Rearrangement Invariant Optimal Domain for Monotone Kernel Operators

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Abstract. For a kernel operator T with values in a Banach function space X, we give monotonicity conditions on the kernel which allow us to describe the rearrangement invariant optimal domain for T (still with values in X). We also study the relation between this optimal domain and the space of integrable functions with respect to the X-valued measure canonically associated to T.

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1. Introduction

Let $K\colon [0,1]\times[0,1]\to[0,\infty]$ be a measurable function such that every $x\in[0,1]$ satisfies $K(x,\cdot)<\infty$ a.e. and consider the *kernel operator* T defined by K as

$$Tf(x)=\int_0^1 f(y)K(x,y)\,dy,\quad x\in[0,1], \tag{1.1}$$

for any $f\in L^0$ (the space of all measurable real functions on $[0,1]$, identifying functions which are equal a.e.) for which the integral exists a.e. x. Given a Banach function space (B.f.s.) X, an important problem is to find the *optimal domain* for T considered with values in X, that is the largest B.f.s. Y such that $T\colon Y\to X$ is well defined (and so continuous, since it is a positive linear operator between Banach lattices, see [11, p. 2]). The "largest" B.f.s. Y may be understood in the following sense: if Z is another B.f.s. such that $T\colon Z\to X$ is well defined then $Z\subset Y$. This problem has been studied for classical operators in numerous works as for instance [2], [4], [9], [12] and [13].

Throughout the paper, we will assume that K satisfies the condition

$$\int_0^1 K(x,y)\,dx>0\quad \text{a.e. } y\in[0,1], \tag{1.2}$$

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that is, $T|f| = 0$ a.e. implies $f = 0$ a.e., or equivalently, there exists no measurable set A of strictly positive Lebesgue measure such that $T(f\chi_A) = 0$ a.e. for all $f \in L^0$. Let us denote by $[T, X]$ the optimal domain for T considered with values in X. This space has been studied in [3], where it is described in a natural way as

$$[T, X] = \{f \in L^0 : T|f| \in X\} \tag{1.3}$$

endowed with the norm $\|f\|_{[T,X]} := \|\, T|f|\, \|_X$. Note that (1.2) guarantees that $\|\cdot\|_{[T,X]}$ is a norm. Moreover, conditions are given for obtaining a more precise description for $[T, X]$ in terms of interpolation spaces. See also [8] for the case $[0, \infty)$ instead of $[0, 1]$.

On other hand, under appropriate conditions on X and K, the set function ν associated to T via $\nu(A) = T(\chi_A)$ is an X-valued vector measure which turns out to be a powerful tool for studying T. The spaces $L^1(\nu)$ and $L^1_w(\nu)$ of integrable and weakly integrable functions with respect to ν respectively, are closed related to the optimal domain $[T, X]$ as shown in [3] and [5]. Indeed, the containments $L^1(\nu) \subset [T, X] \subset L^1_w(\nu)$ always hold.

In this paper we are interested in the *rearrangement invariant (r.i.) optimal domain* for T, that is the largest r.i. B.f.s. contained in $[T, X]$, denoted by $[T, X]_{r.i.}$. This space has been already studied for the kernel operator associated with the Sobolev's inequality ([4], [7]) and the Hardy operator ([9]). In Section 3 we will see that $[T, X]_{r.i.}$ can be described in a similar way as (1.3) provided K satisfies that $K(x, \cdot)$ is a monotone map for every $x \in [0, 1]$. Even more, we give conditions under which $[T, X]_{r.i.}$ can be more precisely described as an interpolation space. Section 4 is devoted to the study of the relation among all the spaces $[T, X]$, $[T, X]_{r.i.}$, $L^1(\nu)$ and $L^1_w(\nu)$.

