Optimal Extensions for pth Power Factorable Operators

O. Delgado and E. A. Sánchez Pérez

Abstract. Let $X(\mu)$ be a function space related to a measure space (Ω, Σ, μ) with $\chi_\Omega \in X(\mu)$ and let $T\colon X(\mu) \to E$ be a Banach space-valued operator. It is known that if T is pth power factorable then the largest function space to which T can be extended preserving pth power factorability is given by the space $L^p(m_T)$ of p-integrable functions with respect to m_T, where $m_T\colon \Sigma \to E$ is the vector measure associated to T via $m_T(A) = T(\chi_A)$. In this paper, we extend this result by removing the restriction $\chi_\Omega \in X(\mu)$. In this general case, by considering m_T defined on a certain δ-ring, we show that the optimal domain for T is the space $L^p(m_T) \cap L^1(m_T)$. We apply the obtained results to the particular case when T is a map between sequence spaces defined by an infinite matrix.

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1. Introduction

Although the concept of pth power factorable operator has previously been used as a tool in operator theory, it was introduced explicitly in [19, Sect. 5]. Given a measure space (Ω, Σ, μ) and a Banach function space $X(\mu)$ of (μ-a.e. classes of) Σ-measurable functions such that $\chi_\Omega \in X(\mu)$, for $1 \le p < \infty$, a Banach space-valued operator $T\colon X(\mu) \to E$ is pth power factorable if there is a continuous extension of T to the $\frac{1}{p}$th power space $X(\mu)^{\frac{1}{p}}$ of $X(\mu)$. This is equivalent to the existence of a constant $C > 0$ satisfying that

$$\|T(f)\| \le C \, \| \, |f|^{\frac{1}{p}} \, \|^p_{X(\mu)} = C \, \|f\|_{X(\mu)^{\frac{1}{p}}}$$

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for all $f \in X(\mu)$. The main characterization of this class of operators establishes that any of them can be extended to an space L^p of a vector measure $m_T\colon \Sigma \to E$ associated to T via $m_T(A) = T(\chi_A)$ and the extension is maximal. Note that the condition $\chi_\Omega \in X(\mu)$ is necessary for a correct definition of pth power factorable operator (i.e. $X(\mu) \subset X(\mu)^{\frac{1}{p}}$) and for m_T to be well defined.

Several applications are shown also in [19, Sect. 6, 7], mainly in factorization of operators through spaces $L^q(\mu)$ (Maurey–Rosenthal type theorems) and in harmonic analysis (Fourier transform and convolution operators). After that pth power factorable operators have turned out to be a useful tool for the study of different problems in mathematical analysis, regarding for example Banach space interpolation theory [6], differential equations [10], description of maximal domains for several classes of operators [12], factorization of kernel operators [13] or adjoint operators [11].

The requirement $\chi_\Omega \in X(\mu)$ excludes basic spaces as $L^q(0,\infty)$ or ℓ^q. Although these spaces can be represented as spaces satisfying the needed requirement (for instance $L^q(0,\infty)$ is isometrically isomorphic to $L^q(e^{-x}\mathrm{d}x)$ via the multiplication operator induced by $e^{\frac{x}{q}}$), to use such a representation provides some kind of factorization for T but not genuine extensions.

The aim of this paper is to extend the results on maximal extensions of pth power factorable operators to quasi-Banach spaces $X(\mu)$ which do not necessarily contain χ_Ω. Also we will consider p to be any positive number removing the restriction $p \geq 1$. The first problem is the definition of pth power factorable operator, as in general the containment $X(\mu) \subset X(\mu)^{\frac{1}{p}}$ does not hold. This can be solved by replacing $X(\mu)^{\frac{1}{p}}$ by the sum $X(\mu)^{\frac{1}{p}} + X(\mu)$. The second problem is the definition of the vector measure m_T associated to T. The technique to overcome this obstacle consists of considering m_T defined on the δ-ring $\Sigma_{X(\mu)} = \big\{A \in \Sigma :\ \chi_A \in X(\mu)\big\}$ instead of the σ-algebra Σ. We will see that actually no topology is needed on $X(\mu)$ to extend $T\colon X(\mu) \to E$, it suffices an ideal structure on $X(\mu)$ and a certain property on T which relates the μ-a.e. pointwise order of $X(\mu)$ and the weak topology of E. This property, called order-w continuity, is the minimal condition for m_T to be a vector measure.

The paper is organized as follows. Section 2 is devoted to establish the notation and to state the results on ideal function spaces, quasi-Banach function spaces and integration with respect to a vector measure defined on a δ-ring, which will be use along this work. For the aim of completeness, we include the proof of some relevant facts. In Sect. 3 we show that every order-w continuous operator T defined on an ideal function space $X(\mu)$, can be extended to the space $L^1(m_T)$ of integrable functions with respect to m_T and this space is the largest one to which T can be extended as an order-w continuous operator (Theorem 3.2). Section 4 deals with operators T which are pth power factorable with an order-w continuous extension, that is, there is an order-w continuous extension of T to the space $X(\mu)^{\frac{1}{p}} + X(\mu)$. We prove that the space $L^p(m_T) \cap L^1(m_T)$ is the optimal domain for T preserving the property of being pth power factorable with an order-w continuous extension

(Theorem 4.2). In Sects. 5 and 6 we endow $X(\mu)$ with a topology (namely $X(\mu)$ will be a σ-order continuous Quasi-Banach function space) and consider T to be continuous. Results on maximal extensions analogous to the ones of the previous sections are obtain for continuity instead of order-w continuity (Theorems 5.1 and 6.2). Finally, as an application of our results, in the last section we study when an infinite matrix of real numbers defines a continuous linear operator from ℓ^p into any given sequence space.

2. Preliminaries

2.1. Ideal Function Spaces

Let (Ω, Σ) be a fixed measurable space. For a measure $\mu\colon \Sigma \to [0,\infty]$, we denote by $L^0(\mu)$ the space of all (μ-a.e. classes of) Σ-measurable real-valued functions on Ω. Given two set functions $\mu, \lambda\colon \Sigma \to [0,\infty]$ we will write $\lambda \ll \mu$ if $\mu(A) = 0$ implies $\lambda(A) = 0$. We will say that μ and λ are *equivalent* if $\lambda \ll \mu$ and $\mu \ll \lambda$. In the case when μ and λ are two measures with $\lambda \ll \mu$, the map $[i]\colon L^0(\mu) \to L^0(\lambda)$ which takes a μ-a.e. class in $L^0(\mu)$ represented by f into the λ-a.e. class represented by the same f, is a well-defined linear map. To simplify notation $[i](f)$ will be denoted again as f. Note that if λ and μ are equivalent then $L^0(\mu) = L^0(\lambda)$ and $[i]$ is the identity map i.

An *ideal function space* (briefly, i.f.s.) is a vector space $X(\mu) \subset L^0(\mu)$ satisfying that if $f \in X(\mu)$ and $g \in L^0(\mu)$ with $|g| \le |f|$ μ-a.e. then $g \in X(\mu)$. We will say that $X(\mu)$ has the *σ-property* if there exists $(\Omega_n) \subset \Sigma$ such that $\Omega = \cup\Omega_n$ and $\chi_{\Omega_n} \in X(\mu)$ for all n. For instance, this happens if there is some $g \in X(\mu)$ with $g > 0$ μ-a.e.

