Heisenberg superalgebras

L.M. Camacho[a], J.R. Gómez[a] and R.M. Navarro[b]

[a]*Departamento de Matemática Aplicada I, Universidad de Sevilla, Sevilla, Spain;* [b]*Departamento de Matemáticas, Universidad de Extremadura, Cáceres, Spain*

Heisenberg algebras are the only Lie algebras $(\mathfrak{g}, [,])$ which verify $[\mathfrak{g}, \mathfrak{g}] = \mathcal{Z}(\mathfrak{g})$ and $\dim(\mathcal{Z}(\mathfrak{g})) = 1$, where $\mathcal{Z}$ denotes the center of the algebra. We classify nilpotent Lie superalgebras that verify the same algebraic conditions in arbitrary finite dimension. We study the geometrical properties with the aid of the software *Mathematica*.

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1. Introduction

In the past years, the theory of Lie superalgebras has suffered a remarkable evolution both in Mathematics and Physics. The Lie superalgebras emerge to Physics in 1974 [3]. Later, in 1975, Kăc [8] offers a comprehensive description of the mathematical theory of Lie superalgebras, and establishes the classification of all finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero.

Some authors have studied Lie superalgebras (nilpotent and non nilpotent) and their cohomology [5–7,10,11,14,15]. But there exist a few results concerning nilpotent Lie superalgebras with even part the Heisenberg algebra (these superalgebras are called Lie superalgebras of Heisenberg type [2]). The simplest non trivial nilpotent Lie algebra is the Heisenberg algebra. Actually there is a family of Heisenberg algebras, $\mathcal{H}_{2n+1}$, one for each odd dimension. Heisenberg algebras play a fundamental role in many fields. For example, we are going to see the central role of Heisenberg algebras in Quantum Mechanics: in the classical description of a system, the observables are functions of $2n$ canonical variables, n coordinates $q_1, \ldots, q_n$ and n momenta $p_1, \ldots, p_n$. If H is the total energy of the system, or so called the Hamiltonian, then Hamilton's equations may be written in terms of Poisson brackets as

$$\dot{q}_i = \{q_i, H\}, \quad \dot{p}_i = \{p_i, H\}.$$

More generally, the evolution in time of an observable F is given by

$$\dot{F} = \{F, H\}.$$

The relationship between symmetries and conservation laws is fundamental. If G is a function of the canonical variables such that $\{G, H\} = 0$, where H is the Hamiltonian, then G generates a symmetry of the system that can be obtained by solving the equations

$$\begin{aligned} \frac{\mathrm{d}q_i}{\mathrm{d}s} &= \{q_i, G\} = \frac{\partial G}{\partial p_i}, \\ \frac{\mathrm{d}p_i}{\mathrm{d}s} &= \{p_i, G\} = \frac{\partial G}{\partial q_i}. \end{aligned}$$

If $\{q(s), p(s)\}$ is the flow generated by the equations, we have that

$$\frac{\mathrm{d}H}{\mathrm{d}s} = \sum_i \left(\frac{\partial H}{\partial q_i} \frac{\mathrm{d}q_i}{\mathrm{d}s} + \frac{\partial H}{\partial p_i} \frac{\mathrm{d}p_i}{\mathrm{d}s} \right) \sum_i \left(\frac{\partial H}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial G}{\partial q_i} \right) = \{H, G\} = 0$$

It follows that the observables form an infinite dimensional Lie algebra with respect to the Poisson bracket

$$\{F, G\} = \sum_i^n \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right)$$

The equations of motion are $\dot{q}_i = \partial H / \partial p_i$ and $\dot{p}_i = \partial H / \partial q_i$ with H, the Hamiltonian of the system. It is easy to check that the symmetries of the Hamiltonian, that is, the observables such that $\{H, G\} = 0$, form a Lie subalgebra of the Lie algebra of all the observables.

A quantum mechanical description of the same system is obtained by finding an algebra of Hermitian operators on a Hilbert space with the Lie product given by the commutator. In that case, if A and B are the operators corresponding to the classical functions a and b, respectively, then $[A, B]$ is an operator corresponding to the classical function $\imath\hbar\{a, b\}$ ($\hbar = h/2\pi$ with h Planck's constant). In particular, since the classical canonical variables q_r, p_s satisfy $\{q_r, q_s\} = 0$, $\{p_r, p_s\} = 0$, $\{q_r, p_s\} = \delta_{rs}$, the corresponding quantum mechanical operators have to satisfy the commutation relations

$$[Q_r, Q_s] = 0, \quad [P_r, P_s] = 0, \quad [Q_r, P_s] = \imath\hbar\delta_{rs}\mathbf{1}.$$

In the case of one variable, we obtain the Heisenberg algebra $\{Q, P, E\}$, where $E = \imath\hbar\mathbf{1}$; with n variables we get the generalized Heisenberg algebra [1,16].