2. Preliminaries

A *Banach function space* (B.f.s.) is a Banach space X contained in L^0 such that if $f \in L^0$, $g \in X$ and $|f| \le |g|$ a.e. then $f \in X$ and $\|f\|_X \le \|g\|_X$. Note that a B.f.s. is a Banach lattice for the pointwise a.e. order. Given two B.f.s.' X and Y, we will write $X \hookrightarrow_c Y$ when X is continuously contained in Y with $\|f\|_Y \le c\|f\|_X$ for all $f \in X$ and $X \hookrightarrow_i Y$ when the containment is isometric. By $X \equiv Y$ we mean that $X = Y$ and the norms coincide. A B.f.s. is *order continuous* (o.c.) if every order bounded increasing sequence is norm convergent. Let m denote the Lebesgue measure on $[0, 1]$. Note that, since m is finite, in the case when X contains the simple functions, X is o.c. if and only if every $f \in X$ satisfies $\|f\chi_A\|_X \to 0$ as $m(A) \to 0$. A B.f.s. X has the *Fatou property* if for every sequence $(f_n) \subset X$ such that $0 \le f_n \uparrow f$ a.e. and $\sup_n \|f_n\|_X < \infty$, it follows that $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$. A B.f.s. X is *rearrangement invariant* (r.i.) whenever $f \in X$ if and only if $f^* \in X$, and in this case $\|f\|_X = \|f^*\|_X$. Here, f^* denotes the *decreasing rearrangement* of f, i.e., $f^*(s) = \inf\big\{r > 0 : m\big(\{x \in [0, 1] : |f(x)| > r\}\big) \le s\big\}$ for all $s \in [0, 1]$. A non trivial r.i. B.f.s. X satisfies $L^\infty \subset X \subset L^1$, see [10, Theorem II.4.1]. Adding to X the Fatou property, we obtain an r.i. B.f.s. in the sense of

Bennett and Sharpley [1, Definition I.1.1]. Then X can be generated by the K-method of interpolation of Peetre as $(L^1, L^\infty)_X$. Let us recall briefly this method. If (X_0, X_1) are Banach spaces continuously embedded in a common Hausdorff topological vector space, then the K–functional of $f \in X_0 + X_1$ is defined as

$$\mathcal{K}(t, f; X_0, X_1) = \inf\big\{\|f_0\| + t\|f_1\| : f = f_0 + f_1;\ f_0 \in X_0,\ f_1 \in X_1\big\}, \quad t > 0.$$

Assume $X_0 \cap X_1$ is dense in X_0. Given an r.i. B.f.s. X having the Fatou property, $(X_0, X_1)_X$ denotes the space of all function $f \in X_0 + X_1$ such that $\mathcal{K}'(\cdot, f; X_0, X_1) \in X$, where $\mathcal{K}'$ is the derivative of the K-functional $\mathcal{K}$. Note that $\mathcal{K}'$ is a decreasing function. The *interpolation space* $(X_0, X_1)_X$ between X_0 and X_1, is a B.f.s. endowed with the norm $\|f\|_{(X_0,X_1)_X} := \|\mathcal{K}'(\cdot, f; X_0, X_1)\|_X$. See [1, Chp. V] for further information.

Given an increasing concave function $\varphi\colon [0,1] \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(0^+) = 0$, the *Lorentz space* $\Lambda_\varphi = \{f \in L^0 :\ \|f\|_{\Lambda_\varphi} = \int_0^1 f^*(t)\varphi'(t)\,dt < \infty\}$ with norm $\|\cdot\|_{\Lambda_\varphi}$, is an o.c. r.i. B.f.s. having the Fatou property, see [10, §II.5].

Let $\mathcal{B}([0,1])$ be the σ-algebra of all Borel subsets of $[0,1]$, X a B.f.s. and $\nu\colon \mathcal{B}([0,1]) \to X$ a *vector measure* (i.e., countably additive). Let us recall briefly the theory of integration of real functions with respect to ν, which will be used in Section 4. A set $A \in \mathcal{B}([0,1])$ is *ν-null* if $\nu(B) = 0$ whenever $B \in \mathcal{B}([0,1]) \cap 2^A$. Assume that ν and m have the same null sets. A function $f \in L^0$ is *weakly integrable* with respect to ν, if $f \in L^1(|x^*\nu|)$ for every element x^* in X^* (the topological dual of X), where $|x^*\nu|$ is the variation of the real measure $x^*\nu$. If moreover f satisfies that for each $A \in \mathcal{B}([0,1])$ there exists $x_A \in X$ such that

$$x^*(x_A) = \textstyle\int_A f\,dx^*m, \quad \text{for every } x^* \in X^*,$$

f is said to be *integrable* with respect to ν. The vector x_A is unique and will be written as $\int_A f\,dm$. Let $L^1_w(\nu)$ denote the space of all weakly integrable function and $L^1(\nu)$ the space of all integrable function with respect to ν. In both spaces, functions which are equal a.e. are identified. The map $\|\cdot\|_\nu$ defined for $f \in L^0$ as

