Complex cyclic Leibniz superalgebras

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Abstract
Since Loday introduction of Leibniz algebras as a generalisation of Lie algebras, many results of the theory of Lie algebras have been extended to Leibniz algebras. Cyclic Leibniz algebras, which are generated by one element, have no equivalent into Lie algebras, though. This fact provides cyclic Leibniz algebras with important properties. Throughout the present paper we extend the concept of cyclic to Leibniz superalgebras, obtaining then the definition, as long as the description and classification of finite-dimensional complex cyclic Leibniz superalgebras. Furthermore, we prove that any cyclic Leibniz superalgebra can be obtained by means of infinitesimal deformations of the null-filiform Leibniz superalgebra. We also obtain a description of irreducible components in the variety of Leibniz algebras and superalgebras.

Keywords Leibniz superalgebra · Cyclic Leibniz superalgebra · Null-filiform superalgebra · Infinitesimal deformation · Irreducible component

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1 Introduction

Leibniz algebras were introduced in the 90's by Loday as a generalization of Lie algebras and since then many results of the theory of Lie algebras have been extended to Leibniz algebras [3,4,7,10,14,15]. The notion of Leibniz superalgebras was introduced in [1] (they were also considered under the name of graded Leibniz algebras in [12]). Furthermore, Leibniz superalgebras are generalizations of Leibniz algebras and, on the other hand, they naturally generalize well-known Lie superalgebras. It should be noted that the study of Leibniz superalgebras is much more complicated since the graded anti-symmetric identity does not hold in Leibniz superalgebras.

The most interesting examples of non-Lie Leibniz algebras are cyclic Leibniz algebras, those Leibniz algebras generated by one element and do not appear in Lie algebras, except one-dimensional abelian algebra. In nilpotent algebras case such algebras are called null-filiform Leibniz algebras [2], while solvable one-generated Leibniz algebras are called cyclic algebras [16].

To the study of Leibniz superalgebras have been devoted a few works, we just mention the papers [6,9] (and reference therein), where the description of nilpotent Leibniz superalgebras of nilindex equal to the dimension of the superalgebra is obtained. In this paper we extend the results on one-generated Leibniz superalgebras (which are always solvable). More precisely, we extend the concept of cyclic Leibniz algebras to Leibniz superalgebras, obtaining then the definition, as long as the description and classification of finite-dimensional complex cyclic Leibniz superalgebras. In fact, we showed that any cyclic Leibniz superalgebra is an infinitesimal deformation of null-filiform superalgebra.

On the other hand, let us consider V an $(n+m)$-dimensional vector space over a field $\mathbb{F}$. Then, the bilinear maps $V \times V \to V$ form an $(n+m)^3$-dimensional affine space. Recall that a set is called irreducible if it cannot be represented as a union of two nontrivial closed subsets (in the sense of Zariski topology), otherwise it is called reducible. The maximal irreducible closed subset of a variety is called an irreducible component. Moreover, from algebraic geometry we know that an algebraic variety is the union of its irreducible components and that closures of open irreducible subsets produce irreducible components. Therefore, for the description of a variety it is very important to find all open sets. Thus, throughout this paper we find an irreducible component of the variety of $(n+m)$-dimensional Leibniz superalgebras, which is a closure of union of orbits of cyclic Leibniz superalgebras.

Henceforth, we will consider vector spaces and algebras over the field of complex numbers $\mathbb{C}$.

2 Basic notions and preliminaries results

Let us start with some basics regarding Leibniz superalgebras. We suppose that the reader is familiarised with the basics of Leibniz algebras.

Definition 2.1 [1]. A $\mathbb{Z}_2$-graded vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$ over a field $\mathbb{F}$ is called a Leibniz superalgebra if it is equipped with a product $[-,-]$ which satisfies the following conditions:

$$[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta\ (mod\ 2)}\ for\ all\ \alpha, \beta \in \mathbb{Z}_2$$
$$[x,[y,z]] = [[x,y],z] - (-1)^{\alpha\beta}[[x,z],y] -\ graded\ Leibniz\ identity$$

for all $x \in L$, $y \in L_\alpha$, $z \in L_\beta$, $\alpha, \beta \in \mathbb{Z}_2$.

Note that if a Leibniz superalgebra L satisfies the identity $[x, y] = -(-1)^{\alpha\beta}[y, x]$ for all $x \in L_\alpha$, $y \in L_\beta$, then the graded Leibniz identity is reduced to the graded Lie identity. Therefore Leibniz superalgebras are a generalization of Lie superalgebras. Note that as usual $L = L_{\bar{0}}$ is a Leibniz algebra. Hence, if we replace $L_{\bar{1}}$ by $\{0\}$ in the next results and definitions we obtain the equivalent results for Leibniz algebras. Recall that isomorphisms between Leibniz superalgebras are assumed to be consistent with the $\mathbb{Z}_2$-graduation.

We call a $\mathbb{Z}_2$-graded vector space $M = M_0 \oplus M_1$ a module over a Leibniz superalgebra L if there are two bilinear maps:

$$[-,-] : L \times M \to M \qquad \text{and} \qquad [-,-] : M \times L \to M$$

satisfying the following three axioms

$$\begin{aligned}
[m,[x,y]] &= [[m,x],y] - (-1)^{|x||y|}[[m,y],x],\\
[x,[m,y]] &= [[x,m],y] - (-1)^{|y||m|}[[x,y],m],\\
[x,[y,m]] &= [[x,y],m] - (-1)^{|m||y|}[[x,m],y],
\end{aligned}$$

for any $m \in M_{|m|}$, $x \in L_{|x|}$, $y \in L_{|y|}$.

