When is the Algebra of Multisymmetric Polynomials Generated by the Elementary Multisymmetric Polynomials ?

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Abstract . Multisymmetric polynomials are the $r-$ fold diagonal invariants of the symmetric group $\mathfrak{S}_n$. Elementary multisymmetric polynomials are analogues of the elementary symmetric polynomials , in the multisymmetric setting . In this paper , we give a necessary and sufficient condition on a r ing A for the algebra of multi - symmetric polynomials with coefficients in A to be generated by the elementary multisymmetric polynomials .

Introduction

It i s well - known that the r ing of symmetric polynomials with integer coefficients i s generated by the elementary symmetric polynomials . This result (with the algebraic independence of the elementary symmetric polynomials) is often referred as " the fundamental theorem of

symmetric *polynomials* "
.

The *multisymmetric polynomials* are the $r-$ fold diagonal invariants of the symmetric group $\mathfrak{S}_n$. When taken with coefficients in a r ing A, they constitute an $A-$ algebra we denote by $\mathfrak{J}_n^r(A)$. Multisymmetric polynomials are generalizations of symmetric polynomials (recovered with $r = 1$). The elementary symmetric polynomials have analogues in the multisymmetric setting , called *elementary multisymmetric polynomials* .

It is quite an o ld theorem that $\mathfrak{J}_n^r(\mathbb{Q})$, the algebra of multisymmetric polynomials with rational coefficients , i s generated by the elementary multisymmetric polynomials . This was established first by Schl ä fli [20] and also by MacMahon [1 5] , and later other proofs were given

by Emmy Noether [1 7] and Hermann Weyl [24] . On the other hand , various authors [1 6 , 2 , 4 , 3 , 5] gave counter - examples showing that the elementary multisymmetric polynomials may not generate $\mathfrak{J}_n^r(A)$, depending on the coefficient r ing A. In particular , John Dalbec proved in [4 , 3] that $\mathfrak{J}_n^r(\mathbb{Z})$, the r ing of multisymmetric polynomials with integer coefficients , is generated by the elementary multisymmetric polynomials only in the trivial cases $n = 1$ or $r = 1$ and in the special case $(n, r) = (2, 2)$.

For which r ings A do the elementary multisymmetric polynomials generate $\mathfrak{J}_n^r(A)$? The purpose of this paper is to demonstrate that we have the fo llowing answer :

Theorem 1 . *Let A be a ring , and n and r be positive integers . The elementary multisym - metric polynomials generate the $A-$ algebra $\mathfrak{J}_n^r(A)$ if and only if $n!$ is invertib le in A, except in the following special cas es :*

$$\text{(i)} \quad (n, r) = (2, 2).$$

$$\text{(i i)} \quad (n, r) = (3, 2) \quad \textit{and} \quad 3 \textit{ is invertib le in } \quad A.$$

$$\text{(iii)} \quad r = 1.$$

In thes e special cas es , the elementary multisymmetric polynomials generate $\mathfrak{J}_n^r(A)$, even if the condition $n!$ invertible does not hold .

Let us give a geometric motivation . Let $\mathbb{K}$ be an algebraically closed field , and V be a $(r+1)-$ dimensional $\mathbb{K}-$ vector space . Let us consider the mapping *formal product* with values in S^nV, the $n-$ th symmetric power of V :

$$\begin{array}{ccc} V^n & \to & S^nV \\ (v_1, \ldots, v_n) & \mapsto & v_1 \cdots v_n. \end{array}$$

It induces an inj ective morphism φ of algebraic varieties from the $n-$ th symmetric product $(\mathbb{P}V)^n/\mathfrak{S}_n$ of the proj ective space $\mathbb{P}V$ of l ines in V, to the projective space $\mathbb{P}(S^nV)$ of lines in S^nV. It s image is a closed algebraic subvariety , called the *Chow variety of multi - s ets of n points in $\mathbb{P}V$*. We denote it by $C\,how(0, n, \mathbb{P}V)$. So we have two algebraic varieties parameterizing the length n multi - sets of points in $\mathbb{P}V$: the symmetric product $(\mathbb{P}V)^n/\mathfrak{S}_n$ and the Chow variety $C\,how(0, n, \mathbb{P}V)$.

Neeman [1 6] showed that a necessary and sufficient condition for these two algebraic varieties to be i somorphic i s that the elementary multisymmetric polynomials generate the $\mathbb{K}-$ algebra $\mathfrak{J}_n^r(\mathbb{K})$ (and i f it is so , then φ is an isomorphism) .

We deduce from Neeman ' s result and Theorem 1 that $(\mathbb{P}V)^n/\mathfrak{S}_n$ and $C\,how(0, n, \mathbb{P}V)$ are i somorphic if and only i f at least one of the following conditions holds :

- the coefficient field $\mathbb{K}$ has characteristic zero ,
- the coefficient field $\mathbb{K}$ has positive characteristic greater than n,
- $(n, r) = (2, 2)$ (two points in the plane) ,
- $(n, r) = (3, 2)$ (three points in the plane) with $\mathbb{K}$ having characteristic two ,
- $r = 1$ (any number of points on the l ine) .

This paper i s organized as fo llows .

In the first s ection , multisymmetric polynomials are introduced .

The s econd section deals with the non - modular case (following the terminology of invari - ant theoretists) , that i s the case when $n!$ i s invertible in the coefficient r ing A. It is proved that in this case , the elementary multisymmetric polynomials generate $\mathfrak{J}_n^r(A)$.

In the third section , it i s demonstrated that when $n!$ i s not invertible , the elementary multisymmetric polynomials don ' t generate $\mathfrak{J}_n^r(A)$, provided $r > 1$ and $(n,r) \neq (2,2), (n,r) \neq$

$$(3,2).$$

The last section is devoted to the two special cases $(n,r) = (2,2)$ and $(n,r) = (3,2)$.