$$\|f\|_\nu = \sup_{x^* \in B_{X^*}} \textstyle\int_\Omega |f|\,d|x^*\nu|,$$

where B_{X^*} denotes the unit ball of X^*, endows of B.f.s. structure the spaces $L^1_w(\nu)$ and $L^1(\nu)$. Of course, $L^1(\nu)$ is a closed subspace of $L^1_w(\nu)$. The space $L^1_w(\nu)$ has the Fatou property and $L^1(\nu)$ is order continuous containing the simple functions as a dense set. For more details see [6], [14, Ch. 3] and the references therein.

3. R.i. optimal domain for T

Let T be the kernel operator given in (1.1) with kernel K satisfying (1.2). Depending on each particular B.f.s. X, the optimal domain $[T, X]$ is or is not r.i. For instance, if T is the Volterra operator (i.e., $K(x,y) = \chi_{[0,x]}(y)$), then $[T, L^\infty] \equiv L^1$ is r.i. while $[T, L^1] \equiv L^1_{1-y}$ is not r.i. The equivalences follow directly from (1.3). So, a natural question arises: which is the largest r.i. space contained in $[T, X]$, in other words, which is the *r.i. optimal domain* for T considered with values in X?

Let us denote by $[T,X]_{r.i.}$ this r.i. optimal domain. A good candidate to describe $[T,X]_{r.i.}$ in a similar way as in (1.3) is

$$\Gamma_X = \{f \in L^0 : Tf^* \in X\},$$

since Γ_X satisfies the r.i. and the ideal properties, and every r.i. space Y contained in $[T,X]$ is inside of Γ_X. Unfortunately, in general, Γ_X is not a linear space.

Example. For $K(x,y) = \big(y(1-y)\big)^{-1}\chi_{[x,1]}(y)$, we have that $f \in \Gamma_{L^1}$ if and only if $\int_0^1 f^*(y)\frac{1}{1-y}\,dy < \infty$. Then, $f = \chi_{[0,\frac{1}{2}]}, g = \chi_{(\frac{1}{2},1]} \in \Gamma_{L^1}$, while $f+g \notin \Gamma_{L^1}$.

However, we can require K to satisfy an appropriate monotonicity condition guaranteeing the linearity of Γ_X. Namely,

$$\text{for every fixed } x \in [0,1], \text{ the map } K(x,\cdot) \text{ is decreasing.} \tag{3.1}$$

In this case, $T(f+g)^* \leq Tf^* + Tg^*$ (see [1, Theorem II.3.4 and Proposition II.3.6]) and $T|f| \leq Tf^*$ (see [1, Theorem II.2.2]). Then, Γ_X is a linear space contained in $[T,X]$ and the functional $\rho(f) = \|Tf^*\|_X$ is a norm on Γ_X satisfying the Riesz Fischer property, see [15, Ch. 15, §64]. So, we obtain the following result.

Proposition 3.1. *If K satisfies (3.1), then $[T,X]_{r.i.} = \{f \in L^0 : Tf^* \in X\}$ with norm $\|f\|_{[T,X]_{r.i.}} = \|Tf^*\|_X$. Moreover, $[T,X]_{r.i.} \hookrightarrow_1 [T,X]$.*

From now on in this section we assume (3.1) holds. Let us see some cases in which $[T,X]_{r.i.}$ can be described more precisely. Consider the decreasing function

$$\omega(y) = \int_0^1 K(x,y)\,dx.$$

Proposition 3.2. *If $\omega \in L^1$, then $[T,L^1]_{r.i.} \equiv \Lambda_{\int_0^y \omega(s)ds}$. If $\omega \notin L^1$, $[T,L^1]_{r.i.} = \{0\}$.*

Proof. Given $f \in L^0$, we have that

$$\int_0^1 Tf^*(x)\,dx = \int_0^1 f^*(y)\int_0^1 K(x,y)\,dx\,dy = \int_0^1 f^*(y)\,\omega(y)\,dy.$$

Then, from Proposition 3.1, the conclusion follows for $\omega \in L^1$. Note that

$$\int_0^1 \omega(y)\,dy = \int_0^1 T\chi_{[0,1]}(x)\,dx,$$

then $\omega \in L^1$ if and only if $\chi_{[0,1]} \in [T,L^1]_{r.i.}$, or equivalently, $[T,L^1]_{r.i.} \neq \{0\}$. $\square$