Given a Leibniz superalgebra L, let $C^n(L, M)$ be the space of all $\mathbb{F}$-linear homogeneous mappings $L^{\otimes n} \to M$, $n \geq 0$ and $C^0(L, M) = M$. This space is graded by $C^n(L, M) = C^n_0(L, M) \oplus C^n_1(L, M)$ with

$$C^n_p(L, M) = \bigoplus_{\substack{n_0+n_1=n \\ n_1+r \equiv p\ (mod\ 2)}} Hom(L_0^{\otimes n_0} \otimes L_1^{\otimes n_1}, M_r)$$

Let $d^n : C^n(L, M) \to C^{n+1}(L, M)$ be an $\mathbb{F}$-homomorphism defined by

$$\begin{aligned}
(d^n f)(x_1, \ldots, x_{n+1}) &:= [x_1, f(x_2, \ldots, x_{n+1})]\\
&+ \sum_{i=2}^{n+1} (-1)^{i+|x_i|(|f|+|x_{i+1}|+\cdots+|x_{n+1}|)}[f(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}), x_i]\\
&+ \sum_{1\leq i<j\leq n+1} (-1)^{j+1+|x_j|(|x_{i+1}|+\cdots+|x_{j-1}|)} f(x_1, \ldots, x_{i-1}, [x_i, x_j],\\
&x_{i+1}, \ldots, \widehat{x_j}, \ldots, x_{n+1}),
\end{aligned}$$

where $f \in C^n(L, M)$ and $x_i \in L$.

Since the differential operator $d = \sum_{i \geq 0} d^i$ satisfies the property $d \circ d = 0$, the cohomology group is well defined and

$$HL_p^n(L, M) = ZL_p^n(L, M)/BL_p^n(L, M), \;\; p \in \{0, 1\}$$

where the elements $ZL_0^n(L, M)$ $(BL_0^n(L, M))$ and $ZL_1^n(L, M)$ $(BL_1^n(L, M))$ are called *even n-cocycles* (*even n-coboundaries*) and *odd n-cocycles* (*odd n-coboundaries*), respectively.

Let us remark that formula for d^n can be obtained from the differential operator for color Leibniz algebras [8].

Note that the space $ZL^1(L, L)$ consists of derivations of the superalgebra L, which are defined by the condition:

$$d([x, y]) = (-1)^{|d||y|}[d(x), y] + [x, d(y)].$$

If we denote by R_x the right multiplication operator, i.e., $R_x : L \to L$, then due to graded Leibniz identity it can be seen that R_x is a derivation. Note that the graded Leibniz identity can be expressed in the following form: $R_{[x,y]} = R_y R_x - (-1)^{\alpha\beta} R_x R_y$ where $x \in L_\alpha$, $y \in L_\beta$. We denote by $R(L)$ the set of all right multiplication operators. It is not difficult to check that $R(L)$ with the multiplication defined as follows $< R_a, R_b >:= R_a R_b - (-1)^{\alpha\beta} R_b R_a$ for $R_a \in R(L)_\alpha$, $R_b \in R(L)_\beta$, becomes a Lie superalgebra.

For a Leibniz superalgebra $L = L_0 \oplus L_1$ we consider the set $Ann_r(L)$, $Ann_r(L) = \{X \in L \;:\; [L, X] = 0\}$ which is called the right annihilator of L. It is easy to see that $Ann_r(L)$ is a two-sided ideal of L and $[X, X] \in Ann_r(L)$ for any $X \in L_0$, this notion is consistent with the right annihilator in Leibniz algebras. If we consider $I = ideal < [X, Y] + (-1)^{\alpha\beta}[Y, X] \;:\; X \in L_\alpha, Y \in L_\beta >$, then $I \subseteq Ann_r(L)$.

A deformation of a Leibniz superalgebra L is a one-parameter family L_t of Leibniz superalgebras with the bracket

$$\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \cdots ,$$

where φ_i are L-valued even 2-cochains, i.e., elements of $Hom(L \otimes L, L)_0 = C^2(L, L)_0$.

Two deformations L_t, L'_t with corresponding laws μ_t, μ'_t are *equivalent* if there exists a linear automorphism $f_t = id + f_1 t + f_2 t^2 + \cdots$ of L, where f_i are elements of $C^1(L, L)_0$ such that the following equation holds

$$\mu'_t(x, y) = f_t^{-1}(\mu_t(f_t(x), f_t(y))) \;\text{ for any } x, y \in L.$$

Applying the graded Leibniz identity for the superalgebras L_t implies that the 2-cochain φ_1 is an even 2-cocycle, i.e. $d^2\varphi_1 = 0$. If φ_1 vanishes identically, the first non vanishing φ_i will be a 2-cocycle.

If μ'_t is an equivalent deformation with cochains φ'_i, then $\varphi'_1 - \varphi_1 = d^1 f_1$, hence every equivalence class of deformations defines uniquely an element of $HL^2(L, L)_0$.

Note that the linear integrable deformation φ satisfies the condition

$$\varphi(x, \varphi(y, z)) - \varphi(\varphi(x, y), z) + (-1)^{|y||z|}\varphi(\varphi(x, z), y) = 0.$$

It should be noted that a Leibniz algebra is a superalgebra with trivial odd part and the definition of cohomology groups of Leibniz superalgebras extends the definition of cohomology groups of Leibniz algebras given in [13].

For a Leibniz superalgebra L consider the following central lower and derived series

$$\begin{aligned} &L^1 = L, \quad L^{k+1} = [L^k, L^1], \quad k \geq 1, \\ &L^{[1]} = 1, \quad L^{[s+1]} = [L^{[s]}, L^{[s]}], \quad s \geq 1. \end{aligned}$$

Definition 2.2 A Leibniz superalgebra L is said to be nilpotent, (respectively, solvable) if there exists $p \in \mathbb{N}$ such that $L^p = 0$ (respectively, $L^{[q]} = 0$).

Now we give the notion of null-filiform Leibniz superalgebra.

Definition 2.3 An n-dimensional Leibniz superalgebra is said to be null-filiform if $\dim L^i = n + 1 - i, \ 1 \leq i \leq n + 1$.

Similarly to the case of nilpotent Leibniz algebras [2] it is easy to check that a Leibniz superalgebra is null-filiform if and only if it is single-generated. Moreover, a null-filiform superalgebra has the maximal nilindex.

Theorem 2.1 [1] Let L be a null-filiform Leibniz superalgebra of the variety $Leib^{n,m}$. Then L is isomorphic to one of the following non-isomorphic superalgebras:

$$NF^n : [x_i, x_1] = x_{i+1}, \ 1 \leq i \leq n-1; \quad NF^{n,m} : \begin{cases} [y_i, y_1] = x_i, & 1 \leq i \leq n, \\ [x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \leq i \leq m-1, \\ [y_j, x_1] = y_{j+1}, & 1 \leq j \leq m-1, \\ [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n-1. \end{cases}$$

where $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_m\}$ are bases of the even and odd parts, respectively.

