The isoperimetric problem in complete annuli of revolution with increasing Gauss curvature

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In this work we describe the isoperimetric regions in complete symmetric annuli of revolution with Gauss curvature non-decreasing from the shortest parallel. This description allows us to complete the classification of isoperimetric regions in quadrics of revolution.

1. Introduction

It is well known that least-perimeter sets of given area in the plane, in hyperbolic planes and in round spheres are geodesic discs [3]. However, even in very simple surfaces, this *isoperimetric problem* has remained open. In 1996, Benjamini and Cao [2] proved that the least-perimeter way to enclose a given area in a *paraboloid of revolution* is by means of a circle of revolution. This result was recovered by different methods by Pansu [9], Topping [12], Morgan *et al*. [8] and Ritoré [10].

In these works, the *isoperimetric regions* were classified for some new types of surfaces. Benjamini and Cao [2] solved the problem for complete planes of revolution with non-increasing curvature from the origin which are *convex at infinity.* Morgan *et al*. [8, § 4.3] removed this convexity assumption, and characterized the isoperimetric regions in real projective planes of revolution with non-increasing Gauss curvature from the origin. In [10], amongst other results, the isoperimetric problem was solved for spheres of revolution with an equatorial symmetry and Gauss curvature either non-increasing or non-decreasing from the equator to the poles. An approach to the classification of isoperimetric regions in tori of revolution has been given by Cañete [4], who has classified the *stable* regions in such surfaces.

Even in this simple class of examples, the geometry of the perimeter-minimizing regions of given area can be quite complex. In some planes [8] and spheres [10] of revolution, these regions can be either discs or annuli, and in annuli of revolution with decreasing curvature from one end of finite area the isoperimetric regions are bounded by a single circle of revolution [8, 10]. On the other hand, in tori of revolution, the boundary of a stable region can be composed of curves of constant geodesic curvature which are not circles of revolution [4]. Moreover, in non-compact surfaces, isoperimetric regions may not exist. In general, a minimizing sequence of sets of a given area whose perimeters converge to the infimum of perimeters for this area may have a convergent part of smaller area and a diverging part of positive area, so that in the limit we obtain an isoperimetric region of smaller area, and possibly some minimizing object at infinity.

This paper is devoted to the classification of isoperimetric regions in a complete annulus of revolution with an equatorial symmetry and Gauss curvature which is non-decreasing from this equator. Examples of such annuli are minimal catenoids and one-sheeted hyperboloids. We shall use techniques from the calculus of variations to treat this problem by classifying the embedded curves with constant geodesic curvature which can be part of the boundary of an isoperimetric region. Our results allow us to complete the classification of isoperimetric regions in *quadrics of revolution*. In the resolution of this problem we shall find all the difficulties mentioned before: non-existence of isoperimetric regions, the break of a minimizing sequence into two parts and the existence of isoperimetric regions of several different types. We prove in our main result, theorem 3.9, that in a complete symmetric annulus of revolution with non-decreasing Gauss curvature from the shortest parallel, the isoperimetric regions may be

(i) a 'disc at infinity',

(ii) a symmetric annulus,

(iii) an asymmetric annulus, or

(iv) an annulus bounded by an unduloid-type curve and a circle of revolution.

All possibilities occur in different annuli, as shown in example 3.8, where we exhibit isoperimetric regions of the last type, and in § 4.

We have organized the remainder of the paper into three sections. In § 2 we establish notation and give preliminary results. In § 3, we shall prove our main results, mainly that a minimizing sequence cannot be broken into two pieces, and that there exist isoperimetric regions in these annuli which are not of revolution. Finally, in § 4, we apply the previous results to the classification of isoperimetric regions in the one-sheeted hyperboloid. This allows us to conclude, in corollary 4.4, the classification for quadrics of revolution (invariant by a one-parameter group of rotations around a line).

2. Preliminaries

2.1. Annuli of revolution with non-decreasing curvature

We shall denote by M the product $\mathbb{S}^1 \times \mathbb{R}$ endowed with a complete warped metric

$$\mathrm{d}s^2 := f(t)^2\, \mathrm{d}\theta^2 + \mathrm{d}t^2,$$

for $t \in \mathbb{R}$, $\theta \in \mathbb{S}^1$ and $f : \mathbb{R} \to \mathbb{R}$ a smooth positive function. The Gauss curvature depends only on the t-coordinate, and is given by

$$K(t) := -\frac{f''(t)}{f(t)}. \tag{2.1}$$

Moreover, the length and the geodesic curvature of the *parallels* $\mathbb{S}^1 \times \{t\}$, are given by