This work aims at finding the Lie superalgebras that verify the same algebraic conditions that Heisenberg algebras verify in the theory of Lie algebras.

We will not presuppose any knowledge of the theory of Lie superalgebras, however, we assume that the reader is familiar with the standard theory of Lie algebras.

2. Preliminaries

2.1. *$\mathbb{Z}_2$-graded algebraic structures. Superalgebras*

The vector space V is said to be $\mathbb{Z}_2$-graded if it admits a decomposition in direct sum $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The elements of $V_{\bar{0}}$ and $V_{\bar{1}}$ are called even odd, respectively.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and $W = W_{\bar{0}} \oplus W_{\bar{1}}$ be two-graded vector spaces. A linear mapping $f : V \longrightarrow W$ is said to be homogeneous of degree γ ($\gamma \in \mathbb{Z}_2$) if $f(V_\alpha) \subset W_{\alpha+\gamma(\mathrm{mod}\,2)}$ for all $\alpha \in \mathbb{Z}_2$. The mapping f is called a homomorphism of the $\mathbb{Z}_2$-graded vector spaces if it is homogeneous of degree 0. Now it is evident how we define an isomorphism or an automorphism of $\mathbb{Z}_2$-graded vector spaces.

A *Lie superalgebra* is a $\mathbf{Z}_2$-graded vector space, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, with a bracket product [,] which verifies

(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta(\mathrm{mod}\,2)} \quad \forall \alpha, \beta \in \mathbf{Z}_2$.
(2) $[X, Y] = -(-1)^{\alpha\beta}[Y, X] \quad \forall X \in \mathfrak{g}_\alpha, \forall Y \in \mathfrak{g}_\beta$.
(3) $(-1)^{\gamma\alpha}[X, [Y, Z]] + (-1)^{\alpha\beta}[Y, [Z, X]] + (-1)^{\beta\gamma}[Z, [X, Y]] = 0$ for all $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta, Z \in \mathfrak{g}_\gamma$ with $\alpha, \beta, \gamma \in \mathbf{Z}_2$.

The last property is called *graded Jacobi identity* and we will call it $J_g(X, Y, Z)$. $\mathfrak{g}_{\bar{0}}$ is called the even part and it is a Lie algebra. $\mathfrak{g}_{\bar{1}}$ is called the odd part and it is a $\mathfrak{g}_{\bar{0}}$-module. We consider $\mathcal{L}^{n+m}$ the set of the Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with $\dim(\mathfrak{g}_{\bar{0}}) = n$ and $\dim(\mathfrak{g}_{\bar{1}}) = m$.

The *descending central sequence* of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is defined by $\mathcal{C}^0(\mathfrak{g}) = \mathfrak{g}$, $\mathcal{C}^{k+1}(\mathfrak{g}) = [\mathcal{C}^k(\mathfrak{g}), \mathfrak{g}]$ for all $k \geq 0$. If $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ for some k, the Lie superalgebra is called *nilpotent*. The smallest integer k such as $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ is called the nilindex of $\mathfrak{g}$.

2.2. *Cohomology of Lie superalgebras*

The cohomology of Lie superalgebras is a 'generalization' of the cohomology of Lie algebras. Such generalization takes into account the $\mathbb{Z}_2$-graduation of the superalgebras and the parity of the isomorphisms amongst them.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a graded-vector space and $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ a Lie superalgebra. $\mathrm{End}(V)$ is an associative superalgebra if we define the $\mathbb{Z}_2$-graduation by:

$$\mathrm{End}(V)_\alpha = \left\{ A \in \frac{\mathrm{End}(V)}{A(V_\beta)} \subset V_{\alpha+\beta}, \beta \in \mathbb{Z}_2 \right\} \quad \forall \alpha \in \mathbb{Z}_2$$

It is easy to prove that $\mathrm{End}(V)_{\bar{0}}$ and $\mathrm{End}(V)_{\bar{1}}$ are the homegeneous linear maps of degree 0 and 1, respectively. We call the associated superalgebra to $\mathrm{End}(V)$ $\mathrm{pl}(V) = \mathrm{pl}_{\bar{0}}(V) \oplus \mathrm{pl}_{\bar{1}}(V)$. This superalgebra plays the same role as the $gl(V)$ in Theory of the Lie algebras.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra and $\mathcal{D}_\alpha(\mathfrak{g})$, $\alpha \in \mathbb{Z}_2$ the subspace of all $f \in \mathrm{pl}_\alpha(\mathfrak{g})$ such that

$$f([X, Y]) = [f(X), Y] + (-1)^{\alpha\xi}[X, f(Y)] \quad \forall X \in \mathfrak{g}_\xi, \ \ \forall Y \in \mathfrak{g}, \ \ \xi \in \mathbb{Z}_2.$$

