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A LINK BETWEEN MENGER'S THEOREM AND INFINITE EULER GRAPHS

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1. Introduction

An infinite graph G is usually called Eulerian if there is an Eulerian trail in G, i.e., a one-way or two-way infinite trail which contains all the edges of G. In both cases, it is obviously necessary that $V(G)$ be countable and that G be, at most, two ended (in the case of one-way trail G must be one ended, see [6]), which means that for each finite $K \subset G$ there is, at most, two infinite components in $G - K$ (one infinite component in the case of one ended).

It is the purpose of this note to extend the definition of an infinite Eulerian graph in such a way that the above restrictions on the number of ends are not required, but such that in the cases previously considered the "old" notion of Eulerian graph appears as a special case.

By an infinite graph we mean a graph G such that its vertex set $V(G)$ is countable and the degree in every vertex is finite. In particular the edge set $E(G)$ is also countable. In other words, G is a locally finite countable graph. We will use the notations and definitions of [5], except vertex instead of point and edge instead of line. And if A is a set we represent its cardinal by $|A|$.

If v is a vertex in a graph G, we denote by $\delta(v, G)$ its degree in G (or by $\delta(v)$ when no confusion is possible), and by $lk(v, G)$ we denote the set of incident vertices with v. The set of vertices with odd degree in G will be denoted by $\mathcal{O}(G)$.

If G and G' are two graphs, and $v \in V(G)$ and $v' \in V(G')$ are two vertices, by $G \cup_{v=v'} G'$ we mean the graph obtained after identifying v with v' in the disjoint union of G and G'.

Finally, we explictly define the notion of end. A Freudenthal end of a non-compact space X is an element of the inverse limit $\mathcal{F}(X) = \varprojlim \pi_0(X - K)$, where K ranges over the family of compact sets of X and π_0 stands for the set of connected components. The cardinal of $\mathcal{F}(X)$ is denoted by $e(X)$. When X is a graph we can use a countable sequence $G_1 \subsetneqq G_2 \subsetneqq \dots$ of finite subgraphs to obtain $\mathcal{F}(X)$ (see [1] for details).

2. Menger's theorems

DEFINITION 1. Let G be an infinite connected graph and $A \subset V(G)$. A is said to be Menger if there exists no subset $M \subset E(G)$ with $k < |A|$ elements such that there are $k+1$ elements of A in finite components of $G - M$.

We can reformulate [3, Satz 3] in the following way.

THEOREM 2. *Let G be an infinite connected graph and $v \in V(G)$. Then there are $\delta(v)$ edge-disjoint 1-paths starting at v if and only if $lk(v)$ is Menger.*

Theorem 2 is trivially equivalent to

THEOREM 3. *Let G be an infinite connected graph and A a finite subset of $V(G)$. Then there exist $|A|$ edge-disjoint 1-paths starting at each vertex of A if and only if A is Menger.*

It is not difficult, by using Zorn's Lemma, to extend Theorem 2 to graphs with $\delta(v) = \infty$. But that technique does not work if we try to extend Theorem 3 to infinite subsets of $V(G)$. However, it is possible to adapt Halin's proof in order to achieve

THEOREM 4. *Let G be an infinite connected graph and $A \subset V(G)$, with A possibly infinite. Then there exist $|A|$ edge-disjoint 1-paths starting at each vertex of A if and only if A is Menger.*

PROOF. Let $\{H_n : n \in N\}$ be a family of finite subgraphs of G satisfying:

1. $H_i \subset H_{i+1}$.
2. $\cup H_n = G$.

As a consequence of Theorem 3, there exist $|A_n|$ edge-disjoint 1-paths starting at each vertex of $A_n = A \cap H_n$. Let P_n be that set of 1-paths and let $P_n^i = P_n \cap H_i$.