$$L(t) := 2\pi f(t), \qquad h(t) := \frac{f'(t)}{f(t)}. \tag{2.2}$$

We shall suppose that the annulus M is *symmetric* with respect to the parallel $\mathbb{S}^1 \times \{0\}$, which is equivalent to the symmetry $f(t) = f(-t)$, and that the Gauss curvature K is a non-decreasing function of the distance t from $\mathbb{S}^1 \times \{0\}$. We note that the monotonicity of $K(t)$ is equivalent to that of the function

$$(4\pi^2)((f')^2 - ff'')(t) = L^2(K + h^2)(t).$$

From the monotonicity of K, since there are no complete ends with positive curvature, it follows that $K \leqslant 0$. Moreover, if K vanishes at some point t_0, then $K \equiv 0$ in $[t_0, +\infty)$, and so our annulus is flat near infinity. Since K is non-decreasing and non-positive, we shall define

$$K_\infty := \lim_{t \to \infty} K(t) \leqslant 0.$$

As $K \leqslant 0$, it follows that $f'' \geqslant 0$, and so f' is non-decreasing. Since $f'(0) = 0$ by the symmetry of f, we deduce that $f' \geqslant 0$ for $t \geqslant 0$ (and non-positive for $t \leqslant 0$). Then f is non-decreasing for $t \geqslant 0$ and, consequently, M is complete and $\mathbb{S}^1 \times \{0\}$ is a shortest parallel of M. We will refer to it as the *shortest geodesic loop* (although it is not necessarily unique).

Given some $\Omega \subset M$, we shall denote the Riemannian area of M by $A(M)$. If C is a rectifiable curve, the length of C will be denoted by $L(C)$. If Ω is a finite perimeter set in M, then its perimeter will be denoted by $\mathcal{P}(\Omega)$.

We shall consider the isoperimetric problem of minimizing perimeter under an area constraint in these symmetric annuli of revolution with non-decreasing Gauss curvature from the shortest geodesic loop.

2.2. Isoperimetric regions

For a surface M, given $a \in (0, A(M))$, we consider the *isoperimetric profile* of M, defined by

$$I(a) = \inf\{L(\partial B) : B \subset M, \text{ smooth with } A(B) = a\}.$$

An *isoperimetric region* $\Omega \subset M$ is a finite perimeter set such that $\mathcal{P}(\Omega) = I(A(\Omega))$. The regularity results by Morgan [7] imply that an isoperimetric region has smooth boundary. The existence of an isoperimetric region for a given area $a > 0$ is not guaranteed in a non-compact surface, since a *minimizing sequence* $\{\Omega_n\}_{n\in\mathbb{N}}$ of sets of area a, and satisfying

$$\lim_{n\to\infty} L(\partial\Omega_n) = I(a),$$

may lose all or part of its area at infinity. However, we have the following result.

LEMMA 2.1 (Ritoré [10, lemma 1.8]). *Let M be a Riemannian surface, $A > 0$, and let $\{\Omega_n\}_n$ be a minimizing sequence for area A. Then Ω_n can be decomposed as*

$\Omega_n = \Omega_n^{\mathrm{c}} \cup \Omega_n^{\mathrm{d}}$, *where*

(i) Ω_n^{c} *converges to a set* $\Omega \subset M$, *with* $A(\Omega) \in [0, A]$,

(ii) Ω_n^{d} *diverges,*

(iii) *if* $L_{\mathrm{c}} = \lim L(\partial \Omega_n^{\mathrm{c}})$ *and* $L_{\mathrm{d}} = \lim L(\partial \Omega_n^{\mathrm{d}})$, *then* $L_{\mathrm{c}} + L_{\mathrm{d}} = I(A)$, *and*

(iv) Ω *is an isoperimetric region of area* $A(\Omega)$.

From this result, we may conclude that if the loss of area at infinity $A_{\mathrm{d}} = \lim_{n\to\infty} \operatorname{area}(\Omega_n^{\mathrm{d}})$ is zero, then Ω is an isoperimetric region for area A.

2.3. Constant geodesic curvature curves

Classical variational formulae for length and area [11], together with the regularity results for isoperimetric regions in surfaces, imply that the boundary of an isoperimetric region has constant geodesic curvature with respect to the inner normal.

Constant geodesic curvature curves in rotationally symmetric surfaces can be classified due to the existence of a first integral coming from the one-parameter group of isometries. From [4, 10] we have the following result.

THEOREM 2.2. *Let M be a symmetric annulus of revolution with non-decreasing Gauss curvature from the shortest geodesic loop. Let C be a curve with constant geodesic curvature and finite length.*

Then C is a parallel, a nodoid-type curve or an unduloid-type curve.

In nodoid-type curves, the tangent vector turns monotonically, presenting points with vertical tangent vector. When these curves are closed, they bound discs in the surface. On the other hand, unduloid-type curves are periodic graphs over θ, symmetric with respect to every critical point of its t-coordinate. We define the *period* of the latter as the θ-distance between two consecutive maxima (or minima) points of the t-coordinate (see figure 1).