In the sequel , the s et of natural numbers i s denoted by $\mathbb{N}$; and for $\alpha \in \mathbb{N}^r$, we set $\mid \alpha \mid = \alpha_1 + \cdots + \alpha_r$. The zero vector of $\mathbb{N}^r$ is denoted with a bold zero : $\mathbf{0}$.

1 . Multisymmetric polynomials

In this s ection we present the multisymmetric polynomials . We introduce only the necessary material . More can be found about these obj ects in the original works of MacMahon [1 5] and Junker [8 , 9 , 1 0 , 1 1 , 1 2 , 1 3] , or in modern accounts by Dalbec [3 , 4] , by Rosas [1 9] and by the author [1] .

Let $a, b, \ldots, z$ be a finite alphabet with n letters . To each of these letters we associate r variables indexed with the integers from 1 to r :

$$a_1, \ldots, a_r$$
$$b_1, \ldots, b_r$$
$$\ldots$$
$$z_1, \ldots, z_r.$$

We will use the fo llowing notations : the bold symbol $\boldsymbol{a}$ will denote the family of variables $(a_1, \ldots, a_r)$, and $\boldsymbol{a}^\alpha$, where $\alpha \in \mathbb{N}^r$, will denote the monomial $a_1^{\alpha_1} \cdots a_r^{\alpha_r}$.

Definition 1 . *The multisymmetric polynomials in the n families of r variables $\boldsymbol{a}, \ldots, \boldsymbol{z}$ are the polynomials in the $n \times r$ variables $a_1, a_2, \ldots, z_r$ that remain unchanged under every permutation of the n letters $a, b, \ldots, z$.*

Let A be a ring . The algebra of the multisymmetric polynomials in n families of r variab les with coefficients in A is denoted $\mathfrak{J}_n^r(A)$.

Example . When $n = r = 2$, the multisymmetric polynomials are those polynomials P in

$$a_1, a_2, b_1, b_2 \text{suchthat :}$$
$$P(a_1, a_2, b_1, b_2) = P(b_1, b_2, a_1, a_2).$$

We provide $\mathfrak{J}_n^r(A)$ with a grading with values in $\mathbb{N}^r$: the one obtained by giving to every variable x_i(for x any letter in the alphabet $a, \ldots, z$ and i any index between 1 and r) as it s multidegree the $i-$ th vector of the canonical basis of $\mathbb{Z}^r$ (denoted with ξi in the sequel) .

The multidegree of an homogeneous multisymmetric polynomial P will be denoted with mdeg P.

As in the case of symmetric polynomials , the algebra of multisymmetric polynomials admits an obvious module basis : the one made of the symmetrizations of the monomials , called *monomial (multisymmetric) functions* . Their symmetric analogues are the monomial symmetric functions , indexed by the integer partitions . Integer partitions are the finished decreasing s equences of positive integers , that we may think as representing the finished multi - sets of integers . Let λ be an integer partition with length at most n. It i s obtained by forgetting the ordering and the possible occurrences of 0 in a length n integer sequence $\boldsymbol{s} = (s_a, \ldots , s_z)$. The monomial symmetric function in the variables $a, b, \ldots , z$ indexed with λ is the symmetrization of the monomial with exponent $\boldsymbol{s}$, that i s :

$$m_\lambda = \sum_{\mathrm{t}} a^{t_a} b^t b \cdots z^{t_z}$$

where $\boldsymbol{t}$ runs over the orbit of $\boldsymbol{s}$ under the permutations of it s n t erms .

In the same way , and mimicking the vocabulary associated to integer partitions , we set the following definitions :

Definition 2 . *A vector partition of* $\mathbb{N}^r$ *is a finished multi - s et of vectors of* $\mathbb{N}^r \setminus \{0\}$. *If the vector partition* $\mathfrak{p}$ *is ob tained from the s equence*

$$(\alpha^{(1)}, \ldots , \alpha^{(k)})$$

of vectors of $\mathbb{N}^r$, *by forgetting the ordering , and the possib le occurrences of the zero vector , we will denote*

$$\mathfrak{p} = [\alpha^{(1)}, \ldots , \alpha^{(k)}].$$

If moreover al l of the $\alpha^{(i)}$ *are non - zero , we will cal l them the parts of* $\mathfrak{p}$, *and say that* k *is the length of* $\mathfrak{p}$, *denoted* $\ell_\mathfrak{p}$.

The vector $\beta = \alpha^{(1)} + \cdots + \alpha^{(k)}$ *is cal led the sum of* $\mathfrak{p}$, *and denoted* $s(\mathfrak{p})$. *It will als o be written* $\mathfrak{p} \vdash \beta$, *and read that* $\mathfrak{p}$ *is a partition of* β.

Let α *in* $\mathbb{N}^r \setminus \{0\}$. *Its multiplicity in* $\mathfrak{p}$ *is denoted* $\mu_\mathfrak{p}(\alpha)$. *It is often us eful to consider the multi - s et of multiplicities of* $\mathfrak{p}$, *that is the multi - s et ob tained from the s equence of the multiplicities of the parts of* $\mathfrak{p}$. *It is denoted* $\mu_\mathfrak{p}$. *Last , the notation* $\mu_\mathfrak{p}!$ *s tands for the product of the* $\mu_\mathfrak{p}(\alpha)!$, *for* α *running in the parts of* $\mathfrak{p}$.

