Vector measures: where are their integrals?

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Abstract. Let ν be a vector measure with values in a Banach space Z. The integration map $I_\nu\colon L^1(\nu) \to Z$, given by $f \mapsto \int f\, d\nu$ for $f \in L^1(\nu)$, always has a formal extension to its bidual operator $I_\nu^{**}\colon L^1(\nu)^{**} \to Z^{**}$. So, we may consider the "integral" of any element f^{**} of $L^1(\nu)^{**}$ as $I_\nu^{**}(f^{**})$. Our aim is to identify when these integrals lie in more tractable subspaces Y of Z^{**}. For Z a Banach function space X, we consider this question when Y is any one of the subspaces of X^{**} given by the corresponding identifications of X, X'' (the Köthe bidual of X) and X'^* (the topological dual of the Köthe dual of X). Also, we consider certain kernel operators T and study the extended operator I_ν^{**} for the particular vector measure ν defined by $\nu(A) := T(\chi_A)$.

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Introduction

The general theory of vector measures and integration with respect to them is well established; see [1, 19, 20, 29], for example. In recent years it has become evident that many classical operators from various branches of analysis can be viewed as integration operators with respect to suitable vector measures; see [7–10, 14, 15, 24, 27] and the references therein, for example. Accordingly, such integration operators are becoming objects of ever finer investigations.

Recall, for a vector measure ν defined on a measurable space (Ω, Σ) and with values in a Banach space Z, that a measurable function $f\colon \Omega \to \mathbb{R}$ is *scalarly*

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ν*–integrable* whenever

$$\int_\Omega |f|\, d|z^*\nu| < \infty, \quad z^* \in Z^*;$$

here Z^* is the dual Banach space of Z and $|z^*\nu|$ denotes the variation of the scalar-valued measure $z^*\nu\colon A \mapsto \langle z^*, \nu(A)\rangle$ for $A \in \Sigma$. This is the case if and only if

$$\|f\|_\nu := \sup_{\|z^*\|\le 1} \int_\Omega |f|\, d|z^*\nu| < \infty. \tag{1}$$

Moreover, the space $L^1_w(\nu)$ of all such scalarly ν–integrable functions (modulo ν–null functions) is a Banach space for the norm (1), [30]. Actually, for the pointwise ν–a.e. order on Ω, the space $L^1_w(\nu)$ is a Banach lattice. Let $f \in L^1_w(\nu)$. If, for each $A \in \Sigma$, there exists a vector in Z (denoted by $\int_A f\, d\nu$) satisfying

$$\left\langle z^*, \int_A f\, d\nu \right\rangle = \int_A f\, dz^*\nu, \quad z^* \in Z^*, \tag{2}$$

then f is called ν*–integrable*. The space of all such ν–integrable functions f is denoted by $L^1(\nu)$ and forms a closed ideal in $L^1_w(\nu)$ which, depending on ν and Z, may be proper. In this case, given $f \in L^1_w(\nu) \setminus L^1(\nu)$, the condition (2) is meaningless because the integral "$\int_A f\, d\nu$" is not available in Z. Of course, via a typical approximation and uniform boundedness argument it follows from (1) that there does exist a vector "$\int_A f\, d\nu$" $\in Z^{**}$ satisfying

$$\left\langle z^*, \text{“}\int_A f\, d\nu\text{”} \right\rangle = \int_A f\, dz^*\nu, \quad z^* \in Z^*.$$

If $I_\nu\colon L^1(\nu) \to Z$ denotes the integration operator $f \mapsto \int_\Omega f\, d\nu$, then its bidual operator $I_\nu^{**}\colon L^1(\nu)^{**} \to Z^{**}$ satisfies

$$\langle I_\nu^{**}(f), z^*\rangle = \int_\Omega f\, dz^*\nu, \quad z^* \in Z^*,$$

whenever $f \in L^1(\nu)$. However, for $f \in L^1_w(\nu) \setminus L^1(\nu)$, what is the connection (if any) between the vector "$\int_\Omega f\, d\nu$" $\in Z^{**}$ and the operator I_ν^{**}, that is, does there exist $\nu_f \in L^1(\nu)^{**}$ satisfying "$\int_\Omega f\, d\nu$" $= I_\nu^{**}(\nu_f)$ and if so, is it unique and what properties does it have? For a finer analysis of the operator I_ν such types of questions become crucial.

In general, the approach via duality alone throws no light on this question. To make some headway, we first note that both $L^1_w(\nu)$ and $L^1(\nu)$ are *Banach function spaces* (briefly, B.f.s.), relative to any control measure for ν; this approach was systematically used and exposed in [3–5] and has been very useful ever since. It makes available the results, methods and techniques of a rich theory developed by W. Luxemburg, A. Zaanen and others in the 1960's and beyond. A second relevant point occurs more recently and makes the connection between $L^1(\nu)$ and $L^1_w(\nu)$ explicit: the *Köthe bidual* of the B.f.s. $L^1(\nu)$, denoted by $L^1(\nu)''$, is precisely $L^1_w(\nu)$, [9]. This merges various aspects of the theory of B.f.s.' with those from the theory of vector measures and integration, thereby providing a combined approach for attacking the above problem. This requires some further explanation.

Suppose that the range space Z of ν is a Banach lattice E. Then the dual Banach lattice E^* has a decomposition $E^* = E^*_n \oplus E^*_s$, where E^*_n (resp. E^*_s) is the band of all order continuous (resp. singular) functionals on E. Passing to the bidual we have the band decomposition

$$E^{**} \simeq (E^*_n)^*_n \oplus (E^*_n)^*_s \oplus (E^*_s)^*_n \oplus (E^*_s)^*_s. \tag{3}$$

More specifically, if E is a B.f.s. X, then X^*_n corresponds to the Köthe dual X' of X via a suitable isometric order isomorphism. Moreover, the Banach lattice dual X'^* of the B.f.s. X' can be naturally identified with the band $[\,X'^*\,] \simeq (X^*_n)^*_n \oplus (X^*_n)^*_s$ in X^{**}. Similarly, the Köthe bidual X'' can be identified in a natural way with the band $[\,X''\,] \simeq (X^*_n)^*_n$ in X^{**}. Finally, there is the standard isometric imbedding j_X of X into X^{**}; its image $j_X(X)$ is denoted by $[\,X\,]$. If X_a denotes the closed ideal of all order continuous (briefly, o.c.) elements of X and $[\,X_a\,] := j_X(X_a)$, then we have the containments

$$[\,X_a\,] \subseteq [\,X''\,] \subseteq [\,X'^*\,] \subseteq X^{**}.$$

As noted above, for a vector measure $\nu\colon \Sigma \to X$ both of the spaces $L^1(\nu)$ and $L^1_w(\nu)$ are B.f.s.'. Moreover, $L^1(\nu)$ is always o.c., that is, $L^1(\nu)_a = L^1(\nu)$. Recalling that $L^1_w(\nu) = L^1(\nu)''$ we see that the previous containments specialize to

$$[\,L^1(\nu)\,] \subseteq [\,L^1_w(\nu)\,] \subseteq [\,L^1(\nu)'^*\,] = L^1(\nu)^{**},$$

where the equality is due to the fact that for any B.f.s. X which is o.c., we have $X^*_s = \{0\}$ and so (from (3)) it follows that $X^{**} \simeq (X^*_n)^*_n \oplus (X^*_n)^*_s \simeq [\,X'^*\,]$. The aim of this paper then becomes clear: where and when, amongst the spaces $[\,X_a\,]$, $[\,X\,]$, $[\,X''\,]$ and $[\,X'^*\,]$, is the image of the spaces $[\,L^1(\nu)\,]$ and $[\,L^1_w(\nu)\,]$ under the bidual map I^{**}_ν? Can the restriction of I^{**}_ν to $[\,L^1_w(\nu)\,]$ be considered as an extension of the integration map $I_\nu\colon L^1(\nu) \to X$? And, so on.

Special cases already give some clues for ν taking values in a general Banach space Z. For instance, if $L^1(\nu)$ is reflexive then, of course, $I^{**}_\nu(L^1(\nu)^{**}) \subseteq [\,Z\,]$. But, I^{**}_ν can then be considered as being equal to I_ν and so, is not an "extension" of I_ν. On the other hand, if I_ν is weakly compact, then Gantmacher's theorem, [17, Theorem VI.4.2], tells us that again $I^{**}_\nu(L^1(\nu)^{**}) \subseteq [\,Z\,]$. However, if $L^1(\nu)$ is *not* reflexive, then I^{**}_ν is a *genuine* extension of I_ν still taking all of its values in $[\,Z\,]$. Moreover, if I_ν is a compact operator, then the variation measure $|\nu|$ of ν is a finite positive measure and $L^1(\nu) = L^1(|\nu|)$, [28]. In this case, $L^1(\nu)$ has the Fatou property. We extend this observation to a general characterization, namely, $L^1(\nu)$ has the Fatou property if and only if $I^{**}_\nu([\,L^1_w(\nu)\,]) \subseteq [\,Z\,]$. And, so on.

A particular (but, important) class of vector measures ν is that generated by certain X–valued kernel operators T via $\nu(A) := T(\chi_A)$. Such operators have associated with them an *optimal domain space* $[\,T, X\,] := \{f : T|f| \in X\}$ with the property that T has a continuous X*–valued* extension to $[\,T, X\,]$ and $[\,T, X\,]$ is the maximal B.f.s. with this property. It is known that $L^1(\nu) \subseteq [\,T, X\,] \subseteq L^1_w(\nu)$, typically with strict inclusions, and that $T(f) = I_\nu(f)$ for $f \in L^1(\nu)$, [7,9]. So, for such ν there is the additional B.f.s. $[\,T, X\,]$ available which has no analogue for general vector measures. In Section 3 we analyze the relationships between I_ν^{**} and the operator $T\colon [\,T, X\,] \to X$ and, in particular, the problem of when I_ν^{**} is an extension of $T\colon [\,T, X\,] \to X$.

1. Preliminaries

For a vector measure ν, the natural setting for $L^1(\nu)$ is the class of B.f.s.'. In this section we introduce B.f.s.' and the spaces $L^1(\nu)$. The latter part is devoted to presenting results (some known) on decompositions of the bidual of a Banach lattice and of a B.f.s. Due to their importance in the paper, their subtle nature, and the difficulty of finding clear references, we have decided to present them in detail. For ease of reading, longer proofs have been transferred to an Appendix.

Recall that a Banach lattice E is *order continuous* if all order bounded, increasing sequences are norm convergent. It has the *Fatou property* if for every upwards directed system $0 \le e_\alpha \uparrow$ in E with $\sup_\alpha \|e_\alpha\|_E < \infty$, there exists $e \in E^+$ (the positive cone of E) such that $e_\alpha \uparrow e$ and $\|e_\alpha\|_E \uparrow \|e\|_E$.

Let (Ω, Σ, μ) be a σ–finite measure space. Denote by $\mathcal{M}$ the space of (classes of) measurable finite real functions. A *Banach function space* relative to μ is a Banach space X contained in $\mathcal{M}$ such that if $f \in \mathcal{M}$ with $|f| \le |g|$ μ–a.e. for some $g \in X$, then $f \in X$ and $\|f\|_X \le \|g\|_X$. Note that a B.f.s. is a Banach lattice for the μ–a.e. pointwise order. A B.f.s. X has the Fatou property if, for every sequence $(f_n) \subseteq X^+$ with $\sup_n \|f_n\|_X < \infty$ that increases μ–a.e. to f, we have that $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$.

