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ON INITIAL AND TERMINAL VALUE PROBLEMS FOR FRACTIONAL NONCLASSICAL DIFFUSION EQUATIONS

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Abstract. In this paper, we consider fractional nonclassical diffusion equations under two forms: initial value problem and terminal value problem. For an initial value problem, we study local existence, uniqueness, and continuous dependence of the mild solution. We also present a result on unique continuation and a blow-up alternative for mild solutions of fractional pseudo-parabolic equations. For the terminal value problem, we show the well-posedness of our problem in the case $0 < \alpha \leq 1$ and show the ill-posedness in the sense of Hadamard in the case $\alpha > 1$. Then, under the a priori assumption on the exact solution belonging to a Gevrey space, we propose the Fourier truncation method for stabilizing the ill-posed problem. A stability estimate of logarithmic-type in L^q norm is first established.

1. Introduction

In this paper, we consider the following nonlinear nonclassical diffusion equation (called pseudo-parabolic equation):

$$\begin{cases} u_t - k\Delta u_t + (-\Delta)^\alpha u &= G(u) \quad \text{in} \quad (0,T] \times \Omega, \\ u(t,x) &= 0 \quad \text{on} \quad (0,T] \times \partial\Omega, \end{cases} \tag{1}$$

where $k > 0$, and $\Omega \subset \mathbb{R}^d, (d \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, the operator $(-\Delta)^\alpha$ is the fractional Laplacian with $\alpha \in (0,1) \cup (1,\infty)$. Let us divide the nonclassical diffusion equation into two different problems as follows:

- *Initial value problem:* This problem consists of finding $u(x,t)$ for $0 < t \leq T$ from the initial state

$$u(x,0) = u^0(x), \ x \in \Omega. \tag{2}$$

- *Terminal value problem:* This problem is related to recovering $u(x,t)$ for $0 \leq t \leq T$ from the terminal value data (or final state)

$$u(x,T) = u_T(x), \ x \in \Omega. \tag{3}$$

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Nonclassical diffusion equations (or pseudo-parabolic equations) are characterized by the occurrence of a time derivative appearing in the highest order term, which describes various important physical processes. It can be regarded as a Sobolev-type equation or a Sobolev-Galpern-type equation. Pseudo-parabolic equations have many applications in science and technology, especially in physical phenomena such as seepage of homogeneous fluids through a fissured rock, aggregation of populations, etc.; see, e.g., [7] and its references. If $\alpha = 1$, then the fractional operator $(-\Delta)^\alpha$ becomes the standard Laplace operator. In this special case, the nonclassical diffusion equation has been studied in [2,4–8,11,14,19–21] with various directions and motivations. Until now, the results on fractional pseudo-parabolic equations are limited and can be mentioned in just a few papers, for example, [7,22,28,29]. Since the fractional operator $(-\Delta)^\alpha$ appearing in the main equation is nonlocal and can be regarded as the infinitesimal generator of Levy stable diffusion processes, many scientists believed that it described some physical phenomena more exactly than integral differential equations. As in [7], equation (1) is of the regularity-gain-type for $\alpha > 1$, and of the regularity-loss-type for $0 < \alpha < 1$. Our new results and main contributions in this paper are described as follows:

- For *the initial value problem* (1)-(2), our main goal is the study of existence, uniqueness, continuous dependence, unique continuation of solutions, and a blow-up alternative under critical nonlinearity of source function G. As we know, nonlinear PDEs with critical nonlinearities are an interesting topic. This is mentioned in [3, 15] and the references therein. Studying the initial value problem for (1)-(2) in the critical case is also a challenging problem. We note that the work on global existence, blow-up criterion, and continuation of solutions for PDEs has recently attracted many authors, for example, T. Issa and W. Chen [16–18], T. Caraballo et al. [19], A. N. Carvalho et al. [15, 26], B. de Andrade et al. [23–25] and the references therein.
- For *the terminal value problem for pseudo-parabolic equation* (1)-(3), to the best of our knowledge, there are not any results about it. Our work is the first study in this direction which is divided into various cases. Under the case $0 < \alpha \leq 1$, we state well-posedness of the terminal value problem. However, the property of the solution in the case $\alpha > 1$ is very different from one in the case $0 < \alpha \leq 1$. The well-posedness of problems (1)-(3) with $\alpha > 1$ is not guaranteed because of the ill-posedness of the backward problem in the sense of Hadamard [13]. An "ill-posed problem" (not well-posed problem) is a problem that either has no solutions in the desired class, or has many (two or more) solutions, or the solution procedure is unstable (i.e., arbitrarily small errors in the measurement data may lead to indefinitely large errors in the solutions). Our main goal in this paper is to provide some regularized solutions that are called regularized solutions for approximating $u(x,t), 0 \leq t < T$. In this paper, we do not investigate the existence and uniqueness of the solution of the backward problem (1)-(3) with $\alpha > 1$. It is also a challenging and open problem, and should be the topic for another paper. In this paper, we assume that the backward problem (1)-(3) has a unique solution u (called a sought solution) that belongs to an appropriate space. So our main purpose in this case is to consider a regularized problem for finding an approximate solution. Furthermore, error estimates with

the speed of convergence between the regularized solution and the sought solution under some a priori assumptions on the sought solution are also our primary purpose.

- This is a first study for the backward problem for PDEs when the noisy data $\varphi_\epsilon \in L^p(\Omega)$ for $1 \le p < 2$. Since $L^p(\Omega) \hookrightarrow L^2(\Omega)$, we know that if $\varphi_\epsilon \in L^p(\Omega)$ for $p \ge 2$, then it belongs to $L^2(\Omega)$. So, the analysis for the noisy data belongs to $L^p(\Omega)$ for $p \ge 2$ is trivial and similar to the $L^2(\Omega)$ setting. However, in some physical practice, in oder to approximate u_T, we only obtain the noisy data φ_ϵ which only belongs to class $L^p(\Omega)$ for $p < 2$. This means that we only get the noisy level ϵ which is the upper bound of the error $\|\varphi_\epsilon - u_T\|_{L^p(\Omega)}$. Section 3 is the first result for a regularized solution and the convergence analysis in L^p norms. The case of L^2 is much easier than the other ones since one can use the Parseval equality in obtaining stability estimates. The technique in the L^p estimate here is more complex than the L^2 estimate, since we do not have the Parseval equality. Our new technique in this section is based on applying some Sobolev embedding. This is a new and strong point of this paper. We also emphasize that our method in the current paper can be applied for various PDEs such as parabolic equations, elliptic equations, etc.

The paper is organized as follows. In Section 1, we introduce our problem and the motivation for our study. In Section 2, we state and prove a local well-posedness and the regularity result for (1)-(2). The existence of a unique continuation and a blow-up alternative for the mild solution of the abstract problems (1)-(2) are also mentioned in Section 2. In Section 3, we establish a regularized solution using a truncation method as in [9, 10, 12] and show the well-posedness of our problem. We also mention our results concerning the error estimate in the L^p norm of the regularized solution and the sought solution.

Let us consider the operator $\mathcal{A} = -\Delta$ on $\mathbb{V} := \mathcal{H}_0^1(\Omega) \cap H^2(\Omega)$, and assume that the operator $\mathcal{A}$ has the eigenvalues λ_j such that $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \ldots$ which approach ∞ as j goes to ∞. The corresponding eigenfunctions are denoted by $e_j \in \mathbb{V}$. Now, let us define fractional powers of $\mathcal{A}$ and its domain. For all $s \ge 0$, we define by $\mathcal{A}^s$ the following operator $\mathcal{A}^s v := \sum_{j=1}^{\infty} \langle v, e_j \rangle \lambda_j^s e_j, \quad v \in D(\mathcal{A}^s) = \{v \in L^2(\Omega) :\ \sum_{j=1}^{\infty} |\langle v, e_j\rangle|^2 \lambda_j^{2s} < \infty\}$. The domain $D(\mathcal{A}^s)$ is a Banach space equipped with the norm $\|v\|_{D(\mathcal{A}^s)} := \left(\sum_{j=1}^{\infty} |\langle v, e_j\rangle|^2 \lambda_j^{2s}\right)^{\frac{1}{2}}, \quad v \in D(\mathcal{A}^s)$. The definition of the negative fractional power $\mathcal{A}^{-s}$ with $s > 0$ can be found in H. Brezis [1]. Its domain $D(\mathcal{A}^{-s})$ is a Hilbert space endowed with the dual inner product $\langle ., . \rangle_{-s,s}$ taken between $D(\mathcal{A}^{-s})$ and $D(\mathcal{A}^s)$. This generates the norm $\|v\|_{D(\mathcal{A}^{-s})} = \left(\sum_{j=1}^{\infty} | \langle v, e_j \rangle_{-s,s} |^2 \lambda_j^{-2s}\right)^{\frac{1}{2}}$. Let $L^\infty(0, T; \mathbf{G}_{\alpha,k}(\Omega))$ (see [27]) be the following space:

(4)
$$L^\infty(0, T; \mathbf{G}_{\alpha,k}(\Omega)) := \left\{ v \in L^\infty(0, T; L^2(\Omega)),\ \sup_{0 \le t \le T} \exp\left(2tk^{-1}\lambda_j^{\alpha-1}\right) \langle v, e_j \rangle^2 < \infty \right\}$$

for any $\alpha > 1$.