Definition 3 . *Let* $\mathfrak{p}$ *be a vector partition of* $\mathbb{N}^r$ *with length at most* n. *There exists a s equence of* n *vectors* $\alpha = (\alpha^{(a)}, \ldots , \alpha^{(z)})$ *(with possib ly s ome of them being zero) such that*

$$\mathfrak{p} = [\alpha^{(a)}, \ldots , \alpha^{(z)}].$$

The monomial symmetric function indexed by $\mathfrak{p}$ *in* $\mathfrak{J}_n^r(A)$ *is*

$$m_\mathfrak{p} = \sum \boldsymbol{a}^{\beta^{(a)}} \cdots \boldsymbol{z}^{\beta^{(z)}}$$

where the sum is carried over the s equences $(\beta^{(a)}, \ldots , \beta^{(z)})$ *in the orbit of* α *under the permutations of its* n *terms .*

Remark . The monomial multisymmetric function $m_{\mathfrak{p}}$ has multidegree $s(\mathfrak{p})$.
Two special cases of monomial functions are , on the first hand , when the indexing vector partition has all it s parts members of the canonical basis $(\xi 1, ... , \xi_r)$, then we obtain an
elementary multisymmetric polynomial[1]; and on the second hand , when the indexing vector partition has only one part , then we get a *multisymmetric power sum* . Precise definitions follow . Note that when $r = 1$ the elementary symmetric polynomials and the symmetric power sums are recovered this way .
Definition 4 . *Let* $\alpha \in \mathbb{N}^r$ *with* $1 \leq \mid \alpha \mid \leq n$. *The e lementary multisymmetric polynomial with multidegree* α *is*

$$e_\alpha = m^{\alpha}_{[\xi_1} 1^{\alpha}_{\xi_2} 2 \cdots \xi^{\alpha_r}_r].$$

Otherwise stated , an elementary multisymmetric polynomial i s the symmetrization of a mono - mial in which every letter appears at most once . It i s coherent to set :

$$e_0 = 1$$

and for $\alpha \in \mathbb{N}^r$ with $\mid \alpha \mid > n$

$$e_\alpha = 0.$$

Definition 5 . *Let* $\alpha \in \mathbb{N}^r \setminus \{0\}$. *The multisymmetric power sum with multidegree* α *is*

$$p\alpha = m_{[\alpha]} = \boldsymbol{a}^\alpha + \boldsymbol{b}^\alpha + ... + \boldsymbol{z}^\alpha.$$

Among them , we will distinguish the first power sums : thos e $p\alpha$ *with* $\mid \alpha \mid \leq n$.
We will also index the products of elementary multisymmetric polynomials and multisym - metric power sums by vector partitions . So if

$$\mathfrak{p} = [\alpha^{(1)}, ... , \alpha^{(k)}]$$

then

$$e_{\mathfrak{p}} = e_{\alpha^{(1)}} \cdots e_{\alpha^{(k)}}$$

and

$$p_{\mathfrak{p}} = p_\alpha(1) \cdots p_\alpha(k).$$

Elementary multisymmetric polynomials and multisymmetric power sums are not only gener - alizations of elementary symmetric polynomials and symmetric power sums . Actually , they are obtained through a systematic process , called *polarization* , from their symmetric ana - logues . We present now this process , fo llowing the presentation given by Larry Smith [2 1] .

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[1] In [23], a different natural multisymmetric generalization of the elementary symmetric polynomials ap - pears .

Let $\alpha \in \mathbb{N}^r$ with $\mid \alpha \mid = k$. The *polarization operator* Δ_α^r i s the function from the homoge - neous component of degree k of $\mathbb{Z}[a, b, \ldots, z]$ to the homogeneous component of multidegree α of $\mathbb{Z}[a_1, a_2, \ldots, z_r]$ obtained first by making the substitutions :

$$\begin{aligned} a &\mapsto a_1 + \ldots + a_r \\ b &\mapsto b_1 + \ldots + b_r \end{aligned}$$

. . .

$$z \mapsto z_1 + \ldots + z_r$$

and next by s electing the homogeneous component of multidegree α in the result . For instance , for a power of the variable a, one has :

$$\Delta_\alpha^r(a^k) = \begin{pmatrix} k \\ \alpha \end{pmatrix} \boldsymbol{a}^\alpha$$

where $\binom{k}{\alpha}$ i s the multinomial coefficient :

$$\frac{k!}{\alpha_1!\alpha_2! \cdots \alpha_r!}.$$

From the definitions it is readily seen that polarization sends symmetric polynomials to multisymmetric polynomials . Better ,

Proposition 2 . *Polarization s ends e lementary symmetric polynomials to elementary multi - symmetric polynomials , and symmetric power sums to integer multiples of multisymmetric power sums . Precisely , for* $\alpha \in \mathbb{N}^r$ *with* $\mid \alpha \mid = k$,

$$\Delta_\alpha^r(e_k) = e_\alpha$$

and if $\alpha \neq 0$,

$$\Delta_\alpha^r(pk) = \begin{pmatrix} k \\ \alpha \end{pmatrix} p\alpha\cdot$$

We will denote the subalgebra of $\mathfrak{J}_n^r(A)$ generated by the polarized symmetric polynomials

$$\text{by} \mathfrak{E}_n^r(A).$$

Remark . Let f be any homogeneous symmetric polynomial with degree k. Then there is a polynomial with integer coefficients P such that

$$f = P(e_1, \ldots, e_k).$$

Let $\alpha \in \mathbb{N}^r$ with $\mid \alpha \mid = k$. Apply the polarization operator $\Delta_{\alpha\cdot}^r$ this yields an expression of $\Delta_\alpha^r(f)$ as a polynomial with integer coefficients in the elementary multisymmetric polynomi - als . This shows that $\mathfrak{E}_n^r(A)$ i s generated by the elementary multisymmetric polynomials , and the problem raised in the present paper i s also the problem of knowing when all the polarized symmetric polynomials generate $\mathfrak{J}_n^r(A)$.

2 . Non - modular case

In this s ection , we prove that i f $n!$ i s invertible in A then the elementary multisymmetric polynomials generate $\mathfrak{J}_n^r(A)$. Our proof is essentially the proof already given by Richman [1 8] , but we think our presentation i s clearer .