Remark 2.1 Note that the first superalgebra is a null-filiform Leibniz algebra [2] and from the assertion of Theorem 2.1 we conclude that in the case of non-trivial odd part of the null-filiform Leibniz superalgebra $NF^{n,m}$ there are two possibilities for m, namely, $m = n$ or $m = n + 1$.

Next, we recall some results about cyclic Leibniz algebras. We transfrom the results in [16] regarding cyclic left-Leibniz algebras to the cyclic right-Leibniz algebras. Thus, let $\mathbf{A}$ be an n-dimensional vector space over $\mathbb{C}$ containing a nonzero element a. Suppose we have a linear operator $T : \mathbf{A} \to \mathbf{A}$ such that a is a cyclic vector for T, i.e., such that $B = \{a, T(a), \ldots, T^{n-1}(a)\}$ is a basis of $\mathbf{A}$. Therefore $T^n(a) =$

$\alpha_1 a + \alpha_2 T(a) + \cdots + \alpha_n T^{n-1}(a)$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Let us define a product $\mathbf{A} \times \mathbb{C}a \to \mathbf{A}$ by $v(ca) = cT(v)$ for all $v \in \mathbf{A}$ and $c \in \mathbb{C}$, therefore T can be seen as the right multiplication operator by a (that is $T(a) = R_a$). For simplicity we denote $a^k := R_a^{k-1}(a)$.

Let us recall now the Leibniz identity $[[x, y], z] = [[x, z], y] + [x, [y, z]]$ which can be rewritten as $R_z([x, y]) = [R_z(x), y] + [x, R_z(y)]$. This latter condition is nothing but the condition for R_z to be a derivation. Therefore we have a general equivalence, i.e.

$$R_z \text{ is a derivation for all } z \in \mathbf{A} \text{ iff } \mathbf{A} \text{ is a Leibniz algebra}$$

Additionally, in our case (one-generated algebras) we have that right multiplication operator is a derivation (that is, $\mathbf{A}$ is a Leibniz algebra) if and only if $R_{a^j} = 0$ for all $j \geq 2$. In fact if $\mathbf{A}$ is a Leibniz algebra is a well-known fact that $R_{a^2} = 0$ and then by induction it can be easily check that $R_{a^j} = 0$ for all $j \geq 2$. For the converse, suppose $R_{a^j} = 0$ for all $j \geq 2$. Then, we consider an arbitray $x \in \mathbf{A}$, then x can be expressed as a linear combination of the basis vectors $\{a, a^2, \ldots, a^n\}$, that is: $x = c_1 a + c_2 a^2 + \cdots + a^n$. Since $R_{a^j} = 0$ for all $j \geq 2$, then $R_x = c_1 R_a$, so it suffices to prove that R_a is a derivation. The condition for being a derivation over the basis vectors remains:

$$R_a([a^i, a^j]) = [R_a(a^i), a^j] + [a^i, R_a(a^j)]$$

this equation trivially holds for all $j \geq 2$ as $R_{a^j} = 0$. For $j = 1$ and since $[a^i, a^2] = 0$ we get $a^{i+2} = a^{i+2}$ which trivially holds.

Let us now remark that the authors in [16] use a different characterization for (left) Leibniz algebras to have (left) multiplication operator as a derivation. Nevertheless, the classificaton Theorem remains valid and adapting the result to right Leibniz algebras we obtain

Theorem 2.2 [16] let $\mathbf{A}$ be an n-dimensional cyclic Leibniz algebra over $\mathbb{C}$. Then $\mathbf{A}$ is isomorphic to a Leibniz algebra spanned by $\{a, a^2, \ldots, a^n\}$ and with bracket products one of the following:

(1) $NF^n : [a^i, a] = a^{i+1}, \ 1 \leq i \leq n-1$

(2) $CNF^n : \begin{cases} [a^i, a] = a^{i+1}, & 1 \leq i \leq n-1 \\ [a^n, a] = a^n \end{cases}$

(3) $CNF^n(0, \ldots, 1, \alpha_{k+1}, \ldots, \alpha_n)$:

$$\begin{cases} [a^i, a] = a^{i+1}, & 1 \leq i \leq n-1 \\ [a^n, a] = a^k + \alpha_{k+1} a^{k+1} & \\ \quad + \cdots + \alpha_n a^n, & 2 \leq k \leq n-1, \quad (\alpha_{k+1}, \cdots, \alpha_n) \in \mathbb{C}^{n-k}/\sim \end{cases}$$

With the equivalence relation $\sim$ defined over $\mathbb{C}^d$, with $d = n - k$, as follows: $(\gamma_1, \gamma_2, \ldots, \gamma_d) \sim (\gamma_1', \gamma_2', \ldots, \gamma_d')$ if $(\gamma_1, \gamma_2, \ldots, \gamma_d) = (\omega^d \gamma_1', \omega^{d-1} \gamma_2', \ldots, \omega \gamma_d')$
for some $(d+1)$-th root of unity ω.

3 Complex cyclic Leibniz superalgebras

Let $\mathbf{A} = \mathbf{A}_{\bar{0}} \oplus \mathbf{A}_{\bar{1}}$ be a $(n+m)$-dimensional $\mathbb{Z}_2$-graded vector space, i.e. $dim(\mathbf{A}_{\bar{0}}) = n$ and $dim(\mathbf{A}_{\bar{1}}) = m$. Suppose we have a non-zero element a and a linear operator $T : \mathbf{A} \to \mathbf{A}$ such that $B = \{a, T(a), \ldots, T^{n+m-1}(a)\}$ is a homogeneous basis for $\mathbf{A}$. Then $T^{n+m}(a) = \alpha_1 a + \alpha_2 T(a) + \cdots + \alpha_{n+m-1} T^{n+m-1}(a)$ for some $\alpha_1, \ldots, \alpha_{n+m-1} \in \mathbb{C}$. We define the following product: $\mathbf{A} \times \mathbb{C}a \to \mathbf{A}$ by $v(ca) = cT(v)$ for all $v \in \mathbf{A}$ and $c \in \mathbb{C}$, therefore T can be seen as the right multiplication by a, i.e. R_a. We denote $R_a^{k-1}(a)$ by a^k. We intend to extend this product linearly such that right multiplication is a derivation, i.e. $\mathbf{A} = \mathbf{A}_{\bar{0}} \oplus \mathbf{A}_{\bar{1}}$ is a Leibniz superalgebra.