Lemma 2.1. *Let $X(\mu)$ be an i.f.s. satisfying the σ-property. For every Σ–measurable function $f\colon \Omega \to [0,\infty)$ there exists $(f_n) \subset X(\mu)$ such that $0 \le f_n \uparrow f$ pointwise.*

Proof. Let $(\Omega_n) \subset \Sigma$ be the sequence given by the σ-property of $X(\mu)$ and let $f\colon \Omega \to [0,\infty)$ be a Σ–measurable function. Taking $A_n = \cup_{j=1}^n \Omega_j \cap \big\{\omega \in \Omega : \ f(\omega) \le n\big\}$, we have that $f_n = f\chi_{A_n} \in X(\mu)$, as $0 \le f_n \le n\chi_{\cup_{j=1}^n \Omega_j}$ pointwise, and that $f_n \uparrow f$ pointwise. □

The *sum* of two i.f.s.' $X(\mu)$ and $Y(\mu)$ is the space defined as

$$X(\mu) + Y(\mu) = \big\{f \in L^0(\mu) : \ f = f_1 + f_2 \ \ \mu\text{-a.e.}, \ f_1 \in X(\mu), \ f_2 \in Y(\mu)\big\}.$$

Proposition 2.2. *The sum $X(\mu) + Y(\mu)$ of two i.f.s.' is an i.f.s.*

Proof. Let $f \in X(\mu)+Y(\mu)$ and $g \in L^0(\mu)$ be such that $|g| \le |f|$ μ-a.e. Write $f = f_1 + f_2$ μ-a.e. with $f_1 \in X(\mu)$ and $f_2 \in Y(\mu)$ and denote $A = \big\{\omega \in \Omega : |g(\omega)| \le |f_1(\omega)|\big\}$. Taking $h_1 = |g|\chi_A + |f_1|\chi_{\Omega\setminus A}$ and $h_2 = (|g| - |f_1|)\chi_{\Omega\setminus A}$, we have that $|g| = h_1 + h_2$ with $h_1 \in X(\mu)$ as $0 \le h_1 \le |f_1|$ pointwise and $h_2 \in Y(\mu)$ as $0 \le h_2 \le |f_2|$ μ-a.e. Now, denote $B = \big\{\omega \in \Omega : \ g(\omega) \ge 0\big\}$ and take $g_1 = h_1\big(\chi_B - \chi_{\Omega\setminus B}\big)$ and $g_2 = h_2\big(\chi_B - \chi_{\Omega\setminus B}\big)$. Then, $g = g_1 + g_2$ with $g_1 \in X(\mu)$ as $|g_1| = h_1$ and $g_2 \in Y(\mu)$ as $|g_2| = h_2$. So, $g \in X(\mu) + Y(\mu)$. □

Let $p \in (0, \infty)$. The *p-power* of an i.f.s. $X(\mu)$ is the i.f.s. defined as

$$X(\mu)^p = \big\{ f \in L^0(\mu) : |f|^p \in X(\mu) \big\}.$$

Lemma 2.3. *Let $X(\mu)$ be an i.f.s. For $s, t \in (0, \infty)$ and $\frac{1}{r} = \frac{1}{s} + \frac{1}{t}$, it follows that if $f \in X(\mu)^s$ and $g \in X(\mu)^t$ then $fg \in X(\mu)^r$. In particular, if $\chi_\Omega \in X(\mu)$ then $X(\mu)^q \subset X(\mu)^p$ for all $0 < p < q < \infty$.*

Proof. For the first part only note that for every $a, b > 0$ it follows

$$a^r b^r \le \frac{r}{s}\, a^s + \frac{r}{t}\, b^t. \tag{2.1}$$

For the second part take $r = p$, $s = q$ and $t = \frac{pq}{q-p}$. Then, if $f \in X(\mu)^q$, since $\chi_\Omega \in X(\mu)^t$, we have that $f = f\chi_\Omega \in X(\mu)^p$. $\square$

Recall that a *quasi-norm* on a real vector space X is a non-negative real map $\|\cdot\|_X$ on X satisfying

(i) $\|x\|_X = 0$ if and only if $x = 0$,
(ii) $\|\alpha x\|_X = |\alpha| \cdot \|x\|_X$ for all $\alpha \in \mathbb{R}$ and $x \in X$, and
(iii) There exists a constant $K \ge 1$ such that $\|x + y\|_X \le K(\|x\|_X + \|y\|_X)$ for all $x, y \in X$.

A quasi-norm $\|\cdot\|_X$ induces a metric topology on X in which a sequence (x_n) converges to x if and only if $\|x - x_n\|_X \to 0$. If X is complete under this topology then it is called a *quasi-Banach space* (*Banach space* if $K = 1$). A linear map $T\colon X \to Y$ between quasi-Banach spaces is continuous if and only if there exists a constant $M > 0$ such that $\|T(x)\|_Y \le M\|x\|_X$ for all $x \in X$. For issues related to quasi-Banach spaces see [14].

A *quasi-Banach function space* (quasi-B.f.s. for short) is a i.f.s. $X(\mu)$ which is also a quasi-Banach space with a quasi-norm $\|\cdot\|_{X(\mu)}$ *compatible with the μ-a.e. pointwise order*, that is, if $f, g \in X(\mu)$ are such that $|f| \le |g|$ μ-a.e. then $\|f\|_{X(\mu)} \le \|g\|_{X(\mu)}$. When the quasi-norm is a norm, $X(\mu)$ is called a *Banach function space* (B.f.s.). Note that every quasi-B.f.s. is a quasi-Banach lattice for the μ-a.e. pointwise order satisfying that if $f_n \to f$ in quasi-norm then there exists a subsequence $f_{n_j} \to f$ μ-a.e. Also note that every positive linear operator between quasi-Banach lattices is continuous, see the argument given in [16, p. 2] for Banach lattices which can be adapted for quasi-Banach spaces. Then all "inclusions" of the type [*i*] between quasi-B.f.s.' are continuous.

A quasi-B.f.s. $X(\mu)$ is said to be *σ-order continuous* if for every $(f_n) \subset X(\mu)$ with $f_n \downarrow 0$ μ-a.e. it follows that $\|f_n\|_X \downarrow 0$.

It is routine to check that the intersection $X(\mu) \cap Y(\mu)$ of two quasi-B.f.s.' (B.f.s.') $X(\mu)$ and $Y(\mu)$ is a quasi-B.f.s. (B.f.s.) endowed with the quasi-norm (norm)

$$\|f\|_{X(\mu)\cap Y(\mu)} = \max\big\{ \|f\|_{X(\mu)}, \|f\|_{Y(\mu)} \big\}.$$

Moreover, if $X(\mu)$ and $Y(\mu)$ are σ-order continuous then $X(\mu) \cap Y(\mu)$ is σ-order continuous.

Proposition 2.4. *The sum $X(\mu)+Y(\mu)$ of two quasi-B.f.s.' (B.f.s.') $X(\mu)$ and $Y(\mu)$ is a quasi-B.f.s. (B.f.s.) endowed with the quasi-norm (norm)*

$$\|f\|_{X(\mu)+Y(\mu)} = \inf\big(\|f_1\|_{X(\mu)} + \|f_2\|_{Y(\mu)}\big),$$

where the infimum is taken over all possible representations $f = f_1 + f_2$ μ-a.e. with $f_1 \in X(\mu)$ and $f_2 \in Y(\mu)$. Moreover, if $X(\mu)$ and $Y(\mu)$ are σ-order continuous then $X(\mu)+Y(\mu)$ is also σ-order continuous.

Proof. From Proposition 2.2 we have that $X(\mu)+Y(\mu)$ is a i.f.s. Even more, looking at the proof we see that for every $f \in X(\mu)+Y(\mu)$ and $g \in L^0(\mu)$ with $|g| \le |f|$ μ-a.e., if $f = f_1+f_2$ μ-a.e. with $f_1 \in X(\mu)$ and $f_2 \in Y(\mu)$ then there exist $g_1 \in X(\mu)$ and $g_2 \in Y(\mu)$ such that $|g_i| \le |f_i|$ μ-a.e. and $g = g_1 + g_2$. Then,

$$\|g\|_{X(\mu)+Y(\mu)} \le \|g_1\|_{X(\mu)} + \|g_2\|_{Y(\mu)} \le \|f_1\|_{X(\mu)} + \|f_2\|_{Y(\mu)}$$

and so, taking infimum over all possible representations $f = f_1 + f_2$ μ-a.e. with $f_1 \in X(\mu)$ and $f_2 \in Y(\mu)$, it follows that $\|g\|_{X(\mu)+Y(\mu)} \le \|f\|_{X(\mu)+Y(\mu)}$. Hence, $\|\cdot\|_{X(\mu)+Y(\mu)}$ is compatible with the μ-a.e. pointwise order.

