-\alpha)}\right]^{1-\alpha}} \\
&\quad =: \Lambda_n(t).
\end{aligned}$$

Choose $t = 1$ and note that since $\lambda_n \to \infty$ then given $0 < \lambda < 1/4$ there is $n_\lambda \in \mathbb{N}$ such that

$$\|Q_{n_\lambda} S_u(1)\|_{\mathscr{L}(X^\alpha)} \leqslant \Lambda_{n_\lambda}(1) \leqslant \lambda, \qquad \forall u \in \mathscr{A}.$$

Finally, $S_u(1) = P_{n_\lambda} S_u(1) + Q_{n_\lambda} S_u(1)$ and since $P_{n_\lambda} S_u(1)$ is a compact operator we can apply Theorem 2.2.8 and then guarantee that $\mathscr{A}$ has finite fractal dimension in X^α.

2.2.7.2 Application 2: 2D Navier–Stokes equations

In this section we show how the smoothing property can be used to bound the dimension of the attractor for the two-dimensional Navier-Stokes equations. For a bounded smooth domain $\mathscr{O} \subset \mathbb{R}^2$, let H and V be the closure of the set

$$\mathscr{V} := \left\{ v \in \left(C_0^\infty(\mathscr{O})\right)^2 \,:\, \nabla \cdot v = 0 \right\}$$

over spaces $\left(L^2(\mathscr{O})\right)^2$ and $\left(H_0^1(\mathscr{O})\right)^2$, respectively. We use $\|\cdot\|$ for the L^2 norm and $(\cdot,\cdot)$ for its inner product. Let A be the Stokes operator defined by

$$Av = -P\Delta v, \qquad \forall v \in D(A) = \left(H^2(\mathscr{O})\right)^2 \cap V,$$

where P is the orthogonal projection onto the divergence-free fields H.

Defining the trilinear form

$$b(u,v,w) := \sum_{i,j=1}^{2} \int_{\mathscr{O}} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,$$

whenever the integrals make sense, we consider the following problem

$$\frac{\partial u}{\partial t} + \mu Au + B(u,u) = f(x), \tag{2.39}$$

$$\nabla \cdot u = 0, \quad u|_{\partial \mathscr{O}} = 0, \quad u|_{t=0} = u_0, \tag{2.40}$$

where $x = (x_1, x_2) \in \mathscr{O}$, $t \geqslant 0$, the unknown $u = u(x,t) = \big(u^1(x,t), u^2(x,t)\big)$ is a velocity vector, $u_0 \in H$, $\mu > 0$ is a constant, $f(x) = \big(f^1(x), f^2(x)\big)$ and $B : V \times V \to V'$ is the bilinear operator defined by

$$\big(B(u,v), w\big) = b(u,v,w), \quad \forall u,v,w \in V.$$

For more details on this standard setting see (TEMAM, 1988) or (ROBINSON, 2001), for example. These equations generate a semigroup $\{S(t)\}_{t \geqslant 0}$ on the space H of divergence-free functions. We use the norm $\|\nabla u\|$ on the space V.

Estimates for this equation are usually given in terms of the quantity

$$G := \|f\|/(\mu^2 \lambda_1),$$

where λ_1 is the first eingenvalue of the laplacian. Although the dimension we will obtain for the attractor here is $\sim G^4$, which is worse than the best known estimates ($\sim G^{2/3}(1+\ln G)^{1/3}$ in the periodic case and $\sim G$ for bounded domains), this polynomial estimate requires only the relatively simple bounds from this section rather than the full Hilbert-space theory in Temam (TEMAM, 1988).

First we recall the following estimates for solutions on the attractor (see (ROBINSON, 2001), Chapter 12) in which we have for some constant $c > 0$ that

$$\|u\|^2 \leqslant c\mu^2 G^2 \qquad \text{and} \qquad \|\nabla u\|^2 \leqslant c\mu^2 \lambda_1 G^2, \qquad u \in \mathscr{A}. \tag{2.41}$$

Theorem 2.2.24. *If $S(t)$ denotes the time-t map of the semigroup generated by the 2D Navier–Stokes equations on H then there exists a time $t_0 > 0$ such that $S := S(t_0)$ satisfies the smoothing property*

$$\|Su_0 - Sv_0\|_{H^1} \leqslant \kappa \|u_0 - v_0\|, \qquad u_0, v_0 \in \mathscr{A},$$

where $\kappa = c\mu^{1/2}\lambda_1^{1/2} G^2$. Consequently

$$\dim_F(\mathscr{A}; H) \leqslant c\mu\lambda_1 G^4.$$

Proof. The equation for the difference $w = u - v$ of two solutions of (2.39) is

$$\frac{\partial w}{\partial t} + \mu Aw + B(u,w) + B(w,v) = 0. \tag{2.42}$$

Take the inner product with w to give (using the fact that $\big(B(u,w),w\big)=0$ and the Ladyzhenskaya's inequality $\|w\|_{L^4}\leq c\|w\|^{1/2}\|\nabla w\|^{1/2}$, see (ROBINSON, 2001), Lemma 5.27)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2+\mu\|\nabla w\|^2 &\leqslant -(B(w,v),w)\\ &\leqslant \int_{\mathscr{O}}|w||\nabla v||w|\\ &\leqslant \|w\|_{L^4}^2\|\nabla v\|\\ &\leqslant c\|w\|\|\nabla w\|\|\nabla v\|\\ &\leqslant \frac{c}{2\mu}\|\nabla v\|^2\|w\|^2+\frac{\mu}{2}\|\nabla w\|^2,\end{aligned}$$

where we have used a Young's inequality. [3] So

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2+\mu\|\nabla w\|^2\leqslant\frac{c}{\mu}\|\nabla v\|^2\|w\|^2\leqslant c\mu\lambda_1G^2\|w\|^2.$$

Drop the second term on the left-hand side, and integrate from $t=0$ to s to obtain

$$\|w(s)\|^2\leqslant e^{c\mu\lambda_1G^2s}\|w(0)\|^2.$$

Now use this to integrate again from $t=0$ to t^*, where $t^*=1/(\mu\lambda_1G^2)$:

$$\begin{aligned}\|w(t^*)\|^2+\mu\int_0^{t^*}\|\nabla w(s)\|^2ds &\leqslant\left[c\mu\lambda_1G^2\int_0^{t^*}e^{c\mu\lambda_1G^2s}ds\right]\|w(0)\|^2\\ &\leqslant c\|w(0)\|^2,\end{aligned}$$

with $c>0$ independent of G. From this we take the estimate

$$\mu\int_0^{t^*}\|\nabla w(s)\|^2ds\leqslant c\|w(0)\|^2. \tag{2.43}$$

Now take the inner product of (2.42) with Aw to give (using Ladyzhenskaya's inequality and Agmon's inequality $\|w\|_{L^\infty}\leqslant c\|w\|^{1/2}\|Aw\|^{1/2}$, see (TEMAM, 1988), page 50)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2+\mu\|Aw\|^2 &\leqslant\int_{\mathscr{O}}\Big(|u||\nabla w||Aw|+|w||\nabla v||Aw|\Big)\\ &\leqslant\|u\|_{L^4}\|\nabla w\|_{L^4}\|Aw\|+c\|w\|^{1/2}\|Aw\|^{1/2}\int_{\mathscr{O}}|\nabla v||Aw|\\ &\leqslant c\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla w\|^{1/2}\|Aw\|^{1/2}\|Aw\|+c\|w\|^{1/2}\|Aw\|^{1/2}\|\nabla v\|\|Aw\|\\ &=c\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla w\|^{1/2}\|Aw\|^{3/2}+c\|w\|^{1/2}\|\nabla v\|\|Aw\|^{3/2}\\ &\leqslant\frac{\mu}{4}\|Aw\|^2+\frac{c}{\mu^3}\|u\|^2\|\nabla u\|^2\|\nabla w\|^2+\frac{\mu}{4}\|Aw\|^2+\frac{c}{\mu^3}\|w\|^2\|\nabla v\|^4\\ &=\frac{\mu}{2}\|Aw\|^2+\frac{c}{\mu^3}\|u\|^2\|\nabla u\|^2\|\nabla w\|^2+\frac{c}{\mu^3}\|w\|^2\|\nabla v\|^4.\end{aligned}$$

[3] [Young's inequality, see (KREYSZIG, 1978), pages 12-13] Let $a,b>0$ and $p,q>1$, with $\frac{1}{p}+\frac{1}{q}=1$. Given $\varepsilon>0$ we obtain (denoting $c_\varepsilon:=(\varepsilon p)^{-q/p}/q$) the inequality $ab\leqslant\varepsilon a^p+c_\varepsilon b^q$. The most frequently case is $p=q=2$, in which we have $ab\leqslant\varepsilon a^2+\frac{b^2}{4\varepsilon}$

Since $\|w\|^2 \leqslant \lambda_1^{-1}\|\nabla w\|^2$ (Poincaré's inequality) this gives using (2.41)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2+\|Aw\|^2 &\leqslant \frac{c}{\mu^3}\big(\|u\|^2\|\nabla u\|^2+\lambda_1^{-1}\|\nabla v\|^4\big)\|\nabla w\|^2\\ &\leqslant \frac{c}{\mu^3}\big(\mu^4\lambda_1 G^4\big)\|\nabla w\|^2\\ &= c\mu\lambda_1 G^4\|\nabla w\|^2.\end{aligned}$$

Now integrate from $t=s$ to $t=t^*$ with $0\leqslant s\leqslant t^*$ to give

$$\|\nabla w(t^*)\|^2 \leqslant \|\nabla w(s)\|^2 + c\mu\lambda_1 G^4\int_s^{t^*}\|\nabla w(\tau)\|^2\,d\tau,$$

and then integrate again with respect to s from $s=0$ to $s=t^*$ to give

$$\begin{aligned}t^*\|\nabla w(t^*)\|^2 &\leqslant \int_0^{t^*}\|\nabla w(s)\|^2\,ds + c\mu\lambda_1 G^4\int_0^{t^*}\int_s^{t^*}\|\nabla w(\tau)\|^2\,d\tau\,ds\\ &\leqslant \big(1+c\mu\lambda_1 G^4 t^*\big)\int_0^{t^*}\|\nabla w(s)\|^2\,ds\\ &\leqslant c\big(1+c\mu\lambda_1 G^4 t^*\big)\|w(0)\|^2. \qquad \text{(by (2.43))}\end{aligned}$$

So

$$\|\nabla w(t^*)\|^2 \leqslant \Big(\frac{c}{t^*}+c\mu\lambda_1 G^4\Big)\|w(0)\|^2$$

and since $t^*=(\mu\lambda_1 G^2)^{-1}$ this is

$$\|\nabla w(t^*)\|^2 \leqslant \big(c\mu\lambda_1 G^2 + c\mu\lambda_1 G^4\big)\|w(0)\|^2,$$

which for G large this gives

$$\|\nabla w(t^*)\|^2 \leqslant \big[c\mu\lambda_1 G^4\big]\,\|w(0)\|^2.$$

This is the (H,V)-smoothing estimate that we need, with $\kappa = c\mu^{1/2}\lambda_1^{1/2}G^2$.

Since the smoothing estimate is from L^2 into H^1, this means from (2.22) ($d=2$, $l_1=1$, $l_2=0$) that $\gamma=2$. Therefore the dimension bound in (2.25) is of the order of κ^2, i.e. of the order of G^4. □

2.2.7.3 Application 3: A problem on a space of sequences

Let $S:\ell^2\to\ell^2$ be a general nonlinear Lipschitz map in ℓ^2 with Lipschitz constant $M>0$ and with a global attractor (you can take for example S as the introductory example in Section 2.2), and for a sequence $\{\lambda_j\}_{j=1}^{\infty}\subset\mathbb{R}$ with $|\lambda_j|\leqslant 1$, $|\lambda_j|\geqslant|\lambda_{j+1}|$ and $a_1 j^{-1}\leqslant|\lambda_j|\leqslant a_2 j^{-1}$ for constants $a_1,a_2>0$, define the bounded linear operator $A:\ell^2\to h^1$ by

$$(A\mathbf{x})_j := \lambda_j x_j,$$

where h^1 is the subspace of ℓ^2 endowed with the norm

$$\|\mathbf{x}\|_1^2 = \sum_{j=1}^{\infty} j|x_j|^2.$$

Then setting $T := A \circ S : \ell^2 \to h^1$, it has a global attractor $\mathscr{A}$ and satisfies the smoothing condition

$$\|T\mathbf{x} - T\mathbf{y}\|_1 \leqslant a_2 M \|\mathbf{x} - \mathbf{y}\|_2, \qquad \mathbf{x}, \mathbf{y} \in \ell^2,$$

so by Theorem 2.2.14 and expression (2.25) we obtain the estimate

$$\dim_F(\mathscr{A}; \ell^2) = \dim_F(\mathscr{A}; h^1) \leqslant c a_2 M,$$

since it holds (with $\gamma = 1$) the Kolmogorov estimate $\mathbf{H}_\varepsilon(h^1, \ell^2) \leqslant c_2 \varepsilon^{-1}$, for all $\varepsilon > 0$, and some positive constant $c_2 > 0$.

On the other hand we can proceed as in the following. With $S : \ell^2 \to \ell^2$ and $A : \ell^2 \to \ell^2$ as before, but this time with S continuously differentiable, given $0 < \varepsilon < 1/(8M)$ we choose a sufficiently large $j_\varepsilon \in \mathbb{N}$ such that $|\lambda_j| < \varepsilon$ for all $j > j_\varepsilon$ and split A as $A = A_1 + A_2$, where

$$A_1 \mathbf{x} = \begin{cases} x_j & , j \leqslant j_\varepsilon \\ 0 & , j > j_\varepsilon \end{cases}$$

is a finite rank operator (with dimension j_ε) and A_2 is a contraction with $\|A_2\|_{\mathscr{L}(\ell^2)} < \varepsilon$. Then setting again $T := A \circ S : \ell^2 \to \ell^2$, it has a global attractor $\mathscr{A}$ and $T = T_1 + T_2$, with T_1 a finite rank mapping (with dimension j_ε) and T_2 a contraction such that

$$\|T_2\mathbf{x} - T_2\mathbf{y}\|_2 < 1/8 \|\mathbf{x} - \mathbf{y}\|_2, \qquad \mathbf{x}, \mathbf{y} \in \ell^2.$$

By Corollary 2.2.13 (see also Theorem 2.2.8) we conclude that $\mathscr{A}$ has finite fractal dimension in ℓ^2.

CHAPTER

3

NON-AUTONOMOUS DYNAMICAL SYSTEMS

In this chapter we introduce the theory of non-autonomous dynamical systems with emphasis on the fractal dimension of uniform attractors. As in the autonomous setting, uniform attractors are the indicated objects to describe the dynamics of a solution of non-autonomous evolution equations with a driving system, trying to capture its information in the future. Actually, the theory of semigroups is understood as a particular case of the theory of non-autonomous dynamical systems and in many aspects the notions and definitions follow the same structure.

We want in this chapter to obtain estimates on the fractal dimension of uniform attractors in Banach spaces. For that, we are based again on a smoothing condition for a family of evolution processes, and the finite-dimensionality of the driving system is required. Besides that we find possible, yet under a smoothing condition, to obtain estimates on the fractal dimension of uniform attractors in more regular spaces. Estimates in both cases are given in terms of a Kolmogorov entropy (related to the smoothing property) plus the fractal dimension of the symbol space.

The condition involved with the symbol space (we mean its finite-dimensionality) is a technical problem which is usually difficult to hold in applications. Up to this point the symbol space was considered basically as a hull of a quasiperiodic function on the space of bounded continuous functions, what restricted the number of problems in which we could apply the theory. Now we are able to construct new symbol spaces with finite fractal dimension on the more general space of continuous functions with a Fréchet metric. More precisely, we are considering hulls of Lipschitz functions eventually exponentially converging to quasiperiodic ones. It allows us for example to consider more general non-autonomous terms in problems.

Finally, as application of our theoretical results we study two problems. First a non-autonomous 2D Navier-Stokes equation is taken into account and we prove the finite-dimensionality of its uniform attractor in spaces L^2 and H_0^1. We also consider a non-autonomous reaction-diffusion equation obtainig estimates on the fractal dimension of the uniform attractor in L^2 and L^p, with $p > 2$.

This chapter is organized into four sections as follows. First in Section 3.1 we introduce the theory of non-autonomous dynamical systems and uniform attractor via a skew-product approach. This is a compilation of results which can be found in more details in (CHEPYZHOV; VISHIK, 2002). Then in Section 3.2 we obtain estimates on the fractal dimension of uniform attractors based on a smoothing condition. In Section 3.3 we revise the construction of symbol spaces with finite fractal dimension on the space of bounded continuous functions and construct new symbol spaces with finite-dimensionality on the space of continuous functions. Lastly, in Section 3.4 we estimate the dimension of uniform attractors for a 2D Navier-Stokes problem and a reaction-diffusion equation. The last three sections, in great part, are summarized in (CUI *et al.*,).

3.1 Non-autonomous dynamical systems: uniform attractors

In Chapter 2 we described the theory of autonomous dynamical systems and showed how it is related to autonomous evolution equations. Now we are interested in studying a family of mappings which describe the dynamics for non-autonomous evolution equations. The aim is to somehow generalize the concepts already studied previously in Section 2.1, giving a mean for what we call the uniform attractor. See (CHEPYZHOV; VISHIK, 2002) for a complete treatment of this subject.

3.1.1 Evolution processes and kernel sections

Let $\big(X, \|\cdot\|_X\big)$ be a Banach space and let $\{U(t,s) : t \geqslant s,\ t,s \in \mathbb{R}\}$ be a two-parameter set of mappings acting on X, i.e., $U(t,s) : X \to X$ for all real numbers $t,s \in \mathbb{R}$ with $t \geqslant s$. Throughout the text we denote $\{U(t,s) : t \geqslant s,\ t,s \in \mathbb{R}\}$ simply by $\{U(t,s)\}$, keeping in mind that $t \geqslant s$. In the following we shall give the basic setting and definitions which are involved with non-autonomous dynamical systems, including the notions of attraction, absorption and kernels.

Definition 3.1.1. *We say that* $\{U(t,s)\}$ *is an* evolution process *(or briefly a* process*) in* X *if it satisfies*

i) $U(s,s) = Id_X$*, for all* $s \in \mathbb{R}$*;*

ii) $U(t,\tau)U(\tau,s) = U(t,s)$*, for all* $t \geqslant \tau \geqslant s$*, with* $t,\tau,s \in \mathbb{R}$*;*

iii) For each $t \geqslant s$ *fixed, the mapping* $U(t,s) : X \to X$ *is continuous.*

Evolution processes will be called sometimes simply by *processes*, without any loss of meaning.

Remark 3.1.2. *Notice that on one hand if* $\{S(t)\}_{t\geqslant 0}$ *is a semigroup then defining a family of mappings* $\{U(t,s)\}$ *as*

$$U(t,s) := S(t-s), \qquad \forall t \geqslant s,$$

we clearly see that $\{U(t,s)\}$ *is an evolution process in* X*. On the other hand, if an evolution process* $\{U(t,s)\}$ *is such that* $U(t,s) = U(t-s,0)$ *for all* $t \geqslant s$*, then the family* $\{S(t)\}_{t\geqslant 0}$ *defined by* $S(t) = U(t,0)$ *is a semigroup, because for all* $t,s \geqslant 0$ *we have*

$$S(t+s) = U(t+s,0) = U(t+s,s)U(s,0) = U(t,0)U(s,0) = S(t)S(s).$$

That indicates to us the most relevant difference between autonomous and non-autonomous dynamical systems: the dependence on initial time. While for autonomous dynamical systems the initial time is useless, for non-autonomous processes it plays an essential role.

For evolution processes we have similar notions of *attraction* and *absorption*. Remember the Hausdorff semi-distance $\operatorname{dist}_X(A,B) := \sup\limits_{a\in A}\inf\limits_{b\in B}\|a-b\|_X$, for $A,B \in 2^X\backslash\emptyset$.

Definition 3.1.3. *Let* $\{U(t,s)\}$ *be an evolution process in* X*. A subset* $\mathscr{B} \subseteq X$ *is said to be an* attracting set *for* $\{U(t,s)\}$ *if for any* $\tau \in \mathbb{R}$ *and* $B \subseteq X$ *bounded it holds*

$$\operatorname{dist}_X\big(U(t,\tau)B,\mathscr{B}\big) \to 0, \qquad \text{as } t \to \infty. \tag{3.1}$$

Definition 3.1.4. *Let* $\{U(t,s)\}$ *be an evolution process in* X*. A subset* $\mathscr{B} \subseteq X$ *is called an* absorbing set *for* $\{U(t,s)\}$ *if given* $\tau \in \mathbb{R}$ *and* $B \subseteq X$ *bounded, there exists a time* $t_0 = t_0(\tau,B) \geqslant \tau$ *such that*

$$U(t,\tau)B \subseteq \mathscr{B}, \qquad \forall t \geqslant t_0.$$

Remark 3.1.5. *Clearly absorbing sets are attracting sets. Conversely, if* $\mathscr{B}$ *is an attracting set then given* $\varepsilon > 0$*,* $\tau \in \mathbb{R}$ *and* $B \subseteq X$ *bounded there exists* $t_0 = t_0(\varepsilon,\tau,B) \geqslant \tau$ *such that*

$$U(t,\tau)B \subseteq B_X\big(\mathscr{B},\varepsilon\big) := \bigcup_{x\in\mathscr{B}} B_X(x,\varepsilon), \qquad \forall t \geqslant t_0,$$

proving that the ε*-neighborhood of* $\mathscr{B}$ *is an absorbing set for* $\{U(t,s)\}$*. More generally, any neighborhood of* $\mathscr{B}$ *is an absorbing set.*

Another previously defined concept is stated in this setting: *global solutions*. As before it tells us some information about the structure of the attrator (the uniform attractor, that we are going to define later) and it is also related to the *kernel* of processes.

Definition 3.1.6. *Let* $\{U(t,s)\}$ *be a process. A mapping* $u : \mathbb{R} \to X$ *taking values in* X *is a* solution *for the process* $\{U(t,s)\}$ *if*

$$U(t,s)u(s) = u(t), \qquad \forall t \geqslant s.$$

In the following we define the kernel and the kernel sections of a process.

Definition 3.1.7. *Let $\{U(t,s)\}$ be a process.*

1. *The* kernel *of process $\{U(t,s)\}$ is defined as*

$$\mathscr{K} := \big\{u(\cdot) : u(\cdot) \text{ is a bounded solution for } U(t,s)\big\}.$$

2. *The* kernel section at moment *s for process $\{U(t,s)\}$ is defined as*

$$\mathscr{K}(s) := \{u(s) : u(\cdot) \in \mathscr{K}\}.$$

Remark 3.1.8. *It is immediate that the kernel sections satisfy the following invariance property:*

$$U(t,s)\mathscr{K}(s) = \mathscr{K}(t), \qquad \forall t \geqslant s.$$

3.1.2 Uniform attractors

Let (Ξ, d_Ξ) be a complete metric space and let $\{\theta_s\}_{s\in\mathbb{R}}$ be a group of continuous operators acting on Ξ, i.e., $\theta_0\sigma = \sigma$ and $\theta_t(\theta_s\sigma) = \theta_{t+s}\sigma$ for all $\sigma \in \Xi$, $t,s \in \mathbb{R}$, and for each $s \in \mathbb{R}$, $\theta_s : \Xi \to \Xi$ is a continuous mapping on Ξ. Let $\Sigma \subseteq \Xi$ be a *compact* subset of Ξ which is invariant under $\{\theta_s\}_{s\in\mathbb{R}}$, i.e., $\theta_s\Sigma = \Sigma$ for all $s \in \mathbb{R}$. The operators θ_s in applications are usually defined as the translations

$$\theta_s\sigma(\cdot) = \sigma(\cdot + s), \quad \forall s \in \mathbb{R},$$

for time-dependent functions σ.

Let us consider now a collection of evolution processes $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ indexed by $\sigma \in \Sigma$, i.e., each $\{U_\sigma(t,s)\}$ is an evolution process in X accordingly to Definition 3.1.1.

Definition 3.1.9. *A collection $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is called* a system of evolution processes *(or simply* a system*) if the* translation-identity

$$U_{\theta_h\sigma}(t,s) = U_\sigma(t+h, s+h), \quad \forall \sigma \in \Sigma, t \geqslant s, h \in \mathbb{R}, \tag{3.2}$$

is satisfied. In this case, the parameter σ is called the symbol *of the process $\{U_\sigma(t,s)\}$ and the set Σ the* symbol space *of the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$.*

Remark 3.1.10. *The translation-identity* (3.2) *indicates that any translation of initial times of a process is equivalent to a corresponding translation of the symbol. It is satisfied if the underlying non-autonomous evolution equation has unique solutions, and we will discuss it in Section 3.1.6.*

Analogously to Definition 3.1.3 and Definition 3.1.4 we can consider uniformly attracting and absorbing sets for a system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$.

Definition 3.1.11. *Let $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a system of processes. A set $\mathscr{B} \subseteq X$ is a* uniformly attracting *set for the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ if for any $\tau \in \mathbb{R}$ and any $B \subset X$ bounded we have*

$$\sup_{\sigma\in\Sigma} \operatorname{dist}_X\big(U_\sigma(t,\tau)B, \mathscr{B}\big) \to 0, \qquad \text{as } t \to \infty.$$

Definition 3.1.12. *Let $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a system of processes. A set $\mathscr{B}$ is a* uniformly absorbing *set for the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ if for any $\tau\in\mathbb{R}$ and any $B\subseteq X$ bounded there exists $t_1 = t_1(\tau,B)\geqslant\tau$ such that*

$$\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B\subseteq\mathscr{B},\qquad \forall t\geqslant t_1.$$

The *uniform attractor* for such systems of processes is defined as follows.

Definition 3.1.13. *A compact subset $\mathscr{A}_\Sigma$ of X is said to be the* uniform (w.r.t. $\sigma\in\Sigma$) attractor *of a system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ if*

(i) $\mathscr{A}_\Sigma$ is uniformly attracting, i.e., for any $\tau\in\mathbb{R}$ and any bounded subset $B\subset X$ it holds

$$\sup_{\sigma\in\Sigma}\operatorname{dist}_X\big(U_\sigma(t,\tau)B,\mathscr{A}_\Sigma\big)\to 0,\qquad \text{as } t\to\infty; \tag{3.3}$$

(ii) (Minimality) If $\mathscr{A}'_\Sigma$ is a closed subset of X uniformly attracting, then $\mathscr{A}_\Sigma\subseteq\mathscr{A}'_\Sigma$.

Remark 3.1.14. *Notice that unlike global attractors, uniform attractors are not supposed to be invariant, and so in order to guarantee a uniqueness for it we consider instead the minimality condition. The lack of invariance for $\mathscr{A}_\Sigma$ is a problem that has to be overcome when estimating the fractal dimension of it.*

Note that, since the symbol space Σ is invariant under translations, by the translation-identity (3.2) we have

$$\begin{aligned}\sup_{\sigma\in\Sigma}\operatorname{dist}_X\big(U_\sigma(t,0)B,\mathscr{A}_\Sigma\big) &= \sup_{\sigma\in\Sigma}\operatorname{dist}_X\big(U_{\theta_\tau\sigma}(t,0)B,\mathscr{A}_\Sigma\big)\\ &= \sup_{\sigma\in\Sigma}\operatorname{dist}_X\big(U_\sigma(t+\tau,\tau)B,\mathscr{A}_\Sigma\big),\quad \forall\tau\in\mathbb{R}.\end{aligned}$$

Hence, the uniformly attracting property (3.3) is equivalent to

$$\sup_{\sigma\in\Sigma}\operatorname{dist}_X\big(U_\sigma(t,0)B,\mathscr{A}_\Sigma\big)\to 0,\qquad \text{as } t\to\infty.$$

Clearly, if a system of processes $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ has a uniform attractor $\mathscr{A}_\Sigma$ then any neighborhood $\mathscr{B}$ of $\mathscr{A}_\Sigma$ is a uniformly absorbing set: for any $B\subseteq X$ bounded, there exists a time $t_0 = t_0(B)\geqslant 0$ such that

$$\bigcup_{\sigma\in\Sigma} U_\sigma(t,0)B\subseteq\mathscr{B},\quad \forall t\geqslant t_0.$$

We are interested in determining classes of systems for which there exists a uniform attractor, and as in the autonomous setting we begin with a presentation of ω-limit sets in the following section.

3.1.3 ω-limit sets

In this section we define ω-limit sets for system of processes. It is natural that these sets should be taken for fixed initial times and uniformly with respect to $\sigma \in \Sigma$.

Definition 3.1.15. *Let $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a system and $B \subseteq X$. The* uniform ω-limit set of B with origin at τ *for the system is defined as*

$$\omega_{\tau,\Sigma}(B) := \bigcap_{t\geqslant\tau} \left[\overline{\bigcup_{\sigma\in\Sigma} \bigcup_{s\geqslant t} U_\sigma(s,\tau)B}^X \right].$$

Next lemma gives us a characterization for ω-limit sets and it is the analogous of Lemma 2.1.19 for autonomous dynamical systems. For any $\tau \in \mathbb{R}$, denote $\mathbb{R}^+_\tau := \{t \in \mathbb{R} : t \geqslant \tau\}$.

Lemma 3.1.16. *Given $\tau \in \mathbb{R}$ and $B \subseteq X$ we have*

$$\omega_{\tau,\Sigma}(B) = \Big\{ x \in X : \textit{there are sequences } \{x_n\} \subseteq B, \{\sigma_n\} \subseteq \Sigma, \{t_n\} \subseteq \mathbb{R}^+_\tau \textit{ with } t_n \to \infty \\ \textit{such that } \lim_{n\to\infty} U_{\sigma_n}(t_n,\tau)x_n = x \Big\}.$$

Proof. For $\tau \in \mathbb{R}$ and $B \subseteq X$ denote

$$\omega'_{\tau,\Sigma}(B) := \Big\{ x \in X : \text{there are sequences } \{x_n\} \subseteq B, \{\sigma_n\} \subseteq \Sigma, \{t_n\} \subseteq \mathbb{R}^+_\tau \text{ with } t_n \to \infty \\ \text{such that } \lim_{n\to\infty} U_{\sigma_n}(t_n,\tau)x_n = x \Big\}.$$

Let $x \in \omega_{\tau,\Sigma}(B)$. So for each $n \in \mathbb{N}$ with $n \geqslant \tau$ we have

$$x \in \overline{\bigcup_{\sigma\in\Sigma} \bigcup_{s\geqslant n} U_\sigma(s,\tau)B}$$

and consequently there is $z_n \in \bigcup_{\sigma\in\Sigma}\bigcup_{s\geqslant n} U_\sigma(s,\tau)B$ such that $\|x - z_n\|_X < 1/n$. Moreover, $z_n = U_{\sigma_n}(t_n,\tau)x_n$, where $x_n \in B$, $\sigma_n \in \Sigma$ and $t_n \geqslant n$, and we conclude that $x = \lim_{n\to\infty} U_{\sigma_n}(t_n,\tau)x_n$. Hence $x \in \omega'_{\tau,\Sigma}(B)$.

Now let $x \in \omega'_{\tau,\Sigma}(B)$. Then there are sequences $\{x_n\} \subseteq B$, $\{\sigma_n\} \subseteq \Sigma$ and $\{t_n\} \subseteq \mathbb{R}^+_\tau$ with $t_n \to \infty$ and such that $x = \lim_{n\to\infty} U_{\sigma_n}(t_n,\tau)x_n$. Given $t \geqslant \tau$ let $n_t \in \mathbb{N}$ be such that for any $n \geqslant n_t$ we have $t_n \geqslant t$. Hence for any $n \geqslant n_t$ it holds $U_{\sigma_n}(t_n,\tau)x_n \in \bigcup_{\sigma\in\Sigma}\bigcup_{s\geqslant t} U_\sigma(s,\tau)B$, and then

$$x \in \overline{\bigcup_{\sigma\in\Sigma} \bigcup_{s\geqslant t} U_\sigma(s,\tau)B}, \qquad \forall t \geqslant \tau.$$

Therefore $x \in \omega_{\tau,\Sigma}(B)$. □

A class of system for which there exist uniform attractors is the class of *uniformly asymptotically compact* systems of processes, defined in the following.

Definition 3.1.17. *A system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is* uniformly asymptotically compact *if there exists a compact uniformly attracting set $K\subset X$, i.e., K is compact and given $\tau\in\mathbb{R}$ and $B\subset X$ bounded it holds*

$$\sup_{\sigma\in\Sigma}\operatorname{dist}_X\big(U_\sigma(t,\tau)B,K\big)\to 0,\qquad \text{as } t\to\infty.$$

For uniformly asymptotically compact systems the uniform omega-limit sets satisfy the following important properties.

Lemma 3.1.18. *Let $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a uniformly asymptotically compact system with K a compact uniformly attracting set. Then for any $\tau\in\mathbb{R}$ and any non-empty bounded subset $B\subset X$ it holds*

i) $\omega_{\tau,\Sigma}(B)$ *is non-empty, compact and* $\omega_{\tau,\Sigma}(B)\subseteq K$*;*

ii) $\omega_{\tau,\Sigma}(B)$ *uniformly attracts B with origin at time τ, i.e,*

$$\sup_{\sigma\in\Sigma}\operatorname{dist}_X\big(U_\sigma(t,\tau)B,\omega_{\tau,\Sigma}(B)\big)\to 0,\qquad \text{as } t\to\infty;$$

iii) If F is closed and uniformly attracts B with origin at time τ, then $\omega_{\tau,\Sigma}(B)\subseteq F$.

Proof. First of all, since K is a uniformly attracting set for the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ then for any bounded subset $D\subset X$ we have

$$\sup_{\sigma\in\Sigma}\operatorname{dist}_X\big(U_\sigma(t,\tau)D,K\big)\to 0,\qquad \text{as } t\to\infty. \tag{3.4}$$

Proof of i): Let $\sigma\in\Sigma$ and $x\in B$ be fixed elements and consider $\{t_n\}\subseteq\mathbb{R}^+_\tau$ with $t_n\to\infty$ as $n\to\infty$. Given $\varepsilon>0$, by (3.4) in particular for $D=\{x\}$ there is $n_0\in\mathbb{N}$ such that for $n\geqslant n_0$ we have

$$\operatorname{dist}_X\big(U_\sigma(t_n,\tau)x,K\big)<\varepsilon.$$

So there is $y_n\in K$ satisfying

$$\|U_\sigma(t_n,\tau)x-y_n\|_X<\varepsilon$$

and so

$$\|U_\sigma(t_n,\tau)x-y_n\|_X\to 0,\ \text{as } n\to\infty.$$

Since K is compact, $\{y_n\}$ has a subsequence (still denoted by $\{y_n\}$) converging to a point $y\in K$ and then $\lim_{n\to\infty}U_\sigma(t_n,\tau)x=y$. But from Lemma 3.1.16, $y\in\omega_{\tau,\Sigma}(B)$, which proves $\omega_{\tau,\Sigma}(B)\neq\emptyset$.

Now let us prove that $\omega_{\tau,\Sigma}(B)$ is compact by proving that $\omega_{\tau,\Sigma}(B)\subseteq K$. Indeed, for any $y\in\omega_{\tau,\Sigma}(B)$ by Lemma 3.1.16 there are $\{x_n\}\subseteq B$, $\{\sigma_n\}\subseteq\Sigma$ and $\{t_n\}\subseteq\mathbb{R}^+_\tau$ with $t_n\to\infty$ such that $\lim_{n\to\infty}U_{\sigma_n}(t_n,\tau)x_n=y$. But by the uniformly attracting property of K we obtain

$$\operatorname{dist}_X\big(U_{\sigma_n}(t_n,\tau)x_n,K\big)\leqslant\sup_{\sigma\in\Sigma}\operatorname{dist}_X\big(U_\sigma(t_n,\tau)B,K\big)\to 0,\quad n\to\infty.$$

By the triangle inequality for the Hausdorff semi-distance we conclude that

$$\operatorname{dist}_X(y,K) \leqslant \operatorname{dist}_X\big(y, U_{\sigma_n}(t_n,\tau)x_n\big) + \operatorname{dist}_X\big(U_{\sigma_n}(t_n,\tau)x_n, K\big) \to 0, \qquad \text{as } n\to\infty,$$

and we obtain $\operatorname{dist}_X(y,K)=0$. But it implies $y\in\overline{K}=K$ and therefore $\omega_{\tau,\Sigma}(B)\subseteq K$. Since K is compact we obtain that $\omega_{\tau,\Sigma}(B)$ is compact as well (note that $\omega_{\tau,\Sigma}(B)$ is closed by definition).

Proof of ii): Suppose it is not true. Then there exist $\varepsilon_0>0$ and a sequence $\{t_n\}\subseteq\mathbb{R}^+_\tau$ with $t_n\to\infty$ such that

$$\sup_{\sigma\in\Sigma}\operatorname{dist}_X\big(U_\sigma(t_n,\tau)B, \omega_{\tau,\Sigma}(B)\big) > \varepsilon_0, \qquad n\in\mathbb{N}.$$

But by the definition of supremum we can find sequences $\{x_n\}\subseteq B$ and $\{\sigma_n\}\subseteq\Sigma$ in such a way that

$$\operatorname{dist}_X\big(U_{\sigma_n}(t_n,\tau)x_n, \omega_{\tau,\Sigma}(B)\big) > \varepsilon_0, \qquad n\in\mathbb{N}. \tag{3.5}$$

Notice that since K uniformly attracts B we have

$$\begin{aligned}\operatorname{dist}_X\big(U_{\sigma_n}(t_n,\tau)x_n, K\big) &\leqslant \operatorname{dist}_X\big(U_{\sigma_n}(t_n,\tau)B, K\big)\\ &\leqslant \sup_{\sigma\in\Sigma}\operatorname{dist}_X\big(U_\sigma(t_n,\tau)B,K\big)\to 0, \qquad \text{as } n\to\infty,\end{aligned}$$

and as in part *i)* there exists a sequence $\{y_n\}\subseteq K$ such that

$$\|U_{\sigma_n}(t_n,\tau)x_n - y_n\|_X \to 0, \qquad \text{as } n\to\infty.$$

Since K is compact we may assume $y_n\to y$, where $y\in K$, and by the last expression we conclude $\lim_{n\to\infty}U_{\sigma_n}(t_n,\tau)x_n = y$, i.e., $y\in\omega_{\tau,\Sigma}(B)$. Moreover, by (3.5) we have

$$0=\operatorname{dist}_X\big(y,\omega_{\tau,\Sigma}(B)\big) = \lim_{n\to\infty}\operatorname{dist}_X\big(U_{\sigma_n}(t_n,\tau)x_n, \omega_{\tau,\Sigma}(B)\big) \geqslant \varepsilon_0 > 0,$$

a contradiction, and *ii)* is proved.

Proof of iii): Let F be a closed set that uniformly attracts B with origin at time τ under the action of $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$. By Lemma 3.1.16, for any $y\in\omega_{\tau,\Sigma}(B)$ there exist sequences $\{x_n\}\subseteq B$, $\{\sigma_n\}\subseteq\Sigma$ and $\{t_n\}\subseteq\mathbb{R}^+_\tau$ with $t_n\to\infty$ such that $\lim_{n\to\infty}U_{\sigma_n}(t_n,\tau)x_n = y$. Hence we have

$$\operatorname{dist}_X\big(U_{\sigma_n}(t_n,\tau)x_n, F\big) \leqslant \sup_{\sigma\in\Sigma}\operatorname{dist}_X\big(U_\sigma(t_n,\tau)B,F\big)\to 0, \qquad \text{as } n\to\infty,$$

and then $\operatorname{dist}_X(y,F)=0$. But it means $y\in\overline{F}=F$ and finally we conclude that $\omega_{\tau,\Sigma}(B)\subseteq F$. $\square$

3.1.4 Existence of uniform attractors

Clearly if a system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ has a uniform attractor $\mathscr{A}_\Sigma$ then it is uniformly asymptotically compact with $K=\mathscr{A}_\Sigma$. We shall prove in the sequence that in fact uniformly asymptotically compactness for a system is a sufficient condition to guarantee the existence of uniform attractors.

Theorem 3.1.19. *Let $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a system of processes. The following conditions are equivalent:*

1. *$\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is uniformly asymptotically compact;*
2. *$\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ has a uniform attractor $\mathscr{A}_\Sigma$.*

Moreover, in this case we have

$$\mathscr{A}_\Sigma = \overline{\bigcup_{\tau\in\mathbb{R}} \bigcup_{B\in\mathscr{B}(X)} \omega_{\tau,\Sigma}(B)},$$

where $\mathscr{B}(X)$ denotes the collection of all bounded subsets of X.

Proof. (1) $\implies$ (2): Let us prove that the set

$$\mathscr{A}_\Sigma := \overline{\bigcup_{\tau\in\mathbb{R}} \bigcup_{B\in\mathscr{B}(X)} \omega_{\tau,\Sigma}(B)}$$

is the uniform attractor of the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$, i.e., $\mathscr{A}_\Sigma$ is compact and it is the minimal closed uniformly attracting set.

Indeed, let K be a compact uniformly attracting set for the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$. By Lemma 3.1.18, *i)*, for any $\tau\in\mathbb{R}$ and any bounded subset $B\subseteq X$ we already know that $\omega_{\tau,\Sigma}(B)\subseteq K$, and this implies $\mathscr{A}_\Sigma\subseteq K$. Since K is compact we conclude that $\mathscr{A}_\Sigma$ is compact as well.

Moreover, if $\tau\in\mathbb{R}$ and $B\subseteq X$ is bounded we have $\omega_{\tau,\Sigma}(B)\subseteq\mathscr{A}_\Sigma$ and since from Lemma 3.1.18, *ii)*, $\omega_{\tau,\Sigma}(B)$ uniformly attracts B, then

$$\sup_{\sigma\in\Sigma} \operatorname{dist}_X\big(U_\sigma(t,\tau)B,\mathscr{A}_\Sigma\big) \leqslant \sup_{\sigma\in\Sigma} \operatorname{dist}_X\big(U_\sigma(t,\tau)B,\omega_{\tau,\Sigma}(B)\big) \to 0, \qquad \text{as } t\to\infty,$$

proving that $\mathscr{A}_\Sigma$ uniformly attracts the bounded subsets of X.

Finally, if F is a closed uniformly attracting set then by Lemma 3.1.18, *iii)*, we obtain $\omega_{\tau,\Sigma}(B)\subseteq F$, for all $\tau\in\mathbb{R}$ and all $B\subseteq X$ bounded. Hence $\mathscr{A}_\Sigma\subseteq F$, showing that $\mathscr{A}_\Sigma$ is minimal among the closed uniformly attracting sets. Therefore $\mathscr{A}_\Sigma$ is the uniform attractor of the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$.

(2) $\implies$ (1): It is immediate and we take $K=\mathscr{A}_\Sigma$. □

Up to this point we realized the same analysis previously developed for autonomous semigroups, ending up with a characterization of which class of system has uniform attractors. But unlike global attractors, uniform attractors do not satisfy any invariance property and this is a drawback of this object because we can lost the information about the dynamic of the system. An attempt to overcome this is presented in the next section.

3.1.5 Skew-product semiflow and reduction to a semigroup

In this section we intend to induce an autonomous semigroup on the extended space $\Sigma \times X$ and describe conditions for the existence of its global attractor, being able to recover some information about the dynamics of our non-autonomous problem.

Let $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a system of evolution processes and let $\{\theta_s\}_{s\in\mathbb{R}}$ be a group acting on Ξ and such that $\theta_s\Sigma = \Sigma$, for all $s \in \mathbb{R}$. Recall the translation identity is valid

$$U_{\theta_h\sigma}(t,s) = U_\sigma(t+h,s+h), \qquad \forall \sigma \in \Sigma,\ t \geqslant s,\ h \in \mathbb{R}. \tag{3.6}$$

Continuity properties for the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ will play an essential role in this section and in the conclusions we are interested in. So let us make it precise.

Definition 3.1.20. *A system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is $\big(\Sigma\times X, X\big)$-continuous if for all fixed $t,s\in\mathbb{R}$, $t\geqslant s$, the mapping $(\sigma,u)\mapsto U_\sigma(t,s)u$ is continuous.*

Define $\mathbb{X} := \Sigma\times X$, endowed with the metric $d_{\mathbb{X}} : \mathbb{X}\times\mathbb{X}\to\mathbb{R}$ defined for $(\sigma_1,u_1),(\sigma_2,u_2)\in\mathbb{X}$ as

$$d_{\mathbb{X}}\big((\sigma_1,u_1),(\sigma_2,u_2)\big) := d_\Xi(\sigma_1,\sigma_2) + \|u_1-u_2\|_X.$$

Clearly $\big(\mathbb{X}, d_{\mathbb{X}}\big)$ is a complete metric space. Notice also that if $\mathbb{B}\subseteq\mathbb{X}$ is bounded then there is $B\subseteq X$ bounded such that $\mathbb{B}\subseteq\Sigma\times B$.

In the sequence we define a semigroup on the extended space $\mathbb{X}$.

Proposition 3.1.21. *If a system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is $(\Sigma\times X, X)$-continuous then the family $\{S(t)\}_{t\geqslant 0}$ of mappings $S(t):\mathbb{X}\to\mathbb{X}$ defined as*

$$S(t)(\sigma,u) := \big(\theta_t\sigma, U_\sigma(t,0)u\big), \qquad t\geqslant 0, \tag{3.7}$$

is a semigroup on $\mathbb{X}$.

Proof. If $t=0$, clearly $S(0)=Id_{\mathbb{X}}$. Let $t,s\geqslant 0$ and $(\sigma,u)\in\mathbb{X}$. Then by the translation identity (3.6) and the definition of $S(\cdot)$ in (3.7) we have

$$\begin{aligned} S(t)S(s)(\sigma,u) &= S(t)\big(\theta_s\sigma, U_\sigma(s,0)u\big) \\ &= \big(\theta_t\theta_s\sigma, U_{\theta_s\sigma}(t,0)U_\sigma(s,0)u\big) \\ &= \big(\theta_{t+s}\sigma, U_\sigma(t+s,s)U_\sigma(s,0)u\big) \\ &= \big(\theta_{t+s}\sigma, U_\sigma(t+s,0)u\big) \\ &= S(t+s)(\sigma,u), \end{aligned}$$

proving the semigroup property.

Finally, for each $t\geqslant 0$, the continuity of $S(t)$ follows from the continuity of $\theta_t:\Xi\to\Xi$ and the continuity of $(\sigma,u)\mapsto U_\sigma(t,0)u$. Therefore, $\{S(t)\}_{t\geqslant 0}$ is a semigroup on $\mathbb{X}$. $\square$

Based on Section 2.1 where we have discussed the theory of semigroup, we can state all the definitions in this new setting, particularly the notions of global solutions and invariance. In the following we reformulate the definition of global solution for it to fit better in this new context we are working on.

Definition 3.1.22. *A mapping $\xi : \mathbb{R} \to \mathbb{X}$, $\xi(s) = \big(\gamma(s), u(s)\big)$, is a* global solution *for the semigroup $\{S(t)\}_{t\geqslant 0}$ defined by (3.7) if*

$$S(t)\big(\gamma(s), u(s)\big) = \big(\gamma(s+t), u(s+t)\big), \qquad \forall t \geqslant 0,\ s \in \mathbb{R}. \tag{3.8}$$

Consequently if the semigroup $\{S(t)\}_{t\geqslant 0}$ has the global attractor $\mathbb{A} \subseteq \mathbb{X}$ then it can be characterized as (see Corollary 2.1.16):

Lemma 3.1.23. *The global attractor $\mathbb{A}$ of the semigroup $\{S(t)\}_{t\geqslant 0}$ is characterized as*

$$\mathbb{A} = \big\{\xi(0) : \xi \textit{ is a bounded solution of } \{S(t)\}_{t\geqslant 0}\big\}. \tag{3.9}$$

We will be interested in recovering information for the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ in space X once we obtain information for the semigroup $\{S(t)\}_{t\geqslant 0}$ and its attractor $\mathbb{A}$ on the extended space $\mathbb{X}$. For this, we consider the following projection mappings. Denote by $\Pi_\Sigma : \mathbb{X} \to \Sigma$ and $\Pi_X : \mathbb{X} \to X$ the projections of $\mathbb{X}$ onto the spaces Σ and X, respectively. Then for any $(\sigma, u) \in \mathbb{X}$ we have

$$\Pi_\Sigma(\sigma, u) = \sigma \qquad \text{and} \qquad \Pi_X(\sigma, u) = u.$$

Next theorem shows to us the relation between the global attractor for semigroup $\{S(t)\}_{t\geqslant 0}$ and the uniform attractor $\mathscr{A}_\Sigma$ of the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$. Besides that, it still gives us a relation between $\mathscr{A}_\Sigma$ and the kernel sections $\{\mathscr{K}_\sigma(\cdot)\}_{\sigma\in\Sigma}$. Recall (see Definition 3.1.7) that the kernel section of a process $\{U_\sigma(t,s)\}$ is defined as

$$\mathscr{K}_\sigma(s) = \big\{u(s) : u(\cdot) \text{ is a bounded solution of } U_\sigma(t,s)\big\}.$$

Theorem 3.1.24. *Let $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a uniformly asymptotically compact system which is $(\Sigma \times X, X)$-continuous. Then*

1. *The semigroup $\{S(t)\}_{t\geqslant 0}$ acting on $\mathbb{X}$ has the global attractor $\mathbb{A}$;*

2. *The global attractor $\mathbb{A}$ satisfies*

$$\mathbb{A} = \bigcup_{\sigma\in\Sigma} \{\sigma\} \times \mathscr{K}_\sigma(r), \quad \forall r \in \mathbb{R}; \tag{3.10}$$

3. *The projection $\Pi_X(\mathbb{A})$ of the global attractor $\mathbb{A}$ onto X satisfies*

$$\Pi_X(\mathbb{A}) = \bigcup_{\sigma\in\Sigma} \mathscr{K}_\sigma(r), \quad \forall r \in \mathbb{R}; \tag{3.11}$$

4. *Moreover*

$$\Pi_X(\mathbb{A}) = \mathscr{A}_\Sigma,$$

where $\mathscr{A}_\Sigma$ *is the uniform attractor of the system* $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$*. Therefore*

$$\mathscr{A}_\Sigma = \bigcup_{\sigma\in\Sigma} \mathscr{K}_\sigma(r), \quad \forall r \in \mathbb{R}; \tag{3.12}$$

5. *The projection* $\Pi_\Sigma(\mathbb{A})$ *of the global attractor* $\mathbb{A}$ *onto* Σ *satisfies*

$$\Pi_\Sigma(\mathbb{A}) = \Sigma.$$

Remark 3.1.25. *Decomposition (3.12) will be crucial in order to prove the finite dimensionality of the uniform attractor* $\mathscr{A}_\Sigma$ *(see Section 3.2 for all the details).*

Proof. Let $K \subseteq X$ be a compact uniformly attracting set for the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$, i.e., for any $\tau \in \mathbb{R}$ and $D \subseteq X$ bounded we have

$$\sup_{\sigma\in\Sigma} \mathrm{dist}_X\big(U_\sigma(t,\tau)D, K\big) \to 0, \qquad \text{as } t \to \infty.$$

Proof of (1): First notice that since $K \subseteq X$ is a compact uniformly attracting set for the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ then $\Sigma \times K$ is a compact attracting set for the semigroup $\{S(t)\}_{t\geqslant 0}$ as well. Indeed, let $\mathbb{B}$ be a bounded subset of $\mathbb{X}$. Then we already know that there exists $B \subseteq X$ bounded such that $\mathbb{B} \subseteq \Sigma \times B$. Hence for $\tau = 0$ and this $B \subseteq X$, given $\varepsilon > 0$ there exists $t_0 > 0$ such that

$$\sup_{\sigma\in\Sigma} \mathrm{dist}_X\big(U_\sigma(t,0)B, K\big) < \varepsilon, \qquad \forall t \geqslant t_0.$$

So for any fixed $\sigma \in \Sigma$, $u \in B$ and $t \geqslant t_0$ we have

$$\mathrm{dist}_X\big(U_\sigma(t,0)u, K\big) < \varepsilon,$$

and there exists $x_0 = x_0(t,\sigma,u) \in K$ such that

$$\|U_\sigma(t,0)u - x_0\|_X < \varepsilon.$$

Then

$$\begin{aligned}
\mathrm{dist}_{\mathbb{X}}\big(S(t)(\sigma,u), \Sigma\times K\big) &\leqslant d_{\mathbb{X}}\big(S(t)(\sigma,u), (\theta_t\sigma, x_0)\big) \\
&= d_{\mathbb{X}}\big(\big(\theta_t\sigma, U_\sigma(t,0)u\big), \big(\theta_t\sigma, x_0\big)\big) \\
&= d_\Xi\big(\theta_t\sigma, \theta_t\sigma\big) + \|U_\sigma(t,0)u - x_0\|_X \\
&< \varepsilon,
\end{aligned}$$

and taking the supremum over $(\sigma,u) \in \Sigma \times B$ we conclude that

$$\mathrm{dist}_{\mathbb{X}}\big(S(t)(\Sigma\times B), \Sigma\times K\big) \leqslant \varepsilon, \qquad \forall t \geqslant t_0.$$

Finally, since $\mathbb{B} \subseteq \Sigma \times B$ we obtain

$$\mathrm{dist}_{\mathbb{X}}\big(S(t)(\mathbb{B}), \Sigma\times K\big) \leqslant \mathrm{dist}_{\mathbb{X}}\big(S(t)(\Sigma\times B), \Sigma\times K\big) \to 0, \qquad \text{as } t \to \infty,$$

and $\Sigma \times K$ is a compact attracting set for the semigroup $\{S(t)\}_{t\geqslant 0}$.

Therefore by Theorem 2.1.24 we conclude that the semigroup $\{S(t)\}_{t\geqslant 0}$ has a global attractor $\mathbb{A}$ which is characterized as the omega limit set $\omega(\Sigma\times K)$, i.e.,

$$\mathbb{A} = \omega(\Sigma\times K) = \bigcap_{t\geqslant 0}\left[\overline{\bigcup_{\tau\geqslant t} S(\tau)(\Sigma\times K)}^{\mathbb{X}}\right].$$

We are going to prove (2) and (3) for $r=0$. For any other $r\neq 0$ the proof follows precisely the same steps.

Proof of (2): Let $(\sigma_0,u_0)\in\mathbb{A}$. From (3.9) there exists a bounded solution $\xi:\mathbb{R}\to\mathbb{X}$ of the semigroup $\{S(t)\}_{t\geqslant 0}$ such that $\xi(0) = \big(\gamma(0),u(0)\big) = (\sigma_0,u_0)$. Furthermore, for any $t\geqslant 0$ and $\tau\in\mathbb{R}$ we have $S(t)\big(\gamma(\tau),u(\tau)\big) = \big(\gamma(\tau+t),u(\tau+t)\big)$, and it follows from the definition of $S(t)$ in (3.7) that

$$\theta_t\gamma(\tau) = \gamma(\tau+t) \qquad \text{and} \qquad U_{\gamma(\tau)}(t,0)u(\tau) = u(\tau+t). \tag{3.13}$$

Clearly from the first equation in last expression we have that $\gamma:\mathbb{R}\to\Xi$ is a bounded solution of the semigroup $\{\theta_s\}_{s\geqslant 0}$.

We shall see now that $u:\mathbb{R}\to X$ is a bounded solution of the particular evolution process $\{U_{\sigma_0}(t,s)\}$, i.e., $U_{\sigma_0}(t,s)u(s) = u(t)$ for all $t\geqslant s$ with $t,s\in\mathbb{R}$. Indeed, if $s\geqslant 0$ then from (3.6) (the translation identity) and (3.13) we have

$$U_{\sigma_0}(t,s)u(s) = U_{\gamma(0)}(t,s)u(s) = U_{\theta_s\gamma(0)}(t-s,0)u(s) = U_{\gamma(s)}(t-s,0)u(s) = u(t).$$

If $s<0$ then

$$U_{\sigma_0}(t,s)u(s) = U_{\gamma(0)}(t,s)u(s) = U_{\theta_{-s}\gamma(s)}(t,s)u(s) = U_{\gamma(s)}(t-s,0)u(s) = u(t).$$

Hence $u:\mathbb{R}\to X$ is a bounded solution of $\{U_{\sigma_0}(t,s)\}$ and in particular $u_0 = u(0)\in\mathscr{K}_{\sigma_0}(0)$. So

$$\mathbb{A}\subseteq\bigcup_{\sigma\in\Sigma}\{\sigma\}\times\mathscr{K}_\sigma(0).$$

Now, let $(\sigma_0,u_0)\in\bigcup_{\sigma\in\Sigma}\{\sigma\}\times\mathscr{K}_\sigma(0)$. Then $u_0\in\mathscr{K}_{\sigma_0}(0)$ and there exists a bounded solution $u:\mathbb{R}\to X$ of process $\{U_{\sigma_0}(t,s)\}$ with $u(0)=u_0$. Since Σ is bounded and invariant under the action of $\{\theta_s\}_{s\geqslant 0}$, by Proposition 2.1.6 there exists a bounded solution $\gamma:\mathbb{R}\to\Xi$ of $\{\theta_s\}_{s\geqslant 0}$ such that $\gamma(0)=\sigma_0$. We claim that $\xi(s) := \big(\gamma(s),u(s)\big)$ is a bounded solution of semigroup $\{S(t)\}_{t\geqslant 0}$. Indeed, if $t\geqslant 0$ and $s\geqslant 0$ we have

$$\begin{aligned} S(t)\big(\gamma(s),u(s)\big) &= \big(\theta_t\gamma(s),U_{\gamma(s)}(t,0)u(s)\big)\\ &= \big(\gamma(t+s),U_{\theta_s\gamma(0)}(t,0)u(s)\big)\\ &= \big(\gamma(t+s),U_{\sigma_0}(t+s,s)u(s)\big)\\ &= \big(\gamma(t+s),u(t+s)\big). \end{aligned}$$

If $t \geqslant 0$ and $s < 0$ then

$$\begin{aligned} S(t)\big(\gamma(s),u(s)\big) &= \big(\theta_t\gamma(s),U_{\gamma(s)}(t,0)u(s)\big) \\ &= \big(\gamma(t+s),U_{\gamma(s)}(t,0)u(s)\big) \\ &= \big(\gamma(t+s),U_{\theta_{-s}\gamma(s)}(t+s,s)u(s)\big) \\ &= \big(\gamma(t+s),U_{\sigma_0}(t+s,s)u(s)\big) \\ &= \big(\gamma(t+s),u(t+s)\big). \end{aligned}$$

Hence ξ is a bounded solution of $\{S(t)\}_{t\geqslant 0}$ and so $(\sigma_0,u_0) = \big(\gamma(0),u(0)\big) \in \mathbb{A}$, proving that

$$\bigcup_{\sigma\in\Sigma} \{\sigma\} \times \mathscr{K}_\sigma(0) \subseteq \mathbb{A}.$$

Proof of (3): On one hand, if $u \in \Pi_X(\mathbb{A})$ then there exists $\sigma \in \Sigma$ such that $(\sigma,u) \in \mathbb{A}$ and by (3.10) we have $u \in \mathscr{K}_\sigma(0)$, proving $\Pi_X(\mathbb{A}) \subseteq \bigcup_{\sigma\in\Sigma}\mathscr{K}_\sigma(0)$. On the other hand, if $u_0 \in \bigcup_{\sigma\in\Sigma}\mathscr{K}_\sigma(0)$ then $u_0 \in \mathscr{K}_{\sigma_0}(0)$ for some $\sigma_0 \in \Sigma$ and $(\sigma_0,u_0) \in \mathbb{A}$. Hence $u_0 = \Pi_X(\sigma_0,u_0) \in \Pi_X(\mathbb{A})$ and we obtain $\bigcup_{\sigma\in\Sigma}\mathscr{K}_\sigma(0) \subseteq \Pi_X(\mathbb{A})$.

Proof of (4): First notice that since the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is uniformly asymptotically compact there exists the uniform attractor $\mathscr{A}_\Sigma$ (see Theorem 3.1.19). So in order to prove that $\Pi_X(\mathbb{A})$ coincides with $\mathscr{A}_\Sigma$ we have to guarantee that $\Pi_X(\mathbb{A})$ is compact and is the minimal closed unifomly attracting set for the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$. We shall prove that $\Pi_X(\mathbb{A})$ is such that for any given $\tau \in \mathbb{R}$ and $B \subseteq X$ bounded it holds

$$\sup_{\sigma\in\Sigma} \mathrm{dist}_X\big(U_\sigma(t,\tau)B,\Pi_X(\mathbb{A})\big) \to 0, \quad \text{as } t\to\infty.$$

Indeed, given $B_1 \subseteq X$ bounded we have that $\Sigma\times B_1$ is bounded in $\mathbb{X}$ and then

$$\mathrm{dist}_{\mathbb{X}}\big(S(t)(\Sigma\times B_1),\mathbb{A}\big) \to 0, \quad \text{as } t\to\infty.$$

So given $\varepsilon > 0$, there exists $t_0 \geqslant 0$ such that

$$\mathrm{dist}_{\mathbb{X}}\big(S(t)(\Sigma\times B_1),\mathbb{A}\big) < \varepsilon, \qquad t \geqslant t_0. \tag{3.14}$$

Let $(\sigma,u) \in \Sigma\times B_1$ be an arbitrary element and $t \geqslant t_0$. By (3.14) we obtain $\mathrm{dist}_{\mathbb{X}}\big(S(t)(\sigma,u),\mathbb{A}\big) < \varepsilon$, and there exists some pair $(\sigma_0,u_0) \in \mathbb{A}$ such that

$$d_{\mathbb{X}}\big(S(t)(\sigma,u),(\sigma_0,u_0)\big) < \varepsilon.$$

Then since $u_0 \in \mathscr{K}_{\sigma_0}(0) \subseteq \Pi_X(\mathbb{A})$ we have

$$\begin{aligned} \varepsilon &> d_{\mathbb{X}}\big(S(t)(\sigma,u),(\sigma_0,u_0)\big) \\ &= d_{\mathbb{X}}\big((\theta_t\sigma,U_\sigma(t,0)u),(\sigma_0,u_0)\big) \\ &= d_{\Xi}\big(\theta_t\sigma,\sigma_0\big) + \|U_\sigma(t,0)u-u_0\|_X \\ &\geqslant \|U_\sigma(t,0)u-u_0\|_X \\ &\geqslant \mathrm{dist}_X\big(U_\sigma(t,0)u,\Pi_X(\mathbb{A})\big), \end{aligned}$$

for any $\sigma \in \Sigma$ and $u \in B_1$. Hence taking the supremum over $u \in B_1$ and then over $\sigma \in \Sigma$ we obtain

$$\sup_{\sigma\in\Sigma} \operatorname{dist}_X\big(U_\sigma(t,0)B_1, \Pi_X(\mathbb{A})\big) \leqslant \varepsilon, \qquad \forall t \geqslant t_0,$$

proving that

$$\sup_{\sigma\in\Sigma} \operatorname{dist}_X\big(U_\sigma(t,0)B_1, \Pi_X(\mathbb{A})\big) \to 0, \quad \text{as } t \to \infty, \tag{3.15}$$

for any $B_1 \subseteq X$ bounded.

Now, let $\tau \in \mathbb{R}$ and $B \subseteq X$ be bounded. Since $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is uniformly asymptotically compact then there exists a time $t_1 \geqslant \tau$ (with $t_1 > 0$), such that the set $B_1 := \bigcup_{\sigma\in\Sigma} U_\sigma(t_1,\tau)B$ is bounded in X. Therefore for any $t \geqslant t_1$ and $\sigma \in \Sigma$

$$\begin{aligned} U_\sigma(t,\tau)B &= U_\sigma(t,t_1)U_\sigma(t_1,\tau)B \\ &\subseteq U_\sigma(t,t_1)B_1 \\ &= U_{\theta_{t_1}\sigma}(t-t_1,0)B_1 \end{aligned}$$

and then

$$\begin{aligned} \sup_{\sigma\in\Sigma} \operatorname{dist}_X\big(U_\sigma(t,\tau)B, \Pi_X(\mathbb{A})\big) &\leqslant \sup_{\sigma\in\Sigma} \operatorname{dist}_X\big(U_{\theta_{t_1}\sigma}(t-t_1,0)B_1, \Pi_X(\mathbb{A})\big) \\ &= \sup_{\sigma\in\Sigma} \operatorname{dist}_X\big(U_\sigma(t-t_1,0)B_1, \Pi_X(\mathbb{A})\big). \end{aligned}$$

Finally, from (3.15) we conclude

$$\sup_{\sigma\in\Sigma} \operatorname{dist}_X\big(U_\sigma(t,\tau)B, \Pi_X(\mathbb{A})\big) \to 0, \quad \text{as } t \to \infty,$$

proving that $\Pi_X(\mathbb{A})$ is a compact set (because Π_X is continuous and $\mathbb{A}$ is compact) uniformly attracting the bounded subsets of X.