The following lemma gives necessary and sufficient conditions for these curves to be closed and embedded.

LEMMA 2.3 (Ritoré [10, proposition 1.3]). *Let C be a curve with constant geodesic curvature in a warped product $\mathbb{S}^1 \times I$.*

(i) *If C is a nodoid-type curve, it yields a closed embedded curve if and only if the maximum and the minimum of $t|_C$ are in the same vertical line.*

(ii) *If C is an unduloid-type curve, it yields a closed embedded curve if and only if the period of C is equal to $2\pi/k$, with $k \in \mathbb{N}$.*

2.4. Stability

We shall say that a smooth curve C of finite length and constant geodesic curvature is *stable* if it is actually a local minimum of the perimeter for variations

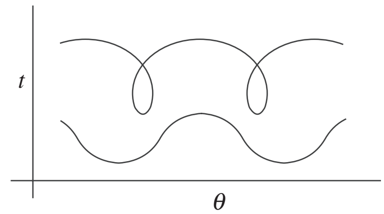

Figure 1. Nodoid-type and unduloid-type curves in the product $\mathbb{S}^1 \times I$.

preserving the area. For such variations, the second derivative of length is given [1] by

$$I(u) = -\int_C u\left\{\frac{\mathrm{d}^2u}{\mathrm{d}s^2} + (K + h^2)u\right\}\mathrm{d}s, \tag{2.3}$$

where $u : C \to \mathbb{R}$ is the normal component of the vector field associated to the variation and s is the arc-length parameter in C. We find that C is stable if and only if

$$I(u) \geqslant 0 \quad \text{for any function } u \text{ such that } \int_C u\,\mathrm{d}s = 0.$$

We will say that a $\Omega \subset M$ is a *stable region* if $\partial\Omega$ is an embedded stable curve with constant geodesic curvature with respect to the inner normal. We remark that any isoperimetric region is stable.

Related to (2.3), we consider the *Jacobi operator* defined by

$$J(u) = \frac{\mathrm{d}^2u}{\mathrm{d}s^2} + (K + h^2)u, \tag{2.4}$$

and, on every connected component $C' \subset C$, the corresponding eigenvalue problem

$$J(u) + \lambda u = 0,$$

for C^2 functions $u : C' \to \mathbb{R}$. The eigenvalues associated to the Jacobi operator will be of interest throughout this paper. We note that a stable curve cannot have more than one connected component with a negative first eigenvalue. We recall some information about these eigenvalues in the following lemma (see [4] for details).

LEMMA 2.4. *Let $C \subset M$ be a connected curve with constant geodesic curvature, and consider $\lambda_1(C)$ the first eigenvalue associated to the Jacobi operator (2.4) in C.*

(i) *If C is a nodoid-type or an unduloid-type curve, then $\lambda_1(C) < 0$.*

(ii) *If C is the parallel $\mathbb{S}^1 \times \{t\}$, then $\lambda_1(C) = -(K + h^2)(t)$.*

The following results discuss the stability of the curves described in theorem 2.2, and of the horizontal annuli bounded by two parallels.

LEMMA 2.5 (Ritoré [10, lemma 1.6]). *A parallel $\mathbb{S}^1 \times \{t\}$ in M is stable if and only if*

$$L^2(K + h^2)(t) \leqslant 4\pi^2, \tag{2.5}$$

or, equivalently, if $((f')^2 - ff'')(t) \leqslant 1$.

LEMMA 2.6 (Ritoré [10, lemma 1.7]). *A horizontal annulus $\mathbb{S}^1 \times [t_1, t_2] \subset M$ is stable if and only if the parallels $\mathbb{S}^1 \times \{t_i\}$ are stable ($i = 1, 2$) and*

$$L^{-1}(K + h^2)(t_1) + L^{-1}(K + h^2)(t_2) \leqslant 0. \tag{2.6}$$

Moreover, in the case of a symmetric annulus $\mathbb{S}^1 \times [-t, t]$, condition (2.6) reduces to

$$(K + h^2)(t) \leqslant 0. \tag{2.7}$$

LEMMA 2.7. *Let $C \subset M$ be a closed embedded nodoid-type curve, not contained in a region with constant Gauss curvature. Then C is unstable.*

Proof. By [10, lemma 2.3], the only closed embedded nodoids will necessarily intersect the parallel $\mathbb{S}^1 \times \{0\}$. But, from [10, lemma 3.4], we conclude that such nodoids are unstable, since $\mathbb{S}^1 \times \{0\}$ is the parallel where K achieves its minimum. □

REMARK 2.8. We recall that closed embedded nodoids contained in regions with constant Gauss curvature are stable, by the classical isoperimetric inequalities.

Applying the same reasoning as in [4, § 2], it can be proved that there exist closed and embedded unduloid-type curves (and even stable ones) in some of our annuli. Moreover, we shall see in § 3 that these curves actually appear as the boundaries of isoperimetric regions. Let us recall a necessary condition for the stability of unduloid-type curves.