If $\alpha \equiv 0(\mathrm{mod}\,2)$, $\mathcal{D}_{\bar{0}}$ is formed by the even derivations of the superalgebra $\mathfrak{g}$ and $\alpha \equiv 1(\mathrm{mod}\,2)$, $\mathcal{D}_{\bar{1}}$ are the odd derivations of the superalgebra $\mathfrak{g}$. Thus we obtain that $\mathcal{D}(\mathfrak{g}) = \mathcal{D}_{\bar{0}} \oplus \mathcal{D}_{\bar{1}}$ is a graded subalgebra of $\mathrm{pl}(\mathfrak{g})$. We call the elements of $\mathcal{D}(\mathfrak{g})$ *superderivations* of $\mathfrak{g}$. Thus, *the Lie superalgebra of the superderivations of* $\mathfrak{g}$ is denoted by $\mathcal{D}(\mathfrak{g})$.

We define the operator $\mathrm{ad}(X)$ following Lie algebras theory. If X is an homogeneous element of degree 0, $\mathrm{ad}(X)$ is an even superderivation and if X is an homogeneous element of degree 1, $\mathrm{ad}(X)$ is an odd superderivation. Thus, $\mathcal{A}\mathrm{d}(\mathfrak{g})$ is the space of all inner superderivations of $\mathfrak{g}$ and $\mathcal{A}d(\mathfrak{g}) = \mathcal{A}\mathrm{d}(\mathfrak{g}_{\bar{0}}) \oplus \mathcal{A}\mathrm{d}(\mathfrak{g}_{\bar{1}})$.

We consider $\mathfrak{g}$ as a $\mathfrak{g}$-module by the adjoint representation. We can identify $Z^1(\mathfrak{g}, \mathfrak{g}) = Z^1_0(\mathfrak{g}, \mathfrak{g}) \oplus Z^1_1(\mathfrak{g}, \mathfrak{g})$ (the space of the cocicles of degree 1) with the space of superderivations of $\mathfrak{g}$ $(\mathcal{D}(\mathfrak{g}) = \mathcal{D}_{\bar{0}}(\mathfrak{g}) \oplus \mathcal{D}_{\bar{1}}(\mathfrak{g}))$. Moreover, $B^1(\mathfrak{g}, \mathfrak{g}) = B^1_0(\mathfrak{g}, \mathfrak{g}) \oplus B^1_1(\mathfrak{g}, \mathfrak{g})$ (the space of the coboundaries of degree 1) can be identified with the space of inner superderivations $(\mathcal{A}\mathrm{d}(\mathfrak{g}) = \mathcal{A}\mathrm{d}(\mathfrak{g}_{\bar{0}}) \oplus \mathcal{A}\mathrm{d}(\mathfrak{g}_{\bar{1}}))$. In particular, $Z^1_0(\mathfrak{g}, \mathfrak{g}) = \mathcal{D}_{\bar{0}}(\mathfrak{g})$ and $B^1_0(\mathfrak{g}, \mathfrak{g}) = \mathcal{A}\mathrm{d}(\mathfrak{g}_{\bar{0}})$.

Thus, the first space of cohomology, $\mathcal{H}^1(\mathfrak{g}, \mathfrak{g})$, can be identified with the space of exterior superderivations $\mathcal{O}\mathrm{ut}(\mathfrak{g})$ (see [4], [12]), that is

$$\mathcal{O}\mathrm{ut}(\mathfrak{g}) = \frac{\mathcal{D}_{\bar{0}}(\mathfrak{g})}{\mathcal{A}\mathrm{d}(\mathfrak{g}_{\bar{0}})} \oplus \frac{\mathcal{D}_{\bar{1}}(\mathfrak{g})}{\mathcal{A}\mathrm{d}(\mathfrak{g}_{\bar{1}})}.$$

If we call $\mathcal{O}(\mathfrak{g})$ the orbit of the law of $\mathfrak{g}$ in the variety of laws of Lie superalgebras, $\mathcal{L}^{n+m}$, we have that

$$\dim(\mathcal{O}(\mathfrak{g})) = \dim(B^2_0(\mathfrak{g}, \mathfrak{g})) = n^2 + m^2 - \dim(\mathcal{D}_{\bar{0}}(\mathfrak{g})),$$

where $B^2_0(\mathfrak{g}, \mathfrak{g})$ is the space of 2-coboundaries.

3. Heisenberg Superalgebras

3.1. *Heisenberg algebras*

DEFINITION 3.1.1 *A Lie algebra $\mathcal{H}_k$ is called Heisenberg algebra of dimension $n = 2k + 1$ if it is defined in the basis $\{e_1, \ldots, e_{2k+1}\}$ by*

$$[e_{2i-1}, e_{2i}] = e_{2k+1}, \quad 1 \leq i \leq k.$$

(The undefined brackets are null).

REMARK 3.1.2 *All Heisenberg algebras verify that*

$$\mathcal{C}^1(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g}) \ \ and \ \ \dim(\mathcal{Z}(\mathfrak{g})) = 1.$$

This is a characteristic of Heisenberg algebras.