Let (Ω, Σ) be a measurable space and Z be a Banach space; its unit ball is denoted by B_Z. A set function $\nu\colon \Sigma \to Z$ is a *vector measure* if $\sum \nu(A_n)$ converges to $\nu(\cup A_n)$ in X for every sequence (A_n) of disjoint sets in Σ. A set $A \in \Sigma$ is *ν–null* if $\nu(B) = 0$ for all $B \in \Sigma$, $B \subseteq A$. A property holds ν–almost everywhere (ν–a.e.) if it holds except on a ν–null set. As noted in the Introduction, the spaces $L^1_w(\nu)$ and $L^1(\nu)$ are Banach spaces when equipped with the norm (1). For simplicity, $\int_\Omega f\,d\nu$ will be denoted by $\int f\,d\nu$. The Σ–simple functions are always dense in $L^1(\nu)$. If Z does not contain a copy of the sequence space c_0, then $L^1(\nu) = L^1_w(\nu)$, [20]. By choosing a Rybakov control measure $\mu = |z_0^*\nu|$ for ν, for a suitable $z_0^* \in Z^*$, [16, Ch. IX, §2], both $L^1_w(\nu)$ and $L^1(\nu)$ can be considered as B.f.s.' relative to μ. The space $L^1_w(\nu)$ always has the Fatou property and $L^1(\nu)$ is always o.c. The integration operator $I_\nu\colon L^1(\nu) \to Z$, defined by $I_\nu(f) := \int f\,d\nu$, is linear and continuous with $\|\int f\,d\nu\|_Z \le \|f\|_\nu$ for $f \in L^1(\nu)$. For more detailed information on the spaces $L^1(\nu)$ and $L^1_w(\nu)$ and the integration map I_ν we refer to [3–5,9,12,13,25,28,29], for example.

For the general theory and basic facts about Banach lattices, see the monographs [21–23,32], for example. The dual space E^* of a Banach lattice E is also a Banach lattice for the order defined via $e^* \ge 0$ iff $\langle e^*, e\rangle \ge 0$ for all

$e \in E^+$. As already noted $E^* = E_n^* \oplus E_s^*$, where E_n^* is the space of all *order continuous functionals* on E (i.e. those $e^* \in E^*$ such that $\inf_\alpha |\langle e^*, e_\alpha\rangle| = 0$ whenever $e_\alpha \downarrow 0$ in E) and E_s^* is the space of all *singular functionals* on E (i.e. those $e^* \in E^*$ such that $|e^*| \wedge |y^*| = 0$ for all $y^* \in E_n^*$).

Given $F \subseteq E^*$, we write ${}^\perp F = \{e \in E : \langle e^*, e\rangle = 0 \text{ for all } e^* \in F\}$ and $F^\perp = \{e^{**} \in E^{**} : \langle e^{**}, e^*\rangle = 0 \text{ for all } e^* \in F\}$. Let E_a denote the ideal consisting of the elements in E which have o.c. norm (i.e. those $e \in E$ such that if $|e| \geq e_\alpha \downarrow 0$ then $\|e_\alpha\|_E \downarrow 0$). Then

$$E_a = {}^\perp(E_s^*).$$

The following conditions are equivalent to E being o.c.:

$$E_a = E \;\Leftrightarrow\; E_n^* = E^* \;\Leftrightarrow\; E_s^* = \{0\} \;\Leftrightarrow\; (E_s^*)^\perp = E^{**}.$$

The following standard fact is recorded for later use.

Lemma 1.1. *Let Z be a Banach space which is the direct sum of two closed subspaces, i.e. $Z = Z_1 \oplus Z_2$. Then*

$$Z^* = \pi_{Z_1^*}(Z_1^*) \oplus \pi_{Z_2^*}(Z_2^*),$$

where $\pi_{Z_i^*} : Z_i^* \to Z^*$ is defined by $\pi_{Z_i^*}(z^*) = z^* \circ P_{Z_i}$ for $z^* \in Z_i^*$, and P_{Z_i} is the projection of Z onto Z_i for $i = 1, 2$.

By applying Lemma 1.1 to $E^* = E_n^* \oplus E_s^*$ we get the following result.

Proposition 1.2. *Let E be a Banach lattice. Then*

$$E^{**} = \pi_{(E_n^*)^*}\left((E_n^*)_n^*\right) \oplus \pi_{(E_n^*)^*}\left((E_n^*)_s^*\right) \oplus \pi_{(E_s^*)^*}\left((E_s^*)_n^*\right) \oplus \pi_{(E_s^*)^*}\left((E_s^*)_s^*\right).$$

Moreover, we also have

$$\begin{aligned}
(E^*)_n^* &= \pi_{(E_n^*)^*}\left((E_n^*)_n^*\right) \oplus \pi_{(E_s^*)^*}\left((E_s^*)_n^*\right), \\
(E^*)_s^* &= \pi_{(E_n^*)^*}\left((E_n^*)_s^*\right) \oplus \pi_{(E_s^*)^*}\left((E_s^*)_s^*\right), \\
\pi_{(E_n^*)^*}\left((E_n^*)_n^*\right) &= (E^*)_n^* \cap (E_s^*)^\perp.
\end{aligned}$$

Less formally, the above decomposition of E^{**} can be written as (3).

A Banach space Z can be considered as a closed subspace of Z^{**}, namely, via the isometric imbedding $j_Z : Z \to Z^{**}$ where, for every $z \in Z$, we have

$$\langle j_Z(z), z^*\rangle \;=\; \langle z^*, z\rangle, \quad z^* \in Z^*.$$

If $Z = E$ is a Banach lattice, then j_E is also an order homomorphism and so the order of E is transferred to the order of E^{**}. Typically $j_E(E)$ need not be an ideal in E^{**}; this is the case if and only if E is o.c., [21, Theorem 1.b.16]. We note, for any Banach lattice E, that

$$j_E(E) \subseteq (E^*)_n^*$$

always holds. Indeed, it suffices to show that $j_E(E^+) \subseteq (E^*)_n^*$. But, if $e \in E^+$ and $e_\alpha^* \downarrow 0$ in E^*, then $\langle j_E(e), e_\alpha^*\rangle = \langle e_\alpha^*, e\rangle \downarrow 0$.

The following useful result shows the relationship amongst various subspaces of E^{**}; for the proof we refer to the Appendix.

Proposition 1.3. *For a Banach lattice E we have*

$$j_E(E_a) \subseteq \pi_{(E_n^*)^*}\left((E_n^*)_n^*\right) \subseteq \pi_{(E_n^*)^*}\left((E_n^*)^*\right) = (E_s^*)^\perp \subseteq E^{**}. \tag{4}$$

Moreover, the following assertions hold.

(i) *Let ${}^\perp(E_n^*) = \{0\}$. Then the equality $j_E(E_a) = \pi_{(E_n^*)^*}\left((E_n^*)_n^*\right)$ holds if and only if E is o.c. and has the Fatou property.*

(ii) $\pi_{(E_n^*)^*}\left((E_n^*)_n^*\right) = (E_s^*)^\perp$ *if and only if E_n^* is o.c.*

(iii) $j_E(E) \cap (E_s^*)^\perp = j_E(E_a)$.

Standard references for B.f.s.' are the monographs [21,22], [31, Ch.15]. Let X be a B.f.s. over (Ω, Σ, μ) in which case X^* is a Banach lattice but, it may fail to be a B.f.s. (e.g. $X = L^\infty([0,1])$). Recall that the *Köthe dual* (or *associate space*) of X is defined by

$$X' = \left\{x' \in \mathcal{M} : \int |\, x \cdot x'|\, d\mu < \infty \ \text{ for all } x \in X\right\}.$$

We assume that X is saturated, i.e. given $A \in \Sigma$ with $\mu(A) > 0$ there exists $B \in \Sigma$ with $B \subseteq A$ such that $\mu(B) > 0$ and $\chi_B \in X$, [31, Ch. 15]. In this case, X' is also a B.f.s. relative to μ (with the Fatou property) when it is endowed with the norm

$$\|x'\|_{X'} := \sup_{x \in B_X} \left| \int x \cdot x'\, d\mu \right|.$$

The *Köthe bidual* of X, denoted by X'', is the Köthe dual of the B.f.s. X' and is equipped with the norm

$$\|x''\|_{X''} := \sup_{x' \in B_{X'}} \left| \int x' \cdot x''\, d\mu \right|.$$

Moreover, we have the following Hölder type inequality

$$\left| \int x' \cdot x\, d\mu \right| \leq \|x\|_X \cdot \|x'\|_{X'}, \quad x \in X,\, x' \in X'.$$

In addition, $\|x\|_{X''} \leq \|x\|_X$, i.e. X is continuously contained in X'' (via the identity map) and X is an ideal in X''. The equality $X = X''$ holds if and only if X has the Fatou property, in which case the norms coincide. If $X' \subseteq X^*$ is norming, then the norms of X and X'' coincide on X, i.e. X is isometrically isomorphic to a closed ideal of X''.

The associate space X' can be identified with the band X_n^* in X^* via the linear isometry $\eta_{X'} \colon X' \to X^*$ where, for $x' \in X'$,

$$\eta_{X'}(x') \colon x \mapsto \langle \eta_{X'}(x'), x \rangle = \int x \cdot x'\, d\mu, \quad x \in X.$$

That is, $\eta_{X'}(X') = X_n^*$ and

$$\|\eta_{X'}(x')\|_{X^*} = \|x'\|_{X'}, \quad x' \in X'.$$

In particular, $\eta_{X'}$ is injective and hence, is an isomorphism of X' onto X_n^*. Even more, $\eta_{X'}$ is an order isomorphism of X' onto X_n^* (i.e. preserves the order), since it is a positive operator. So, X' and X_n^* coincide as Banach lattices. Note that $\eta_{X'}$ is surjective, that is, $\eta_{X'}(X') = X^*$, if and only if $X_n^* = X^*$ or, equivalently, if and only if X is o.c.

Analogously, X'' can be identified with the band $(X')_n^*$ in X'^* via the linear isometry $\eta_{X''}\colon X'' \to X'^*$ where, for $x'' \in X''$,

$$\eta_{X''}(x'')\colon x' \mapsto \langle \eta_{X''}(x''), x' \rangle \ = \int x' \cdot x'' \, d\mu, \quad x' \in X'.$$

Then, we have

$$\|\eta_{X''}(x'')\|_{X'^*} = \|x''\|_{X''}, \quad x'' \in X''.$$

Let us see how X'^* (the Banach lattice dual of the B.f.s. X') can be identified with a band in X^{**}. Recall that $X^* = X_n^* \oplus X_s^*$. In this setting, since B.f.s.' are always super Dedekind complete, [22, pp. 126–127], o.c. functionals on X can be defined via sequences (i.e. $x^* \in X^*$ belongs to X_n^* if $\inf_n |x^*(x_n)| = 0$ whenever $x_n \downarrow 0$ in X), [32, Theorem 84.4(i)]. Let

$$P_{X_n^*}\colon X^* \to X_n^*$$

denote the band projection of X^* onto X_n^*; it is also an order homomorphism. Note that

$$\|P_{X_n^*}(x^*)\|_{X^*} \leq \|x^*\|_{X^*}, \quad x^* \in X^*.$$

Then, the linear map $\Pi_{X'^*}\colon X'^* \to X^{**}$ defined by

$$\Pi_{X'^*}(z) = z \circ \eta_{X'}^{-1} \circ P_{X_n^*}, \quad z \in X'^*, \tag{5}$$

gives the required identification.

The proof of the following result is given in the Appendix.

Proposition 1.4. *Let X be a B.f.s. The following assertions hold for the map $\Pi_{X'^*}\colon X'^* \to X^{**}$.*

(i) *$\Pi_{X'^*}$ is an isometry, i.e.*

$$\|\Pi_{X'^*}(z)\|_{X^{**}} = \|z\|_{X'^*}, \quad z \in X'^*,$$

and hence, $\Pi_{X'^*}(X'^*)$ is a closed subspace of X^{**}.

(ii) *The equality*

$$\Pi_{X'^*}(X'^*) = (X_s^*)^{\perp},$$

holds. In particular, $\Pi_{X'^*}(X'^*)$ is a band (hence, ideal) in X^{**}.

Observe that X'' can also be identified with a closed subspace (actually, a band) in X^{**} because the composition (of isometries)

$$\begin{tikzcd}
X'' \arrow[rr, "\Pi_{X'^*} \circ \eta_{X''}"] \arrow[dr, "\eta_{X''}"'] & & X^{**} \\
& X'^* \arrow[ur, "\Pi_{X'^*}"'] &
\end{tikzcd}$$

is an isometry, that is,

$$\|\Pi_{X'^*} \circ \eta_{_{X''}}(z)\|_{X^{**}} = \|z\|_{X''}, \quad z \in X''.$$

For reasons of clarity, the relevant identifications of each of the above subspaces within X^{**} is indicated by the following notation:

(I) $[\,X'^*\,] := \Pi_{X'^*}(X'^*) = (X_s^*)^\perp$,
(II) $[\,X''\,] := \Pi_{X'^*} \circ \eta_{_{X''}}(X'')$,
(III) $[\,X\,] := j_X(X)$,
(IV) $[\,X_a\,] := j_X(X_a)$.