LEMMA 2.9 (Cañete [4, lemma 2.2]). *Let $C \subset M$ be a closed embedded stable unduloid-type curve, not contained in the region where $(f')^2 - ff'' = 1$. Then the curve C touches the regions where $(f')^2 - ff'' < 1$ and $(f')^2 - ff'' > 1$.*

Stable symmetric annuli of small area satisfy $\mathrm{d}h/\mathrm{d}A > 0$. These annuli grow up to reach the parallels where $K + h^2 = 0$. At this point we obtain stable asymmetric annuli by letting one of these boundary curves approach and the other move away from the shortest parallel with the same geodesic curvature. In this process we obtain asymmetric annuli with $\mathrm{d}h/\mathrm{d}A < 0$. The deformation continues until the largest parallel in the boundary of the asymmetric annulus reaches a parallel with $L^2(K + h^2) = 4\pi^2$, where unduloids appear. This latter phenomenon has been studied in detail in [4].

3. Main results

As in [4], from theorem 2.2 we can classify the stable regions in our surfaces.

THEOREM 3.1. *Let M be a symmetric annulus of revolution with non-decreasing Gauss curvature from the shortest parallel.*

Then the stable regions in M may be

(i) *discs bounded by a nodoid-type curve, contained in a region with constant Gauss curvature,*

(ii) *horizontal annuli symmetric with respect to the shortest parallel, and bounded by two parallels contained in the region $K + h^2 \leqslant 0$,*

(iii) *non-symmetric horizontal annuli bounded by two parallels satisfying the stability condition (2.6), and contained in the region* $L^2(K+h^2) \leqslant 4\pi^2$,

(iv) *annuli bounded by a stable unduloid-type curve, and a parallel contained in* $K+h^2<0$, *or*

(v) *unions of a disc of constant Gauss curvature, and a symmetric annulus contained in* $K+h^2<0$, *with the same geodesic curvature.*

Proof. Let Ω be a stable region in M. Since the first eigenvalue of the Jacobi operator (2.4) associated to a nodoid-type or an unduloid-type curve is negative, it follows that $\partial\Omega$ will contain at most one of these curves. On the other hand, the geodesic curvature $h(t)$ of parallels has a different sign in each half of the annulus and, consequently, $\partial\Omega$ will contain at most one parallel in each half.

If $\partial\Omega$ does not contain either a nodoid-type curve or an unduloid-type curve, then Ω is a horizontal annulus of type (ii) or (iii) satisfying the conditions described in lemma 2.6. If $\partial\Omega$ has a nodoid-type curve, by lemma 2.7 it turns out that Ω is of type (i) or (v). If $\partial\Omega$ contains an unduloid-type curve, then Ω must be of type (iv), satisfying the condition of lemma 2.9. □

From now on we shall denote by $M(K_\infty)$ the complete plane with constant Gauss curvature K_∞.

THEOREM 3.2. *Let M be a symmetric annulus of revolution with non-decreasing Gauss curvature from the shortest geodesic loop. Consider $A>0$, and a minimizing sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ for area A.*

If the area A_{c} of the convergent part and the area A_{d} of the divergent part are both positive, then the value of the isoperimetric profile $I(A)$ is given by the sum of the perimeter of a stable symmetric annulus in M and the perimeter of a disc in $M(K_\infty)$, both with the same geodesic curvature.

Proof. Since L_{d} is finite, the boundary curves of the sets of the divergent part of the minimizing sequence are homotopically trivial for n large enough. Since $K \leqslant K_\infty$, applying the classical isoperimetric inequality to Ω_n^{d} and passing to the limit we get

$$L_{\mathrm{d}}^2 \geqslant 4\pi A_{\mathrm{d}} - K_\infty A_{\mathrm{d}}^2.$$

Consider a disc $D \subset M(K_\infty)$ of area $A_{\mathrm{d}}>0$. As $K_\infty \leqslant 0$, the injectivity radius of the complement of any compact set in M is infinite. From the isoperimetric inequality it follows that

$$L(\partial D)^2 = 4\pi A_{\mathrm{d}} - K_\infty A_{\mathrm{d}}^2 \leqslant L_{\mathrm{d}}^2,$$

where L_{d} denotes the limit length of the divergent part of the minimizing sequence $\{\Omega_n\}_n$. If $L(\partial D) < L_{\mathrm{d}}$, it is easy to get a contradiction from the minimizing character of $\{\Omega_n\}_n$, simply by considering a family of geodesic discs in M of area A_{d} whose centres diverge. Then $L(\partial D) = L_{\mathrm{d}}$.