We shall prove now the minimality condition for the set $\Pi_X(\mathbb{A})$. For this, let $\mathscr{A}_0 \subseteq X$ be a closed uniformly attracting set for $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$. Remember from (3.11) that $\Pi_X(\mathbb{A}) = \bigcup_{\sigma\in\Sigma} \mathscr{K}_\sigma(0)$ and let $u_0 \in \Pi_X(\mathbb{A})$. Then for some $\sigma_0 \in \Sigma$ we have $u_0 \in \mathscr{K}_{\sigma_0}(0)$ and there exists $u : \mathbb{R} \to X$ a bounded solution for the process $\{U_{\sigma_0}(t,s)\}$ with $u(0) = u_0$. Since $\sigma_0 \in \Sigma$ and Σ is invariant we also know that there exists $\gamma : \mathbb{R} \to \Sigma$ a bounded solution for $\{\theta_s\}_{s\geqslant 0}$ such that $\gamma(0) = \sigma_0$. Let $B_0 := \big\{u(-n) : n \in \mathbb{N} \cup \{0\}\big\} \subseteq X$. Clearly B_0 is bounded in X. Note that

$$u_0 = u(0) = U_{\sigma_0}(0,-n)u(-n) = U_{\theta_n\gamma(-n)}(0,-n)u(-n) = U_{\gamma(-n)}(n,0)u(-n)$$

and we have

$$\begin{aligned} \operatorname{dist}_X(u_0, \mathscr{A}_0) &= \operatorname{dist}_X\big(U_{\gamma(-n)}(n,0)u(-n), \mathscr{A}_0\big) \\ &\leqslant \sup_{\sigma\in\Sigma} \operatorname{dist}_X\big(U_\sigma(n,0)B_0, \mathscr{A}_0\big). \end{aligned}$$

Taking the limit as $n \to \infty$ in last inequality and using the fact that $\mathscr{A}_0$ is uniformly attracting we obtain $\operatorname{dist}_X(u_0, \mathscr{A}_0) = 0$. Then $u_0 \in \overline{\mathscr{A}_0} = \mathscr{A}_0$, which implies $\Pi_X(\mathbb{A}) \subseteq \mathscr{A}_0$, and the minimality is proved. Therefore $\Pi_X(\mathbb{A}) = \mathscr{A}_\Sigma$.

Proof of (5): On one hand, clearly $\Pi_\Sigma(\mathbb{A}) \subseteq \Sigma$. On the other hand, since K is bounded in X we have that $\Sigma \times K$ is bounded in $\mathbb{X}$ and then given $\varepsilon > 0$ there is $t_0 \geqslant 0$ such that

$$\mathrm{dist}_{\mathbb{X}}\big(S(t)(\Sigma \times K), \mathbb{A}\big) < \varepsilon, \quad \forall t \geqslant t_0.$$

So for any $\sigma \in \Sigma$ and $u \in K$ we have

$$\mathrm{dist}_{\mathbb{X}}\big(S(t)(\sigma, u), \mathbb{A}\big) < \varepsilon, \quad \forall t \geqslant t_0,$$

and there is a pair $(\sigma_0, u_0) \in \mathbb{A}$ satisfying

$$d_{\mathbb{X}}\big(S(t)(\sigma, u), (\sigma_0, u_0)\big) < \varepsilon, \quad \forall t \geqslant t_0.$$

But it means that

$$\begin{aligned}\varepsilon &> d_{\mathbb{X}}\big(S(t)(\sigma, u), (\sigma_0, u_0)\big)\\ &\geqslant d_{\Xi}(\theta_t\sigma, \sigma_0)\\ &\geqslant \mathrm{dist}_{\Xi}\big(\theta_t\sigma, \Pi_\Sigma(\mathbb{A})\big), \quad \forall t \geqslant t_0,\end{aligned}$$

and taking the supremum over $\sigma \in \Sigma$ we conclude by the invariance of Σ that

$$\mathrm{dist}_{\Xi}\big(\Sigma, \Pi_\Sigma(\mathbb{A})\big) = \mathrm{dist}_{\Xi}\big(\theta_t\Sigma, \Pi_\Sigma(\mathbb{A})\big) \to 0, \quad \text{as } t \to \infty.$$

Therefore $\Sigma \subseteq \overline{\Pi_\Sigma(\mathbb{A})}^{\Xi} = \Pi_\Sigma(\mathbb{A})$, proving that $\Pi_\Sigma(\mathbb{A}) = \Sigma$. □

Remark 3.1.26. *The characterization by kernel sections in Theorem 3.1.24, (3.12), allows us to easily obtain a lifted negative semi-invariance*

$$\begin{aligned}\mathscr{A}_\Sigma &= \bigcup_{\sigma\in\Sigma} \mathscr{K}_\sigma(t) = \bigcup_{\sigma\in\Sigma} U_\sigma(t,0)\mathscr{K}_\sigma(0)\\ &\subseteq \bigcup_{\sigma\in\Sigma} U_\sigma(t,0)\mathscr{A}_\Sigma, \qquad \forall t \geqslant 0,\end{aligned} \tag{3.16}$$

for the uniform attractor of $(\Sigma \times X, X)$-continuous system of processes. Both the characterization by kernel sections and the negative semi-invariance will be crucial for us to study the finite-dimensionality of the uniform attractor in Section 3.2.

To finish this section, we warn the reader that the method of skew product semiflow described here can lead us to a rather complicated phase space since now we are considering the extended space $\mathbb{X} = \Sigma \times X$. But at the same time, and as indicated by Theorem 3.1.24, this approach preserves the main concept of invariance of attractors, and one can use the theory of semigroup already studied.

3.1.6 Evolution processes generated by non-autonomous evolution equations

Analogously to Section 2.1.4 in which we described a typical situation to obtain semigroups generated by autonomous evolution equations, in this section we intend to present the same approach but this time for non-autonomous evolution equations. Instead of semigroups, in this setting we get evolution processes generated by non-autonomous problems.

A non-autonomous evolution equation is usually represented as

$$\frac{\partial u}{\partial t} = f(u,t), \qquad t \in \mathbb{R}, \tag{3.17}$$

where the right-hand side of this expression explicitly depends on time t and $f : X \times \mathbb{R} \to X$, with X a Banach space. Usually the parameters depending on time are external forces, parameters of media, interaction functions, control functions, etc.

We denote by $\sigma(t)$ the collection of all time-dependent coefficients of the equation (3.17) and we call it the *time symbol* of the equation. We then rephrase the problem as

$$\frac{\partial u}{\partial t} = f_{\sigma(t)}(u), \qquad t \in \mathbb{R}, \tag{3.18}$$

where the operator $f_{\sigma(s)}(\cdot) : X \to X$ is given for each $s \in \mathbb{R}$. The time symbol $\sigma(s)$ reflects the dependence on time of the equation. The symbol $\sigma(\cdot)$ is a function of $s \in \mathbb{R}$ with values in some Banach or metric space $\mathscr{M}$, i.e., $\sigma : \mathbb{R} \to \mathscr{M}$. Suppose that the symbol $\sigma(\cdot)$, as a function of $s \in \mathbb{R}$, belongs to a topological space $\Xi := \{\xi(\cdot) : \ \xi : \mathbb{R} \to \mathscr{M}\}$.

For each $h \in \mathbb{R}$ denote by $\theta_h : \Xi \to \Xi$ the translation operator defined for $\xi \in \Xi$ as

$$\theta_h \xi(s) := \xi(s+h), \qquad \forall s,h \in \mathbb{R}.$$

Suppose in addition that for each $h \in \mathbb{R}$, θ_h is a continuous mapping on Ξ. So clearly $\{\theta_h\}_{h\in\mathbb{R}}$ is a group. For a particular and fixed time symbol $\sigma_0 \in \Xi$, define a set $\Sigma \subseteq \Xi$ as

$$\begin{aligned}\Sigma &:= \overline{\{\theta_h \sigma_0(\cdot) : h \in \mathbb{R}\}}^{\Xi} \\ &= \overline{\{\sigma_0(\cdot + h) : h \in \mathbb{R}\}}^{\Xi}.\end{aligned}$$

Then Σ is invariant under the action of the group $\{\theta_h\}_{h\in\mathbb{R}}$. For $\sigma \in \Sigma$ we supplement equation (3.18) with initial data at $t = s$, for $s \in \mathbb{R}$,

$$\frac{\partial u}{\partial t} = f_{\sigma(t)}(u), \qquad t > s, \tag{3.19}$$

$$u|_{t=s} = u_s, \qquad u_s \in X. \tag{3.20}$$

Let us suppose Σ is a compact subset of Ξ and that for each $\sigma \in \Sigma$ problem (3.19)-(3.20) has a unique solution $u_\sigma(\cdot,s;u_s)$ for each $s \in \mathbb{R}$ and arbitrary $u_s \in X$, and that the mapping $u_\sigma(t,s;\cdot) : X \to X$ is continuous for each $\sigma \in \Sigma$, where $t \geqslant s$. Under these conditions define the operator $U_\sigma(t,s) : X \to X$ by

$$U_\sigma(t,s)u_s := u_\sigma(t,s;u_s), \qquad u_s \in X. \tag{3.21}$$

We note that for each fixed $\sigma \in \Sigma$ the operators $\{U_\sigma(t,s)\}$ satisfy *i)* $U_\sigma(s,s) = Id_X$, for all $s \in \mathbb{R}$;

ii) $U_\sigma(t,\tau)U_\sigma(\tau,s) = U_\sigma(t,s)$, for all $t \geqslant \tau \geqslant s$;

iii) $U_\sigma(t,s) : X \to X$ is continuous for $t \geqslant s$.

Indeed, since $u_\sigma\big(t,\tau;u_\sigma(\tau,s;u_s)\big)$ and $u_\sigma(t,s;u_s)$ are both solutions of the initial value problem

$$\frac{\partial u}{\partial t} = f_{\sigma(t)}(u),$$
$$u|_{t=\tau} = u_\sigma(\tau,s;u_s),$$

it follows by the uniqueness of solution that they are equal, i.e., $U_\sigma(t,\tau)U_\sigma(\tau,s)u_s = U_\sigma(t,s)u_s$ for arbitrary $u_s \in X$.

With that we guarantee that for each $\sigma \in \Sigma$, the family $\{U_\sigma(t,s)\}$ is an evolution process. In the following we verify a translation identity for group $\{\theta_h\}_{h\in\mathbb{R}}$ and family $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$:

$$U_{\theta_h\sigma}(t,s) = U_\sigma(t+h,s+h),$$

for all $\sigma \in \Sigma$, $h \in \mathbb{R}$ and $t \geqslant s$. Indeed, for $s \in \mathbb{R}$ and any $u_0 \in X$ let $u_{\theta_h\sigma}(t,s;u_0)$ be the solution at time t of problem

$$\frac{\partial u}{\partial t} = f_{\theta_h\sigma(t)}(u),$$
$$u|_{t=s} = u_0,$$

and with the change of variables $t_1 = t+h$, it is also solution of

$$\begin{aligned} \frac{\partial u}{\partial t_1} &= f_{\sigma(t_1)}(u), \\ u|_{t_1=s+h} &= u_0. \end{aligned} \tag{3.22}$$

Since $u_\sigma(t_1,s+h;u_0)$ is also a solution of (3.22), we have by the uniqueness of solutions that $u_{\theta_h\sigma}(t,s;u_0) = u_\sigma(t+h,s+h;u_0)$, i.e., $U_{\theta_h\sigma}(t,s)u_0 = U_\sigma(t+h,s+h)u_0$. Therefore $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is a system generated by the non-autonomous evolution problem (3.19)-(3.20).

As indicated by the definition of the symbol space Σ, usually the first component of the extended phase space $\mathbb{X}$ is taken as the "hull" of the whole right-hand side $f(u,t)$ on a suitable topological space.

3.2 Finite-dimensionality of uniform attractors

As we could notice in Section 3.1, despite the similarities the uniform attractor theory of non-autonomous dynamical systems is much more elaborated than the global attractor theory of autonomous systems. Due to its complicated internal structure, in order to estimate the fractal dimension of uniform attractors we have two major difficulties: the lack of invariance and the superposition of the symbol space.

Two main techniques are used and applied to obtain estimates on the fractal dimension of uniform attractors. First technique is via an investigation of quasi-differentials of the underlying system, which applies widely to dynamical systems on Hilbert spaces, and it can be seen in details in (CHEPYZHOV; VISHIK, 2002, Theorem IX.2.1). It is also applied to global attractors associated to semigroups and details are found in (TEMAM, 1988) and (MIRANVILLE; ZELIK, 2009).

Second technique is based on a smoothing property of the system and it is assumed with a compact embedding between an auxiliary Banach space Y and the phase space X, and can be seen essentially as a Lipschitz condition on initial data between X and Y. This approach was first proposed by (MÁLEK; RUZICKA; THÄTER, 1994) for global attractors (see also Section 2.2 here, where we have discussed the theme) and then further developed in (EFENDIEV; MIRANVILLE; ZELIK, 2000) and (EFENDIEV; MIRANVILLE; ZELIK, 2003) constructing exponential uniform attractors and estimating their Kolmogorov entropies.

In this section, we use the smoothing approach to study conditions which ensure a uniform attractor to have finite fractal dimension. In Section 3.2.1 we first establish an abstract criterion for a uniform attractor to be finite-dimensional, and then go deeper into these conditions. Our theorem shows that, under the (X,Y)-smoothing property and a Lipschitz property on symbols, the fractal dimension of the uniform attractor is bounded by that of the symbol space plus a Kolmogorov entropy number of Y in X, see Theorem 3.2.1. In Section 3.2.2 we prove that once the attractor has finite dimension on the phase space X, the smoothing condition is also applied to ensure the dimensionality in more regular spaces, despite the lack of invariance of the uniform attractor, see Theorem 3.2.3. For our analysis we suppose the symbol space has finite fractal dimension, which is discussed in Section 3.2.3, along some conclusive remarks about the sufficient conditions for our theorems. The results in this section are presented in (CUI *et al.*,). The analysis in this section will be in great part generalized to random uniform attractors in Section 4.2.

Recall that given a compact subset E of a Banach space X, the *fractal dimension* of E in X is defined as

$$\dim_F(E;X) := \limsup_{\varepsilon\to 0^+} \frac{\ln N_X[E;\varepsilon]}{-\ln \varepsilon},$$

where $N_X[E;r]$ denotes the minimum number of open ε-balls in X centred at points of E that are necessary to cover E.

3.2.1 Finite-dimensionality in X

As before, let X be a Banach space and let Σ be a compact subset of some complete metric space $\left(\Xi, d_\Xi\right)$ and assume that Σ is invariant under translations, i.e., $\theta_s\Sigma = \Sigma$, for all $s \in \mathbb{R}$. Let $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a system of processes in X with uniform attractor $\mathscr{A}_\Sigma$. In this section, we shall study the finite-dimensionality of $\mathscr{A}_\Sigma$ in space X.

Let $\mathscr{B}$ be a closed and bounded uniformly absorbing set of the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$. Then by the minimality of the uniform attractor among closed uniformly attracting sets we know that $\mathscr{A}_\Sigma \subseteq \mathscr{B}$. Recall that if $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is $(\Sigma\times X, X)$-continuous then by Theorem 3.1.24, (4), we have the decomposition

$$\mathscr{A}_\Sigma = \bigcup_{\sigma\in\Sigma} \mathscr{K}_\sigma(\tau), \qquad \tau\in\mathbb{R}, \tag{3.23}$$

and by Remark 3.1.26 the negative semi-invariance

$$\mathscr{A}_\Sigma \subseteq \bigcup_{\sigma\in\Sigma} U_\sigma(t,0)\mathscr{A}_\Sigma, \qquad t \geqslant 0. \tag{3.24}$$

Consider the following conditions.

(H_1) The symbol space Σ has finite fractal dimension in space Ξ, i.e., $\dim_F\left(\Sigma;\Xi\right) < \infty$.

(H_2) The system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is $(\Sigma\times X, X)$-continuous.

(H_3) $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is (Σ, X)-Lipschitz on the absorbing set $\mathscr{B}$, satisfying

$$\|U_{\sigma_1}(t,0)x - U_{\sigma_2}(t,0)x\|_X \leqslant L(t) d_\Xi(\sigma_1,\sigma_2), \qquad \forall t\geqslant 1,\ \sigma_1,\sigma_2\in\Sigma,\ x\in\mathscr{B}, \tag{3.25}$$

where $1 \leqslant L(t) \leqslant c_1 e^{\beta t}$ for some positive constants $c_1, \beta > 0$ for $t \geqslant 1$.

(H_4) There is an auxiliary Banach space Y compactly embedded in X.

(H_5) $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is uniformly (X,Y)-smoothing on the absorbing set $\mathscr{B}$, i.e., for any $t>0$ there exists a $\kappa(t) > 0$ such that

$$\sup_{\sigma\in\Sigma} \|U_\sigma(t,0)x - U_\sigma(t,0)y\|_Y \leqslant \kappa(t)\|x-y\|_X, \quad \forall x,y\in\mathscr{B}. \tag{3.26}$$

Remember that the Kolmogorov ε-entropy of the embedding of Y into X is given as

$$\mathbf{H}_\varepsilon(Y;X) = \log_2 N_\varepsilon, \tag{3.27}$$

where $N_\varepsilon = N_X\left[B_Y(0,1);\varepsilon\right]$.

Our first main theorem shows that, under these hypotheses, the uniform attractor $\mathscr{A}_\Sigma$ has finite fractal dimension in X.

Theorem 3.2.1. *Let $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a system of processes in X with uniform attractor $\mathscr{A}_\Sigma$. If conditions (H_1) - (H_5) hold, then the uniform attractor $\mathscr{A}_\Sigma$ has finite fractal dimension in X. In particular,*

$$\dim_F\big(\mathscr{A}_\Sigma;X\big) \leqslant \mathbf{H}_{\frac{1}{4\kappa}}(Y;X) + \left(\frac{\beta}{\ln 2}+1\right)\dim_F\big(\Sigma;\Xi\big), \tag{3.28}$$

where $\kappa := \kappa(T_{\mathscr{B}})$ is given in (H_5) with $T_{\mathscr{B}} \geqslant 1$ an absorption time after which $\mathscr{B}$ uniformly absorbs itself (i.e., satisfying (3.29) below).

Proof. Let $\nu \in (0,1)$ be given and fixed. Since the uniformly absorbing set $\mathscr{B}$ is bounded, we have

$$\mathscr{B} = B_X(x_0,R)\cap\mathscr{B}$$

for some point $x_0 \in \mathscr{B}$ and radius $R>0$. In addition, since $\mathscr{B}$ uniformly absorbs itself, there is a time $T_{\mathscr{B}} \geqslant 1$ such that

$$\bigcup_{\sigma\in\Sigma} U_\sigma(t,0)\mathscr{B} \subseteq \mathscr{B}, \quad \forall t \geqslant T_{\mathscr{B}}. \tag{3.29}$$

In the following we prove the theorem by making use of (3.25) and (3.26) for $t = T_{\mathscr{B}}$. For ease of notation we write $\kappa := \kappa(T_{\mathscr{B}})$, and, without loss of generality, we choose $T_{\mathscr{B}} = 1$.

Since Y is compactly embedded in X, the unit ball $B_Y(0,1)$ in Y is covered by at least $N_{\frac{\nu}{2\kappa}}$ balls of radius $\frac{\nu}{2\kappa}$ in the space X, i.e.,

$$B_Y(0,1) \subseteq \bigcup_{i=1}^{N_{\frac{\nu}{2\kappa}}} B_X\left(p_i,\frac{\nu}{2\kappa}\right), \qquad p_i \in B_Y(0,1). \tag{3.30}$$

In terms of Kolmogorov entropy (3.27) this is in fact

$$\mathbf{H}_{\frac{\nu}{2\kappa}}(Y;X) = \log_2 N_{\frac{\nu}{2\kappa}}. \tag{3.31}$$

Next we shall construct sets $W^n(\sigma)\subseteq\mathscr{B}$ by induction on $n\in\mathbb{N}$ such that for all $\sigma\in\Sigma$

$$W^n(\sigma)\subseteq\mathscr{B}, \tag{3.32}$$

$$\sharp W^n(\sigma) \leqslant N^n_{\frac{\nu}{2\kappa}}, \tag{3.33}$$

$$U_\sigma(n,0)\mathscr{B} \subseteq \bigcup_{u\in W^n(\sigma)} B_X(u,R\nu^n)\cap\mathscr{B}. \tag{3.34}$$

For $n=1$ and $\sigma\in\Sigma$ we have by the smoothing property (3.26) that

$$\begin{aligned} U_\sigma(1,0)\mathscr{B} &= U_\sigma(1,0)\big[B_X(x_0,R)\cap\mathscr{B}\big] \\ &\subseteq B_Y\big(U_\sigma(1,0)x_0,\kappa R\big)\cap U_\sigma(1,0)\mathscr{B}. \end{aligned}$$

Let $y_0^\sigma := U_\sigma(1,0)x_0$. From (3.29) and (3.30) we note that

$$\begin{aligned}
B_Y\left(y_0^\sigma, \kappa R\right) \cap U_\sigma(1,0)\mathscr{B} &\subseteq \bigcup_{i=1}^{N_{\frac{\nu}{2\kappa}}} B_X\left(y_0^\sigma + \kappa R p_i, \kappa R \frac{\nu}{2\kappa}\right) \cap \mathscr{B} \\
&= \bigcup_{i=1}^{N_{\frac{\nu}{2\kappa}}} B_X\left(y_0^\sigma + \kappa R p_i, \frac{R\nu}{2}\right) \cap \mathscr{B} \\
&\subseteq \bigcup_{i=1}^{N_{\frac{\nu}{2\kappa}}} B_X\left(q_i^\sigma, R\nu\right) \cap \mathscr{B},
\end{aligned}$$

for some $q_i^\sigma \in \mathscr{B}$, so that

$$U_\sigma(1,0)\mathscr{B} \subseteq \bigcup_{i=1}^{N_{\frac{\nu}{2\kappa}}} B_X\left(q_i^\sigma, R\nu\right) \cap \mathscr{B}.$$

Let $W^1(\sigma) := \left\{q_i^\sigma : i = 1, \cdots, N_{\frac{\nu}{2\kappa}}\right\} \subseteq \mathscr{B}$. Then $W^1(\sigma)$ satisfies (3.32)-(3.34) for $n = 1$.

Assuming that the sets $W^k(\sigma)$ have been constructed for all $1 \leqslant k \leqslant n$ and $\sigma \in \Sigma$, we now construct the set $W^{n+1}(\sigma)$. Given $\sigma \in \Sigma$, since $\{U_\sigma(t,s)\}$ is a process we have

$$\begin{aligned}
U_\sigma\left(n+1,0\right)\mathscr{B} &= U_\sigma(n+1,n)U_\sigma(n,0)\mathscr{B} \\
&= U_{\theta_n\sigma}(1,0)U_\sigma(n,0)\mathscr{B},
\end{aligned}$$

and by the induction hypothesis

$$U_\sigma\left(n,0\right)\mathscr{B} \subseteq \bigcup_{u\in W^n(\sigma)} B_X(u, R\nu^n) \cap \mathscr{B},$$

where $W^n(\sigma) \subseteq \mathscr{B}$ and $\sharp W^n(\sigma) \leqslant N^n_{\frac{\nu}{2\kappa}}$. Moreover, for each $u \in W^n(\sigma)$, by the absorption (3.29) and the smoothing property (3.26) we obtain

$$\begin{aligned}
U_{\theta_n\sigma}(1,0)\left[B_X(u, R\nu^n) \cap \mathscr{B}\right] &\subseteq B_Y\left(U_{\theta_n\sigma}(1,0)u, \kappa R\nu^n\right) \cap \mathscr{B} \\
&\subseteq \bigcup_{i=1}^{N_{\frac{\nu}{2\kappa}}} B_X(p_{i,u}, R\nu^{n+1}) \cap \mathscr{B},
\end{aligned}$$

for points $p_{i,u} \in \mathscr{B}$. Then

$$U_\sigma(n+1,0)\mathscr{B} \subseteq \bigcup_{u\in W^n(\sigma)} \bigcup_{i=1}^{N_{\frac{\nu}{2\kappa}}} B_X\left(p_{i,u}, R\nu^{n+1}\right) \cap \mathscr{B},$$

and we define $W^{n+1}(\sigma) := \left\{p_{i,u} : u \in W^n(\sigma) \text{ and } 1 \leqslant i \leqslant N_{\frac{\nu}{2\kappa}}\right\}$, which satisfies $W^{n+1}(\sigma) \subseteq \mathscr{B}$ and $\sharp W^{n+1}(\sigma) \leqslant N^{n+1}_{\frac{\nu}{2\kappa}}$. Hence, the desired sets $\{W^n(\sigma)\}_{n\in\mathbb{N}}$ are constructed.

To find a finite cover for the uniform attractor $\mathscr{A}_\Sigma$ let us make a decomposition of it as a finite union of sets using the known structure (3.23). By the compactness of the symbol space Σ

in Ξ, for any positive number $\eta > 0$ there exists a finite cover of Σ by at least $M_\eta := N_\Xi[\Sigma;\eta]$ balls of radius η, i.e., there are $\sigma_l \in \Sigma$, $l = 1,\cdots,M_\eta$, such that

$$\Sigma = \bigcup_{l=1}^{M_\eta} B_\Xi(\sigma_l,\eta) \cap \Sigma, \qquad \sigma_l \in \Sigma.$$

For each $l = 1,\cdots,M_\eta$, denote by

$$\Sigma_l := B_\Xi(\sigma_l,\eta) \cap \Sigma \qquad \text{and} \qquad \mathscr{K}_{\Sigma_l}(\cdot) := \bigcup_{\sigma\in\Sigma_l} \mathscr{K}_\sigma(\cdot).$$

Then, by (3.23) the uniform attractor $\mathscr{A}_\Sigma$ is decomposed as

$$\mathscr{A}_\Sigma = \bigcup_{l=1}^{M_\eta} \mathscr{K}_{\Sigma_l}(\tau), \qquad \tau \in \mathbb{R}. \tag{3.35}$$

Note that M_η is independent of $\tau \in \mathbb{R}$, and depends only on the symbol space Σ and the corresponding given number η. In the following we shall find a finite cover for $\mathscr{K}_{\Sigma_l}(n)$.

For each l, let $\sigma_l \in \Sigma_l$ be as before. Then for any $\sigma \in \Sigma_l$, $d_\Xi(\sigma,\sigma_l) < \eta$. From the invariance of the kernels $\{\mathscr{K}_\sigma(\cdot)\}_{\sigma\in\Sigma}$ and by hypothesis (H_3) we conclude that

$$\mathscr{K}_{\Sigma_l}(n) \subseteq B_X\big(U_{\sigma_l}(n,0)\mathscr{K}_{\Sigma_l}(0), L(n)\eta\big) \tag{3.36}$$

for each $1 \leqslant l \leqslant M_\eta$ and $n \in \mathbb{N}$. Indeed, if $h \in \mathscr{K}_{\Sigma_l}(n)$ then $h \in \mathscr{K}_\sigma(n)$ for some $\sigma \in \Sigma_l$. But $\mathscr{K}_\sigma(n) = U_\sigma(n,0)\mathscr{K}_\sigma(0)$ and so $h = U_\sigma(n,0)x$, for some $x \in \mathscr{K}_\sigma(0) \subseteq \mathscr{A}_\Sigma \subseteq \mathscr{B}$. Hence

$$\begin{aligned}\|h - U_{\sigma_l}(n,0)x\|_X &= \|U_\sigma(n,0)x - U_{\sigma_l}(n,0)x\|_X \\ &\leqslant L(n)d_\Xi(\sigma,\sigma_l) \\ &< L(n)\eta,\end{aligned}$$

and we obtain (3.36).

Notice that, since $\mathscr{K}_{\Sigma_l}(0) \subseteq \mathscr{A}_\Sigma \subseteq \mathscr{B}$, it follows from (3.33)-(3.34) that

$$N_X\Big[U_{\sigma_l}(n,0)\mathscr{K}_{\Sigma_l}(0); R\nu^n\Big] \leqslant N^n_{\frac{\nu}{2\kappa}}$$

and then

$$N_X\Big[B_X\big(U_{\sigma_l}(n,0)\mathscr{K}_{\Sigma_l}(0), L(n)\eta\big); R\nu^n + L(n)\eta\Big] \leqslant N^n_{\frac{\nu}{2\kappa}}.$$

Hence, from (3.36)

$$N_X\Big[\mathscr{K}_{\Sigma_l}(n); R\nu^n + L(n)\eta\Big] \leqslant N^n_{\frac{\nu}{2\kappa}}, \qquad l = 1,\cdots,M_\eta,$$

and then by (3.35) (with $\tau = n$)

$$N_X\Big[\mathscr{A}_\Sigma; R\nu^n + L(n)\eta\Big] \leqslant N^n_{\frac{\nu}{2\kappa}} M_\eta, \tag{3.37}$$

for all $n \in \mathbb{N}$ and $\eta > 0$.

In the following we stablish by (3.37) a finite ε-cover of $\mathscr{A}_\Sigma$ for small $\varepsilon > 0$. Let

$$\eta_n := \frac{R\nu^n}{L(n)} > 0, \quad n \in \mathbb{N}. \tag{3.38}$$

Then $\eta_n \to 0^+$ as $n \to \infty$, and from (3.37) we have

$$\begin{aligned} N_X\left[\mathscr{A}_\Sigma; 2R\nu^n\right] &= N_X\left[\mathscr{A}_\Sigma; R\nu^n + L(n)\eta_n\right] \\ &\leqslant N^n_{\frac{\nu}{2\kappa}} M_{\eta_n}. \end{aligned}$$

Since $\nu \in (0,1)$, for any $\varepsilon \in (0,1)$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$2R\nu^{n_\varepsilon} < \varepsilon \leqslant 2R\nu^{n_\varepsilon - 1},$$

and the numbers n_ε can be chosen with $n_\varepsilon \to \infty$ as $\varepsilon \to 0^+$. Hence

$$\begin{aligned} N_X\left[\mathscr{A}_\Sigma; \varepsilon\right] &\leqslant N_X\left[\mathscr{A}_\Sigma; 2R\nu^{n_\varepsilon}\right] \\ &\leqslant N^{n_\varepsilon}_{\frac{\nu}{2\kappa}} M_{\eta_{n_\varepsilon}}, \end{aligned}$$

and then

$$\frac{\ln N_X\left[\mathscr{A}_\Sigma; \varepsilon\right]}{-\ln \varepsilon} \leqslant \frac{n_\varepsilon \ln N_{\frac{\nu}{2\kappa}} + \ln M_{\eta_{n_\varepsilon}}}{-\ln\left[2R\nu^{n_\varepsilon - 1}\right]}, \quad \forall \varepsilon \in (0,1). \tag{3.39}$$

Since by (H_1) the symbol space Σ has finite fractal dimension in Ξ and $\eta_{n_\varepsilon} \to 0^+$ as $\varepsilon \to 0^+$, then for any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta) \in (0,1)$ such that

$$N_\Xi\left[\Sigma; \eta_{n_\varepsilon}\right] \leqslant \left(\frac{1}{\eta_{n_\varepsilon}}\right)^{\dim_F(\Sigma;\Xi)+\delta}, \quad \forall \varepsilon \leqslant \varepsilon_0,$$

and then from the definition (3.38) of η_n and the growth condition on $L(n_\varepsilon)$ in (H_3) we obtain

$$\begin{aligned} \ln N_\Xi\left[\Sigma; \eta_{n_\varepsilon}\right] &\leqslant \left(\dim_F(\Sigma;\Xi) + \delta\right) \ln\left(\frac{L(n_\varepsilon)}{R\nu^{n_\varepsilon}}\right) \\ &= \left(\dim_F(\Sigma;\Xi) + \delta\right)\left(\ln c_1 + \beta n_\varepsilon - \ln R - n_\varepsilon \ln \nu\right). \end{aligned} \tag{3.40}$$

From (3.39) and (3.40) it follows that

$$\frac{\ln N_X\left[\mathscr{A}_\Sigma; \varepsilon\right]}{-\ln \varepsilon} \leqslant \frac{n_\varepsilon \ln N_{\frac{\nu}{2\kappa}}}{-\ln\left[2R\nu^{n_\varepsilon - 1}\right]} + \left(\dim_F(\Sigma;\Xi) + \delta\right)\frac{\left(\ln c_1 + \beta n_\varepsilon - \ln R - n_\varepsilon \ln \nu\right)}{-\ln\left[2R\nu^{n_\varepsilon - 1}\right]},$$

and since $n_\varepsilon \to \infty$ as $\varepsilon \to 0^+$ we have

$$\dim_F\left(\mathscr{A}_\Sigma; X\right) \leqslant \frac{\ln N_{\frac{\nu}{2\kappa}}}{-\ln \nu} + \left(\dim_F(\Sigma;\Xi) + \delta\right)\left(\frac{\beta}{-\ln \nu} + 1\right).$$

Since $\delta > 0$ was taken arbitrarily, we have

$$\dim_F\left(\mathscr{A}_\Sigma; X\right) \leqslant \frac{\ln N_{\frac{\nu}{2\kappa}}}{-\ln \nu} + \dim_F(\Sigma;\Xi)\left(\frac{\beta}{-\ln \nu} + 1\right), \qquad \forall \nu \in (0,1). \tag{3.41}$$

Taking particularly $\nu = 1/2$ and by (3.31) we conclude the proof. $\square$

3.2.2 Finite-dimensionality in Y

By Theorem 3.2.1 we have established the finite dimensionality of the uniform attractor $\mathscr{A}_\Sigma$ in the phase space X via some auxiliary Banach space Y compactly embedded in X. Now we are interested in the finite-dimensionality of $\mathscr{A}_\Sigma$ in Y. To be more general, in the following we study a Banach space Z for which $Z = Y$ is a particular case.

Let $(Z, \|\cdot\|_Z)$ be a Banach space and suppose the system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ takes values in Z, i.e., for each $x \in X$ and $\sigma \in \Sigma$, $U_\sigma(t,s)x \in Z$ for $t > s$. Suppose also that the uniform attractor $\mathscr{A}_\Sigma \subseteq X \cap Z$. In the following we shall see that under a $(\Sigma\times X, Z)$-smoothing property the fractal dimension in Z of the uniform attractor can be bounded by the dimension in X plus the dimension of the symbol space Σ in Ξ.

The $(\Sigma\times X, Z)$-smoothing condition we need is as follows.

(H_6) There is a $\bar{t} > 0$ such that for some positive constants $\delta_1, \delta_2, \bar{L} > 0$ it holds

$$\|U_{\sigma_1}(\bar{t},0)x - U_{\sigma_2}(\bar{t},0)y\|_Z \leqslant \bar{L}\Big[\|x-y\|_X^{\delta_1} + \big(d_\Xi(\sigma_1,\sigma_2)\big)^{\delta_2}\Big],$$

for all $\sigma_1, \sigma_2 \in \Sigma$ and $x, y \in \mathscr{A}_\Sigma$.

Remark 3.2.2. *Even for the case $Z = Y$, (H_6) is not implied by (H_3) and (H_5), and vice versa. Nevertheless, (H_6) is often more useful in applications since powers δ_1 and δ_2 are allowed while in (H_5) such powers can not be considered. In fact, it is still open that whether or not Theorem 3.2.1 can be established for a weaker version of (H_5) with condition (3.26) weakened to: there exists some power $\delta \in (0,1]$ such that*

$$\sup_{\sigma\in\Sigma} \big\|U_\sigma(t,0)x - U_\sigma(t,0)y\big\|_Y \leqslant \kappa(t)\|x-y\|_X^{\delta}, \qquad \forall x, y \in \mathscr{B}. \tag{3.26$'$}$$

Theorem 3.2.3. *Let $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a $(\Sigma\times X, X)$-continuous system of evolution processes with a uniform attractor $\mathscr{A}_\Sigma \subseteq X\cap Z$. Suppose that $\dim_F(\mathscr{A}_\Sigma; X) < \infty$ and $\dim_F(\Sigma;\Xi) < \infty$. If (H_6) is satisfied, then $\mathscr{A}_\Sigma$ has finite fractal dimension in Z as well:*

$$\dim_F\big(\mathscr{A}_\Sigma; Z\big) \leqslant \frac{1}{\delta_1}\dim_F\big(\mathscr{A}_\Sigma; X\big) + \frac{1}{\delta_2}\dim_F\big(\Sigma;\Xi\big).$$

Remark 3.2.4. *Note that Z need not be a subset of X and no embeddings from Z into X is required, so the theorem applies to the case of, e.g., $X = L^2(\mathbb{R})$ and $Z = L^p(\mathbb{R})$, with $p > 2$.*

Proof of Theorem 3.2.3. For any $\varepsilon \in (0,1)$, let $M_\varepsilon := N_\Xi[\Sigma;\varepsilon]$ and $\Sigma = \bigcup_{l=1}^{M_{\varepsilon^{1/\delta_2}}} \Sigma_l$, where we denote $\Sigma_l := B_\Xi\big(\sigma_l, \varepsilon^{1/\delta_2}\big)\cap\Sigma$, $\sigma_l \in \Sigma$. Since $\mathscr{A}_\Sigma$ is a compact subset of X, let

$$\mathscr{A}_\Sigma = \bigcup_{i=1}^{N_X[\mathscr{A}_\Sigma;\varepsilon^{1/\delta_1}]} B_X(x_i, \varepsilon^{1/\delta_1}) \cap \mathscr{A}_\Sigma, \qquad x_i \in \mathscr{A}_\Sigma.$$

By the negative semi-invariance (3.24) of the uniform attractor $\mathscr{A}_\Sigma$ we have

$$
\begin{aligned}
\mathscr{A}_\Sigma \subseteq \bigcup_{\sigma\in\Sigma} U_\sigma(\bar{t},0)\mathscr{A}_\Sigma &= \bigcup_{l=1}^{M_{\varepsilon^{1/\delta_2}}} \bigcup_{\sigma\in\Sigma_l} U_\sigma(\bar{t},0)\mathscr{A}_\Sigma \\
&\subseteq \bigcup_{l=1}^{M_{\varepsilon^{1/\delta_2}}} \bigcup_{\sigma\in\Sigma_l} \bigcup_{i=1}^{N_X\left[\mathscr{A}_\Sigma;\varepsilon^{1/\delta_1}\right]} U_\sigma(\bar{t},0)\left[B_X(x_i,\varepsilon^{1/\delta_1})\cap\mathscr{A}_\Sigma\right] \\
&= \bigcup_{l=1}^{M_{\varepsilon^{1/\delta_2}}} \bigcup_{i=1}^{N_X\left[\mathscr{A}_\Sigma;\varepsilon^{1/\delta_1}\right]} U_{\Sigma_l}(\bar{t},0)\left[B_X(x_i,\varepsilon^{1/\delta_1})\cap\mathscr{A}_\Sigma\right],
\end{aligned}
\tag{3.42}
$$

where $U_{\Sigma_l}(\bar{t},0) := \bigcup_{\sigma\in\Sigma_l} U_\sigma(\bar{t},0)$.

Let $v_1, v_2 \in U_{\Sigma_l}(\bar{t},0)\left[B_X(x_i,\varepsilon^{1/\delta_1})\cap\mathscr{A}_\Sigma\right]$. Then for some $\sigma_1,\sigma_2\in\Sigma_l$ and some $u_1,u_2\in B_X(x_i,\varepsilon^{1/\delta_1})\cap\mathscr{A}_\Sigma$ we have $v_1 = U_{\sigma_1}(\bar{t},0)u_1$ and $v_2 = U_{\sigma_2}(\bar{t},0)u_2$ and therefore by (H_6)

$$
\begin{aligned}
\|v_1 - v_2\|_Z &= \|U_{\sigma_1}(\bar{t},0)u_1 - U_{\sigma_2}(\bar{t},0)u_2\|_Z \\
&\leqslant \bar{L}\left[\|u_1-u_2\|_X^{\delta_1} + d_\Xi(\sigma_1,\sigma_2)^{\delta_2}\right] \\
&\leqslant \bar{L}(2^{\delta_1}+2^{\delta_2})\varepsilon \\
&= r\varepsilon,
\end{aligned}
$$

where $r = r(\bar{L},\delta_1,\delta_2) := \bar{L}(2^{\delta_1}+2^{\delta_2})$. Hence,

$$
\operatorname{diam}_Z\left(U_{\Sigma_l}(\bar{t},0)\left[B_X(x_i,\varepsilon^{1/\delta_1})\cap\mathscr{A}_\Sigma\right]\right) \leqslant r\varepsilon,
\tag{3.43}
$$

for all $l = 1,\cdots,M_{\varepsilon^{1/\delta_2}}$ and all $i = 1,\cdots,N_X\left[\mathscr{A}_\Sigma;\varepsilon^{1/\delta_1}\right]$.

From (3.42) and (3.43) we obtain

$$
\begin{aligned}
N_Z\left[\mathscr{A}_\Sigma;2r\varepsilon\right] &\leqslant N_X\left[\mathscr{A}_\Sigma;\varepsilon^{1/\delta_1}\right] M_{\varepsilon^{1/\delta_2}} \\
&= N_X\left[\mathscr{A}_\Sigma;\varepsilon^{1/\delta_1}\right]\cdot N_\Xi\left[\Sigma;\varepsilon^{1/\delta_2}\right],
\end{aligned}
$$

so $\mathscr{A}_\Sigma$ is a precompact subset of Z and, since r is independent of ε and $\mathscr{A}_\Sigma$ is finite-dimensional in X, taking the limit as $\varepsilon\to 0^+$ we finally conclude that

$$
\dim_F\left(\mathscr{A}_\Sigma;Z\right) \leqslant \frac{1}{\delta_1}\dim_F\left(\mathscr{A}_\Sigma;X\right) + \frac{1}{\delta_2}\dim_F\left(\Sigma;\Xi\right). \qquad \square
$$

3.2.3 Some conclusive remarks

In this chapter we established criteria for uniform attractors to have finite fractal dimension on general Banach spaces. The theorems depend heavily on the finite-dimensionality of the symbol space, which is, however, a technical problem in itself as we will see in next section. Then a natural question arises: are these theorems possible to hold for infinite-dimensional symbol spaces? Unfortunately, the answer is generally negative.

To see that, let us consider a system of processes $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ satisfying a contraction property

$$\sup_{\sigma\in\Sigma}\|U_\sigma(1,0)u-U_\sigma(1,0)w\|_X\leqslant\frac{1}{2}\|u-w\|_X,\qquad\forall u,w\in\mathscr{B},\tag{3.44}$$

and suppose that the system admits a uniform attractor $\mathscr{A}_\Sigma$. Then the contraction property makes each kernel section a singleton, i.e., for each $t\in\mathbb{R}$ and $\sigma\in\Sigma$, $\mathscr{K}_\sigma(t)$ is a singleton. Particularly, for $t=0$, there are points $v_\sigma\in X$ such that $\mathscr{K}_\sigma(0)=\{v_\sigma\}$, $\sigma\in\Sigma$, so the uniform attractor $\mathscr{A}_\Sigma=\bigcup_{\sigma\in\Sigma}\mathscr{K}_\sigma(0)=\{v_\sigma\}_{\sigma\in\Sigma}$. Then we obtain a single-valued map

$$v:\Sigma\to\mathscr{A}_\Sigma,\ \sigma\mapsto v_\sigma.$$

This map shows a relationship between the structure of the uniform attractor and that of the symbol space. For instance, if v is in addition Lipschitz and there exist $L_1,L_2>0$ such that

$$L_1\,d_\Xi(\sigma_1,\sigma_2)\leqslant\|v_{\sigma_1}-v_{\sigma_2}\|_X\leqslant L_2\,d_\Xi(\sigma_1,\sigma_2),\quad\forall\sigma_1,\,\sigma_2\in\Sigma,$$

then $N_\Xi[\Sigma;\varepsilon/L_1]\leqslant N_X[\mathscr{A}_\Sigma;\varepsilon]\leqslant N_\Xi[\Sigma;\varepsilon/L_2]$ for all $\varepsilon>0$, and so $\dim_F(\Sigma;\Xi)=\dim_F(\mathscr{A}_\Sigma;X)$. Hence, in this case, even if all the other conditions of Theorem 3.2.1 hold, the uniform attractor can not be finite-dimensional if the symbol space is not.

On the other hand, we may have that the uniform attractor is finite-dimensional but the symbol space is not, as for example if the uniform attractor is a trivial case, i.e., $\mathscr{A}_\Sigma=\{a\}$. In this case, $\mathscr{A}_\Sigma$ is zero-dimensional no matter what the structure of the symbol space is. Nevertheless, it is an open problem how to obtain a finite-dimensional uniform attractor for general applications when the symbol space is infinite-dimensional.

3.3 Construction of finite-dimensional symbol space Σ

In Theorem 3.2.1, the symbol space Σ was required to be finite-dimensional, which is generally a technical issue in applications. Given an evolution equation with non-autonomous term g (also called the symbol of the equation), the symbol space Σ is usually constructed as the *hull* $\mathscr{H}(g)$ of g (see (3.46) bellow). In this section we investigate conditions which ensure a hull $\mathscr{H}(g)$ to have finite fractal dimension.

We start with a revision of almost periodic functions presenting some properties of them and showing that quasiperiodic functions are particular cases of those. It will be essential in order to prove the finite-dimensionality of symbols on the space of bounded continuous functions in Section 3.3.2, in which we guarantee that the hull of quasiperiodic functions has finite fractal dimension (a well-known result already presented in (CHEPYZHOV; VISHIK, 2002)). In Section 3.3.3 we propose then a construction of finite dimensional hulls on the space of continuous functions (not necessarily bounded), allowing in this way that the non-autonomous term g can be considered in more general forms. To be more precise, we prove that the tails of a mapping determines the dimensionality of its hull, and as an example g can be thought as a continuous function converging (at an exponential rate) to quasiperiodic mappings. With that we can take in applications more general non-autononomous terms and the results in Section 3.2 are more applicable. Such results on finite dimensionality of symbol spaces are given in (CUI *et al.*,).

3.3.1 Preliminaries

Let $(\mathscr{X}, d_{\mathscr{X}})$ and (Ξ, d_{Ξ}) be complete metric spaces and $\{\theta_s\}_{s\in\mathbb{R}}$ be the particular family of translation operators $\theta_s : \Xi \to \Xi$ defined for mappings $\zeta : \mathbb{R} \to \mathscr{X}$ by

$$\theta_s\zeta(\cdot) = \zeta(\cdot + s), \quad s \in \mathbb{R}. \tag{3.45}$$

Definition 3.3.1. *For any $g \in \Xi$ the* hull $\mathscr{H}(g)$ *of g in Ξ is defined as*

$$\mathscr{H}(g) := \overline{\{\theta_s g : s \in \mathbb{R}\}}^{\Xi}. \tag{3.46}$$

Definition 3.3.2. *If $\mathscr{H}(g)$ is compact in Ξ we say that g is a* translation compact *(abbrev. tr.c.) mapping in Ξ.*

In this section we will be interested in constructing translation compact mappings such that their hull have finite fractal dimension. For that let us start with some important classes of continuous functions that play an important role for this topic.

Let $\xi : \mathbb{R} \to \mathscr{X}$ be a continuous mapping. For any $\varepsilon > 0$, a number $\tau \in \mathbb{R}$ is said to be an ε-*period* of function ξ if

$$\sup_{s\in\mathbb{R}} d_{\mathscr{X}}\big(\xi(s+\tau), \xi(s)\big) \leqslant \varepsilon.$$

Definition 3.3.3. *A continuous function $\xi : \mathbb{R} \to \mathscr{X}$ is said to be* almost periodic *if for any $\varepsilon > 0$ the ε-periods of ξ form a relatively dense subset of $\mathbb{R}$, i.e., there is a number $l_\varepsilon > 0$ such that for any $\alpha \in \mathbb{R}$ the interval $[\alpha, \alpha + l_\varepsilon]$ contains an ε-period τ of ξ. Denote by $D_\varepsilon(\xi) \subseteq \mathbb{R}$ the set of all ε-periods of the mapping ξ.*

A special case of almost periodic function is the class of *quasiperiodic functions*, which we describe in the following. For $k \in \mathbb{N}$, let $\mathbb{T}^k := [\mathbb{R} \mod 2\pi]^k$ be the k-dimensional torus, that is

$$\mathbb{T}^k = \big\{(x_1, \cdots, x_k) \in [0, 2\pi]^k : 0 \sim 2\pi\big\}.$$

The torus $\mathbb{T}^k$ is composed by equivalence classes $\bar{x}$ (where $x = (x_1, \cdots, x_k) \in \mathbb{R}^k$) in such a way that $y \in \bar{x}$ if and only if for each $i = 1, \cdots, k$ there is $m_i \in \mathbb{Z}$ such that $x_i - y_i = 2m_i\pi$. Here in order to simplify the notation we consider simply $\mathbb{T}^k = [0, 2\pi]^k$ and denote its elements by $x = (x_1, \cdots, x_k)$, $x_i \in [0, 2\pi]$, $i = 1, \cdots, k$.

Let us consider the space $\mathscr{C}(\mathbb{T}^k; \mathscr{X})$ of continuous functions defined on the torus $\mathbb{T}^k$ and taking values in $\mathscr{X}$. Actually, this set is also described as the space of continuous functions $\varphi \in \mathscr{C}(\mathbb{R}^k; \mathscr{X})$ which are 2π-periodic in each argument, i.e., for $(x_1, \cdots, x_k) \in \mathbb{R}^k$ and each $i = 1, \cdots, k$ it holds

$$\varphi(x_1, \cdots, x_i + 2\pi, \cdots, x_k) = \varphi(x_1, \cdots, x_i, \cdots, x_k).$$

Let $\alpha := (\alpha_1, \cdots, \alpha_k) \in \mathbb{R}^k$, where $\{\alpha_i : i = 1, \cdots, k\}$ is a set of rationally independent real numbers, i.e., if $n_1, \cdots, n_k \in \mathbb{Z}$ are integers such that $n_1\alpha_1 + \cdots + n_k\alpha_k = 0$ then $n_1 = \cdots = n_k = 0$. For $\varphi \in \mathscr{C}(\mathbb{T}^k; \mathscr{X})$, a function $\xi : \mathbb{R} \to \mathscr{X}$ given as

$$\xi(t) := \varphi(\alpha_1 t, \cdots, \alpha_k t) = \varphi(\alpha t), \quad t \in \mathbb{R}, \tag{3.47}$$

is said to be *quasiperiodic (with k frequences)* with values in $\mathscr{X}$. Notice that periodic functions (with period 2π) are particular cases ($k = 1$) of quasiperiodic functions.

In order to guarantee that quasiperiodic functions are indeed almost periodic ones let us revise first a Kronecker's result for the solution of systems of real numbers (the proof of this fact will be omitted here).

Theorem 3.3.4 (Kronecker Theorem)**.** *Let $\lambda_1, \cdots, \lambda_n$ and $\gamma_1, \cdots, \gamma_n$ be arbitrary real numbers. Then the following conditions are equivalent:*

I) For arbitrarily small real number $\delta > 0$ there is a solution (over $t \in \mathbb{R}$) for the system of inequalities

$$|\lambda_i t - \gamma_i| < \delta \ (\mathrm{mod}\ 2\pi), \qquad i = 1, \cdots, n. \tag{3.48}$$

II) If it holds for $l_1, \cdots, l_n \in \mathbb{Z}$ the relation

$$l_1\lambda_1 + \cdots + l_n\lambda_n = 0,$$

then we have the congruence

$$l_1\gamma_1 + \cdots + l_n\gamma_n = 0 \ (\mathrm{mod}\ 2\pi).$$

Proof. See (LEVITAN; ZHIKOV, 1982), Kronecker Theorem, Chapter 3, page 37. □

Remark 3.3.5. *First we note that expression* (3.48) *means that there exists for each* $i = 1, \cdots, n$ *an integer* $m_i \in \mathbb{Z}$ *such that*

$$|\lambda_i t - \gamma_i - 2m_i \pi| < \delta.$$

Actually, one obtains more than a unique solution for the system of inequalies (3.48). More precisely, the set of solutions of (3.48) is a relatively dense subset of $\mathbb{R}$ as indicated by the following corollary.

Corollary 3.3.6. *If system* (3.48) *of inequalities admits a solution for any given* $0 < \delta < \pi$ *then there exists a lengh* $l_\delta > 0$ *such that for all* $a \in \mathbb{R}$ *the interval* $[a, a + l_\delta]$ *contains some solution of* (3.48). *In other words, the set of solutions (for each* $\delta < \pi$*) is a relatively dense subset of* $\mathbb{R}$.

Proof. See (AMERIO; PROUSE, 1971), Chapter 2, Section 5, Corollary, page 34. □

A particular case of Kronecker Theorem holds when $\{\lambda_i : i = 1, \cdots, n\}$ is a set of rationally independent numbers.

Corollary 3.3.7. *If* $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ *are rationally independent then the system of inequalities*

$$|\lambda_i t - \gamma_i| < \delta \pmod{2\pi}, \qquad i = 1, \cdots, n,$$

has a solution (over $t \in \mathbb{R}$*) for any choice of real numbers* $\gamma_1, \cdots, \gamma_n$ *and* $\delta > 0$.

Let us show now that quasiperiodic functions (with k frequences) are almost periodic functions. Indeed, let $\xi : \mathbb{R} \to \mathscr{X}$, $\xi(t) = \varphi(\alpha t)$, be a quasiperiodic function with k frequences as in (3.47). Since $\varphi \in \mathscr{C}(\mathbb{T}^k; \mathscr{X})$ then it is uniformly continuous on $\mathbb{R}^k$ and so for any given $\varepsilon > 0$ there exists $0 < \delta_0 < \pi$ such that for any $(t_1, \cdots, t_k), (t_1', \cdots, t_k') \in \mathbb{R}^k$ with $|t_i - t_i'| < \delta_0$ for all $i = 1, \cdots, k$, it implies

$$d_{\mathscr{X}}\left(\varphi(t_1, \cdots, t_k), \varphi(t_1', \cdots, t_k')\right) < \varepsilon.$$

But $\alpha_1, \cdots, \alpha_k \in \mathbb{R}$ are rationally independent real numbers, and so from Corollary 3.3.7 there is $\tau \in \mathbb{R}$ such that it holds the system of inequalities (for $\gamma_i = 0$)

$$|\alpha_i \tau| < \delta_0 \pmod{2\pi}, \qquad i = 1, \cdots, k. \tag{3.49}$$

In other words, there are $m_i \in \mathbb{Z}$, $i = 1, \cdots, k$, such that

$$|\alpha_i \tau - 2m_i \pi| < \delta_0, \qquad i = 1, \cdots, k.$$

Hence, for any $s \in \mathbb{R}$ we have

$$\begin{aligned} d_{\mathscr{X}}\left(\xi(s+\tau), \xi(s)\right) &= d_{\mathscr{X}}\left(\varphi\left(\alpha_1(s+\tau), \cdots, \alpha_k(s+\tau)\right), \varphi(\alpha_1 s, \cdots, \alpha_k s)\right) \\ &= d_{\mathscr{X}}\left(\varphi(\alpha_1 s + \alpha_1 \tau - 2m_1\pi, \cdots, \alpha_k s + \alpha_k \tau - 2m_k \pi), \varphi(\alpha_1 s, \cdots, \alpha_k s)\right) \\ &< \varepsilon, \end{aligned}$$

proving that $\tau \in \mathbb{R}$ is an ε-period of ξ. But from Corollary 3.3.6 we know that the set of solutions for the system of inequalities (3.49) is a relatively dense subset of $\mathbb{R}$, which means that the set of ε-periods of ξ is relatively dense as well. Therefore we conclude that ξ is an almost periodic function.

It is known that if $\xi : \mathbb{R} \to \mathscr{X}$ is quasiperiodic then the hull of ξ, $\mathscr{H}(\xi)$, is a compact set with finite fractal dimension in space $\mathscr{C}_b(\mathbb{R};\mathscr{X})$ of bounded continuous functions, as described in (CHEPYZHOV; VISHIK, 2002). Now we first recall this result in more detail (including some properties of almost periodic functions), and then we consider more general terms ξ in the bigger space $\mathscr{C}(\mathbb{R};\mathscr{X})$ of continuous functions.

3.3.2 Case I: Finite-dimensional symbol spaces in $\mathscr{C}_b(\mathbb{R};\mathscr{X})$

Let $\Xi_b := \mathscr{C}_b(\mathbb{R};\mathscr{X})$ be the space of bounded continuous functions endowed with the metric d_{Ξ_b} given by

$$d_{\Xi_b}(\xi_1,\xi_2) := \sup_{t\in\mathbb{R}} d_{\mathscr{X}}\big(\xi_1(t),\xi_2(t)\big), \qquad \xi_1,\xi_2 \in \Xi_b.$$

Definition 3.3.8. *For any $g \in \Xi_b$ the* hull $\mathscr{H}_b(g)$ *of g in Ξ_b is given as*

$$\mathscr{H}_b(g) := \overline{\{\theta_s g : s \in \mathbb{R}\}}^{\Xi_b}. \tag{3.50}$$

Clearly, $\{\theta_s\}_{s\in\mathbb{R}}$, $\theta_s : \Xi_b \to \Xi_b$, is a group such that $\theta_s\mathscr{H}_b(g) = \mathscr{H}_b(g)$, for all $s \in \mathbb{R}$. In addition, for each $s \in \mathbb{R}$, $\xi \mapsto \theta_s\xi$ is a Lipschitz mapping on Ξ_b with Lipschitz constant $L_{\theta_s} = 1$, while for $\xi \in \Xi_b$ the mapping $s \mapsto \theta_s\xi$ is Lipschitz provided that ξ is Lipschitz.

Remark 3.3.9. *Notice that the hull $\mathscr{H}_b(g)$ of a nonzero g is a sphere in Ξ_b, i.e., $d_{\Xi_b}(\sigma,0) = d_{\Xi_b}(g,0)$ for any $\sigma \in \mathscr{H}_b(g)$. Hence, $\big(\mathscr{H}_b(g), d_{\Xi_b}\big)$ is a complete metric space, but not a linear space and thus it is not a Banach space.*

In this section we guarantee via a Bochner criterion that a hull $\mathscr{H}_b(g)$ is a compact subset of Ξ_b if and only if g is an almost periodic mapping. Moreover, for the particular case of quasiperiodic mappings g it is possible to go further and prove that $\mathscr{H}_b(g)$ has finite fractal dimension as a subset of Ξ_b.

First of all, we recall some basic facts about almost periodic functions on Ξ_b.

Lemma 3.3.10. *Let $\xi : \mathbb{R} \to \mathscr{X}$ be an almost periodic mapping. Then*

a) For any $\varepsilon > 0$ the set $D_\varepsilon(\xi)$ of ε-periods of ξ is a closed subset of $\mathbb{R}$.

b) For any $\varepsilon > 0$ the set $L_\varepsilon(\xi) := \{l > 0 : l \text{ is an } \varepsilon\text{-period of } \xi\} \subset \mathbb{R}^+$ has a minimum.

c) ξ is uniformly continuous on $\mathbb{R}$.

d) The set of values $\{\xi(t) : t \in \mathbb{R}\} \subseteq \mathscr{X}$ is a precompact subset of $\mathscr{X}$.

Proof. Since ξ is almost periodic, for any $\varepsilon > 0$ the set $D_\varepsilon(\xi)$ of all ε-periods of ξ is relatively dense on $\mathbb{R}$, i.e., there exists $l_\varepsilon > 0$ such that for all $a \in \mathbb{R}$ it holds $[a, a+l_\varepsilon] \cap D_\varepsilon(\xi) \neq \emptyset$.

Proof of a): Let $\tau \in \overline{D_\varepsilon(\xi)}$. Then $\tau = \lim_{n\to\infty} \tau_n$, for $\{\tau_n\}_n \subseteq D_\varepsilon(\xi)$. Hence for each $n \in \mathbb{N}$ we have

$$d_{\mathscr{X}}\big(\xi(t+\tau_n), \xi(t)\big) \leqslant \varepsilon, \qquad \forall t \in \mathbb{R},$$

and by the continuity of ξ we conclude

$$d_{\mathscr{X}}\big(\xi(t+\tau), \xi(t)\big) \leqslant \varepsilon, \qquad \forall t \in \mathbb{R}.$$

So $\tau \in D_\varepsilon(\xi)$ and we prove that this set is closed.

Proof of b): Let $L_\varepsilon := \inf L_\varepsilon(\xi)$ and $a \in \mathbb{R}$ be an arbitrary real number. Let us prove that $[a, a+L_\varepsilon] \cap D_\varepsilon(\xi) \neq \emptyset$. Indeed, for each $n \in \mathbb{N}$ let $\tau_n \in [a, a+L_\varepsilon + 1/n] \cap D_\varepsilon(\xi)$. Since $\{\tau_n\}_n \subset [a, a+L_\varepsilon+1]$ it has a subsequence (still denoted by $\{\tau_n\}_n$) converging to some $\tau \in \mathbb{R}$. Clearly $\tau \in [a, a+L_\varepsilon]$, and from a) we conclude that $\tau \in D_\varepsilon(\xi)$, proving that L_ε is a minimal ε-period for ξ.

Proof of c): First notice that given $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that if $\tau \in \mathbb{R}$ with $|\tau| \leqslant \delta$ then $\tau \in D_\varepsilon(\xi)$. Indeed, because if it is not true we can find $\varepsilon_0 > 0$ and sequences $\{\delta_n\}_n, \{t_n\}_n \subset \mathbb{R}$ with $|\delta_n| \leqslant 1$, $\lim_{n\to\infty} \delta_n = 0$ and

$$d_{\mathscr{X}}\big(\xi(t_n+\delta_n), \xi(t_n)\big) > \varepsilon_0.$$

Since ξ is almost periodic we know that there exists $\tau_n \in [-t_n, -t_n + l_{\varepsilon_0/4}] \cap D_{\varepsilon_0/4}(\xi)$ and consequently it holds

$$\begin{aligned}
\varepsilon_0 &< d_{\mathscr{X}}\big(\xi(t_n+\delta_n), \xi(t_n)\big)\\
&\leqslant d_{\mathscr{X}}\big(\xi(t_n+\delta_n), \xi(t_n+\delta_n+\tau_n)\big) + d_{\mathscr{X}}\big(\xi(t_n+\delta_n+\tau_n), \xi(t_n+\tau_n)\big) + d_{\mathscr{X}}\big(\xi(t_n+\tau_n), \xi(t_n)\big)\\
&\leqslant \frac{\varepsilon_0}{2} + d_{\mathscr{X}}\big(\xi(t_n+\delta_n+\tau_n), \xi(t_n+\tau_n)\big)
\end{aligned}$$

that is,

$$\frac{\varepsilon_0}{2} < d_{\mathscr{X}}\big(\xi(t_n+\delta_n+\tau_n), \xi(t_n+\tau_n)\big).$$

Since $\{t_n+\tau_n\}_n$ is a sequence in $[0, l_{\varepsilon_0/4}]$ we can suppose it has a convergent subsequence (still denoted by itself) and then by $\lim_{n\to\infty} \delta_n = 0$ and the continuity of ξ we conclude that

$$\frac{\varepsilon_0}{2} \leqslant \lim_{n\to\infty} d_{\mathscr{X}}\big(\xi(t_n+\delta_n+\tau_n), \xi(t_n+\tau_n)\big) = 0,$$

an absurd.

Hence, given $\varepsilon > 0$ there is $\delta > 0$ such that if $\tau \in \mathbb{R}$ with $|\tau| \leqslant \delta$ then $\tau \in D_\varepsilon(\xi)$. But it implies that for any $t \in \mathbb{R}$ and $|\tau| \leqslant \delta$ we have

$$d_{\mathscr{X}}\big(\xi(t+\tau), \xi(t)\big) \leqslant \varepsilon,$$

which means that ξ is uniformly continuous on $\mathbb{R}$.