The proof of the fact that $\|\cdot\|_{X(\mu)+Y(\mu)}$ is a quasi-norm for which $X(\mu)+Y(\mu)$ is complete is similar to the one given in [1, Sect. 3, Theorem 1.3] for compatible couples of Banach spaces.

Suppose that $X(\mu)$ and $Y(\mu)$ are σ-order continuous. Let $(f_n) \subset X(\mu)+Y(\mu)$ be such that $f_n \downarrow 0$ μ-a.e. Consider $f_1 = g + h$ μ-a.e. with $g \in X(\mu)$ and $h \in Y(\mu)$. We can rewrite $f_1 = f_1^1 + f_1^2$ with $f_1^1 \in X(\mu)$, $f_1^2 \in Y(\mu)$ and $f_1^1, f_1^2 \ge 0$ μ-a.e. This can be done by taking $A = \big\{\omega \in \Omega :\ f_1(\omega) \le |g(\omega)|\big\}$, $f_1^1 = f_1\chi_A + |g|\chi_{\Omega\setminus A}$ and $f_1^2 = (f_1 - |g|)\chi_{\Omega\setminus A}$. Note that $f_1^1 \in X(\mu)$ as $0 \le f_1^1 \le |g|$ μ-a.e. and $f_1^2 \in Y(\mu)$ as $0 \le f_1^2 \le |h|$ μ-a.e. Since $0 \le f_2 \le f_1$ μ-a.e., looking again at the proof of Proposition 2.2 we see that there exist $f_2^1 \in X(\mu)$ and $f_2^2 \in Y(\mu)$ such that $0 \le f_2^i \le f_1^i$ μ-a.e. and $f_2 = f_2^1 + f_2^2$ μ-a.e. By induction we construct two μ-a.e. pointwise decreasing sequences of positive functions $(f_n^1) \subset X(\mu)$ and $(f_n^2) \subset Y(\mu)$ such that $f_n = f_n^1 + f_n^2$. Note that $f_n^i \downarrow 0$ μ-a.e. as $0 \le f_n^i \le f_n$ μ-a.e. Then, since $X(\mu)$ and $Y(\mu)$ are σ-order continuous, we have that

$$\|f_n\|_{X(\mu)+Y(\mu)} \le \|f_n^1\|_{X(\mu)} + \|f_n^2\|_{Y(\mu)} \to 0.$$

$\square$

Let $p \in (0,\infty)$. The p-power $X(\mu)^p$ of a quasi-B.f.s. $X(\mu)$ is a quasi-B.f.s. endowed with the quasi-norm

$$\|f\|_{X(\mu)^p} = \|\,|f|^p\,\|_{X(\mu)}^{\frac{1}{p}}.$$

Moreover, $X(\mu)^p$ is σ-order continuous whenever $X(\mu)$ is so. Note that in the case when $X(\mu)$ is a B.f.s. and $p \ge 1$ it follows that $\|\cdot\|_{X(\mu)^p}$ is a norm and so $X(\mu)^p$ is a B.f.s. An exhaustive study of the space $X(\mu)^p$ can be found in [19, Sect. 2.2] for the case when μ is finite and $\chi_\Omega \in X(\mu)$. This study can be extended to our general case adapting the arguments with the natural modifications (note that our p-powers here are the $\frac{1}{p}$th powers there).

2.2. Integration with Respect to a Vector Measure Defined on a δ-Ring

Let $\mathcal{R}$ be a *δ-ring* of subsets of a set Ω, that is, a ring closed under countable intersections. Measurability will be considered with respect to the σ-algebra $\mathcal{R}^{loc}$ of all subsets A of Ω such that $A \cap B \in \mathcal{R}$ for all $B \in \mathcal{R}$. Let us write $\mathcal{S}(\mathcal{R})$ for the space of all *$\mathcal{R}$-simple* functions, that is, simple functions with support in $\mathcal{R}$.

A set function $m\colon \mathcal{R} \to E$ with values in a Banach space E is said to be a *vector measure* if $\sum m(A_n)$ converges to $m(\cup A_n)$ in E for every sequence of pairwise disjoint sets $(A_n) \subset \mathcal{R}$ with $\cup A_n \in \mathcal{R}$.

Consider first a *real measure* $\lambda\colon \mathcal{R} \to \mathbb{R}$. The *variation* of λ is the measure $|\lambda|\colon \mathcal{R}^{loc} \to [0,\infty]$ defined as

$$|\lambda|(A) = \sup\Big\{ \sum |\lambda(A_j)| : \ (A_j) \text{ finite disjoint sequence in } \mathcal{R} \cap 2^A \Big\}.$$

Note that $|\lambda|$ is finite on $\mathcal{R}$. The space $L^1(\lambda)$ of integrable functions with respect to λ is defined as the classical space $L^1(|\lambda|)$. The integral with respect to λ of $\varphi = \sum_{j=1}^n \alpha_j \chi_{A_j} \in \mathcal{S}(\mathcal{R})$ over $A \in \mathcal{R}^{loc}$ is defined in the natural way by $\int_A \varphi \, \mathrm{d}\lambda = \sum_{j=1}^n \alpha_j \lambda(A_j \cap A)$. The space $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\lambda)$, allowing to define the integral of $f \in L^1(\lambda)$ over $A \in \mathcal{R}^{loc}$ as $\int_A f \, \mathrm{d}\lambda = \lim \int_A \varphi_n \, \mathrm{d}\lambda$ for any sequence $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ converging to f in $L^1(\lambda)$.

Let now $m\colon \mathcal{R} \to E$ be a vector measure. The *semivariation* of m is the set function $\|m\|\colon \mathcal{R}^{loc} \to [0,\infty]$ defined by

$$\|m\|(A) = \sup_{x^* \in B_{E^*}} |x^* m|(A).$$

Here, B_{E^*} is the closed unit ball of the dual space E^* of E and $|x^*m|$ is the variation of the real measure x^*m given by the composition of m with x^*. A set $A \in \mathcal{R}^{loc}$ is *m-null* if $\|m\|(A) = 0$, or equivalently, if $m(B) = 0$ for all $B \in \mathcal{R} \cap 2^A$. From [2, Theorem 3.2], there always exists a measure $\eta\colon \mathcal{R}^{loc} \to [0,\infty]$ equivalent to $\|m\|$, that is, m and η have the same null sets. Let us denote $L^0(m) = L^0(\eta)$.

The space $L^1(m)$ of integrable functions with respect to m is defined as the space of functions $f \in L^0(m)$ satisfying that

(i) $f \in L^1(x^*m)$ for every $x^* \in E^*$, and
(ii) for each $A \in \mathcal{R}^{loc}$ there exists $x_A \in E$ such that

$$x^*(x_A) = \int_A f \, \mathrm{d}x^*m, \qquad \text{for every } x^* \in E^*.$$

The vector x_A is unique and will be denoted by $\int_A f \, \mathrm{d}m$. The space $L^1(m)$ is a σ-order continuous B.f.s. related to the measure space $(\Omega, \mathcal{R}^{loc}, \eta)$, with norm

$$\|f\|_{L^1(m)} = \sup_{x^* \in B_{E^*}} \int_\Omega |f| \, d|x^*m|.$$

Moreover, $\mathcal{S}(\mathcal{R})$ is dense in $L^1(m)$. Note that $\int_A \varphi \, \mathrm{d}m = \sum_{j=1}^n \alpha_j m(A_j \cap A)$ for every $\varphi = \sum_{j=1}^n \alpha_j \chi_{A_j} \in \mathcal{S}(\mathcal{R})$ and $A \in \mathcal{R}^{loc}$.