Remark 1.5. Care should be taken on how subspaces of X^{**} are identified. It can happen that $j_X(X)$ does not coincide with $\Pi_{X'^*} \circ \eta_{_{X''}}(X)$. Indeed, the containments $X_a \subseteq X \subseteq X''$ imply that

$$\Pi_{X'^*} \circ \eta_{_{X''}}(X_a) \subseteq \Pi_{X'^*} \circ \eta_{_{X''}}(X) \subseteq \Pi_{X'^*} \circ \eta_{_{X''}}(X'') = [\,X''\,].$$

Consider any element $x \in X$. Then, for $x^* = x_n^* + x_s^* \in X^*$ with $x_n^* \in X_n^*$ and $x_s^* \in X_s^*$, we have

$$\begin{aligned}\langle \Pi_{X'^*} \circ \eta_{_{X''}}(x), x^* \rangle &= \langle \eta_{_{X''}}(x) \circ \eta_{_{X'}}^{-1} \circ P_{X_n^*}, x^* \rangle \;=\; \langle \eta_{_{X''}}(x), \eta_{_{X'}}^{-1}(x_n^*) \rangle \\ &= \int x \cdot \eta_{_{X'}}^{-1}(x_n^*)\; d\mu \;=\; \langle \eta_{_{X'}}\left(\eta_{_{X'}}^{-1}(x_n^*)\right), x \rangle \;=\; \langle x_n^*, x \rangle \\ &= \langle P_{X_n^*}(x^*), x \rangle \;=\; \langle j_X(x), P_{X_n^*}(x^*) \rangle = \langle j_X(x) \circ P_{X_n^*}, x^* \rangle,\end{aligned}$$

that is, $\Pi_{X'^*} \circ \eta_{_{X''}}(x) = j_X(x) \circ P_{X_n^*}$ as elements of X^{**}. However, if $x \in X_a = {}^\perp(X_s^*)$, then $\Pi_{X'^*} \circ \eta_{_{X''}}(x) = j_X(x)$ since $\langle j_X(x), x^* \rangle = \langle j_X(x), x_n^* \rangle + \langle j_X(x), x_s^* \rangle = \langle j_X(x) \circ P_{X_n^*}, x^* \rangle$ for every $x^* \in X^*$ (as $\langle j_X(x), x_s^* \rangle = \langle x_s^*, x \rangle = 0$). Thus,

$$\Pi_{X'^*} \circ \eta_{_{X''}}(X_a) = [\,X_a\,], \tag{6}$$

and

$$\Pi_{X'^*} \circ \eta_{_{X''}}(X) = \{ j_X(x) \circ P_{X_n^*} : x \in X \}. \tag{7}$$

So, $j_X(X)$ may not equal $\Pi_{X'^*} \circ \eta_{_{X''}}(X)$, that is, the image of X in X^{**} via j_X may *not* coincide with the image of X in X^{**} via $\Pi_{X'^*} \circ \eta_{_{X''}}$. Indeed, from (6) and (7) we have

$$[\,X\,] \cap \left(\Pi_{X'^*} \circ \eta_{_{X''}}(X) \right) = [\,X_a\,].$$

Hence, $j_X(X) = \Pi_{X'^*} \circ \eta_{_{X''}}(X)$ if and only if $X = X_a$, that is, if and only if X is o.c. Actually, these two spaces can even be "disjoint", as in the case of $X = L^\infty([0,1])$ for which $X_a = \{0\}$.

The spaces $[\,X'^*\,]$ and $[\,X''\,]$ can also be described in terms of the spaces forming the band decomposition of X^{**} given in Proposition 1.2. The notation of the following result is that of Lemma 1.1; for the proof we refer to the Appendix.

Proposition 1.6. *For a B.f.s. X the following formulae hold.*

(i) $[\,X''\,] = \pi_{(X_n^*)^*}((X_n^*)_n^*)$.
(ii) $[\,X'^*\,] = [\,X''\,] \oplus \pi_{(X_n^*)^*}((X_n^*)_s^*)$.

(iii) $X^{**} = [\,X'^{*}\,] \oplus \pi_{(X_s^*)^*}\,((X_s^*)^*)$.
(iv) $X^{**} = [\,X''\,] \oplus \pi_{(X_n^*)^*}\,((X_n^*)_s^*) \oplus \pi_{(X_s^*)^*}\,((X_s^*)^*)$.

Some special cases of Proposition 1.6 are not without interest. For instance, if the Banach lattice X_n^* is o.c., but $X_s^* \neq \{0\}$ (e.g. $X = \ell^\infty$ or $X = L^\infty([0,1])$), then $(X_n^*)_s^* = \{0\}$ and so Proposition 1.6 yields

$$[\,X'^{*}\,] = [\,X''\,] = \pi_{(X_n^*)^*}\,((X_n^*)_n^*) \quad \text{and} \quad X^{**} = [\,X''\,] \oplus \pi_{(X_n^*)^*}\,((X_s^*)^*)\,.$$

The following result follows from Propositions 1.3 and 1.6 together with the fact that, for a B.f.s. X, we always have ${}^\perp(X_n^*) = \{0\}$ (which follows from $X' \simeq X_n^*$ and [31, Ch. 15, §69, Theorem 1]).

Proposition 1.7. *For a B.f.s. X the following containments hold.*

$$[\,X_a\,] \subseteq [\,X''\,] \subseteq [\,X'^{*}\,] \subseteq X^{**}. \tag{8}$$

Moreover, we have:

(i) $[\,X_a\,] = [\,X''\,]$ *if and only if X is o.c. and has the Fatou property.*
(ii) $[\,X''\,] = [\,X'^{*}\,]$ *if and only if X' is o.c.*
(iii) $[\,X'^{*}\,] = X^{**}$ *if and only if X is o.c.*
(iv) $[\,X\,] \cap [\,X'^{*}\,] = [\,X_a\,]$.

Even though always $X \subseteq X'' \subseteq X'^{*}$ and $[\,X_a\,] \subseteq [\,X'^{*}\,]$, the space $[\,X\,]$ may *not* be contained in $[\,X'^{*}\,]$. From Proposition 1.7(iv), this is the case if and only if $[\,X_a\,] = [\,X\,]$, that is, if and only if $X_a = X$, i.e. X is o.c.

We now show that various inclusions in (8) can be strict.

Example 1.8. (i) The spaces $X = L^p([0,1])$ for $1 < p < \infty$ are o.c., have the Fatou property, and X' is o.c. Hence,

$$[\,X_a\,] = [\,X\,] = [\,X''\,] = [\,X'^{*}\,] = X^{**}.$$

Of course, for any B.f.s. X the above properties are equivalent to its reflexivity, [31, Ch. 15, §73, Theorem 2].

(ii) The space $X = L^1([0,1])$ is o.c., has the Fatou property, and $X' = L^\infty([0,1])$. So, we have

$$[\,X_a\,] = [\,X\,] = [\,X''\,] \subsetneq [\,X'^{*}\,] = X^{**}.$$

(iii) For $X = L^\infty([0,1])$, we have $X' = L^1([0,1])$ is o.c. and so

$$[\,X''\,] = [\,X'^{*}\,] \subsetneq X^{**}.$$

Moreover, $[\,X_a\,] = \{0\}$ and hence, $[\,X\,]$ is "disjoint" with $[\,X'^{*}\,]$; see Proposition 1.7(iv).

(iv) Let $X = L^{p,\infty}([0,1])$ with $1 < p < \infty$. Then X is not o.c. but, $X' = L^{p',1}[0,1]$ (with $\frac{1}{p} + \frac{1}{p'} = 1$) is o.c. (see [2, §IV.4]), and so

$$[\,X_a\,] \subsetneq [\,X''\,] = [\,X'^{*}\,] \subsetneq X^{**}.$$

Note that $[\,X_a\,] \neq \{0\}$, since X_a coincides with the closure of the simple functions in X, and that $[\,X_a\,] \subsetneq [\,X\,]$, since X is not o.c.

(v) For $1 < p < \infty$, let $X = \{f \in L^{p,\infty}([0,1]) : \lim_{t\to+0} t^{1/p} f^*(t) = 0\}$. Then X is o.c. (because it is the o.c.–part of the B.f.s. in part (iv), [18, §II.5.3]). Moreover, X fails the Fatou property and $X' = L^{p',1}([0,1])$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ is o.c. Hence,

$$[\,X_a\,] = [\,X\,] \subsetneq [\,X''\,] = [\,X'^*\,] = X^{**}.$$

Of course, c_0 exhibits the same features.

2. Where are the integrals?

Let Z be a Banach space and $\nu\colon \Sigma \to Z$ be a vector measure. The integration operator $I_\nu\colon L^1(\nu) \to Z$ is extended by its bidual operator $I_\nu^{**}\colon L^1(\nu)^{**} \to Z^{**}$ in the sense of the following factorization diagram:

$$\begin{tikzcd} L^1(\nu)^{**} \arrow[r, "I_\nu^{**}"] & Z^{**} \\ L^1(\nu) \arrow[u, "j_{L^1(\nu)}"] \arrow[r, "I_\nu"] & Z \arrow[u, "j_Z"'] \end{tikzcd}$$

So, we may define the *generalized integral* of $z \in L^1(\nu)^{**}$ as $I_\nu^{**}(z)$. Note, for every $f \in L^1(\nu)$, that we have

$$I_\nu^{**}\left(j_{L^1(\nu)}(f)\right) = j_Z\left(I_\nu(f)\right). \tag{9}$$

Our aim in this section is to locate where the image of various subspaces of $L^1(\nu)^{**}$, under I_ν^{**}, lie within Z^{**}. For example, $[\,L_w^1(\nu)\,]$ lies between $[\,L^1(\nu)\,]$ and $L^1(\nu)^{**}$. Where does $I_\nu^{**}([\,L_w^1(\nu)\,])$ lie within Z^{**}? When Z is a B.f.s. X, one may ask: when do the generalized integrals lie in the subspaces $[\,X_a\,]$, $[\,X\,]$, $[\,X''\,]$, $[\,X'^*\,]$ of X^{**}?

The first relevant observation is that $L_w^1(\nu) = L^1(\nu)''$, [9], and

$$[\,L^1(\nu)_a\,] = [\,L^1(\nu)\,] \subseteq [\,L^1(\nu)''\,] = [\,L_w^1(\nu)\,] \subseteq [\,L^1(\nu)'^*\,] = L^1(\nu)^{**}, \tag{10}$$

after recalling that $L^1(\nu)$ is o.c. and using Proposition 1.7.

It is time to consider the dual operator $I_\nu^*\colon Z^* \to L^1(\nu)^*$. For every $z^* \in Z^*$ the element $I_\nu^*(z^*) \in L^1(\nu)^*$ is given by

$$\langle I_\nu^*(z^*), f\rangle \;=\; \langle z^*, I_\nu(f)\rangle \;=\int f\,dz^*\nu \;=\int f\,\frac{d(z^*\nu)}{d\mu}\,d\mu \tag{11}$$

for all $f \in L^1(\nu)$, where $\frac{d(z^*\nu)}{d\mu}$ is the Radon-Nikodym derivative of the measure $z^*\nu$ with respect to μ (the Rybakov control measure for ν). Note that $\frac{d(z^*\nu)}{d\mu} \in L^1(\nu)'$ as

$$\int |f|\cdot\Big|\frac{d(z^*\nu)}{d\mu}\Big|\,d\mu = \int |f|\,d|z^*\nu| < \infty, \quad f \in L^1(\nu).$$

According to (11) we have

$$\langle I_\nu^*(z^*), f\rangle = \left\langle \eta_{_{L^1(\nu)'}}\left(\frac{d(z^*\nu)}{d\mu}\right), f \right\rangle, \quad f \in L^1(\nu),$$

that is, the dual operator of I_ν is precisely given by

$$I_\nu^*(z^*) = \eta_{_{L^1(\nu)'}}\left(\frac{d(z^*\nu)}{d\mu}\right), \qquad z^* \in Z^*. \tag{12}$$

The following technical result shows how the generalized integral of an element from $[\,L^1_w(\nu)\,] \subseteq L^1(\nu)^{**}$ acts on Z^*.