The above reasoning shows that $I(A)$ is given by the perimeter of the union $D\cup\Omega$, where Ω is the limit set of the convergent part of $\{\Omega_n\}_{n\in\mathbb{N}}$. The configuration $D\cup\Omega$ in $M(K_\infty)\cup M$ cannot be unstable, since otherwise it could be deformed to a least perimeter configuration with the same area, and the deformation of D could be approximated by a set in M, thus giving a contradiction. Since the first eigenvalue associated to the Jacobi operator (2.4) in D is negative, we conclude that Ω must be a symmetric annulus. □

We shall see in the following results that the possibility of the previous theorem 3.2 cannot hold.

LEMMA 3.3. *Let M be a symmetric annulus of revolution with non-decreasing Gauss curvature from the shortest geodesic loop. If R is a stable symmetric annulus of area A, perimeter L and geodesic curvature h, then*

$$L > hA.$$

Proof. We shall prove the equivalent inequality $L^2 - LhA > 0$. If $R = \mathbb{S}^1 \times [-t, t]$, $t > 0$, it suffices to check that

$$f(t)^2 - f'(t) \int_0^t f(s)\,\mathrm{d}s > 0. \tag{3.1}$$

Let g be the derivative with respect to t of the left-hand term of (3.1). We have

$$g(t) = f(t)f'(t) + K(t)f(t) \int_0^t f(s)\,\mathrm{d}s$$

and

$$g'(t) = h(t)g(t) + m(t),$$

with $m(t) = K'(t)f(t)\int_0^t f(s)\,\mathrm{d}s$.

As $g(0) = 0$, it follows (see [6, corollary 2.1, p. 48]) that

$$g(t) = \exp\biggl(\int_0^t h(s)\,\mathrm{d}s\biggr) \int_0^t \exp\biggl(-\int_0^s h(u)\,\mathrm{d}u\biggr) m(s)\,\mathrm{d}s \geqslant 0.$$

Therefore, the left-hand term in (3.1) is an increasing function, and the desired inequality holds. □

PROPOSITION 3.4. *Let M be a symmetric annulus of revolution with non-decreasing Gauss curvature from the shortest geodesic loop. Let $h > 0$. Then the union of a stable symmetric annulus R_h and a disc D_h in $M(K_\infty)$ with the same geodesic curvature h has a larger perimeter than the disc D in $M(K_\infty)$ of area $A(R_h) + A(D_h)$.*

Proof. Let $-b^2 := K_\infty$ and $A := A(R_h) + A(D_h)$. From the isoperimetric inequality in $M(-b^2)$ we have

$$L(\partial D)^2 = 4\pi A + b^2 A^2.$$

On the other hand, it is easy to check that, for the geodesic disc D_h of geodesic curvature h in $M(-b^2)$, we have

$$hL(\partial D_h) = 2\pi + b^2 A(D_h), \tag{3.2}$$

$$h > b. \tag{3.3}$$

Let us prove that

$$(L(\partial D_h) + L(\partial R_h))^2 > 4\pi A + b^2 A^2. \tag{3.4}$$

By applying the isoperimetric inequality in $M(-b^2)$ to D_h, and lemma 3.3 to R_h, we have

$$\begin{aligned}(L(\partial D_h) + L(\partial R_h))^2 &= L(\partial D_h)^2 + L(\partial R_h)^2 + 2L(\partial D_h)L(\partial R_h) \\ &> 4\pi A(D_h) + b^2 A(D_h)^2 + h^2 A(R_h)^2 + 2hL(\partial D_h)A(R_h).\end{aligned} \tag{3.5}$$

From (3.3) it is clear that

$$h^2 A(R_h)^2 > b^2 A(R_h)^2, \tag{3.6}$$

Finally, by using (3.2) and (3.6), we obtain from (3.5) that

$$(L(\partial D_h) + L(\partial R_h))^2 > 4\pi(A(D_h) + A(R_h)) + b^2(A(D_h) + A(R_h))^2,$$

which proves the statement. □

From proposition 3.4 we obtain two interesting consequences.

COROLLARY 3.5. *Consider a minimizing sequence for area A. Then the areas A_{d}, A_{c} of the divergent and convergent parts cannot be positive simultaneously.*

COROLLARY 3.6. *The value of the isoperimetric profile for some given area $A > 0$ cannot be achieved by the sum of the perimeters of a stable symmetric annulus and a disc in $M(K_\infty)$.*

In other words, the union of a stable symmetric annulus and a disc in $M(K_\infty)$ with the same geodesic curvature 'cannot be an isoperimetric region' in M.