Proof of d): Given $\varepsilon > 0$, ξ is continuous in the interval $[0, l_\varepsilon]$ and therefore it is uniformly continuous. So there are $t_i \in \mathbb{R}$, $i = 1, \cdots, k_\varepsilon$, such that

$$\xi\big([0, l_\varepsilon]\big) \subseteq \bigcup_{i=1}^{k_\varepsilon} B_{\mathscr{X}}\big(\xi(t_i), \varepsilon\big).$$

Let $t \in \mathbb{R}$ be an arbitrary point and $\tau \in [-t, -t + l_\varepsilon] \cap D_\varepsilon(\xi)$. So $t + \tau \in [0, l_\varepsilon]$,

$$d_{\mathscr{X}}\big(\xi(t+\tau), \xi(t)\big) \leqslant \varepsilon,$$

and there is $1 \leqslant i_0 \leqslant k_\varepsilon$ such that $\xi(t+\tau) \in B_{\mathscr{X}}\big(\xi(t_{i_0}), \varepsilon\big)$. Hence

$$d_{\mathscr{X}}\big(\xi(t), \xi(t_{i_0})\big) \leqslant d_{\mathscr{X}}\big(\xi(t), \xi(t+\tau)\big) + d_{\mathscr{X}}\big(\xi(t+\tau), \xi(t_{i_0})\big) < 2\varepsilon,$$

and we conclude that

$$\xi(\mathbb{R}) \subseteq \bigcup_{i=1}^{k_\varepsilon} B_{\mathscr{X}}\big(\xi(t_i), 2\varepsilon\big),$$

proving that the set $\xi(\mathbb{R}) = \{\xi(t) : t \in \mathbb{R}\}$ is precompact in $\mathscr{X}$. $\square$

In the following we give conditions for the mapping $s \mapsto \theta_s f$ from $\mathbb{R}$ to Ξ_b, with $f \in \Xi_b$ fixed, to be almost periodic.

Lemma 3.3.11. *Let $f \in \Xi_b$ be fixed. Then f is $\mathscr{X}$-a.p. if and only if $s \mapsto \theta_s f$ is Ξ_b-a.p.*

Proof. First notice that for any $s, \tau \in \mathbb{R}$ it holds

$$\begin{aligned} d_{\Xi_b}\big(\theta_{s+\tau} f, \theta_s f\big) &= \sup_{t \in \mathbb{R}} d_{\mathscr{X}}\big(f(t+s+\tau), f(t+s)\big) \\ &= \sup_{t \in \mathbb{R}} d_{\mathscr{X}}\big(f(t+\tau), f(t)\big) \\ &= d_{\Xi_b}\big(\theta_\tau f, \theta_0 f\big). \end{aligned}$$

Then the lemma follows immediately from this, showing include that f and $s \mapsto \theta_s f$ have the same ε-periods. $\square$

Lemma 3.3.12. *Let $f \in \Xi_b$ be fixed. The mapping $s \mapsto \theta_s f$ is almost periodic if and only if there exists a relatively dense sequence $\{s_n\} \subset \mathbb{R}$ such that $\{\theta_{s_n} f : n \in \mathbb{N}\} = \{f(\cdot + s_n) : n \in \mathbb{N}\}$ is a precompact subset of Ξ_b.*

Proof. On one hand, suppose that the mapping $s \mapsto \theta_s f$ is Ξ_b-a.p. Then by Lemma 3.3.10, d), we conclude that $\mathscr{H}_b(f) = \overline{\{f(\cdot + s) : s \in \mathbb{R}\}}$ is a compact subset of Ξ_b. Particularly for any sequence $\{s_n\}$ in $\mathbb{R}$ the set $\overline{\{f(\cdot + s_n) : n \in \mathbb{N}\}}$ is a closed subset of $\mathscr{H}_b(f)$ and therefore it must be compact as well.

On the other hand, suppose that there exists $\{s_n\}$ in $\mathbb{R}$ such that $\{f(\cdot + s_n) : n \in \mathbb{N}\}$ is a precompact subset of Ξ_b. We shall prove that $s \mapsto \theta_s f$ is an Ξ_b-a.p. function. First notice that given $\varepsilon > 0$ the set $D_\varepsilon\big(\theta_{(\cdot)}f\big)$ of all ε-periods of $s \mapsto \theta_s f$ is relatively dense in $\mathbb{R}$. Indeed, since $\overline{\{f(\cdot + s_n) : n \in \mathbb{N}\}}$ is a compact subset of Ξ_b there are $s_{n_i} \in \{s_n\}_n$, with $1 \leqslant i \leqslant k$, such that

$$\theta_{s_n} f \in \bigcup_{i=1}^{k} B_{\Xi_b}\big(\theta_{s_{n_i}} f, \varepsilon\big), \qquad \forall n \in \mathbb{N}.$$

For each $1 \leqslant i \leqslant k$, let $B_i := \big\{r \in \{s_n\}_n : \theta_r f \in B_{\Xi_b}\big(\theta_{s_{n_i}} f, \varepsilon\big)\big\}$, from where we obtain $\{s_n\}_n = \bigcup_{1 \leqslant i \leqslant k} B_i$. So B_i is a subsequence of $\{s_n\}_n$ that will be represented as $B_i := \{s_n^i\}_n$. For each $1 \leqslant i \leqslant k$ define $\tau_n^i := s_n^i - s_{n_i}$ and notice that for $s \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} d_{\Xi_b}\big(\theta_{s+\tau_n^i} f, \theta_s f\big) &= \sup_{t \in \mathbb{R}} d_{\mathscr{X}}\big(f(t+s+\tau_n^i), f(t+s)\big) \\ &= \sup_{t \in \mathbb{R}} d_{\mathscr{X}}\big(f(t+s_n^i), f(t+s_{n_i})\big) \\ &= d_{\Xi_b}\big(\theta_{s_n^i} f, \theta_{s_{n_i}} f\big) \\ &< \varepsilon \end{aligned}$$

and then τ_n^i is an ε-period of $s \mapsto \theta_s f$, i.e., $\tau_n^i \in D_\varepsilon\big(\theta_{(\cdot)}f\big)$. Let us now prove that the sequence $\bigcup_{1 \leqslant i \leqslant k}\{\tau_n^i\}_n$ is relatively dense in $\mathbb{R}$. As by hypothesis $\{s_n\}_n$ is relatively dense there is $l > 0$ such that $[a, a+l] \cap \{s_n : n \in \mathbb{N}\} \neq \emptyset$ for any given $a \in \mathbb{R}$. Setting

$$m := \min_{1 \leqslant i \leqslant k}\{-s_{n_i}\}, \qquad M := \max_{1 \leqslant i \leqslant k}\{-s_{n_i}\} \qquad \text{and} \qquad L := l + M - m,$$

there exists $s_n^i \in B_i$, for some $n \in \mathbb{N}$ and $1 \leqslant i \leqslant k$, such that $s_n^i \in [a-m, a-m+l]$. Then

$$\tau_n^i = s_n^i - s_{n_i} \in [a, a+L],$$

proving that $\bigcup_{1 \leqslant i \leqslant k}\{\tau_n^i\}_n$ is relatively dense in $\mathbb{R}$. Therefore, since $\bigcup_{1 \leqslant i \leqslant k}\{\tau_n^i\}_n \subseteq D_\varepsilon\big(\theta_{(\cdot)}f\big)$ it follows directly that $D_\varepsilon\big(\theta_{(\cdot)}f\big)$ is relatively dense as well.

It remains to prove that the mapping $s \mapsto \theta_s f$ is continuous, or equivalently, that f is uniformly continuous on $\mathbb{R}$. For that, if $l > 0$ is given as before, setting $I := [-l, l]$ let $\mathscr{C}(I, \mathscr{X})$ be the restriction of the functions in the space $\mathscr{C}_b(\mathbb{R}, \mathscr{X})$ to the interval I. The metric in $\mathscr{C}(I, \mathscr{X})$ is given by $d_I\big(f_1, f_2\big) := \sup_{t \in I} d_{\mathscr{X}}\big(f_1(t), f_2(t)\big)$ for any $f_1, f_2 \in \mathscr{C}(I, \mathscr{X})$. For each $n \in \mathbb{N}$, if $z_n := \theta_{s_n} f \in \mathscr{C}(I, \mathscr{X})$ we have

$$d_I(z_n, z_m) \leqslant d_{\Xi_b}\big(\theta_{s_n} f, \theta_{s_m} f\big)$$

and since $\{\theta_{s_n} f : n \in \mathbb{N}\}$ is precompact in Ξ_b we conclude that $\{z_n\}_{n \in \mathbb{N}}$ is precompact in $\mathscr{C}(I, \mathscr{X})$. Therefore by Arzelà-Ascoli theorem we have that $\{f(\cdot + s_n)\}_n$ is equicontinuous on I, i.e., for a given $\varepsilon > 0$ there is $\delta > 0$ with $0 < \delta \leqslant l/2$, such that for all $t_1, t_2 \in I$, $|t_1 - t_2| \leqslant \delta$ and all $n \in \mathbb{N}$ we have

$$d_{\mathscr{X}}\big(f(t_1 + s_n), f(t_2 + s_n)\big) < \varepsilon.$$

Let $t_0 \in \mathbb{R}$ be arbitrary and fixed. We know that there exists $s_{n_0} \in [t_0 - l/2, t_0 + l/2]$, from where we can write $t_0 = r_0 + s_{n_0}$, with $|r_0| \leqslant l/2$. Now for any $t \in \mathbb{R}$ such that $|t - t_0| \leqslant \delta$, if we set $t = r + s_{n_0}$ we obtain that $|r| \leqslant l$, $|r - r_0| = |t - t_0| \leqslant \delta$ and

$$d_{\mathscr{X}}\big(f(t), f(t_0)\big) = d_{\mathscr{X}}\big(f(r + s_{n_0}), f(r_0 + s_{n_0})\big) \leqslant \varepsilon,$$

proving that f is uniformly continuous on $\mathbb{R}$. This implies that the mapping $s \mapsto \theta_s f$ is continuous. Therefore, it is a Ξ_b-a.p. function. □

Lemma 3.3.13. *Let $f \in \Xi_b$ be fixed. Then the mapping $s \mapsto \theta_s f$ is Ξ_b-a.p. if and only if the hull $\mathscr{H}_b(f) = \overline{\{\theta_s f : s \in \mathbb{R}\}}$ is a compact subset of Ξ_b.*

Proof. On one hand, suppose that $s \mapsto \theta_s f$ is Ξ_b-a.p. Then by Lemma 3.3.10, d), we have that $\mathscr{H}_b(f)$ is a compact subset of Ξ_b.

On the other hand, suppose $\mathscr{H}_b(f) = \overline{\{f(\cdot + s) : s \in \mathbb{R}\}}$ is a compact subset of Ξ_b and notice that $\mathbb{Z}$ is relatively dense in $\mathbb{R}$ (take $l = 1$). Moreover, $\overline{\{f(\cdot + m) : m \in \mathbb{Z}\}}$ is a closed subset of $\mathscr{H}_b(f)$ and then it is compact. From Lemma 3.3.12 we conclude that the mapping $s \mapsto \theta_s f$ is Ξ_b-a.p., proving the lemma. □

By Lemma 3.3.11 and Lemma 3.3.13 we prove the Bochner's criterion given in the following.

Theorem 3.3.14 (Bochner criterion)**.** *Let $f \in \Xi_b$. Then f is $\mathscr{X}$-a.p. if and only if the hull $\mathscr{H}_b(f)$ is a compact subset of Ξ_b.*

Proof. From Lemma 3.3.11, f is $\mathscr{X}$-a.p. if and only if $s \mapsto \theta_s f$ is Ξ_b-a.p., while from Lemma 3.3.13 the mapping $s \mapsto \theta_s f$ is Ξ_b-a.p. if and only if $\mathscr{H}_b(f)$ is a compact subset of Ξ_b, proving the theorem. □

As it has been mentioned before, the hull of functions often works as the symbol space Σ in applications to partial differential equations, so it plays a crucial role in the study of the dimensionality of uniform attractors accordingly to our results presented in Section 3.2. In the following we prove a well-known fact stating that the hull of quasiperiodic mappings has finite fractal dimension in space $\Xi_b = \mathscr{C}_b(\mathbb{R}; \mathscr{X})$, for $\mathscr{X}$ a complete metric space.

But first of all, let us present a characterization of the hull of quasiperiodic functions, which will be essential in order to prove the desired result. With this characterization we can often identify the hull of a quasiperiodic function as the k-torus $\mathbb{T}^k$.

Proposition 3.3.15. *The hull $\mathscr{H}_b(\xi)$ of a quasiperiodic function ξ in the space $\Xi_b = \mathscr{C}_b(\mathbb{R}; \mathscr{X})$ is represented as*

$$\mathscr{H}_b(\xi) = \big\{\varphi(\alpha \cdot + x) \;:\; x \in \mathbb{T}^k\big\}.$$

Proof. Indeed, on one hand let $\sigma \in \mathscr{H}(\xi)$. Then there exists a sequence $\{s_n\}_n \subset \mathbb{R}$ such that $\xi(\cdot + s_n) \to \sigma(\cdot)$ in Ξ_b as $n \to \infty$, which means that $\varphi\big(\alpha(\cdot + s_n)\big) = \varphi(\alpha \cdot + \ \alpha s_n) \to \sigma(\cdot)$ in Ξ_b. Notice that we can write $\alpha s_n = y_n + 2\pi(m_1^n, \cdots, m_k^n)$, with $y_n \in \mathbb{T}^k$ and $m_i^n \in \mathbb{Z}$, for $i = 1, \cdots, k$; since $\mathbb{T}^k$ is a compact set then we have (refining to a subsequence if necessary) $y_n \to y \in \mathbb{T}^k$. Finally, by the continuity of φ and since it is 2π-periodic in each variable, we obtain $\varphi(\alpha \cdot + \ \alpha s_n) = \varphi(\alpha \cdot + \ y_n) \to \varphi(\alpha \cdot + \ y)$ in Ξ_b and then $\sigma(\cdot) = \varphi(\alpha \cdot + \ y)$, proving that $\mathscr{H}_b(\xi) \subseteq \big\{\varphi(\alpha \cdot + x) : x \in \mathbb{T}^k\big\}$.

On the other hand, let $x \in \mathbb{T}^k$, $x = (x_1, \cdots, x_n)$, be an arbitrary and fixed element. Since $\alpha_1, \cdots, \alpha_k$ are rationally independent numbers then by Corollary 3.3.7, given $\delta > 0$ there is $\tau \in \mathbb{R}$ such that we have

$$|\alpha_i \tau - x_i| < \frac{\delta}{k} \ (\mathrm{mod}\ 2\pi), \qquad i = 1, \cdots, k.$$

Particularly, given $n \in \mathbb{N}$ there are $\tau_n \in \mathbb{R}$ and $m_i^n \in \mathbb{Z}$, $i = 1, \cdots, k$, such that

$$|\alpha_i \tau_n - x_i - 2m_i^n \pi| < \frac{1}{kn}, \qquad i = 1, \cdots, k,$$

and then

$$\|\alpha \tau_n - x - 2\pi(m_1^n, \cdots, m_k^n)\|_{\mathbb{R}^k} < \frac{1}{n}, \qquad n \in \mathbb{N}.$$

Since $\varphi : \mathbb{R} \to \mathscr{X}$ is a continuous function which is 2π-periodic in each variable we conclude that $\varphi(\alpha \cdot + \ \alpha \tau_n) \to \varphi(\alpha \cdot + \ x)$ in Ξ_b as $n \to \infty$. But $\theta_{\tau_n} \xi(\cdot) = \xi(\cdot + \tau_n) = \varphi(\alpha \cdot + \alpha \tau_n)$, and therefore $\big\{\varphi(\alpha \cdot + x) \ : \ x \in \mathbb{T}^k\big\} \subseteq \mathscr{H}_b(\xi)$, proving the proposition. $\square$

Suppose now that $\varphi \in \mathscr{C}(\mathbb{T}^k; \mathscr{X})$ is a Lipschitz continuous mapping, i.e., for $\bar{x}, \bar{y} \in \mathbb{T}^k$, $x = (x_1, \cdots, x_k), y = (y_1, \cdots, y_k) \in \mathbb{R}^k$, we have for some $L > 0$ that

$$d_{\mathscr{X}}\big(\varphi(\bar{x}), \varphi(\bar{y})\big) \leqslant L d_{\mathbb{T}^k}\big(\bar{x}, \bar{y}\big),$$

where $d_{\mathbb{T}^k}\big(\bar{x}, \bar{y}\big) := \|\bar{x} - \bar{y}\|_{\mathbb{R}^k}$, the euclidian norm of $\mathbb{R}^k$.

In the following we prove that the hull of a quasiperiodic mapping with φ Lipschitz continuous has finite fractal dimension on space $\Xi_b = \mathscr{C}_b(\mathbb{R}; \mathscr{X})$ of bounded continuous functions.

Lemma 3.3.16 ((CHEPYZHOV; VISHIK, 2002), Proposition IX.2.1)**.** *If $\xi \in \Xi_b = \mathscr{C}_b(\mathbb{R}; \mathscr{X})$ is a quasiperiodic function with k frequencies and $\xi(t) = \varphi(t\alpha)$ with φ Lipschitz continuous, then* $\dim_F\big(\mathscr{H}_b(\xi); \Xi_b\big) \leqslant k$.

Proof. If $\sigma_1, \sigma_2 \in \mathscr{H}(\xi)$ then by Proposition 3.3.15 there are $\bar{x}, \bar{y} \in \mathbb{T}^k$ such that $\sigma_1(\cdot) = \varphi(\alpha \cdot + \bar{x}) = \varphi\big(\overline{\alpha \cdot} + \bar{x}\big)$ and $\sigma_2(\cdot) = \varphi(\alpha \cdot + \bar{y}) = \varphi(\overline{\alpha \cdot} + \bar{y})$, and therefore since φ is Lipschitz we have

$$\begin{aligned} d_{\Xi_b}(\sigma_1, \sigma_2) &= \sup_{s \in \mathbb{R}} d_{\mathscr{X}}\big(\sigma_1(s), \sigma_2(s)\big) \\ &= \sup_{s \in \mathbb{R}} d_{\mathscr{X}}\big(\varphi(\overline{\alpha s} + \bar{x}), \varphi(\overline{\alpha s} + \bar{y})\big) \\ &\leqslant L \|\bar{x} - \bar{y}\|_{\mathbb{R}^k}. \end{aligned}$$

Hence

$$N_{\Xi_b}[\mathscr{H}_b(\xi);\varepsilon] \leqslant N_{\mathbb{R}^k}[\mathbb{T}^k;\varepsilon/L].$$

But the number of open ε/L-balls needed to cover the torus $\mathbb{T}^k$ is estimated by $\left(\frac{2\pi}{\varepsilon/L}\right)^k$ and therefore

$$N_{\Xi_b}[\mathscr{H}_b(\xi);\varepsilon] \leqslant \left(\frac{2\pi L}{\varepsilon}\right)^k, \qquad \varepsilon > 0.$$

Finally,

$$\begin{aligned}
\dim_F\left(\mathscr{H}_b(\xi);\Xi_b\right) &= \limsup_{\varepsilon\to 0^+} \frac{\ln N_{\Xi_b}[\mathscr{H}_b(\xi);\varepsilon]}{-\ln\varepsilon} \\
&\leqslant \limsup_{\varepsilon\to 0^+} \frac{k\ln(\frac{2\pi L}{\varepsilon})}{-\ln\varepsilon} \\
&= k,
\end{aligned}$$

proving the result. □

Theorem 3.3.14 indicates that almost periodicity is a necessary condition for a hull $\mathscr{H}_b(\xi)$ to be finite dimensional in $\Xi_b = \mathscr{C}_b(\mathbb{R};\mathscr{X})$, while by Lemma 3.3.16 a sufficient condition is ξ to be quasiperiodic. To make our Theorem 3.2.1 applicable to more general non-autonomous terms ξ, we next consider the larger space $\mathscr{C}(\mathbb{R};\mathscr{X})$ of continuous functions.

3.3.3 Case II: Finite-dimensional symbol spaces in $\mathscr{C}(\mathbb{R};\mathscr{X})$

In this section we give conditions for a hull of a mapping in the space $\mathscr{C}(\mathbb{R};\mathscr{X})$ to have finite fractal dimension. Unlike the previous section, now we do not need the analysis to almost periodic functions and we show that the tail of a mapping determines the dimensionality of its hull. It is worth noting that the examples presented here (and represented as graphs) are not finite dimensional on space $\mathscr{C}_b(\mathbb{R};\mathscr{X})$ and so we can infer that the basis space for the symbol space plays an important role in order to determine the finite dimensionality of uniform attractors.

These results were presented in (CUI *et al.*,) together with Section 3.2 and compose part of our contribution to the field of infinite-dimensional dynamical systems.

3.3.3.1 Preliminaries: translation compact functions in $\mathscr{C}(\mathbb{R};\mathscr{X})$

Now we consider $\Xi := \mathscr{C}(\mathbb{R},\mathscr{X})$, the space of all continuous functions $\xi:\mathbb{R}\to\mathscr{X}$, endowed with the Fréchet metric d_Ξ

$$d_\Xi(\xi_1,\xi_2) := \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{d^{(n)}(\xi_1,\xi_2)}{1+d^{(n)}(\xi_1,\xi_2)}, \qquad \xi_1,\xi_2\in\Xi,$$

where

$$d^{(n)}(\xi_1,\xi_2) := \max_{s\in[-n,n]} d_{\mathscr{X}}\left(\xi_1(s),\xi_2(s)\right), \qquad n\in\mathbb{N}.$$

The topology generated by this metric d_Ξ is given as: a set $\mathscr{O} \subset \Xi$ is open if and only if for each $f \in \mathscr{O}$ there exist a compact interval $[t_1,t_2] \subset \mathbb{R}$ and a number $\delta > 0$ such that

$$\left\{\xi \in \Xi : \max_{s\in[t_1,t_2]} d_{\mathscr{X}}\big(f(s),\xi(s)\big) < \delta\right\} \subset \mathscr{O}.$$

Hence, a sequence of functions $\{\xi_m(\cdot)\}_{m\in\mathbb{N}} \subset \Xi$ converges to a function $\xi(\cdot) \in \Xi$ if for any interval $[t_1,t_2] \subset \mathbb{R}$ we have

$$\max_{s\in[t_1,t_2]} d_{\mathscr{X}}\big(\xi_m(s),\xi(s)\big) \to 0, \qquad \text{as } m\to\infty.$$

The space $\Xi = \mathscr{C}(\mathbb{R};\mathscr{X})$ is a complete metric space with respect to the metric d_Ξ (see (CONWAY, 1978), Chapter VII, Section I, for details about the space Ξ). Note that for any $\xi_1,\xi_2 \in \Xi_b = \mathscr{C}_b(\mathbb{R},\mathscr{X})$ we have $d_\Xi(\xi_1,\xi_2) \leqslant d_{\Xi_b}(\xi_1,\xi_2)$ and then $\dim_F\big(A;\Xi\big) \leqslant \dim_F\big(A;\Xi_b\big)$ for any precompact subset $A \subseteq \Xi_b \subseteq \Xi$.

Let $\{\theta_s\}_{s\in\mathbb{R}}$ be the group of translation operators and $\mathscr{H}(g)$ the hull of g in $\Xi = \mathscr{C}(\mathbb{R};\mathscr{X})$ as given in (3.45) and (3.46) (where the closure now is taken under the metric d_Ξ of Ξ), respectively. Now for $\Xi = \mathscr{C}(\mathbb{R};\mathscr{X})$, the mapping $t \mapsto \theta_t\xi$ is Lipschitz from $\mathbb{R}$ to Ξ for Lipschitz ξ. In addition, for each $t\in\mathbb{R}$ the operator $\theta_t : \Xi \to \Xi$ is Lipschitz on $\big(\Xi, d_\Xi\big)$, but with t-dependent Lipschitz constant. More precisely,

Proposition 3.3.17. *For any $t\in\mathbb{R}$ the translation operator θ_t on $\Xi = \mathscr{C}(\mathbb{R};\mathscr{X})$ is Lipschitz, i.e.*

$$d_\Xi(\theta_t\xi_1,\theta_t\xi_2) \leqslant 2^{|t|+1} d_\Xi(\xi_1,\xi_2), \qquad \forall \xi_1,\xi_2\in\Xi.$$

Proof. For any $t\in\mathbb{R}$ we have $|t| = k+\rho$ for some $k\in\mathbb{N}\cup\{0\}$ and $\rho\in[0,1)$. Hence, as $|t| \leqslant k+1$,

$$\begin{aligned} d^{(n)}(\theta_t\xi_1,\theta_t\xi_2) &= \max_{s\in[-n,n]} d_{\mathscr{X}}\big(\theta_t\xi_1(s),\theta_t\xi_2(s)\big) \\ &\leqslant \max_{s\in[-(n+k+1),n+k+1]} d_{\mathscr{X}}\big(\xi_1(s),\xi_2(s)\big) \\ &= d^{(n+k+1)}(\xi_1,\xi_2), \qquad n\in\mathbb{N}, \end{aligned}$$

so

$$\begin{aligned} d_\Xi(\theta_t\xi_1,\theta_t\xi_2) &= \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{d^{(n)}(\theta_t\xi_1,\theta_t\xi_2)}{1+d^{(n)}(\theta_t\xi_1,\theta_t\xi_2)} \\ &\leqslant \sum_{n=1}^{\infty} \frac{2^{k+1}}{2^{n+k+1}}\frac{d^{(n+k+1)}(\xi_1,\xi_2)}{1+d^{(n+k+1)}(\xi_1,\xi_2)} \\ &\leqslant 2^{k+1} d_\Xi(\xi_1,\xi_2) \\ &\leqslant 2^{|t|+1} d_\Xi(\xi_1,\xi_2). \square \end{aligned}$$

Given a function $g\in\Xi = \mathscr{C}(\mathbb{R};\mathscr{X})$, if the hull $\mathscr{H}(g)$ in Ξ is going to be finite-dimensional in Ξ, it should be first compact, i.e., g is translation compact (abbrev. tr.c.) in Ξ. Now let us recall from (CHEPYZHOV; VISHIK, 2002), Section V.2, some known results on tr.c. functions and their hulls on the space of continuous functions Ξ. The first result is also regarded as the well-known Arzelà-Ascoli Theorem (see (CONWAY, 1978), Chapter VII, Section I).

Proposition 3.3.18 (Arzelà-Ascoli). *A set $\Gamma \subset \Xi$ is precompact if and only if*

(i) for every $t \in \mathbb{R}$, the set $\{\sigma(t) : \sigma \in \Gamma\}$ is a precompact subset of $\mathscr{X}$;

(ii) the set Γ is equicontinuous on $\mathbb{R}$, i.e., given $t_0 \in \mathbb{R}$ and $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, t_0) > 0$ such that for $t \in \mathbb{R}$ with $|t - t_0| < \delta$ we have

$$d_{\mathscr{X}}\big(\sigma(t), \sigma(t_0)\big) < \varepsilon, \qquad \forall \sigma \in \Gamma.$$

Proof. On one hand, suppose that $\Gamma \subset \Xi$ is precompact. For a fixed $t \in \mathbb{R}$ define the evaluation map $\phi_t : \Xi \to \mathscr{X}$, $\phi_t(\sigma) = \sigma(t)$, and notice that this map is continuous. Indeed, fix $m \in \mathbb{N}$ such that $t \in [-m, m]$ and let $\sigma_k \to \sigma_0$ in Ξ as $k \to \infty$, and $\varepsilon > 0$. Then there is $k_0 \in \mathbb{N}$ such that for any $k \geqslant k_0$ we have

$$d_\Xi(\sigma_k, \sigma_0) < \frac{1}{2^m} \frac{\varepsilon}{1+\varepsilon}.$$

But

$$\frac{1}{2^m} \frac{d^{(m)}(\sigma_k, \sigma_0)}{1 + d^{(m)}(\sigma_k, \sigma_0)} \leqslant d_\Xi(\sigma_k, \sigma_0),$$

that is, $d^{(m)}(\sigma_k, \sigma_0) < \varepsilon$, and in particular $d_\Xi\big(\phi_t(\sigma_k), \phi_t(\sigma_0)\big) = d_{\mathscr{X}}(\sigma_k(t), \sigma_0(t)) < \varepsilon$, proving that ϕ_t is continuous at $\sigma_0 \in \Xi$. Now since $\overline{\Gamma}$ is compact and ϕ_t is continuous we obtain that $\phi_t\big(\overline{\Gamma}\big)$ is a compact subset of $\mathscr{X}$, and particularly $\{\sigma(t) : \sigma \in \Gamma\}$ has compact closure on $\mathscr{X}$, proving (i).

In order to prove (ii), let $t_0 \in \mathbb{R}$ and $\varepsilon > 0$. Let $m \in \mathbb{N}$ be such that $t_0 \in (-m, m)$. Since $\overline{\Gamma}$ is compact in Ξ there are finitely many functions $\sigma_1, \sigma_2, \cdots, \sigma_p \in \Gamma$ such that

$$\Gamma \subseteq \bigcup_{i=1}^{p} B_\Xi\Big(\sigma_i, \frac{1}{2^m} \frac{\varepsilon/3}{1+\varepsilon/3}\Big).$$

Now by the continuity of each σ_i at t_0 $(i = 1, \cdots, p)$, choose $\delta > 0$ (small enough with $\delta < \min\{|t_0 - m|, |t_0 + m|\}$) such that for any $|t - t_0| < \delta$ it holds

$$d_{\mathscr{X}}\big(\sigma_i(t), \sigma_i(t_0)\big) < \frac{\varepsilon}{3}, \qquad \forall i = 1, \cdots, p.$$

Given $\sigma \in \Gamma$ we have for some $i = 1, \cdots, p$ that

$$\frac{1}{2^m} \frac{d^{(m)}(\sigma, \sigma_i)}{1 + d^{(m)}(\sigma, \sigma_i)} \leqslant d_\Xi(\sigma, \sigma_i) < \frac{1}{2^m} \frac{\varepsilon/3}{1+\varepsilon/3},$$

and so $d^{(m)}(\sigma, \sigma_i) < \varepsilon/3$. For the choise of $\delta > 0$ and for any $|t - t_0| < \delta$ (which implies $t \in [-m, m]$) we obtain

$$\begin{aligned} d_{\mathscr{X}}\big(\sigma(t), \sigma(t_0)\big) &\leqslant d_{\mathscr{X}}\big(\sigma(t), \sigma_i(t)\big) + d_{\mathscr{X}}\big(\sigma_i(t), \sigma_i(t_0)\big) + d_{\mathscr{X}}\big(\sigma_i(t_0), \sigma(t_0)\big) \\ &< d^{(m)}(\sigma, \sigma_i) + \frac{\varepsilon}{3} + d^{(m)}(\sigma, \sigma_i) \\ &< \varepsilon, \end{aligned}$$

for any $\sigma \in \Gamma$, proving (ii), i.e., Γ is an equicontinuous set on $\mathbb{R}$.

On the other hand, suppose that (i) and (ii) hold. To prove that $\Gamma \subset \Xi$ is precompact it suffices to show that every sequence in Γ has a subsequence that converges in Ξ. Let $\{\sigma_n\}_n \subset \Gamma$ be an arbitrary sequence and $\{q_j : j \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$.

By (i) there is a subsequence $\{\sigma_{n,1}\} \subset \{\sigma_n\}$ such that the limit $\lim_{n\to\infty} \sigma_{n,1}(q_1) =: z_1 \in \mathscr{X}$ exists; also there is a subsequence $\{\sigma_{n,2}\} \subset \{\sigma_{n,1}\}$ such that the limit $\lim_{n\to\infty} \sigma_{n,2}(q_2) =: z_2 \in \mathscr{X}$ exists; analogously by induction there is a subsequence $\{\sigma_{n,k}\} \subset \{\sigma_{n,k-1}\}$ such that the limit $\lim_{n\to\infty} \sigma_{n,k}(q_k) =: z_k \in \mathscr{X}$ exists. Setting $\{\sigma_{n,n} : n \in \mathbb{N}\}$, it is a subseqeunce of $\{\sigma_n\}$ in which there exists the limit $\lim_{n\to\infty} \sigma_{n,n}(q_k) = z_k$ for each $k \in \mathbb{N}$. Denote $\gamma_n := \sigma_{n,n}$, for $n \in \mathbb{N}$.

Given $\varepsilon > 0$, choose $m \in \mathbb{N}$ with $2^{-m} < \varepsilon/2$. For the interval $[-m,m]$, from (ii) there is $\delta = \delta(\varepsilon,m) > 0$ such that for any $t,s \in [-m,m]$ with $|t-s| < \delta$, it holds

$$d_{\mathscr{X}}\big(\sigma(t),\sigma(s)\big) < \frac{\varepsilon}{8}, \qquad \forall \sigma \in \Gamma. \tag{3.51}$$

Cover the interval $[-m,m]$ with finitely many open intervals I_j, $j = 1,\cdots,p$, with length δ, and choose points $q_{r_j} \in I_j \cap \mathbb{Q} \cap [-m,m]$, where $r_j \in \mathbb{N}$. Since each sequence $\{\gamma_n(q_{r_j})\}_n$ is Cauchy, then we obtain for $M > 0$ large enough that for every choice of q_{r_j}, $j = 1,\cdots,p$,

$$d_{\mathscr{X}}\big(\gamma_n(q_{r_j}),\gamma_l(q_{r_j})\big) < \frac{\varepsilon}{4}, \qquad \forall n,l \geqslant M. \tag{3.52}$$

Now for any $t \in [-m,m]$, we have $t \in I_j$ for some $j = 1,\cdots,p$, and finally from (3.51) and (3.52) and for $n,l \geqslant M$ we obtain

$$\begin{aligned} d_{\mathscr{X}}\big(\gamma_n(t),\gamma_l(t)\big) &\leqslant d_{\mathscr{X}}\big(\gamma_n(t),\gamma_n(q_{r_j})\big) + d_{\mathscr{X}}\big(\gamma_n(q_{r_j}),\gamma_l(q_{r_j})\big) + d_{\mathscr{X}}\big(\gamma_l(q_{r_j}),\gamma_l(t)\big) \\ &< \frac{\varepsilon}{8} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} = \frac{\varepsilon}{2}, \end{aligned}$$

that is, $d^{(m)}(\gamma_n,\gamma_l) \leqslant \varepsilon/2$. Since $d^{(\tilde{m})}(\gamma_n,\gamma_l) \leqslant d^{(m)}(\gamma_n,\gamma_l)$, for any $1 \leqslant \tilde{m} \leqslant m$, and $2^{-m} < \varepsilon/2$ we conclude that for $n,l \geqslant M$

$$\begin{aligned} d_{\Xi}(\gamma_n,\gamma_l) &= \sum_{i=1}^{m} \frac{1}{2^i} \frac{d^{(i)}(\gamma_n,\gamma_l)}{1+d^{(i)}(\gamma_n,\gamma_l)} + \sum_{i=m+1}^{\infty} \frac{1}{2^i} \frac{d^{(i)}(\gamma_n,\gamma_l)}{1+d^{(i)}(\gamma_n,\gamma_l)} \\ &< d^{(m)}(\gamma_n,\gamma_l) + \frac{\varepsilon}{2} \leqslant \varepsilon, \end{aligned}$$

and hence $\{\gamma_n\}_n$ is Cauchy. Since Ξ is complete, the sequence converges, proving that Γ is a precompact subset of Ξ. □

The last proposition is essential in order to characterize the translation compact mappings on space Ξ of continuous functions.

Corollary 3.3.19 ((CHEPYZHOV; VISHIK, 2002), Proposition V.2.2)**.** *Let $g \in \Xi = \mathscr{C}(\mathbb{R},\mathscr{X})$. Then g is tr.c. in Ξ (i.e. $\mathscr{H}(g)$ is compact) if and only if*

(a) the set $\{g(t) : t \in \mathbb{R}\}$ is a precompact subset of $\mathscr{X}$;

(b) *$g : \mathbb{R} \to \mathscr{X}$ is uniformly continuous on $\mathbb{R}$, i.e., given $\varepsilon > 0$ there is $\delta > 0$ such that for any $t, s \in \mathbb{R}$ with $|t-s| < \delta$ we have $d_{\mathscr{X}}\big(g(t), g(s)\big) < \varepsilon$.*

Proof. Suppose g is tr.c. in Ξ. Then $\Gamma := \{g(\cdot + h) : h \in \mathbb{R}\}$ is a precompact subset of Ξ and by Proposition 3.3.18 we obtain that for each $s \in \mathbb{R}$ fixed, $\{g(s+h) : h \in \mathbb{R}\}$ is a precompact subset of $\mathscr{X}$. It implies (a).

Now from (ii) in Proposition 3.3.18 for $t_0 = 1$, given $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, 1) > 0$ such that for any $r \in \mathbb{R}$ with $|r-1| < \delta$ we have

$$d_{\mathscr{X}}\big(g(r+h), g(1+h)\big) < \varepsilon, \qquad \forall h \in \mathbb{R}.$$

Let $t, s \in \mathbb{R}$ with $|t-s| < \delta$. Choosing $h \in \mathbb{R}$ such that $t = 1+h$ and defining $r := s-h$ we have $|r-1| = |s-t| < \delta$ and therefore

$$d_{\mathscr{X}}\big(g(t), g(s)\big) = d_{\mathscr{X}}\big(g(1+h), g(r+h)\big) < \varepsilon,$$

proving (b).

Conversely, suppose (a) and (b) hold. Denote $\Gamma := \{g(\cdot + h) : h \in \mathbb{R}\} \subset \Xi$ and fix $t_0 \in \mathbb{R}$. Then $\{\sigma(t_0) : \sigma \in \Gamma\} = \{g(t_0+h) : h \in \mathbb{R}\} = \{g(r) : r \in \mathbb{R}\}$ is a precompact subset of $\mathscr{X}$, proving (i) in Proposition 3.3.18.

Let $t_0 \in \mathbb{R}$ be fixed. For $\delta > 0$ as in (b) let $t \in \mathbb{R}$ be such that $|t - t_0| < \delta$. Then for any $h \in \mathbb{R}$ we have $d_{\mathscr{X}}\big(g(t+h), g(t_0+h)\big) < \varepsilon$, i.e.

$$d_{\mathscr{X}}\big(\sigma(t), \sigma(t_0)\big) < \varepsilon, \qquad \forall \sigma \in \Gamma,$$

proving (ii) in Proposition 3.3.18, and therefore we conclude that $\mathscr{H}(g) = \overline{\Gamma}$ is a compact subset of Ξ, that is, g is tr.c. in Ξ. □

Remark 3.3.20. *Note that (see Lemma 3.3.10) any almost periodic function $\xi : \mathbb{R} \to \mathscr{X}$ satisfies (a) and (b) in Corollary 3.3.19, so it is tr.c. in Ξ. Furthermore, if $\mathscr{H}_b(\xi)$ is finite-dimensional in Ξ_b, since $\mathscr{H}_b(\xi) \subseteq \mathscr{H}(\xi)$ densely in Ξ we have*

$$\dim_F\big(\mathscr{H}(\xi); \Xi\big) = \dim_F\big(\mathscr{H}_b(\xi); \Xi\big) \leqslant \dim_F\big(\mathscr{H}_b(\xi); \Xi_b\big) < \infty.$$

Particularly, from Lemma 3.3.16 the hull $\mathscr{H}(\xi)$ of quasiperiodic functions ξ are finite-dimensional on space $\Xi = \mathscr{C}(\mathbb{R}; \mathscr{X})$.

Proposition 3.3.21 ((CHEPYZHOV; VISHIK, 2002), Proposition V.2.3)**.** *Let g be tr.c. in $\Xi = \mathscr{C}(\mathbb{R}; \mathscr{X})$. Then*

(i) *every function $\sigma \in \mathscr{H}(g)$ is tr.c. in Ξ. Moreover, $\mathscr{H}(\sigma) \subseteq \mathscr{H}(g)$;*

(ii) *the set $\mathscr{H}(g)$ is bounded in $\mathscr{C}_b(\mathbb{R}; \mathscr{X})$, that is, there exist $u \in \mathscr{X}$ and $r > 0$ such that*

$$\sup_{s \in \mathbb{R}} d_{\mathscr{X}}\big(\sigma(s), u\big) < r, \qquad \forall \sigma \in \mathscr{H}(g);$$

(iii) the set $\mathscr{H}(g)$ is equicontinuous on $\mathbb{R}$, i.e., given $t_0 \in \mathbb{R}$ and $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, t_0) > 0$ such that for any $t \in \mathbb{R}$ with $|t - t_0| < \delta$ we have

$$d_{\mathscr{X}}\big(\sigma(t), \sigma(t_0)\big) < \varepsilon, \qquad \forall \sigma \in \mathscr{H}(g).$$

(iv) $\theta_h \mathscr{H}(g) = \mathscr{H}(g)$, for all $h \in \mathbb{R}$.

Proof. Let $g \in \Xi$ be a tr. c. function, i.e., $\mathscr{H}(g)$ is a compact subset of Ξ. Denote $\Gamma := \{g(\cdot + h) : h \in \mathbb{R}\} \subset \Xi$.

Proof of (i): Let $\sigma \in \mathscr{H}(g)$. Then $\sigma(\cdot) = \lim_{n\to\infty} \sigma_n(\cdot)$ in Ξ, with $\sigma_n \in \Gamma$. So for each $n \in \mathbb{N}$ we have $\sigma_n(\cdot) = g(\cdot + h_n)$, with $h_n \in \mathbb{R}$. Now $\sigma(\cdot) = \lim_{n\to\infty} g(\cdot + h_n)$ in Ξ and therefore for any fixed $t_0 \in \mathbb{R}$ we have $\sigma(t_0) = \lim_{n\to\infty} g(t_0 + h_n)$ in $\mathscr{X}$, i.e., $\sigma(t_0) \in \overline{\{g(t) : t \in \mathbb{R}\}}^{\mathscr{X}}$. Hence

$$\{\sigma(t) : t \in \mathbb{R}\} \subseteq \overline{\{g(t) : t \in \mathbb{R}\}}^{\mathscr{X}} \subseteq \mathscr{X} \tag{3.53}$$

and so $\{\sigma(t) : t \in \mathbb{R}\}$ is a precompact subset of $\mathscr{X}$.

Now notice that σ is uniformly continuous on $\mathbb{R}$. Indeed, since g is uniformly continuous on $\mathbb{R}$, given $\varepsilon > 0$, there is $\delta > 0$ such that for any $t, s \in \mathbb{R}$, $|t - s| < \delta$, we have $d_{\mathscr{X}}\big(g(t), g(s)\big) < \varepsilon/3$. Then for $n \in \mathbb{N}$ large enough we obtain

$$\begin{aligned} d_{\mathscr{X}}\big(\sigma(t), \sigma(s)\big) &\leqslant d_{\mathscr{X}}\big(\sigma(t), g(t + h_n)\big) + d_{\mathscr{X}}\big(g(t + h_n), g(s + h_n)\big) + d_{\mathscr{X}}\big(g(s + h_n), \sigma(s)\big) \\ &\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

and σ is uniformly continuous on $\mathbb{R}$. Hence by Corollary 3.3.19, σ is a tr. c. mapping.

It remains to prove that $\mathscr{H}(\sigma) \subseteq \mathscr{H}(g)$. For that, notice that since $\sigma(\cdot) = \lim_{n\to\infty} g(\cdot + h_n)$ in Ξ it implies for any $h \in \mathbb{R}$ that $\sigma(\cdot + h) = \lim_{n\to\infty} g(\cdot + h_n + h)$ in Ξ and hence

$$\big\{\sigma(\cdot + h) : h \in \mathbb{R}\big\} \subseteq \mathscr{H}(g),$$

and we conclude that $\mathscr{H}(\sigma) \subseteq \mathscr{H}(g)$.

Proof of (ii): It follows directly from (3.53), since $\overline{\{g(t) : t \in \mathbb{R}\}}$ is compact in $\mathscr{X}$.

Proof of (iii): By Proposition 3.3.18, since $\mathscr{H}(g)$ is compact then given $t_0 \in \mathbb{R}$ and $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, t_0) > 0$ such that for $t \in \mathbb{R}$ with $|t - t_0| < \delta$ it follows $d_{\mathscr{X}}\big(\xi(t), \xi(t_0)\big) < \varepsilon/3$, for all $\xi \in \Gamma$. Let $\sigma \in \mathscr{H}(g)$. So $\sigma(\cdot) = \lim_{n\to\infty} \sigma_n(\cdot)$ in Ξ, where $\sigma_n \in \Gamma$. Therefore, for n large enough we have

$$\begin{aligned} d_{\mathscr{X}}\big(\sigma(t), \sigma(t_0)\big) &\leqslant d_{\mathscr{X}}\big(\sigma(t), \sigma_n(t)\big) + d_{\mathscr{X}}\big(\sigma_n(t), \sigma_n(t_0)\big) + d_{\mathscr{X}}\big(\sigma_n(t_0), \sigma(t_0)\big) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

proving (iii).

Proof of (iv): Let $h\in\mathbb{R}$ be fixed. If $\sigma\in\mathscr{H}(g)$, then $\sigma(\cdot)=\lim_{n\to\infty}g(\cdot+h_n)$ in Ξ, where $h_n\in\mathbb{R}$. By the continuity of $\theta_h:\Xi\to\Xi$ we have

$$\theta_h\sigma(\cdot)=\lim_{n\to\infty}\theta_h g(\cdot+h_n)=\lim_{n\to\infty}g(\cdot+h_n+h),$$

and hence $\theta_h\mathscr{H}(g)\subseteq\mathscr{H}(g)$.

Conversely, $\theta_{-h}\sigma(\cdot)=\lim_{n\to\infty}\theta_{-h}g(\cdot+h_n)=\lim_{n\to\infty}g(\cdot+h_n-h)$ and then $\theta_{-h}\sigma(\cdot)\in\mathscr{H}(g)$. Therefore we obtain $\sigma=\theta_h\theta_{-h}\sigma\in\theta_h\mathscr{H}(g)$, proving that $\mathscr{H}(g)\subseteq\theta_h\mathscr{H}(g)$. $\square$

In the following we establish several criteria for the hull $\mathscr{H}(g)$ of a function g in $\Xi=\mathscr{C}(\mathbb{R};\mathscr{X})$ to be finite-dimensional. We begin with a dynamical criterion via conditions on a complete trajectory that goes through g of the group $\{\theta_t\}_{t\in\mathbb{R}}$ in Ξ, and then we turn back to conditions on g itself that ensure the hull $\mathscr{H}(g)$ is finite-dimensional. The following results were established in (CUI *et al.*,) and extended our understanding about finite dimensional subsets of the space of continuous functions.

3.3.3.2 A dynamical criterion

For $g\in\Xi=\mathscr{C}(\mathbb{R};\mathscr{X})$, define $\xi^g_{(\cdot)}:\mathbb{R}\to\Xi$, $h\mapsto\xi^g_h$, by

$$\xi^g_h=\theta_h g,\qquad h\in\mathbb{R}. \tag{3.54}$$

Then $\xi^g_h(t)=(\theta_h g)(t)=g(t+h)$ for $t,h\in\mathbb{R}$, and $\theta_s\xi^g_h=\xi^g_{h+s}$ for all $s,h\in\mathbb{R}$. Hence, $\{\xi^g_h\}_{h\in\mathbb{R}}$ is a complete trajectory of the group $\{\theta_t\}_{t\in\mathbb{R}}$ on Ξ with $\xi^g_0=g$, called the complete trajectory of $\{\theta_t\}_{t\in\mathbb{R}}$ through g.

Theorem 3.3.22. *Suppose that $g\in\Xi=\mathscr{C}(\mathbb{R};\mathscr{X})$ and the complete trajectory $\xi^g_{(\cdot)}:\mathbb{R}\to\Xi$ through g of $\{\theta_t\}_{t\in\mathbb{R}}$ is given as (3.54). If*

(T1) $\xi^g_{(\cdot)}$ is Lipschitz continuous from $\mathbb{R}$ to Ξ with Lipschitz constant $L>0$;

(T2) there exist two finite-dimensional sets A_+ and A_- in Ξ such that the trajectory $\xi^g_{(\cdot)}$ converges forwards to A_+ and backwards to A_-, and the convergence rate is eventually exponential, i.e., there exist constants $C,\beta>0$ and a time $T_*>0$ such that

$$d_\Xi\left(\xi^g_t,A_+\right)\leqslant Ce^{-\beta t}\qquad\textit{and}\qquad d_\Xi\left(\xi^g_{-t},A_-\right)\leqslant Ce^{-\beta t},\qquad\forall t\geqslant T_*, \tag{3.55}$$

then the hull $\mathscr{H}(g)$ of g is finite dimensional in Ξ with

$$\dim_F\left(\mathscr{H}(g);\Xi\right)\leqslant\max\Big\{1,\,\dim_F(A_+;\Xi),\,\dim_F(A_-;\Xi)\Big\}.$$

Remark 3.3.23. *The upper bound in the dimension is independent of the parameters L,C,β and T_*.*

Remark 3.3.24. *Theorem 3.3.22 is in fact a general result for any complete trajectory $\xi(\cdot)$ of any group $\{\theta_t\}_{t\in\mathbb{R}}$ in a complete metric space Ξ for which we could obtain the finite-dimensionality of the trace $\{\xi(s)\}_{s\in\mathbb{R}}$ in the phase space Ξ.*

Proof of Theorem 3.3.22. Notice that $\dim_F\big(\mathscr{H}(g);\Xi\big) = \dim_F\big(\mathring{\mathscr{H}}(g);\Xi\big)$, where

$$\mathring{\mathscr{H}}(g) := \big\{\theta_s g : s\in\mathbb{R}\big\} \quad \text{(without closure)},$$

and, by (3.54),

$$\mathring{\mathscr{H}}(g) = \big\{\theta_s g : s\in\mathbb{R}\big\} = \big\{\xi_s^g : s\in\mathbb{R}\big\} = \big\{\xi_s^g\big\}_{s\in\mathbb{R}}.$$

Hence, it suffices to prove that

$$\dim_F\big(\{\xi_s^g\}_{s\in\mathbb{R}};\Xi\big) \leqslant \max\Big\{1, \dim_F(A_+;\Xi), \dim_F(A_-;\Xi)\Big\}.$$

For any $\varepsilon>0$, by the convergence (3.55) there exists a $T_\varepsilon>0$ such that

$$d_\Xi\big(\xi_t^g, A_+\big) < \varepsilon, \qquad \forall t > T_\varepsilon,$$

that is,

$$\{\xi_t^g\}_{t>T_\varepsilon} \subset B_\Xi(A_+,\varepsilon).$$

In addition, by (3.55), i.e., since the convergence rate is eventually exponential, T_ε is chosen as

$$T_\varepsilon = \beta^{-1}\ln\frac{C}{\varepsilon}$$

for small $\varepsilon\in\big(0, Ce^{-\beta T_*}\big]$ for which $T_\varepsilon \geqslant T_*$.

Since by the (pre) compactness of A_+ there is an ε-cover of A_+

$$A_+ \subset \bigcup_{j=1}^{N_\Xi[A_+;\varepsilon]} B_\Xi(x_j,\varepsilon), \qquad x_j\in A_+,$$

we have a (2ε)-cover of $\{\xi_t^g\}_{t>T_\varepsilon}$ such that

$$\{\xi_t^g\}_{t>T_\varepsilon} \subset B_\Xi(A_+,\varepsilon) \subset \bigcup_{j=1}^{N_\Xi[A_+;\varepsilon]} B_\Xi(x_j,2\varepsilon), \qquad x_j\in A_+.$$

This indicates that

$$N_\Xi\big[\{\xi_t^g\}_{t>T_\varepsilon};2\varepsilon\big] \leqslant N_\Xi[A_+;\varepsilon].$$

In the same way we have for the backward part that

$$N_\Xi\big[\{\xi_{-t}^g\}_{t>T_\varepsilon};2\varepsilon\big] \leqslant N_\Xi[A_-;\varepsilon].$$

On the other hand, by the Lipschitz condition of $\xi_{(\cdot)}^g$ we have

$$\begin{aligned} N_\Xi\big[\{\xi_t^g\}_{|t|\leqslant T_\varepsilon};2\varepsilon\big] &\leqslant \frac{LT_\varepsilon}{\varepsilon}+1 \\ &= \frac{L\beta^{-1}}{\varepsilon}\ln\frac{C}{\varepsilon}+1, \quad \forall\varepsilon\in\big(0,Ce^{-\beta T_*}\big]. \end{aligned}$$

Hence, for $\varepsilon \in \left(0, Ce^{-\beta T_*}\right]$,

$$N_\Xi\left[\{\xi_t^g\}_{t\in\mathbb{R}};2\varepsilon\right] \leqslant \frac{L\beta^{-1}}{\varepsilon}\ln\frac{C}{\varepsilon}+1+N_\Xi\left[A_+;\varepsilon\right]+N_\Xi\left[A_-;\varepsilon\right]$$
$$\leqslant 3\max\left\{\frac{L\beta^{-1}}{\varepsilon}\ln\frac{C}{\varepsilon}+1,\, N_\Xi\left[A_+;\varepsilon\right],\, N_\Xi\left[A_-;\varepsilon\right]\right\}.$$

Therefore, $\{\xi_t^g\}_{t\in\mathbb{R}}$ is precompact in Ξ, and

$$\begin{aligned}\dim_F\left(\{\xi_t^g\}_{t\in\mathbb{R}};\Xi\right) &= \limsup_{\varepsilon\to0^+}\frac{\ln N_\Xi\left[\{\xi_t^g\}_{t\in\mathbb{R}};2\varepsilon\right]}{-\ln 2\varepsilon}\\ &\leqslant \limsup_{\varepsilon\to0^+}\frac{\ln\left(3\max\left\{\frac{L\beta^{-1}}{\varepsilon}\ln\frac{C}{\varepsilon}+1,N_\Xi\left[A_+;\varepsilon\right],N_\Xi\left[A_-;\varepsilon\right]\right\}\right)}{-\ln 2\varepsilon}\\ &\leqslant \max\left\{1,\,\dim_F(A_+;\Xi),\,\dim_F(A_-;\Xi)\right\}.\Box\end{aligned}$$

3.3.3.3 Criterion based on conditions for g

In the previous section we showed in a dynamical way an abstract result (Theorem 3.3.22) via the trajectory $\xi_{(\cdot)}^g$ of $\{\theta_t\}_{t\in\mathbb{R}}$ through g. Now we are interested in conditions for g itself that can ensure the hull $\mathscr{H}(g)$ to be finite-dimensional. Our idea is looking for conditions for g which imply conditions $(T1)$ and $(T2)$ of the trajectory $\xi_{(\cdot)}^g$ in Theorem 3.3.22.

The following lemma shows that the Lipschitz condition $(T1)$ is satisfied if g is Lipschitz.

Lemma 3.3.25. *If $g \in \Xi = \mathscr{C}(\mathbb{R};\mathscr{X})$ is Lipschitz from $\mathbb{R}$ to $\mathscr{X}$, then the complete trajectory $h \mapsto \xi_h^g$ through g is Lipschitz from $\mathbb{R}$ to Ξ.*

Proof. Suppose that g has Lipschitz constant $L_g > 0$. Then for each $n \in \mathbb{N}$ we have

$$\begin{aligned}\frac{1}{2^n}\frac{d^{(n)}(\xi_h^g,\xi_l^g)}{1+d^{(n)}(\xi_h^g,\xi_l^g)} &= \frac{1}{2^n}\frac{d^{(n)}\left(g(\cdot+h),g(\cdot+l)\right)}{1+d^{(n)}\left(g(\cdot+h),g(\cdot+l)\right)}\\ &\leqslant \frac{1}{2^n}d^{(n)}\left(g(\cdot+h),g(\cdot+l)\right)\\ &= \frac{1}{2^n}\max_{s\in[-n,n]}d_{\mathscr{X}}\left(g(s+h),g(s+l)\right)\\ &\leqslant \frac{1}{2^n}\max_{s\in[-n,n]}L_g|h-l| = \frac{L_g}{2^n}|h-l|,\quad h,l\in\mathbb{R}.\end{aligned}$$

Hence,

$$d_\Xi(\xi_h^g,\xi_l^g) = \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{d^{(n)}(\xi_h^g,\xi_l^g)}{1+d^{(n)}(\xi_h^g,\xi_l^g)} \leqslant L_g|h-l|,$$

so $\xi_{(\cdot)}^g$ is Lipschitz with the same Lipschitz constant as g. $\Box$

Now let us keep our focus on conditions for g ensuring $(T2)$. The following theorem shows that $(T2)$ is satisfied if the tail of g converges exponentially to some admissible functions, such as for example quasiperiodic mappings.

Theorem 3.3.26. *Suppose that $g_+, g_- \in \Xi = \mathscr{C}(\mathbb{R};\mathscr{X})$ are two functions with finite-dimensional hulls $\mathscr{H}(g_+)$ and $\mathscr{H}(g_-)$ in Ξ, respectively. If $g \in \Xi$ is a function such that*

(G1) g is Lipschitz continuous from $\mathbb{R}$ to $\mathscr{X}$;

(G2) g converges forwards to g_+ and backwards to g_- eventually exponentially, i.e., there exist a time $T_* \geqslant 0$ and constants $C, \beta > 0$ such that

$$d_{\mathscr{X}}\big(g(t), g_+(t)\big) \leqslant Ce^{-\beta t} \quad \text{and} \quad d_{\mathscr{X}}\big(g(-t), g_-(-t)\big) \leqslant Ce^{-\beta t}, \qquad \forall t \geqslant T_*. \tag{3.56}$$

Then g is tr.c. in Ξ and the hull $\mathscr{H}(g)$ of g is finite dimensional in Ξ with

$$\dim_F\big(\mathscr{H}(g);\Xi\big) \leqslant \max\Big\{1, \dim_F\big(\mathscr{H}(g_+);\Xi\big), \dim_F\big(\mathscr{H}(g_-);\Xi\big)\Big\}.$$

Remark 3.3.27. *We note that:*

(i) The theorem indicates that the finite-dimensionality of the hull $\mathscr{H}(g)$ of Lipschitz $g \in \Xi$ is fully determined by the properties of the "distant tails" of $g(t)$ (for $|t| \geqslant T_*$).

(ii) Remark 3.3.20 implies that quasiperiodic functions in $\Xi = \mathscr{C}(\mathbb{R};\mathscr{X})$ are examples of g_+ and g_-.

(iii) By (CHEPYZHOV; VISHIK, 2002, Example V.2.2, page 99) the hull $\mathscr{H}(g)$ of such g as in Theorem 3.3.26 with g_+ and g_- being quasiperiodic is compact in Ξ, and has the structure

$$\mathscr{H}(g) = \mathring{\mathscr{H}}(g) \cup \mathscr{H}(g_+) \cup \mathscr{H}(g_-),$$

where $\mathring{\mathscr{H}}(g) := \{\theta_r g : r \in \mathbb{R}\}$. Now we know that it is in fact finite-dimensional in $\Xi = \mathscr{C}(\mathbb{R},\mathscr{X})$.

Proof of Theorem 3.3.26. Since g is Lipschitz, by Lemma 3.3.25 we know that the trajectory $\xi^g_{(\cdot)}$ through g is Lipschitz continuous from $\mathbb{R}$ to Ξ, so $(T1)$ in Theorem 3.3.22 is satisfied. In the following we prove $(T2)$ by condition (G2), and then the theorem follows.

Note that (3.56) is equivalent to that for some $\beta' := (\log_2 e)\cdot\beta > 0$

$$d_{\mathscr{X}}\big(g(t), g_+(t)\big) \leqslant C2^{-\beta' t} \qquad \text{and} \qquad d_{\mathscr{X}}\big(g(-t), g_-(-t)\big) \leqslant C2^{-\beta' t}, \quad \forall t \geqslant T_*. \tag{3.57}$$

Without loss of generality let $\beta' \in (0,1)$; otherwise we could take a smaller $\tilde{\beta} \in (0,1)$ satisfying (3.57). Also, let $T_* \in \mathbb{N}$.

For any $h \geqslant T_* + 1$ let us write it as $h = n_h + \rho$ with $n_h \in \mathbb{N}$ and $\rho \in [0,1)$. Then

$$\begin{aligned} d^{(n)}(\xi^g_h, \theta_h g_+) &= \max_{s\in[-n,n]} d_{\mathscr{X}}\big(\xi^g_h(s), \theta_h g_+(s)\big) \\ &= \max_{s\in[-n,n]} d_{\mathscr{X}}\big(g(h+s), g_+(h+s)\big) \\ &= \max_{s\in[h-n,h+n]} d_{\mathscr{X}}\big(g(s), g_+(s)\big) \\ &\leqslant C2^{-\beta'(h-n)}, \quad \text{for all } n \leqslant n_h - T_* \text{ (so that } h-n \geqslant T_* + \rho). \end{aligned}$$

Hence, since $\beta' \in (0,1)$, for all $h \geqslant T_* + 1$ we have

$$
\begin{aligned}
d_\Xi(\xi_h^g, \theta_h g_+) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d^{(n)}(\xi_h^g, \theta_h g_+)}{1 + d^{(n)}(\xi_h^g, \theta_h g_+)} \\
&= \sum_{n=1}^{n_h - T_*} \frac{1}{2^n} \frac{d^{(n)}(\xi_h^g, \theta_h g_+)}{1 + d^{(n)}(\xi_h^g, \theta_h g_+)} + \sum_{n=n_h-T_*+1}^{\infty} \frac{1}{2^n} \frac{d^{(n)}(\xi_h^g, \theta_h g_+)}{1 + d^{(n)}(\xi_h^g, \theta_h g_+)} \\
&\leqslant \sum_{n=1}^{n_h - T_*} \frac{1}{2^n} \frac{d^{(n)}(\xi_h^g, \theta_h g_+)}{1} + \sum_{n=n_h-T_*+1}^{\infty} \frac{1}{2^n} \\
&\leqslant C \sum_{n=1}^{n_h - T_*} \frac{1}{2^n} 2^{-\beta'(h-n)} + \sum_{n=n_h-T_*+1}^{\infty} \frac{1}{2^n} \\
&= C 2^{-\beta' h} \frac{2^{\beta'-1} - 2^{(\beta'-1)(n_h - T_* + 1)}}{1 - 2^{\beta'-1}} + 2^{\rho - h + T_*}.
\end{aligned}
\tag{3.58}
$$

Now we estimate the last term on the right-hand side. Since $\beta' \in (0,1)$ and $n_h - T^* > 0$ we have $2^{\beta'-1} - 2^{(\beta'-1)(n_h-T_*+1)} \leqslant 1$ and then by (3.58)

$$
\begin{aligned}
d_\Xi(\xi_h^g, \theta_h g_+) &\leqslant C 2^{-\beta' h} \frac{2^{\beta'-1} - 2^{(\beta'-1)(n_h - T_* + 1)}}{1 - 2^{\beta'-1}} + 2^{\rho + T_*} \cdot 2^{-h} \\
&\leqslant C 2^{-\beta' h} \frac{1}{1 - 2^{\beta'-1}} + 2^{T_*+1} \cdot 2^{-\beta' h} \\
&\leqslant C_{\beta'} 2^{-\beta' h}, \qquad \forall h \geqslant T^* + 1,
\end{aligned}
$$

where $C_{\beta'} := \frac{C}{1-2^{\beta'-1}} + 2^{T_*+1}$ is a positive constant, so it follows

$$
\begin{aligned}
d_\Xi\big(\xi_h^g, \mathscr{H}(g_+)\big) &= \inf_{\sigma \in \mathscr{H}(g_+)} d_\Xi\big(\xi_h^g, \sigma\big) \\
&\leqslant d_\Xi(\xi_h^g, \theta_h g_+) \quad \big(\text{since } \theta_h g_+ \in \mathscr{H}(g_+)\big) \\
&\leqslant C_{\beta'} 2^{-\beta' h}, \quad h \geqslant T^* + 1.
\end{aligned}
$$

In the same way we have also

$$
d_\Xi\big(\xi_{-h}^g, \mathscr{H}(g_-)\big) \leqslant C_{\beta'} 2^{-\beta' h}, \quad h \geqslant T^* + 1.
$$

Hence, condition $(T2)$ in Theorem 3.3.22 is satisfied as desired. □

Corollary 3.3.28. *Suppose that $g \in \Xi = \mathscr{C}(\mathbb{R}; \mathscr{X})$ is a Lipschitz continuous function that has a bounded support, i.e., there is a $T_* > 0$ such that $g(t) \equiv 0$ for all $|t| \geqslant T_*$. Then the hull $\mathscr{H}(g)$ of g has fractal dimension less than or equal to 1:*

$$
\dim_F \big(\mathscr{H}(g); \Xi\big) \leqslant 1.
$$

Proof. Take $g_+(t) = g_-(t) \equiv 0$ for all $t \in \mathbb{R}$. Then $\mathscr{H}(g_+) = \mathscr{H}(g_-) = \{0\}$ $(\subset \Xi)$, and so $\dim_F \big(\mathscr{H}(g_+); \Xi\big) = \dim_F \big(\mathscr{H}(g_-); \Xi\big) = 0$. By Theorem 3.3.26 we conclude the proof of the corollary. □

3.3.3.4 Examples

From Remark 3.3.20, the hull of quasiperiodic functions in $\Xi = \mathscr{C}(\mathbb{R};\mathscr{X})$ has finite fractal dimension in Ξ, with estimates given by the frequence of the mapping. So based on Theorem 3.3.26 and Corollary 3.3.28 we give the following examples represented in Figure 1 and Figure 2, portraits of maps which have hull with finite fractal dimension in Ξ as well. We notice that these functions are not almost periodic functions according to Definition 3.3.3, and then by Theorem 3.3.14 (Bochner criterion) the hull of them is not even compact in space $\Xi_b = \mathscr{C}_b(\mathbb{R};\mathscr{X})$. In other words, the hull of these functions represented in the following is finite-dimensional in Ξ, while we can not even calculate their fractal dimension in Ξ_b. It gives us the clear information that in order to estimate the fractal dimension of uniform attractors associated to non-autonomous dynamical systmes it depends a lot in which metric space the symbol space is considered.