Lemma 2.1. *Let Z be a Banach space and $\nu\colon \Sigma \to Z$ be a vector measure. If $g \in L^1_w(\nu)$, then*

$$\langle I_\nu^{**}\left(\Pi_{L^1(\nu)'^*} \circ \eta_{_{L^1(\nu)''}}(g)\right), z^*\rangle \;=\; \int g\, dz^*\nu, \quad z^* \in Z^*.$$

Proof. Fix $g \in L^1_w(\nu) = L^1(\nu)''$. Recall from (II) that $\Pi_{L^1(\nu)'^*} \circ \eta_{_{L^1(\nu)''}}(g) \in L^1(\nu)^{**}$ with $I_\nu^{**}(\Pi_{L^1(\nu)'^*} \circ \eta_{_{L^1(\nu)''}}(g)) \in Z^{**}$. For $z^* \in Z^*$,

$$\langle I_\nu^{**}\left(\Pi_{L^1(\nu)'^*} \circ \eta_{_{L^1(\nu)''}}(g)\right), z^*\rangle \;=\; \langle \Pi_{L^1(\nu)'^*} \circ \eta_{_{L^1(\nu)''}}(g), I_\nu^*(z^*)\rangle.$$

Since $\Pi_{L^1(\nu)'^*} \circ \eta_{_{L^1(\nu)''}}(g) = \eta_{_{L^1(\nu)''}}(g) \circ \eta^{-1}_{_{L^1(\nu)'}} \circ P_{L^1(\nu)^*_n}$ and $P_{L^1(\nu)^*_n}$ is the identity map on $L^1(\nu)^*$ (recall $L^1(\nu)$ is o.c.), it follows from (12) that

$$\begin{aligned}\langle I_\nu^{**}\left(\Pi_{L^1(\nu)'^*} \circ \eta_{_{L^1(\nu)''}}(g)\right), z^*\rangle &= \left\langle \eta_{_{L^1(\nu)''}}(g), \frac{d(z^*\nu)}{d\mu} \right\rangle \\ &= \int g\, \frac{d(z^*\nu)}{d\mu}\, d\mu = \int g\, dz^*\nu.\end{aligned}$$

□

Let us see what Proposition 1.6 says for the particular B.f.s. $X = L^1(\nu)$. Since $L^1(\nu)$ is o.c., we have $L^1(\nu)^*_s = \{0\}$ and so $L^1(\nu)^*_n = L^1(\nu)^*$. Accordingly, $\pi_{(L^1(\nu)^*_n)^*}$ is the identity map on $L^1(\nu)^{**}$. Recalling that $L^1(\nu)'' = L^1_w(\nu)$, Proposition 1.6 and (10) give the decomposition

$$L^1(\nu)^{**} = [\,L^1(\nu)'^*\,] = [\,L^1_w(\nu)\,] \oplus (L^1(\nu)^*)^*_s = [\,L^1_w(\nu)\,] \oplus \left(\eta_{_{L^1(\nu)'}}\left(L^1(\nu)'\right)\right)^*_s.$$

Looking at the generalized integral, with Id denoting the identity map in $L^1(\nu)^{**}$, the situation is as follows:

$$\begin{tikzcd}
{[\,L^1(\nu)\,]} \arrow[r, "Id"] \arrow[d, "I_\nu^{**}"'] & {[\,L^1_w(\nu)\,]} \arrow[r, "Id"] \arrow[d, "I_\nu^{**}"] & L^1(\nu)^{**} = [\,L^1_w(\nu)\,] \oplus (L^1(\nu)^*)^*_s \arrow[d, "I_\nu^{**}"] \\
{[\,Z\,]} & ? & Z^{**}
\end{tikzcd}$$

So, the generalized integral of an element $f+u$ in $L^1(\nu)^{**}$, with $f \in [\,L^1_w(\nu)\,]$ and $u \in (L^1(\nu)^*)^*_s$, is the sum $I^{**}_\nu(f) + I^{**}_\nu(u)$. Where does each member of this sum lie within Z^{**}? Particularly, we are interested in the first member $I^{**}_\nu(f)$.

Proposition 2.2. *Let Z be a Banach space and $\nu\colon \Sigma \to Z$ be a vector measure. Then $I^{**}_\nu\left([\,L^1_w(\nu)\,]\right) \subseteq [\,Z\,]$ if and only if $L^1(\nu)$ has the Fatou property.*

Proof. If $L^1(\nu)$ has the Fatou property, then Proposition 1.7(i) (with $X := L^1(\nu)$) implies that $[\,L^1_w(\nu)\,] = [\,L^1(\nu)''\,] = [\,L^1(\nu)\,]$. Hence, $I^{**}_\nu\left([L^1_w(\nu)]\right) \subseteq [\,Z\,]$.

Suppose that $I^{**}_\nu\left([\,L^1_w(\nu)\,]\right) \subseteq [\,Z\,]$. Given $g \in L^1_w(\nu) = L^1(\nu)''$, we have that $g\chi_A \in L^1(\nu)''$, for every $A \in \Sigma$, and so there exists $z_A \in Z$ such that $I^{**}_\nu\left(\Pi_{L^1(\nu)'^*} \circ \eta_{L^1(\nu)''}(g\chi_A)\right) = j_Z(z_A)$. Then by Lemma 2.1, for all $z^* \in Z^*$ it follows that

$$\langle z^*, z_A\rangle = \langle j_Z(z_A), z^*\rangle = \langle I^{**}_\nu\left(\Pi_{L^1(\nu)'^*} \circ \eta_{L^1(\nu)''}(g\chi_A)\right), z^*\rangle = \int_A g\, dz^*\nu,$$

that is, $g \in L^1(\nu)$. Hence, $L^1(\nu) = L^1(\nu)''$ and so $L^1(\nu)$ has the Fatou property. $\square$

Corollary 2.3. *Let Z be a Banach space and $\nu\colon \Sigma \to Z$ be a vector measure such that $I_\nu\colon L^1(\nu) \to Z$ is weakly compact. Then $L^1(\nu)$ has the Fatou property. In particular, $L^1(\nu) = L^1_w(\nu)$.*

Proof. Since $I^{**}_\nu(L^1(\nu)^{**}) \subseteq [\,Z\,]$ (by Gantmacher's theorem) and $[\,L^1_w(\nu)\,] \subseteq L^1(\nu)^{**}$, we have $I^{**}_\nu([\,L^1_w(\nu)\,]) \subseteq [\,Z\,]$. The conclusion follows from Proposition 2.2. $\square$

Remark 2.4. The converse to Corollary 2.3 is not true in general. Let $Z = L^1([0,1])$ and, for each Borel set $A \subseteq [0,1]$, define $\nu(A) \in Z$ by

$$\nu(A) : t \mapsto \int_0^t \chi_A(s)\, ds, \quad t \in [0,1].$$

Then ν is a vector measure and $L^1(\nu)$ is the weighted L^1–space $L^1((1-s)ds)$, which clearly has the Fatou property. However, I_ν is *not* weakly compact, [26, Example 2].

The diagram prior to Proposition 2.2 poses the question of where $I^{**}_\nu([\,L^1_w(\nu)\,])$ lies within Z^{**}? It is to be expected that extra properties on ν or Z will assist. One such result is Proposition 2.2. Furthermore, if $Z = E$ is a Banach lattice such that E^* has o.c. norm and ν is E–valued, then $E^{**} = (E^*)^*_n$ and so $I^{**}_\nu(L^1(\nu)^{**}) \subseteq (E^*)^*_n$. In particular, $I^{**}_\nu([\,L^1_w(\nu)\,]) \subseteq (E^*)^*_n$. Conditions on ν can also lead to the same conclusion. Recall that if E is an *order separable* Banach lattice (i.e. for any subset D of E there exists an at most countable subset of D with the same upper bounds as D), then order continuous functionals can be characterized purely via sequences, i.e. $e^* \in E^*_n$ if and only if for every decreasing sequence $e_n \downarrow 0$ in E we have $\inf_n |\langle e^*, e_n\rangle| = 0$. Recall that a set A in a Banach lattice E is *L–weakly compact* (also known as *almost order bounded*) if for every $\varepsilon > 0$ there exists $0 \leq x_\varepsilon \in E$ such that $A \subseteq [-x_\varepsilon, x_\varepsilon] + \varepsilon \cdot B_E$. A linear operator

$T\colon Z \to E$, with Z a Banach space, is *L–weakly compact* if $T(B_Z)$ is an L-weakly compact set in E; see [23, Definition 3.6.1 and Proposition 3.6.2].

Theorem 2.5. *Let E be a Banach lattice and $\nu\colon \Sigma \to E$ be a vector measure. The following assertions hold.*

(i) *If ν is positive, then $I_\nu^{**}([\,L^1_w(\nu)\,]) \subseteq (E^*)^*_n$.*

(ii) *If $I_\nu(B_{L^1(\nu)})$ is order bounded, then $I_\nu^{**}(L^1(\nu)^{**}) \subseteq (E^*)^*_n$.*

(iii) *If E^* is order separable and I_ν is L–weakly compact, then $I_\nu^{**}(L^1(\nu)^{**}) \subseteq (E^*)^*_n$.*

Proof. (i) Suppose that ν is positive. Fix $u \in [\,L^1_w(\nu)\,]$ and let $e^*_\alpha \downarrow 0$ in E^*. Let us show that

$$\inf_\alpha |\langle I_\nu^{**}(u), e^*_\alpha\rangle| = 0, \tag{13}$$

which implies (by definition) that $I_\nu^{**}(u) \in (E^*)^*_n$. Note that $|\langle I_\nu^{**}(u), e^*_\alpha\rangle| = |\langle u, I_\nu^*(e^*_\alpha)\rangle|$ with

$$u \in [\,L^1_w(\nu)\,] = \pi_{(L^1(\nu)^*_n)^*}\Big(\big(L^1(\nu)^*_n\big)^*_n\Big) = \big(L^1(\nu)^*\big)^*_n$$

(see Proposition 1.6(i) and recall that $L^1(\nu)$ is o.c.) and that

$$I_\nu^*(e^*_\alpha) = \eta_{L^1(\nu)'}\left(\frac{d(e^*_\alpha\nu)}{d\mu}\right) \in L^1(\nu)^*;$$

see (12). If we prove that $\frac{d(e^*_\alpha\nu)}{d\mu} \downarrow 0$ in $L^1(\nu)'$ then, as $\eta_{L^1(\nu)'}$ is an order isomorphism, we have $\eta_{L^1(\nu)'}\left(\frac{d(e^*_\alpha\nu)}{d\mu}\right) \downarrow 0$ in $L^1(\nu)^*$ and so (13) holds.

Since e^*_α is decreasing in E^* and ν is positive, it follows that $\frac{d(e^*_\alpha\nu)}{d\mu}$ is decreasing in $L^1(\nu)'$. Let $h \in L^1(\nu)'$ satisfy $h \le \frac{d(e^*_\alpha\nu)}{d\mu}$ for all α. Then,

$$\int_A h\,d\mu \le \int_A \frac{d(e^*_\alpha\nu)}{d\mu}\,d\mu = e^*_\alpha\nu(A) = \langle e^*_\alpha, \nu(A)\rangle,$$

for all $A \in \Sigma$ and all α, and so $\int_A h\,d\mu \le \inf_\alpha\langle e^*_\alpha, \nu(A)\rangle = 0$ for all $A \in \Sigma$. So, $h \le 0$ μ–a.e. Hence, $\bigwedge_\alpha \frac{d(e^*_\alpha\nu)}{d\mu} = 0$ and therefore (13) holds.