2. Well-posedness of initial value problems (1)-(2)

In this section, we study the well-posedness of problems (1)-(2). Our main purposes are to ensure sufficient conditions for existence and uniqueness of mild solutions to (1)-(2), analyze the possible continuation of this solution to a maximal interval of existence, and consider the problem of continuous dependence with respect to initial data. Let us set

$$\mathcal{S}_{\alpha,k}(t)w = \sum_{j=1}^{\infty} \exp\left(-t\lambda_j^{\alpha}(1+k\lambda_j)^{-1}\right)\langle w, e_j\rangle e_j,\ \mathcal{B}_{\alpha,k}(t)w = \sum_{j=1}^{\infty} \frac{\exp\left(-t\lambda_j^{\alpha}(1+k\lambda_j)^{-1}\right)}{1+k\lambda_j}\langle w, e_j\rangle e_j.$$

Then from [7], we deduce the mild solution of the following initial value problem:

$$u(t) = \mathcal{S}_{\alpha,k}(t)u^0 + \int_0^t \mathcal{B}_{\alpha,k}(t-s)G(u(s))ds. \tag{5}$$

In this section, for simplicity, we only study problems (1)-(2) with the case $0 < \alpha \le 1$. The case $\alpha > 1$ may be more complex but we omit it here since it can be considered by a similar argument.

2.1. **Local well-posedness and regularity.** In this subsection, we consider the following function with the critical nonlinearity form $G : D(\mathcal{A}^{\eta}) \longrightarrow D(\mathcal{A}^{\nu}),\ \nu < \eta < \nu+\alpha$ and satisfy the following:

$$\begin{cases} \|G(v)-G(w)\|_{D(\mathcal{A}^{\nu})} \le K\Big(1+\|v\|^{p-1}_{D(\mathcal{A}^{\eta})}+\|w\|^{p-1}_{D(\mathcal{A}^{\eta})}\Big)\|v-w\|_{D(\mathcal{A}^{\eta})}, \\ \|G(v)\|_{D(\mathcal{A}^{\nu})} \le K\Big(\|v\|^{p}_{D(\mathcal{A}^{\eta})}+1\Big) \end{cases} \tag{6}$$

for $p > 1$, $K > 0$, and $v, w \in D(\mathcal{A}^{\eta})$.

Lemma 2.1. *Let any $w_0 \in D(\mathcal{A}^{\eta})$ and $u \in \mathscr{C}\big([0,\mathcal{T}_0], D(\mathcal{A}^{\eta})\big)$ such that*

$$\sup_{0\le t\le \mathcal{T}_0} \|u(t)-w_0\|_{D(\mathcal{A}^{\eta})} \le \mathcal{M}.$$

If $\nu < \eta < \nu+\alpha$, then

$$\left\|\int_0^t \mathcal{B}_{\alpha,k}(t-s)G(u(s))ds\right\|_{D(\mathcal{A}^{\eta})} \le \overline{K}\Big[\Big(\|w_0\|_{D(\mathcal{A}^{\eta})}+\mathcal{M}\Big)^p+1\Big]t^{\frac{\alpha+\nu-\eta}{\alpha}}. \tag{7}$$

Proof. We have

$$\begin{aligned} &\left\|\int_0^t \mathcal{B}_{\alpha,k}(t-s)G(u(s))ds\right\|_{D(\mathcal{A}^{\eta})} \\ &\quad\le \int_0^t \Big\|\mathcal{B}_{\alpha,k}(t-s)G(u(s))\Big\|_{D(\mathcal{A}^{\eta})}\mathrm{d}s \\ &\quad\le \int_0^t \left(\sum_{j=1}^{\infty}\frac{\lambda_j^{2\eta}}{(1+k\lambda_j)^2}\exp\left(-2(t-s)\lambda_j^{\alpha}(1+k\lambda_j)^{-1}\right)\Big\langle G(u(s)), e_j\Big\rangle^2\right)^{\frac{1}{2}}\mathrm{d}s. \end{aligned}$$

Using the inequality $e^{-y} \le M_\beta y^{-\beta}$ for any $\beta > 0$ and $M_\beta > 0$ is a positive constant, we derive the estimate

$$\exp\left(-2(t-s)\lambda_j^\alpha(1+k\lambda_j)^{-1}\right) \le |M_\beta|^2\Big(1+k\lambda_1\Big)^{2\beta}\lambda_j^{-2\alpha\beta}(t-s)^{-2\beta},$$

which implies that

$$\begin{aligned}
&\left\|\int_0^t \mathcal{B}_{\alpha,k}(t-s)G(u(s))ds\right\|_{D(\mathcal{A}^\eta)}\\
&\quad\le M_\beta\Big(1+k\lambda_1\Big)^{\beta-1}\int_0^t (t-s)^{-\beta}\Big\|G(u(s))\Big\|_{D(\mathcal{A}^{\eta-\alpha\beta})}ds\\
&\quad\le M_\beta\Big(1+k\lambda_1\Big)^{\beta-1}\int_0^t (t-s)^{-\beta}\Big\|G(u(s))\Big\|_{D(\mathcal{A}^{\nu})}ds\\
&\quad\le KM_\beta\Big(1+k\lambda_1\Big)^{\beta-1}\int_0^t (t-s)^{-\beta}\Big(\|u(s)\|^p_{D(\mathcal{A}^\eta)}+1\Big)ds\\
&\quad\le \overline{K}\Big[\Big(\|w_0\|_{D(\mathcal{A}^\eta)}+\mathcal{M}\Big)^p+1\Big]\int_0^t (t-s)^{-\beta}ds
\end{aligned}$$

$$(8)\qquad = \overline{K}\Big[\Big(\|w_0\|_{D(\mathcal{A}^\eta)}+\mathcal{M}\Big)^p+1\Big]t^{\frac{\alpha+\nu-\eta}{\alpha}}.$$

where we have used $\beta = \frac{\eta-\nu}{\alpha} < 1$, and set $\overline{K} = \frac{K\alpha M_\beta\Big(1+k\lambda_1\Big)^{\beta-1}}{\alpha+\nu-\eta}$. □

Theorem 2.2. *Let G satisfy* (6)*. Then there exists $\mathcal{T}_0 > 0$ such that problem* (1) *has a unique mild solution $u \in \mathscr{C}\big((0,\mathcal{T}_0);\mathcal{H}^\eta(\Omega)\big)$.*

Proof. Let any $0 < \mathcal{M} < 1$ and $\mathcal{R} = (\mathcal{M} + \|w_0\|_{D(\mathcal{A}^\eta)})^{p-1}$. Let us choose $\mathcal{T}_0$ such that

$$(9)\qquad \begin{cases} {\mathcal{T}_0}^{1-\beta} \le \dfrac{1-\beta}{2KM_\beta\Big(1+k\lambda_1\Big)^{\beta-1}\Big(1+2\mathcal{R}\Big)},\\[2ex] \mathcal{T}_0 k\lambda_1^{\alpha-1}\big\|w_0\big\|_{D(\mathcal{A}^\eta)} + \overline{K}\left[\Big(\|w_0\|_{D(\mathcal{A}^\eta)}+\mathcal{M}\Big)^p+1\right]\frac{\mathcal{T}_0^{1-\beta}}{1-\beta} \le \mathcal{M}. \end{cases}$$

Using the inequality $1 - e^{-y} \le y$ and noting that $\alpha \le 1$, we have

$$\begin{aligned}
\Big\|\mathcal{S}_{\alpha,k}(t)w_0 - w_0\Big\|^2_{D(\mathcal{A}^\eta)} &= \sum_{j=1}^\infty \lambda_j^{2\eta}\Big[1-\exp\left(-t\lambda_j^\alpha(1+k\lambda_j)^{-1}\right)\Big]^2\Big\langle w_0, e_j\Big\rangle^2_{L^2(\Omega)}\\
(10)\qquad &\le \sum_{j=1}^\infty \lambda_j^{2\eta}\Big[\frac{t\lambda_j^\alpha}{1+k\lambda_j}\Big]^2\Big\langle w_0, e_j\Big\rangle^2_{L^2(\Omega)} \le t^2k^2\lambda_1^{2\alpha-2}\big\|w_0\big\|^2_{D(\mathcal{A}^\eta)}.
\end{aligned}$$

Let us define the following set:

$$\mathscr{F} = \left\{u \in \mathscr{C}\big([0,\mathcal{T}_0];D(\mathcal{A}^\eta)\big),\ \sup_{0\le t\le \mathcal{T}_0}\|u(t)-w_0\|_{D(\mathcal{A}^\eta)} \le \mathcal{M}\right\}.$$

It is not difficult to see that $\mathscr{F}$ is a complete space. Let us define the operator $\mathscr{J} : \mathscr{F} \to \mathscr{F}$ as follows:

$$(11)\qquad \mathscr{J}u(t) = \mathcal{S}_{\alpha,k}(t)u^0 + \int_0^t \mathcal{B}_{\alpha,k}(t-s)G(u(s))ds.$$

Let us consider $u \in \mathscr{F}$ and $0 < t \le t+h \le \mathcal{T}_0$. Then,

$$
\begin{aligned}
&\Big\| \mathscr{J} u(t+h) - \mathscr{J} u(t) \Big\|_{D(\mathcal{A}^\eta)} \\
&\quad \le \Big\| \Big(\mathcal{S}_{\alpha,k}(t+h) - \mathcal{S}_{\alpha,k}(t) \Big) u^0 \Big\|_{D(\mathcal{A}^\eta)} \\
&\quad + \Big\| \int_t^{t+h} \mathcal{B}_{\alpha,k}(t+h-s) G(u(s)) \mathrm{d}s \Big\|_{D(\mathcal{A}^\eta)} \\
&\quad + \Big\| \int_0^t \Big(\mathcal{B}_{\alpha,k}(t+h-s) - \mathcal{S}_{\alpha,k}(t-s) \Big) G(u(s)) \mathrm{d}s \Big\|_{D(\mathcal{A}^\eta)} \\
&\quad = \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3. \qquad (12)
\end{aligned}
$$