Figure 1 – Lipschitz functions with constant tails have hull with dimension $\leqslant 1$.

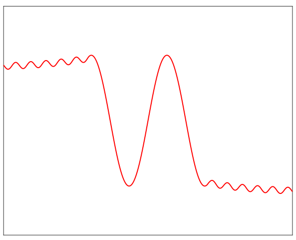

Figure 2 – Lipschitz functions with tails converging exponentially to periodic functions have hull with dimension $\leqslant 1$.

3.4 Application to parabolic equations with translation-compact symbols

In this section, as application of our theoretical analysis, we prove the finite-dimensionality of uniform attractors for two parabolic equations: the 2D Navier-Stokes equation and a reaction-diffusion equation. The non-autonomous terms will be assumed to be continuous in time and translation compact (tr. c.), so the hull could be finite-dimensional under conditions in Section 3.3.3. The results in this section were presented in (CUI *et al.*,).

The following lemma for tr.c. functions is useful for later purposes.

Lemma 3.4.1. *Let* $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ *be a Banach space and* $\Xi := \mathscr{C}(\mathbb{R};\mathscr{X})$. *If* $g \in \Xi$ *is tr.c. in* Ξ *(i.e., the hull* $\mathscr{H}(g)$ *of* g *is compact in* Ξ*), then there is a constant* $c = c(g) > 0$ *such that for any* $\sigma_1, \sigma_2 \in \mathscr{H}(g)$ *we have*

$$\int_\tau^t \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{X}}^q \,\mathrm{d}s \leqslant c\, 2^{q(t-\tau+|\tau|)} \Big(d_\Xi(\sigma_1, \sigma_2)\Big)^q, \qquad \forall t \geqslant \tau,\ q \geqslant 1.$$

Proof. Given arbitrarily $\sigma_1, \sigma_2 \in \mathscr{H}(g)$, denote by $\bar\sigma(s) := \sigma_1(s) - \sigma_2(s)$. Then for all $t \geqslant \tau$ we have

$$\begin{aligned}
&\int_\tau^t \|\bar\sigma(s)\|_{\mathscr{X}}^q \,\mathrm{d}s = \\
&= \int_0^{t-\tau} \|\bar\sigma(s+\tau)\|_{\mathscr{X}}^q \,\mathrm{d}s = \int_0^{t-\tau} 2^{qs-qs}\|\theta_\tau\bar\sigma(s)\|_{\mathscr{X}}^q \,\mathrm{d}s \leqslant 2^{q(t-\tau)} \int_0^{t-\tau} 2^{-qs}\|\theta_\tau\bar\sigma(s)\|_{\mathscr{X}}^q \,\mathrm{d}s \\
&\leqslant 2^{q(t-\tau)} \sum_{i=1}^{[t-\tau]+1} \int_{i-1}^i 2^{-qs}\|\theta_\tau\bar\sigma(s)\|_{\mathscr{X}}^q \,\mathrm{d}s \qquad \Big([t-\tau] = \text{the integer part of } t-\tau\Big) \\
&\leqslant 2^{q(t-\tau)} \sum_{i=1}^{[t-\tau]+1} \Big(2^{-q(i-1)} \max_{s\in[i-1,i]} \|\theta_\tau\bar\sigma(s)\|_{\mathscr{X}}^q\Big) \leqslant 2^{q(t-\tau+1)} \left(\sum_{i=1}^{[t-\tau]+1} 2^{-i} \max_{s\in[i-1,i]} \|\theta_\tau\bar\sigma(s)\|_{\mathscr{X}}\right)^q.
\end{aligned}$$

By Proposition 3.3.21, (ii), $\mathscr{H}(g)$ is bounded in $\mathscr{C}_b(\mathbb{R};\mathscr{X})$, so there is an $r \geqslant 1/2$ such that $\sup_{s\in\mathbb{R}} \|\sigma(s)\|_{\mathscr{X}} \leqslant r$ holds uniformly for all $\sigma \in \mathscr{H}(g)$. Hence, $\sup_{s\in\mathbb{R}} \|\bar\sigma(s)\|_{\mathscr{X}} \leqslant 2r$. Since $\frac{a+\rho}{b+\rho} \geqslant \frac{a}{b}$ for all $0 < a \leqslant b$ and $\rho > 0$, it follows that

$$\begin{aligned}
&\int_\tau^t \|\bar\sigma(s)\|_{\mathscr{X}}^q \,\mathrm{d}s \leqslant \\
&\leqslant 2^{q(t-\tau+1)} \left(\sum_{i=1}^{\infty} 2^{-i} \max_{s\in[i-1,i]} \|\theta_\tau\bar\sigma(s)\|_{\mathscr{X}}\right)^q = 2^{q(t-\tau+1)} \left(2r \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\max_{s\in[i-1,i]} \|\theta_\tau\bar\sigma(s)\|_{\mathscr{X}}}{2r}\right)^q \\
&\leqslant 2^{q(t-\tau+1)} \left(2r \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{2\cdot \max_{s\in[i-1,i]} \|\theta_\tau\bar\sigma(s)\|_{\mathscr{X}}}{2r + \max_{s\in[i-1,i]} \|\theta_\tau\bar\sigma(s)\|_{\mathscr{X}}}\right)^q \leqslant 2^{q(t-\tau+3)} r^q \left(\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\max_{s\in[i-1,i]} \|\theta_\tau\bar\sigma(s)\|_{\mathscr{X}}}{1 + \max_{s\in[i-1,i]} \|\theta_\tau\bar\sigma(s)\|_{\mathscr{X}}}\right)^q \\
&\leqslant 2^{q(t-\tau+3)} r^q \Big(d_\Xi(\theta_\tau\sigma_1, \theta_\tau\sigma_2)\Big)^q \leqslant 2^{q(t-\tau+3)} r^q \cdot 2^{q(|\tau|+1)} \Big(d_\Xi(\sigma_1, \sigma_2)\Big)^q \qquad \text{(by Prop. 3.3.17)},
\end{aligned}$$

which concludes the lemma. ∎

3.4.1 2D Navier-Stokes equation

In this section we consider a non-autonomous version of the 2D Navier-Stokes equation given in Section 2.2.7.2, Chapter 2, and use the smoothing property to bound the dimension of its uniform attractor. For a bounded smooth domain $\mathscr{O} \subset \mathbb{R}^2$, let H and V be the closure of the set

$$\mathscr{V} := \left\{ v \in \left(C_0^\infty(\mathscr{O})\right)^2 \ : \ \nabla \cdot v = 0 \right\}$$

over spaces $\left(L^2(\mathscr{O})\right)^2$ and $\left(H_0^1(\mathscr{O})\right)^2$, respectively. We use $\|\cdot\|$ for the L^2 norm and $(\cdot,\cdot)$ for its inner product. Let A be the Stokes operator defined by

$$Av = -P\Delta v, \qquad \forall v \in D(A) = \left(H^2(\mathscr{O})\right)^2 \cap V,$$

where P is the orthogonal projection onto the divergence-free fields H.

Defining the trilinear form

$$b(u,v,w) := \sum_{i,j=1}^{2} \int_{\mathscr{O}} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,$$

whenever the integrals make sense, we consider the following problem

$$\frac{\partial u}{\partial t} + \gamma A u + B(u,u) = g(x,t), \tag{3.59}$$

$$\nabla \cdot u = 0, \quad u|_{\partial \mathscr{O}} = 0, \quad u|_{t=\tau} = u_\tau, \tag{3.60}$$

where $x = (x_1, x_2) \in \mathscr{O}$, $t \geqslant \tau$, the unknown $u = u(x,t) = \left(u^1(x,t), u^2(x,t)\right)$ is a velocity vector, $u_\tau \in H$, $\gamma > 0$ is a constant, $g(x,t) = \left(g^1(x,t), g^2(x,t)\right)$ is the time-dependent forcing which is considered as the symbol of the equation and $B : V \times V \to V'$ is the bilinear operator defined by

$$\left(B(u,v), w\right) = b(u,v,w), \qquad \forall u,v,w \in V.$$

We have that A^{-1} is a self-adjoint continuous compact operator in H and the following Poincaré's inequality holds

$$\lambda \|u\|^2 \leqslant \|A^{\frac{1}{2}} u\|^2, \qquad \forall u \in V, \tag{3.61}$$

where $\lambda := \sqrt{\lambda_1} > 0$, with λ_1 the first smallest eigenvalue of A. For a detailed interpretation about this setting see, e.g., (TEMAM, 1979; TEMAM, 1988; TEMAM, 1995).

As one of the most important mathematical model of physics, the Navier-Stokes equation has been extensively studied in the literature. Particularly for the uniform attractor theory, (CHEPYZHOV; VISHIK, 2002, Chapter VI.1) showed that with the forcing g being *translation bounded in* $L^2_{loc}(\mathbb{R};H)$, i.e., $g \in L^2_{tr.b}(\mathbb{R};H)$ with

$$\|g\|^2_{L^2_{tr.b}(\mathbb{R};H)} := \sup_{t \in \mathbb{R}} \int_{t-1}^{t} |g(s)|^2 \, ds < \infty, \tag{3.62}$$

the NS equation (3.59) is well-posed and generates a process $U_g = \{U_g(t,\tau) : t \geqslant \tau\}$ in H. Moreover, with the symbol space Σ defined as the hull $\mathscr{H}(g)$ of g in $L^2_{loc}(\mathbb{R};H)$, i.e.,

$$\Sigma := \mathscr{H}(g) = \overline{\left\{ \theta_r g : r \in \mathbb{R} \right\}},$$

where the closure is taken under the topology of $L^2_{loc}(\mathbb{R};H)$ and θ_r is the translation operator on $L^2_{loc}(\mathbb{R};H)$ given by

$$(\theta_r g)(\cdot) = g(\cdot + r),$$

the solutions of the NS equation generate a $(\Sigma \times H, H)$-continuous system $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ of processes in H by

$$U_\sigma(t,\tau;u_\tau) = u_\sigma(t,\tau,u_\tau), \quad \forall t \geqslant \tau,\, u_\tau \in H,\, \sigma \in \Sigma,$$

where $u_\sigma(t,\tau,u_\tau)$ is the unique solution of (3.59)-(3.60) with g replaced by σ. In addition, the system $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ has a uniform attractor $\mathscr{A}_\Sigma$ which is bounded in V and compact in H.

In what follows we are going to study the finite-dimensionality of the uniform attractor $\mathscr{A}_\Sigma$ in both H and V under a stronger condition for the forcing g. Basically, we assume that

$$g \in \Xi := \mathscr{C}(\mathbb{R};H) \text{ and has a finite-dimensional hull } \mathscr{H}(g) \text{ in } \Xi, \tag{3.63}$$

where the metric d_Ξ of Ξ in consideration is the Fréchet metric as in Section 3.3.3. Notice that, when g is time Lipschitz continuous and quasiperiodic, the uniform attractor $\mathscr{A}_\Sigma$ has been shown to be finite-dimensional, see (CHEPYZHOV; VISHIK, 2002, Section IX.2). Now thanks to the analysis in Section 3.3.3, g is allowed to be more general. Theorem 3.3.22 shows that examples of g satisfying (3.63) can be Lipschitz continuous functions with tails eventually exponentially converging to quasiperiodic functions g_+ and g_- satisfying (3.56).

In the following we will consider the symbol space Σ as the hull $\mathscr{H}(g)$ of g in (Ξ, d_Ξ), for which by assumption we have $\dim_F(\Sigma;\Xi) < \infty$. In addition, Proposition 3.3.21, (ii), shows that Σ is bounded in $\mathscr{C}_b(\mathbb{R};H)$, i.e., there exists a constant c_g such that

$$\sup_{\sigma\in\Sigma}\sup_{s\in\mathbb{R}} \|\sigma(s)\|^2 \leqslant c_g. \tag{3.64}$$

3.4.1.1 Uniform estimates of solutions

We now derive some uniform estimates of solutions needed for our later analysis. In the following we denote by $u_\sigma(t,\tau,u_\tau)$ the solution of the NS equation (3.59) with g replaced by σ corresponding to the initial value u_τ at τ.

Lemma 3.4.2. *(ROBINSON, 2001, Proposition 9.2) In the 2D Navier-Stokes equation we have, for some $c_1, c_2 > 0$,*

$$|b(u,v,w)| \leqslant \begin{cases} c_1\|u\|^{\frac12}\|A^{\frac12}u\|^{\frac12}\|A^{\frac12}v\|\|w\|^{\frac12}\|A^{\frac12}w\|^{\frac12}, & u,v,w\in V,\\ c_2\|u\|^{\frac12}\|A^{\frac12}u\|^{\frac12}\|A^{\frac12}v\|^{\frac12}\|Av\|^{\frac12}\|w\|, & u\in V, v\in D(A), w\in H. \end{cases} \tag{3.65}$$

Lemma 3.4.3. *For any $t \geqslant \tau$ and $u_\tau \in H$ the solution $u_\sigma(t,\tau,u_\tau)$ of (3.59) has the uniform (w.r.t. $\sigma \in \Sigma$) estimate*

$$\sup_{\sigma\in\Sigma} \|u_\sigma(t,\tau,u_\tau)\|^2 \leqslant e^{-(t-\tau)}\|u_\tau\|^2 + c, \tag{3.66}$$

$$\sup_{\sigma\in\Sigma} \int_\tau^t \|A^{\frac12}u_\sigma(s,\tau,u_\tau)\|^2 \,\mathrm{d}s \leqslant \|u_\tau\|^2 + ce^{(t-\tau)}, \tag{3.67}$$

where $c > 0$ is an absolute constant independent of $\sigma \in \Sigma$.

Proof. Let $\sigma \in \Sigma$ and denote simply $u = u_\sigma$. Taking the inner product of (3.59) (with g replaced by σ) with u in H we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \gamma\|A^{\frac{1}{2}}u\|^2 = (\sigma, u) \leqslant \frac{\lambda\gamma}{4}\|u\|^2 + c\|\sigma\|^2.$$

Hence, by (3.61) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \lambda\gamma\|u\|^2 + \gamma\|A^{\frac{1}{2}}u\|^2 \leqslant \frac{\lambda\gamma}{2}\|u\|^2 + c\|\sigma\|^2,$$

and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \|u\|^2 + \|A^{\frac{1}{2}}u\|^2 \leqslant c\|\sigma\|^2.$$

By Gronwall's lemma we have

$$\|u(t)\|^2 + \int_\tau^t e^{-(t-s)}\|A^{\frac{1}{2}}u(s)\|^2 \,\mathrm{d}s \leqslant e^{-(t-\tau)}\|u(\tau)\|^2 + c\int_\tau^t e^{-(t-s)}\|\sigma(s)\|^2 \,\mathrm{d}s,$$

and then

$$\|u(t)\|^2 + e^{-(t-\tau)}\int_\tau^t \|A^{\frac{1}{2}}u(s)\|^2 \,\mathrm{d}s \leqslant e^{-(t-\tau)}\|u(\tau)\|^2 + c\int_\tau^t e^{-(t-s)}\|\sigma(s)\|^2 \,\mathrm{d}s. \qquad (3.68)$$

Notice that

$$\begin{aligned}\sup_{\sigma\in\Sigma}\int_\tau^t e^{-(t-s)}\|\sigma(s)\|^2 \,\mathrm{d}s &= \sup_{\sigma\in\Sigma}\int_{\tau-t}^0 e^s\|\sigma(s+t)\|^2 \,\mathrm{d}s \\ &= \sup_{\sigma\in\Sigma}\int_{\tau-t}^0 e^s\|\sigma(s)\|^2 \,\mathrm{d}s \\ &\leqslant c_g \qquad \text{(by (3.64))}.\end{aligned}$$

Therefore, from (3.68) the lemma follows. □

Lemma 3.4.4. *For any bounded set $E \subset H$ with $\|E\| \leqslant R$ (i.e. $\|x\| \leqslant R$ for any $x \in E$) there exists a constant $c_R > 0$ such that the solution $u_\sigma(t,\tau,u_\tau)$ of (3.59) has the estimate*

$$\sup_{\sigma\in\Sigma}\sup_{u_\tau\in E}\left(\|A^{\frac{1}{2}}u_\sigma(t,\tau,u_\tau)\|^2 + \int_{t-\frac{\varepsilon}{3}}^t \|Au_\sigma(s,\tau,u_\tau)\|^2 \,\mathrm{d}s\right) \leqslant \frac{c_R}{\varepsilon} + c_R, \qquad (3.69)$$

whenever $t-\tau \geqslant \varepsilon$, with $\varepsilon \in (0,1]$.

Proof. Let $\sigma \in \Sigma$ and denote simply $u = u_\sigma$. Taking the inner product of (3.59) with Au in H we have by (3.65) that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^{\frac{1}{2}}u\|^2 + \gamma\|Au\|^2 &= (\sigma, Au) - b(u,u,Au) \\ &\leqslant \|\sigma\|\|Au\| + c_2\|u\|^{\frac{1}{2}}\|A^{\frac{1}{2}}u\|\|Au\|^{\frac{3}{2}}.\end{aligned}$$

Since by (3.66) we have the uniform bound $|u_\sigma(t,\tau,u_\tau)| \leqslant c_R$ for some $c_R > 0$ for all $t \geqslant \tau$, $\sigma \in \Sigma$ and $u_\tau \in E$, it follows that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^{\frac{1}{2}}u\|^2 + \gamma\|Au\|^2 &\leqslant \|\sigma\|\|Au\| + c\|A^{\frac{1}{2}}u\|\|Au\|^{\frac{3}{2}} \\ &\leqslant \frac{\gamma}{2}\|Au\|^2 + c\|A^{\frac{1}{2}}u\|^4 + c\|\sigma\|^2,\end{aligned}$$

and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{\frac{1}{2}}u\|^2+\|Au\|^2 \leqslant c\|A^{\frac{1}{2}}u\|^4+c\|\sigma\|^2.$$

By Gronwall's lemma, for all $t \geqslant s > \tau$ we have

$$\begin{aligned}
&\|A^{\frac{1}{2}}u(t)\|^2+\int_s^t e^{\int_\xi^t c\|A^{\frac{1}{2}}u(\eta)\|^2\mathrm{d}\eta}\|Au(\xi)\|^2\,\mathrm{d}\xi \\
&\quad\leqslant e^{\int_s^t c\|A^{\frac{1}{2}}u(\eta)\|^2\mathrm{d}\eta}\|A^{\frac{1}{2}}u(s)\|^2+\int_s^t e^{\int_\xi^t c\|A^{\frac{1}{2}}u(\eta)\|^2\mathrm{d}\eta}\|\sigma(\xi)\|^2\,\mathrm{d}\xi.
\end{aligned}$$

Integrate the above inequality w.r.t. s over $(t-\frac{2\varepsilon}{3},t-\frac{\varepsilon}{3})$ to obtain

$$\begin{aligned}
&\frac{\varepsilon}{3}\|A^{\frac{1}{2}}u(t)\|^2+\frac{\varepsilon}{3}\int_{t-\frac{\varepsilon}{3}}^t e^{\int_\xi^t c\|A^{\frac{1}{2}}u(\eta)\|^2\mathrm{d}\eta}\|Au(\xi)\|^2\,\mathrm{d}\xi \\
&\quad\leqslant \int_{t-\frac{2\varepsilon}{3}}^{t-\frac{\varepsilon}{3}} e^{\int_s^t c\|A^{\frac{1}{2}}u(\eta)\|^2\mathrm{d}\eta}\|A^{\frac{1}{2}}u(s)\|^2\,\mathrm{d}s+\frac{\varepsilon}{3}\int_{t-\frac{2\varepsilon}{3}}^t e^{\int_\xi^t c\|A^{\frac{1}{2}}u(\eta)\|^2\mathrm{d}\eta}\|\sigma(\xi)\|^2\,\mathrm{d}\xi \\
&\quad\leqslant e^{\int_{t-\frac{2\varepsilon}{3}}^t c\|A^{\frac{1}{2}}u(\eta)\|^2\mathrm{d}\eta}\left(\int_{t-\frac{2\varepsilon}{3}}^{t-\frac{\varepsilon}{3}}\|A^{\frac{1}{2}}u(s)\|^2\,\mathrm{d}s+\frac{\varepsilon}{3}\int_{t-\frac{2\varepsilon}{3}}^t\|\sigma(\xi)\|^2\,\mathrm{d}\xi\right).
\end{aligned}\tag{3.70}$$

Hence, since by (3.66) and (3.67) we have the bound

$$\begin{aligned}
\int_{t-\frac{2\varepsilon}{3}}^t \|A^{\frac{1}{2}}u(\eta)\|^2\,\mathrm{d}\eta &\leqslant \left\|u\Big(t-2\varepsilon/3\Big)\right\|^2+ce^{\frac{2\varepsilon}{3}} \qquad (\text{by } (3.67)) \\
&\leqslant e^{-(t-\frac{2\varepsilon}{3}-\tau)}\|u_\tau\|^2+c+ce^{\frac{2\varepsilon}{3}} \qquad (\text{by } (3.66)) \\
&\leqslant R^2+c, \qquad \big(\text{since } t-\tau\geqslant\varepsilon \text{ and } \varepsilon\in(0,1]\big)
\end{aligned}$$

for all $u_\tau \in E$, it follows from (3.70) that

$$\begin{aligned}
\frac{\varepsilon}{3}\|A^{\frac{1}{2}}u(t)\|^2+\frac{\varepsilon}{3}\int_{t-\frac{\varepsilon}{3}}^t\|Au(\xi)\|^2\,\mathrm{d}\xi &\leqslant \frac{\varepsilon}{3}\|A^{\frac{1}{2}}u(t)\|^2+\frac{\varepsilon}{3}\int_{t-\frac{\varepsilon}{3}}^t e^{\int_\xi^t c\|A^{\frac{1}{2}}u(\eta)\|^2\mathrm{d}\eta}\|Au(\xi)\|^2\,\mathrm{d}\xi \\
&\leqslant c_R+\frac{\varepsilon}{3}c_R\int_{t-1}^t\|\sigma(\xi)\|^2\,\mathrm{d}\xi \\
&\leqslant c_R+\frac{\varepsilon}{3}c_Rc_g \qquad (\text{by } (3.64)).
\end{aligned}$$

The proof is complete. □

3.4.1.2 Lipschitz continuity and smoothing on bounded sets

Take arbitrarily $\sigma_j\in\Sigma$ and $u_{\tau,j}\in H$, $j=1,2$. Denote by $\bar\sigma=\sigma_1-\sigma_2$, and for $t\geqslant\tau$

$$\bar u=\bar u(t,\tau):=u_{\sigma_1}(t,\tau,u_{\tau,1})-u_{\sigma_2}(t,\tau,u_{\tau,2}),$$

the difference of the two solutions corresponding to initial data $u_{\tau,j}\in H$, respectively. Denoting $u_i:=u_{\sigma_i}$, for $i=1,2$, then $\bar u$ satisfies

$$\begin{aligned}
&\frac{\partial\bar u}{\partial t}+\gamma A\bar u+B(\bar u,u_1)+B(u_2,\bar u)=\bar\sigma, \\
&\nabla\cdot\bar u=0,\quad \bar u|_{\partial\mathscr{O}}=0,\quad \bar u|_{t=\tau}=u_{\tau,1}-u_{\tau,2}.
\end{aligned}\tag{3.71}$$

Lemma 3.4.5 ($(\Sigma\times H,H)$-Lipschitz)**.** *For any bounded set $E\subset H$ with $\|E\|\leqslant R$ there exists a constant $c_R>0$ such that the difference of two solutions defined above with initial values $u_{\tau,1},u_{\tau,2}\in E$ satisfies*

$$\big\|\bar u\big(t,\tau,\bar u(\tau)\big)\big\|^2\leqslant e^{(c_R+(t-\tau)\frac{c_R}{\varepsilon})}\Big[\|\bar u(\tau)\|^2+c_Re^{c(t-\tau+|\tau|)}\big(d_\Xi(\sigma_1,\sigma_2)\big)^2\Big],$$

whenever $t-\tau\geqslant\varepsilon$, with $\varepsilon\in(0,1]$.

Proof. Take the inner product of (3.71) with $\bar u$ in H to obtain by (3.65) that

$$\begin{aligned}\frac12\frac{\mathrm d}{\mathrm dt}\|\bar u\|^2+\gamma\|A^{\frac12}\bar u\|^2&=-b(\bar u,u_1,\bar u)-b(u_2,\bar u,\bar u)+(\bar\sigma,\bar u)\\&\leqslant c_1\|\bar u\|\|A^{\frac12}\bar u\|\|A^{\frac12}u_1\|+(\bar\sigma,\bar u)\\&\leqslant\frac{\gamma}{2}\|A^{\frac12}\bar u\|^2+c\|\bar u\|^2\|A^{\frac12}u_1\|^2+c\|\bar\sigma\|^2\end{aligned}$$

where we have used the fact that $b(u_2,\bar u,\bar u)=0$ and the Poincaré's inequality (3.61). Hence,

$$\frac{\mathrm d}{\mathrm dt}\|\bar u\|^2+\|A^{\frac12}\bar u\|^2\leqslant c\|A^{\frac12}u_1\|^2\|\bar u\|^2+c\|\bar\sigma\|^2.$$

By Gronwall's inequality, for any $t>\tau$ we have

$$\begin{aligned}&\|\bar u(t)\|^2+\int_\tau^t e^{\int_s^t c\|A^{\frac12}u_1(\eta)\|^2\mathrm d\eta}\|A^{\frac12}\bar u(s)\|^2\,\mathrm ds\leqslant\\&\leqslant e^{\int_\tau^t c\|A^{\frac12}u_1(\eta)\|^2\mathrm d\eta}\|\bar u(\tau)\|^2+c\int_\tau^t e^{\int_s^t c\|A^{\frac12}u_1(\eta)\|^2\mathrm d\eta}\|\bar\sigma(s)\|^2\,\mathrm ds\\&\leqslant e^{\int_\tau^t c\|A^{\frac12}u_1(\eta)\|^2\mathrm d\eta}\bigg(\|\bar u(\tau)\|^2+c\int_\tau^t\|\bar\sigma(s)\|^2\,\mathrm ds\bigg).\end{aligned}\tag{3.72}$$

Since $t-\tau\geqslant\varepsilon$, with $\varepsilon\in(0,1]$, by (3.67) and (3.69) we obtain

$$\begin{aligned}\int_\tau^t c\|A^{\frac12}u_1(\eta)\|^2\,\mathrm d\eta&=\int_\tau^{\tau+\varepsilon}c\|A^{\frac12}u_1(\eta)\|^2\,\mathrm d\eta+\int_{\tau+\varepsilon}^t c\|A^{\frac12}u_1(\eta)\|^2\,\mathrm d\eta\\&\leqslant\Big(c\|u_{\tau,1}\|^2+ce^{\varepsilon}\Big)+c\big(t-(\tau+\varepsilon)\big)\Big(\frac{c_R}{\varepsilon}+c_R\Big)\\&\leqslant c_R+(t-\tau)\frac{c_R}{\varepsilon},\end{aligned}$$

where $c_R>0$ is a constant depending on R but independent of σ, ε, t and τ. Hence, by (3.72),

$$\begin{aligned}\|\bar u(t)\|^2+\int_\tau^t\|A^{\frac12}\bar u(s)\|^2\,\mathrm ds&\leqslant\|\bar u(t)\|^2+\int_\tau^t e^{\int_s^t c\|A^{\frac12}u_1(\eta)\|^2\mathrm d\eta}\|A^{\frac12}\bar u(s)\|^2\,\mathrm ds\\&\leqslant e^{c_R+(t-\tau)\frac{c_R}{\varepsilon}}\bigg(\|\bar u(\tau)\|^2+\int_\tau^t\|\bar\sigma(s)\|^2\,\mathrm ds\bigg),\end{aligned}\tag{3.73}$$

which along with Lemma 3.4.1 concludes the lemma. □

Lemma 3.4.6 ($(\Sigma\times H,V)$-smoothing)**.** *For any bounded set $E\subset H$ with $\|E\|\leqslant R$ there exists a constant $c_R>0$ such that the difference $\bar u$ of two solutions with initial values $u_{\tau,1},u_{\tau,2}\in E$ satisfies*

$$\big\|A^{\frac12}\bar u\big(t,\tau,\bar u(\tau)\big)\big\|^2\leqslant\varepsilon^{-1}e^{(t-\tau+1)\frac{c_R}{\varepsilon}}\Big[\|\bar u(\tau)\|^2+e^{|\tau|}\big(d_\Xi(\sigma_1,\sigma_2)\big)^2\Big],$$

whenever $t-\tau\geqslant\varepsilon$, with $\varepsilon\in(0,1]$.

Proof. Take the inner product of (3.71) with $A\bar{u}$ in H and use (3.65) to obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^{\frac{1}{2}}\bar{u}\|^2+\gamma\|A\bar{u}\|^2 &= -b(\bar{u},u_1,A\bar{u})-b(u_2,\bar{u},A\bar{u})+(\bar{\sigma},A\bar{u})\leqslant\\
&\leqslant c_2\|\bar{u}\|^{\frac{1}{2}}\|A^{\frac{1}{2}}\bar{u}\|^{\frac{1}{2}}\|A^{\frac{1}{2}}u_1\|^{\frac{1}{2}}\|Au_1\|^{\frac{1}{2}}\|A\bar{u}\|+c_2\|u_2\|^{\frac{1}{2}}\|A^{\frac{1}{2}}u_2\|^{\frac{1}{2}}\|A^{\frac{1}{2}}\bar{u}\|^{\frac{1}{2}}\|A\bar{u}\|^{\frac{3}{2}}+(\bar{\sigma},A\bar{u})\\
&\leqslant \gamma\|A\bar{u}\|^2+c\Big(\|\bar{u}\|\|A^{\frac{1}{2}}\bar{u}\|\|A^{\frac{1}{2}}u_1\|\|Au_1\|+\|u_2\|^2\|A^{\frac{1}{2}}u_2\|^2\|A^{\frac{1}{2}}\bar{u}\|^2\Big)+c\|\bar{\sigma}\|^2\\
&\leqslant \gamma\|A\bar{u}\|^2+c\Big(\|A^{\frac{1}{2}}\bar{u}\|^2\|Au_1\|^2+\|u_2\|^2\|Au_2\|^2\|A^{\frac{1}{2}}\bar{u}\|^2\Big)+c\|\bar{\sigma}\|^2.
\end{aligned}
$$

Hence,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|A^{\frac{1}{2}}\bar{u}\|^2\leqslant c\Big(\|Au_1\|^2+\|u_2\|^2\|Au_2\|^2\Big)\|A^{\frac{1}{2}}\bar{u}\|^2+c\|\bar{\sigma}\|^2.
$$

By Gronwall's lemma we have

$$
\begin{aligned}
\|A^{\frac{1}{2}}\bar{u}(t)\|^2 &\leqslant e^{\int_s^t c(\|Au_1(\eta)\|^2+\|u_2(\eta)\|^2\|Au_2(\eta)\|^2)\mathrm{d}\eta}\|A^{\frac{1}{2}}\bar{u}(s)\|^2+\\
&\quad+\int_s^t e^{\int_\xi^t c(\|Au_1(\eta)\|^2+\|u_2(\eta)\|^2\|Au_2(\eta)\|^2)\mathrm{d}\eta}\|\bar{\sigma}(\xi)\|^2\,\mathrm{d}\xi,\qquad \forall t\geqslant s>\tau.
\end{aligned}
$$

Now, for $t-\tau\geqslant\varepsilon$, with $\varepsilon\in(0,1]$, integrate the above inequality w.r.t. s over $(t-\varepsilon/3,t)$ to obtain

$$
\begin{aligned}
\frac{\varepsilon}{3}\|A^{\frac{1}{2}}\bar{u}(t)\|^2 &\leqslant \int_{t-\frac{\varepsilon}{3}}^t e^{\int_s^t c(\|Au_1(\eta)\|^2+\|u_2(\eta)\|^2\|Au_2(\eta)\|^2)\mathrm{d}\eta}\|A^{\frac{1}{2}}\bar{u}(s)\|^2\,\mathrm{d}s+\\
&\quad+\frac{\varepsilon}{3}\int_{t-\frac{\varepsilon}{3}}^t e^{\int_\xi^t c(\|Au_1(\eta)\|^2+\|u_2(\eta)\|^2\|Au_2(\eta)\|^2)\mathrm{d}\eta}\|\bar{\sigma}(\xi)\|^2\,\mathrm{d}\xi\\
&\leqslant e^{\int_{t-\frac{\varepsilon}{3}}^t c(\|Au_1(\eta)\|^2+\|u_2(\eta)\|^2\|Au_2(\eta)\|^2)\mathrm{d}\eta}\left(\int_{t-\frac{\varepsilon}{3}}^t\|A^{\frac{1}{2}}\bar{u}(s)\|^2\,\mathrm{d}s+\int_{t-\frac{\varepsilon}{3}}^t\|\bar{\sigma}(s)\|^2\,\mathrm{d}s\right).
\end{aligned}
\tag{3.74}
$$

Since by (3.66) we know that $|u_2(\eta)|^2$ is uniformly bounded, by (3.69) we have

$$
\begin{aligned}
\int_{t-\frac{\varepsilon}{3}}^t c(\|Au_1(\eta)\|^2+\|u_2(\eta)\|^2\|Au_2(\eta)\|^2)\,\mathrm{d}\eta &\leqslant c_R\int_\tau^t(\|Au_1(\eta)\|^2+\|Au_2(\eta)\|^2)\,\mathrm{d}\eta\\
&\leqslant \frac{c_R}{\varepsilon}+c_R,
\end{aligned}
$$

where $c_R>0$ is a constant depending only on R and independent of σ. Therefore,

$$
\begin{aligned}
\frac{\varepsilon}{3}\|A^{\frac{1}{2}}\bar{u}(t)\|^2 &\leqslant e^{(\frac{c_R}{\varepsilon}+c_R)}\left(\int_\tau^t\|A^{\frac{1}{2}}\bar{u}(s)\|^2\,\mathrm{d}s+\int_\tau^t\|\bar{\sigma}(s)\|^2\,\mathrm{d}s\right) && \text{(by (3.74))}\\
&\leqslant e^{(\frac{c_R}{\varepsilon}+c_R)}\cdot e^{c_R+(t-\tau)\cdot\frac{c_R}{\varepsilon}}\left(\|\bar{u}(\tau)\|^2+c\int_\tau^t\|\bar{\sigma}(s)\|^2\,\mathrm{d}s\right) && \text{(by (3.73))}\\
&\leqslant e^{(t-\tau+3)\frac{c_R}{\varepsilon}}\left(\|\bar{u}(\tau)\|^2+ce^{c(t-\tau+|\tau|)}\big(d_\Xi(\sigma_1,\sigma_2)\big)^2\right), && \text{(as } \varepsilon\in(0,1])
\end{aligned}
$$

where the last inequality follows by Lemma 3.4.1, and then the lemma is concluded. □

3.4.1.3 Finite-dimensionality in H and in V of the uniform attractor

Now we are ready to conclude that the uniform attractor $\mathscr{A}_\Sigma$ of the 2D Navier-Stokes equation (3.59) has finite fractal dimension in both H and V.

Theorem 3.4.7. *Assume that the symbol space Σ is finite-dimensional in $\Xi = \mathscr{C}(\mathbb{R};H)$, i.e.,* (3.63) *holds. Then the fractal dimensions in H and in V of the uniform attractor $\mathscr{A}_\Sigma$ for the 2D Navier-Stokes equation* (3.59) *are both finite.*

Proof. Remember Theorem 3.2.1. With $X := H$ and $Y := V$, (H_4) is clearly satisfied. Lemma 3.4.5 (with $\tau = 0$ and $\varepsilon = 1$) proves (H_2) and (H_3), and finally (H_5) is proved by Lemma 3.4.6 (taking $\sigma_1 = \sigma_2$). Therefore, by Theorem 3.2.1 we conclude that the uniform attractor is finite-dimensional in H.

On the other hand, since Lemma 3.4.6 with $t = \varepsilon = 1$ and $\tau = 0$ proves also (H_6) for $\bar{t} = 1$ and $\delta_1 = \delta_2 = 1$, by Theorem 3.2.3 we have the finite-dimensionality of the uniform attractor in V. □

3.4.2 A reaction-diffusion equation

In this section we study the following reaction-diffusion equation

$$\begin{aligned} &\frac{\partial v}{\partial t} + \lambda v - \Delta v = f(v) + \sigma(x,t), \\ &v(x,t)|_{t=\tau} = v_\tau(x), \quad v(x,t)|_{\partial\mathscr{O}} = 0, \qquad x \in \mathscr{O},\ t \geqslant \tau, \end{aligned} \tag{3.75}$$

where $\mathscr{O} \subset \mathbb{R}^N$, $N \in \mathbb{N}$, is a bounded smooth domain and $\lambda > 0$ is a constant. The nonlinear term $f(\cdot) \in \mathscr{C}^1(\mathbb{R};\mathbb{R})$ is assumed to satisfy the following standard conditions for all $s \in \mathbb{R}$:

$$f(s)s \leqslant -\alpha_1|s|^p + \beta_1, \tag{3.76}$$

$$|f(s)| \leqslant \alpha_2|s|^{p-1} + \alpha_2, \tag{3.77}$$

$$|f'(s)| \leqslant \kappa_2|s|^{p-2} + l_2, \tag{3.78}$$

$$f'(s) \leqslant -\kappa_1|s|^{p-2} + l_1, \tag{3.79}$$

where $p \geqslant 2$ and all the coefficients are positive constants. Note that condition (3.78) is in fact equivalent to the following form commonly used in the literature:

$$|f(s_1) - f(s_2)| \leqslant c|s_1 - s_2|\big(1 + |s_1|^{p-2} + |s_2|^{p-2}\big). \tag{3.80}$$

The following result was obtained by (CUI; KLOEDEN; ZHAO, , Corollary 3.3) and will facilitate our computations later.

Lemma 3.4.8. *(CUI; KLOEDEN; ZHAO, , Corollary 3.3) For any $\mathscr{C}^1$-function f satisfying conditions* (3.76), (3.77) *and* (3.79) *there are positive constants $c_1, c_2 > 0$ such that*

$$-\big(f(s_1) - f(s_2)\big)(s_1 - s_2)|s_1 - s_2|^r \geqslant c_1|s_1 - s_2|^{p+r} - c_2|s_1 - s_2|^{r+2},$$

for any $r \geqslant 0$ and $s_1, s_2 \in \mathbb{R}$.

The non-autonomous symbol σ is in a symbol space Σ constructed as the hull $\mathscr{H}(g)$ of a given non-autonomous function $g \in \Xi := \mathscr{C}\big(\mathbb{R}; L^2(\mathscr{O})\big)$, i.e.,

$$\Sigma = \mathscr{H}(g) := \overline{\big\{\theta_r g : r \in \mathbb{R}\big\}},$$

where the closure is taken under the metric d_Ξ of Ξ as in Section 3.3.3. We suppose that Σ has finite fractal dimension $\dim_F(\Sigma;\Xi) < \infty$. In Section 3.3.3 we gave examples of such functions.

Let

$$H := \big(L^2(\mathscr{O}), \|\cdot\|\big), \qquad V := H_0^1(\mathscr{O}) \qquad \text{and} \qquad Z := \big(L^p(\mathscr{O}), \|\cdot\|_p\big).$$

Then the reaction-diffusion equation (3.75) is well-posed in H and, similarly to the Navier-Stokes equation studied in Section 3.4.1, its solutions generate a system $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ of processes in H by

$$U_\sigma(t,\tau;v_\tau) = v_\sigma(t,\tau,v_\tau), \qquad \forall t \geqslant \tau,\, v_\tau \in H,\, \sigma \in \Sigma,$$

where $v_\sigma(t,\tau,v_\tau)$ is the unique solution of (3.75). In addition, the system $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ has a uniform attractor $\mathscr{A}_\Sigma$.

In the following we shall see that the uniform attractor $\mathscr{A}_\Sigma$ is finite-dimensional in H by Theorem 3.2.1 and, moreover, finite-dimensional in the Banach space $Z = L^p(\mathscr{O})$ by Theorem 3.2.3. It is known that for all $p \geqslant 2$ the solution of the reaction-diffusion equation is (H,V)-continuous in initial data, as one can see in (CUI; KLOEDEN; ZHAO,). However, in order to verify the (H,V)-smoothing property (H_5), which is stronger than the (H,V)-continuity, we need the following additional assumption on the growth order p of the nonlinearity:

$$\begin{cases} p \geqslant 2, & N = 1,2; \\ 2 \leqslant p \leqslant \frac{2N-2}{N-2}, & N \geqslant 3, \end{cases}$$

which ensures the continuous embedding $V \hookrightarrow L^{2p-2}(\mathscr{O})$ with

$$\|u\|_{2p-2} \leqslant c\|\nabla u\|, \qquad \forall u \in V, \tag{3.81}$$

for some $c > 0$, as you can see in (ROBINSON, 2001, Theorem 5.26).

3.4.2.1 Uniform estimates of solutions

We begin with some uniform estimates of solutions.

Lemma 3.4.9 (Uniform estimates in V and Z). *There is an absolute constant $c > 0$ such that any solution of the reaction-diffusion equation (3.75) with initial value $v_\tau \in H$ has the uniform (w.r.t. $\sigma \in \Sigma$) estimate*

$$\sup_{\sigma\in\Sigma}\left(\varepsilon\|v_\sigma(t)\|_p^p + \int_\tau^{\tau+\varepsilon}\int_r^t e^{-(t-s)}\|v_\sigma(s)\|_{2p-2}^{2p-2}\,\mathrm{d}s\mathrm{d}r\right) \leqslant ce^{-\varepsilon}\|v_\tau\|^2 + c(\varepsilon+1), \tag{3.82}$$

$$\sup_{\sigma\in\Sigma}\|\nabla v_\sigma(t)\|^2 \leqslant \frac{ce^{-\varepsilon}}{\varepsilon}\|v_\tau\|^2 + c\left(\frac{1}{\varepsilon}+1\right), \tag{3.83}$$

whenever $t \geqslant \tau + \varepsilon$, with $\varepsilon > 0$.

Proof. Let $\sigma \in \Sigma$ and denote simply $v = v_\sigma$. Taking the inner product of (3.75) with v in H we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \lambda\|v\|^2 + \|\nabla v\|^2 = (f(v), v) + (\sigma, v)$$
$$\leqslant -\alpha_1\|v\|_p^p + c + c\|\sigma\|^2 + \frac{\lambda}{2}\|v\|^2 \quad \text{(by (3.76))},$$

and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \|v\|^2 + \|v\|_p^p + \|\nabla v\|^2 \leqslant c + c\|\sigma\|^2.$$

Hence, by Gronwall's inequality, for any $t \geqslant \tau$ we have

$$\begin{aligned} &\|v(t)\|^2 + \int_\tau^t e^{-(t-s)}\Big(\|v(s)\|_p^p + \|\nabla v(s)\|^2\Big)\,\mathrm{d}s \leqslant \\ &\leqslant e^{-(t-\tau)}\|v(\tau)\|^2 + c\int_\tau^t e^{-(t-s)}\Big(1 + \|\sigma(s)\|^2\Big)\,\mathrm{d}s \\ &\leqslant e^{-(t-\tau)}\|v_\tau\|^2 + c, \qquad \text{(by (3.64))}. \end{aligned} \tag{3.84}$$

Taking the inner product of (3.75) with $|v|^{p-2}v$ in H, we obtain

$$\begin{aligned} \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_p^p + \lambda\|v\|_p^p &\leqslant \big(f(v), |v|^{p-2}v\big) + (\sigma, |v|^{p-2}v) \\ &\leqslant \big(f(v), |v|^{p-2}v\big) + \frac{\alpha_1}{4}\|v\|_{2p-2}^{2p-2} + c\|\sigma\|^2, \end{aligned} \tag{3.85}$$

where we have used the fact that

$$-\int_{\mathscr{O}} \Delta v(|v|^{p-2}v)\,\mathrm{d}x = \int_{\mathscr{O}} \nabla v \cdot \nabla(|v|^{p-2}v)\,\mathrm{d}x \geqslant 0.$$

By (3.76) we have

$$f(v)v \leqslant -\alpha_1|v|^p + \beta_1,$$

and so

$$\begin{aligned} \big(f(v), |v|^{p-2}v\big) &\leqslant \int_{\mathscr{O}} \Big(-\alpha_1|v|^{2p-2} + \beta_1|v|^{p-2}\Big)\,\mathrm{d}x \\ &\leqslant \int_{\mathscr{O}} \Big(-\alpha_1|v|^{2p-2} + \frac{\alpha_1}{2}|v|^{2p-2} + c\Big)\,\mathrm{d}x \\ &\leqslant -\frac{\alpha_1}{2}\|v\|_{2p-2}^{2p-2} + c. \end{aligned}$$

Hence, from (3.85) it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_p^p + \|v\|_p^p + \|v\|_{2p-2}^{2p-2} \leqslant c\|\sigma\|^2 + c. \tag{3.86}$$

By Gronwall's lemma, multiply (3.86) by e^t and then integrate over (r,t) for $r \in (\tau, \tau+\varepsilon)$ to obtain

$$\begin{aligned} &\|v(t)\|_p^p + \int_r^t e^{-(t-s)}\|v(s)\|_{2p-2}^{2p-2}\,\mathrm{d}s \leqslant \\ &\leqslant ce^{-(t-r)}\|v(r)\|_p^p + c\int_r^t e^{-(t-s)}\big(\|\sigma(s)\|^2 + 1\big)\,\mathrm{d}s. \end{aligned} \tag{3.87}$$

Integrating (3.87) over $r \in (\tau, \tau+\varepsilon)$ yields

$$
\begin{aligned}
\varepsilon \|v(t)\|_p^p &+ \int_\tau^{\tau+\varepsilon} \int_r^t e^{-(t-s)} \|v(s)\|_{2p-2}^{2p-2} \, \mathrm{d}s \mathrm{d}r \leqslant \\
&\leqslant c \int_\tau^{\tau+\varepsilon} e^{-(t-r)} \|v(r)\|_p^p \, \mathrm{d}r + \varepsilon c \int_\tau^t e^{-(t-s)} \big(\|\sigma(s)\|^2 + 1 \big) \, \mathrm{d}s \\
&\leqslant c \int_\tau^{\tau+\varepsilon} e^{-(t-r)} \|v(r)\|_p^p \, \mathrm{d}r + \varepsilon c,
\end{aligned}
$$

where the last inequality is because the symbol space Σ is bounded as a subset of $\mathscr{C}_b(\mathbb{R}; H)$ by Proposition 3.3.21, (ii). Since, by (3.84),

$$
\int_\tau^t e^{-(t-r)} \|v(r)\|_p^p \, \mathrm{d}r \leqslant e^{-(t-\tau)} \|v_\tau\|^2 + c,
$$

we have

$$
\varepsilon \|v(t)\|_p^p + \int_\tau^{\tau+\varepsilon} \int_r^t e^{-(t-s)} \|v(s)\|_{2p-2}^{2p-2} \, \mathrm{d}s \mathrm{d}r \leqslant c e^{-\varepsilon} \|v_\tau\|^2 + c(1+\varepsilon).
$$

Since all the estimates above are independent of $\sigma \in \Sigma$, then (3.82) is proved.

Taking the inner product of (3.75) with $-\Delta v$ in H, by (3.77) we have

$$
\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla v\|^2 + \lambda \|\nabla v\|^2 + \|\Delta v\|^2 &= \big(f(v), -\Delta v \big) + (\sigma, -\Delta v) \\
&\leqslant \|\Delta v\|^2 + c\|v\|_{2p-2}^{2p-2} + c + c\|\sigma\|^2,
\end{aligned}
$$

and then

$$
\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla v\|^2 + \|\nabla v\|^2 \leqslant c\|v\|_{2p-2}^{2p-2} + c\|\sigma\|^2 + c.
$$

Hence, by Gronwall's lemma, for all $t \geqslant \tau+\varepsilon$ and $r \in (\tau, \tau+\varepsilon)$, we obtain

$$
\begin{aligned}
\|\nabla v(t)\|^2 &\leqslant e^{-(t-r)} \|\nabla v(r)\|^2 + c \int_r^t e^{-(t-s)} \Big(\|v(s)\|_{2p-2}^{2p-2} + \|\sigma(s)\|^2 + 1 \Big) \, \mathrm{d}s \\
&\leqslant e^{-(t-r)} \|\nabla v(r)\|^2 + c \int_r^t e^{-(t-s)} \|v(s)\|_{2p-2}^{2p-2} \, \mathrm{d}s + c, \qquad \text{(by (3.64))}
\end{aligned}
$$

where $c > 0$ is independent of $\sigma \in \Sigma$. Integrating the above inequality over $r \in (\tau, \tau+\varepsilon)$ we have

$$
\begin{aligned}
\varepsilon \|\nabla v(t)\|^2 &\leqslant \int_\tau^{\tau+\varepsilon} e^{-(t-r)} \|\nabla v(r)\|^2 \, \mathrm{d}r + c \int_\tau^{\tau+\varepsilon} \int_r^t e^{-(t-s)} \|v(s)\|_{2p-2}^{2p-2} \, \mathrm{d}s \mathrm{d}r + c\varepsilon \\
&\leqslant e^{-\varepsilon} \|v_\tau\|^2 + c \int_\tau^{\tau+\varepsilon} \int_r^t e^{-(t-s)} \|v(s)\|_{2p-2}^{2p-2} \, \mathrm{d}s \mathrm{d}r + c(\varepsilon+1) \quad \text{(by (3.84))} \\
&\leqslant c e^{-\varepsilon} \|v_\tau\|^2 + c(1+\varepsilon). \qquad \text{(by (3.82))}
\end{aligned}
$$

The proof is complete. □

3.4.2.2 Finite-dimensionality of the uniform attractor in H

In this section, making use of Theorem 3.2.1 we shall see that the uniform attractor $\mathscr{A}_\Sigma$ of the reaction-diffusion system (3.75) has finite fractal dimension in H. To this end, we now prove a $(\Sigma \times H, H)$-Lipschitz property which implies conditions (H_2) and (H_3) in Theorem 3.2.1.

Denote by $v_\sigma(t,0;v_0)$ the solution of (3.75) corresponding to initial value $v_0 \in H$ at $t = 0$. Given $\sigma_1, \sigma_2 \in \Sigma$, define $\bar{\sigma} := \sigma_1 - \sigma_2$. Then the difference $\bar{v}(t) := v_{\sigma_1}(t,0;v_{1,0}) - v_{\sigma_2}(t,0;v_{2,0})$ of two solutions satisfies

$$\begin{aligned} &\frac{\partial \bar{v}}{\partial t} + \lambda \bar{v} - \Delta \bar{v} = f(v_1) - f(v_2) + \bar{\sigma}, \\ &\bar{v}(x,t)|_{t=0} = v_{1,0}(x) - v_{2,0}(x), \quad \bar{v}(x,t)|_{\partial \mathscr{O}} = 0, \end{aligned} \tag{3.88}$$

for $t \geqslant 0$ and $x \in \mathscr{O}$, where we used the notation $v_j := v_{\sigma_j}(t,0;v_{j,0})$, $j = 1,2$.

Lemma 3.4.10 ($(\Sigma \times H, H)$-Lipschitz)**.** *There exist positive constants $C_1, \beta > 0$ such that for all $\sigma_1, \sigma_2 \in \Sigma$ and $v_{1,0}, v_{2,0} \in H$ the difference of any two solutions satisfies*

$$\|v_{\sigma_1}(t,0;v_{1,0}) - v_{\sigma_2}(t,0;v_{2,0})\|^2 \leqslant C_1 e^{\beta t}\Big[\|v_{1,0} - v_{2,0}\|^2 + \big(d_\Xi(\sigma_1,\sigma_2)\big)^2\Big], \qquad \forall t \geqslant 0.$$

Proof. Taking the inner product of (3.88) with $\bar{v}$ in H we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{v}\|^2 + \lambda\|\bar{v}\|^2 + \|\nabla \bar{v}\|^2 &= \big(f(v_1) - f(v_2), \bar{v}\big) + (\bar{\sigma}, \bar{v}) \\ &\leqslant -c_1\|\bar{v}\|_p^p + c_2\|\bar{v}\|^2 + \|\bar{\sigma}\|\|\bar{v}\| \quad \text{(by Lemma 3.4.8)} \\ &\leqslant -c_1\|\bar{v}\|_p^p + c\|\bar{v}\|^2 + \|\bar{\sigma}\|^2, \end{aligned}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{v}\|^2 + \|\bar{v}\|_p^p + \|\nabla \bar{v}\|^2 \leqslant c\|\bar{v}\|^2 + c\|\bar{\sigma}\|^2.$$

By Gronwall's lemma, for all $t \geqslant 0$ we have

$$\|\bar{v}(t)\|^2 + \int_0^t e^{c(t-s)}\Big(\|\bar{v}(s)\|_p^p + \|\nabla \bar{v}(s)\|^2\Big)\mathrm{d}s \leqslant e^{ct}\|\bar{v}(0)\|^2 + c\int_0^t e^{c(t-s)}\|\bar{\sigma}(s)\|^2\,\mathrm{d}s, \tag{3.89}$$

where $c > 0$ is a constant depending only on c_1, c_2, λ. Since by Lemma 3.4.1 we have

$$\begin{aligned} \int_0^t e^{c(t-s)}\|\bar{\sigma}(s)\|^2\,\mathrm{d}s &\leqslant e^{ct}\int_0^t \|\bar{\sigma}(s)\|^2\,\mathrm{d}s \\ &\leqslant c e^{ct}\Big(d_\Xi(\sigma_1,\sigma_2)\Big)^2, \quad \forall t \geqslant 0, \end{aligned}$$

then from (3.89) we prove the lemma. □

Now we establish a uniform (w.r.t. $\sigma \in \Sigma$) (H,V)-smoothing property on bounded sets by which condition (H_5) of Theorem 3.2.1 is fulfilled. Let $\|D\| := \sup_{u \in D}\|u\|$ for $D \subseteq H$.

Lemma 3.4.11 ((H,V)-smoothing). *For any $T>\varepsilon>0$ and bounded set $D\subseteq H$ with $\|D\|\leqslant R$, there is a constant $C_{\varepsilon,T,R}>0$ such that the difference of any two solutions of the reaction-diffusion equation (3.75) with initial data in D satisfies*

$$\sup_{\sigma\in\Sigma}\|\nabla v_\sigma(t,0;v_{1,0})-\nabla v_\sigma(t,0;v_{2,0})\|^2\leqslant C_{\varepsilon,T,R}\|v_{1,0}-v_{2,0}\|^2,\qquad \forall t\in(\varepsilon,T],$$

whenever $v_{j,0}\in D$, $j=1,2$.

Proof. For any $\sigma\in\Sigma$ we denote by $\bar v_\sigma(t):=v_\sigma(t,0;v_{1,0})-v_\sigma(t,0;v_{2,0})$ the difference of two solutions. Then $\bar v_\sigma$ satisfies

$$\begin{aligned}&\frac{\partial\bar v_\sigma}{\partial t}+\lambda\bar v_\sigma-\Delta\bar v_\sigma=f(v_1)-f(v_2),\\&\bar v_\sigma(x,t)|_{t=0}=v_{1,0}(x)-v_{2,0}(x),\quad \bar v_\sigma(x,t)|_{\partial\mathscr{O}}=0,\end{aligned}\tag{3.90}$$

for $t\geqslant 0$ and $x\in\mathscr{O}$, where we wrote for short $v_j(t):=v_\sigma(t,0;v_{j,0})$, $j=1,2$. Let us drop the subscript "$_\sigma$" since the analysis in the following is independent of it.

Taking the inner product of (3.90) with $-\Delta\bar v$ in H, we obtain

$$\begin{aligned}\frac12\frac{\mathrm d}{\mathrm dt}\|\nabla\bar v\|^2+\lambda\|\nabla\bar v\|^2+\|\Delta\bar v\|^2&=\Big(f(v_1)-f(v_2),-\Delta\bar v\Big)\\&\leqslant c\int_{\mathscr{O}}\Big(|v_1|^{p-2}+|v_2|^{p-2}+1\Big)|\bar v||\Delta\bar v|\,\mathrm dx\qquad\text{(by (3.80))}\\&\leqslant\|\Delta\bar v\|^2+c\int_{\mathscr{O}}(|v_1|^{2p-4}+|v_2|^{2p-4})|\bar v|^2\,\mathrm dx+c\|\bar v\|^2\\&\leqslant\|\Delta\bar v\|^2+c\big(\|v_1\|_{2p-2}^{2p-4}+\|v_2\|_{2p-2}^{2p-4}\big)\|\bar v\|_{2p-2}^2+c\|\bar v\|^2.\end{aligned}$$

Hence,

$$\frac{\mathrm d}{\mathrm dt}\|\nabla\bar v\|^2\leqslant c\big(\|v_1\|_{2p-2}^{2p-4}+\|v_2\|_{2p-2}^{2p-4}\big)\|\bar v\|_{2p-2}^2+c\|\bar v\|^2,\quad t>0.\tag{3.91}$$

Note that, by (3.83) with $\tau=0$, for solutions with initial data in D we have

$$\begin{aligned}\sup_{\sigma\in\Sigma}\|\nabla v\big(t,0;v(0)\big)\|^2&\leqslant\frac{ce^{-\varepsilon}}{\varepsilon}\|v(0)\|^2+c+c\left(\frac1\varepsilon+1\right)\\&\leqslant c_{\varepsilon,R},\qquad\forall t\geqslant\varepsilon,\end{aligned}\tag{3.92}$$

where $c_{\varepsilon,R}>0$ is a constant that depends only on $\varepsilon,R>0$, and the value may change from line to line. Hence, by the continuous embedding $V\hookrightarrow L^{2p-2}(\mathscr{O})$ with (3.81), from (3.91) and Poincaré's inequality it follows

$$\begin{aligned}\frac{\mathrm d}{\mathrm dt}\|\nabla\bar v\|^2&\leqslant c\big(\|\nabla v_1\|^{2p-4}+\|\nabla v_2\|^{2p-4}\big)\|\nabla\bar v\|^2+c\|\bar v\|^2\\&\leqslant c_{\varepsilon,R}\|\nabla\bar v\|^2,\qquad t\geqslant\varepsilon.\qquad\text{(by (3.92))}\end{aligned}$$

By Gronwall's inequality we have

$$\|\nabla\bar v(t)\|^2\leqslant e^{c_{\varepsilon,R}(t-s)}\|\nabla\bar v(s)\|^2,\qquad\forall t\geqslant s>\varepsilon,$$

and then integrate the above inequality over $s \in (\frac{t-\varepsilon}{2}, t)$ to obtain

$$\begin{aligned}\frac{t+\varepsilon}{2}\|\nabla\bar{v}(t)\|^2 &\leqslant \int_{\frac{t-\varepsilon}{2}}^{t} e^{c_{\varepsilon,R}(t-s)}\|\nabla\bar{v}(s)\|^2\,\mathrm{d}s\\ &\leqslant e^{c_{\varepsilon,R}(t+\varepsilon)}\int_{\frac{t-\varepsilon}{2}}^{t}\|\nabla\bar{v}(s)\|^2\,\mathrm{d}s\\ &\leqslant e^{c_{\varepsilon,R}(t+\varepsilon)}\cdot e^{ct}\|\bar{v}(0)\|^2, \qquad \text{(by (3.89) with } \bar{\sigma}=0.)\end{aligned}$$

Hence, for any $t \in (\varepsilon, T]$ with $T > \varepsilon$ we have

$$\begin{aligned}\|\nabla\bar{v}(t)\|^2 &\leqslant \frac{e^{c_{\varepsilon,R}(t+\varepsilon)}\cdot e^{ct}}{(t+\varepsilon)/2}\|\bar{v}(0)\|^2\\ &\leqslant \frac{e^{c_{\varepsilon,R}(T+\varepsilon)}}{\varepsilon}\|\bar{v}(0)\|^2.\end{aligned}$$

Since all the constants are independent of $\sigma \in \Sigma$, the lemma follows. □

Now we are ready to prove the finite-dimensionality of the uniform attractor in H.

Theorem 3.4.12. *The uniform attractor $\mathscr{A}_\Sigma$ of the reaction-diffusion equation* (3.75) *has finite fractal dimension in H, i.e.,*

$$\dim_F(\mathscr{A}_\Sigma; H) < \infty.$$

Proof. Take $X := H$ and $Y := V$. Then clearly Y is compactly embedded into X as required by (H_4) in Theorem 3.2.1. In addition, (H_2) and (H_3) are implied by Lemma 3.4.10 while (H_5) by Lemma 3.4.11. Hence, since (H_1) was given by assumption we have the theorem. □

3.4.2.3 Finite-dimensionality of the uniform attractor in $Z = L^p(\mathscr{O})$

In the previous section we have proved that the uniform attractor $\mathscr{A}_\Sigma$ is finite-dimensional in H. On the other hand, by estimate of solutions (3.82), the uniform attractor $\mathscr{A}_\Sigma$ is also bounded in Banach space $Z = L^p(\mathscr{O})$. Now, using Theorem 3.2.3 we show that it is in fact finite-dimensional in Z. We need the following property which proves (H_6).

Lemma 3.4.13 (($\Sigma \times H, Z$)-smoothing)**.** *For any $T > 0$ there is an absolute constant $C_T > 0$ such that for all $\sigma_1, \sigma_2 \in \Sigma$ and $v_{1,0}, v_{2,0} \in H$ we have*

$$\|v_{\sigma_1}(s,0;v_{1,0}) - v_{\sigma_2}(s,0;v_{2,0})\|_p^p \leqslant C_T\Big[\|v_{1,0}-v_{2,0}\|^2 + \big(d_\Xi(\sigma_1,\sigma_2)\big)^2\Big], \quad \forall s \in (T/2, T].$$

Proof. Denote by $\bar{v}(s) := v_{\sigma_1}(s,0,v_{1,0}) - v_{\sigma_2}(s,0,v_{2,0})$ and $\bar{\sigma} = \sigma_1 - \sigma_2$. Then $\bar{v}$ satisfies (3.88).