The space $[T,L^\infty]_{r.i.}$ can be also described as a Lorentz space in the case of K being decreasing when fixing the second variable, i.e.,

$$K(\cdot,y) \text{ decreases for all } y \in [0,1]. \tag{3.2}$$

In this case, we consider the decreasing function

$$\xi(y) = K(0,y).$$

Proposition 3.3. *Suppose K satisfies (3.2). If $\xi \in L^1$, then $[T,L^\infty]_{r.i.} = \Lambda_{\int_0^y \xi(s)ds}$. In other case $[T,L^\infty]_{r.i.} = \{0\}$.*

Proof. Given $f \in L^0$, we have that

$$\sup_{0\le x\le 1} Tf^*(x) = \sup_{0\le x\le 1} \int_0^1 f^*(y)K(x,y)\,dy = \int_0^1 f^*(y)\xi(y)\,dy.$$

Then, from Proposition 3.1, the conclusion follows for $\xi \in L^1$. Note that

$$\int_0^1 \xi(y)\,dy = \sup_{0\le x\le 1} T\chi_{[0,1]}(x),$$

and so $[T, L^\infty]_{r.i.} \neq \{0\}$ if and only if $\xi \in L^1$. □

Condition (3.2) also allows us to give a precise description for the r.i. optimal domain of T considered with values in a Lorentz space Λ_φ. Under this condition, we consider the decreasing function

$$\theta_\varphi(y) = \int_0^1 \varphi'(x)K(x,y)\,dx.$$

Proposition 3.4. *Suppose that* (3.2) *holds. Given a Lorentz space* Λ_φ*, we have that* $[T,\Lambda_\varphi]_{r.i.} = \Lambda_{\int_0^y \theta_\varphi(s)ds}$ *whenever* $\theta_\varphi \in L^1$ *and* $[T,\Lambda_\varphi]_{r.i.} = \{0\}$ *in other case.*

Proof. Condition (3.2) implies that Tg decreases for all $0 \le g \in L^0$. Given $f \in L^0$,

$$\int_0^1 (Tf^*)^*(x)\varphi'(x)\,dx = \int_0^1 Tf^*(x)\varphi'(x)\,dx = \int_0^1 f^*(y)\,\theta_\varphi(y)\,dy$$

and so the conclusion follows for $\theta_\varphi \in L^1$. Moreover, $[T,\Lambda_\varphi]_{r.i.} \neq \{0\}$ if and only if $\theta_\varphi \in L^1$, as $\int_0^1 \theta_\varphi(y)\,dy = \int_0^1 \big(T\chi_{[0,1]}\big)^*(x)\varphi'(x)\,dx$. □

Example. For $0 < \alpha < 1$, the kernel $K(x,y) = \min\{\frac{1}{x^\alpha}, \frac{1}{y^\alpha}\}$ satisfies (1.2), (3.1) and (3.2). Note that the operator T defined by K is the sum of the kernel operator associated with the Sobolev's inequality (see [4]) with its adjoint operator.

Let us consider now a general r.i. B.f.s. X (non trivial) having the Fatou property. Then $L^\infty \subset X \subset L^1$ and X can be described as an interpolation space between L^1 and L^∞, namely $X = (L^1, L^\infty)_X$. It is clear that

$$[T, L^\infty]_{r.i.} \subset [T, X]_{r.i.} \subset [T, L^1]_{r.i.}\,.$$

The question is the following: can $[T,X]_{r.i.}$ be described as the corresponding interpolation space between $[T,L^1]_{r.i.}$ and $[T,L^\infty]_{r.i.}$, i.e., $\big([T,L^1]_{r.i.},[T,L^\infty]_{r.i.}\big)_X$?