Note that R_a is a (super) derivation if and only if $\mathbf{A} = \mathbf{A}_{\bar{0}} \oplus \mathbf{A}_{\bar{1}}$ is a Leibniz superalgebra. In fact, the condition for being a derivation of a Leibniz superalgebra (for more details see [11]) is $d([x, y]) = (-1)^{|d||y|}[d(x), y] + [x, d(y)]$. Since the degree of R_z as homomorphism between $\mathbb{Z}_2$-graded vector spaces is the same as the degree of the homogeneous element z, that is $|R_z| = |z|$, then the condition for R_z to be a derivation is exactly $R_z([x, y]) = (-1)^{|z||y|}[R_z(x), y] + [x, R_z(y)]$. This last condition can be rewritten $[[x, y], z] = (-1)^{|z||y|}[[x, z], y] + [x, [y, z]]$ which is nothing but the graded Leibniz identity.

Remark 3.1 In the setting defined above, $a \in \mathbf{A}_{\bar{1}}$ in order not to have a degenerate case (Leibniz algebra case). In fact, if $| a |= 0$ then $| a^j |= 0$ for all $2 \leq j \leq n+m$, which means that all the basis vectors are even and $m = 0$, being $\mathbf{A} = \mathbf{A}_{\bar{0}}$ a Leibniz algebra. Note that cyclic Leibniz algebras have been already studied in [16].

Hereafter we consider a an odd element, and consequently a^j is even for j even and odd otherwise.

Remark 3.2 In the setting defined above, we have only two possibilites for m, that is, either $m = n$ or $m = n + 1$.

Proposition 3.1 *Let $\mathbf{A} = \mathbf{A}_{\bar{0}} \oplus \mathbf{A}_{\bar{1}}$ be a $(n+m)$-dimensional $\mathbb{Z}_2$-graded vector space such that we have a non-zero odd element a and a linear operator $T : \mathbf{A} \to \mathbf{A}$ verifying that $B = \{a, T(a), ..., T^{n+m-1}(a)\}$ is a homogeneous basis for $\mathbf{A}$. The right multiplication by a, i.e. R_a is defined as the linear operator T together with the following product: $\mathbf{A} \times \mathbb{C}a \to \mathbf{A}$ with $v(ca) = cT(v)$ for all $v \in \mathbf{A}$ and $c \in \mathbb{C}$.*

Under the above conditions, right multiplication operator is a derivation ($\mathbf{A}$ is a Leibniz superalgebra) if and only if $R_{a^j} = 0$ for all $j \geq 3$ and $R_{a^2} = 2R_a R_a$.

Proof Suppose right multiplication operator is a derivation, then $R(A)$ - the set of all right multiplication operators - has Lie superalgebra structure with the following multiplication:

$$< R_a, R_b >:= R_a R_b - (-1)^{\bar{i}\bar{j}} R_b R_a, \quad R_a \in R(A)_{\bar{i}}, \quad R_b \in R(A)_{\bar{j}}$$

it can be seen then that $R_{a^2} =< R_a, R_a >= 2R_aR_a$. Since $R_a^j(x) = R_a(R_a^{j-1}(x))$ for all $j \geq 2$ and for all $x \in \mathbf{A}$ and $a^j := R_a^{j-1}(a)$, then $a^{j+1} = R_a^j(a) = R_a(R_a^{j-1}(a)) = R_a(a^j) = [a^j, a]$. Therefore, we have

$$R_{a^3} = R_{[a^2,a]} =< R_{a^2}, R_a >= R_{a^2}R_a - R_aR_{a^2} = 2R_aR_aR_a - 2R_aR_aR_a = 0.$$

Since

$$R_{a^{j+1}} = R_{[a^j,a]} =< R_{a^j}, R_a >= R_{a^j}R_a - (-1)^j R_aR_{a^j}, \quad j \geq 3$$

then by induction we have $R_{a^j} = 0$ for all $j \geq 3$. On the other hand, any $x \in \mathbf{A}$ can be expressed as a linear combination of the basis vectors, i.e., $x = \alpha_1 a + \alpha_2 a^2 + \cdots + \alpha_{n+m}a^{n+m}$. This latter fact leads to $R_x = \alpha_1 R_a + \alpha_2 R_{a^2}$ on account of $R_{a^j} = 0$ for all $j \geq 3$. Since it is enough to consider x an homogeneous element, then either $\alpha_1 = 0$ or $\alpha_2 = 0$ and therefore it suffices to show that R_a and R_{a^2} are derivations over $\mathbf{A}$, R_a an odd derivation and R_{a^2} an even one. By linearity we only have to check the condition of being a derivation over the basis vectors $\{a, a^2, \ldots, a^{n+m}\}$. On account of $\mid a^j \mid$ is either 0 or 1 depending on if j is even or odd respectively, then the equation for R_a to be an odd derivation remains

$$R_a([a^i, a^j]) = (-1)^j[R_a(a^i), a^j] + [a^i, R_a(a^j)]$$

this equation trivially holds for all $j \geq 3$ as $R_{a^j} = 0$. For $j = 2$ and since $[a^i, a^3] = 0$ it remains

$$R_a([a^i, a^2]) = [R_a(a^i), a^2]$$

replacing R_{a^2} by $2R_aR_a$ the above equation trivially holds, i.e. $2a^{i+3} = 2a^{i+3}$. For $j = 1$ we get

$$R_a(a^{i+1}) = -[a^{i+1}, a] + [a^i, R_a(a)]$$

but as $R_{a^2} = 2R_aR_a$, then we have from the above equation $a^{i+2} = -a^{i+2} + 2a^{i+2} = a^{i+2}$. Finally, the condition for R_{a^2} to be an even derivation over the basis vectors is

$$R_{a^2}([a^i, a^j]) = [R_{a^2}(a^i), a^j] + [a^i, R_{a^2}(a^j)]$$

this equation trivially holds for $j \geq 3$, and also holds for $j = 2$ and $j = 1$ taking into account $[a^i, a^4] = 0$ and $[a^i, a^3] = 0$, respectively. □

Corollary 3.1 *If A is Leibniz superalgebra with $m = n$, then a^{n+m+1} is an odd element and can be expressed as a linear combination of the odd basis vectors, i.e., $a^{n+m+1} = \alpha_1 a + \alpha_3 a^3 + \ldots \alpha_{n+m-1}a^{n+m-1}$, then since $R_{a^j} = 0$ for all $j \geq 3$ we have*

$$0 = [a, a^{n+m+1}] = [a, (\alpha_1 a + \alpha_3 a^3 + \ldots \alpha_{n+m-1}a^{n+m-1})] = \alpha_1 a^2,$$

which implies that $\alpha_1 = 0$. Analogously, if $m = n + 1$, then a^{n+m+1} is and even element and can be expressed as a linear combination of the even basis vectors, i.e. $a^{n+m+1} = \alpha_2 a^2 + \alpha_4 a^4 + \ldots \alpha_{n+m-1} a^{n+m-1}$, then

$$0 = [a, a^{n+m+1}] = [a, (\alpha_2 a^2 + \alpha_4 a^4 + \ldots \alpha_{n+m-1} a^{n+m-1})] = \alpha_2 a^3$$

obtaining then $\alpha_2 = 0$.