LEMMA 3.1.3 *Every Lie algebra satisfying $\mathcal{C}^1(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g})$ and $\dim(\mathcal{Z}(\mathfrak{g})) = 1$ is isomorphic to the Heisenberg algebra.*

3.2. *Classification of Heisenberg superalgebras*

The superalgebras $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ with $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$ satisfy the two conditions to be a Lie algebra that is: skew-symmetry and Jacobi identity. Thus, the structure of Heisenberg in these cases are similar to Heisenberg algebras. So, in this paper, we will focus the study of Lie superalgebras with $[\mathfrak{g}_1, \mathfrak{g}_1]$ non null.

DEFINITION 3.2.1 *A nilpotent superalgebra $\mathfrak{g}$ ($\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$) is called Heisenberg superalgebra (HSA) if $[\mathfrak{g}_1, \mathfrak{g}_1] \neq 0$, and*

$$\mathcal{C}^1(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g})$$
$$\dim(\mathcal{Z}(\mathfrak{g})) = 1.$$

The complete list of Lie superalgebras ($\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$) with $\mathcal{C}^1(\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}) = \mathcal{Z}(\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}})$ and $\dim(\mathcal{Z}(\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}})) = 1$ is derived from the following results.

LEMMA 3.2.2 *Let $\mathfrak{g}$ be a HSA, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, then $\mathfrak{g}$ admits a basis such that the only non null products as follows:*

$$\begin{cases} [X_i, X_j] = \epsilon_{ij} Z & 1 \leq i < j \leq n-1 \\ [Y_k, Y_l] = \delta_{kl} Z & 1 \leq k \leq l \leq m, \quad \exists \delta_{kl} \neq 0 \, \textit{for some } k, l \end{cases}$$

with $\epsilon_{ij}, \delta_{kl} \in \{0, 1\}$, $\mathcal{C}^1(\mathfrak{g}) = \langle Z \rangle \oplus \{0\}$ and $\mathcal{Z}(\mathfrak{g}) = \langle Z \rangle$.

Proof Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ a HSA we have that $\mathcal{C}^1(\mathfrak{g}) = \mathcal{C}^1(\mathfrak{g})_{\bar{0}} \oplus \mathcal{C}^1(\mathfrak{g})_{\bar{1}}$, $\mathcal{C}^1(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g})$, $\dim(\mathcal{Z}(\mathfrak{g})) = 1$.

As $\mathcal{C}^1(\mathfrak{g}) = \langle Z \rangle \oplus \langle 0 \rangle$, it is easy to prove that $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = 0$ and the characteristic sequence of Lie superalgebra is $(\dots | 1, 1, 1, \dots, 1)$.

The only non null products are the following:

$$\begin{cases} [X_i, X_j] = \epsilon_{ij} Z & 1 \leq i < j \leq n-1 \\ [Y_k, Y_l] = \delta_{kl} Z & 1 \leq k \leq l \leq m \quad \exists \delta_{kl} \neq 0 \text{ for some } k, l \end{cases}$$

with $\epsilon_{ij}, \delta_{kl} \in \{0, 1\}$.

If $\epsilon_{ij} = 0$ for all i, j, we obtain the degenerate case. Then, there exists $\epsilon_{ij} \neq 0$ for some i, j; and $\delta_{kl} \neq 0$ for some k, l. ■

COROLLARY 3.2.3 *Let $\mathfrak{g}$ be under the conditions of the Theorem above, then we have $\mathfrak{g}_{\bar{0}}$ as a Lie algebra with $\mathcal{C}^1(\mathfrak{g}_{\bar{0}}) = \mathcal{Z}(\mathfrak{g}_{\bar{0}})$ and $\dim(\mathcal{Z}(\mathfrak{g}_{\bar{0}})) = 1$.*

THEOREM 3.2.4 (Classification of HSA) *Every HSA, $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \in \mathcal{L}^{n+m}$, $n = 2r+1$, is isomorphic to the following Lie superalgebra. That can be expressed in a suitable basis $\{X_1, \dots, X_{2r+1}, Y_1, \dots, Y_m\}$ by*

$$\begin{cases} [X_{2i-1}, X_{2i}] = X_{2r+1}, & 1 \leq i \leq r \\ [Y_j, Y_j] = X_{2r+1}, & 1 \leq j \leq m. \end{cases}$$

REMARK 3.2.5 *For each pair of dimensions, n and m (n must be odd), there exists only one Lie superalgebra, up to isomorphism, that verifies the same algebraic conditions as the Heisenberg algebra. That is, for each pair of dimensions, there exists a HSA.*

Proof of the Theorem Let $\mathfrak{g}$ be a HSA, by the previous results we have that

$$[X_{2i-1}, X_{2i}] = X_{2r+1} \quad 1 \leq i \leq r$$
$$[Y_j, Y_l] = \delta_{jl} X_{2r+1} \quad 1 \leq j \leq l \leq m$$

with $\delta_{jl} \neq 0$ for some j, l.