Now, we estimate the term $\mathcal{N}_1$. Indeed, using the inequality $e^{-a} - e^{-b} \le |a-b|$, we have

$$
\begin{aligned}
|\mathcal{N}_1|^2 &= \sum_{j=1}^\infty \lambda_j^{2\eta} \Big[\Big[\exp\big(-(t+h)\lambda_j^\alpha (1+k\lambda_j)^{-1} \big) - \exp\big(-t\lambda_j^\alpha (1+k\lambda_j)^{-1} \big) \Big] u_j^0 \Big]^2 \\
&\le k^2 \lambda_1^{2\alpha-2} h^2 \sum_{j=1}^\infty \lambda_j^{2\eta} |u_j^0|^2 = k^2 \lambda_1^{2\alpha-2} h^2 \|u^0\|^2_{D(\mathcal{A}^\eta)}. \qquad (13)
\end{aligned}
$$

Notice that, if $u \in \mathscr{F}$, then for any $0 \le t \le \mathcal{T}_0$

$$
\|G(u(t))\|_{D(\mathcal{A}^\nu)} \le K\Big(\|u(t)\|^p_{D(\mathcal{A}^\eta)} + 1 \Big) \le K\Big[\Big(\|w_0\|_{D(\mathcal{A}^\eta)} + \mathcal{M} \Big)^p + 1 \Big]. \qquad (14)
$$

Using (14), the term $\mathcal{N}_2$ can be estimated as follows:

$$
\begin{aligned}
\mathcal{N}_2 &\le \int_t^{t+h} \Big\| G(u(s)) \Big\|_{D(\mathcal{A}^\eta)} \mathrm{d}s \\
&\le \overline{K} \Big[\Big(\|w_0\|_{D(\mathcal{A}^\eta)} + \mathcal{M} \Big)^p + 1 \Big] \Big(\int_t^{t+h} (t+h-s)^{-\beta} ds \Big) \\
&= \overline{K} \Big[\Big(\|w_0\|_{D(\mathcal{A}^\eta)} + \mathcal{M} \Big)^p + 1 \Big] \frac{h^{1-\beta}}{1-\beta}, \qquad (15)
\end{aligned}
$$

where we have used the fact that $\|\mathcal{B}_{\alpha,k}(t)v\|_{D(\mathcal{A}^\eta)} \le \|v\|_{D(\mathcal{A}^\eta)}$. Using again the inequality $e^{-a} - e^{-b} \le |a-b|$, we deduce that, for any $0 \le s \le t$, $h > 0$,

(16)
$$
\frac{\exp\big(-(t+h-s)\lambda_j^\alpha (1+k\lambda_j)^{-1} \big) - \exp\big(-(t-s)\lambda_j^\alpha (1+k\lambda_j)^{-1} \big)}{1+k\lambda_j} \le hk^{-2}\lambda_1^{\alpha-2}.
$$

This inequality leads to the following bound for the term $\mathcal{N}_3$:

$$
\begin{aligned}
&\Big\| \int_0^t \Big(\mathcal{B}_{\alpha,k}(t+h-s) - \mathcal{B}_{\alpha,k}(t-s) \Big) G(u(s)) \mathrm{d}s \Big\|_{D(\mathcal{A}^\eta)} \\
&\quad \le k^{-2}\lambda_1^{\alpha-2} h \int_0^t \Big\| G(u(s)) \Big\|_{D(\mathcal{A}^\eta)} ds \\
&\quad \le K k^{-2} \lambda_1^{\alpha-2} \Big[\Big(\|w_0\|_{D(\mathcal{A}^\eta)} + \mathcal{M} \Big)^p + 1 \Big] ht. \qquad (17)
\end{aligned}
$$

Combining some of the preceding estimates, we deduce that if $u \in \mathscr{F}$, then $\mathscr{J} u \in \mathscr{C}\big([0,\mathcal{T}_0]; D(\mathcal{A}^\eta)\big)$. It follows from (10) that

$$\Big\| \mathscr{J} u(t) - w_0 \Big\|_{D(\mathcal{A}^\eta)}$$
$$\le \Big\| \mathcal{S}_{\alpha,k}(t) w_0 - w_0 \Big\|_{D(\mathcal{A}^\eta)} + \Big\| \int_0^t \mathcal{B}_{\alpha,k}(t-s) G(u(s)) ds \Big\|_{D(\mathcal{A}^\eta)}$$
$$\le \mathcal{T}_0 k \lambda_1^{\alpha-1} \|w_0\|_{D(\mathcal{A}^\eta)} + \overline{K} \left[\Big(\|w_0\|_{D(\mathcal{A}^\eta)} + \mathcal{M} \Big)^p + 1 \right] \frac{\mathcal{T}_0^{1-\beta}}{1-\beta} \le \mathcal{M}. \tag{18}$$

Now let $w, v \in \mathscr{F}$. For any $t \in [0, \mathcal{T}_0]$, using (8), we deduce the following estimate:

$$\Big\| \mathscr{J} w(t) - \mathscr{J} v(t) \Big\|_{D(\mathcal{A}^\eta)}$$
$$= \Big\| \int_0^t \mathcal{B}_{\alpha,k}(t-s) \Big(G(w(s)) - G(v(s)) \Big) \mathrm{d}s \Big\|_{D(\mathcal{A}^\eta)}$$
$$\le M_\beta \Big(1 + k\lambda_1 \Big)^{\beta-1} \int_0^t (t-s)^{-\beta} \Big\| G(w(s)) - G(v(s)) \Big\|_{D(\mathcal{A}^\nu)} ds$$
$$\le K M_\beta \Big(1 + k\lambda_1 \Big)^{\beta-1} \int_0^t (t-s)^{-\beta} \Big(1 + \|w\|_{D(\mathcal{A}^\eta)}^{p-1} + \|v\|_{D(\mathcal{A}^\eta)}^{p-1} \Big) \big\| w - v \big\|_{D(\mathcal{A}^\eta)} ds$$
$$\le K M_\beta \Big(1 + k\lambda_1 \Big)^{\beta-1} \left[1 + 2 \Big(\mathcal{M} + \|w_0\|_{D(\mathcal{A}^\eta)} \Big)^{p-1} \right] \int_0^t (t-s)^{-\beta} \big\| w - v \big\|_{D(\mathcal{A}^\eta)} ds.$$

In light of (9) and noting that $\int_0^t (t-s)^{-\beta} ds = \dfrac{t^{1-\beta}}{1-\beta}$, we find that

$$\sup_{0\le t\le \mathcal{T}_0} \Big\| \mathscr{J} w(t) - \mathscr{J} v(t) \Big\|_{D(\mathcal{A}^\eta)}$$
$$\le \frac{K M_\beta \Big(1 + k\lambda_1 \Big)^{\beta-1} \mathcal{T}_0^{1-\beta}}{1-\beta} \Big(1 + 2\mathcal{R} \Big) \sup_{0\le t\le \mathcal{T}_0} \|w(t) - v(t)\|_{D(\mathcal{A}^\eta)}$$
$$\le \frac{1}{2} \sup_{0\le t\le \mathcal{T}_0} \|w(t) - v(t)\|_{D(\mathcal{A}^\eta)}. \tag{19}$$

From the Banach fixed point theorem, it turns out that $\mathscr{J}$ has a unique fixed point $u \in \mathscr{F}$. ☐

2.2. **Continuation and blow-up alternative.** In this subsection, we give continuity to our study of problems (1)-(2) proving that the mild solution provided by Theorem 2.2 has a unique continuation to a larger interval of existence.

Definition 2.3. Given a mild solution $u \in C([0,\mathcal{T}_0], D(\mathcal{A}^\eta))$ to problem (1), it is said that $\widetilde{u}$ is a continuation of u in $[0,\mathcal{T}_1]$ if $u \in C([0,\mathcal{T}_1], D(\mathcal{A}^\eta))$ is a mild solution for $\mathcal{T}_1 > \mathcal{T}_0$ and $u(t) = \widetilde{u}(t)$ for any $t \in [0,\mathcal{T}_0]$.

Theorem 2.4. *Let u be a mild solution of problems* (1)-(2) *on* $[0,\mathcal{T}_0]$. *If G satisfies* (6), *then there exist $\mathcal{T}_1 > \mathcal{T}_0$ and a unique continuation $\widetilde{u}$ of u in* $[0,\mathcal{T}_1]$.

Proof. Fix $0 < \mathcal{M} \leq 1$ and take $\mathcal{T}_1 > \mathcal{T}_0$ such that for $t \in [\mathcal{T}_0, \mathcal{T}_1]$, we denote

$$\mathscr{U}_{\mathcal{M}} = \left\{ v \in C\big([0,\mathcal{T}_1], D(\mathcal{A}^\eta)\big) : \|v(t) - u(\mathcal{T}_0)\|_{D(\mathcal{A}^\eta)} \leq \mathcal{M}, v(t) = u(t), t \in [0, \mathcal{T}_0] \right\}. \tag{20}$$

It is easy to see that $\mathscr{U}_{\mathcal{M}}$ is a complete metric space with the norm of supremum in $D(\mathcal{A}^\eta)$. Let us define $\mathscr{G} : \mathscr{U}_{\mathcal{M}} \to \mathscr{U}_{\mathcal{M}}$ by

$$\mathscr{G}v(t) = \mathcal{S}_{\alpha,k}(t)u^0 + \int_0^t \mathcal{B}_{\alpha,k}(t-s)G(v(s))ds. \tag{21}$$