Taking the inner product of (3.88) with $|\bar{v}|^{p-2}\bar{v}$ in H we have

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{v}\|_p^p + \lambda\|\bar{v}\|_p^p - \int_{\mathscr{O}}\Delta\bar{v}(|\bar{v}|^{p-2}\bar{v})\,\mathrm{d}x =\\ &= \int_{\mathscr{O}}\big(f(v_1)-f(v_2)\big)|\bar{v}|^{p-2}\bar{v}\,\mathrm{d}x + \int_{\mathscr{O}}\bar{\sigma}|\bar{v}|^{p-2}\bar{v}\,\mathrm{d}x\\ &\leqslant -c_1\|\bar{v}\|_{2p-2}^{2p-2} + c_2\|\bar{v}\|_p^p + \int_{\mathscr{O}}\bar{\sigma}|\bar{v}|^{p-2}\bar{v}\,\mathrm{d}x \qquad \text{(by Lemma 3.4.8)}\\ &\leqslant -\frac{c_1}{2}\|\bar{v}\|_{2p-2}^{2p-2} + c_2\|\bar{v}\|_p^p + c\|\bar{\sigma}\|^2.\end{aligned}$$

Since

$$-\int_{\mathscr{O}} \Delta\bar{v}(|\bar{v}|^{p-2}\bar{v})\,\mathrm{d}x = \int_{\mathscr{O}} \nabla\bar{v}\cdot\nabla(|\bar{v}|^{p-2}\bar{v})\,\mathrm{d}x \geqslant 0,$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{v}\|_p^p + \|\bar{v}\|_{2p-2}^{2p-2} \leqslant c\|\bar{v}\|_p^p + c\|\bar{\sigma}\|^2, \quad t>0. \tag{3.93}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|t\bar{v}\|_p^p = pt^{p-1}\|\bar{v}\|_p^p + t^p\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{v}\|_p^p.$$

Hence, multiplying (3.93) by t^p we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|t\bar{v}\|_p^p - pt^{p-1}\|\bar{v}\|_p^p + t^p\|\bar{v}\|_{2p-2}^{2p-2} &= t^p\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{v}\|_p^p + t^p\|\bar{v}\|_{2p-2}^{2p-2}\\ &\leqslant ct^p\|\bar{v}\|_p^p + ct^p\|\bar{\sigma}\|^2,\end{aligned}$$

and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|t\bar{v}\|_p^p \leqslant \left(ct^p + pt^{p-1}\right)\|\bar{v}\|_p^p + ct^p\|\bar{\sigma}\|^2.$$

Multiplying both sides by t,

$$t\frac{\mathrm{d}}{\mathrm{d}t}\|t\bar{v}\|_p^p \leqslant c(t+1)\|t\bar{v}\|_p^p + ct^{p+1}\|\bar{\sigma}\|^2, \quad t>0. \tag{3.94}$$

Integrating the above inequality over $t\in(0,s)$, for $s>0$, we obtain

$$\begin{aligned}\int_0^s t\frac{\mathrm{d}}{\mathrm{d}t}\|t\bar{v}(t)\|_p^p\,\mathrm{d}t &= s\|s\bar{v}(s)\|_p^p - \int_0^s \|t\bar{v}(t)\|_p^p\,\mathrm{d}t\\ &\leqslant c(s+1)\int_0^s \|t\bar{v}(t)\|_p^p\,\mathrm{d}t + c\int_0^s t^{p+1}\|\bar{\sigma}(t)\|^2\,\mathrm{d}t.\end{aligned}$$

Then,

$$\begin{aligned}s\|s\bar{v}(s)\|_p^p &\leqslant c(s+1)\int_0^s \|t\bar{v}(t)\|_p^p\,\mathrm{d}t + c\int_0^s t^{p+1}\|\bar{\sigma}(t)\|^2\,\mathrm{d}t\\ &\leqslant c(s+1)s^p\int_0^s \|\bar{v}(t)\|_p^p\,\mathrm{d}t + cs^{p+1}\int_0^s \|\bar{\sigma}(t)\|^2\,\mathrm{d}t.\end{aligned}$$

Since

$$\begin{aligned}\int_0^s \|\bar{v}(t)\|_p^p\,\mathrm{d}t &\leqslant \int_0^s e^{c(s-t)}\|\bar{v}(t)\|_p^p\,\mathrm{d}t\\ &\leqslant e^{cs}\|\bar{v}(0)\|^2 + \int_0^s e^{c(s-t)}\|\bar{\sigma}(t)\|^2\,\mathrm{d}t \qquad \text{(by (3.89))},\end{aligned}$$

we have

$$\begin{aligned}s\|s\bar{v}(s)\|_p^p &\leqslant c(s+1)s^p\left(e^{cs}\|\bar{v}(0)\|^2 + \int_0^s e^{c(s-t)}\|\bar{\sigma}(t)\|^2\,\mathrm{d}t\right) + cs^{p+1}\int_0^s \|\bar{\sigma}(t)\|^2\,\mathrm{d}t\\ &\leqslant c(s+1)s^pe^{cs}\left(\|\bar{v}(0)\|^2 + \int_0^s \|\bar{\sigma}(t)\|^2\,\mathrm{d}t\right), \quad \forall s>0.\end{aligned}$$

Therefore, for any $s\in(\frac{T}{2},T]$ with $T>0$, by Lemma 3.4.1 with $\tau=0$ we have

$$\begin{aligned}\frac{T}{2}\|\bar{v}(s)\|_p^p \leqslant s\|\bar{v}(s)\|_p^p &\leqslant c(s+1)e^{cs}\left(\|\bar{v}(0)\|^2 + \int_0^s \|\bar{\sigma}(t)\|^2\,\mathrm{d}t\right)\\ &\leqslant c(T+1)e^{cT}\left[\|\bar{v}(0)\|^2 + c2^{2T}\left(d_\Xi(\sigma_1,\sigma_2)\right)^2\right].\end{aligned}$$

The proof is complete. $\square$

Theorem 3.4.14. *The uniform attractor $\mathscr{A}_\Sigma$ of the reaction-diffusion equation has finite fractal dimension in $Z = L^p(\mathscr{O})$, i.e.,* $\dim_F(\mathscr{A}_\Sigma; Z) < \infty$.

Proof. By Theorem 3.4.12 the uniform attractor is finite-dimensional in H, Lemma 3.4.10 proves that the system is $(\Sigma \times H, H)$-continuous and Lemma 3.4.13 proves (H_6). Then by Theorem 3.2.3 we conclude the finite-dimensionality of $\mathscr{A}_\Sigma$ in $Z = L^p(\mathscr{O})$. □

CHAPTER

4

NON-AUTONOMOUS RANDOM DYNAMICAL SYSTEMS

In this chapter we want to study non-autonomous random dynamical systems and more especifically to provide criteria in order to estimate the fractal dimension of random uniform attractors. In this case we are considering stochastic perturbations of non-autonomous dynamical systems and trying to capture its dynamical behavior. In this situation we have two flows acting on the system and in which must be considered simultaneously in any approach. Based on the techniques developed in last chapter for non-autonomous dynamical systems and uniform attractors, especially in Section 3.2, in this chapter we adapt them to the setting considering now stochastic perturbations. The point is that due to the stochastic nature of the problem this adaptation must be done carefully and they do not follow immediately step-by-step.

Recently introduced in (CUI; LANGA, 2017), the random uniform attractor is the objetc in the setting of non-autonomous random dynamical systems emulating the uniform attractor for the deterministic setting. However, in this case the attracting and absorbing notions are taken in the pullback sense, and forward versions hold particularly in a probability sense. Our aim in this chapter is to study the finite-dimensionality of the random uniform attractor in Banach spaces obtaining for that explicitly estimates on the fractal dimension (due to the stochastic nature of the problem estimates are expected to depend on random parameters).

Based first on a smoothing property for a non-autonomous random dynamical system (which emulates the respective property in a deterministic setting), we are able to obtain estimates (which surprisingly do not depend on random parameters) on the fractal dimension of random uniform attractors. We are also able to obtain estimates on the dimension in more regular spaces just as in the deterministic setting. Absorbing properties are a problem here and we have to construct an absorbing random set which absorbs itself after a deterministic period of time - what is not expected in random settings and so it must be done carefully.

Besides our smoothing method works well from a theoretical point of view, in applications to stochastic evolution equations we find it difficult to hold because of technical issues related to the boundedness of the expectation of some random variables. We develop then a method based on a squeezing property (which has been used in other settings on the estimate of dimensions) which seems (at least up to this state of the art) more applicable to problems on stochastic differential equations. Finally, we consider a stochastic reaction-diffusion with an additive white noise and obtain estimates on its random uniform attractor in spaces L^2 and H_0^1.

This chapter is organized in three parts as follows. In Section 4.1 we introduce the theory of non-autonomous random dynamical systems and random uniform attractors via the skew-product approach as in (CUI; LANGA, 2017). Then in Section 4.2 we give criteria (based on smoothing and squeezing properties) to estimate the fractal dimension of random uniform attractors. Finally, in Section 4.3 we obtain estimates on the fractal dimension of a random uniform attractor for a stochastic reaction-diffusion equation applying the methods just developed in the previous section. The new results in this chapter are given in (CUI; CUNHA; LANGA,).

4.1 Non-autonomous random dynamical systems: random uniform attractors

In this section we introduce the basic setting of random uniform attractors associated to non-autonomous random dynamical systems, which was recently proposed in (CUI; LANGA, 2017). The approach is similar to that of deterministic dynamical systems but the random nature of the problem brings together additional issues.

4.1.1 Non-autonomous random dynamical systems

Let (Ξ, d_Ξ) be a separable complete metric space and let $\{\theta_s\}_{s\in\mathbb{R}}$ be a group of continuous operators acting on Ξ, i.e., $\theta_0\sigma = \sigma$ and $\theta_t(\theta_s\sigma) = \theta_{t+s}\sigma$ for all $\sigma \in \Xi$, $t, s \in \mathbb{R}$, and for each $s \in \mathbb{R}$, $\theta_s : \Xi \to \Xi$ is a continuous mapping on Ξ. Let $\Sigma \subseteq \Xi$ be a *compact* subset of Ξ which is invariant under $\{\theta_s\}_{s\in\mathbb{R}}$, i.e., $\theta_s\Sigma = \Sigma$, for all $s \in \mathbb{R}$.

For a set A let $\mathscr{B}(A)$ be the Borel sigma-algebra of A. Denote by $(\Omega, \mathscr{F}, \mathscr{P})$ a probability space, which need not be $\mathscr{P}$-complete, endowed also with a flow $\{\vartheta_t\}_{t\in\mathbb{R}}$, $\vartheta_t : \Omega \to \Omega$, satisfying the following conditions:

i) $\vartheta_0 = Id_\Omega$;

ii) $\vartheta_t\Omega = \Omega$, for all $t \in \mathbb{R}$;

iii) $\vartheta_t \circ \vartheta_s = \vartheta_{t+s}$, for all $t, s \in \mathbb{R}$;

iv) $(t, \omega) \mapsto \vartheta_t\omega$ is $\big(\mathscr{B}(\mathbb{R}) \times \mathscr{F}, \mathscr{F}\big)$-measurable;

v) $\{\vartheta_t\}_{t\in\mathbb{R}}$ is $\mathscr{P}$-preserving, i.e., $\mathscr{P}(\vartheta_t F) = \mathscr{P}(F)$, for all $t \leqslant 0$ and $F \in \mathscr{F}$;

Groups $\{\theta_t\}_{t\in\mathbb{R}}$ and $\{\vartheta_t\}_{t\in\mathbb{R}}$ acting on Ξ and Ω, respectively, are called *base flows*. As we do not assume the probability space $(\Omega,\mathscr{F},\mathscr{P})$ is complete, we shall not distinguish between a full measure subspace $\tilde{\Omega}$ and Ω, that is, by saying that a statement holds for all $\omega\in\Omega$ we mean that it holds on $\tilde{\Omega}$ almost surely.

Let $(X,\|\cdot\|_X)$ be a separable Banach space. The definition of non-autonomous random dynamical systems is given as in the following.

Definition 4.1.1. *A mapping $\phi:\mathbb{R}^+\times\Omega\times\Sigma\times X\to X$ is said to be a non-autonomous random dynamical system (abbrev. NRDS) on X (with base flows $\{\vartheta_t\}_{t\in\mathbb{R}}$ on Ω and $\{\theta_t\}_{t\in\mathbb{R}}$ on Σ) if*

(i) *ϕ is $\big(\mathscr{B}(\mathbb{R}^+)\times\mathscr{F}\times\mathscr{B}(\Sigma)\times\mathscr{B}(X),\mathscr{B}(X)\big)$-measurable;*

(ii) *$\phi(0,\omega,\sigma,\cdot)=Id_X$, for all $\sigma\in\Sigma$ and $\omega\in\Omega$;*

(iii) *it holds the cocycle property for each fixed $\sigma\in\Sigma$, $x\in X$ and $\omega\in\Omega$, i.e.,*

$$\phi(t+s,\omega,\sigma,x)=\phi(t,\vartheta_s\omega,\theta_s\sigma)\circ\phi(s,\omega,\sigma,x),\qquad \forall t,s\in\mathbb{R}^+,$$

where $\phi(t,\vartheta_s\omega,\theta_s\sigma)\circ\phi(s,\omega,\sigma,x):=\phi\big(t,\vartheta_s\omega,\theta_s\sigma,\phi(s,\omega,\sigma,x)\big)$.

An NRDS ϕ is said to be $(\Sigma\times X,X)$-continuous if for fixed $t\geqslant 0$ and $\omega\in\Omega$ the mapping $(\sigma,x)\mapsto\phi(t,\omega,\sigma,x)$ is continuous from $\Sigma\times X$ to X.

Remark 4.1.2. *In applications an NRDS ϕ is typically generated by an evolution equation with both a non-autonomous forcing (in space Ξ) and random perturbations, while Σ is formulated via all the time-translations of the forcing. In this case, the forcing is called the* (non-autonomous) symbol *of the equation, and the space Σ is called the* symbol space *of the NRDS ϕ.*

Definition 4.1.3. *A random set $D(\cdot)$ in X is defined as a random mapping D: $\Omega\to 2^X\setminus\emptyset$, $\omega\mapsto D(\omega)$, which is measurable, i.e., the mapping $\omega\mapsto \mathrm{dist}_X\big(x,D(\omega)\big)$ is $(\mathscr{F},\mathscr{B}(\mathbb{R}))$-measurable for each fixed $x\in X$. If each image $D(\omega)$ is a closed (resp. bounded/compact) subset of X, then D is called a closed (resp. bounded/compact) random set in X.*

A random set D is often identified with its image $\{D(\omega)\}_{\omega\in\Omega}$ and given two random sets D_1 and D_2, we say that D_1 is inside D_2 if $D_1(\omega)\subseteq D_2(\omega)$, for all $\omega\in\Omega$. We represent it simply as $D_1\subseteq D_2$.

Denote by $\mathscr{D}$ a collection of random sets in X satisfying:

- $\mathscr{D}$ is neighborhood-closed, i.e., for each $D\in\mathscr{D}$ there exists $\varepsilon_0>0$ such that the closed ε_0-neighborhood $\overline{B_X(D;\varepsilon_0)}$ of D belongs to $\mathscr{D}$;

- $\mathscr{D}$ is inclusion-closed, i.e., if $D\in\mathscr{D}$ then any random set D_1 with $D_1\subseteq D$ is such that $D_1\in\mathscr{D}$.

For us $\mathscr{D}$ works as the collection of elements expected to be attracted by an attractor and we call it an *attraction universe*. In Section 4.2 we shall take particularly $\mathscr{D}$ as the collection of all tempered random sets, see (4.18).

As in the autonomous and non-autonomous deterministic setting, for the random case we have the analogous notions of *attraction* and *absorption*. Remember the Hausdorff semi-distance $\mathrm{dist}_X(A,B) := \sup_{a\in A}\inf_{b\in B}\|a-b\|_X$, for $A,B\in 2^X\backslash\emptyset$.

Definition 4.1.4. *Let ϕ be an NRDS on X. A random set $\mathscr{B}\subseteq X$ is said to be a uniformly $\mathscr{D}$-pullback attracting set under the action of ϕ if for any $D\in\mathscr{D}$ and any $\omega\in\Omega$ we have*

$$\sup_{\sigma\in\Sigma}\mathrm{dist}_X\Big(\phi\left(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\right),\mathscr{B}(\omega)\Big)\to 0,\qquad \text{as } t\to\infty. \tag{4.1}$$

Definition 4.1.5. *Let ϕ be an NRDS on X. A random set $\mathscr{B}\subseteq X$ is said to be a uniformly $\mathscr{D}$ -pullback absorbing set under the action of ϕ if for any $D\in\mathscr{D}$ and any $\omega\in\Omega$ there exists a time $T=T(D,\omega)>0$ such that*

$$\bigcup_{\sigma\in\Sigma}\phi\left(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\right)\subseteq\mathscr{B}(\omega),\qquad \forall t\geqslant T.$$

Remark 4.1.6. *Clearly, uniformly $\mathscr{D}$-pullback absorbing sets are uniformly $\mathscr{D}$-pullback attracting sets. Conversely, suppose $\mathscr{B}$ is a uniformly $\mathscr{D}$-pullback attracting set. Then given $\varepsilon>0$, $D\in\mathscr{D}$ and $\omega\in\Omega$ there exists $T=T(D,\omega,\varepsilon)>0$ such that*

$$\bigcup_{\sigma\in\Sigma}\phi\left(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\right)\subseteq B_X\left(\mathscr{B}(\omega),\varepsilon\right),\qquad \forall t\geqslant T,$$

which means that $B_X\left(\mathscr{B}(\omega),\varepsilon\right)$ is a uniformly $\mathscr{D}$-pullback obsorbing set.

The following lemma on measurability of random sets will be useful in next sections.

Lemma 4.1.7. *Let X be a separable Banach space.*

a) If $\{D_n\}_n$ is a family of random subsets of X, then

$$\omega\mapsto\overline{\bigcup_{n\in\mathbb{N}}D_n(\omega)}$$

is a closed random set. If in addition $\{D_n\}_n$ is decreasing (i.e. for all $\omega\in\Omega$ and $p>n$ we have $D_p(\omega)\subseteq D_n(\omega)$) and every sequence $\{x_n\}_n$, with $x_n\in D_n(\omega)$, is precompact, then

$$\bigcap_{n\in\mathbb{N}}D_n(\omega)$$

is non-empty and measurable.

b) For any closed random set D in X there exist countable many random variables $f_n:\Omega\to X$, $n\in\mathbb{N}$, such that $f_n(\omega)\in D(\omega)$ for all $\omega\in\Omega$ and

$$D(\omega)=\overline{\bigcup_{n\in\mathbb{N}}f_n(\omega)}.$$

Proof. See (CASTAING; VALADIER, 1977), Chapter III. □

4.1.2 Random uniform attractors and cocycle attractors

In this section we present the notions of random uniform attractors and random cocycle attractors associated to a NRDS ϕ on X. Remember that X is a separable Banach space.

Definition 4.1.8. *A compact random set $\mathscr{A} \in \mathscr{D}$ is said to be the* random $\mathscr{D}$-uniform attractor *of an NRDS ϕ, if*

(i) $\mathscr{A}$ is a uniformly $\mathscr{D}$-pullback attracting set, i.e., for any $D \in \mathscr{D}$ and any $\omega \in \Omega$, it holds

$$\sup_{\sigma\in\Sigma} \operatorname{dist}_X \Big(\phi\big(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)\big), \mathscr{A}(\omega)\Big) \to 0, \qquad \text{as } t \to \infty;$$

(ii) (Minimality) $\mathscr{A}$ is inside any closed random set satisfying (i).

In order to define random cocycle attractors let us first establish the notion of non-autonomous random sets.

Definition 4.1.9. *A non-autonomous random set $D = \{D_\sigma(\cdot)\}_{\sigma\in\Sigma}$ in X is defined as a random mapping $D : \Sigma \times \Omega \to 2^X \backslash \emptyset$, $(\sigma, \omega) \mapsto D_\sigma(\omega)$, such that for each $\sigma \in \Sigma$, $D_\sigma(\cdot)$ is measurable, i.e., for any fixed $x \in X$ we have that $\omega \mapsto \operatorname{dist}_X\big(x, D_\sigma(\omega)\big)$ is $\big(\mathscr{F}, \mathscr{B}(\mathbb{R})\big)$-measurable. If for each $\sigma \in \Sigma$, $D_\sigma(\cdot)$ is closed (resp. bounded/compact), then $D = \{D_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is called a closed (resp. bounded/compact) non-autonomous random set.*

Definition 4.1.10. *A non-autonomous random set $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is called a random $\mathscr{D}$-cocycle attractor of an NRDS ϕ if it satisfies the following properties:*

$i)$ For each $\sigma \in \Sigma$, $A_\sigma(\cdot)$ is a compact random set in X;

$ii)$ $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is invariant under the action of ϕ, i.e., for any $t \geqslant 0$, $\omega \in \Omega$ and $\sigma \in \Sigma$ we have

$$\phi\big(t, \omega, \sigma, A_\sigma(\omega)\big) = A_{\theta_t\sigma}(\vartheta_t\omega);$$

$iii)$ $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is a $\mathscr{D}$-pullback attracting set, i.e., given $D \in \mathscr{D}$, $\omega \in \Omega$ and $\sigma \in \Sigma$ it holds

$$\lim_{t\to\infty} \operatorname{dist}_X \Big(\phi\big(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)\big), A_\sigma(\omega)\Big) = 0;$$

$iv)$ $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is the minimal among all closed non-autonomous random sets $F = \{F_\sigma(\cdot)\}_{\sigma\in\Sigma}$ in X satisfying $iii)$.

Remark 4.1.11. *Notice that without the minimal condition $iv)$ it is not possible to prove the uniqueness of $\mathscr{D}$-cocycle attractor A since it does not belong to $\mathscr{D}$ in general.*

4.1.3 Omega-limit sets

In this section we define omega-limit sets for an NRDS ϕ. As in the autonomous and non-autonomous deterministic settings, here omega-limit sets are essential in order to prove the existence of random uniform attractors and random cocycle attractors.

Definition 4.1.12. *Let ϕ be an NRDS on X and $D \in \mathscr{D}$. For any $\omega \in \Omega$ and $\Sigma_1 \subseteq \Sigma$, the omega-limit set $\mathscr{W}(\omega, \Sigma_1, D)$ of D under the action of ϕ is defined as*

$$\mathscr{W}(\omega,\Sigma_1,D) := \bigcap_{s\geqslant 0}\left[\overline{\bigcup_{\sigma\in\Sigma_1}\bigcup_{t\geqslant s}\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\big)}\right].$$

We also have the following characterization.

Lemma 4.1.13. *Let ϕ be an NRDS on X. Given $D \in \mathscr{D}$, $\Sigma_1 \subseteq \Sigma$ and $\omega \in \Omega$ we have*

$$\mathscr{W}(\omega,\Sigma_1,D) = \Big\{x\in X : \textit{there are sequences } x_n\in D(\vartheta_{-t_n}\omega), \{\sigma_n\}\subset\Sigma_1, \{t_n\}\subset\mathbb{R}^+ \\ \textit{with } t_n\to\infty \textit{ such that } \lim_{n\to\infty}\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n) = x\Big\}.$$

Proof. For $D \in \mathscr{D}$, $\Sigma_1 \subset \Sigma$ and $\omega \in \Omega$ denote

$$\mathscr{W}'(\omega,\Sigma_1,D) := \Big\{x\in X : \text{there are sequences } x_n\in D(\vartheta_{-t_n}\omega), \{\sigma_n\}\subset\Sigma_1, \{t_n\}\subset\mathbb{R}^+ \\ \text{with } t_n\to\infty \text{ such that } \lim_{n\to\infty}\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n) = x\Big\}.$$

Let $x \in \mathscr{W}(\omega,\Sigma_1,D)$. So for each $n \in \mathbb{N}$ we have

$$x \in \overline{\bigcup_{\sigma\in\Sigma_1}\bigcup_{t\geqslant n}\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\big)},$$

and consequently there is $z_n \in \bigcup_{\sigma\in\Sigma_1}\bigcup_{t\geqslant n}\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\big)$ such that $\|x-z_n\|_X \leqslant 1/n$. Moreover, $z_n = \phi\big(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n\big)$, where $t_n \geqslant n$, $\sigma_n \in \Sigma_1$, $x_n \in D(\vartheta_{-t_n}\omega)$ and we conclude that $x = \lim_{n\to\infty}\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n)$. Therefore, $x \in \mathscr{W}'(\omega,\Sigma_1,D)$.

Now let $x \in \mathscr{W}'(\omega,\Sigma_1,D)$. Then there are sequences $x_n \in D(\vartheta_{-t_n}\omega)$, $\{\sigma_n\} \subset \Sigma_1$ and $\{t_n\} \subseteq \mathbb{R}^+$, with $t_n \to \infty$ as $n \to \infty$, such that $x = \lim_{n\to\infty}\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n)$. Given $s \geqslant 0$, let $n_s \in \mathbb{N}$ be such that for any $n \geqslant n_s$ we have $t_n \geqslant s$. Hence for any $n \geqslant n_s$ it holds $\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n) \in \bigcup_{\sigma\in\Sigma_1}\bigcup_{t\geqslant s}\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\big)$, and then

$$x \in \overline{\bigcup_{\sigma\in\Sigma_1}\bigcup_{t\geqslant s}\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\big)}, \qquad \forall s\geqslant 0.$$

Therefore, $x \in \mathscr{W}(\omega,\Sigma_1,D)$. □

Lemma 4.1.14. *Let ϕ be an NRDS which is continuous on Σ. If $\Sigma_1 \subseteq \Sigma$ is a dense subset then given $D \in \mathscr{D}$ and $\omega \in \Omega$ we have*

$$\mathscr{W}(\omega,\Sigma_1,D) = \mathscr{W}(\omega,\Sigma,D).$$

Proof. Remember the basic property $\overline{\bigcup_i A_i} = \overline{\bigcup_i \overline{A_i}}$. Then for any $\omega \in \Omega$ and $D \in \mathscr{D}$ we have

$$\mathscr{W}(\omega,\Gamma,D) = \bigcap_{s\geqslant 0}\left[\overline{\bigcup_{t\geqslant s}\overline{\bigcup_{\xi\in\Gamma}\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\xi,D(\vartheta_{-t}\omega)\big)}}\right], \qquad \forall\Gamma\subseteq\Sigma,$$

and so to prove the lemma it suffices to show that

$$\overline{\bigcup_{\xi\in\Sigma_1}\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\xi,D(\vartheta_{-t}\omega)\big)} = \overline{\bigcup_{\sigma\in\Sigma}\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\big)}, \qquad \forall t\geqslant 0. \tag{4.2}$$

If $x \in \overline{\bigcup_{\sigma\in\Sigma}\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\big)}$ then there are sequences $\{\sigma_n\}\subset\Sigma$ and $\{x_n\}\subset D(\vartheta_{-t}\omega)$ such that $x = \lim_{n\to\infty}\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma_n,x_n)$. Since $\sigma\mapsto\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,x)$ is continuous, by the density of Σ_1 in Σ, for each $n\in\mathbb{N}$ there is $\sigma_n'\in\Sigma_1$ with

$$\|\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma_n',x_n) - \phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma_n,x_n)\|_X \leqslant 1/n,$$

which implies the limit $x = \lim_{n\to\infty}\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma_n',x_n)$, and therefore we must have $x \in \overline{\bigcup_{\xi\in\Sigma_1}\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\xi,D(\vartheta_{-t}\omega)\big)}$, which implies the " $\supset$ " inclusion. The reverse inclusion is immediate and we prove the lemma. □

Lemma 4.1.15. *Let ϕ be a $(\Sigma\times X,X)$-continuous NRDS on X and $K\subset X$ be a compact uniformly $\mathscr{D}$-pullback attracting random set for ϕ. Then for any non-empty random set $D\in\mathscr{D}$ it holds*

i) *$\mathscr{W}(\cdot,\Sigma,D)$ is non-empty, compact and $\mathscr{W}(\omega,\Sigma,D)\subseteq K(\omega)$, for all $\omega\in\Omega$;*

ii) *$\mathscr{W}(\cdot,\Sigma,D)$ is semi-invariant in the sense that*

$$\mathscr{W}(\vartheta_t\omega,\Sigma,D) \subseteq \bigcup_{\sigma\in\Sigma}\phi\big(t,\omega,\sigma,\mathscr{W}(\omega,\Sigma,D)\big), \qquad \forall t\geqslant 0,\ \omega\in\Omega;$$

iii) *If F is a closed random set and uniformly pullback attracts D then $\mathscr{W}(\omega,\Sigma,D)\subseteq F(\omega)$, for all $\omega\in\Omega$.*

Proof. *i)* Let $\omega\in\Omega$ be fixed and consider $\{\sigma_n\}\subset\Sigma$, $\{t_n\}\subset\mathbb{R}^+$, with $t_n\to\infty$ as $n\to\infty$, and $x_n\in D(\vartheta_{-t_n}\omega)$. Denoting $y_n := \phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n)$, we have

$$\mathrm{dist}_X\big(y_n,K(\omega)\big) \leqslant \mathrm{dist}_X\Big(\phi\big(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,D(\vartheta_{-t_n}\omega)\big),K(\omega)\Big)\to 0, \qquad n\to\infty,$$

and since $K(\omega)$ is compact we can suppose that $y_n\to y\in K(\omega)$, as $n\to\infty$. Therefore, by Lemma 4.1.13 we obtain $y\in\mathscr{W}(\omega,\Sigma,D)$ and we prove that $\mathscr{W}(\omega,\Sigma,D)\neq\emptyset$.

Now let us prove that $\mathscr{W}(\cdot,\Sigma,D)$ is compact by proving that $\mathscr{W}(\omega,\Sigma,D)\subseteq K(\omega)$, for all $\omega\in\Omega$. Indeed, given $y\in\mathscr{W}(\omega,\Sigma,D)$, by Lemma 4.1.13 there are $x_n\in D(\vartheta_{-t_n}\omega)$, $\{\sigma_n\}\subseteq\Sigma$ and $\{t_n\}\subseteq\mathbb{R}^+$, with $t_n\to\infty$, such that $\lim_{n\to\infty}\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n) = y$. But since $D\in\mathscr{D}$ and K is a uniformly $\mathscr{D}$-pullback attracting set then for $n\to\infty$ we have

$$\mathrm{dist}_X\big(\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n),K(\omega)\big) \leqslant \mathrm{dist}_X\Big(\phi\big(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,D(\vartheta_{-t_n}\omega)\big),K(\omega)\Big)\to 0,$$

and by the triangle inequality for the Hausdorff semi-distance we conclude that

$$\mathrm{dist}_X\big(y,K(\omega)\big)\leqslant \mathrm{dist}_X\big(y,\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n)\big)+\mathrm{dist}_X\big(\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n),K(\omega)\big)\to 0.$$

So $y\in\overline{K(\omega)}=K(\omega)$ and therefore $\mathscr{W}(\omega,\Sigma,D)\subseteq K(\omega)$, for all $\omega\in\Omega$. Consequently, $\mathscr{W}(\cdot,\Sigma,D)$ is compact (note that $\mathscr{W}(\cdot,\Sigma,D)$ is closed by definition).

ii) Now let us prove the semi-invariance property. For that, let $\omega\in\Omega$ and $t\geqslant 0$ be fixed. If $y\in\mathscr{W}(\vartheta_t\omega,\Sigma,D)$ by Lemma 4.1.13 let $\{t_n\}\subset\mathbb{R}^+$, with $t_n\to\infty$, $\{\sigma_n\}\subseteq\Sigma$ and $x_n\in D(\vartheta_{-t_n}\vartheta_t\omega)$ be such that $\lim_{n\to\infty}\phi(t_n,\vartheta_{-t_n}\vartheta_t\omega,\theta_{-t_n}\theta_t\sigma_n,x_n)=y$. We can suppose $t_n\geqslant t$, for all $n\in\mathbb{N}$. Note that

$$\begin{aligned}\phi(t_n,\vartheta_{-t_n}\vartheta_t\omega,\theta_{-t_n}\theta_t\sigma_n,x_n)&=\phi(t+t_n-t,\vartheta_{-(t_n-t)}\omega,\theta_{-(t_n-t)}\sigma_n,x_n)\\&=\phi(t,\omega,\sigma_n)\circ\phi(t_n-t,\vartheta_{-(t_n-t)}\omega,\theta_{-(t_n-t)}\sigma_n,x_n).\end{aligned}$$

Since K uniformly pullback attracts D, we have

$$\begin{aligned}&\mathrm{dist}_X\big(\phi(t_n-t,\vartheta_{-(t_n-t)}\omega,\theta_{-(t_n-t)}\sigma_n,x_n),K(\omega)\big)\leqslant\\&\quad\leqslant \mathrm{dist}_X\Big(\phi\big(t_n-t,\vartheta_{-(t_n-t)}\omega,\theta_{-(t_n-t)}\sigma_n,D(\vartheta_{-(t_n-t)}\omega)\big),K(\omega)\Big)\\&\quad\leqslant\sup_{\sigma\in\Sigma}\mathrm{dist}_X\Big(\phi\big(t_n-t,\vartheta_{-(t_n-t)}\omega,\theta_{-(t_n-t)}\sigma,D(\vartheta_{-(t_n-t)}\omega)\big),K(\omega)\Big)\to 0,\quad n\to\infty,\end{aligned}$$

and since $K(\omega)$ is a compact subset of X we may find $z\in K(\omega)$ such that (up to a subsequence) $\lim_{n\to\infty}\phi(t_n-t,\vartheta_{-(t_n-t)}\omega,\theta_{-(t_n-t)}\sigma_n,x_n)=z$, i.e., $z\in\mathscr{W}(\omega,\Sigma,D)$. Moreover, by the compactness of Σ we can suppose there exists $\sigma_0\in\Sigma$ in such a way that up to a subsequence it holds $\lim_{n\to\infty}\sigma_n=\sigma_0$. Then, by the continuity of ϕ we conclude that

$$\lim_{n\to\infty}\phi(t_n,\vartheta_{-t_n}\vartheta_t\omega,\theta_{-t_n}\theta_t\sigma_n,x_n)=\phi(t,\omega,\sigma_0,z)=y,$$

and so $y\in\phi\big(t,\omega,\sigma_0,\mathscr{W}(\omega,\Sigma,D)\big)$. Therefore,

$$\mathscr{W}(\vartheta_t\omega,\Sigma,D)\subseteq\bigcup_{\sigma\in\Sigma}\phi\big(t,\omega,\sigma,\mathscr{W}(\omega,\Sigma,D)\big).$$

iii) Let F be a closed random set uniformly pullback attracting D under the action of ϕ. For $\omega\in\Omega$, if $y\in\mathscr{W}(\omega,\Sigma,D)$, then there are $x_n\in D(\vartheta_{-t_n}\omega)$, $\{\sigma_n\}\subset\Sigma$ and $\{t_n\}\subset\mathbb{R}^+$, with $t_n\to\infty$, such that $\lim_{n\to\infty}\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n)=y$. So for $n\to\infty$ we have

$$\mathrm{dist}_X\big(\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n),F(\omega)\big)\leqslant\mathrm{dist}_X\Big(\phi\big(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,D(\vartheta_{-t_n}\omega)\big),F(\omega)\Big)\to 0,$$

and therefore $\mathrm{dist}_X(y,F(\omega))=0$, i.e., $y\in\overline{F(\omega)}=F(\omega)$, and finally $\mathscr{W}(\omega,\Sigma,D)\subseteq F(\omega)$. □

Remark 4.1.16. *Note that in last lemma we did not prove that the omega-limit set $\mathscr{W}(\cdot,\Sigma,D)$ is a random set. It will be done in the following.*

Lemma 4.1.17. *Let ϕ be a $(\Sigma\times X,X)$-continuous NRDS on X and $K\subset X$ be a compact uniformly $\mathscr{D}$-pullback attracting random set for ϕ. If $D\in\mathscr{D}$ is a closed random set uniformly pullback attracting itself, then the omega-limit set $\mathscr{W}(\cdot,\Sigma,D)$ is the minimal closed random set uniformly pullback attracting D.*

Proof. Given $D \in \mathscr{D}$ a closed random set uniformly pullback attracting itself we have to prove that $\mathscr{W}(\cdot,\Sigma,D)$ is a random set, it uniformly pullback attracts D and it is the minimal among closed uniformly pullback attracting random sets.

Let us first prove that $\mathscr{W}(\cdot,\Sigma,D)$ uniformly pullback attracts D, i.e., for any $\omega \in \Omega$ we have

$$\lim_{t\to\infty}\left[\sup_{\sigma\in\Sigma}\operatorname{dist}_X\Big(\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\big),\mathscr{W}(\omega,\Sigma,D)\Big)\right]=0. \tag{4.3}$$

For this first part we do not need the condition that D uniformly pullback attracts itself. Suppose (4.3) is not true. Then for some $\omega \in \Omega$ there exist $\varepsilon_0 > 0$ and a sequence $\{t_n\} \subset \mathbb{R}^+$, with $t_n \to \infty$, such that

$$\sup_{\sigma\in\Sigma}\operatorname{dist}_X\Big(\phi\big(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma,D(\vartheta_{-t_n}\omega)\big),\mathscr{W}(\omega,\Sigma,D)\Big) > \varepsilon_0, \qquad n\in\mathbb{N}.$$

By the supremum definition we obtain sequences $\{\sigma_n\} \subset \Sigma$ and $x_n \in D(\vartheta_{-t_n}\omega)$ in such a way that

$$\operatorname{dist}_X\big(\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n),\mathscr{W}(\omega,\Sigma,D)\big) > \varepsilon_0, \qquad n\in\mathbb{N}.$$

Notice that since K uniformly pullback attracts D we have

$$\begin{aligned}\operatorname{dist}_X\big(\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n),K(\omega)\big) &\leqslant \operatorname{dist}_X\Big(\phi\big(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,D(\vartheta_{-t_n}\omega)\big),K(\omega)\Big)\\ &\leqslant \sup_{\sigma\in\Sigma}\operatorname{dist}_X\Big(\phi\big(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma,D(\vartheta_{-t_n}\omega)\big),K(\omega)\Big)\to 0,\end{aligned}$$

and since $K(\omega)$ is a compact subset of X, there is $y \in K(\omega)$ such that up to a subsequence we have $\lim_{n\to\infty}\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n) = y$, i.e., $y \in \mathscr{W}(\omega,\Sigma,D)$. Moreover,

$$0=\operatorname{dist}_X\big(y,\mathscr{W}(\omega,\Sigma,D)\big)=\lim_{n\to\infty}\operatorname{dist}_X\big(\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n),\mathscr{W}(\omega,\Sigma,D)\big)\geqslant\varepsilon_0>0,$$

an absurd and (4.3) is proved.

Now we prove that $\mathscr{W}(\cdot,\Sigma,D)$ is a random set if $D \in \mathscr{D}$ uniformly pullback attracts itself. Indeed, we have to prove that given $x \in X$ fixed the mapping $\omega \mapsto \operatorname{dist}_X\big(x,\mathscr{W}(\omega,\Sigma,D)\big)$ is $(\mathscr{F},\mathscr{B}(\mathbb{R}^+))$-measurable. First we see that

$$\mathscr{W}(\omega,\Sigma,D)=\bigcap_{n\in\mathbb{N}}\overline{\bigcup_{m\geqslant n}\bigcup_{\sigma\in\Sigma}\phi\big(m,\vartheta_{-m}\omega,\theta_{-m}\sigma,D(\vartheta_{-m}\omega)\big)}, \qquad \forall\omega\in\Omega. \tag{4.4}$$

Indeed, since D is a closed random set uniformly pullback attracting itself then by Lemma 4.1.15, *iii)*, we have that $\mathscr{W}(\omega,\Sigma,D) \subseteq D(\omega)$, for all $\omega \in \Omega$, and by the semi-invariance in *i)* we obtain

$$\begin{aligned}\mathscr{W}(\omega,\Sigma,D) &\subseteq \bigcup_{\sigma\in\Sigma}\phi\big(m,\vartheta_{-m}\omega,\theta_{-m}\sigma,\mathscr{W}(\vartheta_{-m}\omega,\Sigma,D)\big)\\ &\subseteq \bigcup_{\sigma\in\Sigma}\phi\big(m,\vartheta_{-m}\omega,\theta_{-m}\sigma,D(\vartheta_{-m}\omega)\big), \qquad m\in\mathbb{N}.\end{aligned}$$

Therefore,

$$\mathscr{W}(\omega,\Sigma,D)\subseteq\bigcap_{n\in\mathbb{N}}\overline{\bigcup_{m\geqslant n}\bigcup_{\sigma\in\Sigma}\phi\big(m,\vartheta_{-m}\omega,\theta_{-m}\sigma,D(\vartheta_{-m}\omega)\big)}, \qquad \forall\omega\in\Omega;$$

since the reverse inequality is immediate we prove (4.4).

Now as Σ is separable let $\Sigma_1 := \{\sigma_i\}_i$ be a dense subset of Σ and denote

$$D_n(\omega) := \overline{\bigcup_{m\geqslant n}\bigcup_{i\in\mathbb{N}} \phi\left(m, \vartheta_{-m}\omega, \theta_{-m}\sigma_i, D(\vartheta_{-m}\omega)\right)}, \qquad n\in\mathbb{N},\ \omega\in\Omega.$$

Then, each $D_n(\omega)$ is closed and by (4.2) and (4.4) we conclude that $\mathscr{W}(\omega,\Sigma,D) = \bigcap_{n\in\mathbb{N}} D_n(\omega)$, for all $\omega\in\Omega$. We shall prove in the following that D_n is a random set.

Since D is a non-empty closed random set we have by Lemma 4.1.7, *b)*, that there exists a sequence $\{f_j\}_j$ of random variables $f_j : \Omega \to X$, such that $D(\vartheta_{-m}\omega) = \overline{\bigcup_{j\in\mathbb{N}} f_j(\vartheta_{-m}\omega)}$. So by the continuity of $y \mapsto \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma_i, y)$ we have

$$\begin{aligned}
\phi\left(m, \vartheta_{-m}\omega, \theta_{-m}\sigma_i, D(\vartheta_{-m}\omega)\right) &= \phi\left(m, \vartheta_{-m}\omega, \theta_{-m}\sigma_i, \overline{\bigcup_{j\in\mathbb{N}} f_j(\vartheta_{-m}\omega)}\right)\\
&\subseteq \overline{\bigcup_{j\in\mathbb{N}} \phi\left(m, \vartheta_{-m}\omega, \theta_{-m}\sigma_i, f_j(\vartheta_{-m}\omega)\right)}\\
&\subseteq \overline{\phi\left(m, \vartheta_{-m}\omega, \theta_{-m}\sigma_i, D(\vartheta_{-m}\omega)\right)}
\end{aligned}$$

and therefore

$$\overline{\phi\left(m, \vartheta_{-m}\omega, \theta_{-m}\sigma_i, D(\vartheta_{-m}\omega)\right)} = \overline{\bigcup_{j\in\mathbb{N}} \phi\left(m, \vartheta_{-m}\omega, \theta_{-m}\sigma_i, f_j(\vartheta_{-m}\omega)\right)}. \tag{4.5}$$

We remember that since X is a separable Banach space, if $g, h : \Omega \to X$ are random variables then the mapping $\omega \mapsto \mathrm{dist}_X\left(g(\omega), h(\omega)\right)$ is $\left(\mathscr{F}, \mathscr{B}(\mathbb{R}^+)\right)$-measurable. So for fixed $x\in X$, denoting $h(\omega) := \phi\left(m, \vartheta_{-m}\omega, \theta_{-m}\sigma_i, f_j(\vartheta_{-m}\omega)\right)$ we have that $\omega \mapsto \mathrm{dist}_X\left(x, h(\omega)\right)$ is $\left(\mathscr{F}, \mathscr{B}(\mathbb{R}^+)\right)$-measurable. Therefore, the right-hand side of (4.5) is measurable and so is the left-hand side. Hence by Lemma 4.1.7, *a)*, we conclude that

$$\overline{\bigcup_{m\geqslant n}\bigcup_{i\in\mathbb{N}} \overline{\phi\left(m, \vartheta_{-m}\omega, \theta_{-m}\sigma_i, D(\vartheta_{-m}\omega)\right)}} = \overline{\bigcup_{m\geqslant n}\bigcup_{i\in\mathbb{N}} \phi\left(m, \vartheta_{-m}\omega, \theta_{-m}\sigma_i, D(\vartheta_{-m}\omega)\right)} = D_n(\omega)$$

is measurable, i.e., D_n is a closed random set for each $n\in\mathbb{N}$.

Notice now that $\{D_n\}_n$ is decreasing. Moreover, let $\{z_n\}_n$ be a sequence with $z_n \in D_n(\omega)$ and $z \in \overline{\{z_n\}_n}$. Then $z \in \bigcap_{n\in\mathbb{N}} D_n(\omega) = \mathscr{W}(\omega,\Sigma,D)$. So for a sequence $\{y_k\}_k \subseteq \overline{\{z_n\}_n}$ we have $\{y_k\}_k \subseteq \mathscr{W}(\omega,\Sigma,D)$, and since $\mathscr{W}(\omega,\Sigma,D)$ is compact, then up to a subsequence we have $\lim_{k\to\infty} y_k = y \in \mathscr{W}(\omega,\Sigma,D)$, i.e., $\{z_n\}_n$ is a precompact sequence. Therefore, by Lemma 4.1.7, *a)*, we conclude that $\mathscr{W}(\omega,\Sigma,D)$ is measurable, and then $\mathscr{W}(\cdot,\Sigma,D)$ is a closed random set.

Finally, by Lemma 4.1.15, *iii)*, we conclude that $\mathscr{W}(\cdot,\Sigma,D)$ is the minimal closed random set uniformly pullback attracting D. □

4.1.4 Existence of random uniform attractors and random cocycle attractors

In the following we give sufficient conditions for a $(\Sigma\times X, X)$-continuous NRDS ϕ to have a random uniform attractor and a random cocycle attractor. As usual omega-limit sets will play an essential role in this task.

Theorem 4.1.18. *Let ϕ be a $(\Sigma \times X, X)$-continuous NRDS on X, $K \subset X$ be a compact uniformly $\mathscr{D}$-pullback attracting random set and $D \in \mathscr{D}$ a closed uniformly $\mathscr{D}$-pullback absorbing random set. Then, ϕ has a unique random uniform attractor $\mathscr{A} \in \mathscr{D}$ and it is given by*

$$\mathscr{A}(\omega) = \mathscr{W}(\omega, \Sigma, D), \qquad \forall \omega \in \Omega.$$

Moreover, the random uniform attractor $\mathscr{A}$ is negatively semi-invariant in the sense that

$$\mathscr{A}(\vartheta_t \omega) \subseteq \bigcup_{\sigma \in \Sigma} \phi\big(t, \omega, \sigma, \mathscr{A}(\omega)\big), \qquad \forall t \geqslant 0,\ \omega \in \Omega.$$

Proof. Let us prove that $\mathscr{A} : \Omega \to X$, defined as

$$\mathscr{A}(\omega) := \mathscr{W}(\omega, \Sigma, D), \qquad \omega \in \Omega,$$

is the random uniform attractor for the NRDS ϕ, i.e., $\mathscr{A}$ is a compact random set which is the minimal among closed random sets uniformly $\mathscr{D}$-pullback attracting. Indeed, by Lemma 4.1.15, *i*), we have that $\mathscr{A}$ is non-empty and compact while by Lemma 4.1.17 we have that $\mathscr{A}$ is a random set uniformly pullback attracting D. So for $\omega \in \Omega$ and $\varepsilon > 0$ there is $T = T(\varepsilon, \omega) > 0$ such that

$$\sup_{\sigma \in \Sigma} \operatorname{dist}_X \Big(\phi\big(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)\big), \mathscr{A}(\omega) \Big) < \varepsilon, \qquad \forall t \geqslant T(\omega).$$

Since D is a uniformly $\mathscr{D}$-pullback absorbing set then given $B \in \mathscr{D}$ there is $T_B(\omega) > 0$ such that

$$\sup_{\sigma \in \Sigma} \phi\big(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, B(\vartheta_{-t}\omega)\big) \subseteq D(\omega), \qquad \forall t \geqslant T_B(\omega),$$

and therefore

$$\begin{aligned}
\sup_{\sigma \in \Sigma} \operatorname{dist}_X & \Big(\phi\big(t+T, \vartheta_{-(t+T)}\omega, \theta_{-(t+T)}\sigma, B(\vartheta_{-(t+T)}\omega)\big), \mathscr{A}(\omega) \Big) = \\
&= \sup_{\sigma \in \Sigma} \operatorname{dist}_X \Big(\phi\big(T, \vartheta_{-T}\omega, \theta_{-T}\sigma, \phi(t, \vartheta_{-t}\vartheta_{-T}\omega, \theta_{-t}\theta_{-T}\sigma, B(\vartheta_{-t}\vartheta_{-T}\omega))\big), \mathscr{A}(\omega) \Big) \\
&\leqslant \sup_{\sigma \in \Sigma} \operatorname{dist}_X \Big(\phi\big(T, \vartheta_{-T}\omega, \theta_{-T}\sigma, D(\vartheta_{-T}\omega)\big), \mathscr{A}(\omega) \Big) \qquad \forall t \geqslant T_B(\vartheta_{-T}\omega) \\
&< \varepsilon,
\end{aligned}$$

proving that $\mathscr{A}$ is a uniformly $\mathscr{D}$-pullback attracting set. Since $\mathscr{A}$ is the minimal closed set uniformly pullback attracting D then we conclude easily that $\mathscr{A}$ is the minimal closed uniformly $\mathscr{D}$-pullback attracting set. Finally, since $\mathscr{A} \subseteq D$ and $D \in \mathscr{D}$, by the inclusion-closed property of $\mathscr{D}$ we have $\mathscr{A} \in \mathscr{D}$, proving the theorem. ☐

Theorem 4.1.19. *Let ϕ be a (X, X)-continuous NRDS on X, $K \subset X$ be a compact uniformly $\mathscr{D}$-pullback attracting random set and $D \in \mathscr{D}$ a closed uniformly $\mathscr{D}$-pullback absorbing random set. Then, ϕ has a unique random $\mathscr{D}$-cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma \in \Sigma}$. Moreover, A is given by*

$$A_\sigma(\cdot) = \mathscr{W}(\cdot, \sigma, D), \qquad \forall \sigma \in \Sigma.$$

Proof. Given $\sigma \in \Sigma$, denote
$$A_\sigma(\cdot) := \mathscr{W}(\cdot, \sigma, D).$$
By Theorem 4.1.18 there exists a random uniform attractor $\mathscr{A} \in \mathscr{D}$ for ϕ which is given by $\mathscr{A}(\cdot) = \mathscr{W}(\cdot, \Sigma, D)$. So as for a fixed $\sigma \in \Sigma$ we have $A_\sigma(\cdot) \subseteq \mathscr{A}(\cdot) \in \mathscr{D}$ then $A_\sigma(\cdot)$ is compact and by the inclusion-closed property we also have that $A_\sigma(\cdot) \in \mathscr{D}$, i.e., $A_\sigma(\cdot)$ is a compact random set.

Let us prove that $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is invariant under the action of ϕ. Indeed, let $t \geqslant 0$, $\sigma \in \Sigma$ and $\omega \in \Omega$. For $y \in A_{\theta_t\sigma}(\vartheta_t\omega)$ there are $\{t_n\}_n \subseteq \mathbb{R}^+$ and $x_n \in D(\vartheta_{-t_n}\vartheta_t\omega)$ such that $\lim_{n\to\infty} \phi(t_n, \vartheta_{-t_n}\vartheta_t\omega, \theta_{-t_n}\theta_t\sigma, x_n) = y$. We can suppose $t_n \geqslant t$, for all $n \in \mathbb{N}$. Note that
$$\begin{aligned}\phi(t_n, \vartheta_{-t_n}\vartheta_t\omega, \theta_{-t_n}\theta_t\sigma, x_n) &= \phi(t + t_n - t, \vartheta_{-(t_n-t)}\omega, \theta_{-(t_n-t)}\sigma, x_n)\\ &= \phi(t, \omega, \sigma) \circ \phi(t_n - t, \vartheta_{-(t_n-t)}\omega, \theta_{-(t_n-t)}\sigma, x_n).\end{aligned}$$

Since K is a uniformly $\mathscr{D}$-pullback attracting set and $D \in \mathscr{D}$ we have
$$\lim_{n\to\infty} \operatorname{dist}_X\Big(\phi(t_n - t, \vartheta_{-(t_n-t)}\omega, \theta_{-(t_n-t)}\sigma, x_n), K(\omega)\Big) = 0,$$
and so since $K(\omega)$ is a compact subset of X we find $z \in K(\omega)$ such that (up to a subsequence) $\lim_{n\to\infty} \phi\big(t_n - t, \vartheta_{-(t_n-t)}\omega, \theta_{-(t_n-t)}\sigma, x_n\big) = z$, i.e., $z \in A_\sigma(\omega)$. Finally, we conclude by the continuity of ϕ on X that
$$\lim_{n\to\infty} \phi\big(t_n, \vartheta_{-t_n}\vartheta_t\omega, \theta_{-t_n}\theta_t\sigma, x_n\big) = \phi(t, \omega, \sigma, z) = y,$$
and therefore $A_{\theta_t\sigma}(\vartheta_t\omega) \subseteq \phi\big(t, \omega, \sigma, A_\sigma(\omega)\big)$.

Conversely, let $\tilde{z} \in A_\sigma(\omega)$. Then there are $\{t_n\}_n \subset \mathbb{R}^+$ and $x_n \in D(\vartheta_{-t_n}\omega)$ such that $\lim_{n\to\infty} \phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma, x_n) = \tilde{z}$. But
$$\begin{aligned}\phi\big(t, \omega, \sigma, \phi(t_n, \vartheta_{-t_n}, \theta_{-t_n}\sigma, x_n)\big) &= \phi(t + t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma, x_n)\\ &= \phi(t + t_n, \vartheta_{-(t+t_n)}\vartheta_t\omega, \theta_{-(t+t_n)}\theta_t\sigma, x_n).\end{aligned}$$
As before, since K is a compact uniformly $\mathscr{D}$-pullback attracting set and $D \in \mathscr{D}$ then we find $\tilde{y} \in K(\vartheta_t\omega)$ such that $\lim_{n\to\infty} \phi(t + t_n, \vartheta_{-(t+t_n)}\vartheta_t\omega, \theta_{-(t+t_n)}\theta_t\sigma, x_n) = \tilde{y}$, i.e., $\tilde{y} \in A_{\theta_t\sigma}(\vartheta_t\omega)$. Finally, by the continuity of ϕ we obtain
$$\phi(t, \omega, \sigma, \tilde{z}) = \lim_{n\to\infty} \phi\big(t, \omega, \sigma, \phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma, x_n)\big) = \tilde{y},$$
proving that $\phi\big(t, \omega, \sigma, A_\sigma(\omega)\big) \subseteq A_{\theta_t\sigma}(\vartheta_t\omega)$.

To guarantee that $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is a $\mathscr{D}$-pullback attracting set we proceed just as in the first part of Lemma 4.1.17 and Theorem 4.1.18.

Finally, let $F = \{F_\sigma(\cdot)\}_{\sigma\in\Sigma}$ be a closed non-autonomous random set which is $\mathscr{D}$-pullback attracting. So in particular, since $D \in \mathscr{D}$ it follows that F is a closed random set pullback attracting D. But by Lemma 4.1.15, *iii)*, $A_\sigma(\cdot) \subseteq F_\sigma(\cdot)$, for all $\sigma \in \Sigma$, proving the minimality of $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ and finishing the proof. □

4.1.5 Skew-product semiflow and reduction to a random dynamical system

In this section we are going to define random dynamical systems on an extended phase space and establish as a consequence of that an essential decomposition for the random uniform attractor in terms of the cocycle attractor associated to an NRDS. This decomposition will play one of the central role in order to obtain bounds on the fractal dimension of random uniform attractors in Section 4.2.

Let ϕ be a $(\Sigma\times X,X)$-continuous NRDS on a separable Banach space X, i.e., the mapping $(\sigma,x)\mapsto\phi(t,\omega,\sigma,x)$ is continuous from $\Sigma\times X$ to X, for fixed $t\geqslant 0$ and $\omega\in\Omega$. Define $\mathbb{X}:=\Sigma\times X$, endowed with the metric $d_{\mathbb{X}}:\mathbb{X}\times\mathbb{X}\to\mathbb{R}$ defined for $(\sigma_1,u_1),(\sigma_2,u_2)\in\mathbb{X}$ as

$$d_{\mathbb{X}}\big((\sigma_1,u_1),(\sigma_2,u_2)\big):=d_{\Xi}(\sigma_1,\sigma_2)+\|u_1-u_2\|_X.$$

In the following we define a random dynamical system (abbrev. RDS) on the extended space $\mathbb{X}$.

Proposition 4.1.20. *Let ϕ be a $(\Sigma\times X,X)$-continuous NRDS on X. Then $S:\mathbb{R}^+\times\Omega\times\mathbb{X}\to\mathbb{X}$ defined by*

$$S\big(t,\omega,(\sigma,x)\big):=\big(\theta_t\sigma,\phi(t,\omega,\sigma,x)\big)$$

satisfies

i) S is $\big(\mathscr{B}(\mathbb{R}^+)\times\mathscr{F}\times\mathscr{B}(\mathbb{X}),\mathscr{B}(\mathbb{X})\big)$-measurable;

ii) $S\big(0,\omega,(\sigma,x)\big)=(\sigma,x)$, for all $\omega\in\Omega$ and all $(\sigma,x)\in\mathbb{X}$;

iii) it holds the cocycle property for each fixed $\omega\in\Omega$ and $(\sigma,x)\in\mathbb{X}$, i.e.,

$$S\big(t+s,\omega,(\sigma,x)\big)=S\big(t,\vartheta_s\omega,S(s,\omega,(\sigma,x))\big),\qquad\forall t,s\in\mathbb{R}^+.$$

Proof. Note that $i)$ and $ii)$ are immediate. Now given $\omega\in\Omega$, $(\sigma,x)\in\mathbb{X}$ and $t,s\in\mathbb{R}^+$ we have

$$\begin{aligned}S\big(t+s,\omega,(\sigma,x)\big)&=\big(\theta_{t+s}\sigma,\phi(t+s,\omega,\sigma,x)\big)\\&=\big(\theta_t\theta_s\sigma,\phi(t,\vartheta_s\omega,\theta_s\sigma,\phi(s,\omega,\sigma,x))\big)\\&=S\big(t,\vartheta_s\omega,(\theta_s\sigma,\phi(s,\omega,\sigma,x))\big)\\&=S\big(t,\vartheta_s\omega,S(s,\omega,(\sigma,x))\big),\end{aligned}$$

proving $iii)$ and the proposition. □

The map S is called the skew-product generated by ϕ (and θ). Note that $S(t,\omega,\cdot)$ is $(\mathbb{X},\mathbb{X})$-continuous since ϕ is $(\Sigma\times X,X)$-continuous.

Denote by $\Pi_\Sigma : \mathbb{X} \to \Sigma$ and $\Pi_X : \mathbb{X} \to X$ the projections of $\mathbb{X}$ onto the spaces Σ and X, respectively. Then for any $(\sigma, u) \in \mathbb{X}$ we have

$$\Pi_\Sigma(\sigma, u) = \sigma \qquad \text{and} \qquad \Pi_X(\sigma, u) = u.$$

For a set $\mathbb{B} \subseteq \mathbb{X}$ we have $\mathbb{B} = \bigcup_{\sigma\in\Sigma}\{\sigma\} \times B(\sigma)$, where each $B(\sigma)$ is a (possibly empty) subset of X. In this way we obtain

$$\Pi_X(\mathbb{B}) = \bigcup_{\sigma\in\Sigma} B(\sigma) = \big\{x \in X : \exists\, \sigma \in \Sigma, (\sigma, x) \in \mathbb{B}\big\}.$$

In the following we define the concept of random sets in the extended phase space $\mathbb{X}$.

Definition 4.1.21. *A random set $\mathbb{B}(\cdot)$ in $\mathbb{X}$ is defined as a random mapping $\mathbb{B} : \Omega \to 2^{\mathbb{X}}\backslash\emptyset$, $\omega \mapsto \mathbb{B}(\omega)$, which is measurable, i.e., the mapping $\omega \mapsto \operatorname{dist}_X\big((\sigma,x),\mathbb{B}(\omega)\big)$ is $\big(\mathscr{F}, \mathscr{B}(\mathbb{R}^+)\big)$-measurable for each fixed $(\sigma,x) \in \mathbb{X}$. If each $\mathbb{B}(\omega)$ is a closed (resp. bounded/compact) subset of $\mathbb{X}$ then $\mathbb{B}$ is called a closed (resp. bounded/compact) random subset of $\mathbb{X}$.*

For $\mathbb{B}(\cdot) \subseteq \mathbb{X}$ we have for each $\omega \in \Omega$ that

$$\mathbb{B}(\omega) = \bigcup_{\sigma\in\Sigma} \{\sigma\} \times B(\sigma, \omega).$$

Remark 4.1.22. *Note that if $\mathbb{B}(\cdot)$ is closed we can suppose without loss of generality that $B(\sigma, \omega)$ is closed for any $\sigma \in \Sigma$ and any $\omega \in \Omega$. Indeed, we have*

$$\mathbb{B}(\omega) = \bigcup_{\sigma\in\Sigma} \{\sigma\} \times B(\sigma, \omega) = \overline{\bigcup_{\sigma\in\Sigma} \overline{\{\sigma\} \times B(\sigma, \omega)}} = \bigcup_{\sigma\in\Sigma} \{\sigma\} \times \overline{B(\sigma, \omega)}.$$

With that we define the proper random sets.

Definition 4.1.23. *A random set $\mathbb{B}$ in $\mathbb{X}$ is called proper if it satisfies that*

$$B(\sigma, \omega) \neq \emptyset, \qquad \forall \omega \in \Omega, \sigma \in \Sigma, \tag{4.6}$$

and

$$\Pi_X(\mathbb{B}) \in \mathscr{D}. \tag{4.7}$$

Remark 4.1.24. *Notice that condition (4.6) is equivalent to*

$$\Pi_\Sigma\big(\mathbb{B}(\omega)\big) = \Sigma, \qquad \forall \omega \in \Omega. \tag{4.8}$$

Denote by $\mathscr{D}_{\mathbb{X}}$ the collection of all proper random subsets of $\mathbb{X}$, i.e.,

$$\mathscr{D}_{\mathbb{X}} := \big\{\mathbb{D} : \mathbb{D} \text{ is a proper random subset of } \mathbb{X}\big\}.$$

Clearly, random sets in the form $\Sigma \times D = \big\{\Sigma \times D(\omega)\big\}_{\omega\in\Omega}$, with $D \in \mathscr{D}$, belong to $\mathscr{D}_{\mathbb{X}}$.

In the following we define the notion of random attractor associated to the skew-product S.

Definition 4.1.25. *A random set $\mathbb{A} \in \mathscr{D}_{\mathbb{X}}$ is called a $\mathscr{D}_{\mathbb{X}}$-random attractor of the skew-product S if it satisfies the following properties:*

i) *$\mathbb{A}$ is compact;*

ii) *$\mathbb{A}$ is invariant under S, i.e., for any $t \geqslant 0$ and $\omega \in \Omega$ it holds*

$$S\big(t, \omega, \mathbb{A}(\omega)\big) = \mathbb{A}(\vartheta_t \omega);$$

iii) *$\mathbb{A}$ is a $\mathscr{D}_{\mathbb{X}}$-pullback attracting set, i.e., for any $\omega \in \Omega$ and $\mathbb{D} \in \mathscr{D}_{\mathbb{X}}$ we have*

$$\lim_{t\to\infty} \mathrm{dist}_{\mathbb{X}}\Big(S\big(t, \vartheta_{-t}\omega, \mathbb{D}(\vartheta_{-t}\omega)\big), \mathbb{A}(\omega)\Big) = 0.$$

Remark 4.1.26. *Note that since a compact random attractor $\mathbb{A}$ belongs to $\mathscr{D}_{\mathbb{X}}$ by definition, then it must be unique.*

Theorem 4.1.27. *Let ϕ be a $(\Sigma \times X, X)$-continuous NRDS on X, $K \subseteq X$ be a compact uniformly $\mathscr{D}$-pullback attracting set and $D \in \mathscr{D}$ be a closed uniformly $\mathscr{D}$-pullback absorbing set. If S is the skew-product generated by ϕ then the omega-limit set of $\mathbb{D} := \Sigma \times D$ given by*

$$\Omega_{\mathbb{D}}(\omega) := \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} S\big(t, \vartheta_{-t}\omega, \mathbb{D}(\vartheta_{-t}\omega)\big)}, \qquad \omega \in \Omega,$$

is the $\mathscr{D}_{\mathbb{X}}$-random attractor for S.

Proof. First notice that $\mathbb{K} := \Sigma \times K$ is a compact $\mathscr{D}_{\mathbb{X}}$-pullback attracting random set and $\mathbb{D} := \Sigma \times D \in \mathscr{D}_{\mathbb{X}}$ is a closed $\mathscr{D}_{\mathbb{X}}$-pullback absorbing random set. Indeed, since K is a compact random set it clearly follows that $\mathbb{K}$ is a compact random set as well. Now given $\mathbb{B} \in \mathscr{D}_{\mathbb{X}}$ we have for $\omega \in \Omega$ and $t \geqslant 0$ that $\mathbb{B}(\vartheta_{-t}\omega) = \bigcup_{\gamma\in\Sigma}\{\theta_{-t}\gamma\} \times B(\theta_{-t}\gamma, \vartheta_{-t}\omega)$; so $\Pi_X\big(\mathbb{B}(\cdot)\big) = \bigcup_{\gamma\in\Sigma} B(\gamma, \cdot) \in \mathscr{D}$. Therefore, for a fixed $\omega \in \Omega$, given $\varepsilon > 0$ there exists $T(\omega) = T(\omega, \varepsilon) > 0$ such that

$$\sup_{\sigma\in\Sigma} \mathrm{dist}_X\Big(\phi\big(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \cup_{\gamma\in\Sigma} B(\theta_{-t}\gamma, \vartheta_{-t}\omega)\big), K(\omega)\Big) < \varepsilon, \qquad \forall t \geqslant T(\omega).$$

Then for any fixed $\sigma \in \Sigma$ and $u \in \bigcup_{\gamma\in\Sigma} B(\theta_{-t}\gamma, \vartheta_{-t}\omega)$ we have

$$\mathrm{dist}_X\Big(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, u), K(\omega)\Big) < \varepsilon, \qquad \forall t \geqslant T(\omega),$$

and there exists $x_u \in K(\omega)$ such that

$$\big\|\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, u) - x_u\big\|_X < \varepsilon, \qquad \forall t \geqslant T(\omega).$$

Now for any $(\theta_{-t}\sigma, u) \in \mathbb{B}(\vartheta_{-t}\omega)$ it holds

$$\begin{aligned}
\mathrm{dist}_{\mathbb{X}}\Big(S\big(t, \vartheta_{-t}\omega, (\theta_{-t}\sigma, u)\big), \mathbb{K}(\omega)\Big) &\leqslant \mathrm{dist}_{\mathbb{X}}\Big(\big(\sigma, \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, u)\big), (\sigma, x_u)\Big) \\
&= \big\|\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, u) - x_u\big\|_X \\
&< \varepsilon, \qquad \forall t \geqslant T(\omega),
\end{aligned}$$

and therefore

$$\mathrm{dist}_{\mathbb{X}}\Big(S\big(t,\vartheta_{-t}\omega,\mathbb{B}(\vartheta_{-t}\omega)\big),\mathbb{K}(\omega)\Big)\leqslant\varepsilon,\qquad\forall t\geqslant T(\omega),$$

proving that $\mathbb{K}$ is a $\mathscr{D}_{\mathbb{X}}$-pullback attracting random set. Analogously we prove that $\mathbb{D}\in\mathscr{D}_{\mathbb{X}}$ is a closed $\mathscr{D}_{\mathbb{X}}$-pullback absorbing random set.