For each j, $1 \leq j \leq m$, there exists l ($l \geq j$) such that $\delta_{jl} \neq 0$. By finite induction, we prove that $\delta_{jj} = 1$, with $1 \leq j \leq m$ and $\delta_{jl} = 0 \; \forall l > j$.

The change of basis $Y_1' = 1/\sqrt{\delta_{11}} Y_1$ if $\delta_{11} \neq 0$, or the change $Y_1' = (1 - \delta_{ll})/(2\delta_{1l}) Y_1 + Y_l$ if $\delta_{11} = 0$ and $\delta_{1l} \neq 0$ permits to suppose $\delta_{11} = 1$. If $\delta_{1l} = 0$, $\forall l > 1$, we have arrived at the result and if $\delta_{1l} \neq 0$ for any l, the change $Y_1' = Y_1,\ Y_l' = -\delta_{1l} Y_1 + Y_l$ for $l > 1$ permits to obtain the result.

In the rest of the cases, the situation is analogous. ■

4. Cohomology

In this section, we study the space of superderivations of a HSA. First, we compute the dimension of the above spaces with the aid of the following program. Finally, and as application, some geometrical properties are studied (the first cohomology space).

4.1. *Effective computing*

The following program allows us to obtain the dimension of the space of superalgebras of superderivations. We compute this dimension for a HSA in particular dimension. Next, by an induction processing, we obtain the dimension for a HSA in arbitrary finite dimension. The program allows us to obtain the even superderivations. For the computing of odd superderivations, we use a similar program as the one presented here, but making some modifications. For example, the conditions of odd superderivations.

Step 1 This step introduces the conditions of the HSA.

```
r1 = 1;
dim = 2r1 + 1;
dim1 = 2;
basemu0 = Table[x[i], {i, 1, dim}];
basemu1 = Table[y[j], {j, 1, dim1}];

mu0[0, x_] := 0;
mu0[x_, 0] := 0;
mu0[x_, x_] := 0;
mu0[x_, y_] := Simplify[-mu0[y, x]] /; OrderedQ[{y, x}];
mu0[x_ + y_, z_] := Simplify[mu0[x, z] + mu0[y, z]];
mu0[z_, x_ + y_] := Simplify[mu0[z, x] + mu0[z, y]];
mu0[x_, a_ y_] := a mu0[x, y];
mu0[a_ x_, y_] := a mu0[x, y];

mu1[0, x_] := 0;
mu1[x_, 0] := 0;
mu1[x_, y_] := Simplify[mu1[y, x]] /; OrderedQ[{y, x}];
mu1[x_ + y_, z_] := Simplify[mu1[x, z] + mu1[y, z]];
mu1[z_, x_ + y_] := Simplify[mu1[z, x] + mu1[z, y]];
mu1[x_, a_ y_] := a mu1[x, y];
mu1[a_ x_, y_] := a mu1[x, y];

mu2[0, x_] := 0;
mu2[x_, 0] := 0;
mu2[x_, y_] := Simplify[-mu2[y, x]] /; OrderedQ[{y, x}];
mu2[x_ + y_, z_] := Simplify[mu2[x, z] + mu2[y, z]];
mu2[z_, x_ + y_] := Simplify[mu2[z, x] + mu2[z, y]];
mu2[x_, a_ y_] := a mu2[x, y];
mu2[a_ x_, y_] := a mu2[x, y];
```

```
For[i = 1, i <= r1, i++, mu0[x[2i - 1], x[2i]] = x[2r1 + 1]];
   mu0[x[i_], x[j_]] := 0;
For[i = 1, i <= 2r1 + 1, i++,For[j = 1, j <= dim1, j++,
      mu2[x[i], y[j]] = 0]];
For[i = 1, i <= dim1, i++, mu1[y[i], y[i]] = x[2r1 + 1]];
    mu1[y[j_], y[i_]] = 0;
```