If $v \in \mathscr{U}_{\mathcal{M}}$, then it is obvious to obtain that $\mathscr{G}v(t) = u(t)$ for any $t \in [0, \mathcal{T}_0]$. Let any $t \in [\mathcal{T}_0, \mathcal{T}_1]$ and let any $v \in \mathscr{U}_{\mathcal{M}}$. By some simple computations, we have

$$\begin{aligned}
&\mathscr{G}v(t) - u(\mathcal{T}_0) \\
&\quad= \Big(\mathcal{S}_{\alpha,k}(t) - \mathcal{S}_{\alpha,k}(\mathcal{T}_0)\Big)u_0 \\
&\quad\quad+ \int_0^{\mathcal{T}_0} \Big(\mathcal{S}_{\alpha,k}(t-s) - \mathcal{S}_{\alpha,k}(\mathcal{T}_0 - s)\Big)G(u(s))ds + \int_{\mathcal{T}_0}^t \mathcal{B}_{\alpha,k}(t-s)G(u(s))ds \\
&\quad= \mathcal{S}_{\alpha,k}(t-\mathcal{T}_0)u(\mathcal{T}_0) - u(\mathcal{T}_0) + \int_{\mathcal{T}_0}^t \mathcal{B}_{\alpha,k}(t-s)G(v(s))ds, \qquad (22)
\end{aligned}$$

where we have used the fact that $u(\mathcal{T}_0) = \mathcal{S}_{\alpha,k}(\mathcal{T}_0)u^0 + \int_0^{\mathcal{T}_0} \mathcal{S}_{\alpha,k}(\mathcal{T}_0 - s)G(u(s))\mathrm{d}s$. In view of (8) and (6),

$$\begin{aligned}
&\|\mathscr{G}v(t) - u(\mathcal{T}_0)\|_{D(\mathcal{A}^\eta)} \\
&\quad\leq \Big\|\mathcal{S}_{\alpha,k}(t-\mathcal{T}_0)u(\mathcal{T}_0) - u(\mathcal{T}_0)\Big\|_{D(\mathcal{A}^\eta)} \\
&\quad\quad\quad+ M_\beta\Big(1 + k\lambda_1\Big)^{\beta-1} \int_{\mathcal{T}_0}^t (t-s)^{-\beta} \Big\|G(v(s))\Big\|_{D(\mathcal{A}^\nu)} ds \\
&\quad\leq (\mathcal{T}_1 - \mathcal{T}_0)k\lambda_1^{\alpha-1}\Big\|u(\mathcal{T}_0)\Big\|_{D(\mathcal{A}^\eta)} \\
&\quad\quad\quad+ KM_\beta\Big(1 + k\lambda_1\Big)^{\beta-1} \int_{\mathcal{T}_0}^t (t-s)^{-\beta}\Big(\|v(s)\|^p_{D(\mathcal{A}^\eta)} + 1\Big)ds. \qquad (23)
\end{aligned}$$

Since $v \in \mathscr{U}_{\mathcal{M}}$, we deduce that $\|v(t)\|_{D(\mathcal{A}^\eta)} \leq \|u(\mathcal{T}_0)\|_{D(\mathcal{A}^\eta)} + \mathcal{M}$ for any $t \in [\mathcal{T}_0, \mathcal{T}_1]$. Therefore, we have the following estimate:

$$\begin{aligned}
\int_{\mathcal{T}_0}^t (t-s)^{-\beta}\Big(\|v(s)\|^p_{D(\mathcal{A}^\eta)} + 1\Big)ds &\leq \left[\Big(\|u(\mathcal{T}_0)\|_{D(\mathcal{A}^\eta)} + \mathcal{M}\Big)^p + 1\right] \int_{\mathcal{T}_0}^t (t-s)^{-\beta} ds \\
&\leq \left[\Big(\|u(\mathcal{T}_0)\|_{D(\mathcal{A}^\eta)} + \mathcal{M}\Big)^p + 1\right] \frac{(\mathcal{T}_1 - \mathcal{T}_0)^{1-\beta}}{1-\beta}. \qquad (24)
\end{aligned}$$

Let us choose $\mathcal{T}_1$ such that

$$\begin{cases} (\mathcal{T}_1 - \mathcal{T}_0)k\lambda_1^{\alpha-1}\Big\|u(\mathcal{T}_0)\Big\|_{D(\mathcal{A}^\eta)} \leq \frac{\mathcal{M}}{2}, \\ KM_\beta\Big(1 + k\lambda_1\Big)^{\beta-1}\left[\Big(\|u(\mathcal{T}_0)\|_{D(\mathcal{A}^\eta)} + \mathcal{M}\Big)^p + 1\right] \frac{(\mathcal{T}_1 - \mathcal{T}_0)^{1-\beta}}{1-\beta} \leq \frac{\mathcal{M}}{2}. \end{cases} \tag{25}$$

Take any $v, w \in \mathscr{U}_{\mathcal{M}}$. We have

$$\begin{aligned}
&\|\mathscr{G}v(t) - \mathscr{G}w(t)\|_{D(\mathcal{A}^\eta)} \\
&\quad = \Big\| \int_{\mathcal{T}_0}^t \mathcal{B}_{\alpha,k}(t-s)\Big(G(v(s)) - G(w(s))\Big)ds\Big\|_{D(\mathcal{A}^\eta)} \\
&\quad \le M_\beta\Big(1+k\lambda_1\Big)^{\beta-1}\int_{\mathcal{T}_0}^t (t-s)^{-\beta}\Big\|G(v(s)) - G(w(s))\Big\|_{D(\mathcal{A}^\nu)}ds \\
&\quad \le KM_\beta\Big(1+k\lambda_1\Big)^{\beta-1}\int_{\mathcal{T}_0}^t (t-s)^{-\beta}\Big(1+\|v\|^{p-1}_{D(\mathcal{A}^\eta)}+\|w\|^{p-1}_{D(\mathcal{A}^\eta)}\Big)\|v-w\|_{D(\mathcal{A}^\eta)}ds.
\end{aligned}$$

Since $v, w \in \mathscr{U}_{\mathcal{M}}$, we deduce $1 + \|v(t)\|^{p-1}_{\mathcal{H}^\eta(\Omega)} + \|w(t)\|^{p-1}_{\mathcal{H}^\eta(\Omega)} \le 1 + 2(\|u(\mathcal{T}_0)\|_{\mathcal{H}^\eta(\Omega)} + \mathcal{M})^{p-1}$ for any $t \in [\mathcal{T}_0, \mathcal{T}_1]$. From the preceding estimates,

$$\begin{aligned}
&\Big\|\mathscr{G}v(t) - \mathscr{G}w(t)\Big\|_{D(\mathcal{A}^\eta)} \\
&\quad \le KM_\beta\Big(1+k\lambda_1\Big)^{\beta-1}\Big(1+2\Big(\|u(\mathcal{T}_0)\|_{\mathcal{H}^\eta(\Omega)}+\mathcal{M}\Big)^{p-1}\Big) \\
&\quad\quad \times \int_{\mathcal{T}_0}^t (t-s)^{-\beta}\|v-w\|_{D(\mathcal{A}^\eta)}ds \\
&\quad \le KM_\beta\Big(1+k\lambda_1\Big)^{\beta-1}\Big(1+2\Big(\|u(\mathcal{T}_0)\|_{\mathcal{H}^\eta(\Omega)}+\mathcal{M}\Big)^{p-1}\Big) \\
&\quad\quad \times \frac{(\mathcal{T}_1-\mathcal{T}_0)^{1-\beta}}{1-\beta}\sup_{0\le t\le \mathcal{T}_1}\Big\|w(t)-v(t)\Big\|_{D(\mathcal{A}^\eta)}. \qquad (26)
\end{aligned}$$

□

The next theorem is our result on global existence or noncontinuation by blow-up.

Theorem 2.5. *Assume that G satisfies* (6)*. Let u be the mild solution of problem* (1) *defined on* $[0, \mathcal{T}_{\max})$*, where $T_{\max}$ is the maximal time of existence of u. Then we have $\mathcal{T}_{max} = +\infty$ or* $\limsup_{t\to\mathcal{T}_{max}^-}\|u(t)\|_{D(\mathcal{A}^\eta)} = \infty$*.*

Proof. Suppose that $\mathcal{T}_{\max} < \infty$ and there exists a constant $\mathcal{B}$ such that

$$\max\Big(\|u^0\|_{D(\mathcal{A}^\eta)}, \sup_{0\le t\le T}\|u(t)\|_{\mathcal{H}^\eta(\Omega)}\Big) \le \mathcal{B}$$

for any $t \in [0, \mathcal{T}_{\max})$. Let us pick a sequence of positive numbers $t_n \to \mathcal{T}_{\max}^-$; we consider the sequence $\{u(t_n)\}$ in $D(\mathcal{A}^\eta)$. We will prove that this sequence is a Cauchy sequence in the space $D(\mathcal{A}^\eta)$. For $t_m, t_n \in [0, \mathcal{T}_{\max})$ such that $t_m < t_n$, we obtain after some simple caculations

$$u(t_n) - u(t_m) = \Big[\mathcal{S}_{\alpha,k}(t_n) - \mathcal{S}_{\alpha,k}(t_m)\Big]u^0 + \int_{t_m}^{t_n}\mathcal{B}_{\alpha,k}(t_n-s)G(u(s))ds. \qquad (27)$$

Therefore,

(28)

$$\begin{aligned}
&\Big\|u(t_n)-u(t_m)\Big\|_{D(\mathcal{A}^\eta)} \\
&\quad \le \underbrace{\Big\|\Big[\mathcal{S}_{\alpha,k}(t_n)-\mathcal{S}_{\alpha,k}(t_m)\Big]u^0\Big\|_{D(\mathcal{A}^\eta)}}_{\mathcal{A}_1} + \underbrace{\Big\|\int_{t_m}^{t_n}\mathcal{B}_{\alpha,k}(t_n-s)G(u(s))ds\Big\|_{D(\mathcal{A}^\eta)}}_{\mathcal{A}_2}.
\end{aligned}$$

Step 1 (Estimation of $\mathcal{A}_1$.). Using the inequality $|e^{-a} - e^{-b}| \le |a-b|$ for any a, b, we bound the term $\mathcal{A}_1$ as follows:

$$|\mathcal{A}_1|^2 \le k^{-2} \lambda_1^{2\alpha-2} (t_n - t_m)^2 \sum_{j=1}^{\infty} \lambda_j^{2\eta} \langle u^0, e_j \rangle_{L^2(\Omega)}^2. \tag{29}$$