4 Classification of cyclic Leibniz superalgebras over $\mathbb{C}$

Since throughout this section we consider Leibniz superalgebras, we denote by $[\ ,\ ]$ the standard bracket product as usual. Next, we start with the nilpotent case.

Proposition 4.1 *The only one, up to isomorphism, non-degenerate nilpotent cyclic Leibniz superalgebra is exactly $NF^{n,m}$ with $m = n$ or $m = n + 1$.*

Proof Note that the only one, up to isomorphism, null-filiform Leibniz superalgebra (non Leibniz algebra) is $NF^{n,m}$. Moreover, in order to have a non-trivial odd part we have only two possibilites for m ($m = n$ or $m = n + 1$). For more details it can be consulted [1]. After a very carefully chosen isomorphism, change of scale, we have a new law for $NF^{n,m}$ verifying in particular $[y'_i, y'_1] = x'_i,\ 1 \le i \le n$, and $[x'_i, y'_1] = y'_{i+1},\ 1 \le i \le m - 1$ and with some non-null scalars in the other two set of brackets. Replacing then y'_1 by a (and then $x'_1 = a^2, y'_2 = a^3, x'_2 = a^4$, and so on) we obtain $[a^k, a] = R_a(a^k) = a^{k+1}$ which concludes the proof. □

Thus, hereafter we consider the non-nilpotent case, i.e. $[a^{n+m}, a] \neq 0$

Proposition 4.2 *Any finite dimensional non-nilpotent and non-degenerate cyclic Leibniz superalgebra, $L = L_{\bar{0}} \oplus L_{\bar{1}}$, with $dim(L_{\bar{0}}) = n$ and $dim(L_{\bar{0}}) = m$, can be expressed in a homogeneous basis $\{a, a^2, \ldots a^{n+m}\}$ by one of the following laws*

- **If** $m = n$:
  - $[a^i, a] = a^{i+1}, \quad 1 \le i \le n + m - 1,$
  - $[a^{n+m}, a] = \sum \alpha_k a^k, \quad 3 \le k \le n + m - 1,$
  - $[a^i, a^2] = 2a^{i+2}, \quad 1 \le i \le n + m - 2,$
  - $[a^{n+m-1}, a^2] = 2\sum \alpha_k a^k, \quad 3 \le k \le n + m - 1,$
  - $[a^{n+m}, a^2] = 2\sum \alpha_k a^{k+1}, \quad 3 \le k \le n + m - 1,$

  with some $\alpha_k \neq 0$ and k odd,
- **If** $m = n + 1$:
  - $[a^i, a] = a^{i+1}, \quad 1 \le i \le n + m - 1,$
  - $[a^{n+m}, a] = \sum \alpha_k a^k, \quad 4 \le k \le n + m - 1,$
  - $[a^i, a^2] = 2a^{i+2}, \quad 1 \le i \le n + m - 2,$
  - $[a^{n+m-1}, a^2] = 2\sum \alpha_k a^k, \quad 4 \le k \le n + m - 1,$
  - $[a^{n+m}, a^2] = 2\sum \alpha_k a^{k+1}, \quad 4 \le k \le n + m - 1,$

  with some $\alpha_k \neq 0$ and k even.

where the omitted products are equal to zero. Moreover, a^i for all i odd is the basis of the odd part and a^i for all i even is the basis of the even part of the Leibniz superalgebra.

Henceforth, the aforementioned laws of Leibniz superalgebras will be noted by $CNF^{n,n}(\alpha_3, \alpha_5, \ldots, \alpha_{2n-1})$ and $CNF^{n,n+1}(\alpha_4, \alpha_6, \ldots, \alpha_{2n})$ respectively.

Proof The proof is a consequence of Proposition 3.1 together with Corollary 3.1 and the structure of a Leibniz superalgebra. In particular, thanks to Proposition 3.1 all the bracket products $[a^i, a^2]$ for all i, $1 \le i \le n+m$, are exactly $2[[a^i, a], a]$. Furthermore, from Corollary 3.1 we have the general expression of $a^{n+m+1} = [a^{n+m}, a]$ depending on the two possibilities for m. □

From the above Theorem we have then the following result.

Proposition 4.3 *Any finite-dimensional non-nilpotent and non-degenerate cyclic Leibniz superalgebra is solvable with index of solvability less or equal to* 4.

Lemma 4.1 *Let L be an $(n+m)$-dimensional cyclic Leibniz superalgebra and a be a cyclic generator of L. Then every generator $b = c_1a + c_3a^3 + c_5a^5 + \cdots + c_da^d$ for L satisfies*

$$\begin{aligned}
[b^{n+m}, b] &= c_1^{n+m-2}\alpha_3 b^3 + c_1^{n+m-4}\alpha_5 b^5 + \cdots + c_1^4\alpha_{n+m-3}b^{n+m-3} \\
&\quad + c_1^2\alpha_{n+m-1}b^{n+m-1},\ if\ m = n \\
[b^{n+m}, b] &= c_1^{n+m-4}\alpha_4 b^3 + c_1^{n+m-5}\alpha_6 b^5 + \cdots + c_1^4\alpha_{n+m-3}b^{n+m-3} \\
&\quad + c_1^2\alpha_{n+m-1}b^{n+m-1},\ if\ m = n+1
\end{aligned}$$

Proof Let $b = c_1a + c_3a^3 + c_5a^5 + \cdots + c_da^d$ be a cyclic generator of L with $d = n+m-1$ if $m = n$ and $d = n+m$ if $m = n+1$. Note that $c_1 \neq 0$. We distinguish two cases:

• **Case** $m = n$**.** Since a is a cyclic vector of R_a, it follows that R_a has characteristic polynomial

$$f(t) = t^{n+m} - \alpha_{n+m-1}t^{n+m-2} - \alpha_{n+m-3}t^{n+m-4} - \cdots - \alpha_7t^6 - \alpha_5t^4 - \alpha_3t^2.$$

By the Cayley-Hamilton theorem $f(R_a) = 0$. That is,

$$\begin{aligned}
R_a^{n+m}(x) &= \alpha_{n+m-1}R_a^{n+m-2}(x) + \alpha_{n+m-3}R_a^{n+m-4}(x) + \cdots + \alpha_7R_a^6(x) \\
&\quad + \alpha_5R_a^4(x) + \alpha_3R_a^2(x)
\end{aligned}$$

for all x in L. We multiply by c_1^{n+m} and take $x = b$.

$$c_1^{n+m}R_a^{n+m}(b) = c_1^{n+m}\alpha_{n+m-1}R_a^{n+m-2}(b) + \cdots + c_1^{n+m}\alpha_5R_a^4(b) + c_1^{n+m}\alpha_3R_a^2(b).$$

From the Proposition 3.1 and considering that the cyclic generator b is an odd element we know that $R_b = c_1R_a$ then $R_b^{n+m}(b) = c_1^{n+m}R_a^{n+m}(b)$ and

$$R_b^{n+m}(b) = c_1^2\alpha_{n+m-1}R_b^{n+m-2}(b) + \cdots + c_1^{n+m-4}\alpha_5R_b^4(b) + c_1^{n+m-2}\alpha_3R_b^2(b)$$

which we may also write as

$$[b^{n+m}, b] = c_1^{n+m-2}\alpha_3 b^3 + c_1^{n+m-4}\alpha_5 b^5 + \cdots + c_1^4\alpha_{n+m-3}b^{n+m-3} \\ +c_1^2\alpha_{n+m-1}b^{n+m-1}.$$

• **Case** $m = n + 1$**.** In this case, the characteristic polynomial is

$$f(t) = t^{n+m} - \alpha_{n+m-1}t^{n+m-2} - \alpha_{n+m-3}t^{n+m-4} - \cdots - \alpha_8 t^7 - \alpha_6 t^5 - \alpha_4 t^3.$$

By the Cayley-Hamilton theorem $f(R_a) = 0$. That is,

$$R_a^{n+m}(x) = \alpha_{n+m-1}R_a^{n+m-2}(x) + \alpha_{n+m-3}R_a^{n+m-4}(x) + \cdots + \alpha_8 R_a^7(x) \\ +\alpha_6 R_a^5(x) + \alpha_4 R_a^3(x)$$

for all x in L. We multiply by c_1^{n+m} and take $x = b$.

$$c_1^{n+m}R_a^{n+m}(b) = c_1^{n+m}\alpha_{n+m-1}R_a^{n+m-2}(b) + \cdots + c_1^{n+m}\alpha_6 R_a^5(b) \\ +c_1^{n+m}\alpha_4 R_a^3(b).$$

Analogously to the case $m = n$, we know that $R_b = c_1 R_a$ then $R_b^{n+m}(b) = c_1^{n+m}R_a^{n+m}(b)$ and

$$R_b^{n+m}(b) = c_1^2\alpha_{n+m-1}R_b^{n+m-2}(b) + \cdots + c_1^{n+m-5}\alpha_6 R_b^5(b) \\ +c_1^{n+m-4}\alpha_4 R_b^4(b)$$

which we may also write as

$$[b^{n+m}, b] = c_1^{n+m-4}\alpha_4 b^3 + c_1^{n+m-5}\alpha_6 b^5 + \cdots + c_1^4\alpha_{n+m-3}b^{n+m-3} \\ +c_1^2\alpha_{n+m-1}b^{n+m-1}.$$

□

We have the following classification

Theorem 4.1 *(Classification). Let L be an $(n+m)$-dimensional non-nilpotent cyclic Leibniz superalgebra over $\mathbb{C}$. Then L is isomorphic to a Leibniz superalgebra spanned by $\{a, a^2, \ldots, a^{n+m}\}$, with a^i for all i odd the basis of $L_{\bar{1}}$ and a^i for all i even the basis of the $L_{\bar{0}}$, with bracket products one of the following:*

• **Case** $m = n$

(1) $CNF^{n,n}$:

$$\begin{cases} [a^i, a] = a^{i+1}, & 1 \le i \le n+m-1, \\ [a^{n+m}, a] = a^{n+m-1}, & \\ [a^i, a^2] = 2a^{i+2}, & 1 \le i \le n+m-2, \\ [a^{n+m-1}, a^2] = 2a^{n+m-1}, & \\ [a^{n+m}, a^2] = 2a^{n+m}, & \end{cases}$$

(2) $CNF^{n,n}(0,\dots,1,\alpha_{k+2},\dots,\alpha_{n+m-3}),\ k\ odd:$

$$\begin{cases} [a^i,a]=a^{i+1}, & 1\le i\le n+m-1,\\ [a^{n+m},a]=a^k+\alpha_{k+2}a^{k+2}+\cdots+\alpha_{n+m-3}a^{n+m-3}, & 3\le k\le n+m-3,\\ [a^i,a^2]=2a^{i+2}, & 1\le i\le n+m-2,\\ [a^{n+m-1},a^2]=2(a^k+\alpha_{k+2}a^{k+2}+\cdots+\alpha_{n+m-3}a^{n+m-3}), & 3\le k\le n+m-3,\\ [a^{n+m},a^2]=2(a^{k+1}+\alpha_{k+2}a^{k+3}+\cdots+\alpha_{n+m-3}a^{n+m-2}), & 3\le k\le n+m-3, \end{cases}$$

with $(\alpha_{k+2},\dots,\alpha_{n+m-3})\in\mathbb{C}^{n-k-1}/\sim$, *being the equivalence relation* $\sim$ *defined over* $\mathbb{C}^{n-k-1}$ *as follows:*

$$(\alpha_{k+2},\dots,\alpha_{n+m-3})\sim(\alpha'_{k+2},\dots,\alpha'_{n+m-3})\ \mathit{if}\ (\alpha_{k+2},\dots,\alpha_{n+m-3}) = (c_1^{2n-k-1}\alpha'_{k+2},\dots,c_1^4\alpha'_{n+m-3})$$

for some $(2n-k+1)$*-th root of unity* c_1.