(ii) Fix $u \in L^1(\nu)^{**}$ and let $e^*_\alpha \downarrow 0$ in E^*. Then

$$|\langle I_\nu^{**}(u), e^*_\alpha\rangle| = |\langle u, I_\nu^*(e^*_\alpha)\rangle| \le \|u\|_{L^1(\nu)^{**}}\|I_\nu^*(e^*_\alpha)\|_{L^1(\nu)^*}.$$

Since $I_\nu(B_{L^1(\nu)})$ is order bounded, there exists $x \in E^+$ such that $|I_\nu(f)| \le x$ for all $f \in B_{L^1(\nu)}$. Using (12), we have

$$\begin{aligned}
\|I_\nu^*(e^*_\alpha)\|_{L^1(\nu)^*} &= \left\|\eta_{L^1(\nu)'}\left(\frac{d(e^*_\alpha\nu)}{d\mu}\right)\right\|_{L^1(\nu)^*} = \left\|\frac{d(e^*_\alpha\nu)}{d\mu}\right\|_{L^1(\nu)'} \\
&= \sup_{f \in B_{L^1(\nu)}}\left|\int f\,\frac{d(e^*_\alpha\nu)}{d\mu}\,d\mu\right| = \sup_{f \in B_{L^1(\nu)}}\left|\int f\,d(e^*_\alpha\nu)\right| \\
&= \sup_{f \in B_{L^1(\nu)}}|\langle e^*_\alpha, I_\nu(f)\rangle| \le \langle e^*_\alpha, x\rangle,
\end{aligned}$$

where the last inequality is due to the fact that $|\langle e^*_\alpha, I_\nu(f)\rangle| \le \langle |e^*_\alpha|, |I_\nu(f)|\rangle = \langle e^*_\alpha, |I_\nu(f)|\rangle \le \langle e^*_\alpha, x\rangle$. Hence,

$$\inf_\alpha |\langle I_\nu^{**}(u), e^*_\alpha\rangle| \le \|u\|_{L^1(\nu)^{**}} \inf_\alpha \langle e^*_\alpha, x\rangle = 0.$$

(iii) Let $u \in L^1(\nu)^{**}$ and $e^*_n \downarrow 0$ in E^*. Then

$$|\langle I_\nu^{**}(u), e^*_n\rangle| = |\langle u, I^*_\nu(e^*_n)\rangle| \le \|u\|_{L^1(\nu)^{**}} \|I^*_\nu(e^*_n)\|_{L^1(\nu)^*}.$$

Since $I_\nu(B_{L^1(\nu)})$ is almost order bounded, given any $\varepsilon > 0$ there exists $x_\varepsilon \in E^+$ such that $I_\nu(B_{L^1(\nu)}) \subseteq [-x_\varepsilon, x_\varepsilon] + \varepsilon \cdot B_E$. Then, as in the proof of (ii), we have

$$\|I^*_\nu(e^*_n)\|_{L^1(\nu)^*} = \sup_{f\in B_{L^1(\nu)}} |\langle e^*_n, I_\nu(f)\rangle| \le \langle e^*_n, x_\varepsilon\rangle + \varepsilon\|e^*_n\|_{E^*} \le \langle e^*_n, x_\varepsilon\rangle + \varepsilon\|e^*_1\|_{E^*}.$$

The conclusion then follows as in (ii). □

If the B.f.s. X is o.c., then $\pi_{(X^*_n)^*}$ is the identity map and so, by Proposition 1.6(i), we have $(X^*)^*_n = [\,X''\,]$. Moreover, $X^* = \eta_{X'}(X')$ is then order separable (since X' is a B.f.s.). Hence, from Theorem 2.5 we have the following result.

Corollary 2.6. *Let X be a B.f.s. which is o.c. and $\nu\colon \Sigma \to X$ be a vector measure.*

- (i) *If ν is positive, then $I_\nu^{**}([\,L^1_w(\nu)\,]) \subseteq [\,X''\,]$.*
- (ii) *If $I_\nu(B_{L^1(\nu)})$ is order bounded, then $I_\nu^{**}(L^1(\nu)^{**}) \subseteq [\,X''\,]$.*
- (iii) *If I_ν is L–weakly compact, then $I_\nu^{**}(L^1(\nu)^{**}) \subseteq [\,X''\,]$.*

Concerning Corollary 2.6, note that condition (iii) always implies (ii), since order bounded sets are necessarily L–weakly compact.

The following result presents criteria for I_ν^{**} to map $[\,L^1_w(\nu)\,]$ into subspaces of the bidual other than $(E^*)^*_n$, such as $(E^*_s)^\perp$, for example.

Theorem 2.7. *Let E be a Banach lattice and $\nu\colon \Sigma \to E$ be a vector measure. The following assertions are equivalent.*

- (i) *$e^*\nu \equiv 0$ for all $e^* \in E^*_s$.*
- (ii) *$\nu(\Sigma) \subseteq E_a$.*
- (iii) *$I_\nu^{**}\big([\,L^1(\nu)\,]\big) \subseteq j_E(E_a)$.*
- (iv) *$I_\nu^{**}\big([\,L^1_w(\nu)\,]\big) \subseteq (E^*_s)^\perp$.*
- (v) *$I_\nu^{**}\big(L^1(\nu)^{**}\big) \subseteq (E^*_s)^\perp$.*

Proof. (i) ⇔ (ii) is clear from $E_a = {}^\perp(E^*_s)$.

(ii) ⇒ (iii) Since E_a is closed in E, we have $I_\nu(f) \in E_a$ for $f \in L^1(\nu)$. From (9) it then follows that $I_\nu^{**}(j_{L^1(\nu)}(f)) = j_E(I_\nu(f)) \in j_E(E_a)$.

(iii) ⇒ (v) Let $z \in L^1(\nu)^{**}$ and $e^* \in E^*_s$. Fix $f \in L^1(\nu)$. Since $j_E(I_\nu(f)) \in j_E(E_a)$ and j_E is injective, it follows that $I_\nu(f) \in E_a = {}^\perp(E^*_s)$. Hence, $\langle I^*_\nu(e^*), f\rangle = \langle e^*, I_\nu(f)\rangle = 0$. Thus, $I^*_\nu(e^*) = 0$. Accordingly, $\langle I_\nu^{**}(z), e^*\rangle = \langle z, I^*_\nu(e^*)\rangle = 0$.

(v) ⇒ (iv) is obvious.

(iv) $\Rightarrow$ (i) Let $A \in \Sigma$ and $e^* \in E_s^*$. Then $I_\nu^{**}\left(\Pi_{L^1(\nu)'^*} \circ \eta_{_{L^1(\nu)''}}(\chi_A)\right) \in (E_s^*)^\perp$, since $\chi_A \in L^1_w(\nu)$. Lemma 2.1 (with $g = \chi_A$) yields

$$e^*\nu(A) = \int \chi_A \, d\, e^*\nu = \langle I_\nu^{**}\left(\Pi_{L^1(\nu)'^*} \circ \eta_{_{L^1(\nu)''}}(\chi_A)\right), e^*\rangle \ = \ 0.$$

$\square$

Remark 2.8. Let E be a Banach lattice satisfying $E_a = \{0\}$ (e.g. $L^\infty([0,1])$). It is clear that *no* non-zero E–valued vector measure ν can satisfy (ii) of Theorem 2.7. In particular, neither of the containments (iv), (v) is then valid.

In the case when E is a B.f.s. X, recall that $(X_s^*)^\perp = [\,X'^*\,]$; see (I).

Corollary 2.9. *Let X be a B.f.s. and $\nu\colon \Sigma \to X$ be a vector measure. The following assertions are equivalent.*

- (i) $x^*\nu \equiv 0$ *for all* $x^* \in X_s^*$.
- (ii) $\nu(\Sigma) \subseteq X_a$.
- (iii) $I_\nu^{**}\left([\,L^1(\nu)\,]\right) \subseteq [\,X_a\,]$.
- (iv) $I_\nu^{**}\left([\,L^1_w(\nu)\,]\right) \subseteq [\,X'^*\,]$.
- (v) $I_\nu^{**}\left(L^1(\nu)^{**}\right) \subseteq [\,X'^*\,]$.

For a Banach lattice E, the following result is a consequence of Proposition 2.2, Theorems 2.5, 2.7, Gantmacher's theorem and the facts that $[\,E_a\,] = [\,E\,] \cap (E_s^*)^\perp$ and $\pi_{(E_n^*)^*}\left((E_n^*)_n^*\right) = (E^*)_n^* \cap (E_s^*)^\perp$.

Corollary 2.10. *Let E be a Banach lattice and $\nu\colon \Sigma \to E$ be a vector measure.*

- (i) $I_\nu^{**}\left([\,L^1_w(\nu)\,]\right) \subseteq [\,E_a\,]$ *if and only if* $L^1(\nu)$ *has the Fatou property and* $\nu(\Sigma) \subseteq E_a$.
- (ii) $I_\nu^{**}\left(L^1(\nu)^{**}\right) \subseteq [\,E_a\,]$ *if and only if* I_ν *is weakly compact and* $\nu(\Sigma) \subseteq E_a$.
- (iii) *If* ν *is positive and* $\nu(\Sigma) \subseteq E_a$, *then* $I_\nu^{**}([\,L^1_w(\nu)\,]) \subseteq \pi_{(E_n^*)^*}\left((E_n^*)_n^*\right)$.
- (iv) *If* $I_\nu(B_{L^1(\nu)})$ *is order bounded and* $\nu(\Sigma) \subseteq E_a$, *then* $I_\nu^{**}(L^1(\nu)^{**}) \subseteq \pi_{(E_n^*)^*}\left((E_n^*)_n^*\right)$.
- (v) *If* E^* *is order separable,* I_ν *is* L–*weakly compact and* $\nu(\Sigma) \subseteq E_a$, *then* $I_\nu^{**}(L^1(\nu)^{**}) \subseteq \pi_{(E_n^*)^*}\left((E_n^*)_n^*\right)$.

We end this section with a relevant example.

Example 2.11. Let $K\colon \mathbb{N} \times \mathbb{N} \to [0,\infty)$ satisfy:

$$\sup_{n\geq 1} \sum_{m=1}^{\infty} K(n,m) < \infty. \tag{14}$$

Then, for every $A \subseteq \mathbb{N}$, the sequence $\nu(A) := (\nu(A)(n))_{n=1}^\infty$ given by

$$\nu(A)(n) := \sum_{m\in A} K(n,m), \quad n \in \mathbb{N},$$

is finite-valued and the set function $\nu\colon A \mapsto \nu(A) \in \ell^\infty$, for $A \subseteq \mathbb{N}$, is well defined and finitely additive. In order that ν is σ−additive, it is necessary and sufficient that K satisfies

$$\lim_{j\to\infty} \left\{ \sup_{n\geq 1} \sum_{m\geq j} K(n,m) \right\} = 0. \tag{15}$$

Of course, ν is then a positive vector measure in ℓ^∞. For $x^* \in (\ell^\infty)^*$ and $m \in \mathbb{N}$, we have $|x^*\nu|(\{m\}) = |x^*\nu(\{m\})| = |\langle x^*, K(\cdot, m)\rangle|$. Consequently, for $A \subseteq \mathbb{N}$, we have

$$|x^*\nu|(A) = \sum_{m\in A} |\langle x^*, K(\cdot, m)\rangle| \,.$$

Given any function $f\colon \mathbb{N} \to \mathbb{R}$, we have

$$\int |f|\, d|x^*\nu| = \sum_{m=1}^{\infty} |f(m)|\cdot |\langle x^*, K(\cdot, m)\rangle| \,,$$

and

$$\|f\|_\nu = \sup_{\|x^*\|\leq 1} \sum_{m=1}^{\infty} |f(m)|\cdot |\langle x^*, K(\cdot, m)\rangle| \,. \tag{16}$$

Recall that $f \in L^1_w(\nu)$ precisely when $\|f\|_\nu < \infty$. For a simple function $\varphi\colon \mathbb{N} \to \mathbb{R}$, we have $\int \varphi\, d\nu = \sum_{m=1}^{\infty} \varphi(m)\cdot K(\cdot, m)$. It follows, via the dominated convergence theorem for vector measures that, if $f \in L^1(\nu)$, then

$$\int f\, d\nu = \sum_{m=1}^{\infty} f(m)\cdot K(\cdot, m) \in \ell^\infty.$$

From (10), the general situation regarding the space $L^1(\nu)$ is that

$$[\, L^1(\nu) \,] \subseteq [\, L^1_w(\nu) \,] \subseteq [\, L^1(\nu)'^* \,] = L^1(\nu)^{**}.$$