This implies the following bound:

$$\mathcal{A}_1 \le k^{-1} \lambda_1^{\alpha-1} (t_n - t_m) \|u^0\|_{D(\mathcal{A}^\eta)} \le k^{-1} \lambda_1^{\alpha-1} \mathcal{B} (t_n - t_m). \tag{30}$$

Step 2 (Estimation of $\mathcal{A}_2$.). First, using (7), and noting that the integral $\int_{t_m}^{t_n} (t_n - s)^{-\beta} ds$ is convergent, we obtain the following estimate:

$$\begin{aligned}
\mathcal{A}_2 &\le M_\beta \Big(1 + k\lambda_1\Big)^{\beta-1} \int_{t_m}^{t_n} (t_n - s)^{-\beta} \Big\| G(u(s)) \Big\|_{D(\mathcal{A}^\nu)} ds \\
&\le K M_\beta \Big(1 + k\lambda_1\Big)^{\beta-1} \int_{t_m}^{t_n} (t_n - s)^{-\beta} \Big(\|u(s)\|_{D(\mathcal{A}^\nu)}^p + 1 \Big) ds \\
&\le K M_\beta \Big(1 + k\lambda_1\Big)^{\beta-1} \Big(|\mathcal{B}|^p + 1 \Big) \int_{t_m}^{t_n} (t_n - s)^{-\beta} ds \\
&= \frac{K M_\beta \Big(1 + k\lambda_1\Big)^{\beta-1} \Big(|\mathcal{B}|^p + 1 \Big)}{1-\beta} (t_n - t_m)^{1-\beta}.
\end{aligned}$$

From some previous observations, we deduce that

$$\begin{aligned}
\|u(t_n) - u(t_m)\|_{D(\mathcal{A}^\eta)} &\le k^{-1} \lambda_1^{\alpha-1} \mathcal{B} (t_n - t_m) \\
&\quad + \frac{K M_\beta \Big(1 + k\lambda_1\Big)^{\beta-1} \Big(|\mathcal{B}|^p + 1 \Big)}{1-\beta} (t_n - t_m)^{1-\beta}.
\end{aligned} \tag{31}$$

Let $\epsilon > 0$. Since (t_n) is convergent and noting that

$$k^{-1} \lambda_1^{\alpha-1} \mathcal{B} \quad \text{and} \quad \frac{K M_\beta \Big(1 + k\lambda_1\Big)^{\beta-1} \Big(|\mathcal{B}|^p + 1 \Big)}{1-\beta}$$

are independent of n, m, we can take N^* such that the right hand side of (31) is less than or equal to ϵ for any $n \ge m \ge N^*$. This implies that the sequence $\{u(t_n)\}$ is a Cauchy sequence in $D(\mathcal{A}^\eta)$. Hence, $\{u(t_n)\}$ converges to $\widetilde{u} \in \mathcal{H}^\eta(\Omega)$ as $n \to +\infty$. Since (t_n) is arbitrary, we deduce that $\lim_{t \to \mathcal{T}_{\max}^-} \|u(t)\|_{D(\mathcal{A}^\eta)} = \|\widetilde{u}\|_{D(\mathcal{A}^\eta)}$. Then, we can extend u over $[0, \mathcal{T}_{\max}]$. Therefore, we obtain a contradiction with the maximality of $\mathcal{T}_{\max}$. □

3. Terminal value problems (1)-(3)

In this section, we study the terminal value problems (1)-(3) in two cases $0 < \alpha \le 1$ and $\alpha > 1$. When $0 < \alpha \le 1$, we show existence and regularity of the mild solution. When $\alpha > 1$, we show that problems (1)-(3) are not well-posed in the space L^p. We also give a regularized solution and investigate the error estimate between the regularized solution and the sought solution in L^p norm.

Now, we establish a representation formula for the solution of problems (1)-(3). From (5), we find that

$$u^0 = \mathcal{S}^{-1}_{\alpha,k}(T)\left[u_T - \int_0^T \mathcal{P}_{\alpha,k}(T-s)G(u(s))ds\right].$$

Hence, by replacing the latter equality into (5) and then by a simple computation,

$$\begin{aligned} u(t) &= \mathcal{S}_{\alpha,k}(t)\mathcal{S}^{-1}_{\alpha,k}(T)\Big[u_T - \int_0^T \mathcal{B}_{\alpha,k}(T-s)G(u(s))ds\Big] + \int_0^t \mathcal{B}_{\alpha,k}(t-s)G(u(s))ds \\ &= \mathcal{S}^{-1}_{\alpha,k}(T-t)u_T - \int_t^T \mathcal{B}^{-1}_{\alpha,k}(s-t)G(u(s))ds, \end{aligned} \tag{32}$$

where we have the following definitions for any $0 \le t \le T$ and $w \in L^2(\Omega)$:

$$\begin{aligned} \mathcal{S}^{-1}_{\alpha,k}(t)w &= \sum_{j=1}^{\infty} \exp\left(t\lambda_j^{\alpha}(1+k\lambda_j)^{-1}\right)\langle w, e_j\rangle e_j,\ \mathcal{B}^{-1}_{\alpha,k}(t)w \\ &= \sum_{j=1}^{\infty} \frac{\exp\left(t\lambda_j^{\alpha}(1+k\lambda_j)^{-1}\right)}{1+k\lambda_j}\langle w, e_j\rangle e_j. \end{aligned}$$

Lemma 3.1. *The following inclusions hold true:*

$$\left.\begin{aligned} L^p(\Omega) &\hookrightarrow D(\mathcal{A}^{\sigma}) \quad \text{if} \quad -\tfrac{d}{4} < \sigma \le 0, \quad p \ge \tfrac{2d}{d-4\sigma}, \\ D(\mathcal{A}^{\sigma}) &\hookrightarrow L^p(\Omega) \quad \text{if} \quad 0 \le \sigma < \tfrac{d}{4}, \quad p \le \tfrac{2d}{d-4\sigma} \end{aligned}\right\}. \tag{33}$$

3.1. **Well-posedness of problems** (1)-(3) **under the case** $0 < \alpha \le 1$**.** . We have the following well-posedness result.

Theorem 3.2. *Assume that G satisfies a globally Lipschitz condition, i.e., there exists a constant $\mathcal{K} > 0$ such that*

$$|G(u) - G(v)| \le \mathcal{K}|u - v|.$$

- a) *If the final state $u_T \in L^2(\Omega)$, then problems* (1)-(3) *have a unique global solution $u \in L^{\infty}(0,T; L^2(\Omega))$.*
- b) *If the final state $u_T \in L^p(\Omega)$ for $1 \le p \le 2$, then problem* (1) *has a unique local solution $u \in L^q(0,T; L^{\frac{2d}{d-4\gamma}}(\Omega))$ where $1 \le q < \frac{\alpha}{\gamma-\sigma}$ and $\max(-\alpha, -\frac{d}{4}) < \sigma \le \frac{(p-2)d}{4p},\ 0 < \gamma \le \min(\sigma+\alpha, \frac{d}{4})$.*

Proof of part a)*.*

Let us consider the function $\mathcal{Y}w = \mathcal{S}^{-1}_{\alpha,k}(T-t)u_T - \int_t^T \mathcal{B}^{-1}_{\alpha,k}(s-t)G(w(s))ds$. Consider also the set

$$\mathcal{Z}_p := \left\{w : [0,T] \to L^2(\Omega), \left\|\exp(-p(T-t))w(.,t)\right\|_{L^2(\Omega)} < \infty,\ 0 \le t \le T\right\} \tag{34}$$

associated with the norm $\|w\|_{\mathcal{Z}_p} := \max_{0\le t\le T} \Big\| \exp(-p(T-t))w(.,t)\Big\|_{L^2(\Omega)}$. First, we deduce that

$$\Big\|\mathcal{S}^{-1}_{\alpha,k}(T-t)u_T\Big\|^2_{L^2(\Omega)} = \sum_{j=1}^{\infty} \exp\left(2(T-t)\lambda_j^\alpha(1+k\lambda_j)^{-1}\right)\langle u_T, e_j\rangle^2$$
$$\le \exp\left(2(T-t)k^{-1}\lambda_1^{\alpha-1}\right)\|u_T\|^2_{L^2(\Omega)}. \tag{35}$$

Since $u_T \in L^2(\Omega)$, we deduce from (35) that $\mathcal{S}^{-1}_{\alpha,k}(T-t)u_T \in \mathcal{Z}_p$ for any $p>0$. We continue now to estimate $\|\mathcal{Y}w_1 – \mathcal{Y}w_2\|_{\mathcal{Z}_{p,\sigma}}$. Indeed, noting $\exp\big(2(s-t)\lambda_j^\alpha(1+k\lambda_j)^{-1}\big)$ $\le \exp\left(2(s-t)k^{-1}\lambda_1^{\alpha-1}\right)$ for any $0\le t\le s\le T$, we obtain

(36)
$$\|\mathcal{Y}w_1 - \mathcal{Y}w_2\|_{\mathcal{Z}_p}$$
$$= \max_{0\le t\le T} \Big\| \exp(-p(T-t)) \int_t^T \mathcal{B}^{-1}_{\alpha,k}(s-t)\Big(G(w_1(s)) - G(w_2)(s)\Big)ds\Big\|_{L^2(\Omega)}$$
$$\le \mathcal{K} \max_{0\le t\le T}\int_t^T \Big\|\exp(-p(T-t))\exp\left((s-t)k^{-1}\lambda_1^{\alpha-1}\right)\Big(w_1(s)-w_2(s)\Big)\Big\|_{L^2(\Omega)} ds$$
$$\le \mathcal{K}\left(\int_t^T e^{(t-s)(p-k^{-1}\lambda_1^{\alpha-1})}ds\right)\|w_1-w_2\|_{\mathcal{Z}_p}.$$

Let us choose $p > k^{-1}\lambda_1^{\alpha-1}$. Then the latter inequality implies that

$$\|\mathcal{Y}w_1 - \mathcal{Y}w_2\|_{\mathcal{Z}_p} \le \frac{\mathcal{K}}{p-k^{-1}\lambda_1^{\alpha-1}}\|w_1-w_2\|_{\mathcal{Z}_p}.$$

Let us choose $p > \mathcal{K} + k^{-1}\lambda_1^{\alpha-1}$; then we deduce that $\mathcal{Y}$ is a contractive mapping in the space $\mathcal{Z}_p$. Applying the Banach fixed point theorem, we derive that $\mathbf{J}$ has a fixed point $u \in L^\infty(0,T;L^2(\Omega))$.