Since H_1 is finite, in the sequence $\{P_1^1, P_2^1, P_3^1, \dots\}$ there must be an element repeated infinitely many times; let P^1 be that element. In the same way, in the sequence $\{P_1^2, P_2^2, P_3^2, \dots\}$ there must be an element repeated infinitely many times and such that the intersection of that element with H_1 agrees with P^1; let P^2 be that element. By iterating this process, we get an increasing sequence of sets of edges in G that defines $|A|$ edge-disjoint 1-paths starting at each vertex of A. □

3. Eulerian graphs

We call a 1-path to a graph homeomorphic to the half-line and a 2-path to a graph homeomorphic to the line. Thus, we can say that a graph G is i-Eulerian ($i = 1, 2$) if there exists a morphism $\phi : P \to G$ inducing a bijection on the edges, where P is an i-path ($i = 1, 2$) (cf. [6]).

Following [4], we give the following definitions:

DEFINITION 5. We define a tree T to be a purely infinite tree if either it is a 1-path or an infinite tree without vertex of degree 1.

In [4] it is pointed out that 1-paths and 2-paths are the only purely infinite trees T with $e(T) = 1, 2$. We can now give the following definition.

DEFINITION 6. *An infinite connected graph G is said to be Eulerian, if there exists a morphism from a purely infinite tree T, $\phi : T \to G$ inducing a bijection on the edges.*

Obviously, an Eulerian graph is connected because a morphism conserves the number of connected components.

REMARK 7. As it was said above, if a graph is 1-Eulerian or 2-Eulerian, then it is Eulerian in the sense of Definition 6. In [2] a definition of n-Eulerian graphs is given, namely those graphs for which there exists a morphism $\phi : W \to G$ inducing a bijection on the edges, where W represents the wedge of n 1-way paths. It is straightforward from that definition to check that n-Eulerian graphs are also Eulerian in the sense of Definition 6.

In order to characterize Eulerian graphs, we are going to give the following results:

LEMMA 8. *If G is an infinite Eulerian graph and G' is either another infinite Eulerian graph or a finite graph with $\mathcal{O}(G') = \emptyset$; and $v \in V(G)$, $v' \in V(G')$. Then $G \cup_{v=v'} G'$ is Eulerian.*

PROOF. The proof is a direct consequence of the definition. □

PROPOSITION 9. *If G is an infinite connected graph with $\mathcal{O}(G) = \emptyset$ then G is Eulerian.*

PROOF. Let v be a vertex in G. We denote by E_n the subgraph induced by those vertices which are, at most, at a distance n from v.

We are going to give a sequence of purely infinite trees rooted at v and morphisms $\{(T_1, \phi_1), (T_2, \phi_2), \dots\}$ such that

1. $T_j \subsetneqq T_i$ for any $i > j$ and the distance from v to $T_n - T_{n-1}$ is, at least, n.
2. ϕ_n induces a bijection on the edges of E_n.
3. $\phi_i|T_j = \phi_j$ for any $i > j$.
4. $\mathcal{O}\big(G - \phi_n(T_n)\big) = \emptyset$.
5. $G - \phi(T_n)$ has not finite components.

Considering $T_0 = \big(\{v\}, \emptyset\big)$, we are going to give the method to get (T_{n+1}, ϕ_{n+1}) from (T_n, ϕ_n).

If there are no edges in $E_{n+1} - \phi_n(T_n)$, then $(T_{n+1}, \phi_{n+1}) = (T_n, \phi_n)$. Otherwise, let $\{u, w\}$ be an edge in $E_{n+1} - \phi_n(T_n)$ with $u \in E_n$. As a consequence of Theorem 3 and the characterization of finite Eulerian graphs, there exists a 2-path P in $E_{n+1} - \phi_n(T_n)$ containing $\{u, w\}$ such that $E_{n+1} - (\phi_n(T_n) \cup P)$ has no finite components. Thus, we add two 1-paths at a vertex of T_n which is mapped in u and we extend ϕ_n to the new tree so obtained. We repeat the same process with all the edges in $E_{n+1} - \phi_n(T_n)$ in order to get (T_{n+1}, ϕ_{n+1}). □

By using Lemma 8 we get as a corollary of Proposition 9.