Proposition 3.5. *Suppose that* $[T, L^\infty]_{r.i.} \neq \{0\}$*. Then*

$$\big([T,L^1]_{r.i.},[T,L^\infty]_{r.i.}\big)_X \hookrightarrow_1 [T,X]_{r.i.}$$

Proof. Since $L^\infty \subset [T,L^\infty]_{r.i.}$ and L^∞ is dense in $[T,L^1]_{r.i.}$ (see Proposition 3.2), from [1, Proposition V.1.15] we have that

$$\mathcal{K}(t,f;[T,L^1]_{r.i.},[T,L^\infty]_{r.i.}) = \int_0^t \mathcal{K}'(s,f;[T,L^1]_{r.i.},[T,L^\infty]_{r.i.})\,ds$$

for every $f \in [T,L^1]_{r.i.}$ and $t > 0$. Then, from [1, Theorem II.4.7] and since $\mathcal{K}(t,h;L^1,L^\infty) = \int_0^t h^*(s)ds$ for every $h \in L^1$ (see [1, Proposition V.1.6]), it is enough to prove that, for every $f \in \big([T,L^1]_{r.i.},[T,L^\infty]_{r.i.}\big)_X$ and $t > 0$,

$$\mathcal{K}(t,Tf^*;L^1,L^\infty) \le \mathcal{K}(t,f;[T,L^1]_{r.i.},[T,L^\infty]_{r.i})\,. \tag{3.3}$$

Take $f \in \big([T,L^1]_{r.i.},[T,L^\infty]_{r.i.}\big)_X$ and $t>0$. For every $f_0 \in [T,L^1]_{r.i.}$ and $f_1 \in [T,L^\infty]_{r.i.}$ such that $f=f_0+f_1$, it follows

$$\begin{aligned} \|f_0\|_{[T,L^1]_{r.i.}} + t\|f_1\|_{[T,L^\infty]_{r.i.}} &= \|Tf_0^*\|_{L^1} + t\|Tf_1^*\|_{L^\infty} \\ &\ge \mathcal{K}(t, T(f_0^*+f_1^*); L^1, L^\infty) \\ &\ge \mathcal{K}(t, Tf^*; L^1, L^\infty)\,, \end{aligned}$$

where the last inequality holds as $Tf^* \le T(f_0^*+f_1^*)$. Taking infimum on f_0, f_1 we obtain the inequality (3.3). □

From Propositions 3.2, 3.3 and 3.5 we obtain the following corollary.

Corollary 3.6. *If K satisfies* (3.2) *and $\xi \in L^1$, then*

$$\big(\Lambda_{\int_0^y \omega(s)ds}, \Lambda_{\int_0^y \xi(s)ds}\big)_X \hookrightarrow_1 [T,X]_{r.i.}$$

We can require K to satisfy extra conditions under which the two spaces in Corollary 3.6 coincide.

Theorem 3.7. *Suppose that K satisfies* (3.2), *$\xi \in L^1$ and there exists a constant $C>0$ such that*

$$\int_0^y \int_0^t K(x,s)\,dx\,ds \ge C \min\Big\{ \int_0^y \int_0^1 K(x,s)\,dx\,ds\,,\; t\cdot \int_0^y K(0,s)\,ds \Big\} \tag{3.4}$$

holds for all $0<t,y<1$. Suppose also that $h(y) = \big(\int_0^y \omega(s)ds\big)\cdot\big(\int_0^y \xi(s)ds\big)^{-1}$ is a monotone map. Then,

$$[T,X]_{r.i.} = \big(\Lambda_{\int_0^y \omega(s)ds}, \Lambda_{\int_0^y \xi(s)ds}\big)_X.$$

Proof. Since h is monotone, arguments similar to the used in the proof of [10, Theorem II.5.9] lead to

$$\mathcal{K}(t,f;\Lambda_{\int_0^y \omega(s)ds}, \Lambda_{\int_0^y \xi(s)ds}) = \int_0^1 f^*(y)\,d\phi_t(y) \tag{3.5}$$

for all $f \in \Lambda_{\int_0^y \omega(s)ds}$ and $0<t<1$, where

$$\phi_t(y) = \min\big\{ \int_0^y \omega(s)\,ds\,,\, t\cdot\int_0^y \xi(s)\,ds \big\}.$$

Let $f \in [T,X]_{r.i.}$ and $0<t<1$. From (3.4) and (3.5) it follows

$$\begin{aligned} \mathcal{K}(t,Tf^*;L^1,L^\infty) &= \int_0^t (Tf^*)^*(x)\,dx = \int_0^t Tf^*(x)\,dx \\ &= \int_0^1 f^*(y) \int_0^t K(x,y)\,dx\,dy \\ &= \int_0^1 f^*(y)\,d\big(\int_0^y \int_0^t K(x,s)\,dx\,ds\big)(y). \\ &\ge C\cdot \mathcal{K}(t,f;\Lambda_{\int_0^y \omega(s)ds}, \Lambda_{\int_0^y \xi(s)ds}). \end{aligned}$$