Proof of part b). Since $u_T \in L^p(\Omega)$, we find that $u_T \in D(\mathcal{A}^\sigma)$ for $\max(-\alpha, -\frac{d}{4}) < \sigma \le \frac{(p-2)d}{4p}$. This implies that

$$\Big\|\mathcal{S}^{-1}_{\alpha,k}(T-t)u_T\Big\|^2_{D(\mathcal{A}^\gamma)} = \sum_{j=1}^{\infty} \exp\left(2(T-t)\lambda_j^\alpha(1+k\lambda_j)^{-1}\right)\lambda_j^{2\gamma-2\sigma}\lambda_j^{2\sigma}\langle u_T, e_j\rangle^2$$
$$\le \sum_{j=1}^{\infty} \exp\left(2(T-t)\lambda_j^\alpha(1+k\lambda_1)^{-1}\right)\lambda_j^{2\gamma-2\sigma}\lambda_j^{2\sigma}\langle u_T, e_j\rangle^2. \tag{37}$$

On the other hand, using the inequality $e^{-z} \le D_m z^{-m}$ for any $m>0$, we have for any $t\ge 0,\ \gamma>0, \sigma\le 0$

$$\lambda_j^{\gamma-\sigma}\exp\left(-tk^{-1}\lambda_j^{\alpha-1}\right) \le D_{\alpha,\gamma,\sigma}\Big(t\lambda_j^\alpha(k\lambda_1)^{-1}\Big)^{\frac{-\gamma+\sigma}{\alpha}}\lambda_j^{\gamma-\sigma} \le D_{\alpha,\gamma,\sigma,k}t^{\frac{\gamma-\sigma}{-\alpha}}. \tag{38}$$

We note that

$$\lambda_j^{\gamma-\sigma}\exp\left((T-t)\lambda_j^\alpha(1+k\lambda_j)^{-1}\right) \le \lambda_j^{\gamma-\sigma}\exp\left(Tk^{-1}\lambda_j^{\alpha-1}\right)\exp\left(-tk^{-1}\lambda_j^{\alpha-1}\right)$$
$$\le \underbrace{\exp\Big(Tk^{-1}\lambda_1^{\alpha-1}\Big)D_{\alpha,\gamma,\sigma,k}}_{\overline{D}}\, t^{\frac{\gamma-\sigma}{-\alpha}}. \tag{39}$$

Using the Sobolev embedding $D(\mathcal{A}^\gamma) \hookrightarrow L^{\frac{2d}{d-4\gamma}}(\Omega)$, we find that

$$\begin{aligned}(40)\qquad &\left\|\mathcal{S}^{-1}_{\alpha,k}(T-t)u_T\right\|_{L^{\frac{2d}{d-4\gamma}}(\Omega)} \\ &\lesssim \left\|\mathcal{S}^{-1}_{\alpha,k}(T-t)u_T\right\|_{D(\mathcal{A}^\gamma)} \le \overline{D}t^{\frac{\gamma-\sigma}{-\alpha}}\|u_T\|_{D(\mathcal{A}^\sigma)} \lesssim t^{\frac{\gamma-\sigma}{-\alpha}}\|u_T\|_{L^p(\Omega)}.\end{aligned}$$

Since $\sigma > -\alpha$ and $\gamma < \sigma+\alpha$ we can choose q such that $1 \le q < \frac{\alpha}{\gamma-\sigma}$. Therefore, we deduce that $\mathcal{S}^{-1}_{\alpha,k}(T-t)u_T \in L^q(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega))$. By a similar argument as above, we find that

$$\begin{aligned}&\left\|\mathcal{B}^{-1}_{\alpha,k}(s-t)\Big(G(w_1(s))-G(w_2)(s)\Big)\right\|^2_{D(\mathcal{A}^\gamma)} \\ &\quad= \sum_{j=1}^{\infty} \frac{\exp\left(2(s-t)\lambda_j^\alpha(1+k\lambda_j)^{-1}\right)\lambda_j^{2\gamma}}{(1+k\lambda_j)^2}\left\langle G(w_1(s))-G(w_2)(s),e_j\right\rangle^2 \\ (41)\qquad &\quad\le \frac{\exp\Big(2Tk^{-1}\lambda_1^{\alpha-1}\Big)D^2_{\alpha,\gamma,0,k}}{(1+k\lambda_1)^2}t^{-2\gamma/\alpha}\sum_{j=1}^{\infty}\left\langle G(w_1(s))-G(w_2)(s),e_j\right\rangle^2,\end{aligned}$$

which implies

$$\begin{aligned}(42)\qquad &\int_t^T \left\|\mathcal{B}^{-1}_{\alpha,k}(s-t)\Big(G(w_1(s))-G(w_2)(s)\Big)\right\|_{D(\mathcal{A}^\gamma)}ds \\ &\quad\le \frac{\mathcal{K}\exp\Big(Tk^{-1}\lambda_1^{\alpha-1}\Big)D_{\alpha,\gamma,0,k}}{(1+k\lambda_1)}t^{-\gamma/\alpha}\int_t^T \|w_1-w_2\|_{L^2(\Omega)}ds.\end{aligned}$$

This implies that

(43)

$$\begin{aligned}&\left\|\mathcal{Y}w_1(t)-\mathcal{Y}w_2(t)\right\|^q_{D(\mathcal{A}^\gamma)} \\ &\quad\le \underbrace{\left(\frac{\mathcal{K}\exp\Big(Tk^{-1}\lambda_1^{\alpha-1}\Big)D_{\alpha,\gamma,0,k}}{(1+k\lambda_1)}\right)^q}_{\mathcal{L}} t^{-\frac{\gamma q}{\alpha}}\left(\int_t^T\|w_1-w_2\|_{L^2(\Omega)}ds\right)^q \\ &\quad\le \mathcal{L}t^{-\frac{\gamma q}{\alpha}}(T-t)\left(\int_t^T\|w_1-w_2\|^q_{L^2(\Omega)}ds\right) \le \mathcal{L}Tt^{-\frac{\gamma q}{\alpha}}\|w_1-w_2\|^q_{L^q(0,T;L^2(\Omega))}\end{aligned}$$

The latter inequality leads to

$$(44)\qquad \|\mathcal{Y}w_1-\mathcal{Y}w_2\|_{L^q(0,T;D(\mathcal{A}^\gamma))} \le \frac{\alpha T^{\frac{\alpha-\gamma q}{\alpha q}}}{\alpha-\gamma q}(\mathcal{L}T)^{\frac{1}{q}}\|w_1-w_2\|_{L^q(0,T;L^2(\Omega))}.$$

The Sobolev embeddings $L^q(0,T;D(\mathcal{A}^\gamma)) \hookrightarrow L^q(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega)) \hookrightarrow L^q(0,T;L^2(\Omega))$ imply that

$$(45)\quad \|\mathcal{Y}w_1-\mathcal{Y}w_2\|_{L^q(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega))} \le \mathcal{L}_1\frac{\alpha T^{\frac{\alpha-\gamma q}{\alpha q}}}{\alpha-\gamma q}(\mathcal{L}T)^{\frac{1}{q}}\|w_1-w_2\|_{L^q(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega))}$$

where $\mathcal{L}_1$ depends only on γ, d, q. By choosing T small enough, we deduce that $\mathcal{L}_1\frac{\alpha T^{\frac{\alpha-\gamma q}{\alpha q}}}{\alpha-\gamma q}(\mathcal{L}T)^{\frac{1}{q}} < 1$, which implies that $\mathcal{Y}$ is a contracting mapping on the space $L^q(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega))$. $\square$

Remark 3.1. Using the techniques in Section 2, we can show the global existence for a mild solution which is given in part b) of Theorem 3.2.

3.2. **Ill-posedness and regularization under the case $\alpha > 1$.** Let us first prove the following theorem.