The proof has three steps .

First , it i s shown that if $n!$ i s invertible , then $\mathfrak{J}_n^r(A)$ i s generated by the power sums .

Next , an inductive formula (the *reduction formula for multisymmetric power sums* , Propo - s it ion 4) is established ; this formula proves that any power sum i s in the $\mathfrak{E}_n^r(A)-$ module generated by the first power sums , under the condition $n!$ invertible (Corollary 5) .

The last step demonstrates that the first power sums are in $\mathfrak{E}_n^r(A)$, under the condition

$n!\text{invertible(Lemma6)}.$

Theorem 3 . *Let A be a ring in which $n!$ is invertib le . Then the multisymmetric power sums generate $\mathfrak{J}_n^r(A)$ as an $A-$ algebra .*

This theorem appears in [2 1] , but with a gap in the proof . This gap i s closed the proof that follows . Another short proof of the theorem is given in [6] .

Proof . Let $\mathfrak{p} = [\alpha^{(1)}, \ldots , \alpha^{(k)}]$ be a vector partition with length $k \leq n$. We want to show that $m_{\mathfrak{p}}$ is in the subalgebra of $\mathfrak{J}_n^r(A)$ generated by the power sums .

It is known that there is a polynomial P with integer coefficients such that :

$$(\underset{\text{expression}}{\overset{\text{see [14]}}{}} \text{for} \underset{\text{asapolynomial}}{\text{of}}{}^{\text{formulas}} k!e_{1,1,\ldots,1}\text{giving} P). \quad \underset{\text{coefficients}}{\text{Apply}_{\text{with}}\text{thepolarizationinteger}}\text{operator}_{\text{inthe}}\Delta^k_{(1,1,\ldots,1)}, \quad \text{this}$$

sums . Now evaluate every variable x_i to $\boldsymbol{x}^{\alpha^{(i)}}$. The image of $e_{1,1,\ldots,1}$ i s readily seen to be an integer multiple of $m_{\mathfrak{p}}$. To determine the multiplicative coefficient , note that $m_{\mathfrak{p}}$ i s the sum of $k!/\mu_{\mathfrak{p}}!$ monomials , while $e_{1,1,\ldots,1}$ is the sum of $k!$ monomials . So the multiplicative coefficient is $\mu_{\mathfrak{p}}!$.

This map also sends multisymmetric power sums to multisymmetric power sums .

So it i s established that $k!\mu_{\mathfrak{p}}!m_{\mathfrak{p}}$ i s a polynomial with integer coefficients in the multi - symmetric power sums . Note that $k!\mu_{\mathfrak{p}}!$ is invertible , s ince $\mu_{\mathfrak{p}}!$ i s a product of integers less than or equal to k, and $k \leq n$. □

The fo llowing formula was suggested to me by Nicolas Thi é ry [22] .

Proposition 4 . (Reduction formula) *Let n be a positive integer . Let $\omega \in \mathbb{N}^r$ with $\mid \omega \mid = n$. Let $\alpha \in \mathbb{N}^r$. Then , in $\mathfrak{J}_n^r(\mathbb{Z})$,*

$$\binom{n}{\omega} p\omega + \alpha = \sum_{\substack{(-\\ \in_{\gamma,\beta}\mathbb{N}^r, \gamma \neq 0 \\ \beta+\gamma=\omega}} 1)^{|\gamma|-1} e_\gamma \binom{\mid \beta \mid}{\beta} p_{\beta+\alpha\cdot} \tag{1}$$

Note that $\binom{n}{\omega}$ is invertible as so on as $n!$ i s invertible .

: $\mathbb{Z}[a, b, \ldots, z]$ one *Proof .* In has :

$$\sum_{i+j=n} (-1)^i e_i a^j = 0$$

where the e_i ' s are the elementary symmetric polynomials in the variables $a, \ldots, z$. Let us apply $\Delta^r_{\omega,}$ this yields :

$$\sum_{+_\gamma \beta = \omega}^{(-} 1)^{|\gamma|} e_\gamma \left(\begin{array}{c} |\beta| \\ \beta \end{array} \right) \boldsymbol{a}^\beta = 0.$$

Let us multiply by $\boldsymbol{a}^\alpha$:

$$\sum_{+_\gamma \beta = \omega}^{(-} 1)^{|\gamma|} e_\gamma \left(\begin{array}{c} |\beta| \\ \beta \end{array} \right) \boldsymbol{a}^{\beta+\alpha} = 0.$$

A s imilar formula , with a replaced by any letter $b, \ldots, z$, can be obtained . Let us sum over the expressions obtained this way and i so late the t erm with $\gamma = 0$ to get formula (1). □

Corollary 5 . *Let A be a ring in which n! is invertible . Then any power sum is in the module over* $\mathfrak{E}^r_n(A)$ *generated by the first power sums .*

Proof . This i s a direct consequence of the reduction formula above , by induction . □

Lemma 6 . *Let A be a ring in which n! is invertib le . Then any of the first power sums is in*

$$\mathfrak{E}^r_n(A).$$

Proof . Let $\alpha \in \mathbb{N}^r$ with $|\alpha| = k$. There is a polynomial P with integer coefficients such that :

$$pk = P(e_1, \ldots, e_k).$$

Apply the polarization operator $\Delta^r_{\alpha,}$ this yields an expression of $\binom{k}{\alpha} p\alpha$ as a polynomial with integer coefficients in the elementary multi - symmetric polynomials .

Now , s ince $k \leq n$, the multinomial coefficient $\binom{k}{\alpha}$ i s invertible in A. □

3 . Modular case

We now study the modular case , that i s the case when $n!$ is not invertible in A.