EXAMPLE 3.7. Let M be a minimal catenoid defined by

$$x^2 + y^2 = \lambda^2 \cosh^2\left(\frac{z}{\lambda}\right), \quad \lambda > 0.$$

This surface is included in our family of annuli, and the corresponding warped function is

$$f(t) = (t^2 + \lambda^2)^{1/2}, \quad t \in \mathbb{R}.$$

As $(f')^2 - ff'' < 1$, there are no stable closed unduloid-type curves in M. By a comparison argument, it follows that the discs in $M(K_\infty) = M(0)$ have a smaller perimeter than horizontal annuli, and so the isoperimetric profile of the catenoids is given by the planar isoperimetric inequality $I(a) = (4\pi a)^{1/2}$. The validity of the planar isoperimetric inequality in minimal surfaces is an extremely interesting subject (see [5]).

We now give an example of a symmetric annulus of revolution with non-decreasing Gauss curvature from the shortest parallel, where the largest area stable asymmetric annulus has a smaller perimeter than a disc in $M(K_\infty)$. Then, by the continuity of the isoperimetric profile, it turns out that annuli bounded by an unduloid-type curve and a parallel are also isoperimetric regions.

Example 3.8. Consider the function

$$f(t) := a - r\cos(t/r), \quad t \in [-\pi r, \pi r].$$

with $a > r > 0$. The largest area stable asymmetric annulus is $R = \mathbb{S}^1 \times [t_0, t_1]$, with $t_1 = \frac{1}{2}\pi r$, and

$$t_0 = -r \arccos\left(\frac{2ar}{a^2+r^2}\right).$$

The area A and the perimeter L of this annulus are given by

$$A = \pi a r\left(\pi - \frac{4ar}{a^2+r^2} + 2\arccos\left(\frac{2ar}{a^2+r^2}\right)\right),$$
$$L = 4\pi \frac{a^3}{a^2+r^2}.$$

When a tends to r, it easily follows from the above expressions that

$$L^2 < 4\pi A. \tag{3.7}$$

Fix a, r satisfying (3.7), and consider now, for $s < r$, the function

$$f_s(t) := a - r\cos\left(\frac{t}{s}\right), \quad t \in [-\pi s, \pi s],$$

and the associated surface of revolution with boundary M_s. The Gauss curvature K_s of M_s is increasing for $t > 0$, and the largest area stable asymmetric annulus $R_s = \mathbb{S}^1 \times [t_0^s, t_1^s]$, satisfying $[(f_s')^2 - f_s f_s''](t_1^s) = 1$, corresponds to

$$t_1^s = s\arccos\left(\frac{r^2-s^2}{ar}\right).$$

For s sufficiently close to r, we have

$$K_s(t_1^s) = \frac{-r^2+s^2}{s^2(a^2-r^2+s^2)} < 0$$

and

$$L^2(\partial R_s) < 4\pi A(R_s), \tag{3.8}$$

since the annulus R_s is close to R.

Now fix s close to r satisfying (3.8), and extend the Gauss curvature $K_s : [-t_1^s, t_1^s] \to \mathbb{R}$ to a smooth symmetric negative function $\tilde{K} : \mathbb{R} \to \mathbb{R}$ increasing for $t > 0$, and with $\lim_{t\to\infty} \tilde{K}(t) = 0$. Solving the ordinary differential equation $u'' + \tilde{K}u = 0$ with initial conditions $u(0) = f_s(0)$, $u'(0) = 0$, we obtain a symmetric annulus of revolution $\tilde{M}$ so that $M_s \subset \tilde{M}$.

Then, in $\tilde{M}$, the stable asymmetric annulus with the largest area is R_s, which, in view of (3.8), has a smaller perimeter than the disc of the same area contained in $\tilde{M}(\tilde{K}_\infty) = M(0)$. By the continuity of the isoperimetric profile, we deduce that there are isoperimetric regions in $\tilde{M}$ which are annuli bounded by an unduloid-type curve and a parallel.

In view of theorem 3.1 and the previous examples, we have the following theorem.

THEOREM 3.9. *Let M be a symmetric annulus of revolution with non-decreasing Gauss curvature from the shortest geodesic loop. Then the value of the isoperimetric profile in M is achieved by the perimeter of the following sets.*

(i) *Geodesic discs in $M(K_\infty)$; in this case, if K_∞ is not attained in M, there is non-existence of isoperimetric regions.*

(ii) *Horizontal annuli bounded by two circles of revolution, symmetric with respect to the shortest geodesic loop.*

(iii) *Horizontal asymmetric annuli, bounded by two circles of revolution.*

(iv) *Annuli bounded by an unduloid-type curve and a circle of revolution, with the same geodesic curvature.*

If $K + h^2 \leqslant 0$, then cases (iii) and (iv) can be excluded. If $L^2(K + h^2) \leqslant 4\pi^2$, then (iv) cannot happen.

All the possibilities described in theorem 3.9 can actually occur.

4. Classification of isoperimetric regions in quadrics of revolution

A particular family of surfaces included in our study consists of the one-sheeted hyperboloids. The following result describes the isoperimetric regions in these surfaces.