The measure ν takes values in $X = \ell^\infty$ with $X_a = c_0$, $X' = \ell^1$ and so $X'' = \ell^\infty$, $X'^* = \ell^\infty$. Then, the following containments hold

$$[\, X_a \,] \subsetneq [\, X'' \,] = [\, X'^* \,] \subsetneq X^{**}.$$

As explained in Section 2, the way of viewing these spaces as closed subspaces of $(\ell^\infty)^{**}$ is precise and given via exact imbeddings. Let, for example, $a = (a_n)$ be a bounded sequence. Considering $a \in X$, we have $j_X(a) \in (\ell^\infty)^{**}$. To see how $j_X(a)$ acts on $X^* = (\ell^\infty)^*$, let $x^* \in (\ell^\infty)^* = X^*_n \oplus X^*_s$. Then, $x^* = \eta_{X'}(b) + \xi$ with $b = (b_n) \in X' = \ell^1$ and $\xi \in X^*_s$. Of course, ξ can be identified with a bounded, finitely additive measure on $\mathbb{N}$ (denoted also by ξ) vanishing on the standard unit coordinate vectors of ℓ^∞. Then

$$\langle j_X(a), x^* \rangle = \langle \eta_{X'}(b), a \rangle + \langle \xi, a \rangle = \sum_{n=1}^{\infty} a_n b_n + \int a\, d\xi.$$

However, if we consider that $a \in X''$, then a is interpreted as an element of $(\ell^\infty)^{**}$ via $\Pi_{X'^*} \circ \eta_{_{X''}}(a)$. That is, with the same notation for $x^* \in X^*$ as before, we have

$$\langle \Pi_{X'^*} \circ \eta_{_{X''}}(a), x^* \rangle = \langle \eta_{_{X''}}(a), b \rangle = \sum_{n=1}^{\infty} a_n b_n.$$

Different choices of K (always assumed to satisfy (14) and (15)) give different properties of the measure ν, of the space $L^1(\nu)$, and of the integration map I_ν.

(i) Suppose that K also satisfies

$$\lim_{n\to\infty} K(n,m) = 0, \quad m \in \mathbb{N}.$$

This is the case, for example, for $K(n,m) = 1/(n+m^2)$. Then the measure ν takes its values in $X_a = c_0$. Thus, via Corollary 2.9(iv), we have that

$$I_\nu^{**}(L^1(\nu)^{**}) \subseteq [\, X'^* \,] = [\, X'' \,] = \Pi_{X'^*} \circ \eta_{_{X''}}(\ell^\infty),$$

with $\Pi_{X'^*} \circ \eta_{_{X''}}(\ell^\infty)$ a subspace of $(\ell^\infty)^{**}$ (different from $j_{\ell^\infty}(\ell^\infty)$) which is isometric to ℓ^∞.

(ii) Suppose now that $K(\cdot, m)$ is decreasing on $\mathbb{N}$, for every $m \in \mathbb{N}$. In order to identify $L^1(\nu)$, we consider the variation measure $|\nu|$ of ν. Now, $\|K(\cdot,m)\|_\infty = \sup_{n\geq 1} K(n,m) = K(1,m)$, for $m \in \mathbb{N}$, and so $|\nu|$ is a finite measure; see (14). Moreover, for $A \subseteq \mathbb{N}$, we have $|\nu|(A) = \sum_{m\in A} K(1,m)$. Accordingly, a function $f\colon \mathbb{N} \to \mathbb{R}$ belongs to $L^1(|\nu|)$ if and only if

$$\int |f|\, d|\nu| = \sum_{m=1}^{\infty} |f(m)|{\cdot}K(1,m) < \infty.$$

Let $e_1 = (1,0,0,\dots) \in \ell^1 = X'$. Since $K(1,m) = \langle \eta_{_{X'}}(e_1), \nu(m) \rangle$ and $|\langle x^*, \nu(m) \rangle| \leq \|K(\cdot,m)\|_\infty$, for $x^* \in B_{X^*}$, it follows from (16) that

$$\|f\|_\nu = \int |f|\, d|\nu|.$$

This implies (since always $L^1(|\nu|) \subseteq L^1(\nu)$, [20]) that $L^1(|\nu|) = L^1(\nu) = L^1_w(\nu)$ with equal norms and, because $L^1(\nu)' = L^\infty(|\nu|)$ is not o.c., that $[\, L^1_w(\nu) \,] \subsetneq [\, L^1(\nu)'^* \,]$; see Proposition 1.7(ii). Hence, in this case, (10) reduces to

$$[\, L^1(\nu) \,] = [\, L^1_w(\nu) \,] \subsetneq [\, L^1(\nu)'^* \,] = L^1(\nu)^{**}.$$

This situation can even occur for measures ν (i.e. with K still satisfying (14) and (15)) such that $\nu(A) \in c_0$, for every $A \subseteq \mathbb{N}$. Indeed, example (i) above with $K(n,m) = 1/(n+m^2)$ has the desired properties. Of course, the same features can occur for measures such that $\nu(A) \in \ell^\infty \setminus c_0$, for some $A \subseteq \mathbb{N}$; e.g. $K(n,m) = n/(n+m^2)$. In this latter case, Corollary 2.9(v) implies that

$$I_\nu^{**}(L^1(\nu)^{**}) \nsubseteq [\, X'^* \,] = \Pi_{X'^*} \circ \eta_{_{X''}}(\ell^\infty).$$

3. Kernel operators and optimal domains

Let $K\colon [0,1]\times[0,1]\to[0,\infty)$ be a measurable function. The following standing assumptions on K are assumed throughout this section:

(i) $K_x \in L^1([0,1])$ for all $x\in[0,1]$, where K_x is the function defined by $K_x(y) := K(x,y)$ for $y\in[0,1]$,

(ii) $\sup_{x\in[0,1]} \int K(x,y)\,dy = \sup_{x\in[0,1]} \|K_x\|_{L^1([0,1])} < \infty$,

(iii) $\lim_{\lambda(A)\to 0} \sup_{x\in[0,1]} \int_A K(x,y)\,dy = 0$, where λ is Lebesgue measure on $[0,1]$.

These conditions guarantee, for every set $A\in\mathcal{B}$ (the Borel σ-algebra of $\Omega=[0,1]$), that the function $\nu(A)(\cdot)=\int_A K(\cdot,y)\,dy$ is well defined with $\nu(A)\in L^\infty([0,1])$ and that the so defined set function $\nu\colon\mathcal{B}\to L^\infty([0,1])$ is a vector measure, i.e. is σ–additive. Let T be the operator associated to K via the formula

$$T(f)(x) := \int_0^1 f(y)K(x,y)\,dy, \quad x\in[0,1], \tag{17}$$

for any measurable function f for which it is meaningful to do so for λ–a.e. $x\in[0,1]$. Clearly, $T(f)\geq 0$ whenever $f\geq 0$ and $T(f)$ is defined. Examples include the kernels of the Volterra operator and the fractional integral operator, [7], and of the Sobolev imbedding operator for certain domains in $\mathbb{R}^n$, [8]. Further examples, arising in classical analysis can be found in [6,7,10,11,15,29].

Throughout this section X will be a B.f.s. over the measure space $([0,1],\mathcal{B},\lambda)$ for which $L^\infty([0,1])\subseteq X\subseteq L^1([0,1])$. Under the above conditions on K, we have $T\colon L^\infty([0,1])\to X$ continuously with $T\geq 0$. We denote by $[T,X]$ the maximal B.f.s. to which T can be extended as a continuous linear operator, still with values in X. This maximality is to be understood as follows: there exists a continuous linear extension of T (denoted by T again) $T\colon [T,X]\to X$ and if T has a continuous linear extension $\tilde{T}\colon \tilde{X}\to X$, where $\tilde{X}$ is a B.f.s. over $([0,1],\mathcal{B},\lambda)$ containing $L^\infty([0,1])$, then $\tilde{X}$ is continuously imbedded in $[T,X]$. Then $[T,X]$ is the *optimal* (*lattice*) *domain* for T. In order to ensure that the definition of $[T,X]$ is meaningful there should not exist any set $A\in\mathcal{B}$ with $\lambda(A)>0$ for which $T(f\chi_A)=0$ λ–a.e. for every function f. This condition corresponds to the requirement that $\int_0^1 K_y(x)\,dx>0$ for λ–a.e. $y\in[0,1]$, where $K_y := K(\cdot,y)$ for each $y\in[0,1]$. Under these conditions, it turns out that

$$[T,X]=\{f : T(|f|)\in X\},$$

[7, Proposition 5.2], and that it is a B.f.s. when endowed with the norm

$$\|f\|_{[T,X]} := \|T(|f|)\|_X, \quad f\in[T,X].$$

Of course, with this notation,

$$\nu(A)=T(\chi_A), \quad A\in\mathcal{B}. \tag{18}$$

It is known, with continuous inclusions of norm at most 1, that

$$L^1(\nu)\subseteq[T,X]\subseteq[T,X]''=L^1(\nu)''=L^1_w(\nu). \tag{19}$$

Moreover, if X' is a norming subspace of X^*, then

$$\|f\|_{L^1_w(\nu)} = \|f\|_{[T,X]}, \quad f \in [T,X].$$

All these facts can be found in [9]. It is important to note that

$$T(f) = I_\nu(f) = \int f\,d\nu, \quad f \in L^1(\nu), \tag{20}$$

that is, if $f \in L^1(\nu)$, then $T(f)$ exists in the sense of (17) and coincides with the function $\int f\,d\nu \in X$. It is instructive to see why this is the case, for which it suffices to consider $f \in L^1(\nu)^+$. It is clear from (18) that (20) holds for all simple functions. Now, choose simple functions $0 \le \varphi_n \uparrow f$. By the monotone convergence theorem we see that $T(\varphi_n) \uparrow g$ pointwise λ–a.e. where $g(x) := \int_0^1 f(y)K(x,y)\,dy$ for a.e. $x \in [0,1]$. By the dominated convergence theorem for vector measures, [19], we have $T(\varphi_n) = \int \varphi_n\,d\nu \to \int f\,d\nu$ in X and hence, $T(\varphi_n) \to \int f\,d\nu$ pointwise λ–a.e. Accordingly, $\int f\,d\nu \in X$ coincides with the function $g := T(f)$. In particular, since $L^\infty([0,1]) \subseteq L^1(\nu)$, it follows from (20) that $I_\nu \colon L^1(\nu) \to X$ is a continuous extension of $T \colon L^\infty([0,1]) \to X$. By the optimality property it follows that $L^1(\nu) \subseteq [T,X]$.

The question arises of how I_ν^{**} acts in relation to the extended operator $T \colon [T,X] \to X$, that is, is the diagram

$$\begin{tikzcd}
{[T,X]} \arrow[r, "T"] \arrow[d, "\Pi_{L^1(\nu)'^*} \circ \eta_{L^1(\nu)''}"'] & X \arrow[d, "j_X"] \\
{[\,L^1_w(\nu)\,]} \arrow[r, "I_\nu^{**}"] & X^{**}
\end{tikzcd}$$

commutative? Equivalently, is it the case that

$$j_X \circ T(g) = I_\nu^{**}\left(\Pi_{L^1(\nu)'^*} \circ \eta_{L^1(\nu)''}(g)\right), \quad g \in [T,X]? \tag{21}$$

The following result characterizes precisely when I_ν^{**} is an "extension" of T (in the sense of the above diagram commuting).

Proposition 3.1. *The operator I_ν^{**} is an extension of T (i.e. (21) holds) if and only if $[T,X] = L^1(\nu)$.*

Proof. Suppose that (21) holds for every function in $[T,X]$. Let $f \in [T,X] \subseteq L^1_w(\nu)$. Since $[T,X]$ is an ideal, it follows that $f\chi_A \in [T,X]$ for all $A \in \Sigma$. In particular, $T(f\chi_A) \in X$ and, for every $x^* \in X^*$, (21) yields

$$\begin{aligned}\langle x^*, T(f\chi_A)\rangle &= \langle j_X \circ T(f\chi_A)\,,\, x^*\rangle \\ &= \langle I_\nu^{**}\left(\Pi_{L^1(\nu)'^*} \circ \eta_{L^1(\nu)''}(f\chi_A)\right)\,,\, x^*\rangle = \int_A f\,dx^*\nu\end{aligned}$$

(see Lemma 2.1). Accordingly, $f \in L^1(\nu)$. Combining this with (19) gives $[T,X] = L^1(\nu)$.