Just as in Lemma 4.1.15 and Theorem 4.1.18 we prove that $\Omega_{\mathbb{D}}(\cdot)$ is a non-empty compact random subset of $\mathbb{X}$ which is invariant under S and $\mathscr{D}_{\mathbb{X}}$-pullback attracting. It remains to prove that $\Omega_{\mathbb{D}}(\cdot)\in\mathscr{D}_{\mathbb{X}}$, i.e., $\Omega_{\mathbb{D}}(\cdot)$ is a proper set. Let $\sigma\in\Sigma$, $\omega\in\Omega$ and notice that since $\Pi_X(\mathbb{D})=D\in\mathscr{D}$ we have

$$\lim_{n\to\infty}\mathrm{dist}_X\Big(\phi\left(n,\vartheta_{-n}\omega,\theta_{-n}\sigma,\Pi_X(\mathbb{D}(\vartheta_{-n}\omega))\right),K(\omega)\Big)=0.$$

Then for any sequence $\{x_n\}_n$, with $x_n\in\Pi_X\big(\mathbb{D}(\vartheta_{-n}\omega)\big)$, by the compactness of $K(\omega)$ we find $y_\sigma\in K(\omega)$ such that up to a subsequence we obtain

$$\lim_{n\to\infty}\phi(n,\vartheta_{-n}\omega,\theta_{-n}\sigma,x_n)=y_\sigma.$$

But it means that

$$\lim_{n\to\infty}S\big(n,\vartheta_{-n}\omega,(\theta_{-n}\sigma,x_n)\big)=(\sigma,y_\sigma),$$

and therefore by the characterization of omega-limit sets it holds $(\sigma,y_\sigma)\in\Omega_{\mathbb{D}}(\omega)$, proving (4.6).

Now let $\omega\in\Omega$ and suppose $z\in\Pi_X\big(\Omega_{\mathbb{D}}(\omega)\big)$. For some $\sigma\in\Sigma$ we have $(\sigma,z)\in\Omega_{\mathbb{D}}(\omega)$ and there are sequences $\{t_n\}_n\subset\mathbb{R}^+$, $\{\sigma_n\}_n\subset\Sigma$, $x_n\in D(\vartheta_{-t_n}\omega)$ with $t_n\to\infty$ and such that

$$\lim_{n\to\infty}S\big(t_n,\vartheta_{-t_n}\omega,(\theta_{-t_n}\sigma_n,x_n)\big)=(\sigma,z).$$

Then $\lim_{n\to\infty}\sigma_n=\sigma$ and $\lim_{n\to\infty}\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,x_n)=z$, i.e., $z\in\mathscr{W}\big(\omega,\Sigma,D\big)$. Hence $\Pi_X\big(\Omega_{\mathbb{D}}(\omega)\big)\subseteq\mathscr{W}(\omega,\Sigma,D)$. Analogously we prove the reverse inclusion and so

$$\Pi_X\big(\Omega_{\mathbb{D}}(\omega)\big)=\mathscr{W}(\omega,\Sigma,D),\qquad\forall\omega\in\Omega.\tag{4.9}$$

Remember by Theorem 4.1.18 that $\mathscr{W}\big(\cdot,\Sigma,D\big)\in\mathscr{D}$. Therefore we conclude (4.7), proving that $\Omega_{\mathbb{D}}(\cdot)$ is a proper random set, i.e., $\Omega_{\mathbb{D}}(\cdot)\in\mathscr{D}_{\mathbb{X}}$. $\square$

Now we show a relation between the random attractor $\mathbb{A}(\cdot)$ for a skew-product S associated to an NRDS ϕ and the random uniform attractor $\mathscr{A}(\cdot)$ of ϕ. Besides that, we give a decomposition of $\mathscr{A}(\cdot)$ in terms of the cocycle attractor $A=\{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$.

Theorem 4.1.28. *Let ϕ be a $(\Sigma\times X,X)$-continuous NRDS on X, $K\subset X$ be a compact uniformly $\mathscr{D}$-pullback attracting set and $D\in\mathscr{D}$ be a closed uniformly $\mathscr{D}$-pullback absorbing set. Then:*

1. *The NRDS ϕ has a random uniform attractor $\mathscr{A}\in\mathscr{D}$, a $\mathscr{D}$-cocycle attractor $A=\{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ and the skew-product S generated by ϕ (and θ) has a random $\mathscr{D}_{\mathbb{X}}$-attractor $\mathbb{A}\in\mathscr{D}_{\mathbb{X}}$. Moreover, for any $\omega\in\Omega$ and $\sigma\in\Sigma$ we have*

$$\mathscr{A}(\omega)=\mathscr{W}(\omega,\Sigma,D),\qquad A_\sigma(\omega)=\mathscr{W}(\omega,\sigma,D)\qquad\textit{and}\qquad\mathbb{A}(\omega)=\Omega_{\mathbb{D}}(\omega),$$

where $\mathbb{D}:=\Sigma\times D$.

2. *The projection $\Pi_X(\mathbb{A})$ is the random $\mathscr{D}$-uniform attractor $\mathscr{A}$ for ϕ, i.e.,*

$$\mathscr{A}(\omega) = \Pi_X\big(\mathbb{A}(\omega)\big), \qquad \forall \omega \in \Omega.$$

3. *The random $\mathscr{D}_{\mathbb{X}}$-attractor $\mathbb{A}$ satisfies*

$$\mathbb{A}(\omega) = \bigcup_{\sigma\in\Sigma} \{\sigma\} \times A_\sigma(\omega), \qquad \forall \omega \in \Omega; \tag{4.10}$$

Therefore,

$$\mathscr{A}(\omega) = \bigcup_{\sigma\in\Sigma} A_\sigma(\omega), \qquad \forall \omega \in \Omega; \tag{4.11}$$

4. *The projection $\Pi_\Sigma(\mathbb{A})$ satisfies*

$$\Pi_\Sigma\big(\mathbb{A}(\omega)\big) = \Sigma, \qquad \forall \omega \in \Omega.$$

Proof. 1) By Theorem 4.1.18, Theorem 4.1.19 and Theorem 4.1.27 there are, respectively, a random uniform attractor $\mathscr{A} \in \mathscr{D}$, a $\mathscr{D}$-cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ and a random $\mathscr{D}_{\mathbb{X}}$-attractor $\mathbb{A} \in \mathscr{D}_{\mathbb{X}}$. Moreover, for any $\omega \in \Omega$ and $\sigma \in \Sigma$ we have $\mathscr{A}(\omega) = \mathscr{W}(\omega, \Sigma, D)$, $A_\sigma(\omega) = \mathscr{W}(\omega, \sigma, D)$ and $\mathbb{A}(\omega) = \Omega_{\mathbb{D}}(\omega)$, where $\mathbb{D} := \Sigma \times D$.

2) Since $\mathbb{A}(\cdot) = \Omega_{\mathbb{D}}(\cdot)$ it follows by (4.9) in Theorem 4.1.27 that

$$\Pi_X\big(\mathbb{A}(\omega)\big) = \mathscr{W}(\omega, \Sigma, D) = \mathscr{A}(\omega), \qquad \forall \omega \in \Omega.$$

3) For a given $\omega \in \Omega$ we have $\mathbb{A}(\omega) = \bigcup_{\sigma\in\Sigma}\{\sigma\} \times A(\sigma,\omega)$, and since $\mathbb{A} \in \mathscr{D}_{\mathbb{X}}$ then each $A(\sigma,\omega) \neq \emptyset$ and $A(\sigma,\cdot)$ is a random set for each $\sigma \in \Sigma$. It suffices to prove that $A_\sigma(\omega) = A(\sigma,\omega)$. Indeed, let us prove first that the non-autonomous random set $\{A(\sigma,\cdot)\}_{\sigma\in\Sigma}$ is invariant under the action of ϕ. On one hand let $t \geqslant 0$ and take $y \in A(\theta_t\sigma, \vartheta_t\omega)$. Then $(\theta_t\sigma, y) \in \mathbb{A}(\vartheta_t\omega)$ and since $\mathbb{A}$ is invariant under S there exists $(\sigma', x) \in \mathbb{A}(\omega)$ such that

$$(\theta_t\sigma, y) = S\big(t, \omega, (\sigma', x)\big) = \big(\theta_t\sigma', \phi(t, \omega, \sigma', x)\big).$$

Therefore, $\sigma = \sigma'$ and $y = \phi(t, \omega, \sigma, x)$, with $x \in A(\sigma, \omega)$, and we obtain $A(\theta_t\sigma, \vartheta_t\omega) \subseteq \phi\big(t, \omega, \sigma, A(\sigma, \omega)\big)$.

On the other hand, note that

$$\begin{aligned}
\{\theta_t\sigma\} \times \phi\big(t, \omega, \sigma, A(\sigma, \omega)\big) &= S\big(t, \omega, \{\sigma\} \times A(\sigma, \omega)\big) \\
&\subseteq S\big(t, \omega, \mathbb{A}(\omega)\big) \\
&= \mathbb{A}(\vartheta_t\omega),
\end{aligned}$$

and therefore $\phi\big(t, \omega, \sigma, A(\sigma, \omega)\big) \subseteq A(\theta_t\sigma, \vartheta_t\omega)$, proving the invariance.

Now we prove that $\{A(\sigma, \cdot)\}_{\sigma\in\Sigma}$ is a $\mathscr{D}$-cocycle attracting set, i.e., given $B \in \mathscr{D}$, $\omega \in \Omega$ and $\sigma \in \Sigma$ we have

$$\lim_{t\to\infty} \mathrm{dist}_X\Big(\phi\big(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, B(\vartheta_{-t}\omega)\big), A(\sigma, \omega)\Big) = 0. \tag{4.12}$$

If it is not true, there are $B \in \mathscr{D}$, $\omega \in \Omega$, $\sigma \in \Sigma$, $\varepsilon_0 > 0$ and sequences $t_n \to \infty$ and $y_n \in \phi\big(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma, B(\vartheta_{-t_n}\omega)\big)$ such that

$$\mathrm{dist}_X\big(y_n, A(\sigma,\omega)\big) > \varepsilon_0, \qquad \forall n \in \mathbb{N}.$$

Then, denoting $\mathbb{B} := \Sigma \times B \in \mathscr{D}_{\mathbb{X}}$ we obtain

$$\begin{aligned}\mathrm{dist}_{\mathbb{X}}\Big((\sigma,y_n),\mathbb{A}(\omega)\Big) &\leqslant \mathrm{dist}_{\mathbb{X}}\Big(\{\sigma\}\times\phi\big(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma,B(\vartheta_{-t_n}\omega)\big),\mathbb{A}(\omega)\Big)\\ &= \mathrm{dist}_{\mathbb{X}}\Big(S\big(t_n,\vartheta_{-t_n}\omega,\{\theta_{-t_n}\sigma\}\times B(\vartheta_{-t_n}\omega)\big),\mathbb{A}(\omega)\Big)\\ &\leqslant \mathrm{dist}_{\mathbb{X}}\Big(S\big(t_n,\vartheta_{-t_n}\omega,\mathbb{B}(\vartheta_{-t_n}\omega)\big),\mathbb{A}(\omega)\Big) \to 0, \quad \text{as } n\to\infty,\end{aligned}$$

and since $\mathbb{A}(\omega)$ is compact we may find $(\sigma', y) \in \mathbb{A}(\omega)$ such that up to a subsequence we have $\lim_{n\to\infty}(\sigma, y_n) = (\sigma', y)$. Therefore, $\sigma = \sigma'$ and $\lim_{n\to\infty} y_n = y \in A(\sigma,\omega)$, showing that

$$0 = \mathrm{dist}_X\big(y, A(\sigma,\omega)\big) = \lim_{n\to\infty}\mathrm{dist}_X\big(y_n, A(\sigma,\omega)\big) \geqslant \varepsilon_0 > 0,$$

a contradiction and (4.12) is proved.

Finally we are going to prove that $A(\sigma,\omega) = A_\sigma(\omega)$, for all $\omega\in\Omega$ and $\sigma\in\Sigma$. On one hand, note that since $\{A(\sigma,\cdot)\}_{\sigma\in\Sigma}$ is invariant under ϕ and $\mathscr{A} \in \mathscr{D}$ we have

$$\begin{aligned}\mathrm{dist}_X\big(A(\sigma,\omega),A_\sigma(\omega)\big) &= \mathrm{dist}_X\Big(\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,A(\theta_{-t}\sigma,\vartheta_{-t}\omega)\big),A_\sigma(\omega)\Big)\\ &\leqslant \mathrm{dist}_X\Big(\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\mathscr{A}(\vartheta_{-t}\omega)\big),A_\sigma(\omega)\Big) \to 0, \quad \text{as } t\to\infty,\end{aligned}$$

and then $A(\sigma,\omega) \subseteq \overline{A_\sigma(\omega)} = A_\sigma(\omega)$.

On the other hand since $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is the $\mathscr{D}$-cocycle attractor for ϕ then in particular $A_\sigma(\cdot)$ is a compact random set. By Lemma 4.1.7, *a)*, for fixed $\sigma\in\Sigma$ we conclude that $\omega \mapsto \overline{\bigcup_{j\in\mathbb{N}} A_{\theta_{-j}\sigma}(\omega)}$ is a closed random set in X. But by the minimality of $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ among the closed non-autonomous random sets we conclude for all $\omega\in\Omega$ and all $\sigma\in\Sigma$ that $A_\sigma(\omega) \subseteq \mathscr{A}(\omega)$, and in particular, $\overline{\bigcup_{j\in\mathbb{N}} A_{\theta_{-j}\sigma}(\omega)} \subseteq \mathscr{A}(\omega)$. With that $\overline{\bigcup_{j\in\mathbb{N}} A_{\theta_{-j}\sigma}(\cdot)} \in \mathscr{D}$ and since $\{A(\sigma,\cdot)\}_{\sigma\in\Sigma}$ is a $\mathscr{D}$-cocycle attracting set we obtain

$$\begin{aligned}\mathrm{dist}_X\big(A_\sigma(\omega),A(\sigma,\omega)\big) &= \mathrm{dist}_X\Big(\phi\big(n,\vartheta_{-n}\omega,\theta_{-n}\sigma,A_{\theta_{-n}\sigma}(\vartheta_{-n}\omega)\big),A(\sigma,\omega)\Big)\\ &\leqslant \mathrm{dist}_X\Big(\phi\big(n,\vartheta_{-n}\omega,\theta_{-n}\sigma,\overline{\bigcup_{j\in\mathbb{N}} A_{\theta_{-j}\sigma}(\vartheta_{-n}\omega)}\big),A(\sigma,\omega)\Big)\\ &\to 0, \quad \text{as } n\to\infty,\end{aligned}$$

proving that $A_\sigma(\omega) \subseteq \overline{A(\sigma,\omega)} = A(\sigma,\omega)$, and so $A_\sigma(\omega) = A(\sigma,\omega)$, for all $\sigma\in\Sigma$ and $\omega\in\Omega$. We have then (4.10) and (4.11).

4) It is immediate since $\mathbb{A} \in \mathscr{D}_{\mathbb{X}}$. $\square$

Remark 4.1.29. *The decomposition* (4.11) *allows us to prove the following semi-invariance property: for all $\omega \in \Omega$ we have*

$$\begin{aligned}\mathscr{A}(\omega) &= \bigcup_{\sigma\in\Sigma} A_\sigma(\omega)\\ &= \bigcup_{\sigma\in\Sigma} \phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,A_{\theta_{-t}\sigma}(\vartheta_{-t}\omega)\big)\\ &\subseteq \bigcup_{\sigma\in\Sigma} \phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\mathscr{A}(\vartheta_{-t}\omega)\big),\end{aligned}$$

i.e.,

$$\mathscr{A}(\omega) \subseteq \bigcup_{\sigma\in\Sigma} \phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\mathscr{A}(\vartheta_{-t}\omega)\big), \qquad \forall t \geqslant 0,\ \omega\in\Omega. \tag{4.13}$$

Moreover, if B is a uniformly $\mathscr{D}$-pullback absorbing set then for $t = t(\omega) > 0$ large enough we obtain

$$\mathscr{A}(\omega) \subseteq \bigcup_{\sigma\in\Sigma} \phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\mathscr{A}(\vartheta_{-t}\omega)\big) \subseteq B(\omega), \qquad \forall \omega\in\Omega.$$

4.1.6 Conjugate attractors and their structural relationship

The idea of conjugate dynamical systems has been widely used to turn a stochastic partial differential equation into a deterministic partial differential equation with random parameters, as for example in (CHUESHOV, 2002), (FLANDOLI; LISEI, 2004) and (CUI; LI; YIN, 2016). Now we study in a more abstract framework the attractors under this transformation.

Suppose that X and $\tilde{X}$ are two separable Banach spaces (where $X = \tilde{X}$ is allowed), and that ϕ and $\tilde{\phi}$ are two NRDS with the same base flows (θ,Σ) and (ϑ,Ω) on spaces X and $\tilde{X}$, respectively.

Definition 4.1.30. *The NRDS ϕ and $\tilde{\phi}$ are said to be conjugate NRDS if there is a mapping $\mathsf{T}:\Omega\times X\to\tilde{X}$, which is called a* cohomology *of ϕ and $\tilde{\phi}$, satisfying:*

i) The mapping $x\mapsto \mathsf{T}_\omega(x) := \mathsf{T}(\omega,x)$ is a homeomorphism from X onto $\tilde{X}$, for each fixed $\omega\in\Omega$;

ii) The mappings $\omega\mapsto \mathsf{T}(\omega,x)\in\tilde{X}$ and $\omega\mapsto \mathsf{T}_\omega^{-1}(y)\in X$ are measurable for each fixed $x\in X$ and $y\in\tilde{X}$;

iii) For any $t>0$, $\omega\in\Omega$, $\sigma\in\Sigma$, $x\in X$, we have

$$\tilde{\phi}\big(t,\omega,\sigma,\mathsf{T}(\omega,x)\big) = \mathsf{T}\big(\vartheta_t\omega,\phi(t,\omega,\sigma,x)\big). \tag{4.14}$$

Let $\mathscr{D}$ and $\tilde{\mathscr{D}}$ be attraction universes in X and $\tilde{X}$, respectively. We say that a cohomology T is a bijection between $\mathscr{D}$ and $\tilde{\mathscr{D}}$ if for each $D\in\mathscr{D}$ there is a unique $\tilde{D}\in\tilde{\mathscr{D}}$ such that $\tilde{D}(\omega) = \mathsf{T}\big(\omega,D(\omega)\big)$, for all $\omega\in\Omega$, and conversely, for each $\tilde{B}\in\tilde{\mathscr{D}}$ there is a unique $B\in\mathscr{D}$ such that $\tilde{\mathscr{B}}(\omega) = \mathsf{T}\big(\omega,B(\omega)\big)$, for all $\omega\in\Omega$. A particular example of such a cohomology T is given later in Section 4.3, expression (4.80).

Theorem 4.1.31. *Suppose that ϕ and $\tilde{\phi}$ are conjugate NRDS with cohomology $\mathsf{T}:\Omega\times X\to\tilde{X}$ which is a bijection between $\mathscr{D}$ and $\tilde{\mathscr{D}}$. If ϕ has a random $\mathscr{D}$-uniform attractor $\mathscr{A}$ in X, then $\tilde{\phi}$ has a random $\tilde{\mathscr{D}}$-uniform attractor $\tilde{\mathscr{A}}$ in $\tilde{X}$, and vice versa. Moreover, the attractors satisfy*

$$\tilde{\mathscr{A}}(\omega)=\mathsf{T}\big(\omega,\mathscr{A}(\omega)\big),\qquad \forall\omega\in\Omega. \tag{4.15}$$

Proof. Suppose that ϕ has a random $\mathscr{D}$-uniform attractor $\mathscr{A}$, and let us prove by definition that $\tilde{\mathscr{A}}(\omega):=\mathsf{T}\big(\omega,\mathscr{A}(\omega)\big)$ defines the random $\tilde{\mathscr{D}}$-uniform attractor of $\tilde{\phi}$. Clearly, $\tilde{\mathscr{A}}$ is compact and measurable, since so is $\mathscr{A}$. Given $\tilde{D}\in\tilde{\mathscr{D}}$, since T is a bijection there is $D\in\mathscr{D}$ with $\tilde{D}(\omega)=\mathsf{T}\big(\omega,D(\omega)\big)$, and so

$$\begin{aligned}
&\sup_{\sigma\in\Sigma}\operatorname{dist}_{\tilde{X}}\Big(\tilde{\phi}\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\tilde{D}(\vartheta_{-t}\omega)\big),\tilde{\mathscr{A}}(\omega)\Big)=\\
&\quad=\sup_{\sigma\in\Sigma}\operatorname{dist}_{\tilde{X}}\Big(\tilde{\phi}\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\mathsf{T}(\vartheta_{-t}\omega,D(\vartheta_{-t}\omega))\big),\tilde{\mathscr{A}}(\omega)\Big)\\
&\quad=\sup_{\sigma\in\Sigma}\operatorname{dist}_{\tilde{X}}\Big(\mathsf{T}\Big(\omega,\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\big)\Big),\mathsf{T}\big(\omega,\mathscr{A}(\omega)\big)\Big)\\
&\quad=\operatorname{dist}_{\tilde{X}}\Big(\mathsf{T}\Big(\omega,\bigcup_{\sigma\in\Sigma}\phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\big)\Big),\mathsf{T}\big(\omega,\mathscr{A}(\omega)\big)\Big)\to 0,
\end{aligned}$$

where the convergence follows from the fact that

$$\lim_{t\to\infty}\operatorname{dist}_X\big(\bigcup_{\sigma\in\Sigma}\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)),\mathscr{A}(\omega)\big)=0$$

and $\mathsf{T}_\omega(\cdot)$ is a homeomorphism. Hence, $\tilde{\mathscr{A}}$ is a uniformly $\tilde{\mathscr{D}}$-pullback attracting set under $\tilde{\phi}$. In the same way, the minimality of $\tilde{\mathscr{A}}$ follows from that of $\mathscr{A}$.

The reverse claim is similar and follows the same paths. □

Analogously, the same corresponding conjugate theorem holds for cocycle attractors.

Theorem 4.1.32. *Suppose that ϕ and $\tilde{\phi}$ are conjugate NRDS with cohomology $\mathsf{T}:\Omega\times X\to\tilde{X}$ which is a bijection between $\mathscr{D}$ and $\tilde{\mathscr{D}}$. If ϕ has a $\mathscr{D}$-cocycle attractor $A=\{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ in X, then $\tilde{\phi}$ has a $\tilde{A}=\{\tilde{A}_\sigma(\cdot)\}_{\sigma\in\Sigma}$-cocycle attractor $\tilde{\mathscr{A}}$ in $\tilde{X}$, and vice versa. Moreover, the attractors satisfy*

$$\tilde{A}_\sigma(\omega)=\mathsf{T}\big(\omega,A_\sigma(\omega)\big),\qquad \forall\sigma\in\Sigma,\,\omega\in\Omega. \tag{4.16}$$

Remark 4.1.33. *The structural relationships* (4.15) *and* (4.16) *allow one to learn the structure of an attractor from that of its conjugate attractor. For instance, conjugate attractors could share the same fractal dimension, e.g., when the cohomology $\mathsf{T}(\omega,x)$ is linear in x.*

4.2 Finite-dimensionality of random uniform attractors

In this section we present a random extension of our previous results in Section 3.2 giving two general criteria to estimate the fractal dimension of random uniform attractors. One is based on a *smoothing* property of the system and requires an auxiliary Banach space compactly embedded into the phase space, see Theorem 4.2.3; the other is based on a *squeezing* property of the system, where no auxiliary space is needed, but the phase space, in applications, should be Hilbert, see Theorem 4.2.6. Neither of the two theorems implies the other. We mention that smoothing and squeezing properties have already been used in the literature in many problems in dynamics, as for instance in (CARABALLO; SONNER, 2017), (CARVALHO; SONNER, 2013), (CZAJA; EFENDIEV, 2011), (EFENDIEV; MIRANVILLE; ZELIK, 2000), (EFENDIEV; MIRANVILLE; ZELIK, 2003), (EFENDIEV; ZELIK, 2008), (EFENDIEV; YAMAMOTO; YAGI, 2011), (SHIRIKYAN; ZELIK, 2013) and (ZHAO; ZHOU, 2016) for the smoothing and (FOIAS; TEMAM, 1979), (DEBUSSCHE, 1997), (EDEN *et al.*, 1995), (FLANDOLI; LANGA, 1999), (KLOEDEN; LANGA, 2007), (COTI-ZELATI; KALITA, 2015) and (CUI; FREITAS; LANGA, 2018) for the squeezing. In the random setting we need to overcome the difficulty arising jointly from three features of the problem: the lack of invariance of random uniform attractors; the superposition of the base flow on the symbol space and the stochastic nature of the problem.

For the first two problems our previous results in Section 3.2 (see also (CUI *et al.*,)) provide some inspiration of solution. We carefully use the relationship (see (4.11))

$$\mathscr{A}(\omega)=\bigcup_{\sigma\in\Sigma}A_\sigma(\omega),\qquad \omega\in\Omega, \tag{4.17}$$

between the random uniform attractor $\mathscr{A}$ and the cocycle attractor $A=\{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ of the underlying system, where Σ is the symbol space and $(\Omega,\mathscr{F},\mathscr{P})$ is a probability space. This allows us to decompose the random uniform attractor into sets of cocycle attractor sections, and then the invariance of the cocycle attractor A is useful. Nevertheless, since the absorption time of the random absorbing set is usually random, the analysis in this section is more technical than in Section 3.2. Our solution to this is restricting ourselves to the dynamics within the absorbing set and requiring the absorbing set to absorb itself after a *deterministic* period of time. This condition, i.e., the existence of a deterministic absorbing time, is in fact slightly stronger than really needed, but facilitates our analysis when covering the random uniform attractor by a finite number of balls. This condition has been used in (SHIRIKYAN; ZELIK, 2013) previously in a construction of random exponential attractors, and our application of a reaction-diffusion equation in Section 4.3 shows that it is admissible especially for additive noises.

The third point, i.e., the stochastic nature of the problem, causes difficulties in applying the abstract criteria. Basically, Birkhoff's ergodic theorem is frequently needed and for that it is required some coefficients to have finite expectation which is, however, a difficult task in applications. It is a special problem for the smoothing property, and which we do not face in relation to the squeezing approach, as we will see in Section 4.3. It gives us an indication that the squeezing may be more applicable than the smoothing in the random setting.

However, as in the deterministic case in Section 3.2, here the smoothing approach also applies in order to prove the finite dimensionality of random uniform attractors in more regular spaces once it has finite fractal dimension on the phase space, see Theorem 4.2.8. In this case there is no need to estimate the expectation of coefficients and the problem related before does not appear here. Therefore we can see that both approaches (smoothing and squeezing) are useful in order to estimate the fractal dimension of uniform attractors in different spaces and that we can work with both for the same problem attacking different aspects, as for example in Section 4.3 where we prove for a stochastic reaction-diffusion equation the finite-dimensionality of its random uniform attractor in L^2 by the squeezing method and in H_0^1 by the smoothing method. An absorbing set with a deterministic absorption time for the system is also constructed, which is crucial for the analysis.

Finally, we note that (HAN; ZHOU, 2019) recently constructed a random uniform exponential attractor for a stochastic reaction-diffusion equation with quasiperiodic forcings. This result was derived by taking into account an extended phase space and then studying its respective skew-product semiflow, by which the problem is then reduced to a random autonomous problem. Here, since we are abstract and consider more general non-autonomous terms than quasiperiodic forcings such that the symbol space is no longer a linear space, the skew-product semiflow approach fails. Although we are not constructing random uniform exponential attractors in this present work, we have the advantage of a method that applies to Banach spaces in general and more general non-autonomous terms are allowed. The results in this section are presented in (CUI; CUNHA; LANGA,) and are part of our contribution to the field of random dynamical systems.

Recall that given a compact subset E of a Banach space X, the *fractal dimension* of E in X is defined as

$$\dim_F(E;X) := \limsup_{\varepsilon\to 0^+} \frac{\ln N_X[E;\varepsilon]}{-\ln\varepsilon},$$

where $N_X[E;r]$ denotes the minimum number of open ε-balls in X centred at points of E that are necessary to cover E.

As the attraction universe $\mathscr{D}$ we consider the collection of all tempered random sets in X, i.e.,

$$\mathscr{D} := \Big\{ D : D \text{ is a tempered random set in } X \Big\},$$

where a random set D in X is said to be *tempered* if $\|D(\omega)\|_X := \sup_{x\in D(\omega)} \|x\|_X \leqslant R(\omega)$ for some random variable $R(\cdot) : \Omega \to \mathbb{R}$ which is tempered, i.e.,

$$\lim_{t\to\pm\infty} \frac{\ln R(\vartheta_t\omega)}{|t|} = 0, \qquad \forall \omega\in\Omega. \tag{4.18}$$

4.2.1 Smoothing approach

In this section we present a smoothing method in order to estimate the fractal dimension of random uniform attractors. Let ϕ be an NRDS on a separable Banach space X and $\mathscr{A}$ its random uniform attractor. Suppose Y is a separable Banach space which is compactly embedded in X, i.e., the embedding $I : Y \hookrightarrow X$ is compact. The following lemma of Sobolev compactness embedding gives us examples of such Banach spaces.

Lemma 4.2.1. *(TEMAM, 1997) Let $\mathscr{O} \subset \mathbb{R}^N$ be a $\mathscr{C}^1$-domain which is bounded (or at least bounded in one direction), $N \in \mathbb{N}$. Then the embedding $W^{1,p}(\mathscr{O}) \hookrightarrow L^{q_1}(\mathscr{O})$ is compact for any q_1 with $q_1 \in [1,\infty)$ if $p \geqslant N$ and $q_1 \in [1,q)$, $q^{-1} = p^{-1} - N^{-1}$, if $1 \leqslant p < N$.*

Remember that the Kolmogorov ε-entropy of the embedding $I : Y \hookrightarrow X$ is given as

$$\mathbf{H}_\varepsilon(Y;X) = \log_2 N_\varepsilon, \tag{4.19}$$

where $N_\varepsilon = N_X\big[B_Y(0,1);\varepsilon\big]$. The following estimate is useful. The following lemma gives estimates on the Kolmogorov entropy for particular cases of Sobolev spaces.

Lemma 4.2.2. *(TRIEBEL, 1978, Section 4.10.3) Let $\mathscr{O} \subset \mathbb{R}^N$ be a bounded $\mathscr{C}^\infty$-domain, $N \in \mathbb{N}$. If $1 < p,q < \infty$, $s - \frac{N}{q} > -\frac{N}{p}$, and $s > 0$, then there is a positive constant $\alpha > 0$ such that*

$$\mathbf{H}_\varepsilon\big(W^{s,q}(\mathscr{O});L^p(\mathscr{O})\big) \leqslant \alpha\varepsilon^{-\frac{N}{s}}.$$

As a particular case, for some $\alpha > 0$,

$$\mathbf{H}_\varepsilon\big(W^{1,2}(\mathscr{O});L^2(\mathscr{O})\big) \leqslant \alpha\varepsilon^{-N}. \tag{4.20}$$

Now we give our main criterion for a random uniform attractor to have finite fractal dimension. Suppose that

(R_1) The symbol space Σ has finite fractal dimension

$$\dim_F(\Sigma;\Xi) < \infty,$$

and the driving system $\{\theta_t\}_{t\in\mathbb{R}}$ on Σ is Lipschitz, satisfying

$$d_\Xi(\theta_t\sigma_1,\theta_t\sigma_2) \leqslant M(t)d_\Xi(\sigma_1,\sigma_2), \qquad \forall t \in \mathbb{R},\ \sigma_1,\sigma_2 \in \Sigma, \tag{4.21}$$

where $M(\cdot)$ is a function with $1 \leqslant M(t) \leqslant c_1 e^{\mu|t|}$, $t \in \mathbb{R}$, for some constants $c_1, \mu > 0$;

(R_2) ϕ is $(\Sigma \times X, X)$-continuous;

(R_3) ϕ has a tempered uniformly $\mathscr{D}$-pullback absorbing set $\mathscr{B} = \{\mathscr{B}(\omega)\}_{\omega\in\Omega}$ which pullback absorbs itself after a deterministic period of time, i.e., there exists a deterministic time $T_\mathscr{B} > 0$ such that for all $t \geqslant T_\mathscr{B}$ we have

$$\bigcup_{\sigma\in\Sigma} \phi\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\mathscr{B}(\vartheta_{-t}\omega)\big) \subseteq \mathscr{B}(\omega), \qquad \forall\omega\in\Omega; \tag{4.22}$$

(R_4) ϕ is Lipschitz continuous in symbols within the absorbing set $\mathscr{B}$, i.e.,

$$\|\phi(t,\omega,\sigma_1,u)-\phi(t,\omega,\sigma_2,u)\|_X \leqslant e^{\int_0^t L(\vartheta_s\omega)ds} d_\Xi(\sigma_1,\sigma_2), \quad \forall t \geqslant T_{\mathscr{B}}, \sigma_1,\sigma_2 \in \Sigma, u \in \mathscr{B}(\omega),$$

for a random variable $L(\cdot):\Omega \to \mathbb{R}^+$ with finite expectation $\mathbb{E}(L) < \infty$;

(R_5) Y is a separable Banach space densely and compactly embedded into X, and for any $\varepsilon > 0$ the Kolmogorov ε-entropy of Y in X satisfies

$$\mathbf{H}_\varepsilon(Y;X) = \log_2 N_X[B_Y(0,1);\varepsilon] \leqslant \alpha\varepsilon^{-\gamma},$$

for positive constants $\alpha, \gamma > 0$;

(R_6) ϕ is (X,Y)-smoothing within the absorbing set $\mathscr{B}$, i.e., there exist $\tilde{t} \geqslant T_{\mathscr{B}}$ and a random variable $\kappa(\cdot):\Omega \to \mathbb{R}^+$ with finite expectation $\mathbb{E}(\kappa^\gamma) < \infty$ such that

$$\sup_{\sigma\in\Sigma} \|\phi(\tilde{t},\omega,\sigma,u)-\phi(\tilde{t},\omega,\sigma,v)\|_Y \leqslant \kappa(\omega)\|u-v\|_X, \qquad \forall u,v \in \mathscr{B}(\omega),\ \omega \in \Omega. \tag{4.23}$$

Under these hypotheses the random uniform attractor $\mathscr{A} = \{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ of ϕ has finite fractal dimension which can be bounded by a deterministic number. More precisely,

Theorem 4.2.3. *Suppose that ϕ is an NRDS on X with random $\mathscr{D}$-uniform attractor $\mathscr{A}$ and $\mathscr{D}$-cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$. If conditions (R_1) - (R_6) hold, then $\mathscr{A}$ has finite fractal dimension in X: for any $\nu \in (0,1)$,*

$$\dim_F\left(\mathscr{A}(\omega);X\right) \leqslant \frac{2^\gamma\alpha\,\mathbb{E}(\kappa^\gamma)}{-\nu^\gamma\log_2\nu} + \left(\frac{\mathbb{E}(L)+\mu}{-\ln\nu}+1\right)\dim_F\left(\Sigma;\Xi\right), \qquad \forall\omega\in\Omega. \tag{4.24}$$

In particular, taking $\nu = 1/2$,

$$\dim_F\left(\mathscr{A}(\omega);X\right) \leqslant 4^\gamma\alpha\,\mathbb{E}(\kappa^\gamma) + \left(\frac{\mathbb{E}(L)+\mu}{\ln 2}+1\right)\dim_F\left(\Sigma;\Xi\right), \qquad \forall\omega\in\Omega.$$

Remark 4.2.4. *Notice that*

(i) the upper bound given in the theorem is deterministic and uniform w.r.t. $\omega\in\Omega$;

(ii) the entropy condition (R_5) depends only on the spaces X and Y, and it is independent of the system ϕ. Lemma 4.2.2 is useful in order to obtain such a property.

Proof of Theorem 3.2.1. Let $\nu \in (0,1)$ be given and fixed, and suppose without loss of generality that $\tilde{t} = T_{\mathscr{B}} = 1$ in hypotheses (R_3) and (R_6). Since the absorbing random set $\mathscr{B}$ is tempered, we have

$$\mathscr{B}(\omega) = B_X\big(x_\omega, R(\omega)\big) \cap \mathscr{B}(\omega),$$

for points $x_\omega \in \mathscr{B}(\omega)$ and some tempered random variable $R(\cdot)$ satisfying (4.18). Since Y is compactly embedded into X the unit ball $B_Y(0,1)$ in Y is covered by a finite number of $\frac{\nu}{2\kappa(\omega)}$-balls in X, and we denote by $N(\omega)$ the minimum number of such balls that are necessary for this, i.e.,

$$B_Y(0,1) \subseteq \bigcup_{i=1}^{N(\omega)} B_X\left(p_i^\omega, \frac{\nu}{2\kappa(\omega)}\right), \quad p_i^\omega \in B_Y(0,1). \tag{4.25}$$

Next, for each $\omega \in \Omega$ and $\sigma \in \Sigma$ we construct sets $U^n(\omega,\sigma) \subseteq \mathscr{B}(\omega)$ by induction on $n \in \mathbb{N}$ such that

$$U^n(\omega,\sigma) \subseteq \mathscr{B}(\omega), \tag{4.26}$$

$$\sharp U^n(\omega,\sigma) \leqslant \prod_{j=1}^{n} N(\vartheta_{-j}\omega), \tag{4.27}$$

$$\phi\left(n, \vartheta_{-n}\omega, \theta_{-n}\sigma, \mathscr{B}(\vartheta_{-n}\omega)\right) \subseteq \bigcup_{u\in U^n(\omega,\sigma)} B_X\left(u, R(\vartheta_{-n}\omega)\nu^n\right) \cap \mathscr{B}(\omega). \tag{4.28}$$

Note that the bound (4.27) of cardinality is independent of σ.

For $n=1$, by the smoothing property (4.23) in hypothesis (R_6) we have

$$\begin{aligned}
\phi\left(1, \vartheta_{-1}\omega, \theta_{-1}\sigma, \mathscr{B}(\vartheta_{-1}\omega)\right) &= \phi\left(1, \vartheta_{-1}\omega, \theta_{-1}\sigma, B_X\left(x_{\vartheta_{-1}\omega}, R(\vartheta_{-1}\omega)\right) \cap \mathscr{B}(\vartheta_{-1}\omega)\right) \\
&\subseteq B_Y\left(\phi(1, \vartheta_{-1}\omega, \theta_{-1}\sigma, x_{\vartheta_{-1}\omega}), \kappa(\vartheta_{-1}\omega)R(\vartheta_{-1}\omega)\right) \cap \phi\left(1, \vartheta_{-1}\omega, \theta_{-1}\sigma, \mathscr{B}(\vartheta_{-1}\omega)\right).
\end{aligned}$$

Let $y_{\omega,\sigma} := \phi(1, \vartheta_{-1}\omega, \theta_{-1}\sigma, x_{\vartheta_{-1}\omega})$. From (4.22) (since $1 = \tilde{t} = T_{\mathscr{B}}$) and (4.25) we note that

$$\begin{aligned}
& B_Y\left(y_{\omega,\sigma}, \kappa(\vartheta_{-1}\omega)R(\vartheta_{-1}\omega)\right) \cap \phi\left(1, \vartheta_{-1}\omega, \theta_{-1}\sigma, \mathscr{B}(\vartheta_{-1}\omega)\right) \\
&\quad \subseteq \bigcup_{i=1}^{N(\vartheta_{-1}\omega)} B_X\left(y_{\omega,\sigma} + \kappa(\vartheta_{-1}\omega)R(\vartheta_{-1}\omega)p_i^{\vartheta_{-1}\omega}, \frac{\kappa(\vartheta_{-1}\omega)R(\vartheta_{-1}\omega)\nu}{2\kappa(\vartheta_{-1}\omega)}\right) \cap \mathscr{B}(\omega) \\
&\quad = \bigcup_{i=1}^{N(\vartheta_{-1}\omega)} B_X\left(y_{\omega,\sigma} + \kappa(\vartheta_{-1}\omega)R(\vartheta_{-1}\omega)p_i^{\vartheta_{-1}\omega}, \frac{R(\vartheta_{-1}\omega)\nu}{2}\right) \cap \mathscr{B}(\omega) \\
&\quad \subseteq \bigcup_{i=1}^{N(\vartheta_{-1}\omega)} B_X\left(q_i^{\omega,\sigma}, R(\vartheta_{-1}\omega)\nu\right) \cap \mathscr{B}(\omega)
\end{aligned}$$

for some $q_i^{\omega,\sigma} \in \mathscr{B}(\omega)$, and with this we have

$$\phi\left(1, \vartheta_{-1}\omega, \theta_{-1}\sigma, \mathscr{B}(\vartheta_{-1}\omega)\right) \subseteq \bigcup_{i=1}^{N(\vartheta_{-1}\omega)} B_X\left(q_i^{\omega,\sigma}, R(\vartheta_{-1}\omega)\nu\right) \cap \mathscr{B}(\omega).$$

Let $U^1(\omega,\sigma) := \left\{q_i^{\omega,\sigma} : i = 1, \cdots, N(\vartheta_{-1}\omega)\right\} \subseteq \mathscr{B}(\omega)$, then $U^1(\omega,\sigma)$ satisfies (4.26) - (4.28) for $n=1$.

Assuming that the sets $U^k(\omega,\sigma)$ have been constructed for all $1 \leqslant k \leqslant n$, $\omega \in \Omega$ and $\sigma \in \Sigma$, we now construct the sets $U^{n+1}(\omega,\sigma)$. Given $\omega \in \Omega$ and $\sigma \in \Sigma$, by the cocycle property of ϕ we have

$$\begin{aligned}&\phi\big(n+1,\vartheta_{-(n+1)}\omega,\theta_{-(n+1)}\sigma,\mathscr{B}(\vartheta_{-(n+1)}\omega)\big)\\&\quad=\phi\big(1,\vartheta_{-1}\omega,\theta_{-1}\sigma\big)\circ\phi\big(n,\vartheta_{-(n+1)}\omega,\theta_{-(n+1)}\sigma,\mathscr{B}(\vartheta_{-(n+1)}\omega)\big),\end{aligned}$$

and by the induction hypothesis

$$\begin{aligned}&\phi\big(n,\vartheta_{-(n+1)}\omega,\theta_{-(n+1)}\sigma,\mathscr{B}(\vartheta_{-(n+1)}\omega)\big)\\&\quad=\phi\big(n,\vartheta_{-n}\vartheta_{-1}\omega,\theta_{-n}\theta_{-1}\sigma,\mathscr{B}(\vartheta_{-n}\vartheta_{-1}\omega)\big)\\&\quad\subseteq\bigcup_{u\in U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)}B_X\big(u,R(\vartheta_{-(n+1)}\omega)\nu^n\big)\cap\mathscr{B}(\vartheta_{-1}\omega),\end{aligned}$$

where $U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)\subseteq\mathscr{B}(\vartheta_{-1}\omega)$ and $\sharp U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)\leqslant\prod_{j=1}^{n}N\big(\vartheta_{-j}(\vartheta_{-1}\omega)\big)=\prod_{j=2}^{n+1}N(\vartheta_{-j}\omega)$. Moreover, for each $u\in U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)$, by hypothesis (R_3) and the smoothing property (R_6) we obtain

$$\begin{aligned}&\phi\Big(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,B_X\big(u,R(\vartheta_{-(n+1)}\omega)\nu^n\big)\cap\mathscr{B}(\vartheta_{-1}\omega)\Big)\\&\quad\subseteq B_Y\big(\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,u),\kappa(\vartheta_{-1}\omega)R(\vartheta_{-(n+1)}\omega)\nu^n\big)\cap\mathscr{B}(\omega)\\&\quad\subseteq\bigcup_{i=1}^{N(\vartheta_{-1}\omega)}B_X\big(p^{\omega}_{i,u},R(\vartheta_{-(n+1)}\omega)\nu^{n+1}\big)\cap\mathscr{B}(\omega)\qquad\text{for points } p^{\omega}_{i,u}\in\mathscr{B}(\omega),\end{aligned}$$

so

$$\begin{aligned}&\phi\big(n+1,\vartheta_{-(n+1)}\omega,\theta_{-(n+1)}\sigma,\mathscr{B}(\vartheta_{-(n+1)}\omega)\big)\\&\quad\subseteq\bigcup_{u\in U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)}\phi\Big(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,B_X\big(u,R(\vartheta_{-(n+1)}\omega)\nu^n\big)\cap\mathscr{B}(\vartheta_{-1}\omega)\Big)\\&\quad\subseteq\bigcup_{u\in U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)}\ \bigcup_{i=1}^{N(\vartheta_{-1}\omega)}B_X\big(p^{\omega}_{i,u},R(\vartheta_{-(n+1)}\omega)\nu^{n+1}\big)\cap\mathscr{B}(\omega).\end{aligned}$$

Define $U^{n+1}(\omega,\sigma):=\big\{p^{\omega}_{i,u}:u\in U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma),\,1\leqslant i\leqslant N(\vartheta_{-1}\omega)\big\}$. Then $U^{n+1}(\omega,\sigma)\subseteq\mathscr{B}(\omega)$ and $\sharp U^{n+1}(\omega,\sigma)\leqslant\sharp U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)\cdot N(\vartheta_{-1}\omega)=\prod_{j=1}^{n+1}N(\vartheta_{-j}\omega)$. Hence, the desired sets $\{U^n(\omega,\sigma)\}_{n\in\mathbb{N}}$ are constructed.

Now, to find a finite cover of the random uniform attractor $\mathscr{A}$ let us make a decomposition of it using the structure (4.17). By the compactness of the symbol space Σ, for any positive number $\eta>0$ there exists a finite cover of Σ by at least $M_\eta:=N_\Xi[\Sigma;\eta]$ balls of radius η, i.e., there are centers $\sigma_l\in\Sigma$, $l=1,2,\cdots,M_\eta$, such that

$$\Sigma=\bigcup_{l=1}^{M_\eta}B_\Xi(\sigma_l,\eta)\cap\Sigma.$$

For each $l=1,\cdots,M_\eta$, denote by

$$\Sigma_l:=B_\Xi(\sigma_l,\eta)\cap\Sigma\qquad\text{and}\qquad A_{\Sigma_l}(\omega):=\bigcup_{\sigma\in\Sigma_l}A_\sigma(\omega),\quad\omega\in\Omega,$$

where $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is the $\mathscr{D}$-cocycle attractor of ϕ. Then by (4.17) the random uniform attractor $\mathscr{A}$ is decomposed as

$$\mathscr{A}(\omega) = \bigcup_{l=1}^{M_\eta} A_{\Sigma_l}(\omega), \quad \omega \in \Omega. \tag{4.29}$$

In the following we shall find finite covers for each $A_{\Sigma_l}(\omega)$. Note that the constant M_η is independent of $\omega \in \Omega$, and depends only on the symbol space Σ and the corresponding given number η.

For each l, let $\sigma_l \in \Sigma_l$ be given as above. Then for any $\sigma \in \Sigma_l$, $d_\Xi(\sigma,\sigma_l) < \eta$. From the invariance of the cocyle attractor $\{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ under ϕ, by hypotheses (R_1) and (R_4) we claim that

$$A_{\Sigma_l}(\omega) \subseteq B_X\Big(\phi\big(n,\vartheta_{-n}\omega,\theta_{-n}\sigma_l,A_{\theta_{-n}\Sigma_l}(\vartheta_{-n}\omega)\big), M(-n)e^{\int_{-n}^0 L(\vartheta_s\omega)\mathrm{d}s}\eta\Big) \tag{4.30}$$

for each $1 \leqslant l \leqslant M_\eta$, $n \in \mathbb{N}$ and $\omega \in \Omega$. Indeed, if $h \in A_{\Sigma_l}(\omega)$ then $h \in A_\sigma(\omega)$ for some $\sigma \in \Sigma_l$. Since $A_\sigma(\omega) = \phi\big(n,\vartheta_{-n}\omega,\theta_{-n}\sigma,A_{\theta_{-n}\sigma}(\vartheta_{-n}\omega)\big)$, we have $h = \phi(n,\vartheta_{-n}\omega,\theta_{-n}\sigma,u)$ for some $u \in A_{\theta_{-n}\sigma}(\vartheta_{-n}\omega) \subseteq \mathscr{A}(\vartheta_{-n}\omega) \subseteq \mathscr{B}(\vartheta_{-n}\omega)$. Hence,

$$\begin{aligned}\|h-\phi(n,\vartheta_{-n}\omega,\theta_{-n}\sigma_l,u)\|_X &= \|\phi(n,\vartheta_{-n}\omega,\theta_{-n}\sigma,u)-\phi(n,\vartheta_{-n}\omega,\theta_{-n}\sigma_l,u)\|_X\\ &\leqslant e^{\int_0^n L(\vartheta_{s-n}\omega)\mathrm{d}s} d_\Xi(\theta_{-n}\sigma,\theta_{-n}\sigma_l)\\ &\leqslant e^{\int_{-n}^0 L(\vartheta_s\omega)\mathrm{d}s} M(-n) d_\Xi(\sigma,\sigma_l)\\ &< M(-n)e^{\int_{-n}^0 L(\vartheta_s\omega)\mathrm{d}s}\eta,\end{aligned}$$

and thus (4.30) holds. Notice that, since $A_{\theta_{-n}\Sigma_l}(\vartheta_{-n}\omega) \subseteq \mathscr{B}(\vartheta_{-n}\omega)$, from (4.28) it follows

$$N_X\Big[\phi\big(n,\vartheta_{-n}\omega,\theta_{-n}\sigma_l,A_{\theta_{-n}\Sigma_l}(\vartheta_{-n}\omega)\big); R(\vartheta_{-n}\omega)\nu^n\Big] \leqslant \prod_{j=1}^n N(\vartheta_{-j}\omega),$$

and then

$$\begin{aligned}N_X\Big[B_X\Big(\phi\big(n,\vartheta_{-n}\omega,\theta_{-n}\sigma_l,A_{\theta_{-n}\Sigma_l}(\vartheta_{-n}\omega)\big), M(-n)e^{\int_{-n}^0 L(\vartheta_s\omega)\mathrm{d}s}\eta\Big); R(\vartheta_{-n}\omega)\nu^n\\ +M(-n)e^{\int_{-n}^0 L(\vartheta_s\omega)\mathrm{d}s}\eta\Big] \leqslant \prod_{j=1}^n N(\vartheta_{-j}\omega).\end{aligned}$$

Hence, from (4.30),

$$N_X\Big[A_{\Sigma_l}(\omega);\ R(\vartheta_{-n}\omega)\nu^n + M(-n)e^{\int_{-n}^0 L(\vartheta_s\omega)\mathrm{d}s}\eta\Big] \leqslant \prod_{j=1}^n N(\vartheta_{-j}\omega), \qquad l = 1,2,\cdots,M_\eta,$$

and then by (4.29) we conclude that

$$N_X\Big[\mathscr{A}(\omega);\ R(\vartheta_{-n}\omega)\nu^n + M(-n)e^{\int_{-n}^0 L(\vartheta_s\omega)\mathrm{d}s}\eta\Big] \leqslant \left(\prod_{j=1}^n N(\vartheta_{-j}\omega)\right) M_\eta. \tag{4.31}$$

Given $\tau > 0$, by Birkhoff's ergodic theorem there is $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$ we have

$$e^{\int_{-n}^{0} L(\vartheta_s\omega)\mathrm{d}s} \leqslant e^{(\mathbb{E}(L)+\tau)n},$$

and then from (4.31)

$$N_X\left[\mathscr{A}(\omega);\ R(\vartheta_{-n}\omega)\nu^n + M(-n)e^{(\mathbb{E}(L)+\tau)n}\eta\right] \leqslant \left(\prod_{j=1}^{n} N(\vartheta_{-j}\omega)\right) M_\eta \tag{4.32}$$

for all $n \geqslant n_0$ and $\eta > 0$.

In the following we establish by (4.32) a finite ε-cover of $\mathscr{A}(\omega)$ for any small $\varepsilon > 0$. Let

$$\eta_n := \frac{R(\vartheta_{-n}\omega)\nu^n}{M(-n)e^{(\mathbb{E}(L)+\tau)n}}, \qquad n \in \mathbb{N}. \tag{4.33}$$

Then $\eta_n \to 0^+$ as $n \to \infty$, and from (4.32) we have for n sufficiently large that

$$\begin{aligned} N_X\left[\mathscr{A}(\omega);\ 2R(\vartheta_{-n}\omega)\nu^n\right] &= N_X\left[\mathscr{A}(\omega);R(\vartheta_{-n}\omega)\nu^n + M(-n)e^{(\mathbb{E}(L)+\tau)n}\eta_n\right] \\ &\leqslant \left(\prod_{j=1}^{n} N(\vartheta_{-j}\omega)\right) M_{\eta_n}. \end{aligned}$$

Since the random variable $R(\cdot)$ is tempered, for any $\varepsilon \in (0,1)$ there exists an $n_\varepsilon \in \mathbb{N}$ such that

$$2R(\vartheta_{-n_\varepsilon}\omega)\nu^{n_\varepsilon} < \varepsilon \leqslant 2R(\vartheta_{-(n_\varepsilon-1)}\omega)\nu^{n_\varepsilon-1}, \tag{4.34}$$

and the numbers n_ε can be chosen such that $n_\varepsilon \to \infty$ as $\varepsilon \to 0^+$. Hence, for $\varepsilon > 0$ sufficiently small

$$\begin{aligned} N_X\left[\mathscr{A}(\omega);\varepsilon\right] &\leqslant N_X\left[\mathscr{A}(\omega);2R(\vartheta_{-n_\varepsilon}\omega)\nu^{n_\varepsilon}\right] \\ &\leqslant \left(\prod_{j=1}^{n_\varepsilon} N(\vartheta_{-j}\omega)\right) M_{\eta_{n_\varepsilon}}, \end{aligned}$$

and then

$$\frac{\log_2 N_X\left[\mathscr{A}(\omega);\varepsilon\right]}{-\log_2\varepsilon} \leqslant \frac{\sum_{j=1}^{n_\varepsilon}\left(\log_2 N(\vartheta_{-j}\omega)\right) + \log_2 M_{\eta_{n_\varepsilon}}}{-\log_2\left[2R(\vartheta_{-(n_\varepsilon-1)}\omega)\nu^{n_\varepsilon-1}\right]}. \tag{4.35}$$

Now we estimate the fractal dimension of $\mathscr{A}(\omega)$ by studying the limit as $\varepsilon \to 0^+$. To begin with, let us handle carefully each term involved in the right-hand side of (4.35) to obtain (4.38) bellow. Firstly, by the entropy hypothesis (R_5), $\mathbf{H}_\varepsilon(Y;X) = \log_2 N_X\left[B_Y(0,1);\varepsilon\right] \leqslant \alpha\varepsilon^{-\gamma}$ for all $\varepsilon > 0$, so

$$\log_2 N(\vartheta_{-j}\omega) = \log_2\left(N_X\left[B_Y(0,1);\ \frac{\nu}{2\kappa(\vartheta_{-j}\omega)}\right]\right) \leqslant \frac{\alpha\left(\kappa(\vartheta_{-j}\omega)\right)^\gamma}{(\nu/2)^\gamma}. \tag{4.36}$$

Secondly, for any $\beta \in (\nu,1)$ fixed, since the random variable $R(\cdot)$ is tempered there exists $n_1 \in \mathbb{N}$ such that

$$R(\vartheta_{-n}\omega)\left(\frac{\nu}{\beta}\right)^n < \frac{1}{2}, \qquad \forall n \geqslant n_1,$$

that is,

$$2R(\vartheta_{-n}\omega)\nu^n < \beta^n, \qquad \forall n \geqslant n_1. \tag{4.37}$$

Hence, for all $\varepsilon > 0$ small enough such that $n_\varepsilon \geqslant \max\{n_0; n_1 + 1\}$, it follows from (4.35), (4.36) and (4.37) that

$$\begin{aligned}
\frac{\log_2 N_X\left[\mathscr{A}(\omega);\varepsilon\right]}{-\log_2 \varepsilon} &\leqslant \frac{\sum_{j=1}^{n_\varepsilon}\left(\log_2 N(\vartheta_{-j}\omega)\right) + \log_2 M_{\eta_{n_\varepsilon}}}{-\log_2\left[2R(\vartheta_{-(n_\varepsilon-1)}\omega)\nu^{n_\varepsilon-1}\right]} \\
&\leqslant \frac{\frac{\alpha}{(\nu/2)^\gamma}\sum_{j=1}^{n_\varepsilon}\left(\kappa(\vartheta_{-j}\omega)\right)^\gamma + \log_2 M_{\eta_{n_\varepsilon}}}{-\log_2 \beta^{n_\varepsilon-1}} \\
&= \frac{\frac{\alpha}{(\nu/2)^\gamma}\sum_{j=1}^{n_\varepsilon}\left(\kappa(\vartheta_{-j}\omega)\right)^\gamma}{-(n_\varepsilon-1)\log_2\beta} + \frac{\log_2 N_\Xi\left[\Sigma;\eta_{n_\varepsilon}\right]}{-(n_\varepsilon-1)\log_2\beta}.
\end{aligned} \tag{4.38}$$

Now we take the limit as $\varepsilon \to 0^+$ (which leads to $n_\varepsilon \to \infty$ and $\eta_{n_\varepsilon} \to 0^+$). By Birkhoff's ergodic theorem since $\mathbb{E}(\kappa^\gamma) < \infty$, we first obtain

$$\begin{aligned}
\dim_F\left(\mathscr{A}(\omega);X\right) &= \limsup_{\varepsilon\to 0^+}\frac{\log_2 N_X[\mathscr{A}(\omega);\varepsilon]}{-\log_2\varepsilon} \\
&\leqslant \limsup_{\varepsilon\to 0^+}\frac{\frac{\alpha}{(\nu/2)^\gamma}\sum_{j=1}^{n_\varepsilon}\left(\kappa(\vartheta_{-j}\omega)\right)^\gamma}{-(n_\varepsilon-1)\log_2\beta} + \limsup_{\varepsilon\to 0^+}\frac{\log_2 N_\Xi\left[\Sigma;\eta_{n_\varepsilon}\right]}{-(n_\varepsilon-1)\log_2\beta} \\
&= \frac{\alpha\mathbb{E}(\kappa^\gamma)}{-(\nu/2)^\gamma\log_2\beta} + \limsup_{\varepsilon\to 0^+}\frac{\log_2 N_\Xi\left[\Sigma;\eta_{n_\varepsilon}\right]}{-(n_\varepsilon-1)\log_2\beta}.
\end{aligned} \tag{4.39}$$

Then we consider the last limit in (4.39). Since by (R_1) the symbol space Σ has finite fractal dimension in Ξ and that $\eta_{n_\varepsilon} \to 0^+$ as $\varepsilon \to 0^+$, for any $\chi > 0$ there exists $\varepsilon_0 = \varepsilon_0(\chi) \in (0,1)$ such that

$$N_\Xi\left[\Sigma;\eta_{n_\varepsilon}\right] \leqslant \left(\frac{1}{\eta_{n_\varepsilon}}\right)^{\dim_F(\Sigma;\Xi)+\chi}, \qquad \forall \varepsilon \leqslant \varepsilon_0. \tag{4.40}$$

From (4.40) and the definition (4.33) of η_n we obtain

$$\begin{aligned}
\frac{\log_2 N_\Xi\left[\Sigma;\eta_{n_\varepsilon}\right]}{-(n_\varepsilon-1)\log_2\beta} &= \frac{\ln N_\Xi\left[\Sigma;\eta_{n_\varepsilon}\right]}{-(n_\varepsilon-1)\ln\beta} \\
&\leqslant \frac{\left(\dim_F(\Sigma;\Xi)+\chi\right)\ln\frac{1}{\eta_{n_\varepsilon}}}{-(n_\varepsilon-1)\ln\beta} \\
&= \left(\dim_F(\Sigma;\Xi)+\chi\right)\frac{\ln\left[\frac{M(-n_\varepsilon)e^{(\mathbb{E}(L)+\tau)n_\varepsilon}}{R(\vartheta_{-n_\varepsilon}\omega)\nu^{n_\varepsilon}}\right]}{-(n_\varepsilon-1)\ln\beta}, \quad \forall \varepsilon \leqslant \varepsilon_0,
\end{aligned}$$

while from (R_1) we have

$$\frac{\ln\left[\dfrac{M(-n_\varepsilon)e^{(\mathbb{E}(L)+\tau)n_\varepsilon}}{R(\vartheta_{-n_\varepsilon}\omega)v^{n_\varepsilon}}\right]}{-(n_\varepsilon-1)\ln\beta} \leqslant \frac{\ln\left[\dfrac{c_1e^{(\mathbb{E}(L)+\tau+\mu)n_\varepsilon}}{R(\vartheta_{-n_\varepsilon}\omega)v^{n_\varepsilon}}\right]}{-(n_\varepsilon-1)\ln\beta} \quad (\text{by } (R_1))$$
$$= \frac{\ln c_1+\big(\mathbb{E}(L)+\tau+\mu\big)n_\varepsilon}{-(n_\varepsilon-1)\ln\beta} - \frac{\ln R(\vartheta_{-n_\varepsilon}\omega)}{-(n_\varepsilon-1)\ln\beta} - \frac{n_\varepsilon\ln v}{-(n_\varepsilon-1)\ln\beta}, \quad \forall\varepsilon\leqslant\varepsilon_0.$$

Hence, since $n_\varepsilon\to\infty$ as $\varepsilon\to0^+$,

$$\limsup_{\varepsilon\to0^+}\frac{\log_2 N_\Xi[\Sigma;\eta_{n_\varepsilon}]}{-(n_\varepsilon-1)\log_2\beta} \leqslant \big(\dim_F(\Sigma;\Xi)+\chi\big)\left(\frac{\mathbb{E}(L)+\tau+\mu}{-\ln\beta}+\frac{\ln v}{\ln\beta}\right),$$

which along with (4.39) we conclude that

$$\dim_F\big(\mathscr{A}(\omega);X\big) \leqslant \frac{\alpha\mathbb{E}(\kappa^\gamma)}{-(v/2)^\gamma\log_2\beta} + \big(\dim_F(\Sigma;\Xi)+\chi\big)\left(\frac{\mathbb{E}(L)+\tau+\mu}{-\ln\beta}+\frac{\ln v}{\ln\beta}\right). \tag{4.41}$$

Since the estimate (4.41) holds for all $\chi,\tau>0$ and all $\beta\in(v,1)$ we finally obtain

$$\dim_F\big(\mathscr{A}(\omega);X\big) \leqslant \frac{\alpha\mathbb{E}(\kappa^\gamma)}{-(v/2)^\gamma\log_2 v} + \dim_F(\Sigma;\Xi)\left(\frac{\mathbb{E}(L)+\mu}{-\ln v}+1\right). \qquad \square$$

4.2.2 Squeezing approach

Theorem 4.2.3 gives a criterion on the finite-dimensionality of random uniform attractors where, however, the finiteness of the expectation of the coefficient $\kappa(\omega)$ in the smoothing condition (R_6) is usually not easy to obtain in real applications. To overcome this, we next propose an alternative method using a squeezing condition instead. The *squeezing*, in applications, applies mainly to Hilbert phase spaces X, but allows the coefficients to be an exponential with only the order having finite expectation (the expectation of the entire exponential need not be finte, see (S)).