THEOREM 4.1. *Consider a right circular one-sheeted hyperboloid given by*

$$x^2 + y^2 - \lambda^2 z^2 - \mu^2 = 0, \quad \lambda, \mu \neq 0.$$

Then the value of the isoperimetric profile is achieved by

(i) *a disc in $M(K_\infty) = M(0)$,*

(ii) *a symmetric horizontal annulus or*

(iii) *an asymmetric horizontal annulus.*

Proof. Consider the hyperboloid as a surface of revolution, given by

$$x^2 + y^2 = g(z)^2,$$

with $g(z) := (\mu^2 + \lambda^2 z^2)^{1/2}$. Then the Gauss curvature and the geodesic curvature of the circle with constant height z are given by

$$K(z) = \frac{-g''(z)}{g(z)} \frac{1}{(1 + g'(z)^2)^2}, \qquad h(z) = \frac{g'(z)}{g(z)} \frac{1}{\sqrt{1 + g'(z)^2}}.$$

In the one-sheeted hyperboloid

$$K(z) = \frac{-\lambda^2\mu^2}{(\mu^2 + (\lambda^2 + \lambda^4) z^2)^2},$$

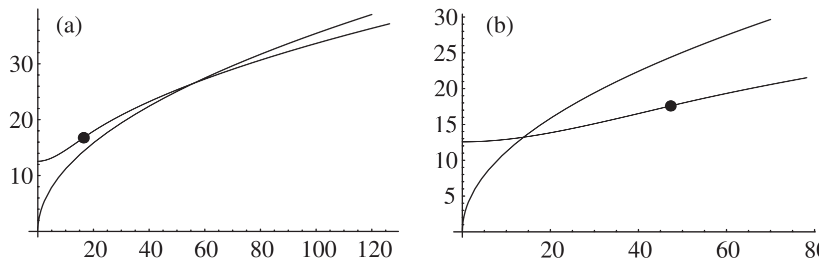

Figure 2. Two possible behaviours in one-sheeted hyperboloids:
(a) $\lambda = 0.8$, $\mu = 1$; (b) $\lambda = 0.3$, $\mu = 1$.

and so $K_\infty = 0$ and

$$(4\pi^2)^{-1} L^2 (K + h^2)(z) = \frac{-\lambda^2\mu^4 + (\lambda^6 + \lambda^8)z^4}{(\mu^2 + (\lambda^2 + \lambda^4)z^2)^2},$$

which is *strictly* less than 1 everywhere. Consequently, stable closed unduloid-type curves do not exist in these surfaces. Therefore, the 'isoperimetric candidates' are discs in $M(K_\infty) = M(0)$ (which actually occur for small areas), symmetric or asymmetric annuli. □

Corollary 4.2. *The isoperimetric profile of a right circular one-sheeted hyperboloid*

$$x^2 + y^2 - \lambda^2 z^2 - \mu^2 = 0, \quad \lambda, \mu \neq 0,$$

is given by

$$I(a) = \min\{(4\pi a)^{1/2}, J(a)\}, \tag{4.1}$$

where $J(a)$ is the perimeter of the only stable annulus (symmetric or asymmetric) of area a. Moreover, for small values of a we have $I(a) = (4\pi a)^{1/2}$ and, for large enough values, $I(a) = J(a)$.

Remark 4.3. The function $J(a)$ can be computed in the following way: let z_0 be the height of the largest stable symmetric annulus, i.e. the only value for which $(K + h^2)(z_0) = 0$, which is given by

$$z_0 = \frac{\mu}{(\lambda^4 + \lambda^6)^{1/4}}.$$

The area of this annulus is given by

$$a_0 = \frac{2\pi\mu^2}{\lambda\sqrt{1+\lambda^2}}(z_0 + \tfrac{1}{2}\sinh(2z_0)).$$

It is easy to check that the perimeter and the area of the stable symmetric annuli $\mathbb{S}^1 \times [-z, z]$, with $z \in [0, z_0]$, is given by

$$L_s(z) = 4\pi g(z),$$
$$A_s(z) = \frac{2\pi\mu^2}{\lambda\sqrt{1+\lambda^2}}(z + \tfrac{1}{2}\sinh(2z)).$$

Table 1. *Non-degenerate quadrics of revolution*

surface type	equation	parameters
right circular cylinder	$x^2+y^2=\mu^2$	$\mu\neq 0$
sphere	$x^2+y^2+z^2=\mu^2$	$\mu\neq 0$
prolate ellipsoid	$x^2+y^2+\lambda^2z^2=\mu^2$	$\lambda^2<1$, $\mu\neq 0$
oblate ellipsoid	$x^2+y^2+\lambda^2z^2=\mu^2$	$\lambda^2>1$, $\mu\neq 0$
paraboloid	$x^2+y^2=\lambda z$	$\lambda\neq 0$
one-sheeted hyperboloid	$x^2+y^2-\lambda^2z^2-\mu^2=0$	$\lambda,\mu\neq 0$
two-sheeted hyperboloid	$x^2+y^2-\lambda^2z^2+\mu^2=0$	$\lambda,\mu\neq 0$

Hence, $J(a)$, for $a\in[0,a_0]$, is the function whose graph coincides with the parametric curve $z\mapsto(A_s(z),L_s(z))$ for $z\in[0,z_0]$.