Conversely, if $[T,X] = L^1(\nu)$, then every $g \in [T,X]$ is ν–integrable with $\int g\,d\nu = T(g) \in X$; see (20). So, for $x^* \in X^*$, we have (using Lemma 2.1 and the fact in (19) that $[T,X] \subseteq L^1_w(\nu) = L^1(\nu)'')$ that

$$\begin{aligned}\langle j_X \circ T(g), x^*\rangle = \langle x^*, T(g)\rangle = \left\langle x^*, \int g\,d\nu \right\rangle \\ = \int g\,dx^*\nu = \left\langle I_\nu^{**}\left(\Pi_{L^1(\nu)'^*} \circ \eta_{L^1(\nu)''}(g)\right), x^* \right\rangle,\end{aligned}$$

and so (21) holds. ☐

Remark 3.2. (i) If X has o.c. norm, then necessarily $L^1(\nu) = [T,X]$, [9, Theorem 3.6(i)]. However, order continuity of X is not necessary for $L^1(\nu) = [T,X]$, [9, Example 3.8].

(ii) Of course, (19) shows that $L^1(\nu) = [T,X]$ also holds whenever $L^1(\nu) = L^1_w(\nu)$. As already noted in Section 1, this is the case whenever X does not contain a copy of c_0 or $L^1(\nu)$ is weakly sequentially complete, [9], (i.e. $L^1(\nu)$ does not contain a copy of c_0, [23, Theorem 2.5.6]). The same is true whenever $I_\nu \colon L^1(\nu) \to X$ is weakly compact; see Corollary 2.3.

Both parts (i) and (ii) give sufficient conditions for $L^1(\nu) = [T,X]$ whereas Proposition 3.1 *characterizes* precisely when this equality holds.

The sufficient conditions listed in Remark 3.2 do not always hold; there exist classical examples for which $L^1(\nu) \neq [T,X]$, [7, Remark 5.3]. Therefore, in general, I_ν^{**} is not an extension of T. However, for $g \in [T,X]$, the functionals $j_X \circ T(g)$ and $I_\nu^{**}\left(\Pi_{L^1(\nu)'^*} \circ \eta_{L^1(\nu)''}(g)\right)$ do always coincide over the order continuous part of X^*.

Proposition 3.3. *For each $g \in [T,X]$, we have*

$$\left\langle I_\nu^{**}\left(\Pi_{L^1(\nu)'^*} \circ \eta_{L^1(\nu)''}(g)\right), x^* \right\rangle = \langle j_X \circ T(g), x^*\rangle, \quad x^* \in X^*_n.$$

Proof. Let $g \in [T,X]$. Given $x^* \in X^*_n$ there exists a function $h \in X'$ such that $x^* = \eta_{X'}(h)$. Then, for $A \in \mathcal{B}$, we have

$$\begin{aligned}x^*\nu(A) = \langle x^*, \nu(A)\rangle = \langle \eta_{X'}(h), \nu(A)\rangle = \int h\cdot\nu(A)\,d\lambda \\ = \int h(x) \int_A K(x,y)\,dy\,dx = \int_A \int h(x)K(x,y)\,dx\,dy.\end{aligned}$$

It follows from Lemma 2.1 that

$$\begin{aligned}\left\langle I_\nu^{**}\left(\Pi_{L^1(\nu)'^*} \circ \eta_{L^1(\nu)''}(g)\right), x^* \right\rangle &= \int g\,dx^*\nu = \int g(y) \int h(x)K(x,y)\,dx\,dy \\ &= \int h(x) \int g(y)K(x,y)\,dy\,dx = \int h(x)\,T(g)(x)\,dx \\ &= \langle \eta_{X'}(h), T(g)\rangle = \langle x^*, T(g)\rangle = \langle j_X \circ T(g), x^*\rangle.\end{aligned}$$

☐

Let $f \in L^1_w(\nu)^+$. Fix $0 \le h \in X'$. Then $\eta_{X'}(h) \in X^*_n \subseteq X^*$. Since $\eta_{X'} \ge 0$ we have $\eta_{X'}(h) \ge 0$ in X^*_n. Moreover,

$$\left(\eta_{X'}(h) \circ \nu\right)(A) = \int_A \int h(x)K(x,y)\,dx\,dy, \quad A \in \mathcal{B}. \tag{22}$$

Consider the non-negative function $\Phi\colon x \mapsto \int f(y)K(x,y)\,dy$ on $[0,1]$. An application of Fubini's theorem yields (via (22))

$$\begin{aligned}\int h(x)\Phi(x)\,dx &= \int f(y) \int h(x)K(x,y)\,dx\,dy \\ &= \int f(y)\,d\left(\eta_{X'}(h) \circ \nu\right) \le \|f\|_{L^1_w(\nu)} \|\eta_{X'}(h)\|_{X^*}.\end{aligned}$$

Hence, $\Phi \in X''$. In particular, Φ is finite a.e. and, according to (17), we write $\Phi = T(f)$. So, whenever $f \in L^1_w(\nu)$ we have that $T(f)$ exists in the sense of (17) and $T(f) \in X''$; see also [9, Proposition 3.2(iii)].

Given $f \in L^1_w(\nu) = L^1(\nu)''$ (identified with the element $\Pi_{L^1(\nu)'^*} \circ \eta_{L^1(\nu)''}(f)$ of $[\,L^1(\nu)''\,]$), in which case $T(f) \in X''$ is identified with the element $\Pi_{X'^*} \circ \eta_{X''}(T(f))$ of $[\,X''\,]$, one may ask whether

$$I^{**}_\nu\left(\Pi_{L^1(\nu)'^*} \circ \eta_{L^1(\nu)''}(f)\right) = \Pi_{X'^*} \circ \eta_{X''}(T(f))? \tag{23}$$

That is, does the following diagram commute

$$\begin{tikzcd} L^1_w(\nu) \arrow[r, "T"] \arrow[d, "\Pi_{L^1(\nu)'^*} \circ \eta_{L^1(\nu)''}"'] & X'' \arrow[d, "\Pi_{X'^*} \circ \eta_{X''}"] \\ {[\,L^1_w(\nu)\,]} \arrow[r, "I^{**}_\nu"] & X^{**} \end{tikzcd}$$

Less formally, is $I^{**}_\nu\colon [\,L^1_w(\nu)] \to X^{**}$ an extension of $T\colon L^1_w(\nu) \to X''$?

In general, the answer is again no! For, if (23) is valid for *all* $f \in L^1_w(\nu)$, then it follows that $I^{**}_\nu([\,L^1_w(\nu)\,]) \subseteq [\,X''\,]$. This implies, via (8) and Corollary 2.9, that $\nu(\mathcal{B}) \subseteq X_a$. So, whenever $\nu(\mathcal{B}) \not\subseteq X_a$ the formula (23) cannot hold for all $f \in L^1_w(\nu)$. That is, there exist $f \in L^1_w(\nu)$ and $x^* \in X^*$ (see Lemma 2.1) such that

$$\left\langle \Pi_{X'^*} \circ \eta_{X''}(T(f)), x^* \right\rangle \ne \int f\,dx^*\nu.$$

4. Appendix

Proof of Proposition 1.3. We begin by establishing (4). The first containment in (4) follows from

$$j_E(E_a) \subseteq \{ j_E(e)|_{E^*_n} \circ P_{E^*_n} : e \in E \} \subseteq \pi_{(E^*_n)^*}\left((E^*_n)^*_n\right). \tag{24}$$

Here, $j_E(e)|_{E_n^*}$ denotes the restriction of $j_E(e)$ to E_n^* and $P_{E_n^*}$ the projection from E^* onto E_n^*. In order to prove the first containment in (24), fix $e \in E_a$. For each $e^* \in E^*$ we have that $e^* = e_n^* + e_s^*$, with $e_n^* \in E_n^*$ and $e_s^* \in E_s^*$. Then, since $E_a = {}^{\perp}(E_s^*)$, it follows that

$$\begin{aligned}\langle j_E(e)|_{E_n^*} \circ P_{E_n^*}, e^* \rangle &= \langle j_E(e)|_{E_n^*}, P_{E_n^*}(e^*) \rangle = \langle j_E(e), e_n^* \rangle \\ &= \langle e_n^*, e \rangle = \langle e^*, e \rangle = \langle j_E(e), e^* \rangle.\end{aligned}$$

This implies that $j_E(e) = j_E(e)|_{E_n^*} \circ P_{E_n^*}$. Concerning the second containment in (24), fix $e \in E$. Then $j_E(e)|_{E_n^*} \in (E_n^*)^*$. However, more is true; in fact $j_E(e)|_{E_n^*} \in (E_n^*)_n^*$. In order to verify this, let $e_\alpha^* \downarrow 0$ in E_n^*. Then

$$\inf_\alpha |\langle j_E(e)|_{E_n^*}, e_\alpha^* \rangle| = \inf_\alpha |\langle j_E(e), e_\alpha^* \rangle| = \inf_\alpha |\langle e_\alpha^*, e \rangle| = 0.$$

The containment is then established by noting that $\pi_{(E_n^*)^*}(j_E(e)|_{E_n^*}) = j_E(e)|_{E_n^*} \circ P_{E_n^*}$. Hence, (24) is proved.

The second containment in (4) follows from $(E_n^*)_n^* \subseteq (E_n^*)^*$.

Next we establish the equality $\pi_{(E_n^*)^*}((E_n^*)^*) = (E_s^*)^{\perp}$ in (4). Fix $z \in (E_n^*)^*$. For each $e_s^* \in E_s^*$ we have

$$\langle \pi_{(E_n^*)^*}(z), e_s^* \rangle = \langle z, P_{E_n^*}(e_s^*) \rangle = 0,$$

that is, $\pi_{(E_n^*)^*}(z) \in (E_s^*)^{\perp}$. Conversely, fix $e^{**} \in (E_s^*)^{\perp}$. Note that $e^{**}|_{E_n^*} \in (E_n^*)^*$ and hence, $\pi_{(E_n^*)^*}(e^{**}|_{E_n^*}) \in E^{**}$. Let $e^* \in E^*$. Then $e^* = e_n^* + e_s^*$ with $e_n^* \in E_n^*$ and $e_s^* \in E_s^*$. Since $\langle e^{**}, e_s^* \rangle = 0$, we have

$$\langle \pi_{(E_n^*)^*}(e^{**}|_{E_n^*}), e^* \rangle = \langle e^{**}|_{E_n^*}, P_{E_n^*}(e^*) \rangle = \langle e^{**}, e_n^* \rangle = \langle e^{**}, e^* \rangle.$$

Hence, $e^{**} = \pi_{(E_n^*)^*}(e^{**}|_{E_n^*})$ and so $e^{**} \in \pi_{(E_n^*)^*}((E_n^*)^*)$.

The last containment in (4) is clear.

In order to establish (i), we require the map $\phi \colon e \in E \mapsto \phi(e) := j_E(e)|_{E_n^*} \in (E_n^*)_n^*$. This map has the following properties: it is injective iff ${}^{\perp}(E_n^*) = \{0\}$ (since ${}^{\perp}(E_n^*) = \mathrm{Ker}(\phi)$) and it is surjective iff E has the weak Fatou property (i.e. if $e_\alpha \uparrow$ in E^+ and $\sup_\alpha \|e_\alpha\| < \infty$, then there exists $e \in E^+$ such that $e_\alpha \uparrow e$). Note that the Fatou property implies the weak Fatou property and, if E is o.c. and has the weak Fatou property, then it has the Fatou property.

Under the hypothesis that ${}^{\perp}(E_n^*) = \{0\}$, it is clear that (i) will follow if we establish the following two facts: (a) the first containment in (24) is an equality iff E is o.c.; (b) the second containment in (24) is an equality iff E has the weak Fatou property.

Concerning (a), let E be o.c. Then $E_a = E$ and $E_n^* = E^*$ with $P_{E_n^*}$ reducing to the identity map. Direct inspection then shows that the first containment in (24) is indeed an equality. Conversely, if this equality holds, then $\phi(E_a) = \phi(E)$ and hence, by injectivity of ϕ, we have $E_a = E$, that is, E is o.c.