The integration operator $I_m\colon L^1(m) \to E$ defined by $I_m(f) = \int_\Omega f\,\mathrm{d}m$ is a continuous linear operator with $\|I_m(f)\|_E \le \|f\|_{L^1(m)}$. Even more,

$$\frac{1}{2}\|f\|_{L^1(m)} \le \sup_{A\in\mathcal{R}} \|I_m(f\chi_A)\|_E \le \|f\|_{L^1(m)} \tag{2.2}$$

for all $f \in L^1(m)$.

Let $p \in (0,\infty)$. We denote by $L^p(m)$ the p-power of $L^1(m)$, that is,

$$L^p(m) = \{f \in L^0(m) : |f|^p \in L^1(m)\}.$$

Then $L^p(m)$ is a quasi-B.f.s. with the quasi-norm $\|f\|_{L^p(m)} = \|\,|f|^p\,\|_{L^1(m)}^{1/p}$. In the case when $p \ge 1$, we have that $\|\cdot\|_{L^p(m)}$ is a norm and so $L^p(m)$ is a B.f.s.

These and other issues concerning integration with respect to a vector measure defined on a δ-ring can be found in [3,5,7,15,17,18].

3. Optimal Domain for Order-w Continuous Operators on a i.f.s.

Let $X(\mu)$ be a i.f.s. satisfying the σ-property (recall: $\Omega = \cup\Omega_n$ with $\chi_{\Omega_n} \in X(\mu)$ for all n) and consider the δ-ring

$$\Sigma_{X(\mu)} = \{A \in \Sigma : \chi_A \in X(\mu)\}.$$

The σ-property guarantees that $\Sigma_{X(\mu)}^{loc} = \Sigma$. Given a Banach space-valued linear operator $T\colon X(\mu) \to E$, we define the finitely additive set function $m_T\colon \Sigma_{X(\mu)} \to E$ by $m_T(A) = T(\chi_A)$.

We will say that T is *order-w continuous* if $T(f_n) \to T(f)$ weakly in E whenever $f_n, f \in X(\mu)$ are such that $0 \le f_n \uparrow f$ μ-a.e.

Proposition 3.1. *If T is order-w continuous, then m_T is a vector measure satisfying that $[i]\colon X(\mu) \to L^1(m_T)$ is well defined and $T = I_{m_T} \circ [i]$.*

Proof. Let $(A_n) \subset \Sigma_{X(\mu)}$ be a pairwise disjoint sequence with $\cup A_n \in \Sigma_{X(\mu)}$. Since T is order-w continuous, for any subsequence (A_{n_j}) we have that

$$\sum_{j=1}^{N} m_T(A_{n_j}) = T(\chi_{\cup_{j=1}^N A_{n_j}}) \to T(\chi_{\cup A_{n_j}}) = m_T(\cup A_{n_j})$$

weakly in E. From the Orlicz–Pettis theorem (see [9, Corollary I.4.4]), it follows that $\sum m_T(A_n)$ is unconditionally convergent in norm to $m_T(\cup A_n)$. Thus, m_T is a vector measure.

Note that $\|m_T\| \ll \mu$ and so $[i]\colon L^0(\mu) \to L^0(m_T)$ is well defined. In addition, note that for every $\varphi \in \mathcal{S}(\Sigma_{X(\mu)})$ we have that $I_{m_T}(\varphi) = T(\varphi)$.

Let $f \in X(\mu)$ be such that $f \ge 0$ μ-a.e. and take a sequence of Σ-simple functions $0 \le \varphi_n \uparrow f$ μ-a.e. For each n we can write $\varphi_n = \sum_{j=1}^m \alpha_j \chi_{A_j}$ with $(A_j)_{j=1}^m \subset \Sigma$ being a pairwise disjoint sequence and $\alpha_j > 0$ for all j. Since $\chi_{A_j} \le \alpha_j^{-1}\varphi_n \le \alpha_j^{-1} f$ μ-a.e., we have that $\chi_{A_j} \in X(\mu)$ and so $\varphi_n \in \mathcal{S}(\Sigma_{X(\mu)})$. Fix $x^* \in E^*$. For every $A \in \Sigma$ it follows that $x^*T(\varphi_n\chi_A) \to x^*T(f\chi_A)$ as T is order-w continuous. Note that $x^*T(\varphi_n\chi_A) = \int_A \varphi_n\,\mathrm{d}x^*m_T$

and that $0 \le \varphi_n \uparrow f$ x^*m_T-a.e. as $|x^*m_T| \ll \|m_T\| \ll \mu$. From [7, Proposition 2.3], we have that $f \in L^1(x^*m_T)$ and

$$\int_A f \, \mathrm{d}x^*m_T = \lim_{n\to\infty} \int_A \varphi_n \, \mathrm{d}x^*m_T = \lim_{n\to\infty} x^*T(\varphi_n \chi_A) = x^*T(f\chi_A).$$

Therefore, $f \in L^1(m_T)$ and $I_{m_T}(f) = T(f)$.

For a general $f \in X(\mu)$, the result follows by taking the positive and negative parts of f. $\square$

For the case when $X(\mu)$ is a B.f.s., Proposition 3.1 and the next Theorem 3.2 can be deduced from [8, Proposition 2.3] and [4, Proposition 4]. The proofs given here are more direct and are valid for general i.f.s.'.

Theorem 3.2. *Suppose that T is order-w continuous. Then, T factors as*

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "{[i]}"'] & & E \\ & L^1(m_T) \arrow[ur, dotted, "I_{m_T}"'] & \end{tikzcd} \tag{3.1}$$

with I_{m_T} being order-w continuous. Moreover, the factorization is optimal *in the sense:*

If $Z(\xi)$ is a i.f.s. such that $\xi \ll \mu$ and

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "{[i]}"'] & & E \\ & Z(\xi) \arrow[ur, dotted, "S"'] & \end{tikzcd}$$

with S being an order-w continuous linear operator

$$\Bigg\} \Longrightarrow \quad [i]\colon Z(\xi) \to L^1(m_T) \text{ is well defined and } S = I_{m_T} \circ [i]. \tag{3.2}$$

Proof. The factorization (3.1) follows from Proposition 3.1. Note that the integration operator $I_{m_T}\colon L^1(m_T) \to E$ is order-w continuous, as it is continuous and $L^1(m_T)$ is σ-order continuous.

Let $Z(\xi)$ satisfy (3.3). In particular, $Z(\xi)$ satisfies the σ-property, as if $\chi_A \in X(\mu)$ then $\chi_A \in Z(\xi)$. From Proposition 3.1 applied to the operator $S\colon Z(\xi) \to E$, we have that $[i]\colon Z(\xi) \to L^1(m_S)$ is well defined and $S = I_{m_S} \circ [i]$. Note that $\Sigma_{X(\mu)} \subset \Sigma_{Z(\xi)}$ and $m_S(A) = S(\chi_A) = T(\chi_A) = m_T(A)$ for all $A \in \Sigma_{X(\mu)}$, that is, m_T is the restriction of $m_S\colon \Sigma_{Z(\xi)} \to E$ to $\Sigma_{X(\mu)}$. Then, from [4, Lemma 3], it follows that $L^1(m_S) = L^1(m_T)$ and $I_{m_S} = I_{m_T}$. $\square$

We can rewrite Theorem 3.2 in terms of optimal domain.