Then $f \in \big(\Lambda_{\int_0^y \omega(s)ds}, \Lambda_{\int_0^y \xi(s)ds}\big)_X$ with $C\cdot\|f\|_{(\Lambda_{\int_0^y \omega(s)ds},\Lambda_{\int_0^y \xi(s)ds})_X} \le \|f\|_{[T,X]_{r.i.}}$. From this and Corollary 3.6 the conclusion follows. □

Example. For $0<\alpha<1<\beta$, the kernel $K(x,y) = \min\{\frac{1}{x^\beta}, \frac{1}{y^\alpha}\}$ satisfies the hypothesis of Theorem 3.7.

Remark 3.8. If K satisfies that $K(\cdot, y)$ increases for all $y \in [0,1]$ instead of (3.2), Proposition 3.3, 3.4, Corollary 3.6 and Theorem 3.7 hold replacing $K(x,y)$ by $K(1-x,y)$ in the definition of ξ, θ_φ and in (1.2). Note that under this condition, Tg increases for all $0 \le g \in L^0$ and so $(Tg)^*(x) = Tg(1-x)$ for all $x \in [0,1]$.

Moreover, in the case when $K(x,\cdot)$ is increasing for all $x \in [0,1]$, the r.i. optimal domain for T can be described as

$$[T,X]_{r.i.} = \{f \in L^0 : T(\tau f^*) \in X\},$$

where τ is the operator which takes $f \in L^0$ into the function defined by $\tau f(t) = f(1-t)$. Similar results can be obtained by taking $\tau\omega$ and $\tau\xi$.

4. Vector integral representation for T

Let ν be the set function given by $A \in \mathcal{B}([0,1]) \to \nu(A) = T(\chi_A)$, where T is as in (1.1) with K satisfying (1.2). Depending on the B.f.s. on which ν takes values, ν will be or not a vector measure. Consider a B.f.s. X satisfying

$$\int_0^1 K(\cdot,y)\,dy \in X \quad \text{and} \quad \lim_{m(A)\to 0} \Big\| \int_A K(\cdot,y)\,dy \Big\|_X = 0. \tag{4.1}$$

Then, $\nu\colon \mathcal{B}([0,1]) \to X$ is a vector measure which will be denoted by ν_X to indicate the space where values are taken. Indeed, the first condition in (4.1) guarantees that ν_X is well defined (as $T(\chi_{[0,1]}) = \int_0^1 K(\cdot,y)\,dy$) and the second one implies that ν_X is countably additive. Note that actually the conditions in (4.1) are equivalent to ν_X being a well-defined vector measure. The next result which has been proved in [3] and [4] under stronger conditions on X and K, remains hold in our context.

Proposition 4.1. *The following containments always hold:*

$$L^1(\nu_X) \hookrightarrow_i [T,X] \hookrightarrow_1 L^1_w(\nu_X).$$

Moreover,

(a) $Tf = \int f\,d\nu_X$ *for all* $f \in L^1(\nu_X)$.
(b) $L^1(\nu_X)$ *is the largest o.c. B.f.s. contained in* $[T,X]$.
(c) $L^1_w(\nu_X)$ *is the smallest B.f.s. with the Fatou property containing* $[T,X]$.

Assume (3.1) holds. Then $[T,X]_{r.i.}$ is described as in Proposition 3.1 and

$$[T,X]_{r.i.} \hookrightarrow_1 [T,X] \hookrightarrow_1 L^1_w(\nu_X). \tag{4.2}$$

But what is the relation between $L^1(\nu_X)$ and $[T,X]_{r.i.}$?

Proposition 4.2. *The containment* $L^1(\nu_X) \subset [T,X]_{r.i.}$ *holds if and only if* $f \in L^1(\nu_X)$ *implies* $f^* \in L^1(\nu_X)$. *Moreover, in this case,* $L^1(\nu_X)$ *is r.i. endowed with the norm* $\|\cdot\|_{[T,X]_{r.i.}}$, *which is equivalent to* $\|\cdot\|_{\nu_X}$.