- **Case** $m=n+1$

(1) $CNF^{n,n+1}:$

$$\begin{cases} [a^i,a]=a^{i+1}, & 1\le i\le n+m-1,\\ [a^{n+m},a]=a^{n+m-1}, & \\ [a^i,a^2]=2a^{i+2}, & 1\le i\le n+m-2,\\ [a^{n+m-1},a^2]=2a^{n+m-1}, & \\ [a^{n+m},a^2]=2a^{n+m}, & \end{cases}$$

(2) $CNF^{n,n+1}(0,\dots,1,\alpha_{k+2},\dots,\alpha_{n+m-3}),\ k\ even:$

$$\begin{cases} [a^i,a]=a^{i+1}, & 1\le i\le n+m-1,\\ [a^{n+m},a]=a^k+\alpha_{k+2}a^{k+2}+\cdots+\alpha_{n+m-3}a^{n+m-3}, & 4\le k\le n+m-3,\\ [a^i,a^2]=2a^{i+2}, & 1\le i\le n+m-2,\\ [a^{n+m-1},a^2]=2(a^k+\alpha_{k+2}a^{k+2}+\cdots+\alpha_{n+m-3}a^{n+m-3}), & 4\le k\le n+m-3,\\ [a^{n+m},a^2]=2(a^{k+1}+\alpha_{k+2}a^{k+3}+\cdots+\alpha_{n+m-3}a^{n+m-2}), & 4\le k\le n+m-3, \end{cases}$$

with $(\alpha_{k+2},\dots,\alpha_{n+m-3})\in\mathbb{C}^{n-k}/\sim$, *being the equivalence relation* $\sim$ *defined over* $\mathbb{C}^{n-k}$ *as follows:*

$$(\alpha_{k+2},\dots,\alpha_{n+m-3})\sim(\alpha'_{k+2},\dots,\alpha'_{n+m-3})\ \mathit{if}\ (\alpha_{k+2},\dots,\alpha_{n+m-3}) = (c_1^{2n-k}\alpha'_{k+2},\dots,c_1^4\alpha'_{n+m-3})$$

for some $(2n-k+2)$*-th root of unity* c_1.

Remark 4.1 Note that all the superalgebras of Theorem 4.1 are pairwise non-isomorphic. In particular, if $k\neq k'$ then

$$CNF^{n,m}(0,\dots,1,\alpha_{k+2},\dots,\alpha_{n+m-3})\ncong CNF^{n,m}(0,\dots,1,\gamma_{k'+2},\dots,\gamma_{n+m-3})$$

in both cases, $m = n$ and $m = n + 1$. Moreover, for the same k we have

$$CNF^{n,m}(0, \ldots, 1, \alpha_{k+2}, \ldots, \alpha_{n+m-3}) \cong CNF^{n,m}(0, \ldots, 1, \gamma_{k+2}, \ldots, \gamma_{n+m-3})$$

if and only if $(\alpha_{k+2}, \ldots, \alpha_{n+m-3}) \sim (\gamma_{k+2}, \ldots, \gamma_{n+m-3})$, with $\sim$ the equivalence relation defined in each case.

Proof We consider b a cyclic generator of L with $b = c_1 a + c_3 a^3 + \cdots + c_d a^d$ where $d = n+m-1$ if $m = n$ and $d = n+m$ in the order case. Two cases are distinguished.

- **Case** $m = n$**.** From Lemma 4.1 we know that

$$\begin{aligned}[b^{n+m}, b] = c_1^{n+m-2}\alpha_3 b^3 + c_1^{n+m-4}\alpha_5 b^5 + \cdots + c_1^4 \alpha_{n+m-3} b^{n+m-3} \\ + c_1^2 \alpha_{n+m-1} b^{n+m-1}.\end{aligned}$$

Then, the new parameters are the following $\gamma_j = c_1^{n+m-j+1}\alpha_j$ with $3 \leq j \leq n+m-1$ and j odd.

Let k be the first value with $3 \leq k \leq n+m-1$ and k odd and $\alpha_k \neq 0$. We can choose $c_1 = \alpha_k^{\frac{1}{k-n-m-1}}$ and we obtain $\gamma_k = 1$. Then $\gamma_j = \alpha_k^{\frac{n+m-j+1}{k-n-m-1}}\alpha_j$ with $k+2 \leq j \leq n+m-1$ and j odd. Thus, we get the superalgebra $CNF^{n,n}(0, \ldots, 1, \alpha_{k+2}, \ldots, \alpha_{n+m-1})$.

- **Case** $m = n + 1$**.** From Lemma 4.1 we know that

$$\begin{aligned}[b^{n+m}, b] = c_1^{n+m-4}\alpha_4 b^3 + c_1^{n+m-5}\alpha_6 b^5 + \cdots + c_1^4 \alpha_{n+m-3} b^{n+m-3} \\ + c_1^2 \alpha_{n+m-1} b^{n+m-1}.\end{aligned}$$

Then, the new parameters are the following $\gamma_j = c_1^{n+m-j+1}\alpha_j$ with $3 \leq j \leq n+m-1$ and j odd.

Let k be the first value with $4 \leq k \leq n+m-1$ and k even and $\alpha_k \neq 0$. We can choose $c_1 = \alpha_k^{\frac{1}{k-n-m-1}}$ and we obtain $\gamma_k = 1$. Then $\gamma_j = \alpha_k^{\frac{n+m-j+1}{k-n-m-1}}\alpha_j$ with $k+2 \leq j \leq n+m-1$ and j even. Thus, we get the superalgebra $CNF^{n,n+1}(0, \ldots, 1, \alpha_{k+2}, \ldots, \alpha_{n+m-1})$.

In summary, in both cases the first non-null parameter can be always normalised to 1 and the remaining are under the corresponding equivalence relation defined in the statement of the theorem. Note that if the position of the first non-null parameter is different then the superalgebras are clearly non-isomorphic. Otherwise, two superalgebras with the same position for the first non-null parameter are isomorphic if and only if their sets of paramenters are related with respect to the equivalence relation. □

5 On irreducible components of Leibniz algebras and superalgebras

Let us note that the set of cyclic Leibniz algebras is an open set in the variety of Leibniz algebras $Leib^n$. In fact, if a q-generated ($q > 1$) Leibniz algebra has a basis $\{e_1, e_2, \ldots, e_n\}$ then for any e_i the powers $e_i, e_i^2, \ldots, e_i^n$ are linearly dependent.