Step 2 We introduce the graduation we will use in the computing of superderivations. Next, we also introduce the conditions of even superderivations.

```
g[j_] := Which[1 <= j <= r1, x[2j - 1], r1 + 2 <= j <= 2r1 + 1,
      x[4r1 + 4 - 2j], True, 0];
g[r1 + 1] := basemu1;
g[2r1 + 2] := x[2r1 + 1];

For[i = 1, i <= 2r1 + 2, i++, Print["g[", i, "]->", g[i]]];

der0[i_Integer, j_Integer, k_Integer] := Collect[d[i][mu0[x[j], x[k]]]
     - mu0[d[i][x[j]], x[k]] - mu0[x[j], d[i][x[k]]],basemu0]

der1[i_Integer, j_Integer, k_Integer] := Collect[d[i][mu1[y[j], y[k]]]
     - mu1[d[i][y[j]], y[k]] - mu1[y[j], d[i][y[k]]],basemu0]

der2[i_Integer, j_Integer, k_Integer] := Collect[d[i][mu2[x[j], y[k]]]
     - mu2[d[i][x[j]], y[k]] - mu2[x[j], d[i][y[k]]],basemu1]

For[i = -dim, i <= dim, i++, For[j = 1, j <= 2r1 + 2, j++,
  Which[i + j == r1 + 1, d[i][g[j]] =Sum_{k = 1}^{dim1}b[i, k] y[k],
   i + j > 2r1 + 2, d[i][g[j]] = 0, True, d[i][g[j]] = a[i, j] g[i + j]]]];

For[i = -dim, i <= dim, i++,Which[g[j] == 0, d[i][g[j]] = 0]];

For[i = -dim, i <= dim, i++,  For[s = 1, s <= dim1, s++,
 Which[i + r1 + 1 == r1 + 1, d[i][y[s]] =Sum_{k = 1}^{dim1} b[i, k, s] y[k],
  i + r1 + 1 > 2r1 + 2, d[i][y[s]] = 0, True, d[i][y[s]] = 0]

For[i = -dim, i <= dim, i++,For[j = 1, j <= 2r1 + 2, j++,
 For[s = 1, s <= dim1, s++,Which[i != 0, d[i][y[s]] = 0, True, d[0][y[s]]
  Sum_{k = 1}^{dim1} b[0, k, s] y[k]]]]];

For[i = -dim, i <= dim, i++,For[j = 1, j <= r1, j++, Which[i + j!=
r1 + 1, d[i][g[j]] = a[i, j] g[i + j], i + j == r1 + 1,
d[i][g[j]]= 0]]];

For[i = -dim, i <= dim, i++,For[j = r1 + 2, j <= 2r1 + 2, j++,
 Which[i + j != r1 + 1, d[i][g[j]] = a[i, j] g[i + j], i + j == r1 + 1,
 d[i][g[j]] = 0]]];
```

Step 3 We compute the conditions amongst the above coefficients, and we obtain the free coefficients.

```
For[i = -dim, i
<= dim, i++,For[j = 1, j <dim, j++, For[k = j + 1, k <= dim,
k++,deriv0[i_, j_, k_] := Coefficient[der0[i, j, k], basemu0]]]]

For[i = -dim, i <= dim, i++,For[j = 1, j <= dim1, j++, For[k = 1, k
<= dim1, k++,deriv1[i_, j_, k_] := Coefficient[der1[i, j, k],
basemu0]]]]
```

```
For[i = -dim, i <=
dim, i++,For[j = 1, j <= dim, j++, For[k = 1, k <= dim1,
k++,deriv2[i_, j_, k_] := Coefficient[der2[i, j, k], basemu1]]]]

For[v = -dim, v <= dim, v++,funcion0[v_] := Module[{j, k, Lec}, Lec = {};
 For [j = 1, j <= dim, j++, For[k = j + 1, k <= dim , k++,
  Lec = Union[Lec, deriv0[v, j, k]];];];Lec]];

For[v = -dim, v <= dim, v++,funcion1[v_] := Module[{j, k, Lec}, Lec = {};
 For [j = 1, j <= dim1, j++,For[k = 1, k <= dim1 , k++,
  Lec = Union[Lec, deriv1[v, j, k]];];];Lec]];

For[v = -dim, v <= dim, v++,funcion2[v_] := Module[{j, k, Lec}, Lec = {};
 For [j = 1, j <= dim, j++,For[k = 1, k <= dim1 , k++,
  Lec = Union[Lec, deriv2[v, j, k]];];];Lec]];

For[v = -dim, v <= dim, v++,funcion[v_] := Union[funcion0[v],
  funcion1[v], funcion2[v]]];For[v = -dim, v <= dim, v++,
    sol[v_] := Solve[funcion[v] == 0]];

Off[Solve::"svars"]; Off[General::"stop"];

For[i = -dim, i <= dim, i++,For[j = 1, j <= r1, j++,
 Which[i + j != r1 + 1, der[i][g[j]] = d[i][ g[j]] /. sol[i][[1]],
  i + j == r1 + 1, der[i][g[j]] = 0]]];

For[i = -dim, i <= dim, i++,For[j = r1 + 2, j <= 2r1 + 2, j++,
 Which[i + j != r1 + 1, der[i][g[j]] = d[i] [g[j]] /. sol[i][[1]],
  i + j == r1 + 1, der[i][g[j]] = 0]]];

For[i = -dim, i <= dim, i++,Which[g[j] == 0, der[i][g[j]] = 0]];

For[i = -dim, i <= dim, i++,For[j = 1, j <= 2r1 + 2, j++,
 For[s = 1, s <= dim1, s++,Which[i != 0, der[i][y[s]] = 0, True,
  der[0][y[s]] = d[0][y[s]] /. sol[0][[1]]]]]];
Print["===================="]

Module[{i, j, k},For[i = -dim, i <= dim, i++,For[j = 1, j <= dim, j++,
 Print["d", i, "(x[", j, "])=", der[i][x[j]]]]; For[
   k = 1, k <= dim1, k++, Print["d", i, "(y[", k, "])=", der[i][y[k]]]]]];

Module[{u}, For[u = -dim, u <= dim, u++,parameterder[u_] :
Select[Select[Variables[Join[Table[der[u][x[i]], {i, 1, dim}],
  Table[der[u][y[j]], {j, 1, dim1}]]], FreeQ[#, x] &],FreeQ[#, y] &]]];

For[p = -dim, p <= dim, p++,dimder[p_Integer] := Length[parameterder[p]]];
Print["===================="]

Module[{t}, For[t = -dim, t <= dim, t++, Print["dimension(d", t,
")--> ", dimder[t]]]]; Print["===================="]

dimensioneven =Sum_{u=-dim}^{dim}dimder[u];
Print["DIMENSIONEVEN-----> ", dimensioneven]
```