Proof. Suppose that $L^1(\nu_X) \subset [T,X]_{r.i.}$. Then, there exists a constant $c > 0$ such that $L^1(\nu_X) \hookrightarrow_c [T,X]_{r.i.}$. Given $f \in L^1(\nu_X)$, it follows that $f^* \in [T,X]_{r.i.}$. Taking simple functions φ_n such that $0 \le \varphi_n \uparrow |f|$, we have that $\varphi_n \to |f|$ in norm $\|\cdot\|_{\nu_X}$ (as $L^1(\nu_X)$ is o.c.) and $0 \le \varphi_n^* \uparrow f^*$, where φ_n^* are also simple functions. From

(4.2) and since $T(f^*-g^*)\le T(|f|-|g|)^*$ (see the comment after (3.1) and note that $f^*=|f|^*$), we have that

$$\|f^*-\varphi_n^*\|_{\nu_X}\le\|f^*-\varphi_n^*\|_{[T,X]}\le\|\,|f|-\varphi_n\|_{[T,X]_{r.i.}}\le c\,\|\,|f|-\varphi_n\|_{\nu_X}\to 0$$

and so $f^*\in L^1(\nu_X)$.

Conversely, suppose that $f^*\in L^1(\nu_X)$ whenever $f\in L^1(\nu_X)$. Then, given $f\in L^1(\nu_X)$, from Proposition 4.1, $f^*\in[T,X]$ and so $f\in[T,X]_{r.i.}$ (as $Tf^*\in X$).

Note that, in the case when $L^1(\nu_X)\subset[T,X]_{r.i.}$, there exists $c>0$ such that

$$\|f\|_{\nu_X}=\|f\|_{[T,X]}\le\|f\|_{[T,X]_{r.i.}}\le c\,\|f\|_{\nu_X},$$

for all $f\in L^1(\nu_X)$. That is, $\|\cdot\|_{[T,X]_{r.i.}}$ is equivalent to $\|\cdot\|_{\nu_X}$ on $L^1(\nu_X)$. Moreover, $L^1(\nu_X)$ is an r.i. B.f.s. with the norm $\|\cdot\|_{[T,X]_{r.i.}}$, since for every $f\in L^0$ with $f^*\in L^1(\nu_X)$, we have that $f\in[T,X]_{r.i.}$ and, from (4.2) and Proposition 4.1,

$$\|f\chi_A\|_{\nu_X}\le\|f\chi_A\|_{[T,X]_{r.i.}}\le\|f^*\chi_{[0,m(A))}\|_{[T,X]}=\|f^*\chi_{[0,m(A))}\|_{\nu_X}\to 0$$

as $m(A)\to 0$, from which it follows that $f\in L^1(\nu_X)$. □

Remark 4.3. The space $L^1(\nu_X)$ is r.i. if and only if $L^1(\nu_X)\hookrightarrow_i[T,X]_{r.i.}$. Indeed, if $L^1(\nu_X)$ is r.i., from Proposition 4.2, $L^1(\nu_X)\subset[T,X]_{r.i.}$ and for every $f\in L^1(\nu_X)$,

$$\|f\|_{[T,X]_{r.i.}}=\|f^*\|_{[T,X]}=\|f^*\|_{\nu_X}=\|f\|_{\nu_X}.$$

Conversely, if $L^1(\nu_X)\hookrightarrow_i[T,X]_{r.i.}$, by Proposition 4.2, we have that $f\in L^1(\nu_X)$ if and only if $f^*\in L^1(\nu_X)$. Moreover, in this case, $\|f^*\|_{\nu_X}=\|f^*\|_{[T,X]_{r.i.}}=\|f\|_{[T,X]_{r.i.}}=\|f\|_{\nu_X}$.

Proposition 4.4. *The containment $[T,X]_{r.i.}\subset L^1(\nu_X)$ holds if and only if $[T,X]_{r.i.}$ is o.c. Moreover, in this case, $[T,X]_{r.i.}\hookrightarrow_1 L^1(\nu_X)$.*

Proof. From Proposition 4.1(b), it follows that if $[T,X]_{r.i.}$ is o.c. then $[T,X]_{r.i.}\subset L^1(\nu_X)$. Moreover, $\|f\|_{\nu_X}=\|f\|_{[T,X]}\le\|f\|_{[T,X]_{r.i.}}$ for every $f\in[T,X]_{r.i.}$.