That is, determinants of the matrices A_i, $1 \le i \le n$, which are composed by the rows $e_i, e_i^2, \dots, e_i^n$ are zero, obtaining then n polynomials with structure constants of the algebra. Consequently, q-generated ($q > 1$) Leibniz algebras form a closed set for the Zariski topology. Furthermore, the complemented set to this closed set, and then open, is the set of all the cyclic Leibniz algebras. Taking into account that in the case of complex field the connectedness with respect to Zariski and Euclidean topologies coincide together with the fact that this complemented set is connected in the Euclidean topology as the complement to the subset of real codimension 2, allow us to conclude that the aforementioned complemented set is also irreducible. Thus, its closure determines an irreducible component of $Leib^n$. Also, it can be seen by using similar arguments to the ones used in [5] that two cyclic Leibniz algebras non-isomorphic named μ_1 and μ_2 verify $\mu_1 \notin \overline{\mathbf{Orb}(\mu_2)}$.

All of the above can be extended for Leibniz superalgebras. On the other hand, in Proposition 3.6 of [11] the authors prove that an arbitray single-generated Leibniz algebra can be obtain by infinitesimal deformations of the null-filiform Leibniz algebra NF^n. This result together with the classification of all the cyclic (single-generated) Leibniz algebras (Theorem 2.2) allow us to improve the expression of the irreducible component found in [11]. Thus, we have

Theorem 5.1

$$\overline{\bigcup_{\substack{2 \le k \le n \\ (\alpha_{k+1}, \cdots, \alpha_n) \in \mathbb{C}^{n-k}/\sim}} \mathbf{Orb}(CNF^n(0, \dots, 1, \alpha_{k+1}, \dots, \alpha_n))}$$

is an irreducible component of the variety of Leibniz algebras $Leib^n$. Note that for simplicity CNF^n has been denoted by $CNF^n(0, \dots, 0, 1)$.

Regarding cyclic Leibniz superalgebras, the families obtained here together with the infinitesimal deformations of $NF^{n,m}$ obtained in [11] allow us to assert the following results.

Theorem 5.2 *(1) Any arbitrary complex cyclic Leibniz superalgebra $CNF^{n,m}$ is isomorphic to $NF^{n,m} + \varphi$, where φ is an infinitesimal deformation of $NF^{n,m}$.*

(2) Conversely, if φ is an infinitesimal deformation of the null-filiform Leibniz superalgebra $NF^{n,m}$, then the law $NF^{n,m} + \varphi$ is a cyclic Leibniz superalgebra iff φ is linearly integrable in the following sense:

$$\varphi(x, \varphi(y, x)) - \varphi(\varphi(x, y), z) + (-1)^{|y||z|}\varphi(\varphi(x, z), y) = 0$$

Proof (2) was proven in [11]. For (1) we re-name the basis vectors $a^j = y_{\frac{j+1}{2}}$ for j odd and $a^i = x_{\frac{i}{2}}$ for i even in the families of Proposition 4.2. Then, by applying the change of scale defined by $x_i' = 2^{i-1}x_i$ and $y_j' = 2^{j-1}y_j$ for all i, j, $1 \le i \le n$ and $1 \le j \le m$, we obtain the superalgebras $NF^{n,n} + \sum_{k=2}^{n} b_k \psi_{n,k}$ and $NF^{n,n+1} + \sum_{k=2}^{n} c_k \varphi_{n+1,k}$

with the multiplication tables

$$
\begin{cases}
[y_i, y_1] = x_i, & 1 \le i \le n, \\
[x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \le i \le n-1, \\
[x_n, y_1] = \sum\limits_{k=2}^{n} b_k y_k, & \\
[y_i, x_1] = y_{i+1}, & 1 \le i \le n-1, \\
[y_n, x_1] = 2 \sum\limits_{k=2}^{n} b_k y_k, & \\
[x_i, x_1] = x_{i+1}, & 1 \le i \le n-1, \\
[x_n, x_1] = 2 \sum\limits_{k=2}^{n} b_k x_k, &
\end{cases}
$$

$$
\text{and} \begin{cases}
[x_i, x_1] = x_{i+1}, & 1 \le i \le n-1, \\
[x_n, x_1] = \sum\limits_{k=2}^{n} c_k x_k, & \\
[y_i, x_1] = y_{i+1}, & 1 \le j \le n, \\
[y_{n+1}, x_1] = \sum\limits_{k=2}^{n} c_k y_{k+1}, & \\
[y_i, y_1] = x_i, & 1 \le i \le n, \\
[y_{n+1}, y_1] = \sum\limits_{k=2}^{n} c_k x_k, & \\
[x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \le i \le n.
\end{cases}
$$

respectively. Note that $\sum_{k=2}^{n} b_k \psi_{n,k}$ and $\sum_{k=2}^{n} c_k \varphi_{n+1,k}$ are infinitesimal deformations of the corresponding null-filiform superalgebras. For more details regarding these infinitesimal deformations it can be consulted [11]. □

Theorem 5.3

$$
\overline{\bigcup_{\substack{2 \le k \le n \\ (\alpha_{2k+1}, \ldots, \alpha_{2n-1}) \in \mathbb{C}^{n-k}/\sim}} \mathbf{Orb}(CNF^{n,n}(0, \ldots, 1, \alpha_{2k+1}, \ldots, \alpha_{2n-1}))}
$$

and

$$
\overline{\bigcup_{\substack{3 \le k \le n+1 \\ (\alpha_{2k}, \ldots, \alpha_{2n}) \in \mathbb{C}^{n-k+1}/\sim}} \mathbf{Orb}(CNF^{n,n+1}(0, \ldots, 1, \alpha_{2k}, \ldots, \alpha_{2n}))}
$$

are irreducible components of the variety of Leibniz superalgebras $Leib^{n,n}$ and $Leib^{n,n+1}$, respectively. Note that for simplicity $CNF^{n,n}$ and $CNF^{n,n+1}$ have been denoted by $CNF^{n,n}(0, \ldots, 0, 1)$ and $CNF^{n,n}(0, \ldots, 0, 1)$.

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