Corollary 3.3. *Suppose that T is order-w continuous. Then $L^1(m_T)$ is the largest i.f.s. to which T can be extended as an order-w continuous operator still with values in E. Moreover, the extension of T to $L^1(m_T)$ is given by the integration operator I_{m_T}.*

4. Optimal Domain for pth Power Factorable Operators on a i.f.s. with an Order-w Continuous Extension

Let $X(\mu)$ be a i.f.s. satisfying the σ-property and let $T\colon X(\mu) \to E$ be a linear operator with values in a Banach space E.

For $p \in (0, \infty)$, we call T pth *power factorable with an order-w continuous extension* if there is an order-w continuous linear extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$, i.e. T factors as

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "i"'] & & E \\ & X(\mu)^{\frac{1}{p}} + X(\mu) \arrow[ur, dotted, "S"'] & \end{tikzcd}$$

with S being an order-w continuous linear operator.

Note that in the case when $\chi_\Omega \in X(\mu)$, from Lemma 2.3, if $1 < p$ we have that $X(\mu) \subset X(\mu)^{\frac{1}{p}}$ and so $X(\mu)^{\frac{1}{p}} + X(\mu) = X(\mu)^{\frac{1}{p}}$. Similarly, if $p \leq 1$ then $X(\mu)^{\frac{1}{p}} + X(\mu) = X(\mu)$, but hence to say that T is pth power factorable with an order-w continuous extension is just to say that T is order-w continuous.

Proposition 4.1. *The following statements are equivalent:*

(a) *T is pth power factorable with an order-w continuous extension.*

(b) *T is order-w continuous and $[i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to L^1(m_T)$ is well defined.*

(c) *T is order-w continuous and $[i]\colon X(\mu) \to L^p(m_T) \cap L^1(m_T)$ is well defined.*

Moreover, if (a)–(c) holds, the extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$ coincides with integration operator $I_{m_T} \circ [i]$.

Proof. (a) $\Rightarrow$ (b) Note that T is order-w continuous as it has an order-w continuous extension. Let $S\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to E$ be an order-w continuous linear operator extending T. Then, from Theorem 3.2, it follows that $[i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to L^1(m_T)$ is well defined and $S = I_{m_T} \circ [i]$.

(b) $\Leftrightarrow$ (c) Since T is is order-w continuous, by Proposition 3.1 we always have that $[i]\colon X(\mu) \to L^1(m_T)$ is well defined. Suppose that $[i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to L^1(m_T)$ is well defined. If $f \in X(\mu)$, since $|f|^p \in X(\mu)^{\frac{1}{p}} \subset X(\mu)^{\frac{1}{p}} + X(\mu)$, we have that $|f|^p \in L^1(m_T)$ and so $f \in L^p(m_T)$. Then $f \in L^p(m_T) \cap L^1(m_T)$. Conversely, suppose that $[i]\colon X(\mu) \to L^p(m_T) \cap L^1(m_T)$ is well defined. Let $f \in X(\mu)^{\frac{1}{p}} + X(\mu)$ and write $f = f_1 + f_2$ μ-a.e. with $f_1 \in X(\mu)^{\frac{1}{p}}$ and $f_2 \in X(\mu)$. Since $|f_1|^{\frac{1}{p}} \in X(\mu)$ we have that $|f_1|^{\frac{1}{p}} \in L^p(m_T) \cap L^1(m_T) \subset L^p(m_T)$ and so $f_1 \in L^1(m_T)$. Then, $f \in L^1(m_T)$ as $f_2 \in L^1(m_T)$.

(b) $\Rightarrow$ (a) From Proposition 3.1 and since $[i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to L^1(m_T)$ is well defined, we have that the operator $I_{m_T} \circ [i]$ extends T to $X(\mu)^{\frac{1}{p}} + X(\mu)$. Moreover, the extension $I_{m_T} \circ [i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to E$ is order-w continuous as the integration operator $I_{m_T}\colon L^1(m_T) \to E$ is so. $\square$

In the case when $\chi_\Omega \in X(\mu)$ and T is order-w continuous, from Proposition 3.1, we have that $\chi_\Omega \in L^1(m_T)$. So, from Lemma 2.3, if $p > 1$ then $L^p(m_T) \subset L^1(m_T)$ and hence $L^p(m_T) \cap L^1(m_T) = L^p(m_T)$. If $p \leq 1$ then $L^p(m_T) \cap L^1(m_T) = L^1(m_T)$, but hence, as commented before, T being pth power factorable with an order-w continuous extension is just T being order-w continuous.

Theorem 4.2. *Suppose that T is pth power factorable with an order-w continuous extension. Then, T factors as*

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "{[i]}"'] & & E \\ & L^p(m_T) \cap L^1(m_T) \arrow[ur, dotted, "I_{m_T}"'] & \end{tikzcd} \tag{4.1}$$

with I_{m_T} being pth power factorable with an order-w continuous extension. Moreover, the factorization is optimal *in the sense:*

$$\left.\begin{array}{c} \textit{If } Z(\xi) \textit{ is a i.f.s. such that } \xi \ll \mu \textit{ and} \\ \begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "{[i]}"'] & & E \\ & Z(\xi) \arrow[ur, dotted, "S"'] & \end{tikzcd} \\ \textit{with } S \textit{ being a pth power factorable} \\ \textit{linear operator with an order-w} \\ \textit{continuous extension} \end{array}\right\} \Longrightarrow \begin{array}{l} [i]\colon Z(\xi) \to L^p(m_T) \cap L^1(m_T) \\ \textit{is well defined and } S = I_{m_T} \circ [i]. \end{array} \tag{4.2}$$

Proof. The factorization (4.1) follows from Propositions 3.1 and 4.1. Note that $L^p(m_T) \cap L^1(m_T)$ satisfies the σ-property as $X(\mu)$ does. Let us see that the operator $I_{m_T}\colon L^p(m_T) \cap L^1(m_T) \to E$ is pth power factorable with an order-w continuous extension by using Proposition 4.1(c). This operator is order-w continuous as the integration operator $I_{m_T}\colon L^1(m_T) \to E$ is so. On other hand, since $\Sigma_{X(\mu)} \subset \Sigma_{L^p(m_T)\cap L^1(m_T)}$ and $m_{I_{m_T}}(A) = I_{m_T}(\chi_A) = T(\chi_A) = m_T(A)$ for all $A \in \Sigma_{X(\mu)}$ (i.e. m_T is the restriction of $m_{I_{m_T}}\colon \Sigma_{L^p(m_T)\cap L^1(m_T)} \to E$ to $\Sigma_{X(\mu)}$), from [4, Lemma 3], it follows that $L^1(m_{I_{m_T}}) = L^1(m_T)$. Then,

$$[i]\colon L^p(m_T) \cap L^1(m_T) \to L^p(m_{I_{m_T}}) \cap L^1(m_{I_{m_T}}) = L^p(m_T) \cap L^1(m_T)$$

is well defined.

Let $Z(\xi)$ satisfy (4.3). In particular, $Z(\xi)$ has the σ-property. Applying Proposition 4.1 to the operator $S\colon Z(\xi) \to E$, we have that $[i]\colon Z(\xi) \to L^p(m_S) \cap L^1(m_S)$ is well defined and $S = I_{m_S} \circ [i]$. Since $\Sigma_{X(\mu)} \subset \Sigma_{Z(\xi)}$ and $m_S(A) = m_T(A)$ for all $A \in \Sigma_{X(\mu)}$, from [4, Lemma 3], it follows that $L^1(m_S) = L^1(m_T)$ and $I_{m_S} = I_{m_T}$. $\square$

Rewriting Theorem 4.2 in terms of optimal domain we obtain the following conclusion.