First , several counter - examples for which $\mathfrak{E}^r_n(A) \neq \mathfrak{J}^r_n(A)$ are presented (Lemmas 7 , 8 , 9) . Next , it is shown that these counter - examples are sufficient to prove that , under the condition $n!$ non - invertible , $\mathfrak{E}^r_n(A) \neq \mathfrak{J}^r_n(A)$ for nearly every value of (n, r), except when $r = 1$ and two special cases whose study is reported to the last section .

Lemma 7

. *Let* $k > 2$ *be an odd number . Let* A *be a ring in which* k *is not invertib le . Let*
fk *be the monomial function :*

$$fk = m_{[line-bracehtipdownright-bracehtipdownleft-line(1,0)(1,0)\cdots(1,0)(0,2)]\cdot}$$
$$k-1$$

Then fk *is not inside the sub - algebra of* $\mathfrak{J}_k^2(A)$ *generated by the elementary multisymmetric polynomials .*

This lemma is inspired by a counter - example due to Fleischmann ([5] , proof of Theorem 4 . 7) .

Proof . Let us suppose that fk l ies in the subalgebra generated by the elementary multi - symmetric polynomials . Let us consider the proj ection from $\mathfrak{J}_k^2(A)$ to $\mathfrak{J}_k^1(A)$, the algebra of symmetric polynomials in $a_1, b_1, \ldots, z_1$, obtained by sending every variable $a_2, b_2, \ldots, z_2$ to 1 and leaving unchanged the variables $a_1, b_1, \ldots, z_1$. It sends fk to e_{k-1}. It also sends every elementary multisymmetric polynomial $e_{i,j}$ to an integer multiple of e_i. Specially , the only $e_{\mathfrak{p}}$ whose image contributes to e_{k-1} are :

$$e_{k-1,1}e_{0,1}, \quad e_{k-1,0}e_{0,1}^2, \quad e_{k-1,0}e_{0,2}.$$

Their images are respectively :

$$ke_{k-1}, \quad k^2e_{k-1}, \quad \left(\begin{array}{c} k \\ 2 \end{array} \right) e_{k-1}.$$

$\mathrm{All}_{\mathrm{binomial}}^{\mathrm{ofthem}}\mathrm{coefficient}_{\binom{k}{2}}^{\mathrm{lieintheideal}}$ i s a $\mathrm{of}\mathfrak{J}_k^1(A)\mathrm{multiplegenerated}_{\mathrm{of}k).}^{\mathrm{bythe}}$ $\mathrm{Therefore},\mathrm{integer}_{1\mathrm{liesin}}\mathrm{the}^k$ $(\mathrm{this}\mathrm{integer}_{\mathrm{idealof}A\mathrm{generated}\mathrm{by}}^{\mathrm{beingodd}}$, $\mathrm{the}\,k$ otherwise stated , k i s invertible in A. □

Lemma 8 . *Let* A *be a ring in which* 2 *is not invertible . The the power sum* $p_(1,1,1)$ *is not*
inside the subalgebra of $\mathfrak{J}_2^3(A)$ *generated by the elementary multisymmetric polynomials .*

This counter - example was known by Campbell , Hughes and Pollack ([2] , Section 6) and
Dalbec ([4] , S ection 1 . 4 and [3] , Section 2 . 1 . 4) .

Proof . We suppose the contrary , that is that $p_(1,1,1)$ i s a $A-$ linear combination of :

$$e_{1,0,0}e_{0,1,0}e_{0,0,1}, \quad e_{1,0,0}e_{0,1,1}, \quad e_{0,1,0}e_{1,0,1}, \quad e_{0,0,1}e_{1,1,0}. \tag{2}$$

Let us consider the proj ection from $\mathfrak{J}_2^3(A)$ to the algebra of symmetric polynomials $\mathfrak{J}_2^1(A)$ obtained by substituting 1 to every one of the variables a_2, b_2, a_3, b_3. It sends $p_(1,1,1)$ to $p1 = e_1$. It also sends the polynomials in (2) respectively to $4e_1, 2e_1, 2e_1$ and $2e_1$. As a result , we get that 2 is invertible in A. □

Lemma 9 . *Let* A *be a ring in which* 2 *is not invertible . Then the power sum* $p_(3,2)$ *is not*
inside the subalgebra of $\mathfrak{J}_4^2(A)$ *generated by the elementary multisymmetric polynomials .*

Proof . Let us consider the algebra morphism from $\mathfrak{J}_4^2(A)$ to $\mathfrak{J}_4^1(A)$, the algebra of symmetric polynomials in the variables a_1, b_1, c_1, d_1, obtained by leaving a_1, b_1, c_1, d_1 unchanged and sending a_2, b_2, c_2, d_2 to 1 . The elementary multisymmetric polynomials have their images given by the following table :

$$e_{0,1} \mapsto 4 \quad e_{0,2} \mapsto 6 \quad e_{0,3} \mapsto 4 \quad e_{0,4} \mapsto 1$$
$$e_{1,0} \mapsto e_1 \quad e_{1,1} \mapsto 3e_1 \quad e_{1,2} \mapsto 3e_1 \quad e_{1,3} \mapsto e_1$$
$$e_{2,0} \mapsto e_2 \quad e_{2,1} \mapsto 2e_2 \quad e_{2,2} \mapsto \quad e_2$$
$$e_{3,0} \mapsto e_3 \quad e_{3,1} \mapsto \quad e_3$$
$$e_{4,0} \mapsto e_4$$

The image of the multisymmetric power sum $p_(3,2)$ i s the symmetric power sum $p3 = a_1^3 + b_1^3 + c_1^3 + d_1^3$. This power sum has a unique decomposition in t erms of elementary symmetric polynomials :

$$p3 = 3e_3 - 3e_2e_1 + e_1^3.$$

Let us suppose that $p_(3,2)$ lies in the subalgebra generated by the elementary multisymmetric polynomials . The only monomials in the e_α with proj ection contributing e_3 are

$$e_{3,0}e_{0,1}^2, \quad e_{3,0}e_{0,2}, \quad e_{3,1}e_{0,1}.$$

But their images are actually

$$16e_3, \quad 6e_3, \quad 4e_3.$$

This implies that the coefficient , 3 , of e_3, l ies inside the ideal generated by 2 in A, and so that 2 i s invertible in A. □