Suppose that $[T,X]_{r.i.}\subset L^1(\nu_X)$. For every $f\in[T,X]_{r.i.}$, it follows that $(f\chi_A)^*\in L^1(\nu_X)$, and then

$$\|f\chi_A\|_{[T,X]_{r.i.}}=\|(f\chi_A)^*\|_{[T,X]}=\|(f\chi_A)^*\|_{\nu_X}\le\|f^*\chi_{[0,m(A))}\|_{\nu_X}\to 0$$

as $m(A)\to 0$. Hence, $[T,X]_{r.i.}$ is o.c. □

Note that if $[T,X]$ is o.c. then $[T,X]_{r.i.}$ is also o.c., since for $f\in[T,X]_{r.i.}$,

$$\|f\chi_A\|_{[T,X]_{r.i.}}=\|(f\chi_A)^*\|_{[T,X]}\le\|f^*\chi_{[0,m(A))}\|_{[T,X]}\to 0$$

as $m(A)\to 0$. In this case, from Proposition 4.1(b), $[T,X]_{r.i.}\hookrightarrow_1 L^1(\nu_X)\equiv[T,X]$. The space $[T,X]$ is o.c. for instance if X is o.c., see [5, Proposition 3.1(i)]. Another interesting fact is that $L^\infty\subsetneq[T,X]_{r.i.}$. Indeed, it is not difficult to prove that an r.i. B.f.s. Y coincides with L^∞ if and only if there exists $c>0$ such that $\|\chi_A\|_Y\ge c$

for all $A \in \mathcal{B}([0,1])$ with $m(A) > 0$, and by (4.1),

$$\|\chi_A\|_{[T,X]_{r.i.}} = \|T\chi_{[0,m(A))}\|_X = \left\| \int_0^{m(A)} K(\cdot,y)\,dy \right\|_X \to 0 \quad \text{as} \quad m(A) \to 0.$$

This also follows from Proposition 4.4, since $L^\infty \subset L^1(\nu_X)$ but L^∞ is not o.c.

The next result shows simple conditions on K and X guaranteeing that $[T,X]_{r.i.}$ is the whole of the space L^1, in particular it is o.c.

Lemma 4.5. *Suppose that K is strictly bounded, i.e., there exists $C > 0$ such that $K(x,y) \le C$ for all $x, y \in [0,1]$, and $L^\infty \subset X$. Then, $[T,X]_{r.i.} = L^1 \hookrightarrow_1 L^1(\nu_X)$.*

Proof. We only have to see that $L^1 \subset [T,X]_{r.i.}$. Given $f \in L^1$, for every $x \in [0,1]$,

$$Tf^*(x) = \int_0^1 f^*(y)K(x,y)\,dy \le C\int_0^1 f^*(y)dy = C\int_0^1 |f(y)| < \infty,$$

that is, $Tf^* \in L^\infty \subset X$. So, $f \in [T,X]_{r.i.}$. □

Example. Let $\mathcal{V}$ be the *Volterra operator*, i.e., $\mathcal{V}f(x) = \int_0^x f(y)\,dy$. Its kernel $K(x,y) = \chi_{[0,x]}(y)$ satisfies (1.2), (3.1) and (4.1) for X containing the simple functions. Moreover, K is strictly bounded. So, $[\mathcal{V},X]_{r.i.} = L^1 \hookrightarrow_1 L^1(\nu_X)$.

In general, there is no containment relation between $[T,X]_{r.i.}$ and $L^1(\nu_X)$.

Example. Let $\mathcal{H}$ be the *Hardy operator*, i.e., $\mathcal{H}f(x) = \frac{1}{x}\int_0^x f(y)\,dy$. Its kernel $K(x,y) = \frac{1}{x}\chi_{[0,x]}(y)$ satisfies (1.2), (3.1) and (4.1) for X being a Lorentz space $L^{p,\infty}$ with $1 < p < \infty$ (see for instance [1, Definition 4.4.1]). Consider the functions $f(y) = y^{-1/p}$ and $g(y) = (1-y)^{-\alpha}$ for $\frac{1}{p} < \alpha < 1$. It can be checked that $f \in [\mathcal{H}, L^{p,\infty}]_{r.i.} \setminus L^1(\nu_{L^{p,\infty}})$ and $g \in L^1(\nu_{L^{p,\infty}}) \setminus [\mathcal{H}, L^{p,\infty}]_{r.i.}$.

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