The program allows us to compute the dimension of the space of superderivations for the mentioned superalgebras in concrete dimensions. These results lead to conjecture the structure of such spaces of superderivations in generic dimension.

4.2. *Computing of superderivations of HSA*

In this section, we compute the dimension of the space of superderivations of HSA. The law of a HSA is the following:

$$\begin{cases} [X_{2i-1}, X_{2i}] = X_{2r+1}, & 1 \leq i \leq r \\ [Y_j, Y_j] = X_{2r+1}, & 1 \leq j \leq m. \end{cases}$$

THEOREM 4.2.1 *Let* $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ *HSA of dimension* $(2r+1, m)$ $(\dim(\mathfrak{g}_{\overline{0}}) = 2r+1$ *and* $\dim(\mathfrak{g}_{\overline{1}}) = m)$. *We have that*

$$\dim(\mathcal{D}_{\overline{0}}(\mathfrak{g})) = 2r^2 + 3r + \frac{m^2 - m + 2}{2}$$

and

$$\dim(\mathcal{D}_{\overline{1}}(\mathfrak{g})) = (2r+1)m$$

Proof We consider the following graduation

$$\begin{aligned} g_i &= \langle X_{2i-1} \rangle \quad \text{if } 1 \leq i \leq r \\ g_{r+1} &= \langle Y_1, Y_2, \ldots, Y_m \rangle \\ g_i &= \langle X_{4r+4-2i} \rangle \quad \text{if } r+2 \leq i \leq 2r+1 \\ g_{2r+2} &= \langle X_{2r+1} \rangle. \end{aligned}$$

First, we compute the even superderivations.

If $d \in (\mathcal{D}_{\overline{0}}(\mathfrak{g}))$, we have that $d(\mathfrak{g}_{\overline{\alpha}}) \subset \mathfrak{g}_{\overline{\alpha}}$ with $\alpha \in \mathbf{Z}_2$, and if $d \in \mathcal{D}(\mathfrak{g})$,

$$d = \sum_{-2r-1}^{2r+1} d_i$$

where $d_i(g_j) \subset g_{i+j}$.

Thus, we compute the subspaces d_i, $-2r-1 \leq i \leq 2r+1$. The computing of the dimension of d_i is easy but laborious.

d_0, we have that

$$\begin{aligned} d_0(X_{2i-1}) &= a(0,i)X_i \quad 1 \leq i \leq r, \\ d_0(Y_j) &= \sum_{k=1}^{m} b(0,k,j)Y_k \quad 1 \leq j \leq m, \\ d_0(X_{2i}) &= a(0, 2r+2-i) \quad 1 \leq i \leq r, \\ d_0(X_{2r+1}) &= a(0, 2r+2)X_{2r+1}. \end{aligned}$$

We compute all the conditions of even superderivations, and we obtain the following restrictions:

$$\begin{aligned} a(0, 2r+2) &= a(0,i) + a(0, 2r+2-i), \quad 1 \leq i \leq r \\ a(0, 2r+2) &= 2b(0,j,j), \quad 1 \leq j \leq m \\ b(0,j,i) + b(0,i,j) &= 0, \quad 1 \leq i < j \leq m. \end{aligned}$$

Thus, we deduce that $\dim(d_0) = r + (m^2 - m + 2)/2$.

In this process, it is necessary to use the program above in order to prove the results. The other cases are more complicate. It is necessary to differentiate some values of r.