We first recall the following lemma of finite-coverings of balls in Euclidian spaces.

Lemma 4.2.5. *(DEBUSSCHE, 1997, Lemma 1.2) Let E be an Euclidean space with algebraic dimension equals to $m\in\mathbb{N}$ and $R\geqslant r>0$ be positive numbers. Then for any $x\in E$ it holds*

$$N_E\big[B_E(x,R);r\big]\leqslant k(R,r)\leqslant\left(\frac{R\sqrt{m}}{r}+1\right)^m.$$

In other words, any ball in E with radius $R>0$ can be covered by $k(R,r)$ balls of radius $r>0$.

Let ϕ be an NRDS, $T_{\mathscr{B}}>0$ be as in (R_3) and suppose in addition the following *squeezing* property:

(S) ϕ satisfies a random uniformly squeezing property on $\mathscr{B}$, i.e., there exist $\tilde{t}\geqslant T_{\mathscr{B}}$, $\delta\in(0,1/4)$, an m-dimensional orthogonal projection $P:X\to PX$ ($\dim(PX)=m$) and a random variable $\zeta(\cdot):\Omega\to\mathbb{R}$ with finite expectation $\mathbb{E}(\zeta)<-\ln(4\delta)$ such that

$$\sup_{\sigma\in\Sigma}\big\|P\big(\phi(\tilde{t},\omega,\sigma,u)-\phi(\tilde{t},\omega,\sigma,v)\big)\big\|_X\leqslant e^{\int_0^{\tilde{t}}\zeta(\vartheta_s\omega)\mathrm{d}s}\|u-v\|_X \tag{4.42}$$

and

$$\sup_{\sigma\in\Sigma}\left\|Q\big(\phi(\tilde{t},\omega,\sigma,u)-\phi(\tilde{t},\omega,\sigma,v)\big)\right\|_X \leqslant \delta e^{\int_0^{\tilde{t}}\zeta(\vartheta_s\omega)\mathrm{d}s}\|u-v\|_X, \tag{4.43}$$

for all $u,v\in\mathscr{B}(\omega)$, $\omega\in\Omega$, where $Q:=I-P$.

We have then the following criterion for the finite-dimensionality of random uniform attractors.

Theorem 4.2.6. *Suppose that ϕ is an NRDS on X with random $\mathscr{D}$-uniform attractor $\mathscr{A}$ and $\mathscr{D}$-cocycle attractor $A=\{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$. If conditions (R_1) - (R_4) and (S) hold, then $\mathscr{A}$ has finite fractal dimension in X: for any $0<\rho<\ln(1/4\delta)-\mathbb{E}(\zeta)$,*

$$\dim_F\big(\mathscr{A}(\omega);X\big)\leqslant\frac{2m\ln\left(\frac{\sqrt{m}}{\delta}+1\right)}{\rho}+\left(\frac{2\big(\mathbb{E}(L)+\mu\big)}{\rho}+1\right)\dim_F(\Sigma;\Xi),\quad \forall\omega\in\Omega. \tag{4.44}$$

Proof. Suppose without loss of generality that $\tilde{t}=T_{\mathscr{B}}=1$ in hypotheses (R_3) and (S). Since the random absorbing set $\mathscr{B}$ is tempered, we have

$$\mathscr{B}(\omega)=B_X\big(x_\omega,R(\omega)\big)\cap\mathscr{B}(\omega),$$

for points $x_\omega\in\mathscr{B}(\omega)$ and some tempered random variable $R(\cdot)$ satisfying (4.18).

Next, for each $\omega\in\Omega$ and $\sigma\in\Sigma$ we construct sets $U^n(\omega,\sigma)\subseteq\mathscr{B}(\omega)$ by induction on $n\in\mathbb{N}$ such that

$$U^n(\omega,\sigma)\subseteq\mathscr{B}(\omega), \tag{4.45}$$

$$\sharp U^n(\omega,\sigma)\leqslant k_0^n,\ \text{where } k_0:=\text{the integral part of }\left(\frac{\sqrt{m}}{\delta}+1\right)^m, \tag{4.46}$$

$$\phi\big(n,\vartheta_{-n}\omega,\theta_{-n}\sigma,\mathscr{B}(\vartheta_{-n}\omega)\big)\subseteq\bigcup_{u\in U^n(\omega,\sigma)}B_X\Big(u,(4\delta)^n e^{\int_{-n}^0\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-n}\omega)\Big)\cap\mathscr{B}(\omega). \tag{4.47}$$

Note that the inclusion (4.45) is independent of σ and the bound (4.46) of cardinality is independent of both σ and ω.

Let $n=1$, $\omega\in\Omega$ and $\sigma\in\Sigma$. Since $\dim(PX)=m$, from Lemma 4.2.5 we have

$$N_{PX}\left[B_{PX}\Big(P\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,x_{\vartheta_{-1}\omega}),e^{\int_{-1}^0\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-1}\omega)\Big);\delta e^{\int_{-1}^0\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-1}\omega)\right]\\ \leqslant\left(\frac{\sqrt{m}}{\delta}+1\right)^m,$$

so there exist k_0 $\big(:=$ the integral part of $\big(\frac{\sqrt{m}}{\delta}+1\big)^m\big)$ centers $x^1_{\omega,\sigma},\cdots,x^{k_0}_{\omega,\sigma}\in X$ such that

$$B_{PX}\Big(P\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,x_{\vartheta_{-1}\omega}),e^{\int_{-1}^0\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-1}\omega)\Big)\\ \subseteq\bigcup_{i=1}^{k_0}B_{PX}\Big(x^i_{\omega,\sigma},\delta e^{\int_{-1}^0\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-1}\omega)\Big).$$

Hence, for any $u \in \mathscr{B}(\vartheta_{-1}\omega)$, since by (4.42)

$$\left\|P\big(\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,u)-\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,x_{\vartheta_{-1}\omega})\big)\right\|_X \leqslant e^{\int_{-1}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-1}\omega),$$

we have

$$\begin{aligned} P\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,u) &\in B_{PX}\Big(P\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,x_{\vartheta_{-1}\omega}),e^{\int_{-1}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-1}\omega)\Big)\\ &\subseteq \bigcup_{i=1}^{k_0} B_{PX}\Big(x_{\omega,\sigma}^i,\delta e^{\int_{-1}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-1}\omega)\Big), \end{aligned}$$

and for some particular $i_0 \in \{1,2,\cdots,k_0\}$

$$P\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,u) \in B_{PX}\Big(x_{\omega,\sigma}^{i_0},\delta e^{\int_{-1}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-1}\omega)\Big). \tag{4.48}$$

Setting

$$y_{\omega,\sigma}^i := x_{\omega,\sigma}^i + Q\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,x_{\vartheta_{-1}\omega}) \in X, \quad i=1,2,\cdots,k_0,$$

we obtain from (4.43) and (4.48) that

$$\begin{aligned} \|\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,u)-y_{\omega,\sigma}^{i_0}\|_X &\leqslant \|P\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,u)-x_{\omega,\sigma}^{i_0}\|_X\\ &\quad+\|Q\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,u)-Q\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,x_{\vartheta_{-1}\omega})\|_X\\ &\leqslant \delta e^{\int_{-1}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-1}\omega)+\delta e^{\int_{-1}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-1}\omega)\\ &=(2\delta)e^{\int_{-1}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-1}\omega). \end{aligned}$$

Hence, since u was taken arbitrarily in $\mathscr{B}(\vartheta_{-1}\omega)$,

$$\phi\left(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,\mathscr{B}(\vartheta_{-1}\omega)\right) \subseteq \bigcup_{i=1}^{k_0} B_X\Big(y_{\omega,\sigma}^i,(2\delta)e^{\int_{-1}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-1}\omega)\Big).$$

In addition, as $\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,\mathscr{B}(\vartheta_{-1}\omega)) \subseteq \mathscr{B}(\omega)$, we finally have

$$\phi\left(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,\mathscr{B}(\vartheta_{-1}\omega)\right) \subseteq \bigcup_{i=1}^{k_0} B_X\Big(\tilde{y}_{\omega,\sigma}^i,(4\delta)e^{\int_{-1}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-1}\omega)\Big)\cap\mathscr{B}(\omega)$$

for some points $\tilde{y}_{\omega,\sigma}^i \in \mathscr{B}(\omega)$. Let $U^1(\omega,\sigma) := \left\{\tilde{y}_{\omega,\sigma}^i : i=1,\cdots,k_0\right\} \subseteq \mathscr{B}(\omega)$. Then $U^1(\omega,\sigma)$ satisfies (4.45)-(4.47) for $n=1$.

Assuming that the sets $U^k(\omega,\sigma)$ have been constructed for all $1 \leqslant k \leqslant n$, $\omega \in \Omega$ and $\sigma \in \Sigma$, we now construct the sets $U^{n+1}(\omega,\sigma)$. Given $\omega \in \Omega$ and $\sigma \in \Sigma$, by the cocycle property of ϕ we have

$$\begin{aligned} &\phi\left(n+1,\vartheta_{-(n+1)}\omega,\theta_{-(n+1)}\sigma,\mathscr{B}(\vartheta_{-(n+1)}\omega)\right)\\ &\quad=\phi\left(1,\vartheta_{-1}\omega,\theta_{-1}\sigma\right)\circ\phi\left(n,\vartheta_{-(n+1)}\omega,\theta_{-(n+1)}\sigma,\mathscr{B}(\vartheta_{-(n+1)}\omega)\right), \end{aligned} \tag{4.49}$$

and by the induction hypothesis

$$\begin{aligned}\phi\big(n,\vartheta_{-(n+1)}\omega,\theta_{-(n+1)}\sigma,\mathscr{B}(\vartheta_{-(n+1)}\omega)\big)&=\phi\big(n,\vartheta_{-n}\vartheta_{-1}\omega,\theta_{-n}\theta_{-1}\sigma,\mathscr{B}(\vartheta_{-n}\vartheta_{-1}\omega)\big)\\&\subseteq\bigcup_{u\in U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)}B_X\Big(u,(4\delta)^n e^{\int_{-(n+1)}^{-1}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\Big)\cap\mathscr{B}(\vartheta_{-1}\omega),\end{aligned}\tag{4.50}$$

where $U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)\subseteq\mathscr{B}(\vartheta_{-1}\omega)$ and $\sharp U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)\leqslant k_0^n$. Combine (4.49) and (4.50) to obtain

$$\begin{aligned}&\phi\big(n+1,\vartheta_{-(n+1)}\omega,\theta_{-(n+1)}\sigma,\mathscr{B}(\vartheta_{-(n+1)}\omega)\big)\\&\subseteq\bigcup_{u\in U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)}\phi\Big(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,B_X\Big(u,(4\delta)^n e^{\int_{-(n+1)}^{-1}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\Big)\cap\mathscr{B}(\vartheta_{-1}\omega)\Big).\end{aligned}\tag{4.51}$$

Now we cover each term in the right-hand side to obtain (4.53). For each $u\in U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)\subseteq\mathscr{B}(\vartheta_{-1}\omega)$ we have

$$\begin{aligned}&B_{PX}\Big(P\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,u),(4\delta)^n e^{\int_{-(n+1)}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\Big)\\&\subseteq\bigcup_{i=1}^{k_0}B_{PX}\Big(x_u^i,\delta(4\delta)^n e^{\int_{-(n+1)}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\Big),\end{aligned}$$

where $x_u^i\in P(\mathscr{B}(\omega))$ since $\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,u)\subseteq\mathscr{B}(\omega)$ by (R_3), and k_0 is given by (4.46). Hence, for any $v\in B_X\Big(u,(4\delta)^n e^{\int_{-(n+1)}^{-1}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\Big)\cap\mathscr{B}(\vartheta_{-1}\omega)$ we obtain by (4.42)

$$\begin{aligned}P\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,v)&\in B_{PX}\Big(P\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,u),(4\delta)^n e^{\int_{-(n+1)}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\Big)\\&\subseteq\bigcup_{i=1}^{k_0}B_{PX}\Big(x_u^i,\delta(4\delta)^n e^{\int_{-(n+1)}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\Big)\end{aligned}$$

and then

$$P\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,v)\in B_{PX}\Big(x_u^{i_0},\delta(4\delta)^n e^{\int_{-(n+1)}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\Big)\tag{4.52}$$

for some $i_0\in\{1,\cdots,k_0\}$. Setting

$$y_u^i:=x_u^i+Q\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,u)\in X,\quad i=1,2,\cdots,k_0,$$

we have from (4.43) and (4.52) that

$$\begin{aligned}\|\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,v)-y_u^{i_0}\|_X&\leqslant\|P\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,v)-x_u^{i_0}\|_X\\&\quad+\|Q\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,v)-Q\phi(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,u)\|_X\\&\leqslant\delta(4\delta)^n e^{\int_{-(n+1)}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\\&\quad+\delta(4\delta)^n e^{\int_{-(n+1)}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\\&=2\delta(4\delta)^n e^{\int_{-(n+1)}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega),\end{aligned}$$

so

$$\phi\Big(1,\vartheta_{-1}\omega,\theta_{-1}\sigma,B_X\Big(u,(4\delta)^n e^{\int_{-(n+1)}^{-1}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\Big)\cap\mathscr{B}(\vartheta_{-1}\omega)\Big)$$
$$\subseteq\bigcup_{i=1}^{k_0}B_X\Big(y_u^i,2\delta(4\delta)^n e^{\int_{-(n+1)}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\Big)\cap\mathscr{B}(\omega)$$
$$\subseteq\bigcup_{i=1}^{k_0}B_X\Big(\tilde{y}_u^i,(4\delta)^{n+1} e^{\int_{-(n+1)}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\Big)\cap\mathscr{B}(\omega)$$

for some points $\tilde{y}_u^i\in\mathscr{B}(\omega)$. Hence, by (4.51) we finally conclude that

$$\begin{aligned}&\phi\big(n+1,\vartheta_{-(n+1)}\omega,\theta_{-(n+1)}\sigma,\mathscr{B}(\vartheta_{-(n+1)}\omega)\big)\subseteq\\&\subseteq\bigcup_{u\in U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)}\bigcup_{i=1}^{k_0}B_X\Big(\tilde{y}_u^i,(4\delta)^{n+1}e^{\int_{-(n+1)}^{0}\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-(n+1)}\omega)\Big)\cap\mathscr{B}(\omega).\end{aligned}\tag{4.53}$$

Define $U^{n+1}(\omega,\sigma):=\big\{\tilde{y}_u^i:u\in U^n(\vartheta_{-1}\omega,\theta_{-1}\sigma)\text{ and }1\leqslant i\leqslant k_0\big\}$. Then $U^{n+1}(\omega,\sigma)\subseteq\mathscr{B}(\omega)$ and $\sharp U^{n+1}(\omega,\sigma)\leqslant k_0^{n+1}$. The desired sets $\{U^n(\omega,\sigma)\}_{n\in\mathbb{N}}$ are constructed.

To find a finite cover of the random uniform attractor $\mathscr{A}$, using the idea of Theorem 4.2.3 we make the decomposition

$$\mathscr{A}(\omega)=\bigcup_{l=1}^{M_\eta}A_{\Sigma_l}(\omega),\quad \omega\in\Omega,\tag{4.54}$$

where for $\eta>0$ we are denoting $M_\eta=N_\Xi[\Sigma;\eta]$, $\Sigma=\cup_{l=1}^{M_\eta}\Sigma_l$, $\Sigma_l=B_\Xi(\sigma_l,\eta)\cap\Sigma$ and $A_{\Sigma_l}(\omega)=\cup_{\sigma\in\Sigma_l}A_\sigma(\omega)$. Moreover,

$$A_{\Sigma_l}(\omega)\subseteq B_X\Big(\phi\big(n,\vartheta_{-n}\omega,\theta_{-n}\sigma_l,A_{\theta_{-n}\Sigma_l}(\vartheta_{-n}\omega)\big),M(-n)e^{\int_{-n}^0 L(\vartheta_s\omega)\mathrm{d}s}\eta\Big)$$

for each $1\leqslant l\leqslant M_\eta$, $\eta>0$, $n\in\mathbb{N}$ and $\omega\in\Omega$, see (4.30). Given $\tau>0$, by Birkhoff's ergodic theorem we have for $n\in\mathbb{N}$ great that

$$e^{\int_{-n}^0 L(\vartheta_s\omega)\mathrm{d}s}\leqslant e^{(\mathbb{E}(L)+\tau)n},\tag{4.55}$$

getting

$$A_{\Sigma_l}(\omega)\subseteq B_X\Big(\phi\big(n,\vartheta_{-n}\omega,\theta_{-n}\sigma_l,A_{\theta_{-n}\Sigma_l}(\vartheta_{-n}\omega)\big),M(-n)e^{(\mathbb{E}(L)+\tau)n}\eta\Big).\tag{4.56}$$

Now fix $0<\rho<\ln(1/4\delta)-\mathbb{E}(\zeta)$ and let $\gamma>0$ be sufficiently small such that

$$\mathbb{E}(\zeta)+\rho+\gamma<\ln(1/4\delta).$$

Since $R(\omega)$ is a tempered random variable we have for n large enough that

$$\begin{aligned}(4\delta)^n e^{\int_{-n}^0\zeta(\vartheta_s\omega)\mathrm{d}s}R(\vartheta_{-n}\omega)&\leqslant(4\delta)^n e^{(\mathbb{E}(\zeta)+\gamma)n}R(\vartheta_{-n}\omega)\\&=e^{\big(\mathbb{E}(\zeta)+\gamma+\rho-\ln(1/4\delta)\big)n}e^{(-\rho/2)n}e^{(-\rho/2)n}R(\vartheta_{-n}\omega)\\&\leqslant e^{(-\rho/2)n},\end{aligned}$$

and so (4.47) gives

$$\phi\left(n,\vartheta_{-n}\omega,\theta_{-n}\sigma,\mathscr{B}(\vartheta_{-n}\omega)\right)\subseteq\bigcup_{u\in U^n(\omega,\sigma)}B_X\left(u,e^{(-\rho/2)n}\right)\cap\mathscr{B}(\omega). \tag{4.57}$$

Notice that $A_{\theta_{-n}\Sigma_l}(\vartheta_{-n}\omega)\subseteq\mathscr{B}(\vartheta_{-n}\omega)$, so from (4.57) it follows for $n\in\mathbb{N}$ sufficiently large that

$$N_X\left[\phi\left(n,\vartheta_{-n}\omega,\theta_{-n}\sigma_l,A_{\theta_{-n}\Sigma_l}(\vartheta_{-n}\omega)\right);\,e^{(-\rho/2)n}\right]\leqslant k_0^n,$$

and then

$$N_X\Big[B_X\Big(\phi\left(n,\vartheta_{-n}\omega,\theta_{-n}\sigma_l,A_{\theta_{-n}\Sigma_l}(\vartheta_{-n}\omega)\right),M(-n)e^{(\mathbb{E}(L)+\tau)n}\eta\Big);e^{-(\rho/2)n}+M(-n)e^{(\mathbb{E}(L)+\tau)n}\eta\Big]$$
$$\leqslant k_0^n.$$

Hence, from (4.56),

$$N_X\left[A_{\Sigma_l}(\omega);\;e^{(-\rho/2)n}+M(-n)e^{(\mathbb{E}(L)+\tau)n}\eta\right]\leqslant k_0^n,\qquad l=1,2,\cdots,M_\eta,$$

and then by (4.54) we conclude that

$$N_X\left[\mathscr{A}(\omega);e^{(-\rho/2)n}+M(-n)e^{(\mathbb{E}(L)+\tau)n}\eta\right]\leqslant k_0^nM_\eta \tag{4.58}$$

for all $\eta>0$ and $n\in\mathbb{N}$ large.

In the following we establish by (4.58) a finite ε-cover of $\mathscr{A}(\omega)$ for any small $\varepsilon>0$. Let

$$\eta_n:=\frac{e^{(-\rho/2)n}}{M(-n)e^{(\mathbb{E}(L)+\tau)n}},\qquad n\in\mathbb{N}\ \text{ large}. \tag{4.59}$$

Then $\eta_n\to0^+$ as $n\to\infty$, and from (4.58) we have

$$\begin{aligned}N_X\left[\mathscr{A}(\omega);\;2e^{(-\rho/2)n}\right]&=N_X\left[\mathscr{A}(\omega);\;e^{(-\rho/2)n}+M(-n)e^{(\mathbb{E}(L)+\tau)n}\eta_n\right]\\&\leqslant k_0^nM_{\eta_n}.\end{aligned}$$

Notice that for any $\varepsilon\in(0,1)$ there exists an $n_\varepsilon\in\mathbb{N}$ such that

$$2e^{(-\rho/2)n_\varepsilon}<\varepsilon\leqslant2e^{(-\rho/2)(n_\varepsilon-1)}, \tag{4.60}$$

and the numbers n_ε can be chosen such that $n_\varepsilon\to\infty$ as $\varepsilon\to0^+$. Hence,

$$\begin{aligned}N_X\left[\mathscr{A}(\omega);\varepsilon\right]&\leqslant N_X\left[\mathscr{A}(\omega);2e^{(-\rho/2)n_\varepsilon}\right]\\&\leqslant k_0^{n_\varepsilon}M_{\eta_{n_\varepsilon}},\end{aligned}$$

and then $\big($recall that $M_{\eta_{n_\varepsilon}}=N_\Xi[\Sigma;\eta_{n_\varepsilon}]\big)$

$$\begin{aligned}\frac{\ln N_X\left[\mathscr{A}(\omega);\varepsilon\right]}{-\ln\varepsilon}&\leqslant\frac{n_\varepsilon\ln k_0+\ln N_\Xi[\Sigma;\eta_{n_\varepsilon}]}{-\ln\left[2e^{(-\rho/2)(n_\varepsilon-1)}\right]}\\&=\frac{n_\varepsilon\ln k_0+\ln N_\Xi[\Sigma;\eta_{n_\varepsilon}]}{-\ln2+(\rho/2)(n_\varepsilon-1)},\qquad\forall\varepsilon\in(0,1).\end{aligned} \tag{4.61}$$

Since by (R_1) the symbol space Σ has finite fractal dimension in Ξ and that $\eta_{n_\varepsilon} \to 0^+$ as $\varepsilon \to 0^+$, for any $\chi > 0$ there exists $\varepsilon_0 = \varepsilon_0(\chi) \in (0,1)$ such that

$$N_\Xi\big[\Sigma; \eta_{n_\varepsilon}\big] \leqslant \left(\frac{1}{\eta_{n_\varepsilon}}\right)^{\dim_F(\Sigma;\Xi)+\chi}, \quad \forall \varepsilon \leqslant \varepsilon_0. \tag{4.62}$$

From (4.62) and the definition (4.59) of η_n we obtain

$$\begin{aligned}\frac{\ln N_\Xi\big[\Sigma; \eta_{n_\varepsilon}\big]}{-\ln 2 + (\rho/2)(n_\varepsilon - 1)} &\leqslant \frac{\big(\dim_F(\Sigma;\Xi)+\chi\big)\ln \dfrac{1}{\eta_{n_\varepsilon}}}{-\ln 2 + (\rho/2)(n_\varepsilon - 1)} \\ &= \big(\dim_F(\Sigma;\Xi)+\chi\big)\,\frac{\ln\left[\dfrac{M(-n_\varepsilon)e^{(\mathbb{E}(L)+\tau)n_\varepsilon}}{e^{-(\rho/2)n_\varepsilon}}\right]}{-\ln 2 + (\rho/2)(n_\varepsilon - 1)}, \quad \forall \varepsilon \leqslant \varepsilon_0,\end{aligned}$$

while from (R_1) we have

$$\begin{aligned}\frac{\ln\left[\dfrac{M(-n_\varepsilon)e^{(\mathbb{E}(L)+\tau)n_\varepsilon}}{e^{-(\rho/2)n_\varepsilon}}\right]}{-\ln 2 + (\rho/2)(n_\varepsilon - 1)} &\leqslant \frac{\ln\left[\dfrac{c_1 e^{(\mathbb{E}(L)+\tau+\mu)n_\varepsilon}}{e^{-(\rho/2)n_\varepsilon}}\right]}{-\ln 2 + (\rho/2)(n_\varepsilon - 1)} \qquad \big(\text{by } (R_1)\big) \\ &= \frac{\ln c_1 + \big(\mathbb{E}(L)+\tau+\mu\big)n_\varepsilon}{-\ln 2 + (\rho/2)(n_\varepsilon - 1)} + \frac{(\rho/2)n_\varepsilon}{-\ln 2 + (\rho/2)(n_\varepsilon - 1)}\end{aligned}$$

for all $\varepsilon \leqslant \varepsilon_0$. Hence, since $n_\varepsilon \to \infty$ as $\varepsilon \to 0^+$,

$$\limsup_{\varepsilon\to 0^+} \frac{\ln N_\Xi[\Sigma; \eta_{n_\varepsilon}]}{-\ln 2 + (\rho/2)(n_\varepsilon - 1)} \leqslant \big(\dim_F(\Sigma;\Xi)+\chi\big)\left(\frac{2\big(\mathbb{E}(L)+\tau+\mu\big)}{\rho} + 1\right).$$

Therefore, by taking the limit in (4.61) as $\varepsilon \to 0^+$ we conclude that

$$\begin{aligned}\dim_F\big(\mathscr{A}(\omega); X\big) &= \limsup_{\varepsilon\to 0^+} \frac{\ln N_X[\mathscr{A}(\omega);\varepsilon]}{-\ln \varepsilon} \\ &\leqslant \frac{2\ln k_0}{\rho} + \big(\dim_F(\Sigma;\Xi)+\chi\big)\left(\frac{2\big(\mathbb{E}(L)+\tau+\mu\big)}{\rho} + 1\right).\end{aligned} \tag{4.63}$$

Since the estimate (4.63) holds for all $\chi, \tau > 0$ we finally obtain

$$\dim_F\big(\mathscr{A}(\omega); X\big) \leqslant \frac{2\ln k_0}{\rho} + \dim_F(\Sigma;\Xi)\left(\frac{2\big(\mathbb{E}(L)+\mu\big)}{\rho} + 1\right),$$

which implies (4.44) since $k_0 \leqslant \big(\frac{\sqrt{m}}{\delta} + 1\big)^m$ by definition (4.46). $\square$

4.2.3 Fractal dimension in more regular spaces

By Theorem 4.2.3 and Theorem 4.2.6 we have established the finite-dimensionality of the random uniform attractor $\mathscr{A} = \{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ on the phase space X. Now we are interested in proving it for more regular spaces $Y \subset X$. But to be more general, in the following we study the problem in a Banach space Z for which $Z = Y$ is a particular case.

Let $(Z, \|\cdot\|_Z)$ be a separable Banach space and suppose the NRDS ϕ takes values in Z, i.e., for each $u \in X$, $\omega \in \Omega$ and $\sigma \in \Sigma$ we have $\phi(t, \omega, \sigma, u) \in Z$ for $t > 0$. Suppose also that $\mathscr{A}(\omega) \subseteq X \cap Z$, for all $\omega \in \Omega$. In the following we shall prove that under a $(\Sigma \times X, Z)$-smoothing property the fractal dimension in Z of the random uniform attractor can be bounded by the dimension of it in X plus the dimension of the symbol space Σ in Ξ.

The $(\Sigma \times X, Z)$-smoothing condition is stated as follows:

(R_7) There is a $\bar{t} > 0$ such that for some positive constants $\delta_1, \delta_2 > 0$ and a random variable $\bar{L}(\omega) > 0$ it holds

$$\|\phi(\bar{t}, \omega, \sigma_1, u) - \phi(\bar{t}, \omega, \sigma_2, v)\|_Z \leqslant \bar{L}(\omega)\Big[\big(d_\Xi(\sigma_1, \sigma_2)\big)^{\delta_1} + \|u_1 - u_2\|_X^{\delta_2}\Big],$$

for all $\sigma_1, \sigma_2 \in \Sigma$, $u, v \in \mathscr{A}(\omega)$, $\omega \in \Omega$.

Remark 4.2.7. *Notice that*

(i) We do not suppose any condition related to the expectation of $\tilde{L}(\omega)$ in (R_7);

(ii) Even for the case $Z = Y$, (R_7) is not implied by (R_4) and (R_6), and vice versa. Nevertheless, (R_7) is often more useful in applications since powers δ_1 and δ_2 are allowed while in (R_6) such powers can not be considered. In fact, it is an open problem whether or not Theorem 4.2.3 can be established using a weaker version of (R_6) with condition (4.23) weakened to: there exists some power $\delta \in (0, 1]$ such that

$$\sup_{\sigma \in \Sigma} \|\phi(\tilde{t}, \omega, \sigma, u) - \phi(\tilde{t}, \omega, \sigma, v)\|_Y \leqslant \kappa(\omega)\|u - v\|_X^{\delta}, \qquad \forall u, v \in \mathscr{B}(\omega),\ \omega \in \Omega. \tag{4.23'}$$

Theorem 4.2.8. *Let ϕ be an NRDS which is $(\Sigma \times X, X)$-continuous and has a random $\mathscr{D}$-uniform attractor $\mathscr{A} \subseteq X \cap Z$ which has finite fractal dimension in X, i.e., $\dim_F(\mathscr{A}(\omega); X) < c(\omega) < \infty$. Then if (R_1) and (R_7) are satisfied, $\mathscr{A}$ has finite fractal dimension in Z as well:*

$$\dim_F\big(\mathscr{A}(\omega); Z\big) \leqslant \frac{1}{\delta_1}\dim_F\big(\Sigma; \Xi\big) + \frac{1}{\delta_2}\dim_F\big(\mathscr{A}(\vartheta_{-\bar{t}}\omega); X\big), \qquad \omega \in \Omega.$$

Remark 4.2.9. *Notice that*

(i) *Z is not necessarily a subset of X and any embedding from Z into X was required, so the theorem applies to the case of, e.g., $X = L^2(\mathbb{R})$ and $Z = L^p(\mathbb{R})$ with $p > 2$;*

(ii) *If the fractal dimension of $\mathscr{A}$ in X is uniformly (w.r.t. $\omega \in \Omega$) bounded, i.e., $\dim_F\big(\mathscr{A}(\omega); X\big) \leqslant c$, for all $\omega \in \Omega$, where $c > 0$ is a deterministic constant, then the fractal dimension of $\mathscr{A}$ in Z is also uniformly bounded by a deterministic number:*

$$\dim_F\big(\mathscr{A}(\omega); Z\big) \leqslant \frac{1}{\delta_1}\dim_F\big(\Sigma; \Xi\big) + \frac{c}{\delta_2}.$$

Proof of Theorem 3.3.22. For any $\varepsilon \in (0,1)$ let $M_\varepsilon := N_\Xi[\Sigma;\varepsilon]$. Then there exists a sequence $\{\sigma_l\}_{l=1}^{M_{\varepsilon^{1/\delta_1}}}$ of centers in Σ such that $\Sigma = \bigcup_{l=1}^{M_{\varepsilon^{1/\delta_1}}} \Sigma_l$, where $\Sigma_l := B_\Xi\big(\sigma_l, \varepsilon^{1/\delta_1}\big) \cap \Sigma$.

Let $\omega \in \Omega$. Since $\mathscr{A}(\omega)$ is a compact subset of X we have

$$\mathscr{A}(\omega) = \bigcup_{i=1}^{N_X[\mathscr{A}(\omega);\varepsilon^{1/\delta_2}]} B_X\big(x_i^\omega, \varepsilon^{1/\delta_2}\big) \cap \mathscr{A}(\omega), \qquad x_i^\omega \in \mathscr{A}(\omega),$$

and by the negatively semi-invariance of $\mathscr{A}$ (see (4.13)) we obtain

$$\begin{aligned}
\mathscr{A}(\omega) &\subseteq \bigcup_{\sigma\in\Sigma} \phi\big(\bar{t}, \vartheta_{-\bar{t}}\omega, \theta_{-\bar{t}}\sigma, \mathscr{A}(\vartheta_{-\bar{t}}\omega)\big)\\
&= \bigcup_{l=1}^{M_{\varepsilon^{1/\delta_1}}} \bigcup_{\sigma\in\Sigma_l} \phi\big(\bar{t}, \vartheta_{-\bar{t}}\omega, \theta_{-\bar{t}}\sigma, \mathscr{A}(\vartheta_{-\bar{t}}\omega)\big)\\
&= \bigcup_{l=1}^{M_{\varepsilon^{1/\delta_1}}} \phi\big(\bar{t}, \vartheta_{-\bar{t}}\omega, \theta_{-\bar{t}}\Sigma_l, \mathscr{A}(\vartheta_{-\bar{t}}\omega)\big)\\
&= \bigcup_{l=1}^{M_{\varepsilon^{1/\delta_1}}} \bigcup_{i=1}^{N_X[\mathscr{A}(\vartheta_{-\bar{t}}\omega);\varepsilon^{1/\delta_2}]} \phi\Big(\bar{t}, \vartheta_{-\bar{t}}\omega, \theta_{-\bar{t}}\Sigma_l, B_X\big(x_i^{\vartheta_{-\bar{t}}\omega}, \varepsilon^{1/\delta_2}\big) \cap \mathscr{A}(\vartheta_{-\bar{t}}\omega)\Big),
\end{aligned} \tag{4.64}$$

where $\phi\big(\bar{t}, \vartheta_{-\bar{t}}\omega, \theta_{-\bar{t}}\Sigma_l, \mathscr{A}(\vartheta_{-\bar{t}}\omega)\big) := \bigcup_{\sigma\in\Sigma_l} \phi\big(\bar{t}, \vartheta_{-\bar{t}}\omega, \theta_{-\bar{t}}\sigma, \mathscr{A}(\vartheta_{-\bar{t}}\omega)\big)$.

Let $u_1, u_2 \in \phi\Big(\bar{t}, \vartheta_{-\bar{t}}\omega, \theta_{-\bar{t}}\Sigma_l, B_X\big(x_i^{\vartheta_{-\bar{t}}\omega}, \varepsilon^{1/\delta_2}\big) \cap \mathscr{A}(\vartheta_{-\bar{t}}\omega)\Big)$. Then $u_1 = \phi\big(\bar{t}, \vartheta_{-\bar{t}}\omega, \theta_{-\bar{t}}\sigma_1, v_1\big)$ and $u_2 = \phi\big(\bar{t}, \vartheta_{-\bar{t}}\omega, \theta_{-\bar{t}}\sigma_2, v_2\big)$ for some $\sigma_1, \sigma_2 \in \Sigma_l$ and $v_1, v_2 \in B_X\big(x_i^{\vartheta_{-\bar{t}}\omega}, \varepsilon^{1/\delta_2}\big) \cap \mathscr{A}(\vartheta_{-\bar{t}}\omega)$, and

$$\begin{aligned}
\|u_1 - u_2\|_Z &= \|\phi\big(\bar{t}, \vartheta_{-\bar{t}}\omega, \theta_{-\bar{t}}\sigma_1, v_1\big) - \phi\big(\bar{t}, \vartheta_{-\bar{t}}\omega, \theta_{-\bar{t}}\sigma_2, v_2\big)\|_Z\\
&\leqslant \bar{L}(\vartheta_{-\bar{t}}\omega)\Big[\big(d_\Xi(\theta_{-\bar{t}}\sigma_1, \theta_{-\bar{t}}\sigma_2)\big)^{\delta_1} + \|v_1 - v_2\|_X^{\delta_2}\Big] \quad (\text{by } (R_7))\\
&\leqslant \bar{L}(\vartheta_{-\bar{t}}\omega)\Big[M(-\bar{t})^{\delta_1}\big(d_\Xi(\sigma_1, \sigma_2)\big)^{\delta_1} + \|v_1 - v_2\|_X^{\delta_2}\Big] \quad (\text{by } (4.21))\\
&\leqslant \bar{L}(\vartheta_{-\bar{t}}\omega)\Big[M(-\bar{t})^{\delta_1} 2^{\delta_1}\varepsilon + 2^{\delta_2}\varepsilon\Big].
\end{aligned}$$

Let $r(\bar{t}, \omega, \delta_1, \delta_2) := \bar{L}(\vartheta_{-\bar{t}}\omega)\big[M(-\bar{t})^{\delta_1} 2^{\delta_1} + 2^{\delta_2}\big]$. Then

$$\operatorname{diam}_Z\Big(\phi\big(\bar{t}, \vartheta_{-\bar{t}}\omega, \theta_{-\bar{t}}\Sigma_l, B_X\big(x_i^{\vartheta_{-\bar{t}}\omega}, \varepsilon^{1/\delta_2}\big) \cap \mathscr{A}(\vartheta_{-\bar{t}}\omega)\big)\Big) \leqslant r(\bar{t}, \omega, \delta_1, \delta_2)\varepsilon, \tag{4.65}$$

for all $l = 1, \cdots, M_{\varepsilon^{1/\delta_1}}$ and $i = 1, \cdots, N_X\big[\mathscr{A}(\vartheta_{-\bar{t}}\omega); \varepsilon^{1/\delta_2}\big]$, so by (4.64) we obtain

$$\begin{aligned}
N_Z\big[\mathscr{A}(\omega); r(\bar{t}, \omega, \delta_1, \delta_2)\varepsilon\big] &\leqslant M_{\varepsilon^{1/\delta_1}} \cdot N_X\big[\mathscr{A}(\vartheta_{-\bar{t}}\omega); \varepsilon^{1/\delta_2}\big]\\
&= N_\Xi\big[\Sigma; \varepsilon^{1/\delta_1}\big] \cdot N_X\big[\mathscr{A}(\vartheta_{-\bar{t}}\omega); \varepsilon^{1/\delta_2}\big].
\end{aligned}$$

Since $r(\bar{t}, \omega, \delta_1, \delta_2)$ is independent of ε and $\mathscr{A}(\vartheta_{-\bar{t}}\omega)$ is finite dimensional in X then taking the limit as $\varepsilon \to 0^+$ we conclude

$$\dim_F\big(\mathscr{A}(\omega); Z\big) \leqslant \frac{1}{\delta_1}\dim_F\big(\Sigma; \Xi\big) + \frac{1}{\delta_2}\dim_F\big(\mathscr{A}(\vartheta_{-\bar{t}}\omega); X\big) < \infty, \qquad \forall \omega \in \Omega. \qquad \square$$

4.3 Application to a stochastic reaction-diffusion equation

In this section we study a stochastic reaction-diffusion equation as an application of our theoretical analysis in Section 4.2. In fact, it is a random pertubation of problem (3.75) in Section 3.4, where we consider now an additive white noise. Existence of solutions and some preliminary results related to this random perturbation have been established recently in (CUI; LANGA, 2017), and now with the non-autonomous term strengthened such that the symbol space is finite-dimensional, we prove the finite-dimensionality of the random uniform attractor.

In Section 4.3.1 we present the preliminary setting and state the problem. In Section 4.3.2 we show how to generate a non-autonomous random dynamical system from the equation and consider a conjugate problem. We see in Section 4.3.3 the existence of an admissible uniformly absorbing set $\mathscr{B}$ which absorbs itself after a deterministic period of time. As we could note in Section 4.2 it is an essential condition in order to prove the finite dimensionality of random uniform attractors. Then in Section 4.3.4 we estimate the fractal dimension in space L^2 via the squeezing method and in H_0^1 via the smoothing method. The calculations and analysis in this section is found in (CUI; CUNHA; LANGA,).

4.3.1 Preliminary settings and the symbol space

We consider the following reaction-diffusion equation with additive scalar white noise

$$\begin{aligned} &\mathrm{d}u + (\lambda u - \Delta u)\mathrm{d}t = f(u)\mathrm{d}t + g(x,t)\mathrm{d}t + h(x)\mathrm{d}\omega, \qquad x \in \mathscr{O},\ t > \tau, \\ &u(x,t)|_{t=\tau} = u_\tau(x), \quad u(x,t)|_{\partial\mathscr{O}} = 0, \end{aligned} \tag{4.66}$$

where $\mathscr{O} \subset \mathbb{R}^N$, $N \in \mathbb{N}$, is a bounded smooth domain and $\lambda > 0$ is a constant. The nonlinear term $f \in \mathscr{C}^1(\mathbb{R},\mathbb{R})$ is assumed to satisfy the same standard conditions

$$f(s)s \leqslant -\alpha_1|s|^p + \beta_1, \tag{4.67}$$

$$|f(s)| \leqslant \alpha_2|s|^{p-1} + \alpha_2, \tag{4.68}$$

$$|f'(s)| \leqslant \kappa_2|s|^{p-2} + l_2, \tag{4.69}$$

$$f'(s) \leqslant -\kappa_1|s|^{p-2} + l_1, \tag{4.70}$$

where all the coefficients are positive constants and the growth order $p \geqslant 2$. Let $h(x) \in W^{2,2p-2}(\mathscr{O})$. To establish the smoothing property (R_6) we will also need the growth order p to satisfy

$$\begin{cases} p \geqslant 2, & N = 1,2; \\ 2 \leqslant p \leqslant \frac{2N-2}{N-2}, & N \geqslant 3. \end{cases} \tag{4.71}$$

This ensures the continuous embedding $H_0^1(\mathscr{O}) \hookrightarrow L^{2p-2}(\mathscr{O})$, with

$$\|u\|_{2p-2} \leqslant c\|\nabla u\|, \qquad \forall u \in H_0^1(\mathscr{O}), \tag{4.72}$$

for some constant $c > 0$, where $\|\cdot\| := \|\cdot\|_2$, see (ROBINSON, 2001, Theorem 5.26).

The probability space $(\Omega, \mathscr{F}, \mathscr{P})$ is defined as usual. Let

$$\Omega := \{\omega \in \mathscr{C}(\mathbb{R};\mathbb{R}) : \omega(0) = 0\},$$

$\mathscr{F}$ be the Borel sigma-algebra induced by the compact-open topology of Ω and $\mathscr{P}$ be the two-sided Wiener measure on $(\Omega, \mathscr{F})$. Define the translation operators ϑ_t on Ω by

$$\vartheta_t \omega = \omega(\cdot + t) - \omega(t), \qquad \forall t \in \mathbb{R},\ \omega \in \Omega.$$

Then $\mathscr{P}$ is ergodic and invariant under ϑ (see (FLANDOLI; SCHMALFUSS, 1996)). Setting

$$z(\omega) := -\lambda \int_{-\infty}^{0} e^{\lambda\tau}\omega(\tau)\, d\tau, \qquad \forall \omega \in \Omega, \tag{4.73}$$

we have that $z(\omega)$ is a stationary solution of the one-dimensional Ornstein-Uhlenbeck equation

$$\mathrm{d}z(\vartheta_t\omega) + \lambda z(\vartheta_t\omega)\mathrm{d}t = \mathrm{d}\omega. \tag{4.74}$$

Moreover, there is a ϑ-invariant subset $\tilde{\Omega} \subseteq \Omega$ with full measure such that $z(\vartheta_t\omega)$ is continuous in t for every $\omega \in \tilde{\Omega}$ and the random variable $|z(\cdot)|$ is tempered (see (FAN, 2006, Lemma 1)). Hereafter, we will not distinguish between $\tilde{\Omega}$ and Ω.

In order to study the finite-dimensionality of the random uniform attractor we need the non-autonomous forcing g to have finite-dimensional hull in some metric space Ξ.

By the analysis in Section 3.3, and more specifically in Section 3.3.3, for the current stochastic reaction-diffusion equation we take $\Xi := \mathscr{C}\big(\mathbb{R}; L^2(\mathscr{O})\big)$ and assume that

(G) $g \in \Xi = \mathscr{C}\big(\mathbb{R}; L^2(\mathscr{O})\big)$ and the hull of g,

$$\mathscr{H}(g) = \overline{\{\theta_r g : r \in \mathbb{R}\}}^{d_\Xi},$$

has finite fractal dimension in Ξ, i.e., $\dim_F\big(\mathscr{H}(g);\Xi\big) < \infty$.

By Lemma 3.3.16, quasiperiodic functions are examples of such functions satisfying condition (G), and Theorem 3.3.26 indicates that Lipschitz continuous functions with tails eventually exponentially converging to quasiperiodic functions satisfy condition (G) as well. Some concrete examples were given in Section 3.3.3.

Now for the stochastic reaction-diffusion equation (4.66) we define the symbol space as the hull of the forcing g in Ξ, i.e.

$$\Sigma := \mathscr{H}(g).$$

Then condition (G) ensures that Σ is a finite-dimensional compact subset of Ξ. Moreover, the group $\{\theta_t\}_{t\in\mathbb{R}}$ of translation operators forms a base flow on Σ.

Remember the auxiliary result from Section 3.4, Lemma 3.4.1.

Lemma 4.3.1. *Let $\big(\mathscr{X},\|\cdot\|_{\mathscr{X}}\big)$ be a Banach space. If $g\in\Xi:=\mathscr{C}(\mathbb{R};\mathscr{X})$ and the hull $\mathscr{H}(g)$ is a compact subset of Ξ, then there is a constant $c=c(g)>0$ such that for any $\sigma_1,\sigma_2\in\mathscr{H}(g)$ we have*

$$\int_\tau^t \|\sigma_1(s)-\sigma_2(s)\|_{\mathscr{X}}^q \,\mathrm{d}s \leqslant c\,2^{q(t-\tau+|\tau|)}\Big(d_\Xi(\sigma_1,\sigma_2)\Big)^q, \qquad \forall t\geqslant\tau,\ q\geqslant 1.$$

In particular,

$$\int_0^t \|\sigma_1(s)-\sigma_2(s)\|_{\mathscr{X}}^2 \,\mathrm{d}s \leqslant c\,4^t\Big(d_\Xi(\sigma_1,\sigma_2)\Big)^2, \qquad \forall t\geqslant 0. \tag{4.75}$$

4.3.2 Generating an NRDS and the random uniform attractor

Now for $\sigma\in\Sigma$ we consider the following stochastic reaction-diffusion equation

$$\begin{cases} \mathrm{d}u+(\lambda u-\Delta u)\mathrm{d}t=f(u)\mathrm{d}t+\sigma(x,t)\mathrm{d}t+h(x)\mathrm{d}\omega, & x\in\mathscr{O},\ t>0,\\ u(x,t)|_{t=0}=u_0(x), \quad u(x,t)|_{\partial\mathscr{O}}=0. \end{cases} \tag{4.76}$$

Consider also the following conjugate deterministic problem with random coefficients:

$$\begin{cases} \dfrac{\partial v}{\partial t}+\lambda v-\Delta v=f\big(v+hz(\vartheta_t\omega)\big)+\sigma(x,t)+z(\vartheta_t\omega)\Delta h(x), & x\in\mathscr{O},\ t>0,\\ v(x,t)|_{t=0}=v_0(x), \qquad v(x,t)|_{\partial\mathscr{O}}=0. \end{cases} \tag{4.77}$$

Denote by

$$H:=\big(L^2(\mathscr{O}),\|\cdot\|\big), \qquad V:=H_0^1(\mathscr{O}), \qquad Z:=L^p(\mathscr{O}),$$

and let $\mathscr{D}$ be the collection of all tempered random subsets of H, i.e.,

$$\mathscr{D}=\Big\{D: D \text{ is a tempered random subset of } H\Big\}.$$

Then $I:V\hookrightarrow H$ is compact, and by Lemma 4.2.2 we have the Kolmogorov ε-entropy condition

$$\mathbf{H}_\varepsilon(V;H)\leqslant \alpha\varepsilon^{-N}, \qquad \forall\varepsilon>0,$$

for some constant $\alpha>0$.

Following a standard method by (TEMAM, 1997) and (CHEPYZHOV; VISHIK, 2002) we have that for each initial data $v_0\in H$, problem (4.77) has a unique solution $v(\cdot,\omega,\sigma,v_0)\in\mathscr{C}([0,\infty);H)\cap L^p_{loc}((0,\infty),Z)\cap L^2_{loc}((0,\infty);V)$, with $v(0,\omega,\sigma,v_0)=v_0$. Moreover, v is $(\mathscr{F},\mathscr{B}(H))$-measurable in ω, as one can see in (CUI; LANGA; LI, 2018). Hence, setting for each $t\geqslant 0$, $\omega\in\Omega$, $\sigma\in\Sigma$ and $v_0\in H$ that

$$\phi(t,\omega,\sigma,v_0):=v(t,\omega,\sigma,v_0), \tag{4.78}$$

then ϕ, generated by solutions of (4.77), is a $(\Sigma\times H,H)$-continuous NRDS on H (In fact, ϕ is Lipschitz in both initial data and symbols as indicated by Lemma 4.3.8).

Now, for each $t\geqslant 0$, $\omega\in\Omega$, $\sigma\in\Sigma$ and $u_0\in H$, set

$$u(t,\omega,\sigma,u_0):=v\big(t,\omega,\sigma,u_0-hz(\omega)\big)+hz(\vartheta_t\omega). \tag{4.79}$$

Then $u(t,\omega,\sigma,u_0)$ is the solution of (4.76) at time t with initial data u_0 (at time $t=0$) satisfying Definition 4.1.1. Hence, $\tilde{\phi}(t,\omega,\sigma,u_0) := u(t,\omega,\sigma,u_0)$, generated by the solutions of the stochastic reaction-diffusion equation (4.76), is also a $(\Sigma \times H, H)$-continuous NRDS on H. In fact, ϕ and $\tilde{\phi}$ are conjugate NRDS, satisfying (4.14) with cohomology

$$\mathsf{T}(\omega,u) = u + hz(\omega), \qquad \omega \in \Omega,\, u \in H. \tag{4.80}$$

Since the cohomology T is a bijection from $\mathscr{D}$ onto $\mathscr{D}$, Theorem 4.1.31 shows that the conjugate $\mathscr{D}$-uniform attractors $\mathscr{A}$ of ϕ and $\tilde{\mathscr{A}}$ of $\tilde{\phi}$ satisfy the relation

$$\tilde{\mathscr{A}}(\omega) = \mathsf{T}\big(\omega, \mathscr{A}(\omega)\big) = \mathscr{A}(\omega) + hz(\omega), \qquad \omega \in \Omega, \tag{4.81}$$

which implies that the two attractors have the same fractal dimension.

In the following we shall study the finite-dimensionality of $\mathscr{A}$ in H by checking conditions (R_1) - (R_4) and (S) in Theorem 4.2.6, and in V by checking (R_7) in Theorem 4.2.8. The existence of the random uniform attractor was proved previously in (CUI; LANGA, 2017), using Theorem 4.1.18.

4.3.3 An admissible uniformly $\mathscr{D}$-absorbing set $\mathscr{B}$

4.3.3.1 Estimates of solutions

The estimates of solutions in this section will be achieved in a standard way as in (CUI; LANGA, 2017), but we need details which are crucial for us to bound the fractal dimension of the random uniform attractor $\mathscr{A}$ associated to problem (4.77). As we just saw, with this we find estimates on the fractal dimension of $\tilde{\mathscr{A}}$, the random uniform attractor for the original problem (4.76). We note that condition (4.71) on p is not needed in this section.

Lemma 4.3.2 (Estimate in H)**.** *Let conditions (4.67) - (4.70) hold. Then any solution v of (4.77) with initial value $v_0 \in H$ satisfies*

$$\begin{aligned}&\|v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 + \int_0^t e^{\lambda(s-t)}\Big(\|\nabla v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 + \|v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|_p^p\Big)\,\mathrm{d}s\\ &\quad \leqslant e^{-\lambda t}\|v_0\|^2 + C\int_{-t}^0 e^{\lambda s}\Big(|z(\vartheta_s\omega)|^p + 1 + \|\sigma(s)\|^2\Big)\,\mathrm{d}s, \qquad t \geqslant 0,\end{aligned}$$

where $C \geqslant 1$ is a positive constant independent of $v_0 \in H$, $\sigma \in \Sigma$ and $\omega \in \Omega$.

Proof. Take the inner product of (4.77) with v in H to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \lambda\|v\|^2 + \|\nabla v\|^2 = \int_{\mathscr{O}} v\Big(f(v+hz(\vartheta_t\omega)) + \sigma(t) + z(\vartheta_t\omega)\Delta h\Big)\,\mathrm{d}x. \tag{4.82}$$

By (4.67), (4.68) and Young's inequality we have, since $u = v + hz(\vartheta_t\omega)$,

$$
\begin{aligned}
\int_{\mathscr{O}} vf\big(v + hz(\vartheta_t\omega)\big)\,\mathrm{d}x &= \int_{\mathscr{O}} uf(u)\,\mathrm{d}x - \int_{\mathscr{O}} hz(\vartheta_t\omega)f(u)\,\mathrm{d}x \\
&\leqslant -\alpha_1\|u\|_p^p + c + \int_{\mathscr{O}} \alpha_2|hz(\vartheta_t\omega)|\Big(|u|^{p-1} + 1\Big)\,\mathrm{d}x \\
&\leqslant -\alpha_1\|u\|_p^p + c + \frac{\alpha_1}{2}\|u\|_p^p + c|z(\vartheta_t\omega)|^p + c|z(\vartheta_t\omega)| \\
&\leqslant -\frac{\alpha_1}{2}\|v\|_p^p + c\big(|z(\vartheta_t\omega)|^p + 1\big).
\end{aligned}
\tag{4.83}
$$

As

$$
\int_{\mathscr{O}} v\big(\sigma(t) + z(\vartheta_t\omega)\Delta h\big)\,\mathrm{d}x \leqslant \frac{\lambda}{2}\|v\|^2 + \frac{1}{\lambda}\|\sigma\|^2 + c(|z(\vartheta_t\omega)|^p + 1), \tag{4.84}
$$

by (4.82) - (4.84) we conclude that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \lambda\|v\|^2 + \|\nabla v\|^2 + \|v\|_p^p \leqslant c\Big(|z(\vartheta_t\omega)|^p + 1 + \|\sigma\|^2\Big). \tag{4.85}
$$

Multiply (4.85) by $e^{\lambda t}$ and then integrate over $(0,t)$ to obtain

$$
\begin{aligned}
&\|v(t,\omega,\sigma,v_0)\|^2 + \int_0^t e^{\lambda(s-t)}\Big(\|\nabla v(s,\omega,\sigma,v_0)\|^2 + \|v(s,\omega,\sigma,v_0)\|_p^p\Big)\,\mathrm{d}s \\
&\leqslant e^{-\lambda t}\|v_0\|^2 + c\int_0^t e^{\lambda(s-t)}\Big(|z(\vartheta_s\omega)|^p + 1 + \|\sigma(s)\|^2\Big)\,\mathrm{d}s.
\end{aligned}
\tag{4.86}
$$

Replacing ω and σ by $\vartheta_{-t}\omega$ and $\theta_{-t}\sigma$, we have

$$
\begin{aligned}
&\|v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 + \int_0^t e^{\lambda(s-t)}\Big(\|\nabla v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 + \|v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|_p^p\Big)\,\mathrm{d}s \\
&\leqslant e^{-\lambda t}\|v_0\|^2 + c\int_0^t e^{\lambda(s-t)}\Big(|z(\vartheta_{s-t}\omega)|^p + 1 + \|\sigma(s-t)\|^2\Big)\,\mathrm{d}s \\
&= e^{-\lambda t}\|v_0\|^2 + c\int_{-t}^0 e^{\lambda s}\Big(|z(\vartheta_s\omega)|^p + 1 + \|\sigma(s)\|^2\Big)\,\mathrm{d}s,
\end{aligned}
$$

and the proof is complete. □

Lemma 4.3.3 (Estimate in V)**.** *Let conditions (4.67) - (4.70) hold. Then for any $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that any solution v of (4.77) with initial value $v_0 \in H$ satisfies*

$$
\|\nabla v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 \leqslant c_\varepsilon e^{-\lambda t}\|v_0\|^2 + c_\varepsilon\int_{-t}^0 e^{\lambda s}\Big(|z(\vartheta_s\omega)|^p + 1 + \|\sigma(s)\|^2\Big)\,\mathrm{d}s, \quad \forall t \geqslant \varepsilon,
$$

where $c_\varepsilon > 0$ depends only on ε.

Proof. Multiply (4.77) by $-\Delta v$ and then integrate over $\mathscr{O}$ to obtain

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2 + \lambda\|\nabla v\|^2 + \|\Delta v\|^2 = -\int_{\mathscr{O}} \Delta v\Big(f\big(v + hz(\vartheta_t\omega)\big) + \sigma(t) + z(\vartheta_t\omega)\Delta h\Big)\mathrm{d}x. \tag{4.87}
$$

By (4.68) - (4.70) we have

$$
\begin{aligned}
-\int_{\mathscr{O}} \Delta v f\big(v + hz(\vartheta_t\omega)\big)\,\mathrm{d}x &= -\int_{\mathscr{O}} \Delta u f(u)\,\mathrm{d}x + \int_{\mathscr{O}} \Delta h z(\vartheta_t\omega)f(u)\,\mathrm{d}x \\
&= \int_{\mathscr{O}} |\nabla u|^2\frac{\mathrm{d}f(u)}{\mathrm{d}u}\,\mathrm{d}x + \int_{\mathscr{O}} \Delta h z(\vartheta_t\omega)f(u)\mathrm{d}x \\
&\leqslant l_1\|\nabla u\|^2 + \int_{\mathscr{O}} |\Delta h z(\vartheta_t\omega)|\Big(\alpha_2|u|^{p-1} + \alpha_2\Big)\,\mathrm{d}x \\
&\leqslant l_1\|\nabla v\|^2 + c\|v\|_p^p + c|z(\vartheta_t\omega)|^p + c.
\end{aligned}
\tag{4.88}
$$

Since

$$-\int_{\mathscr{O}} \Delta v\big(\sigma(t)+z(\vartheta_t\omega)\Delta h\big)\mathrm{d}x \leqslant \|\Delta v\|^2+\frac{1}{2}\|\sigma\|^2+c|z(\vartheta_t\omega)|^2, \tag{4.89}$$

from (4.87) - (4.89) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v(t,\omega,\sigma,v_0)\|^2 \leqslant c\|\nabla v\|^2+c\|v\|_p^p+c|z(\vartheta_t\omega)|^p+\|\sigma\|^2+c, \qquad t>0. \tag{4.90}$$

For $\varepsilon>0$ and $s\in(t-\varepsilon,t)$, integrate (4.90) over (s,t) to obtain

$$\begin{aligned}\|\nabla v(t,\omega,\sigma,v_0)\|^2-\|\nabla v(s,\omega,\sigma,v_0)\|^2 &\leqslant c\int_{t-\varepsilon}^{t}\Big(\|\nabla v(\tau)\|^2+\|v(\tau)\|_p^p\Big)\mathrm{d}\tau\\ &\quad +c\int_{t-\varepsilon}^{t}\Big(|z(\vartheta_\tau\omega)|^p+1+\|\sigma(\tau)\|^2\Big)\mathrm{d}\tau.\end{aligned} \tag{4.91}$$

Then integrating (4.91) with respect to s over $(t-\varepsilon,t)$ we have

$$\begin{aligned}\|\nabla v(t,\omega,\sigma,v_0)\|^2 &\leqslant \Big(c+\frac{1}{\varepsilon}\Big)\int_{t-\varepsilon}^{t}\Big(\|\nabla v(\tau)\|^2+\|v(\tau)\|_p^p\Big)\mathrm{d}\tau\\ &\quad +c\int_{t-\varepsilon}^{t}\Big(|z(\vartheta_\tau\omega)|^p+1+\|\sigma(\tau)\|^2\Big)\mathrm{d}\tau,\end{aligned}$$

and replacing ω and σ by $\vartheta_{-t}\omega$ and $\theta_{-t}\sigma$, respectively, we have

$$\begin{aligned}\|\nabla v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 &\leqslant \Big(c+\frac{1}{\varepsilon}\Big)\int_{t-\varepsilon}^{t}\Big(\|\nabla v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2+\|v(s)\|_p^p\Big)\,\mathrm{d}s\\ &\quad +c\int_{-\varepsilon}^{0}\Big(|z(\vartheta_\tau\omega)|^p+1+\|\sigma(\tau)\|^2\Big)\,\mathrm{d}\tau.\end{aligned}$$

Since, for $t\geqslant\varepsilon$,

$$\begin{aligned}&\int_{t-\varepsilon}^{t}\Big(\|\nabla v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2+\|v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|_p^p\Big)\,\mathrm{d}s\leqslant\\ &\quad\leqslant e^{\lambda\varepsilon}\int_{t-\varepsilon}^{t}e^{\lambda(s-t)}\Big(\|\nabla v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2+\|v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|_p^p\Big)\,\mathrm{d}s\\ &\quad\leqslant e^{\lambda\varepsilon}\int_{0}^{t}e^{\lambda(s-t)}\Big(\|\nabla v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2+\|v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|_p^p\Big)\,\mathrm{d}s\\ &\quad\leqslant ce^{\lambda\varepsilon-\lambda t}\|v_0\|^2+ce^{\lambda\varepsilon}\int_{-t}^{0}e^{\lambda s}\Big(|z(\vartheta_s\omega)|^p+1+\|\sigma(s)\|^2\Big)\,\mathrm{d}s \quad \text{(by Lemma 4.3.2)},\end{aligned}$$

we conclude that

$$\begin{aligned}\|\nabla v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 &\leqslant c\Big(1+\frac{1}{\varepsilon}\Big)e^{\lambda\varepsilon-\lambda t}\|v_0\|^2\\ &\quad +ce^{\lambda\varepsilon}\Big(1+\frac{1}{\varepsilon}\Big)\int_{-t}^{0}e^{\lambda s}\Big(|z(\vartheta_s\omega)|^p+1+\|\sigma(s)\|^2\Big)\,\mathrm{d}s,\end{aligned}$$

which completes the proof. □

To establish the (H,V)-smoothing property of the system we need the following estimates.

Lemma 4.3.4 (Estimate in Z)**.** *Let conditions* (4.67) - (4.70) *hold. Then any solution v of* (4.77) *with initial value $v_0 \in H$ satisfies*

$$\begin{aligned}\varepsilon\|v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|_p^p+\int_0^\varepsilon\int_r^t e^{\lambda(s-t)}\|v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|_{2p-2}^{2p-2}\,\mathrm{d}s\mathrm{d}r \leqslant \\ \leqslant ce^{-\lambda t}\|v_0\|^2+c(\varepsilon+1)\int_{-\infty}^0 e^{\lambda s}\Big(|z(\vartheta_s\omega)|^{2p-2}+1+\|\sigma(s)\|^2\Big)\mathrm{d}s, \qquad \forall t\geqslant\varepsilon>0,\end{aligned} \tag{4.92}$$

where $c>0$ is a positive constant independent of $v_0\in H$, $\sigma\in\Sigma$ and $\omega\in\Omega$.