For statement (b), note that an equality for the second containment in (24) is precisely equivalent to $\phi(E) = (E_n^*)_n^*$, that is, to ϕ being surjective or, as noted above, to E having the weak Fatou property.

To establish (ii), note that $\pi_{(E_n^*)^*}\colon (E_n^*)^* \to E^{**}$ is injective. We have already proved the equality in (4), that is, $\pi_{(E_n^*)^*}\left((E_n^*)^*\right) = (E_s^*)^\perp$ holds. So, if we assume that $\pi_{(E_n^*)^*}\left((E_n^*)_n^*\right) = (E_s^*)^\perp$, then it follows that $\pi_{(E_n^*)^*}\left((E_n^*)_n^*\right) = \pi_{(E_n^*)^*}\left((E_n^*)^*\right)$. So, the injectivity of $\pi_{(E_n^*)^*}$ implies that $(E_n^*)_n^* = (E_n^*)^*$, that is, E_n^* is o.c. The reverse implication follows from the equality in (4) together with the fact that $(E_n^*)_n^* = (E_n^*)^*$ whenever E_n^* is o.c.

For proving (iii), note first that always $j_E(E_a) \subseteq j_E(E)$ and second that $j_E(E_a) \subseteq (E_s^*)^\perp$; see (4). Hence, we have $j_E(E_a) \subseteq j_E(E) \cap (E_s^*)^\perp$. Conversely, let $z^{**} \in j_E(E) \cap (E_s^*)^\perp$. Then, there exists $e \in E$ such that $z^{**} = j_E(e) \in (E_s^*)^\perp$. Let $e_s^* \in E_s^*$. Then $\langle e_s^*, e\rangle = \langle j_E(e), e_s^*\rangle = 0$ and hence, $e \in {}^\perp(E_s^*)$. Since always ${}^\perp(E_s^*) = E_a$, it follows that $e \in E_a$. So, $z^{**} = j_E(e) \in j_E(E_a)$. $\square$

Proof of Proposition 1.4. The map $\Pi_{X'^*}$ is well defined since, for every $z \in X'^*$, we have

$$\Pi_{X'^*}(z) : X^* \xrightarrow{\quad P_{X_n^*}\quad} X_n^* \xrightarrow{\quad \eta_{X'}^{-1}\quad} X' \xrightarrow{\quad z\quad} \mathbb{R}\,.$$

So, $\Pi_{X'^*}(z)$ is a linear functional on X^*. Moreover,

$$\begin{aligned}\|\Pi_{X'^*}(z)\|_{X^{**}} &= \sup_{x^* \in B_{X^*}} \left|\langle \Pi_{X'^*}(z)\,,\, x^*\rangle\right| = \sup_{x^* \in B_{X^*}} \left|\langle z \circ \eta_{X'}^{-1} \circ P_{X_n^*}\,,\, x^*\rangle\right| \\ &= \sup_{x^* \in B_{X^*}} \left|\langle z\,,\, \eta_{X'}^{-1} \circ P_{X_n^*}(x^*)\rangle\right| \le \sup_{x' \in B_{X'}} |\langle z, x'\rangle| = \|z\|_{X'^*},\end{aligned}$$

since $\eta_{X'}^{-1} \circ P_{X_n^*}(x^*) \in X'$ for every $x^* \in X^*$ and, since $\eta_{X'}$ is an isometry,

$$\|\eta_{X'}^{-1} \circ P_{X_n^*}(x^*)\|_{X'} = \|\eta_{X'}\left(\eta_{X'}^{-1} \circ P_{X_n^*}(x^*)\right)\|_{X^*} = \|P_{X_n^*}(x^*)\|_{X^*} \le \|x^*\|_{X^*}.$$

Conversely, the fact that $\eta_{X'}$ is an isometry implies that

$$\eta_{X'}(x') \in X_n^* \cap B_{X^*}, \quad x' \in B_{X'}. \tag{25}$$

Then, since $P_{X_n^*}$ is the identity map on X_n^*, we can conclude that

$$|\langle z, x'\rangle| = \left|\langle z \circ \eta_{X'}^{-1} \circ P_{X_n^*}\,,\, \eta_{X'}(x')\rangle\right| = \left|\langle \Pi_{X'^*}(z)\,,\, \eta_{X'}(x')\rangle\right|.$$

Accordingly, (25) yields

$$\begin{aligned}\|z\|_{X'^*} &= \sup_{x' \in B_{X'}} |\langle z, x'\rangle| = \sup_{x' \in B_{X'}} \left|\langle \Pi_{X'^*}(z)\,,\, \eta_{X'}(x')\rangle\right| \\ &\le \sup_{x^* \in B_{X^*}} \left|\langle \Pi_{X'^*}(z)\,, x^*\rangle\right| = \|\Pi_{X'^*}(z)\|_{X^{**}}.\end{aligned}$$

Therefore, $\Pi_{X'^*}$ is an isometry, from which it is immediate that $\Pi_{X'^*}$ is injective and $\Pi_{X'^*}(X'^*)$ is a closed subspace of X^{**}.

Concerning (ii), given $z \in X'^*$, for every $x_s^* \in X_s^*$ we have

$$\langle \Pi_{X'^*}(z)\,, x_s^*\rangle \;=\; \langle z \circ \eta_{X'}^{-1} \circ P_{X_n^*}\,, x_s^*\rangle \;=\; 0,$$

since $P_{X_n^*}(x_s^*) = 0$. That is, $\Pi_{X'^*}(z) \in (X_s^*)^\perp$. Conversely, given any $z \in (X_s^*)^\perp$, we have

$$X' \xrightarrow{\ \eta_{X'}\ } X_n^* \xrightarrow{\ J\ } X^* \xrightarrow{\ z\ } \mathbb{R}\,,$$

that is, $z \circ J \circ \eta_{X'} \in X'^*$, where $J\colon X_n^* \to X^*$ is the natural inclusion (with $\|J\| = 1$). Then

$$\Pi_{X'^*}(z \circ J \circ \eta_{X'}) = z \circ J \circ \eta_{X'} \circ \eta_{X'}^{-1} \circ P_{X_n^*} = z \circ J \circ P_{X_n^*} = z,$$

where the last equality is due to z vanishing on elements of X_s^*. Then, $z \in \Pi_{X'^*}(X'^*)$. So, we can conclude that $\Pi_{X'^*}(X'^*) = (X_s^*)^\perp$, from which it is also immediate that $\Pi_{X'^*}(X'^*)$ is a band in X^{**}. $\square$

Proof of Proposition 1.6. (ii) Recall that $[\,X'^*\,] = \Pi_{X'^*}(X'^*)$ and, for each $u \in (X_n^*)^*$, that $\pi_{(X_n^*)^*}(u) = u \circ P_{X_n^*}$, as elements of X^{**}. Fix $z \in X'^*$. Since

$$X_n^* \xrightarrow{\ \eta_{X'}^{-1}\ } X' \xrightarrow{\ z\ } \mathbb{R},$$

that is, $z \circ \eta_{X'}^{-1} \in (X_n^*)^*$, we have by (5) and the definition of $\pi_{(X_n^*)^*}$ (c.f. Lemma 1.1) that

$$\Pi_{X'^*}(z) = z \circ \eta_{X'}^{-1} \circ P_{X_n^*} = \pi_{(X_n^*)^*}(z \circ \eta_{X'}^{-1}).$$

Since $z \in X'^*$ is arbitrary, it follows that

$$[\,X'^*\,] \subseteq \pi_{(X_n^*)^*}\left((X_n^*)^*\right).$$

Conversely, fix $z \in (X_n^*)^*$. Since

$$X' \xrightarrow{\ \eta_{X'}\ } X_n^* \xrightarrow{\ z\ } \mathbb{R},$$

i.e. $z \circ \eta_{X'} \in X'^*$, we have by the definitions of $\pi_{(X_n^*)^*}$ and $\Pi_{X'^*}$ that

$$\pi_{(X_n^*)^*}(z) = z \circ P_{X_n^*} = z \circ \eta_{X'} \circ \eta_{X'}^{-1} \circ P_{X_n^*} = \Pi_{X'^*}(z \circ \eta_{X'}).$$

Since $z \in (X_n^*)^*$ is arbitrary, this implies that

$$\pi_{(X_n^*)^*}\left((X_n^*)^*\right) \subseteq [\,X'^*\,].$$

Accordingly, since $(X_n^*)^* = (X_n^*)_n^* \oplus (X_n^*)_s^*$, we conclude that

$$[\,X'^*\,] = \pi_{(X_n^*)^*}\left((X_n^*)^*\right) = \pi_{(X_n^*)^*}\left((X_n^*)_n^*\right) \oplus \pi_{(X_n^*)^*}\left((X_n^*)_s^*\right).$$

(i) Recall that $[\,X''\,] = \Pi_{X'^*} \circ \eta_{X''}(X'')$. For every $z \in X''$, we have $\eta_{X''}(z) \in (X')_n^* \subseteq X'^*$, and so

$$X_n^* \xrightarrow{\quad \eta_{X'}^{-1} \quad} X' \xrightarrow{\quad \eta_{X''}(z) \quad} \mathbb{R},$$

that is, $\eta_{X''}(z) \circ \eta_{X'}^{-1} \in (X_n^*)^*$. Accordingly,

$$\Pi_{X'^*} \circ \eta_{X''}(z) = \eta_{X''}(z) \circ \eta_{X'}^{-1} \circ P_{X_n^*} = \pi_{(X_n^*)^*}(\eta_{X''}(z) \circ \eta_{X'}^{-1}).$$

This shows already that $[\, X''\,] \subseteq \pi_{(X_n^*)^*}((X_n^*)^*)$. To improve this, let us verify that actually $\eta_{X''}(z) \circ \eta_{X'}^{-1} \in (X_n^*)_n^*$. Let $x_\alpha^* \downarrow 0$ in the order of X_n^*. Since $\eta_{X'}$ (and so $\eta_{X'}^{-1}$) is an order isomorphism, $\eta_{X'}^{-1}(x_\alpha^*) \downarrow 0$ in the order of X'. Then,

$$\inf_\alpha |\langle \eta_{X''}(z) \circ \eta_{X'}^{-1}\, ,\, x_\alpha^* \rangle| = \inf_\alpha |\langle \eta_{X''}(z)\, ,\, \eta_{X'}^{-1}(x_\alpha^*) \rangle| = 0,$$

since $\eta_{X''}(z) \in (X')_n^*$. So, we really do have that $\eta_{X''}(z) \circ \eta_{X'}^{-1} \in (X_n^*)_n^*$ (rather than just an element of $(X_n^*)^*$) and hence,

$$[\, X''\,] \subseteq \pi_{(X_n^*)^*}\left((X_n^*)_n^*\right).$$

For every $z \in (X_n^*)_n^*$, we have seen in the proof of part (ii) that $z \circ \eta_{X'} \in X'^*$ and $\pi_{(X_n^*)^*}(z) = \Pi_{X'^*}(z \circ \eta_{X'})$. Let us now verify that actually $z \circ \eta_{X'} \in (X')_n^*$. Let $x'_\alpha \downarrow 0$ in the order of X'. Since $\eta_{X'}$ is an order isomorphism, $\eta_{X'}(x'_\alpha) \downarrow 0$ in the order of X_n^*. Then,

$$\inf_\alpha |\langle z \circ \eta_{X'}\, ,\, x'_\alpha \rangle| = \inf_\alpha |\langle z\, ,\, \eta_{X'}(x'_\alpha) \rangle| = 0,$$

since $z \in (X_n^*)_n^*$. It follows that

$$\pi_{(X_n^*)^*}(z) = \Pi_{X'^*}(z \circ \eta_{X'}) = \Pi_{X'^*} \circ \eta_{X''}\left(\eta_{X''}^{-1}(z \circ \eta_{X'})\right)$$

with $\eta_{X''}^{-1}(z \circ \eta_{X'}) \in X''$. So, by definition of $[\, X''\,]$ in (II), we have

$$\pi_{(X_n^*)^*}\left((X_n^*)_n^*\right) \subseteq [\, X''\,].$$

Accordingly, (i) holds.

Parts (iii) and (iv) then follow from Proposition 1.2. □

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