Finally, we obtain that

$$\begin{aligned}
\dim(d_{-2r-1}) &= 0,\\
\dim(d_i) &= \left\lfloor \frac{2r+2+i}{2} \right\rfloor, \quad -2r \le i \le -r-1,\\
\dim(d_i) &= r+i+\left\lfloor \frac{-i}{2} \right\rfloor, \quad -r \le i \le -1,\\
\dim(d_0) &= r+\frac{m^2-m+2}{2},\\
\dim(d_i) &= r-i+1+\left\lfloor \frac{i}{2} \right\rfloor, \quad 1 \le i \le r,\\
\dim(d_{r+1}) &= \left\lfloor \frac{r+1}{2} \right\rfloor,\\
\dim(d_i) &= \left\lfloor \frac{2r+4-i}{2} \right\rfloor, \quad r+2 \le i \le 2r\\
\dim(d_{2r+1}) &= 1.
\end{aligned}$$

Thus it is easy to prove that $\dim(\mathcal{D}_{\overline{0}}(\mathfrak{g})) = 2r^2+3r+(m^2-m+2)/2$.

Next, we compute the odd superderivations. In this case, the computing is easier.

If $d \in \mathcal{D}_{\overline{1}}(\mathfrak{g})$, we have that $d(\mathfrak{g}_{\overline{0}}) \subset \mathfrak{g}_{\overline{1}}$ and $d(\mathfrak{g}_{\overline{1}}) \subset \mathfrak{g}_{\overline{0}}$. Analogous to case even, if $d \in \mathcal{D}(\mathfrak{g})$

$$d = \sum_{-2r-1}^{2r+1} d_i,$$

where $d_i(g_j) \subset g_{i+j}$.

We compute all the subspaces d_i, and we have that

$$\begin{aligned}
\dim(d_{-2r-1}) &= 0,\\
\dim(d_i) &= 0, \quad -2r \le i \le -r-1,\\
\dim(d_{-r-1}) &= 0,\\
\dim(d_i) &= m, \quad -r \le i \le -1,\\
\dim(d_0) &= 0,\\
\dim(d_i) &= m, \quad 1 \le i \le r,\\
\dim(d_{r+1}) &= m,\\
\dim(d_i) &= 0 \quad r+2 \le i \le 2r,\\
\dim(d_{2r+1}) &= 0.
\end{aligned}$$

Thus it is easy to prove that $\dim(\mathcal{D}_{\overline{1}}(\mathfrak{g})) = (2r+1)m$. ■

COROLLARY 4.2.2 *Let $\mathfrak{g}$ be a HSA. We have that*

$$\dim(\mathcal{D}(\mathfrak{g})) = 2r^2+3r+\frac{m^2-m+2}{2}+(2r+1)m.$$

REMARK 4.2.3 *Even derivations of the HSA can also be found by using the description of derivations of the Heisenberg algebra* [9,13].

4.3. *Application*

In this section, we compute some geometrical properties as application of the space of superderivations.

LEMMA 4.3.1 *Let $\mathfrak{g}$ be a HSA, then we obtain that*

$$\dim(\mathcal{A}\mathrm{d}(\mathfrak{g}_{\bar{0}})) = 2r \quad \textit{and} \quad \dim(\mathcal{A}\mathrm{d}(\mathfrak{g}_{\bar{1}})) = m.$$

COROLLARY 4.3.2 *Let $\mathfrak{g}$ be a HSA. We obtain that*

$$\dim(B^1(\mathfrak{g}, \mathfrak{g})) = 2r + m,$$
$$\dim(\mathcal{H}^1(\mathfrak{g}, \mathfrak{g})) = 2r^2 + r + \frac{m^2 - m + 2}{2} + 2rm.$$

Proof Remember that

$$\dim(B^1(\mathfrak{g}, \mathfrak{g})) = \dim(\mathcal{A}\mathrm{d}(\mathfrak{g}_{\bar{0}})) + \dim(\mathcal{A}\mathrm{d}(\mathfrak{g}_{\bar{1}}))$$

and

$$\dim(\mathcal{H}^1(\mathfrak{g}, \mathfrak{g})) = \dim(\mathcal{D}_{\bar{0}}(\mathfrak{g})) - \dim(\mathcal{A}\mathrm{d}(\mathfrak{g}_{\bar{0}})) + \dim(\mathcal{D}_{\bar{1}}(\mathfrak{g})) - \dim(\mathcal{A}\mathrm{d}(\mathfrak{g}_{\bar{1}})) \qquad \blacksquare$$

COROLLARY 4.3.3 *Let $\mathfrak{g}$ be a HSA. We obtain that*

$$\dim(\mathcal{O}(\mathfrak{g})) = \dim(B_0^2(\mathfrak{g}, \mathfrak{g})) = 2r^2 + r + \frac{m^2 + m}{2} + 2rm.$$

Proof Remember that

$$\dim(\mathcal{O}(\mathfrak{g})) = \dim(B_0^2(\mathfrak{g}, \mathfrak{g})) = (2r + 1)^2 + m^2 - \dim(\mathcal{D}_{\bar{0}}(\mathfrak{g})) \qquad \blacksquare$$

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