Theorem 3.3. *Let any $\varphi_\epsilon \in L^p(\Omega)$. Then the nonlinear integral equation*

$$u_\epsilon(t) = \mathcal{S}^{-1}_{\alpha,k}(T-t)\mathbb{P}_{\mathcal{N}_\epsilon}\varphi_\epsilon - \int_t^T \mathcal{B}^{-1}_{\alpha,k}(s-t)\mathbb{P}_{\mathcal{N}_\epsilon}G(u_\epsilon(s))ds \tag{46}$$

has a unique solution in $L^\infty(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega))$. Moroever, there exists a postive $\mathscr{B}_\epsilon > 0$ such that

$$\|u_\epsilon(t)\|_{L^{\frac{2d}{d-4\gamma}}(\Omega)} \le \mathscr{B}_\epsilon \|\varphi_\epsilon\|_{L^p(\Omega)}. \tag{47}$$

Proof. For any $\mathcal{N} > 0$, let $\mathbb{P}_\mathcal{N}$ be the orthogonal projection onto the eigenspace span $\{e_j, \lambda_j \le \mathcal{N}\}$. Set the regularized solution

$$u_\epsilon(t) = \mathcal{S}^{-1}_{\alpha,k}(T-t)\mathbb{P}_{\mathcal{N}_\epsilon}\varphi_\epsilon - \int_t^T \mathcal{B}^{-1}_{\alpha,k}(s-t)\mathbb{P}_{\mathcal{N}_\epsilon}G(u_\epsilon(s))ds. \tag{48}$$

Let $v \in D(\mathcal{A}^\nu)$. Then, for any $\nu' \ge \nu$, we have

$$\begin{aligned} &\left\| \mathcal{S}^{-1}_{\alpha,k}(T-t)\mathbb{P}_{\mathcal{N}_\epsilon} w \right\|_{D(\mathcal{A}^{\nu'})} \\ &= \left(\sum_{\lambda_j \le \mathcal{N}_\epsilon} \lambda_j^{2\nu'-2\nu} \exp\left(2(T-t)\lambda_j^\alpha (1+k\lambda_j)^{-1}\right) \lambda_j^{2\nu} \langle w, e_j\rangle^2 \right)^{\frac{1}{2}} \\ &\le (\mathcal{N}_\epsilon)^{\nu'-\nu} \exp\Big((T-t)k^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big) \|w\|_{D(\mathcal{A}^\nu)}. \end{aligned} \tag{49}$$

By a similar argument as above, we also find that

$$\left\| \mathcal{B}^{-1}_{\alpha,k}(s-t)\mathbb{P}_{\mathcal{N}_\epsilon} w \right\|_{D(\mathcal{A}^{\nu'})} \le \frac{(\mathcal{N}_\epsilon)^{\nu'-\nu} \exp\Big((s-t)k^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big)}{1+k\lambda_1} \|w\|_{D(\mathcal{A}^\nu)}. \tag{50}$$

For any $p > 0$, denote by $L^\infty_m(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega))$ the function space $L^\infty(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega))$ associated with the norm

$$\|w\|_m := \max_{0\le t\le T} \Big\| \exp(-m(T-t))w(.,t) \Big\|_{L^{\frac{2d}{d-4\gamma}}(\Omega)}, \quad \forall\, v \in L^\infty(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega)).$$

Let us define a nonlinear map $L^\infty_m(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega)) \to L^\infty_m(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega))$ by

$$\mathbf{J}w(t) = \mathcal{S}^{-1}_{\alpha,k}(T-t)\mathbb{P}_{\mathcal{N}_\epsilon}\varphi_\epsilon - \int_t^T \mathcal{B}^{-1}_{\alpha,k}(s-t)\mathbb{P}_{\mathcal{N}_\epsilon}G(w(s))ds.$$

If $\overline{w} = 0$, then $\mathbf{J}\overline{w}(t) = \mathcal{S}^{-1}_{\alpha,k}(T-t)\mathbb{P}_{\mathcal{N}_\epsilon}\varphi_\epsilon$. Using (63) and letting $\nu' = 0, \nu = \sigma < 0$, and noting the Sobolev embedding $L^p(\Omega) \hookrightarrow D(\mathcal{A}^\sigma)$,

$$\begin{aligned} \left\| \mathcal{S}^{-1}_{\alpha,k}(T-t)\mathbb{P}_{\mathcal{N}_\epsilon}\varphi_\epsilon \right\|_{D(\mathcal{A}^\gamma)} &\le (\mathcal{N}_\epsilon)^{\gamma-\sigma} \exp\Big((T-t)k^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big) \|\varphi_\epsilon\|_{D(\mathcal{A}^\sigma)} \\ &\le C_{1,\sigma,p,d}(\mathcal{N}_\epsilon)^{-\sigma} \exp\Big((T-t)k^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big) \|\varphi_\epsilon\|_{L^p(\Omega)} \end{aligned} \tag{51}$$

where we note that $\varphi_\epsilon \in L^p(\Omega),\ p \geq 1$. Using the Sobolev embedding $D(\mathcal{A}^\gamma) \hookrightarrow L^{\frac{2d}{d-4\gamma}}(\Omega)$, we deduce that $\mathbf{J}\overline{w} \in L^\infty_m(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega))$. From the definition of $\mathbf{J}$ and using Lemma 2.1, we have for any $w_1, w_2 \in L^\infty_m(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega))$ that

$$\begin{aligned} &\Big\| \exp(-m(T-t))\,(\mathbf{J}w_1(t) - \mathbf{J}w_2(t))\Big\|_{L^{\frac{2d}{d-4\gamma}}(\Omega)} \\ &\leq C_{2,\gamma,d}\Big\| \exp(-m(T-t))\,(\mathbf{J}w_1(t) - \mathbf{J}w_2(t))\Big\|_{D(\mathcal{A}^\gamma)} \end{aligned} \tag{52}$$

where we have used that $\|v\|_{L^{\frac{2d}{d-4\gamma}}(\Omega)} \leq \|v\|_{D(\mathcal{A}^\gamma)}$, since the Sobolev embedding $D(\mathcal{A}^\gamma) \hookrightarrow L^{\frac{2d}{d-4\gamma}}(\Omega)$. We have

$$\begin{aligned} &\Big\| \exp(-m(T-t))\,(\mathbf{J}w_1(t) - \mathbf{J}w_2(t))\Big\|_{D(\mathcal{A}^\gamma)} \\ &= \Big\| e^{-m(T-t)} \int_t^T \mathcal{B}^{-1}_{\alpha,k}(s-t)\mathbb{P}_{\mathcal{N}_\epsilon}\left(G(w_1(s)) - G(w_2(s))\right) ds \Big\|_{D(\mathcal{A}^\gamma)} \\ &\leq \int_t^T e^{-m(T-t)} \Big\| \mathcal{B}^{-1}_{\alpha,k}(s-t)\mathbb{P}_{\mathcal{N}_\epsilon}\left(G(w_1(s)) - G(w_2(s))\right)\Big\|_{D(\mathcal{A}^\gamma)} ds. \end{aligned} \tag{53}$$

Letting $\gamma > 0$ and $\sigma = 0$ into (50) and using the globally Lipschitz property of G, we find that

$$\begin{aligned} &\underline{\textit{The right hand side of } (53)} \\ &\leq \frac{(\mathcal{N}_\epsilon)^\gamma}{1+k\lambda_1} \int_t^T e^{-m(T-t)} \exp\Big((s-t)k^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big) \big\|G(w_1(s)) - G(w_2(s))\big\|_{L^2(\Omega)} \\ &\leq \frac{\mathcal{K}C_{3,d,\gamma}(\mathcal{N}_\epsilon)^\gamma}{1+k\lambda_1} \int_t^T e^{-m(s-t)} \exp\Big((s-t)k^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big) e^{-m(T-s)} \\ &\quad \times \big\|w_1(s) - w_2(s)\big\|_{L^{\frac{2d}{d-4\gamma}}(\Omega)} ds \end{aligned} \tag{54}$$

where we have used the Sobolev embedding $L^{\frac{2d}{d-4\gamma}}(\Omega) \hookrightarrow L^2(\Omega)$. This implies that

$$\begin{aligned} &\underline{\textit{The left hand side of } (52)} \\ &\leq \frac{\mathcal{K}C_{2,\gamma,d}C_{3,d,\gamma}(\mathcal{N}_\epsilon)^\gamma}{1+k\lambda_1} \left(\int_t^T e^{-m(s-t)} \exp\Big((s-t)k^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big) ds \right) \|w_1 - w_2\|_m \\ &\leq \frac{\mathcal{K}C_{2,\gamma,d}C_{3,d,\gamma}(\mathcal{N}_\epsilon)^\gamma}{1+k\lambda_1} \frac{1}{m - k^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}} \|w_1 - w_2\|_m. \end{aligned} \tag{55}$$

With a suitable choice for m, the last inequality implies

$$\|\mathbf{J}w_1 - \mathbf{J}w_2\|_m \leq \tfrac{1}{2}\|w_1 - w_2\|_m, \quad \forall\, w_1, w_2 \in L^\infty_m(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega)).$$

Thus, $\mathbf{J}$ is contractive on $L^\infty_m(0,T;L^{\frac{2d}{d-4\gamma}}(\Omega))$. Applying the Banach fixed point theorem, we obtain that $\mathbf{J}$ has a fixed point u_ϵ. Now, we will show the regularity of u_ϵ. Using (51), we obtain

$$\|u_\epsilon(t)\|_{L^2(\Omega)} \le C_{1,\sigma,p,d}(\mathcal{N}_\epsilon)^{-\sigma}\exp\Big((T-t)k^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big)\|\varphi_\epsilon\|_{L^p(\Omega)}$$
$$+\mathcal{K}(1+k\lambda_1)^{-1}\int_t^T \exp\left((s-t)\mathcal{N}_\epsilon^{\alpha-1}(1+k\mathcal{N}_\epsilon)^{-1}\right)\|u_\epsilon(s)\|_{L^2(\Omega)}ds. \tag{56}$$

By applying Gronwall's inequality,

$$\exp\Big(tk^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big)\|u_\epsilon(t)\|_{L^2(\Omega)} \le H_1H_2(\epsilon)C_{1,\sigma,p,d}\|\varphi_\epsilon\|_{L^p(\Omega)} \tag{57}$$

which implies that

$$\|u_\epsilon(t)\|_{L^2(\Omega)} \le H_1C_{1,\sigma,p,d}H_2(\epsilon)\exp\Big(-tk^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big)\|\varphi_\epsilon\|_{L^p(\Omega)}. \tag{58}$$