Corollary 4.3. *Suppose that T is pth power factorable with an order-w continuous extension. Then $L^p(m_T) \cap L^1(m_T)$ is the largest i.f.s. to which T can be* *extended as a pth power factorable operator with an order-w continuous extension, still with values in E. Moreover, the extension of T to $L^p(m_T) \cap L^1(m_T)$ is given by the integration operator I_{m_T}.*

5. Optimal Domain for Continuous Operators on a Quasi-B.f.s.

Let $X(\mu)$ be a quasi-B.f.s. satisfying the σ-property and let $T\colon X(\mu) \to E$ be a linear operator with values in a Banach space E.

Theorem 5.1. *Suppose that $X(\mu)$ is σ-order continuous and T is continuous. Then, T factors as*

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "{[i]}"'] & & E \\ & L^1(m_T) \arrow[ur, dotted, "I_{m_T}"'] & \end{tikzcd} \tag{5.1}$$

with I_{m_T} being continuous. Moreover, the factorization is optimal *in the sense:*

If $Z(\xi)$ is a σ-order continuous quasi-B.f.s. such that $\xi \ll \mu$ and

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "{[i]}"'] & & E \\ & Z(\xi) \arrow[ur, dotted, "S"'] & \end{tikzcd}$$

with S being a continuous linear operator $\Longrightarrow$ *$[i]\colon Z(\xi) \to L^1(m_T)$ is well defined and $S = I_{m_T} \circ [i]$.* (5.2)

Proof. Since $X(\mu)$ is σ-order continuous and T is continuous, we have that T is order-w continuous and so the factorization (5.1) follows from Theorem 3.2. Recall that $L^1(m_T)$ is σ-order continuous and I_{m_T} is continuous.

Let $Z(\xi)$ satisfy (5.3). In particular, S is order-w continuous. From Theorem 3.2 we have that $[i]\colon Z(\xi) \to L^1(m_T)$ is well defined and $S = I_{m_T} \circ [i]$. $\square$

Corollary 5.2. *Suppose that $X(\mu)$ is σ-order continuous and T is continuous. Then $L^1(m_T)$ is the largest σ-order continuous quasi-B.f.s. to which T can be extended as a continuous operator still with values in E. Moreover, the extension of T to $L^1(m_T)$ is given by the integration operator I_{m_T}.*

6. Optimal Domain for pth Power Factorable Operators on a Quasi-B.f.s. with a Continuous Extension

Let $X(\mu)$ be a quasi-B.f.s. satisfying the σ-property and let $T\colon X(\mu) \to E$ be a linear operator with values in a Banach space E.

For $p \in (0,\infty)$, we call T *pth power factorable with a continuous extension* if there is a continuous linear extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$, i.e. T

factors as

$$\begin{tikzcd}
X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "i"'] & & E \\
& X(\mu)^{\frac{1}{p}} + X(\mu) \arrow[ur, dotted, "S"'] &
\end{tikzcd}$$

with S being a continuous linear operator.

Note that in the case when $\chi_\Omega \in X(\mu)$ and $1 < p$, from Lemma 2.3, it follows that $X(\mu)^{\frac{1}{p}} + X(\mu) = X(\mu)^{\frac{1}{p}}$. Then our definition of pth power factorable operator with a continuous extension coincides with the one given in [19, Definition 5.1]. If $p \leq 1$, since $X(\mu)^{\frac{1}{p}} + X(\mu) = X(\mu)$, to say that T is pth power factorable with a continuous extension is just to say that T is continuous.

Proposition 6.1. *Suppose that $X(\mu)$ is σ-order continuous. Then, the following statements are equivalent:*

(a) *T is pth power factorable with a continuous extension.*
(b) *T is pth power factorable with an order-w continuous extension.*
(c) *T is order-w continuous and $[i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to L^1(m_T)$ is well defined.*
(d) *T is order-w continuous and $[i]\colon X(\mu) \to L^p(m_T) \cap L^1(m_T)$ is well defined.*
(e) *There exists $C > 0$ such that $\|T(f)\|_E \leq C \|f\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}$ for all $f \in X(\mu)$.*

Moreover, if (a)–(e) holds, the extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$ coincides with the integration operator $I_{m_T} \circ [i]$.

Proof. (a) $\Rightarrow$ (b) Let $S\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to E$ be a continuous linear operator extending T. From Proposition 2.4 we have that $X(\mu)^{\frac{1}{p}} + X(\mu)$ is σ-order continuous and so S is order-w continuous. Then, T is pth power factorable with an order-w continuous extension.

(b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) And the fact that the extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$ coincides with the integration operator $I_{m_T} \circ [i]$ follows from Proposition 4.1.

(c) $\Rightarrow$ (e) The operator $[i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to L^1(m_T)$ is continuous as it is positive. Then, there exists a constant $C > 0$ satisfying that

$$\|f\|_{L^1(m_T)} \leq C \|f\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}$$

for all $f \in X(\mu)^{\frac{1}{p}} + X(\mu)$. Since I_{m_T} extends T to $L^1(m_T)$, it follows that

$$\|T(f)\|_E = \|I_{m_T}(f)\|_E \leq \|f\|_{L^1(m_T)} \leq C \|f\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}$$

for all $f \in X(\mu)$.

(e) $\Rightarrow$ (a) Let $0 \leq f \in X(\mu)^{\frac{1}{p}} + X(\mu)$. From Lemma 2.1, there exists $(f_n) \subset X(\mu)$ such that $0 \leq f_n \uparrow f$ μ-a.e. Since $X(\mu)^{\frac{1}{p}} + X(\mu)$ is σ-order continuous, it follows that $f_n \to f$ in the quasi-norm of $X(\mu)^{\frac{1}{p}} + X(\mu)$. Then, since

$$\|T(f_n) - T(f_m)\|_E = \|T(f_n - f_m)\|_E \leq C\,\|f_n - f_m\|_{X(\mu)^{\frac{1}{p}}+X(\mu)},$$

we have that $\big(T(f_n)\big)$ converges to some element $e \in E$. Define $S(f) = e$. Note that if $(g_n) \subset X(\mu)$ is another sequence such that $0 \leq g_n \uparrow f$ μ-a.e., then

$$\begin{aligned}\|T(f_n) - T(g_n)\|_E &\leq C\,\|f_n - g_n\|_{X(\mu)^{\frac{1}{p}}+X(\mu)} \\ &\leq CK\left(\|f_n - f\|_{X(\mu)^{\frac{1}{p}}+X(\mu)} + \|f - g_n\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}\right),\end{aligned}$$

where K is the constant satisfying the property (iii) of the quasi-norm $\|\cdot\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}$, and so S is well defined. Also note that

$$\begin{aligned}\|S(f)\|_E &\leq \|S(f) - T(f_n)\|_E + \|T(f_n)\|_E \\ &\leq \|S(f) - T(f_n)\|_E + C\,\|f_n\|_{X(\mu)^{\frac{1}{p}}+X(\mu)} \\ &\leq \|S(f) - T(f_n)\|_E + C\,\|f\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}\end{aligned}$$

for all $n \geq 1$, and thus $\|S(f)\|_E \leq C\,\|f\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}$.

For a general $f \in X(\mu)^{\frac{1}{p}} + X(\mu)$, define $S(f) = S(f^+) - S(f^-)$ where f^+ and f^- are the positive and negative parts of f, respectively. It follows that S is linear and $S(f) = T(f)$ for all $f \in X(\mu)$. Moreover, for every $f \in X(\mu)^{\frac{1}{p}} + X(\mu)$ we have that

$$\begin{aligned}\|S(f)\|_E &\leq \|S(f^+)\|_E + \|S(f^-)\|_E \\ &\leq C\,\|f^+\|_{X(\mu)^{\frac{1}{p}}+X(\mu)} + C\,\|f^-\|_{X(\mu)^{\frac{1}{p}}+X(\mu)} \\ &\leq 2C\,\|f\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}.\end{aligned}$$

an so S is continuous. Hence, T is pth power factorable with a continuous extension. $\square$

In the case when μ is finite, $\chi_\Omega \in X(\mu)$ and $p \geq 1$, the equivalences (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) of Proposition 6.1 are proved in [19, Theorem 5.7]. Here, we have included a more detailed proof for the general case.