Now these counter - examples are used to produce counter - examples in nearly every $\mathfrak{J}_n^r(A)$, thanks to the fo llowing lemma :

Lemma 1 0 . *Let A be a ring . Let r, n be integers . If the algebra $\mathfrak{J}_n^r(A)$ is not generated by the elementary multisymmetric polynomials , then neither is $\mathfrak{J}_{n+1}^r(A)$, nor $\mathfrak{J}_n^{r+1}(A)$.*

Proof . We prove the contrapositive .

There i s an algebra epimorphism from $\mathfrak{J}_{n+1}^r(A)$ to $\mathfrak{J}_n^r(A)$ that sends the elementary mul - t i symmetric polynomials of $\mathfrak{J}_{n+1}^r(A)$ to elementary multisymmetric polynomials of $\mathfrak{J}_n^r(A)$. It i s the one obtained by s ending the variables associated to the last letter to 0 . So if $\mathfrak{J}_{n+1}^r(A)$ i s generated by its elementary multisymmetric polynomials , so is $\mathfrak{J}_n^r(A)$.

Let us consider the algebra epimorphism from $\mathfrak{J}_n^{r+1}(A)$ to $\mathfrak{J}_n^r(A)$ obtained by annihilating the variables $a_{r+1}, b_{r+1}, \ldots, z_{r+1}$. It s ends the elementary multisymmetric polynomials of $\mathfrak{J}_n^{r+1}(A)$ to zero , or to elementary multisymmetric polynomials of $\mathfrak{J}_n^r(A)$. Therefore i f $\mathfrak{J}_n^{r+1}(A)$ i s generated by its elementary multisymmetric polynomials , so is $\mathfrak{J}_n^r(A)$. □

The final product of this section i s the following proposition .

Proposition 1 1 . *If $n!$ is not invertible in A, and $r > 1$, and $(n, r) \neq (2, 2)$, and $(n, r) \neq (3, 2)$, then the e lementary multisymmetric polynomials do not generate $\mathfrak{J}_n^r(A)$.*

Proof . If $n!$ is not invertible in A, then there exists a prime number $k \leq n$, not invertible in A. If k i s odd and $r > 1$, then Lemma 7 and Lemma 1 0 show that the elementary multisymmetric polynomials do not generate $\mathfrak{J}_n^r(A)$. If $k = 2$ then the same conclusion is reached thanks to Lemma 8 and Lemma 1 0 when $r \geq 3$, and thanks to Lemma 9 and Lemma 1 0 when $r = 2$ with $n \geq 4$. □

4 . Special cases

The previous reasonings gave , up to now , no necessary and sufficient condition on A for the elementary multisymmetric polynomials to generate $\mathfrak{J}_2^2(A)$ and $\mathfrak{J}_3^2(A)$.

In the case $\mathfrak{J}_2^2(A)$ we have no condition at all , and in the case $\mathfrak{J}_3^2(A)$ we j ust know , after Lemma 7 , the necessary condition that 3 has to be invertible .

In this section we show that there are no further conditions .

The main tool for these proofs i s a reduction algorithm due to Fleischmann [5] .

4 . 1 . Fleischmann ' s reduction algorithm

Before presenting Fleischmann ' s algorithm , we need a lemma about the product of monomial functions .

Lemma 1 2 . (Product formula) . *Let* $\mathfrak{p}$, $\mathfrak{q}$ *be vector partitions of* $\mathbb{N}^r$. *Then :*

$$m_{\mathfrak{p}} \cdot m_{\mathfrak{q}} = \sum_{\mathfrak{r}} c(\mathfrak{p}, \mathfrak{q}; \mathfrak{r}) m_{\mathfrak{r}}.$$

where the coefficient $c(\mathfrak{p}, \mathfrak{q}; \mathfrak{r})$ *is defined as fol lows : a s equence of vectors* $, \gamma = (\gamma^{(1)}, \ldots , \gamma^{(\ell)})$, *such that* $[\gamma] = \mathfrak{r}$ *and that* ℓ *be no less than* $\max(\ell_{\mathfrak{p}}, \ell_{\mathfrak{q}})$, *is arbitrarily chosen . Then this coefficient is the number of decompositions :*

$$\begin{pmatrix} \gamma^{(1)} \\ \gamma^{(2)} \\ . \\ . \\ . \\ \gamma^{(\ell)} \end{pmatrix} = \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ . \\ . \\ . \\ \alpha^{(\ell)} \end{pmatrix} + \begin{pmatrix} \beta^{(1)} \\ \beta^{(2)} \\ . \\ . \\ . \\ \beta^{(\ell)} \end{pmatrix}$$

where

$$= \mathfrak{p}$$
$$[(\beta^{(1)}, \ldots , \beta^{(\ell)})] = \mathfrak{q}.$$

Example . Let $\delta \in \mathbb{N}^r \setminus \{0\}$. Then $m_{[\delta,0]} m_{[\delta,0]} = m_{[2\delta,0]} + 2m_{[\delta,\delta]}$. Indeed :

$$\begin{pmatrix} 2\delta \\ 0 \end{pmatrix} = \begin{pmatrix} \delta \\ 0 \end{pmatrix} + \begin{pmatrix} \delta \\ 0 \end{pmatrix},$$

(only one decomposition , thus the co efficient of $m_{[2\delta,0]}$ i s 1) and

$$\begin{pmatrix} \delta \\ \delta \end{pmatrix} = \begin{pmatrix} \delta \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ \delta \end{pmatrix} + \begin{pmatrix} \delta \\ 0 \end{pmatrix}$$

(two decompositions , thus the co efficient of $m_{[\delta,\delta]}$ is 2) .