COROLLARY 10. *If G is an infinite connected graph with an Eulerian subgraph G' such that $\mathcal{O}(G - G') = \emptyset$, then G is Eulerian.*

Now, we are going to give a generalization of Definition 1 which will be used in the characterization of Eulerian graphs.

DEFINITION 11. Let G be an infinite connected graph and $A \subset V(G)$. A is said to be i-Menger if there exists no subset $M \subset E(G)$ with $k < |A|$ elements such that there are $k - i + 1$ elements of A in finite components of $G - M$.

It is not difficult to check that if T is a purely infinite tree then $\mathcal{O}(T)$ is 2-Menger.

Now, we can set up

THEOREM 12. *An infinite connected graph G is Eulerian if and only if $\mathcal{O}(G)$ is 2-Menger.*

PROOF. Let G be an Eulerian graph. There exists a purely infinite tree T and a morphism $\phi : T \to G$ such that ϕ induces a bijection on the edges. If $\mathcal{O}(G)$ is not 2-Menger then there exists a set M of edges of G, with $|M| < |\mathcal{O}(G)|$, such that there are $|M| - 1$ vertices of $\mathcal{O}(G)$ in the union C of finite components of $G - M$. $\phi^{-1}(C)$ is the union of finite components of $T - \phi^{-1}(M)$. As the number of elements of $\mathcal{O}(T) \cap \phi^{-1}(C)$ is at least the number of elements of $\mathcal{O}(G) \cap C$ and $|\phi^{-1}(M)| = |M|$, $\mathcal{O}(T)$ is not 2-Menger, but this is not possible, thus, $\mathcal{O}(G)$ must be 2-Menger.

Conversely, let G be an infinite connected graph such that $\mathcal{O}(G)$ is 2-Menger. By using the same technique as in Theorem 4, we are going to give an Eulerian subgraph G' of G such that $\mathcal{O}(G - G') = \emptyset$, thus by Corollary 10 we will achieve our result.

As in the proof of Theorem 4, let $\{H_n : n \in N\}$ be a family of finite subgraphs of G satisfying:

1. $H_i \subset H_{i+1}$.
2. $\cup H_n = G$.

As a consequence of Theorem 4, there exist edge-disjoint 1-paths starting at each vertex of $\mathcal{O}_n$ and a 2-path starting at a vertex v of H_n where $\mathcal{O}_n = \mathcal{O}(G) \cap H_n$. Let P be the subgraph of G defined by union of those 1-paths and the 2-path; it is clear that $\mathcal{O}(G - P) = \emptyset$. Now, we are going to "grow" P in order to get an Eulerian subgraph G' with $P \subseteqq G'$.

Let P_n be the union of the 2-path with the 1-paths starting at $\mathcal{O}_n$. Since $\mathcal{O}(H_n - P_n) = \emptyset$, it is possible to apply [7, Theorem 2.2] in order to get an

Eulerian graph D_n such that $P_n \subseteqq D_n$. Basically, D_n is obtained by adding to P_n either cycles or 2-paths in such a way that we get a connected graph. As far as we keep adding only either Eulerian graphs – 2-paths – or cycles, it is possible to apply Lemma 8 to assure that D_n is Eulerian.

Now let $D_n^i = D_n \cap H_i$. And, as in Theorem 4, we know that in the sequence $\{D_1^1, D_2^1, D_3^1, \ldots\}$ there must be an element repeated infinitely many times; let D^1 be that element. In the same way, in the sequence $\{D_1^2, D_2^2, D_3^2, \ldots\}$ there must be an element repeated infinitely many times and such that the intersection of that element with H_1 agrees with D^1; let D^2 be that element. By iterating this process, we get an increasing sequence $D^1 \subseteqq D^2 \subseteqq \ldots$ of Eulerian subgraphs, and so $G' = \bigcup D^n$ is trivially Eulerian. Now it only remains to apply Corollary 10 to assure that G is Eulerian. $\square$

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