Since the Sobolev embedding $D(\mathcal{A}^\gamma)\hookrightarrow L^{\frac{2d}{d-4\gamma}}(\Omega)$, we deduce for any $0<\gamma<\frac{d}{4}$

$$\|u_\epsilon(t)\|_{L^{\frac{2d}{d-4\gamma}}(\Omega)} \le C_{2,\gamma,d}\|u_\epsilon(t)\|_{D(\mathcal{A}^\gamma)} = C_{2,\gamma,d}\Bigg(\sum_{\lambda_j\le\mathcal{N}_\epsilon}\lambda_j^{2\gamma}\left\langle u_\epsilon(t),e_j\right\rangle^2\Bigg)^{\frac12}$$
$$\le C_{2,\gamma,d}(\mathcal{N}_\epsilon)^\gamma\|u_\epsilon(t)\|_{L^2(\Omega)} \le \mathscr{B}_\epsilon\|\varphi_\epsilon\|_{L^p(\Omega)}, \tag{59}$$

where $\mathscr{B}_\epsilon$ depends on ϵ. □

Theorem 3.4. *Let us assume problem* (1) *has a unique solution* $u\in L^\infty(0,T;\mathbf{G}_{\alpha,k}(\Omega))$ *such that* $\|u\|_{L^\infty(0,T;\mathbf{G}_{\alpha,k}(\Omega))}\le\overline{M}$ *. Let us choose* $\mathcal{N}_\epsilon$ *such that for any* $t>0$

$$\lim_{\epsilon\to0}\mathcal{N}_\epsilon=\infty,\ \lim_{\epsilon\to0}(\mathcal{N}_\epsilon)^{-\sigma}\exp\left(Tk^{-1}\mathcal{N}_\epsilon^{\alpha-1}\right)\epsilon=\lim_{\epsilon\to0}(\mathcal{N}_\epsilon)^\gamma\exp\left(-tk^{-1}\mathcal{N}_\epsilon^{\alpha-1}\right)=0. \tag{60}$$

Then $\|u_\epsilon(t)-u(t)\|_{L^{\frac{2d}{d-4\gamma}}(\Omega)}$ *is of order* $\max\Big((\mathcal{N}_\epsilon)^\gamma\exp\Big(-tk^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big)\,,\ (\mathcal{N}_\epsilon)^{-p}\Big)$.

Remark 3.2. Let us choose $\mathcal{N}_\epsilon=\left[\frac{1}{rTk^{-1}}\log(\frac{1}{\epsilon})\right]^{\frac{1}{\alpha-1}}$ for any $0<r<1$. Then it is easy to check that (60) holds. The error $\|u_\epsilon(t)-u(t)\|_{L^{\frac{2d}{d-4\gamma}}(\Omega)}$ is of order $\max\Big(\big(\frac{1}{Tk^{-1}r}\log(\frac{1}{\epsilon})\big)^{\frac{\gamma}{\alpha-1}}\epsilon^{\frac{r}{T}},\big(\frac{1}{Tk^{-1}r}\log(\frac{1}{\epsilon})\big)^{\frac{-p}{\alpha-1}}\Big)$.

Proof. Note that

$$\|u(t)-\mathbb{P}_{\mathcal{N}_\epsilon}u(t)\|_{L^2(\Omega)} = \Bigg(\sum_{\lambda_j>\mathcal{N}_\epsilon}\exp\left(-2tk^{-1}\lambda_j^{\alpha-1}\right)\exp\left(2tk^{-1}\lambda_j^{\alpha-1}\right)\left\langle u(t),e_j\right\rangle^2\Bigg)^{\frac12}$$
$$\le \exp\left(-tk^{-1}\mathcal{N}_\epsilon^{\alpha-1}\right)\overline{M}.$$

Since $\|\varphi_\epsilon-u_T\|_{L^p(\Omega)}\le\epsilon$, and observing that $-\frac{d}{4}<\sigma\le\frac{(p-2)d}{4}$ we have the Sobolev embedding $L^p(\Omega)\hookrightarrow D(\mathcal{A}^\sigma)$, and we then find that $\|\varphi_\epsilon-u_T\|_{D(\mathcal{A}^\sigma)}\le C_{1,\sigma,p,d}\|\varphi_\epsilon-u_T\|_{L^p(\Omega)}\le C_{1,\sigma,p,d}\epsilon$. The latter inequality leads to

$$\left\|\mathcal{S}_{\alpha,k}^{-1}(T-t)\mathbb{P}_{\mathcal{N}_\epsilon}(\varphi_\epsilon-u_T)\right\|_{L^2(\Omega)} \le (\mathcal{N}_\epsilon)^{-\sigma}\exp\Big((T-t)k^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big)\|\varphi_\epsilon-u_T\|_{\mathcal{A}^\sigma}.$$

Using the triangle inequality and combining some previous estimates, we find that

$$\begin{aligned}(61)\qquad &\|u_\epsilon(t) - u(t)\|_{L^2(\Omega)}\\ &\leq \left\|\mathcal{S}^{-1}_{\alpha,k}(T-t)\mathbb{P}_{\mathcal{N}_\epsilon}(\varphi_\epsilon - u_T)\right\|_{L^2(\Omega)} + \|u(t) - P_{\mathcal{N}_\epsilon}u(t)\|_{L^2(\Omega)}\\ &\quad + \int_t^T \left\|\mathcal{B}^{-1}_{\alpha,k}(s-t)\mathbb{P}_{\mathcal{N}_\epsilon}\Big(G(u_\epsilon(s)) - G(u(s))\Big)\right\|_{L^2(\Omega)}\\ &\leq \exp\Big(-tk^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big)\Big[\overline{M} + C_{1,\sigma,p,d}(\mathcal{N}_\epsilon)^{-\sigma}\exp\Big(Tk^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big)\epsilon\Big]ds\\ &\quad + \mathcal{K}(1+k\lambda_1)^{-1}\int_t^T \exp\big((s-t)k^{-1}\mathcal{N}_\epsilon^{\alpha-1}\big)\|u_\epsilon(s) - u(s)\|_{L^2(\Omega)}ds.\end{aligned}$$

Multiplying both sides by $\exp(tk^{-1}(\mathcal{N}_\epsilon)^{\alpha-1})$ and then applying Gronwall's inequality,

$$\begin{aligned}(62)\qquad &\exp\Big(tk^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big)\|u_\epsilon(t) - u(t)\|_{L^2(\Omega)}\\ &\leq \Big[\overline{M} + C_{1,\sigma,p,d}(\mathcal{N}_\epsilon)^{-\sigma}\exp\Big(Tk^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\Big)\epsilon\Big]\exp\big(\mathcal{K}(1+k\lambda_1)^{-1}T\big).\end{aligned}$$

Since the Sobolev embedding $D(\mathcal{A}^\gamma) \hookrightarrow L^{\frac{2d}{d-4\gamma}}(\Omega)$ holds true for any $0 < \gamma < \frac{d}{4}$, then

$$(63)\qquad \|u_\epsilon(t) - u(t)\|_{L^{\frac{2d}{d-4\gamma}}(\Omega)} \leq C_{2,\gamma,d}\|u_\epsilon(t) - u(t)\|_{D(\mathcal{A}^\gamma)}.$$

Now, we continue to estimate the term $\|u_\epsilon(t) - u(t)\|_{D(\mathcal{A}^\gamma)}$. Indeed,

$$(64)\qquad \|u_\epsilon(t) - \mathbb{P}_{\mathcal{N}_\epsilon}u(t)\|^2_{D(\mathcal{A}^\gamma)} = \sum_{\lambda_j \leq \mathcal{N}_\epsilon} \lambda_j^{2\gamma}\langle u_\epsilon(t) - u(t), e_j\rangle^2 \leq (\mathcal{N}_\epsilon)^{2\gamma}\|u_\epsilon(t) - u(t)\|^2_{L^2(\Omega)}$$

and

$$\begin{aligned}(65)\qquad \|u(t) - \mathbb{P}_{\mathcal{N}_\epsilon}u(t)\|^2_{D(\mathcal{A}^\gamma)} &= \sum_{\lambda_j > \mathcal{N}_\epsilon}\lambda_j^{-2p}\lambda_j^{2\gamma+2p}\langle u(t), e_j\rangle^2\\ &\leq (\mathcal{N}_\epsilon)^{-2p}\|u(t)\|^2_{D(\mathcal{A}^{p+\gamma})} \leq (\mathcal{N}_\epsilon)^{-2p}\|u\|^2_{L^\infty(0,T;D(\mathcal{A}^{p+\gamma}))}.\end{aligned}$$

From the two preceding estimates, we find that

$$\begin{aligned}(66)\qquad &\big\|u_\epsilon(t) - u(t)\big\|_{D(\mathcal{A}^\gamma)}\\ &\leq \|u_\epsilon(t) - \mathbb{P}_{\mathcal{N}_\epsilon}u(t)\|_{D(\mathcal{A}^\gamma)} + \|u(t) - \mathbb{P}_{\mathcal{N}_\epsilon}u(t)\|_{D(\mathcal{A}^\gamma)}\\ &\leq (\mathcal{N}_\epsilon)^\gamma\|u_\epsilon(t) - u(t)\|_{L^2(\Omega)} + (\mathcal{N}_\epsilon)^{-p}\|u\|_{L^\infty(0,T;D(\mathcal{A}^{p+\gamma}))}\\ &\leq \Big[\overline{M} + C_{1,\sigma,p,d}H_2(\epsilon)\Big]H_1(\mathcal{N}_\epsilon)^\gamma\exp\big(-tk^{-1}(\mathcal{N}_\epsilon)^{\alpha-1}\big)\\ &\quad + (\mathcal{N}_\epsilon)^{-p}\|u\|_{L^\infty(0,T;D(\mathcal{A}^{p+\gamma}))}.\end{aligned}$$

where we set $H_1 = \exp\big(\mathcal{K}(1+k\lambda_1)^{-1}T\big)$, $H_2(\epsilon) = (\mathcal{N}_\epsilon)^{-\sigma}\exp\big(Tk^{-1}\mathcal{N}_\epsilon^{\alpha-1}\big)\,\epsilon$. From (63) and (66), we obtain the desired result. □

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