Let us recall that for any vector partition $\mathfrak{p}$ one has :

$$m_{\mathfrak{p}} = \sum_{[\alpha^{(a)}, \alpha^{(b)}, ..., \alpha^{(z)}] = \mathfrak{p}} a^{\alpha^{(a)}} b^{\alpha^{(b)}} \cdots z^{\alpha^{(z)}}.$$

Let $\mathfrak{p}$ and $\mathfrak{q}$ be vector partitions , then :

$$m_{\mathfrak{p}} \cdot m_{\mathfrak{q}} = \sum a^{\alpha^{(a)}+\beta^{(a)}} b^{\alpha^{(b)}+\beta^{(b)}} \cdots z^{\alpha^{(z)}+\beta^{(z)}}$$

the sum being carried over all couples of sequences

$$(\alpha^{(a)}, ... \quad, \alpha^{(z)})$$
$$(\beta^{(a)}, ... \quad, \beta^{(z)})$$

such that $[(\alpha^{(a)}, ... \quad, \alpha^{(z)})] = \mathfrak{p}$ and $[(\beta^{(a)}, ... \quad, \beta^{(z)})] = \mathfrak{q}$. So we are done . □

We will say that $\alpha \in \mathbb{N}^r$ *dominates* $\beta \in \mathbb{N}^r$ if for all $i, \alpha_i \geq \beta i$.

Fleischmann ' s algorithm is presented in the proof of the following lemma .

Lemma 13 . ([5] , Theorem 4 . 6) *The multisymmetric polynomials with multidegree dominated by* $(n-1, n-1, ... \quad, n-1)$ *generate* $\mathfrak{J}_n^r(\mathbb{Z})$ *as an* $\mathfrak{E}_n^r(\mathbb{Z})-$ *module .*

Proof . The proof consists in a reduction algorithm , expressing any monomial function with multidegree not dominated by $(n-1, ... \quad, n-1)$ as a l inear combination , with coefficients in $\mathfrak{E}_n^r(\mathbb{Z})$, in the monomial functions with multidegree strict ly lower (according to the dominance ordering) .

Let us choose a coordinate $i \in \{1, ... \quad, r\}$. To every $\mathfrak{p}$, vector partition of $\mathbb{N}^r$, we associate $\lambda(i; \mathfrak{p})$: the s equence of the $i-$ th coordinates of the parts of $\mathfrak{p}$, in the decreasing order . For instance , to $\mathfrak{p} = [(3,1)(2,0)(1,1)]$, it i s associated $\lambda(1; \mathfrak{p}) = (3,2,1)$ and $\lambda(2; \mathfrak{p}) = (1,1,0)$.

We define a partial ordering $\preceq_i$ on the set of vector partitions of $\mathbb{N}^r$ with length at most n in the fo llowing way : $\mathfrak{p} \preceq_i \mathfrak{q}$ if and only i f $\lambda(i; \mathfrak{p})$ is smaller than $\lambda(i; \mathfrak{q})$ in lexicographic order .

Let $\mathfrak{p}$ be a vector partition whose sum of it s $i-$ th coordinates i s not less than n.

- If $\lambda(i; \mathfrak{p}) = (t_1, t_2, ... \quad, t_s, k, ... \quad, k, 0, ... \quad, 0)$ with $t_1 \geq t_2 \geq ... \geq t_s > k > 0$, then we set $\mathfrak{r}$ the vector partition obtained from $\mathfrak{p}$ by changing in the $i-$ th coordinates of it s parts t_j into $t_j - 1$.

The product formula shows that :

$$m_{\mathfrak{p}} = m_{\mathfrak{r}} e_{s\xi i} - \sum m_{\mathfrak{q}}$$

for vector partitions $\mathfrak{q} \prec_i \mathfrak{p}$.

- If $\lambda(i; \mathfrak{p}) = (k, k, ... \quad, k, 0, ... \quad, 0)$ with s occurrences of $k > 1$, then we s et $\mathfrak{r}$ the vector partition obtained from $\mathfrak{p}$ by changing in the $i-$ th coordinates of it s parts k into $k-1$. The product formula shows that :

$$m_{\mathfrak{p}} = m_{\mathfrak{r}} e_{s\xi i} - \sum m_{\mathfrak{q}}$$

for vector partitions $\mathfrak{q} \prec_i \mathfrak{p}$.

- If

$\lambda(i;\mathfrak{p}) = (1, 1, \ldots\ , 1)$ with n o ccurrences of 1 , then one has the factorization :

$$m_{\mathfrak{p}} = m_{\mathfrak{r}} e_{n\xi i}$$

where $\mathfrak{r}$ is the vector partition obtained form $\mathfrak{p}$ by vanishing the $i-$ th coordinates of it s parts . By applying these three types of reduction , any monomial function with multidegree α is expressed as a l inear combination , with coefficients in $\mathfrak{E}_n^r(\mathbb{Z})$, in the monomial functions with multidegree β, with β dominated by α, distinct from α, and $\beta i \leq n-1$.