Theorem 6.2. *Suppose that $X(\mu)$ is σ-order continuous and T is pth power factorable with a continuous extension. Then, T factors as*

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "{[i]}"'] & & E \\ & L^p(m_T) \cap L^1(m_T) \arrow[ur, dotted, "I_{m_T}"'] & \end{tikzcd} \tag{6.1}$$

with I_{m_T} being pth power factorable with a continuous extension. Moreover, the factorization is optimal *in the sense:*

$$\left.\begin{array}{l} \textit{If } Z(\xi) \textit{ is a } \sigma\textit{-order continuous quasi-B.f.s.} \\ \textit{such that } \xi \ll \mu \textit{ and} \\ \begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "{[i]}"'] & & E \\ & Z(\xi) \arrow[ur, dotted, "S"'] & \end{tikzcd} \\ \textit{with } S \textit{ being a pth power factorable linear} \\ \textit{operator with a continuous extension} \end{array}\right\} \Longrightarrow \begin{array}{l} [i]\colon Z(\xi) \to L^p(m_T) \cap L^1(m_T) \\ \textit{is well defined and } S = I_{m_T} \circ [i]. \end{array} \tag{6.2}$$

Proof. From Proposition 6.1 we have that T is pth power factorable with an order-w continuous extension. Then, from Theorem 4.2, the factorization (6.1) holds and $I_{m_T}\colon L^p(m_T) \cap L^1(m_T) \to E$ is pth power factorable with an order-w continuous extension. Noting that the space $L^p(m_T) \cap L^1(m_T)$ is σ-order continuous (as $L^1(m_T)$ is so) and satisfies the σ-property (as $X(\mu)$ does), from Proposition 6.1 it follows that $I_{m_T}\colon L^p(m_T) \cap L^1(m_T) \to E$ is pth power factorable with a continuous extension.

Let $Z(\xi)$ satisfy (6.3), in particular it satisfies the σ-property. Again Proposition 6.1 gives that S is pth power factorable with an order-w continuous extension. So, from Theorem 4.2, it follows that $[i]\colon Z(\xi) \to L^p(m_T) \cap L^1(m_T)$ is well defined and $S = I_{m_T} \circ [i]$. □

Corollary 6.3. *Suppose that $X(\mu)$ is σ-order continuous and T is pth power factorable with a continuous extension. Then $L^p(m_T) \cap L^1(m_T)$ is the largest σ-order continuous quasi-B.f.s. to which T can be extended as a pth power factorable operator with a continuous extension, still with values in E. Moreover, the extension of T to $L^p(m_T) \cap L^1(m_T)$ is given by the integration operator I_{m_T}.*

In the case when μ is finite, $\chi_\Omega \in X(\mu)$ and $p \geq 1$, Corollary 6.3 is proved in [19, Theorem 5.11].

7. Application: Extension for Operators Defined on ℓ^1

Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), c)$ where c is the counting measure on $\mathbb{N}$. Note that a property holds c-a.e. if and only if it holds pointwise and that the space $L^0(c)$ coincides with the space ℓ^0 of all real sequences. Consider the space $\ell^1 = L^1(c)$, which is σ-order continuous and has the σ-property. The δ-ring $\mathcal{P}(\mathbb{N})_{\ell^1}$ is just the set $\mathcal{P}_F(\mathbb{N})$ of all finite subsets of $\mathbb{N}$.

Let $T\colon \ell^1 \to E$ be a continuous linear operator with values in a Banach space E. Denote $e_n = \chi_{\{n\}}$ and assume that $T(e_n) \neq 0$ for all n. This assumption seems to be natural since if $T(e_n) = 0$ then the nth coordinate is not involved in the action of T. Hence, the vector measure $m_T\colon \mathcal{P}_F(\mathbb{N}) \to E$ associated to T by $m_T(A) = T(\chi_A)$ is equivalent to c and so $L^1(m_T) \subset \ell^0$. We will write $\ell^1(m_T) = L^1(m_T)$.

Remark 7.1. By Theorem 5.1 we have that T can be extended as

$$\begin{tikzcd} \ell^1 \arrow[rr, "T"] \arrow[dr, dotted, "i"'] & & E \\ & \ell^1(m_T) \arrow[ur, dotted, "I_{m_T}"'] & \end{tikzcd}$$

and $\ell^1(m_T)$ is the largest σ-order continuous quasi-B.f.s. to which T can be extended as a continuous operator.

Let $p > 1$. We have that T is $\frac{1}{p}$th power factorable with a continuous extension if there is an extension S as

$$\begin{tikzcd} \ell^1 \arrow[rr, "T"] \arrow[dr, dotted, dash, "i"'] & & E \\ & \ell^p \arrow[ur, dotted, "S"'] & \end{tikzcd}$$

with S being a continuous linear operator. Note that $p \le 1$ is not considered as in this case $\ell^p \subset \ell^1$ and so the extension of T to the sum $\ell^p + \ell^1$ is just the same operator T. Applying Proposition 6.1 in the context of this section, we obtain the following result.

Proposition 7.2. *The following statements are equivalent:*

(a) *T is $\frac{1}{p}$th power factorable with a continuous extension.*

(b) $\ell^p \subset \ell^1(m_T)$.

(c) $\ell^1 \subset \ell^{\frac{1}{p}}(m_T) \cap \ell^1(m_T)$.

(d) *There exists $C > 0$ such that*

$$\left\| \sum_{j \in M} x_j T(e_j) \right\|_E \le C \left(\sum_{j \in M} x_j^p \right)^{\frac{1}{p}}$$

for all $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \subset [0, \infty)$.

Proof. From Proposition 6.1, we only have to prove that condition (d) is equivalent to the following condition:

(d') There exists $C > 0$ such that $\|T(x)\|_E \le C \|x\|_{\ell^p}$ for all $x \in \ell^1$.

If (d') holds, we obtain (d) by taking in (d') the element $x = \sum_{j \in M} x_j e_j \in \ell^1$ for every $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \subset [0, \infty)$.

Suppose that (d) holds. Let $0 \le x = (x_n) \in \ell^1$ and take $y^k = \sum_{j=1}^k x_j e_j$. Since $y^k \uparrow x$ pointwise, ℓ^1 is σ-order continuous and T is continuous, we have that

$$\|T(x)\|_E = \lim \|T(y^k)\|_E = \lim \left\| \sum_{j=1}^k x_j T(e_j) \right\|_E \le C \lim \left(\sum_{j=1}^k x_j^p \right)^{\frac{1}{p}} = C \|x\|_{\ell^p}.$$

For a general $x \in \ell^1$, (d') follows by taking the positive and negative parts of x. $\square$

Remark 7.3. Note that if T is $\frac{1}{p}$th power factorable with a continuous extension then the integration operator I_{m_T} extends T to ℓ^p and, from Theorem 6.2, T factors optimally as

$$\begin{tikzcd}
\ell^1 \arrow[rr, "T"] \arrow[dr, dotted, "i"'] & & E \\
& \ell^{\frac{1}{p}}(m_T) \cap \ell^1(m_T) \arrow[ur, dotted, dash, "I_{m_T}"'] &
\end{tikzcd}$$

with I_{m_T} being $\frac{1}{p}$th power factorable with a continuous extension.

Now a natural question arises: when $\ell^{\frac{1}{p}}(m_T) \cap \ell^1(m_T)$ is equal to $\ell^{\frac{1}{p}}(m_T)$ or $\ell^1(m_T)$? For asking this question we introduce the following class of operators.