For a stable asymmetric annulus $\mathbb{S}^1\times[z_1,z_2]$, we have that $z_2>z_0$ and

$$z_1=\frac{-\mu^2}{\lambda^2z_2\sqrt{1+\lambda^2}}.$$

The perimeter and the area of this asymmetric annulus are given by

$$\begin{aligned}L_a(z_2)&=2\pi(g(z_1)+g(z_2)),\\ A_a(z_2)&=\frac{\pi\mu^2}{\lambda\sqrt{1+\lambda^2}}(z_2-z_1+\tfrac{1}{2}(\sinh(2z_2)-\sinh(2z_1))).\end{aligned}$$

Hence, for $a\in[a_0,+\infty)$, $J(a)$ is given by the parametric curve $z\mapsto(A_a(z_2),L_a(z_2))$, for $z_2\in[z_0,+\infty)$.

Proof. We need only to check that, for large values of z_2, the quotient $L_a^2(z_2)/A_a(z_2)$ of the stable asymmetric annulus $\mathbb{S}^1\times[z_1,z_2]$ is strictly smaller than 4π. Since $g(z_1)$ is bounded and $g(z_2)=(\lambda^2z_2^2+\mu^2)^{1/2}$, we have $z_2^{-1}L_a(z_2)\to 2\pi\lambda$ when $z_2\to+\infty$. On the other hand, $z_2^{-2}A_a(z_2)\to+\infty$ when $z_2\to+\infty$. This implies that

$$\lim_{z_2\to+\infty}\frac{L_a^2(z_2)}{A_a(z_2)}=0,$$

which proves the claim. □

We have numerically checked two different behaviours (see figure 2) depending on the parameters λ, μ. In some situations the solutions are given by discs and some asymmetric annuli (for instance, when $\lambda=0.8$, $\mu=1$). In other cases the isoperimetric regions are discs, some symmetric annuli and *all* stable asymmetric annuli (for instance, when $\lambda=0.3$, $\mu=1$).

Theorem 4.1 allows *completion* of the classification of isoperimetric regions in *non-degenerate* quadrics of revolution, which are invariant by a one-parameter group of rotations in Euclidean space. A complete list of such quadrics is given by table 1.

Corollary 4.4. *The isoperimetric regions in the quadrics of revolution are as follows:*

(i) *in the right circular cylinder, either geodesic discs or tubular neighbourhoods of closed geodesics;*

(ii) *in paraboloids and connected components of two-sheeted hyperboloids, geodesic discs centred at the point of maximum curvature;*

(iii) *in spheres, geodesic discs;*

(iv) *in prolate ellipsoids, discs centred at the poles and their complements;*

(v) *in oblate ellipsoids, a family of discs bounded by constant geodesic curvature curves symmetric with respect to the equator, and their complements;*

(vi) *in the one-sheeted hyperboloid, either discs at infinity, symmetric annuli or asymmetric annuli.*

Proof. The case of the right circular cylinder is trivial since $K \equiv 0$. The isoperimetric problem in the paraboloid of revolution was studied in [2]. The same arguments can be applied in each connected component of the two-sheeted hyperboloids, since the Gauss curvature is positive and decreasing from a pole. In both cases, the solutions are geodesic disc centred at the pole.

The case of spheres follows from the classical isoperimetric inequality. Ellipsoids were treated in [10, § 3], and also partially in [8]. Finally, for the one-sheeted hyperboloids, the solutions are described in theorem 4.1. $\square$

Finally, we note that it is possible to describe the isoperimetric profile in the right hyperbolic paraboloid of equation

$$z = \lambda xy, \quad \lambda > 0.$$

Although these surfaces are not invariant by a one-parameter group of rotations in Euclidean space, they exhibit a one-parameter group of intrinsic isometries induced by the rotations in the plane xy with respect to the origin. The Gauss curvature in these surfaces is given by

$$K(x,y) = -\frac{\lambda^2}{(1+\lambda^2(x^2+y^2))^2},$$

and so it is negative, increasing and $K_\infty = 0$. These surfaces were treated in [10], where it was shown that there are no isoperimetric regions for any value of area, and that the isoperimetric profile is given by the planar isoperimetric inequality.

Theorem 4.5. *The isoperimetric profile of the right hyperbolic paraboloid M_λ of equation $z = \lambda xy$ is given by $I(a) = (4\pi a)^{1/2}$. The value of the profile is never attained, so there are no isoperimetric regions in M_λ for any value of the area.*

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