By applying successively this procedure for $i = 1$, and next $i = 2$, ... , $i = r$, it is obtained an expression as a polynomial with integer coefficients , in the elementary mul - t i symmetric polynomials and in the monomial functions with multidegree dominated by

$$(n-1, n-1, \ldots\ , n-1). \quad \square$$

4.2. **Case** $(\boldsymbol{n}, \boldsymbol{r}) = (2, 2)$

Here we want to prove that the elementary multisymmetric polynomials generate $\mathfrak{J}_2^2(\mathbb{Z})$. After Lemma 1 3 , this is reduced to prove that every monomial function with multidegree (1 , 0) , (0 , 1) or (1 , 1) can be expressed as a l inear combination with integer co efficients of products of elementary multisymmetric polynomials . The multisymmetric polynomials of multidegree (1 , 0) (or more generally $(k, 0)$ for any $k \in \mathbb{N}$) are actually symmetric polynomials in the variables a_1, b_1, and thus can be expressed as polynomials with integer coefficients in the elementary symmetric polynomials in a_1, b_2. The latter are elementary multisymmetric
polynomials . The same reasoning holds for the multisymmetric polynomials of multidegree $(0, k)$: they are symmetric polynomials in the variables a_2, b_2.

So we have j ust to deal with the multidegree (1 , 1) case . There are two monomial functions
of multidegree (1 , 1) in $\mathfrak{J}_2^2$, that are $m_{[(1,1)]}$ and $m_{[(1,0),(0,1)]}$. It is easy to check that :

$$m_{[(1,1)]} = e_{1,0}e_{0,1} - e_{1,1}$$
$$m_{[(1,0),(0,1)]} = e_{1,1}.$$

Therefore :

Proposition 14 . *The e lementary multisymmetric polynomials generate the ring* $\mathfrak{J}_2^2(\mathbb{Z})$.
This proposition 1 4 was proved by John Dalbec in [4] . Dalbec ' s argument i s the following : the elementary multisymmetric functions and the monomial functions $m_{[(n,0)(0,1)]}$ form a SAGBI basis of $\mathfrak{J}_2^2(\mathbb{Z})$ for a certain ordering on the variables a_1, a_2, b_1, b_2.

4.3. **Case** $(\boldsymbol{n}, \boldsymbol{r}) = (3, 2)$

We want to prove that the elementary multisymmetric polynomials generate the algebra $\mathfrak{J}_3^2(A)$, provided 3 i s invertible in A. After Proposition 1 3 , this i s reduced to prove that any monomial function with multidegree among :

$$(1,0), \quad (2,0), \quad (0,1), \quad (0,2),$$
$$(1,1), \quad (2,1), \quad (1,2), \quad (2,2)$$

l ies inside the subalgebra generated by the elementary multisymmetric polynomials .

The case of a monomial function with multidegree (1 , 0) , (2 , 0) , (0 , 1) or (0 , 2) is trivial , as explained in the case $(n, r) = (2, 2)$. Moreover , any result valid for the case (2 , 1) is also valid for the case (1 , 2) , by permutation of the coordinates . So we are done once we have established the result for multidegrees (1 , 1) , (2 , 1) and (2 , 2) .

There are two vector partitions indexing the monomial functions with multidegree (1 , 1) , as well as the products of elementary multisymmetric polynomials of this multidegree . These vector partitions are [(1 , 1)] and [(1 , 0) , (0 , 1)] , and we have :

$$\begin{aligned} m_{[(1,1)]} &= e_{1,0}e_{0,1} - e_{1,1} \\ m_{[(1,0)(0,1)]} &= e_{1,1}. \end{aligned}$$

There are four vector partitions indexing the monomial functions with multidegree (2 , 1) , as well as the products of elementary multisymmetric polynomials of this multidegree . These vector partitions are :

$$\begin{aligned} \mathfrak{p}1 &= [(2,1)], \quad \mathfrak{p}2 = [(2,0)(0,1)], \\ \mathfrak{p}3 &= [(1,1)(1,0)], \quad \mathfrak{p}4 = [(1,0)(1,0)(0,1)]. \end{aligned}$$

Let us introduce the *conversion matrix* from the family e to the family m in the multidegree (2 , 1) component of $\mathfrak{J}_3^2(\mathbb{Z})$. This is the matrix whose entry in column $e_{\mathfrak{p}}$ and l ine $m_{\mathfrak{q}}$ i s the coefficient of $m_{\mathfrak{q}}$ in the decomposition of $e_{\mathfrak{p}}$ as a l inear combination of monomial functions . We computed it :

Table ignored!

It is invertible over $\mathbb{Z}$. Thus every monomial function with multidegree (2 , 1) lies ins ide the subring generated by the elementary multisymmetric polynomials .

There are eight vector partitions indexing the monomial functions with multidegree (2 , 2) , that are :

$$\begin{aligned} \mathfrak{q}1 &= [(2,2)], \quad \mathfrak{q}2 = [(2,1)(0,1)], \\ \mathfrak{q}3 &= [(2,0)(0,2)], \quad \mathfrak{q}4 = [(2,0)(0,1)(0,1)], \\ \mathfrak{q}5 &= [(1,2)(1,0)], \quad \mathfrak{q}6 = [(1,1)(1,1)], \\ \mathfrak{q}7 &= [(1,1)(1,0)(0,1)], \quad \mathfrak{q}8 = [(1,0)(1,0)(0,2)]. \end{aligned}$$

There are also eight vector partitions indexing the products of elementary multisymmetric polynomials with multidegree (2 , 2) . These are

$$\begin{aligned} &\mathfrak{q}2, \mathfrak{q}3, \mathfrak{q}4, \mathfrak{q}5, \mathfrak{q}6, \mathfrak{q}7, \mathfrak{q}8 \\ \text{and} &\mathfrak{q}9 = [(1,0)(1,0)(0,1)(0,1)]. \end{aligned}$$

conversion matrix i The
s :

Table ignored!

It s determinant is 3 . Therefore , the elementary multisymmetric polynomials generate the multidegree (2 , 2) component of $\mathfrak{J}_3^2(A)$ if and only i f 3 is invertible in A.

We have finally proved the following proposition :

Proposition 1 5 . *The e lementary multisymmetric polynomials generate* $\mathfrak{J}_3^2(A)$ *if and only if*
3 *is invertible in* A.

This completes the proof of Theorem 1 .

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