Proof. Taking the inner product of (4.77) with $|v|^{p-2}v$ in H, we obtain

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_p^p+\lambda\|v\|_p^p &\leqslant \big(f(v+hz(\vartheta_t\omega)),|v|^{p-2}v\big)+\big(\sigma,|v|^{p-2}v\big)+\big(z(\vartheta_t\omega)\Delta h,|v|^{p-2}v\big)\\ &\leqslant \big(f(v+hz(\vartheta_t\omega)),|v|^{p-2}v\big)+\frac{\alpha_1}{8}\|v\|_{2p-2}^{2p-2}+c\|\sigma\|^2+c|z(\vartheta_t\omega)|^2.\end{aligned} \tag{4.93}$$

By (4.67) we see that, since $u=v+hz(\vartheta_t\omega)$,

$$\begin{aligned}f(v+hz(\vartheta_t\omega))v &= f(u)(u-hz(\vartheta_t\omega))\\ &\leqslant -\alpha_1|u|^p+\beta_1-f(u)hz(\vartheta_t\omega) \qquad \text{(by (4.67))}\\ &\leqslant -\alpha_1|u|^p+\beta_1+\big(\alpha_2|u|^{p-1}+\alpha_2\big)|hz(\vartheta_t\omega)| \qquad \text{(by (4.68))}\\ &\leqslant -\alpha_1|u|^p+\beta_1+\frac{3\alpha_1}{4}|u|^p+c|hz(\vartheta_t\omega)|^p+\alpha_2|hz(\vartheta_t\omega)|\\ &\leqslant -\frac{\alpha_1}{4}|v|^p+c|hz(\vartheta_t\omega)|^p+\beta_1+\alpha_2|hz(\vartheta_t\omega)|.\end{aligned}$$

Hence,

$$\begin{aligned}\big(f(v+hz(\vartheta_t\omega)),|v|^{p-2}v\big) &= \big(f(v+hz(\vartheta_t\omega))v,|v|^{p-2}\big)\\ &\leqslant -\frac{\alpha_1}{4}\|v\|_{2p-2}^{2p-2}+c|z(\vartheta_t\omega)|^{2p-2}+c.\end{aligned}$$

Then from (4.93) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_p^p+\lambda\|v\|_p^p+\|v\|_{2p-2}^{2p-2}\leqslant c|z(\vartheta_t\omega)|^{2p-2}+c\|\sigma\|^2+c. \tag{4.94}$$

Multiply (4.94) by $e^{\lambda t}$ and then integrate over (r,t) for $r\in(0,\varepsilon)$ to obtain

$$\begin{aligned}\|v(t)\|_p^p+\int_r^t e^{\lambda(s-t)}\|v(s)\|_{2p-2}^{2p-2}\,\mathrm{d}s \leqslant \\ \leqslant ce^{-\lambda(t-r)}\|v(r)\|_p^p+c\int_r^t e^{\lambda(s-t)}\Big(|z(\vartheta_s\omega)|^{2p-2}+1+\|\sigma(s)\|^2\Big)\,\mathrm{d}s.\end{aligned} \tag{4.95}$$

Integrating (4.95) over $r\in(0,\varepsilon)$ yields

$$\begin{aligned}&\varepsilon\|v(t,\omega,\sigma,v_0)\|_p^p+\int_0^\varepsilon\int_r^t e^{\lambda(s-t)}\|v(s,\omega,\sigma,v_0)\|_{2p-2}^{2p-2}\,\mathrm{d}s\mathrm{d}r\leqslant\\ &\leqslant c\int_0^\varepsilon e^{-\lambda(t-r)}\|v(r)\|_p^p\,\mathrm{d}r+c\int_0^\varepsilon\int_r^t e^{\lambda(s-t)}\Big(|z(\vartheta_s\omega)|^{2p-2}+1+\|\sigma(s)\|^2\Big)\,\mathrm{d}s\mathrm{d}r\\ &\leqslant c\int_0^t e^{-\lambda(t-r)}\|v(r)\|_p^p\,\mathrm{d}r+\varepsilon c\int_0^t e^{\lambda(s-t)}\Big(|z(\vartheta_s\omega)|^{2p-2}+1+\|\sigma(s)\|^2\Big)\,\mathrm{d}s\\ &\leqslant ce^{-\lambda t}\|v_0\|^2+(1+\varepsilon)c\int_0^t e^{\lambda(s-t)}\Big(|z(\vartheta_s\omega)|^{2p-2}+1+\|\sigma(s)\|^2\Big)\,\mathrm{d}s \qquad \text{(by (4.86))}.\end{aligned} \tag{4.96}$$

Replacing ω and σ by $\vartheta_{-t}\omega$ and $\theta_{-t}\sigma$, respectively, we have the lemma. □

For later purpose we state the following corollary.

Corollary 4.3.5. *Let conditions (4.67) - (4.70) hold. Then any solution v of (4.77) with initial value $v_0 \in H$ satisfies*

$$\int_1^t \|v(s,\omega,\sigma,v_0)\|_{2p-2}^{2p-2}\,\mathrm{d}s \leqslant c\|v_0\|^2 + c\int_0^t e^{\lambda s}\left(|z(\vartheta_s\omega)|^{2p-2} + 1 + \|\sigma(s)\|^2\right)\mathrm{d}s, \tag{4.97}$$

for all $t \geqslant 1$, where $c > 0$ is a positive constant independent of $v_0 \in H$, $\sigma \in \Sigma$ and $\omega \in \Omega$.

Proof. By (4.96) with $\varepsilon = 1$, for $t \geqslant 1$ we have

$$\begin{aligned}\int_1^t \|v(s,\omega,\sigma,v_0)\|_{2p-2}^{2p-2}\,\mathrm{d}s &\leqslant \int_0^1\int_r^t \|v(s,\omega,\sigma,v_0)\|_{2p-2}^{2p-2}\,\mathrm{d}s\mathrm{d}r\\ &\leqslant e^{\lambda t}\int_0^1\int_r^t e^{\lambda(s-t)}\|v(s,\omega,\sigma,v_0)\|_{2p-2}^{2p-2}\,\mathrm{d}s\mathrm{d}r\\ &\leqslant c\|v_0\|^2 + c\int_0^t e^{\lambda s}\left(|z(\vartheta_s\omega)|^{2p-2} + 1 + \|\sigma(s)\|^2\right)\mathrm{d}s. \qquad \square\end{aligned}$$

4.3.3.2 The absorbing set $\mathscr{B}$ with deterministic absorption time

Now we construct an admissible uniformly $\mathscr{D}$-absorbing set $\mathscr{B}$ satisfying (R_3). Since $\Sigma = \mathscr{H}(g)$ is compact in $\Xi = \mathscr{C}(\mathbb{R};H)$ we know from (CHEPYZHOV; VISHIK, 2002, Proposition V.2.3) that it is bounded in $\mathscr{C}_b(\mathbb{R};H)$, i.e., for some constant $C_\Sigma > 0$

$$\sup_{\sigma\in\Sigma}\|\sigma\|_{\mathscr{C}_b(\mathbb{R};H)} = \sup_{\sigma\in\Sigma}\left(\sup_{t\in\mathbb{R}}\|\sigma(t)\|\right) \leqslant C_\Sigma. \tag{4.98}$$

Hence, there exists a uniform bound $C_b > 0$ such that

$$\sup_{\sigma\in\Sigma}\int_{-\infty}^0 e^{\lambda s}\|\sigma(s)\|^2\,\mathrm{d}s \leqslant \sup_{\sigma\in\Sigma}\left(\|\sigma\|^2_{\mathscr{C}_b(\mathbb{R};H)}\int_{-\infty}^0 e^{\lambda s}\,\mathrm{d}s\right) \leqslant C_b. \tag{4.99}$$

Let us define a random set $\mathscr{B} = \{\mathscr{B}(\omega)\}_{\omega\in\Omega}$ in H by

$$\mathscr{B}(\omega) := \left\{u \in H : \|u\|^2 \leqslant \rho(\omega) := 1 + C\int_{-\infty}^0 e^{\lambda s}|z(\vartheta_s\omega)|^p\,\mathrm{d}s + \frac{C}{\lambda} + CC_b\right\},\quad \omega\in\Omega, \tag{4.100}$$

where $C \geqslant 1$ is the constant given in Lemma 4.3.2.

Proposition 4.3.6. *Let conditions (4.67) - (4.70) hold. Then the random set $\mathscr{B}$ defined by (4.100) is a tempered uniformly $\mathscr{D}$-pullback absorbing set for the NRDS ϕ generated by the reaction-diffusion equation (4.77). In addition, $\mathscr{B}$ uniformly absorbs itself after a deterministic period of time, i.e., there exists a deterministic time $T_{\mathscr{B}} > 0$ such that for any $t \geqslant T_{\mathscr{B}}$ we have*

$$\bigcup_{\sigma\in\Sigma} v\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\mathscr{B}(\vartheta_{-t}\omega)\big) \subseteq \mathscr{B}(\omega),\quad \forall\omega\in\Omega.$$

Proof. Since $\rho(\cdot)$ is tempered it follows that $\mathscr{B}$ is tempered as well. Now we show the uniformly $\mathscr{D}$-pullback absorbing property. By Lemma 4.3.2 we know that for any tempered random set $D \in \mathscr{D}$ (i.e., there is some tempered random variable $R(\cdot)$ such that $\|D(\omega)\|^2 \leqslant R(\omega)$), the solutions with initial data in D satisfy

$$\begin{aligned}
&\sup_{\sigma\in\Sigma}\left\|v\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\big)\right\|^2 \leqslant \\
&\quad\leqslant \sup_{\sigma\in\Sigma}\left[e^{-\lambda t}R(\vartheta_{-t}\omega)+C\int_{-t}^{0}e^{\lambda s}\Big(|z(\vartheta_s\omega)|^p+1+\|\sigma(s)\|^2\Big)\,\mathrm{d}s\right] \\
&\quad\leqslant e^{-\lambda t}R(\vartheta_{-t}\omega)+C\int_{-t}^{0}e^{\lambda s}|z(\vartheta_s\omega)|^p\,\mathrm{d}s+\frac{C}{\lambda}+CC_b \quad (\text{by } (4.99)), \quad \forall t>0.
\end{aligned} \tag{4.101}$$

In addition, since the random variable $R(\cdot)$ is tempered, there exists a random variable $T_D(\cdot)>0$ such that $e^{-\lambda t}R(\vartheta_{-t}\omega)\leqslant 1$, for all $t\geqslant T_D(\omega)$. Hence,

$$\sup_{\sigma\in\Sigma}\left\|v\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,D(\vartheta_{-t}\omega)\big)\right\|^2 \leqslant 1+C\int_{-\infty}^{0}e^{\lambda s}|z(\vartheta_s\omega)|^p\,\mathrm{d}s+\frac{C}{\lambda}+CC_b=\rho(\omega), \tag{4.102}$$

for all $t\geqslant T_D(\omega)$, so $\mathscr{B}$ is a uniformly $\mathscr{D}$-pullback absorbing set.

Now we show that $\mathscr{B}$ uniformly attracts itself after a deterministic periodic of time $T_{\mathscr{B}}$. By (4.101),

$$\begin{aligned}
\sup_{\sigma\in\Sigma}\left\|v\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\mathscr{B}(\vartheta_{-t}\omega)\big)\right\|^2 &\leqslant e^{-\lambda t}\rho(\vartheta_{-t}\omega)+C\int_{-t}^{0}e^{\lambda s}|z(\vartheta_s\omega)|^p\,\mathrm{d}s+\frac{C}{\lambda}+CC_b \\
&= e^{-\lambda t}\left(1+C\int_{-\infty}^{0}e^{\lambda s}|z(\vartheta_{s-t}\omega)|^p\,\mathrm{d}s+\frac{C}{\lambda}+CC_b\right) \\
&\quad +C\int_{-t}^{0}e^{\lambda s}|z(\vartheta_s\omega)|^p\,\mathrm{d}s+\frac{C}{\lambda}+CC_b \\
&= e^{-\lambda t}\left(1+\frac{C}{\lambda}+CC_b\right)+C\int_{-\infty}^{-t}e^{\lambda s}|z(\vartheta_s\omega)|^p\,\mathrm{d}s \\
&\quad +C\int_{-t}^{0}e^{\lambda s}|z(\vartheta_s\omega)|^p\,\mathrm{d}s+\frac{C}{\lambda}+CC_b.
\end{aligned}$$

Hence, take $T_{\mathscr{B}}>0$ such that

$$e^{-\lambda T_{\mathscr{B}}}\left(1+\frac{C}{\lambda}+CC_b\right)=1, \qquad \text{i.e.,} \qquad T_{\mathscr{B}}:=\frac{1}{\lambda}\ln\left(1+\frac{C}{\lambda}+CC_b\right).$$

Then, for all $t\geqslant T_{\mathscr{B}}$,

$$\sup_{\sigma\in\Sigma}\left\|v\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\mathscr{B}(\vartheta_{-t}\omega)\big)\right\|^2 \leqslant 1+C\int_{-\infty}^{0}e^{\lambda s}|z(\vartheta_s\omega)|^p\,\mathrm{d}s+\frac{C}{\lambda}+CC_b=\rho(\omega),$$

which by the definition (4.100) indicates that

$$\bigcup_{\sigma\in\Sigma} v\big(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\mathscr{B}(\vartheta_{-t}\omega)\big)\subseteq\mathscr{B}(\omega), \qquad \forall t\geqslant T_{\mathscr{B}},$$

as desired. □

4.3.4 Finite fractal dimension of the random uniform attractor in H and in V

4.3.4.1 $(\Sigma\times H,H)$-Lipschitz continuity and $(\Sigma\times H,V)$-smoothing

Notice that condition (4.69) is in fact equivalent to the following form commonly used in the literature:

$$|f(s_1)-f(s_2)|\leqslant c|s_1-s_2|\big(1+|s_1|^{p-2}+|s_2|^{p-2}\big),\qquad s_1,s_2\in\mathbb{R}. \tag{4.103}$$

In addition, the following result is obtained via a decomposition of f in (CUI; KLOEDEN; ZHAO,) and will facilitate our computations later.

Lemma 4.3.7. *(CUI; KLOEDEN; ZHAO, , Corollary 3.3) For any $\mathscr{C}^1$-function f satisfying (4.67), (4.68) and (4.70) there are positive constants $c_1,c_2>0$ such that*

$$-\big(f(s_1)-f(s_2)\big)(s_1-s_2)|s_1-s_2|^r\geqslant c_1|s_1-s_2|^{p+r}-c_2|s_1-s_2|^{r+2},$$

for any $r\geqslant 0$ and $s_1,s_2\in\mathbb{R}$.

Now we derive a joint Lipschitz continuity of solutions in symbols and initial data. For any two solutions v_j of (4.77) corresponding to initial data $v_{j,0}\in H$ and symbols $\sigma_j\in\Sigma$, $j=1,2$, respectively, with $\bar\sigma:=\sigma_1-\sigma_2$, the difference $\bar v(t,\omega,\bar\sigma,\bar v_0):=v_1(t,\omega,\sigma_1,v_{1,0})-v_2(t,\omega,\sigma_2,v_{2,0})$ of them satisfies

$$\frac{\mathrm{d}\bar v}{\mathrm{d}t}+\lambda\bar v-\Delta\bar v=f\big(v_1+hz(\vartheta_t\omega)\big)-f\big(v_2+hz(\vartheta_t\omega)\big)+\bar\sigma. \tag{4.104}$$

Lemma 4.3.8 ($(\Sigma\times H,H)$-Lipschitz continuity)**.** *The NRDS ϕ is Lipschitz continuous from $\Sigma\times H$ to H with time-dependent Lipschitz constant. More precisely, there exist deterministic constants $C_1=C_1(\|\Sigma\|_{\mathscr{C}_b(\mathbb{R};H)})>0$ and $\beta=\beta(c_1,c_2,\lambda)>0$ such that for any two solutions $v_j(t,\omega,\sigma_j,v_{j,0})$ of (4.77) with $\sigma_j\in\Sigma$ and $v_{j,0}\in H$, $j=1,2$, we have*

$$\|v_1(t,\omega,\sigma_1,v_{1,0})-v_2(t,\omega,\sigma_2,v_{2,0})\|^2\leqslant C_1e^{\beta t}\Big[\|v_{1,0}-v_{2,0}\|^2+\big(d_\Xi(\sigma_1,\sigma_2)\big)^2\Big],$$

for all $t>0$ and $\omega\in\Omega$.

Proof. Take the inner product of (4.104) with $\bar v\,(=\bar u)$ in H and we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar v\|^2+\lambda\|\bar v\|^2+\|\nabla\bar v\|^2&=\Big(f\big(v_1+hz(\vartheta_t\omega)\big)-f\big(v_2+hz(\vartheta_t\omega)\big),\bar u\Big)+(\bar\sigma,\bar v)\\&\leqslant -c_1\|\bar u\|_p^p+c_2\|\bar u\|^2+\|\bar\sigma\|\|\bar v\|\qquad\text{(by Lemma 3.4.8)}\\&\leqslant -c_1\|\bar v\|_p^p+c\|\bar v\|^2+\|\bar\sigma\|^2,\end{aligned}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar v\|^2+\|\bar v\|_p^p+\|\nabla\bar v\|^2\leqslant c\|\bar v\|^2+\|\bar\sigma\|^2.$$

By Gronwall's lemma we have

$$\|\bar{v}(t)\|^2+\int_0^t e^{c(t-s)}\Big(\|\bar{v}(s)\|_p^p+\|\nabla\bar{v}(s)\|^2\Big)\mathrm{d}s \leqslant e^{ct}\|\bar{v}_0\|^2+\int_0^t e^{c(t-s)}\|\bar{\sigma}(s)\|^2\,\mathrm{d}s. \tag{4.105}$$

Since from (4.75) it follows

$$\begin{aligned}\int_0^t e^{c(t-s)}\|\bar{\sigma}(s)\|^2\,\mathrm{d}s &\leqslant e^{ct}\int_0^t \|\bar{\sigma}(s)\|^2\,\mathrm{d}s\\ &\leqslant ce^{(c+\ln 4)t}\big(d_\Xi(\sigma_1,\sigma_2)\big)^2,\end{aligned}$$

the proof is complete. □

Lemma 4.3.9 (($\Sigma\times H,V$)-smoothing)**.** *Let $\mathscr{B}$ be the tempered uniformly $\mathscr{D}$-pullback absorbing set defined by* (4.100)*. Then for all $\omega\in\Omega$ the difference of two solutions of* (4.77) *with initial data $v_{j,0}\in\mathscr{B}(\omega)$, $j=1,2$, satisfies*

$$\begin{aligned}&\|\nabla v_1(t,\omega,\sigma_1,v_{1,0})-\nabla v_2(t,\omega,\sigma_2,v_{2,0})\|^2\leqslant\\ &\quad\leqslant ce^{c\rho(\omega)+ce^{\lambda t}\int_0^t|z(\vartheta_s\omega)|^{2p-2}\mathrm{d}s+ce^{c\lambda t}}\Big(\|v_{1,0}-v_{2,0}\|^2+\big(d_\Xi(\sigma_1,\sigma_2)\big)^2\Big),\qquad \forall t\geqslant 2,\end{aligned}$$

where $\rho(\cdot)$ is the random variable given in (4.100)*.*

Proof. Taking the inner product of (4.104) with $-\Delta\bar{v}$ in H, by (4.103) we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\bar{v}\|^2+\lambda\|\nabla\bar{v}\|^2+\|\Delta\bar{v}\|^2 &= \Big(f(v_1+hz(\vartheta_t\omega))-f(v_2+hz(\vartheta_t\omega)),-\Delta\bar{v}\Big)+(\bar{\sigma},-\Delta\bar{v})\\ &\leqslant c\int_{\mathscr{O}}\Big(|u_1|^{p-2}+|u_2|^{p-2}+1\Big)|\bar{v}||\Delta\bar{v}|\,\mathrm{d}x+(\bar{\sigma},-\Delta\bar{v})\\ &\leqslant \|\Delta\bar{v}\|^2+c\int_{\mathscr{O}}\big(|u_1|^{2p-4}+|u_2|^{2p-4}\big)|\bar{v}|^2\,\mathrm{d}x+c\|\bar{v}\|^2+c\|\bar{\sigma}\|^2\\ &\leqslant \|\Delta\bar{v}\|^2+c\big(\|u_1\|_{2p-2}^{2p-4}+\|u_2\|_{2p-2}^{2p-4}\big)\|\bar{v}\|_{2p-2}^2+c\|\bar{v}\|^2+c\|\bar{\sigma}\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\bar{v}\|^2 &\leqslant c\big(\|u_1\|_{2p-2}^{2p-4}+\|u_2\|_{2p-2}^{2p-4}\big)\|\bar{v}\|_{2p-2}^2+c\|\bar{v}\|^2+c\|\bar{\sigma}\|^2\\ &\leqslant c\Big(\|u_1\|_{2p-2}^{2p-2}+\|u_2\|_{2p-2}^{2p-2}+1\Big)\|\bar{v}\|_{2p-2}^2+c\|\nabla\bar{v}\|^2+c\|\bar{\sigma}\|^2,\end{aligned} \tag{4.106}$$

and then, by the continuous embedding $V\hookrightarrow L^{2p-2}$ in (4.72),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\bar{v}\|^2\leqslant c\Big(\|u_1\|_{2p-2}^{2p-2}+\|u_2\|_{2p-2}^{2p-2}+1\Big)\|\nabla\bar{v}\|^2+c\|\bar{\sigma}\|^2.$$

By Gronwall's lemma we have

$$\begin{aligned}\|\nabla\bar{v}(t)\|^2 &\leqslant e^{\int_s^t c\left(\|u_1(\eta)\|_{2p-2}^{2p-2}+\|u_2(\eta)\|_{2p-2}^{2p-2}+1\right)\mathrm{d}\eta}\|\nabla\bar{v}(s)\|^2\\ &\quad+\int_s^t e^{\int_1^\tau c\left(\|u_1(\eta)\|_{2p-2}^{2p-2}+\|u_2(\eta)\|_{2p-2}^{2p-2}+1\right)\mathrm{d}\eta}\|\bar{\sigma}(\tau)\|^2\,\mathrm{d}\tau,\quad \forall t\geqslant s>1,\end{aligned}$$

and integrating over $s \in (1,2)$ we obtain

$$\|\nabla \bar{v}(t)\|^2 \leqslant e^{\int_1^t c\left(\|u_1(\eta)\|_{2p-2}^{2p-2}+\|u_2(\eta)\|_{2p-2}^{2p-2}+1\right)\mathrm{d}\eta} \int_1^2 \|\nabla \bar{v}(s)\|^2 \,\mathrm{d}s$$
$$+ e^{\int_1^t c\left(\|u_1(\eta)\|_{2p-2}^{2p-2}+\|u_2(\eta)\|_{2p-2}^{2p-2}+1\right)\mathrm{d}\eta} \int_1^t \|\bar{\sigma}(\tau)\|^2 \,\mathrm{d}\tau, \quad \forall t \geqslant 2.$$

Since

$$\begin{aligned}\int_1^2 \|\nabla \bar{v}(s)\|^2 \,\mathrm{d}s &\leqslant \int_1^2 e^{c(2-s)} \|\nabla \bar{v}(s)\|^2 \,\mathrm{d}s \\ &\leqslant c\|\bar{v}(0)\|^2 + \int_0^2 e^{c(2-s)} \|\bar{\sigma}(s)\|^2 \,\mathrm{d}s \quad \text{(by (4.105))} \\ &\leqslant c\|\bar{v}(0)\|^2 + c\int_0^2 \|\bar{\sigma}(s)\|^2 \,\mathrm{d}s,\end{aligned}$$

we have

$$\|\nabla \bar{v}(t)\|^2 \leqslant c e^{\int_1^t c\left(\|u_1(\eta)\|_{2p-2}^{2p-2}+\|u_2(\eta)\|_{2p-2}^{2p-2}+1\right)\mathrm{d}\eta} \left(\|\bar{v}(0)\|^2 + \int_0^t \|\bar{\sigma}(s)\|^2 \,\mathrm{d}s\right), \quad t \geqslant 2.$$

Notice that for all $t \geqslant 2$

$$\begin{aligned}&\int_1^t c\left(\|u_1(\eta,\omega,\sigma_1,u_{1,0})\|_{2p-2}^{2p-2} + \|u_2(\eta,\omega,\sigma_2,u_{2,0})\|_{2p-2}^{2p-2} + 1\right) \mathrm{d}\eta \leqslant \\ &\leqslant \int_1^t c\left(\|v_1(\eta,\omega,\sigma_1,v_{1,0})\|_{2p-2}^{2p-2} + \|v_2(\eta,\omega,\sigma_2,v_{2,0})\|_{2p-2}^{2p-2} + |z(\vartheta_\eta\omega)|^{2p-2} + 1\right) \mathrm{d}\eta \\ &\leqslant c\|\mathscr{B}(\omega)\|^2 + c\int_0^t e^{\lambda s}\left(|z(\vartheta_s\omega)|^{2p-2} + 1 + C_\Sigma\right)\mathrm{d}s \qquad \text{(by (4.97) and (4.98))} \\ &\leqslant c\rho(\omega) + ce^{\lambda t}\int_0^t |z(\vartheta_s\omega)|^{2p-2}\,\mathrm{d}s + ce^{\lambda t},\end{aligned}$$

and by (4.75) we have

$$c\int_0^t \|\bar{\sigma}(s)\|^2 \,\mathrm{d}s \leqslant ce^{(\ln 4)t}\left(d_\Xi(\sigma_1,\sigma_2)\right)^2,$$

so, for all $t \geqslant 2$,

$$\|\nabla \bar{v}(t)\|^2 \leqslant ce^{c\rho(\omega)+ce^{\lambda t}\int_0^t |z(\vartheta_s\omega)|^{2p-2}\mathrm{d}s + ce^{c\lambda t}}\left(\|\bar{v}(0)\|^2 + \left(d_\Xi(\sigma_1,\sigma_2)\right)^2\right).$$

The proof is complete. □

4.3.4.2 Squeezing

Now we prove the squeezing property on the uniformly absorbing set $\mathscr{B}$ defined by (4.100), i.e.,

$$\mathscr{B}(\omega) = \left\{u \in H : \|u\|^2 \leqslant \rho(\omega)\right\},$$

with $\rho(\omega)$ a tempered random variable given by

$$\rho(\omega) := C\int_{-\infty}^0 e^{\lambda s}|z(\vartheta_s\omega)|^p \,\mathrm{d}s + \frac{C}{\lambda} + CC_b + 1, \qquad \forall \omega \in \Omega, \tag{4.107}$$

for some positive constant $C \geqslant 1$. Before giving the desired squeezing property, we prove a useful lemma for the random variable $\rho(\omega)$.

Lemma 4.3.10. *For the random variable $\rho(\omega)$ defined above, let*

$$\hat{\rho}(\omega) := \max_{t\in[-1,0]} \rho(\vartheta_t\omega), \qquad \forall \omega \in \Omega.$$

Then, for all $\omega \in \Omega$, we have

$$\hat{\rho}(\omega) \leqslant e^{\lambda}\rho(\omega), \tag{4.108}$$

$$\big[\rho(\omega)\big]^{\gamma} \leqslant \int_0^1 \big[\hat{\rho}(\vartheta_s\omega)\big]^{\gamma}\, \mathrm{d}s, \quad \forall \gamma \geqslant 1. \tag{4.109}$$

Proof. By definition,

$$\begin{aligned}
\hat{\rho}(\omega) &= \max_{t\in[-1,0]} C\int_{-\infty}^{0} e^{\lambda s}|z(\vartheta_{s+t}\omega)|^p\, \mathrm{d}s + \frac{C}{\lambda} + CC_b + 1\\
&= \max_{t\in[-1,0]} Ce^{-\lambda t}\int_{-\infty}^{0} e^{\lambda (s+t)}|z(\vartheta_{s+t}\omega)|^p\, \mathrm{d}s + \frac{C}{\lambda} + CC_b + 1\\
&= \max_{t\in[-1,0]} Ce^{-\lambda t}\int_{-\infty}^{t} e^{\lambda s}|z(\vartheta_{s}\omega)|^p\, \mathrm{d}s + \frac{C}{\lambda} + CC_b + 1\\
&\leqslant Ce^{\lambda}\int_{-\infty}^{0} e^{\lambda s}|z(\vartheta_{s}\omega)|^p\, \mathrm{d}s + e^{\lambda}\left(\frac{C}{\lambda} + CC_b + 1\right)\\
&= e^{\lambda}\rho(\omega),
\end{aligned}$$

so (4.108) follows. Since for any $\gamma \geqslant 1$ we have

$$\begin{aligned}
\int_0^1 \big[\hat{\rho}(\vartheta_s\omega)\big]^{\gamma}\, \mathrm{d}s &\geqslant \left[\int_0^1 \hat{\rho}(\vartheta_s\omega)\, \mathrm{d}s\right]^{\gamma}\\
&\geqslant \left[\int_0^1 \rho(\omega)\, \mathrm{d}s\right]^{\gamma} = \big[\rho(\omega)\big]^{\gamma},
\end{aligned}$$

(4.109) is proved. □

Now we prove the squeezing property needed for condition (S). To this end, notice that for any two solutions v_j of (4.77) corresponding to initial data $v_{j,0} \in \mathscr{B}(\omega)$, the difference $y(t,\omega,\sigma,\bar{v}_0) := v_1(t,\omega,\sigma,v_{1,0}) - v_2(t,\omega,\sigma,v_{2,0})$ of them satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \lambda y - \Delta y = f\big(v_1 + hz(\vartheta_t\omega)\big) - f\big(v_2 + hz(\vartheta_t\omega)\big). \tag{4.110}$$

In addition, since $A := -\Delta$ is a self-adjoint positive operator on $D(A) = H^2(\mathscr{O}) \cap H_0^1(\mathscr{O})$ with compact inverse, there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$ of spectra satisfying

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to \infty,$$

and a sequence of eigenvectors $\{e_j\}_{j=1}^{\infty}$ forms an orthonormal basis of H which is such that

$$-\Delta e_j = \lambda_j e_j, \qquad j = 1,2,\cdots.$$

For $n \in \mathbb{N}$, set

$$H_n := \mathrm{span}\{e_1, e_2, \cdots, e_n\},$$

and let $P_n : H \to H_n$ and $Q_n := I - P_n$ be the orthonormal projectors on H. Then

$$\lambda_{n+1}\|Q_n v\|^2 \leqslant \|\nabla v\|^2, \qquad \forall v \in V,\ n \in \mathbb{N}. \tag{4.111}$$

Proposition 4.3.11 (Squeezing property)**.** *There exist $m \in \mathbb{N}$, $\delta \in (0,1/4)$ and a tempered random variable $C(\omega)$ with $\mathbb{E}\big(C(\omega)\big) < -\ln(4\delta)$ such that for any two solutions v_1 and v_2 of (4.77) corresponding to initial data $v_{1,0}, v_{2,0} \in \mathscr{B}(\omega)$, respectively, we have*

$$\sup_{\sigma\in\Sigma} \left\| P_m\big(v_1(T_{\mathscr{B}},\omega,\sigma,v_{1,0}) - v_2(T_{\mathscr{B}},\omega,\sigma,v_{2,0})\big)\right\| \leqslant e^{\int_0^{T_{\mathscr{B}}} C(\vartheta_s\omega)\mathrm{d}s}\|v_{1,0}-v_{2,0}\|, \tag{4.112}$$

$$\sup_{\sigma\in\Sigma} \left\| Q_m\big(v_1(T_{\mathscr{B}},\omega,\sigma,v_{1,0}) - v_2(T_{\mathscr{B}},\omega,\sigma,v_{2,0})\big)\right\| \leqslant \delta e^{\int_0^{T_{\mathscr{B}}} C(\vartheta_s\omega)\mathrm{d}s}\|v_{1,0}-v_{2,0}\|, \tag{4.113}$$

where $T_{\mathscr{B}} > 0$ is the deterministic absorption time of $\mathscr{B}$ in Proposition 4.3.6.

Proof. Without loss of generality we let $T_{\mathscr{B}} = 1$. Take arbitrarily $\sigma \in \Sigma$ (the following proof is independent of the choice of σ). Then for two initial data $v_{j,0} \in \mathscr{B}(\omega)$, by Lemma 4.3.8 the difference $y(t,\omega,\sigma,\bar{v}_0) = v_1(t,\omega,\sigma,v_{1,0}) - v_2(t,\omega,\sigma,v_{2,0})$ of solutions satisfies

$$\|y(t,\omega,\sigma,\bar{v}_0)\|^2 \leqslant C_1 e^{\beta t}\|\bar{v}_0\|^2 \qquad (\text{where } \bar{v}_0 := v_{1,0}-v_{2,0} = y(0)), \tag{4.114}$$

for positive constants $C_1, \beta > 0$ and all $t > 0$, and, particularly for $t = 1$,

$$\|y(1,\omega,\sigma,\bar{v}_0)\|^2 \leqslant C_1 e^{\beta}\|\bar{v}_0\|^2 = e^{\ln C_1 + \beta}\|\bar{v}_0\|^2. \tag{4.115}$$

In addition, for any initial value $v(0) \in \mathscr{B}(\omega)$, by Lemma 4.3.3 the solution $v(t)$ of (4.77) for $t > 1/4$ is bounded by

$$\begin{aligned}
\sup_{\sigma\in\Sigma} \|\nabla v(t,\omega,\sigma,v(0))\|^2 &\leqslant \sup_{\sigma\in\Sigma}\left[c e^{-\lambda t}\|v(0)\|^2 + c\int_{-t}^0 e^{\lambda s}\Big(|z(\vartheta_{s+t}\omega)|^p + 1 + \|\theta_t\sigma(s)\|^2\Big)\,\mathrm{d}s\right] \\
&\leqslant c\|v(0)\|^2 + c\int_{-t}^0 e^{\lambda s}|z(\vartheta_{s+t}\omega)|^p\,\mathrm{d}s + \frac{c}{\lambda} + cC_b \quad (\text{by } (4.99)) \\
&\leqslant c\|v(0)\|^2 + c\rho(\vartheta_t\omega), \qquad \forall t > \frac{1}{4},
\end{aligned}$$

where $c > 0$ is an absolute constant, and $\rho(\cdot)$ is the tempered random variable given by (4.107) which also indicates the radius of the absorbing set $\mathscr{B}$ in H. Analogously, since the solution u of (4.76) is in the form $u(t) = v(t) + hz(\vartheta_t\omega)$, with v the solution of (4.77), we have

$$\begin{aligned}
\sup_{\sigma\in\Sigma} \|\nabla u(t,\omega,\sigma,u(0))\|^2 &\leqslant 2\sup_{\sigma\in\Sigma}\|\nabla v(t,\omega,\sigma,v(0))\|^2 + c|z(\vartheta_t\omega)|^2 \\
&\leqslant c\|v(0)\|^2 + c\rho(\vartheta_t\omega) + c|z(\vartheta_t\omega)|^2 \\
&\leqslant c\rho(\omega) + c\tilde{\rho}(\vartheta_t\omega), \qquad \forall t > \frac{1}{4},
\end{aligned} \tag{4.116}$$

where

$$\tilde{\rho}(\omega) := \rho(\omega) + |z(\omega)|^2, \qquad \forall \omega \in \Omega. \tag{4.117}$$

Taking the inner product of (4.110) with $y_n := Q_n y$ in H we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y_n\|^2 + \lambda\|y_n\|^2 + \|\nabla y_n\|^2 = \Big(f\big(v_1 + hz(\vartheta_t\omega)\big) - f\big(v_2 + hz(\vartheta_t\omega)\big), y_n\Big).$$

Since by Hölder's and Young's inequalities we have the formula

$$\begin{aligned}\int_{\mathscr{O}} abc \, \mathrm{d}x &\leqslant \left(\int_{\mathscr{O}} |a|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathscr{O}} |b|^{\frac{2p-2}{p-2}} \, \mathrm{d}x\right)^{\frac{p-2}{2p-2}} \left(\int_{\mathscr{O}} |c|^{2p-2} \, \mathrm{d}x\right)^{\frac{1}{2p-2}} \\ &\leqslant \varepsilon \left(\int_{\mathscr{O}} |c|^{2p-2} \, \mathrm{d}x\right)^{\frac{2}{2p-2}} + c_\varepsilon \left(\int_{\mathscr{O}} |a|^2 \, \mathrm{d}x\right) \left(\int_{\mathscr{O}} |b|^{\frac{2p-2}{p-2}} \, \mathrm{d}x\right)^{\frac{p-2}{p-1}}, \quad \forall \varepsilon > 0,\end{aligned} \tag{4.118}$$

where $c_\varepsilon > 0$ is a constant depending on ε, then for the nonlinearity term it holds

$$\begin{aligned}&\Big(f\big(v_1 + hz(\vartheta_t\omega)\big) - f\big(v_2 + hz(\vartheta_t\omega)\big), y_n\Big) \leqslant \\ &\leqslant c \int_{\mathscr{O}} \Big(|u_1|^{p-2} + |u_2|^{p-2} + 1\Big)|y||y_n| \, \mathrm{d}x \qquad \text{(by (4.103))} \\ &\leqslant \frac{1}{2}\|y_n\|_{2p-2}^2 + c\Big(\|u_1\|_{2p-2}^{2p-4} + \|u_2\|_{2p-2}^{2p-4} + 1\Big)\|y\|^2 \qquad \text{(by (4.118))},\end{aligned}$$

and finally by the continuous embedding (4.72) of $V \hookrightarrow L^{2p-2}$ we conclude that

$$\Big(f(v_1 + hz(\vartheta_t\omega)) - f(v_2 + hz(\vartheta_t\omega)), y_n\Big) \leqslant \frac{1}{2}\|\nabla y_n\|^2 + c\Big(\|\nabla u_1\|^{2p-4} + \|\nabla u_2\|^{2p-4} + 1\Big)\|y\|^2.$$

Hence, for $t > 1/4$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|y_n\|^2 + \|\nabla y_n\|^2 &\leqslant c\Big(\|\nabla u_1\|^{2p-4} + \|\nabla u_2\|^{2p-4} + 1\Big)\|y\|^2 \\ &\leqslant c\Big(\big[\rho(\omega)\big]^{p-2} + \big[\tilde\rho(\vartheta_t\omega)\big]^{p-2} + 1\Big)\|y\|^2 \qquad \text{(by (4.116))} \\ &\leqslant c\Big(\big[\rho(\omega)\big]^{p-2} + \big[\tilde\rho(\vartheta_t\omega)\big]^{p-2}\Big)\|y\|^2 \qquad (\text{since } \rho(\omega) \geqslant 1),\end{aligned}$$

and then, by (4.111),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y_n\|^2 + \lambda_{n+1}\|y_n\|^2 \leqslant c\Big(\big[\rho(\omega)\big]^{p-2} + \big[\tilde\rho(\vartheta_t\omega)\big]^{p-2}\Big)\|y\|^2, \quad t > 1/4.$$

For $r \in (1/4, t)$, by Gronwall's lemma we have

$$\|y_n(t)\|^2 \leqslant e^{-\lambda_{n+1}(t-r)}\|y_n(r)\|^2 + \int_r^t ce^{\lambda_{n+1}(s-t)}\Big(\big[\rho(\omega)\big]^{p-2} + \big[\tilde\rho(\vartheta_s\omega)\big]^{p-2}\Big)\|y(s)\|^2 \, \mathrm{d}s, \tag{4.119}$$

and then for $t \geqslant 1/2$ integrate (4.119) over $r \in (1/4, 1/2)$ to have

$$\|y_n(t)\|^2 \leqslant 4\int_{\frac{1}{4}}^{\frac{1}{2}} e^{-\lambda_{n+1}(t-r)}\|y_n(r)\|^2 \, \mathrm{d}r + \int_0^t ce^{\lambda_{n+1}(s-t)}\Big(\big[\rho(\omega)\big]^{p-2} + \big[\tilde\rho(\vartheta_s\omega)\big]^{p-2}\Big)\|y(s)\|^2 \, \mathrm{d}s.$$

Since

$$\begin{aligned}\int_{\frac{1}{4}}^{\frac{1}{2}} e^{-\lambda_{n+1}(t-r)}\|y_n(r)\|^2 \, \mathrm{d}r &\leqslant e^{-\lambda_{n+1}t}\int_{\frac{1}{4}}^{\frac{1}{2}} e^{\lambda_{n+1}r}\|y(r)\|^2 \, \mathrm{d}r \\ &\leqslant ce^{-\lambda_{n+1}t}\int_{\frac{1}{4}}^{\frac{1}{2}} e^{\lambda_{n+1}r}\Big(e^{\beta r}\|\bar v_0\|^2\Big)\mathrm{d}r \qquad \text{(by (4.114))} \\ &\leqslant ce^{-\lambda_{n+1}t} \cdot \frac{e^{\frac{1}{2}(\lambda_{n+1}+\beta)}}{\lambda_{n+1}+\beta}\|\bar v_0\|^2 \\ &\leqslant \frac{c}{\lambda_{n+1}}\|\bar v_0\|^2, \qquad t \geqslant \frac{1}{2},\end{aligned}$$

we have

$$
\begin{aligned}
\|y_n(t)\|^2 &\leqslant \frac{c}{\lambda_{n+1}}\|\bar{v}_0\|^2 + \int_0^t ce^{\lambda_{n+1}(s-t)}\Big(\big[\rho(\omega)\big]^{p-2} + \big[\tilde{\rho}(\vartheta_s\omega)\big]^{p-2}\Big)\|y(s)\|^2\,\mathrm{d}s \\
&\leqslant \frac{c}{\lambda_{n+1}}\|\bar{v}_0\|^2 + \int_0^t ce^{\lambda_{n+1}(s-t)}\Big(\big[\rho(\omega)\big]^{p-2} + \big[\tilde{\rho}(\vartheta_s\omega)\big]^{p-2}\Big)e^{\beta s}\|\bar{v}_0\|^2\,\mathrm{d}s \quad \text{(by (4.114))} \\
&\leqslant \frac{c}{\lambda_{n+1}}\|\bar{v}_0\|^2 + ce^{\beta t}\|\bar{v}_0\|^2\int_0^t e^{\lambda_{n+1}(s-t)}\Big(\big[\rho(\omega)\big]^{p-2} + \big[\tilde{\rho}(\vartheta_s\omega)\big]^{p-2}\Big)\,\mathrm{d}s. \qquad (4.120)
\end{aligned}
$$

Since for the last term we have

$$
\begin{aligned}
&\int_0^t e^{\lambda_{n+1}(s-t)}\Big(\big[\rho(\omega)\big]^{p-2} + \big[\tilde{\rho}(\vartheta_s\omega)\big]^{p-2}\Big)\,\mathrm{d}s \\
&\quad\leqslant \left[\int_0^t e^{2\lambda_{n+1}(s-t)}\,\mathrm{d}s\right]^{\frac{1}{2}}\left[\int_0^t \Big(\big[\rho(\omega)\big]^{p-2} + \big[\tilde{\rho}(\vartheta_s\omega)\big]^{p-2}\Big)^2\,\mathrm{d}s\right]^{\frac{1}{2}} \\
&\quad\leqslant \frac{1}{\sqrt{2\lambda_{n+1}}}\left[\int_0^t 2\Big(\big[\rho(\omega)\big]^{2p-4} + \big[\tilde{\rho}(\vartheta_s\omega)\big]^{2p-4}\Big)\,\mathrm{d}s\right]^{\frac{1}{2}} \\
&\quad= \frac{1}{\sqrt{\lambda_{n+1}}}\left[\int_0^t \Big(\big[\rho(\omega)\big]^{2p-4} + \big[\tilde{\rho}(\vartheta_s\omega)\big]^{2p-4}\Big)\,\mathrm{d}s\right]^{\frac{1}{2}} \\
&\quad\leqslant \frac{1}{\sqrt{\lambda_{n+1}}}e^{\int_0^t([\rho(\omega)]^{2p-4}+[\tilde{\rho}(\vartheta_s\omega)]^{2p-4})\mathrm{d}s} \qquad \big(\text{since } s^{\frac{1}{2}} \leqslant e^s, \text{ for } s>0\big),
\end{aligned}
$$

taking in particular $t=1$ in (4.120) we obtain

$$
\begin{aligned}
\|y_n(1)\|^2 &\leqslant \frac{c}{\lambda_{n+1}}\|\bar{v}_0\|^2 + c\|\bar{v}_0\|^2\left(\frac{1}{\sqrt{\lambda_{n+1}}}e^{\int_0^1([\rho(\omega)]^{2p-4}+[\tilde{\rho}(\vartheta_s\omega)]^{2p-4})\mathrm{d}s}\right) \\
&= \frac{c}{\lambda_{n+1}}\|\bar{v}_0\|^2 + \frac{c\|\bar{v}_0\|^2}{\sqrt{\lambda_{n+1}}}e^{[\rho(\omega)]^{2p-4}+\int_0^1[\tilde{\rho}(\vartheta_s\omega)]^{2p-4}\mathrm{d}s} \\
&\leqslant \frac{c}{\lambda_{n+1}}\|\bar{v}_0\|^2 + \frac{c\|\bar{v}_0\|^2}{\sqrt{\lambda_{n+1}}}e^{[\rho(\omega)]^{2p-2}+\int_0^1[\tilde{\rho}(\vartheta_s\omega)]^{2p-2}\mathrm{d}s} \qquad (4.121)\\
&\leqslant \frac{c}{\lambda_{n+1}}\|\bar{v}_0\|^2 + \frac{c\|\bar{v}_0\|^2}{\sqrt{\lambda_{n+1}}}e^{\int_0^1[\hat{\rho}(\vartheta_s\omega)]^{2p-2}\mathrm{d}s+\int_0^1[\tilde{\rho}(\vartheta_s\omega)]^{2p-2}\mathrm{d}s} \quad \text{(by (4.109))} \\
&\leqslant \frac{c}{\lambda_{n+1}}\|\bar{v}_0\|^2 + \frac{c\|\bar{v}_0\|^2}{\sqrt{\lambda_{n+1}}}e^{\int_0^1(e^{\lambda(2p-2)}[\rho(\vartheta_s\omega)]^{2p-2}+[\tilde{\rho}(\vartheta_s\omega)]^{2p-2})\mathrm{d}s} \quad \text{(by (4.108))}.
\end{aligned}
$$

By the definition of $\tilde{\rho}$ in (4.117) we have

$$
\begin{aligned}
e^{\lambda(2p-2)}[\rho(\omega)]^{2p-2} + [\tilde{\rho}(\omega)]^{2p-2} &= e^{\lambda(2p-2)}[\rho(\omega)]^{2p-2} + \big[\rho(\omega)+|z(\omega)|^2\big]^{2p-2} \\
&\leqslant \Big(e^{\lambda(2p-2)}+1\Big)\big[\rho(\omega)+|z(\omega)|^2\big]^{2p-2}, \quad \omega\in\Omega,
\end{aligned}
$$

and for later purpose define $k := e^{\lambda(2p-2)} + 1 + (\ln C_1 + \beta)$ and

$$
C(\omega) := k\big[\rho(\omega)+|z(\omega)|^2\big]^{2p-2}, \quad \forall\omega\in\Omega.
$$

Then $C(\cdot)$ is a tempered random variable with finite expectation, and, by (4.121),

$$\begin{aligned}\|y_n(1)\|^2 &\leqslant \frac{c}{\lambda_{n+1}}\|\bar{v}_0\|^2 + \frac{c\|\bar{v}_0\|^2}{\sqrt{\lambda_{n+1}}} e^{\int_0^1 C(\vartheta_s\omega)\mathrm{d}s} \\ &\leqslant \left(\frac{c}{\lambda_{n+1}} + \frac{c}{\sqrt{\lambda_{n+1}}}\right) e^{\int_0^1 C(\vartheta_s\omega)\mathrm{d}s}\|\bar{v}_0\|^2.\end{aligned}$$

Clearly, there exists a $\delta \in (0,1/4)$ such that $4\delta < e^{-\mathbb{E}(C(\omega))}$, or equivalently $\mathbb{E}(C(\omega)) < -\ln(4\delta)$. In addition, since $\lambda_n \to \infty$ as $n \to \infty$, we have an $m \in \mathbb{N}$ large enough such that

$$\frac{c}{\lambda_{m+1}} + \frac{c}{\sqrt{\lambda_{m+1}}} \leqslant \delta^2.$$

In this way we obtain

$$\|y_m(1)\|^2 \leqslant \delta^2 e^{2\int_0^1 C(\vartheta_s\omega)\mathrm{d}s}\|\bar{v}_0\|^2, \qquad \text{with } \mathbb{E}(C(\omega)) < -\ln(4\delta);$$

so (4.113) is verified. Since $2C(\omega) \geqslant \ln C_1 + \beta$ for all $\omega \in \Omega$, by (4.115) we have

$$\begin{aligned}\|y(1,\omega,\sigma,\bar{v}_0)\|^2 &\leqslant e^{\ln C_1 + \beta}\|\bar{v}_0\|^2 \\ &\leqslant e^{2\int_0^1 C(\vartheta_s\omega)\mathrm{d}s}\|\bar{v}_0\|^2,\end{aligned}$$

and (4.112) is also clear. □

4.3.4.3 Finite-dimensionality of the random uniform attractor in H and in V

Now we are ready to conclude that the random uniform attractor $\mathscr{A}$ of (4.77) has finite fractal dimension in H and in V.

Theorem 4.3.12. *Suppose the symbol space $\Sigma = \mathscr{H}(g)$ has finite fractal dimension in $\Xi = \mathscr{C}(\mathbb{R};H)$, i.e. (G) holds. Then the fractal dimension in H and that in V of the random uniform attractor $\mathscr{A}$ of the NRDS ϕ generated by* (4.77) *are both finite.*

Proof. Let us prove first the finite dimensionality of $\mathscr{A}$ in the phase space H. Set $X := H$ and $Y := V$. From Lemma 4.3.8 we have conditions (R_2) and (R_4); from Proposition 4.3.6 it follows (R_3) and finally by Proposition 4.3.11 we obtain (S). Then by Theorem 4.2.6 we conclude that the fractal dimension of the random uniform attractor $\mathscr{A}$ is uniformly bounded in H, i.e. $\dim_F\big(\mathscr{A}(\omega);H\big) \leqslant c_0$, for all $\omega \in \Omega$, for some deterministic constant $c_0 > 0$.

On the other hand, from Lemma 4.3.9 it follows the smoothing property (R_7) (with $\delta_1 = \delta_2 = 1$) and since $\mathscr{A}$ has fractal dimension uniformly bounded in H we obtain by Theorem 4.2.8 that $\mathscr{A}$ is finite-dimensional in V as well: $\dim_F\big(\mathscr{A}(\omega);V\big) \leqslant \dim_F(\Sigma;\Xi) + c_0$ for all $\omega \in \Omega$. □

Finally, from (4.81) we conclude that the random uniform attractor $\tilde{\mathscr{A}}$ associated to the stochastic reaction-diffusion equation (4.66) has finite fractal dimension in spaces H and V.

CHAPTER

5

CONCLUSION AND FUTURE WORKS

In this work we were interested in the study of fractal dimension of attractors in three different settings obtaining estimates on the dimension for global attractors, uniform attractors and random uniform attractors. The main technique we used is based on a compact embedding between Banach spaces and on a regularization condition for the system under consideration. In general we called it a *smoothing property*. Besides that we also established a comparison between different methods (to estimate the dimension) applied to global attractors showing that a smoothing condition is better than the qualitative compactness property assumed for example by Mañé in (MAÑÉ, 1981).

We studied for non-autonomous (random) dynamical systems the construction of new symbol spaces with finite fractal dimension, especially on the space of bounded continuous functions endowed with a Fréchet metric. It could help us in order to apply our previous theoretical results to some problems associated to non-autonomous (stochastic) evolution equations. Our contribution to the theory of nonlinear dynamical systems are summarized and presented in the submitted papers (CARVALHO *et al.*,), (CUI *et al.*,) and (CUI; CUNHA; LANGA,).

This technical condition on the fractal dimension of symbol spaces is an interesting point to be investigated. An important problem can be guarantee the finite-dimensionality of (random) uniform attractors without the assumption on Σ that it has finite fractal dimension. It is usually difficult in applications to find examples of symbol spaces Σ with this property and so we can look for conditions on a system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ (or on an NRDS ϕ) that ensure the dimensionality of (random) uniform attractors avoiding the restriction imposed upon Σ. This problem can be first studied in the deterministic setting for uniform attractors in which the random parameters are not involved. If well-succeed it is expected that the generalization to random problems can be done taking into account some ergodicity properties.

In this work we focused our attention on the study of attractors. A well-known problem of these objects is that despite they attract some expected collection of sets, the rate of attraction is usually unknown and it can happen extremely slowly what can imply a non-stability (under perturbation) for the system under consideration. So in order to overcome this drawback, in (EDEN *et al.*, 1995) was developed the notion of exponential attractor, i.e., a positively invariant compact set which attracts all bounded sets at an exponential rate. In this direction, we talk about some problems that could be investigated in the future. Using techniques developed in this work along with the expertise maybe it is possible to construct exponential attractors in more general settings as for example the construction of uniform exponential attractors and random uniform exponential attractors. Let us describe them in more details.

For non-autonomous dynamical systems we can think at a first glance as in the following. We start with a definition of uniform exponential attractors.

Definition 5.0.1. *Let $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a system of evolution processes on a Banch space X. A compact set $\mathscr{E}_\Sigma \subseteq X$ is a uniform exponential attractor for $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ if*

(i) $\mathscr{E}_\Sigma$ has finite fractal dimension in X, i.e., $\dim_X(\mathscr{E}_\Sigma;X) < \infty$;

(ii) There exists $\alpha > 0$ such that for every $B \subseteq X$ bounded we have

$$\sup_{\sigma\in\Sigma} \operatorname{dist}_X\left(U_\sigma(t,s)B, \mathscr{E}_\Sigma\right) \leqslant C(B)e^{-\alpha(t-s)}, \qquad \forall t \geqslant s+T(B),\ s\in\mathbb{R},$$

for some positive constants $C(B), T(B) > 0$, which only depend on B.

Based on the theory developed in (CHUESHOV, 2015) for global and pullback attractors, the idea is to replace the smoothing property (H_4)-(H_5) (Chapter 3, Section 3.2) with a more general quasi-stability condition for a system of evolution processes and then construct a uniform exponential attractor in this new setting. In this case the smoothing condition is seen as a particular case. We note that the problem proposed is substantially different of what we did here, since now we are proposing to construct uniform exponential attractors, which is a task much more involving than obtaining estimates on the fractal dimension of the attractor. Let us make it more precise what that quasi-stability means.

Definition 5.0.2. *A real-valued function $\mathscr{N}_Z : Z \to \mathbb{R}$ is a seminorm on a linear space Z if*

(i) $\mathscr{N}_Z(x+y) \leqslant \mathscr{N}_Z(x) + \mathscr{N}_Z(y)$, for any $x,y \in Z$;

(ii) $\mathscr{N}_Z(\lambda x) = |\lambda|\mathscr{N}_Z(x)$, for any $x \in Z$ and $\lambda \in \mathbb{R}$.

Definition 5.0.3. *A seminorm $\mathscr{N}_Z : Z \to \mathbb{R}$ on a Banach space Z is compact if any bounded sequence $\{x_m\}_m \subset Z$ has a subsequence $\{x_{m_k}\}_k$ which is Cauchy with respect to the seminorm $\mathscr{N}_Z$, i.e.,*

$$\lim_{k,l\to\infty} \mathscr{N}_Z(x_{m_k} - x_{m_l}) = 0.$$

Let $\mathscr{B}$ be a closed and bounded uniformly attracting set for a system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ and $T_{\mathscr{B}} > 0$ be an absorption time in which $\mathscr{B}$ uniformly absorbs itself. The quasi-stability condition is expressed as follows.

(*Q*) There exist a Banach space Z, a compact seminorm $\mathscr{N}_Z$ on Z and operators $V_\sigma : \mathscr{B} \to Z$, for all $\sigma \in \Sigma$, such that for some $\kappa > 0$, $\eta \in (0, 1/2)$ and $\tilde{t} \geqslant T_{\mathscr{B}}$ we have

$$\sup_{\sigma\in\Sigma} \|V_\sigma u - V_\sigma v\|_Z \leqslant \kappa\|u - v\|_X, \qquad \forall u, v \in \mathscr{B},$$

and

$$\|U_\sigma(\tilde{t},0)u - U_\sigma(\tilde{t},0)v\|_X \leqslant \eta\|u - v\|_X + \mathscr{N}_Z(V_\sigma u - V_\sigma v), \qquad \forall \sigma \in \Sigma,\ u, v \in \mathscr{B}.$$

A conjecture is that supposing (H_1)-(H_3) and (Q) we can construct a uniform exponential attractor for a system $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ of evolution processes obtaining estimates on its fractal dimension.

For the non-autonomous random dynamical setting we can also think about the contruction of a random uniform exponential attractor associated to some NRDS ϕ on Banach space X. A defintion of that object was given in (HAN; ZHOU, 2019).

Definition 5.0.4. *A compact random set $\mathscr{E} \subseteq X$ is said to be a random $\mathscr{D}$-uniform exponential attractor for an NRDS ϕ if it satisfies the following properties:*

1. *There is a random variable $d(\cdot) : \Omega \to \mathbb{R}^+$ such that*

$$\dim_F\big(\mathscr{E}(\omega); X\big) \leqslant d(\omega) < \infty, \qquad \forall \omega \in \Omega;$$

2. *There exists a constant $\alpha > 0$ such that for any given $B \in \mathscr{D}$ there are random variables $T_B(\omega), C_B(\omega) > 0$ satisfying*

$$\sup_{\sigma\in\Sigma} \mathrm{dist}_X\big(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, B(\vartheta_{-t}\omega)), \mathscr{E}(\omega)\big) \leqslant C_B(\omega)e^{-\alpha t}, \qquad \forall t \geqslant T_B(\omega), \omega \in \Omega.$$

A construction of a random uniform exponential attractor was already given in (HAN; ZHOU, 2019). However it is mainly applied for Hilbert phase spaces X and limited to the case in which the symbol space Σ is identified with a k-torus (in this case Σ has fractal dimension bounded by k). As we saw in Section 3.3 we can construct more general symbol spaces with finite-dimensionality.

Based on the approach developed in Section 4.2 (the smoothing method summarized in Theorem 4.2.3; see also (CUI; CUNHA; LANGA,)) to prove the finite-dimensionality of random uniform attractors we intend to extend that method and construct also random uniform exponential attractors. This time the phase space X can be a general Banach space and the symbol space Σ is only supposed to have finite fractal dimension (and it is not necessarily a torus). The smoothing method was also shown to be useful in proving the finite-dimensionality of attractors in more regular Banach spaces than the phase space of the problem. In this way we can expect the existence of random uniform exponential attractors in spaces with more regularization properties and look for results in this direction.

As application of these expected theoretical results (and also the ones we have obtained throughout this complete work) we can possibly work on deterministic (and stochastic perturbations of) Navier-Stokes, Navier-Stokes-Coriolis and Boussinesq-Coriolis equations in Besov and Besov-Fourier type spaces as well as in other fluid dynamics models such as quasi-geostrophic equations constructing for such problems (random) uniform exponential attractors and estimating their fractal dimension. Other Banach spaces to be considered in our analysis can be Sobolev-type spaces, measure and Morrey-type spaces. Non-autonomous reaction-diffusion problems and extensible beam equations can also be considered.

CONCLUSÃO

Neste trabalho estávamos interessados no estudo da dimensão fractal de atratores em três diferentes contextos, obtendo estimativas na dimensão de atratores globais, atratores uniformes e atratores uniformes aleatórios. A principal técnica utilizada baseia-se em uma imersão compacta entre espaços de Banach e em uma condição de regularização para o sistema em estudo. Em geral nós chamamos essa propriedade de *propriedade smoothing*. Além disso, estabelecemos uma comparação entre diferentes métodos de estimativas de dimensão fractal de atratores globais mostrando que a condição smoothing atua melhor do que a compacidade qualitativa assumida por exemplpo por Mañé em (MAÑÉ, 1981).

Estudamos para sistemas dinâmicos (aleatórios) não-autônomos a construção de novos espaços símbolo com dimensão fractal finita, especificamente no espaço das funções contínuas munido de uma métrica de Fréchet. Isto nos ajudou na aplicação dos nossos novos resultados teóricos em alguns problemas associados à equações de evolução (estocásticas) não-autônomas. Nossa contribuição à teoria dos sistemas dinâmicos não-lineares estão agrupados e submetidos para publicação nos trabalhos (CARVALHO *et al.*,), (CUI *et al.*,) and (CUI; CUNHA; LANGA,).

A condição técnica sobre a dimensão fractal do espaço símbolo é um problema interessante para ser investigado. Um passo importante pode ser o de garantir estimativas para a dimensão fractal de atratores uniformes (aleatórios) sem a hipótese sobre Σ de que este tenha dimensão fractal finita. É geralmente difícil nas aplicações encontrarmos espaços símbolos Σ com tal propriedade e então podemos buscar por condições sobre um sistema $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ (ou sobre um sistema dinâmico aleatório não-autônomo ϕ) que nos garanta estimativas na dimensão fractal de atratores uniformes (aleatórios) evitando as restrições sobre Σ. Este problema pode ser estudado primeiramente no caso determinístico para atratores uniformes nos quais os parâmetros aleatórios não são considerados. Se bem sucedido, espera-se que uma generalização em problemas estocásticos possa ser realizada, levando em consideração algumas propriedades de ergodicidade.

Neste trabalho focamos no estudo de atratores. Porém um fato conhecido sobre estes objetos é que, apesar de sua propriedade de atrair determinada coleção de conjuntos, sua taxa de atração é geralmente desconhecida podendo ocorrer de maneira extremamente lenta, o que pode implicar em uma não estabilidade (sob perturbação) do sistema em estudo. Na tentativa de superar essa desvantagem, em (EDEN *et al.*, 1995) foi introduzida a noção de atrator exponencial, i.e., um compacto positivamente invariante que atrai todos os limitados a uma taxa exponencial. Nesta direção, citamos os seguintes problemas que podem ser investigados. Com base nas técnicas desenvolvidas no nosso trabalho talvez seja possível a construção de atratores exponenciais em contextos mais gerais tais como atratores exponenciais uniformes (com adaptações em uma propriedade de quasi-estabilidade, inspirado por exemplo em (CHUESHOV, 2015)) e atratores exponencias uniformes aleatórios (com métodos smoothing/squeezing).

Como aplicação destes resultados teóricos esperados (e também dos nossos resultados neste trabalho) podemos trabalhar com problemas determinísticos (e perturbações estocásticas destes) de equações de Navier-Stokes, Navier-Stokes-Coriolis e Boussinesq-Coriolis em espaços de Besov e Besov-Fourier bem como outros modelos de dinâmica de fluidos tais como problemas quasi-geostróficos, construindo para estes modelos atratores exponenciais uniformes (aleatórios) e estimando suas dimensões fractais. Outros espaços de Banach que podem ser considerados são espaços de Sobolev, de medida e de Morrey. Equações de reação-difusão e problemas de vigas elásticas também podem ser estudados.

CONCLUSIÓN

En este trabajo estamos interesados en el estudio de la dimensión fractal de atractores en tres contextos distintos, obteniendo estimaciones en la dimensión de atractores globales, atractores uniformes y atractores uniformes aleatorios. La principal técnica utilizada se basa en una inmersión compacta entre espacios de Banach y en una condición de regularización para el sistema bajo estudio. En general llamamos a esta propiedad de *propiedad smoothing*. Además, hemos establecido una comparación entre diferentes métodos para estimaciones de dimensión fractal de atractores globales demostrando que la condición smoothing actúa mejor que la compacidad cualitativa supuesta por ejemplo por Mañé en (MAÑÉ, 1981).

Estudiamos para sistemas dinámicos (aleatorios) no-autónomos la construcción de nuevos espacios símbolo con dimensión fractal finita, más específicamente en el espacio de las funciones continuas dotado de una métrica de Fréchet. Esto nos ha ayudado en la aplicación de nuestros nuevos resultados teóricos a algunos problemas asociados a las ecuaciones de evolución (estocásticas) no-autónomas. Nuestra contribución a la teoría de los sistemas dinámicos nolineales están agrupados y enviados para publicación en los trabajos (CARVALHO *et al.*,), (CUI *et al.*,) y (CUI; CUNHA; LANGA,).

La condición técnica sobre la dimensión fractal del espacio símbolo es un problema interesante a investigar. Un paso importante puede ser el de garantizar estimaciones para la dimensión fractal de atractores uniformes (aleatorios) sin la hipótesis sobre Σ que éste tenga dimensión fractal finita. Suele ser difícil en las aplicaciones encontrar espacios símbolo Σ con esta propriedad y entonces podemos buscar condiciones sobre un sistema $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ (o sobre un sistema dinámico aleatorio no-autónomo ϕ) que nos garantiza estimaciones en la dimensión fractal de atractores uniformes (aleatorios) evitando las restricciones impuestas sobre Σ. Este problema puede ser estudiado primero en el caso determinista para atractores uniformes los cuales los parámetros aleatorios no son considerados. Si tenemos éxito, esperamos que una generalización en problemas estocásticos pueda ser realizada, teniendo en cuenta algunas propiedades de ergodicidad.

En este trabajo nos centramos en el estudio de atractores. Sin embargo, un hecho conocido sobre estos objetos es que, a pesar de su propiedad de atraer cierta colección de conjuntos, su tasa de atracción es generalmente desconocida y puede ocurrir muy lentamente, lo que puede implicar una no-estabilidad (bajo perturbación) del sistema en estudio. En un intento por superar esta desventaja, en (EDEN *et al.*, 1995) se introdujo la noción de atractor exponencial, i.e., un compacto positivamente invariante que atrae a todos los conjuntos acotados a una tasa exponencial. En esta dirección, mencionamos los siguientes problemas que se pueden investigar. Con base en las técnicas desarrolladas en nuestro trabajo, puede ser posible la construcción de atractores exponenciales en contextos más generales tales como atractores exponenciales uniformes (con adaptaciones en una propiedad de quasi-estabilidad, inspirado por ejemplo en (CHUESHOV, 2015)) y atractores exponenciales uniformes aleatorios (con métodos smoothing/squeezing).

Como aplicaciones de estos resultados teóricos esperados (y también de nuestros resultados en este trabajo) podemos trabajar con problemas deterministas (y perturbaciones estocásticas de estos) de ecuaciones de Navier-Stokes, Navier-Stokes-Coriolis y Boussinesq-Coriolis en espacios de Besov y Besov-Fourier así como otros modelos de dinámica de fluidos tales como problemas quasi-geostróficos, construyendo para estos modelos atractores exponenciales uniformes (aleatorios) y estimando sus dimensiones fractales. Otros espacios de Banach que pueden ser considerados son espacios de Sobolev, de medida y de Morrey. Ecuaciones de reacción-difusión y problemas de vigas elásticas también pueden ser estudiados.

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