Let $0 < r < \infty$. We say that T is *r-power dominated* if there exists $C > 0$ such that

$$\left\| \sum_{j \in M} x_j^r \, T(e_j) \right\|_E^{\frac{1}{r}} \leq C \sup_{N \subset M} \left\| \sum_{j \in N} x_j T(e_j) \right\|_E$$

for every $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \in [0, \infty)$. Note that in the case when E is a Banach lattice and T is positive we have that

$$\sup_{N \subset M} \left\| \sum_{j \in N} x_j T(e_j) \right\|_E = \left\| \sum_{j \in M} x_j T(e_j) \right\|_E .$$

Lemma 7.4. *The containment $\ell^1(m_T) \subset \ell^r(m_T)$ holds if and only if T is r-power dominated.*

Proof. Suppose that $\ell^1(m_T) \subset \ell^r(m_T)$. Since the containment is continuous (as it is positive), there exists $C > 0$ such that $\|x\|_{\ell^r(m_T)} \leq C \|x\|_{\ell^1(m_T)}$ for all $x \in \ell^1(m_T)$. For every $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \in [0, \infty)$, we consider $x = \sum_{j \in M} x_j e_j \in \ell^1$. Noting that $x^r = \sum_{j \in M} x_j^r e_j \in \ell^1$, it follows that

$$\begin{aligned}
\left\| \sum_{j \in M} x_j^r \, T(e_j) \right\|_E^{\frac{1}{r}} &= \|T(x^r)\|_E^{\frac{1}{r}} = \|I_{m_T}(x^r)\|_E^{\frac{1}{r}} \leq \|x^r\|_{\ell^1(m_T)}^{\frac{1}{r}} = \|x\|_{\ell^r(m_T)} \\
&\leq C \|x\|_{\ell^1(m_T)} \leq 2C \sup_{A \in \mathcal{P}_F(\mathbb{N})} \|I_{m_T}(x\chi_A)\|_E,
\end{aligned}$$

where in the last inequality we have used (2.2). For every $A \in \mathcal{P}_F(\mathbb{N})$ we have that $x\chi_A = \sum_{j \in A \cap M} x_j e_j \in \ell^1$ and so $I_{m_T}(x\chi_A) = T(x\chi_A) =$

$\sum_{j\in A\cap M} x_jT(e_j)$. Then,

$$\Big\|\sum_{j\in M} x_j^r\, T(e_j)\Big\|_E^{\frac{1}{r}} \le 2C \sup_{A\in\mathcal{P}_F(\mathbb{N})} \Big\|\sum_{j\in A\cap M} x_jT(e_j)\Big\|_E$$
$$= 2C \sup_{N\subset M} \Big\|\sum_{j\in N} x_jT(e_j)\Big\|_E .$$

Conversely, suppose that T is r-power dominated and let $x=(x_n)\in \ell^1(m_T)$. Taking $y^k=\sum_{j=1}^k |x_j|^r e_j\in\ell^1$, for every $k>\tilde{k}$ and $A\in\mathcal{P}_F(\mathbb{N})$, we have that $(y^k-y^{\tilde{k}})\chi_A=\sum_{j\in A\cap\{\tilde{k}+1,\dots,k\}}|x_j|^r e_j$ and so

$$\begin{aligned}
\big\|T\big((y^k-y^{\tilde{k}})\chi_A\big)\big\|_E &= \Big\|\sum_{j\in A\cap\{\tilde{k}+1,\dots,k\}} |x_j|^r\, T(e_j)\Big\|_E \\
&\le C^r \sup_{N\subset A\cap\{\tilde{k}+1,\dots,k\}} \Big\|\sum_{j\in N}|x_j|T(e_j)\Big\|_E^r \\
&= C^r \sup_{N\subset A\cap\{\tilde{k}+1,\dots,k\}} \Big\| I_{m_T}\Big(\sum_{j\in N}|x_j|e_j\Big)\Big\|_E^r \\
&\le C^r \sup_{N\subset A\cap\{\tilde{k}+1,\dots,k\}} \Big\|\sum_{j\in N}|x_j|e_j\Big\|_{\ell^1(m_T)}^r \\
&\le C^r\big\|(y^k)^{\frac{1}{r}}-(y^{\tilde{k}})^{\frac{1}{r}}\big\|_{\ell^1(m_T)}^r.
\end{aligned}$$

For the last inequality note that $(y^k)^{\frac{1}{r}}=\sum_{j=1}^k|x_j|e_j$ and so

$$\sum_{j\in N}|x_j|e_j \le \sum_{j=\tilde{k}+1}^{k}|x_j|e_j=(y^k)^{\frac{1}{r}}-(y^{\tilde{k}})^{\frac{1}{r}}$$

for every $N\subset A\cap\{\tilde{k}+1,\dots,k\}$. Then, using (2.2), we have that

$$\begin{aligned}
\big\|y^k-y^{\tilde{k}}\big\|_{\ell^1(m_T)} &\le 2\sup_{A\in\mathcal{P}_F(\mathbb{N})}\big\|I_{m_T}\big((y^k-y^{\tilde{k}})\chi_A\big)\big\|_E \\
&= 2\sup_{A\in\mathcal{P}_F(\mathbb{N})}\big\|T\big((y^k-y^{\tilde{k}})\chi_A\big)\big\|_E \\
&\le 2C^r\big\|(y^k)^{\frac{1}{r}}-(y^{\tilde{k}})^{\frac{1}{r}}\big\|_{\ell^1(m_T)}^r \to 0
\end{aligned}$$

as $k,\tilde{k}\to\infty$ since $(y^k)^{\frac{1}{r}}\uparrow|x|$ pointwise and $\ell^1(m_T)$ is σ-order continuous. Hence, $y^k\to z$ in $\ell^1(m_T)$ for some $z\in\ell^1(m_T)$. In particular, $y^k\to z$ pointwise and so $|x|^r=z\in\ell^1(m_T)$ as $y^k\uparrow|x|^r$ pointwise. Therefore, $x\in\ell^r(m_T)$. □

Lemma 7.5. *Let $p > 1$. If T is $\frac{1}{p}$-power dominated then it is $\frac{1}{p}$th power factorable with a continuous extension.*

Proof. Let us use Proposition 7.2(d). Given $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j\in M} \subset [0,\infty)$, denoting by K the continuity constant of T, we have that

$$\begin{aligned}\left\|\sum_{j\in M} x_j T(e_j)\right\|_E &= \left\|\sum_{j\in M} (x_j^p)^{\frac{1}{p}} T(e_j)\right\|_E \le C^{\frac{1}{p}} \sup_{N\subset M} \left\|\sum_{j\in N} x_j^p T(e_j)\right\|_E^{\frac{1}{p}} \\ &= C^{\frac{1}{p}} \sup_{N\subset M} \left\|T\left(\sum_{j\in N} x_j^p e_j\right)\right\|_E^{\frac{1}{p}} \le C^{\frac{1}{p}} K^{\frac{1}{p}} \sup_{N\subset M} \left\|\sum_{j\in N} x_j^p e_j\right\|_{\ell^1}^{\frac{1}{p}} \\ &= C^{\frac{1}{p}} K^{\frac{1}{p}} \sup_{N\subset M} \left(\sum_{j\in N} x_j^p\right)^{\frac{1}{p}} \le C^{\frac{1}{p}} K^{\frac{1}{p}} \left(\sum_{j\in M} x_j^p\right)^{\frac{1}{p}}.\end{aligned}$$

□

As a consequence of Remark 7.3, Lemmas 7.4 and 7.5, we obtain the following conclusion.

Corollary 7.6. *For $p > 1$ we have that:*

(a) *If T is p-power dominated and $\frac{1}{p}$th power factorable with a continuous extension, then T factors optimally as*

$$\begin{tikzcd} \ell^1 \arrow[rr, "T"] \arrow[dr, dotted, "i"'] & & E \\ & \ell^{\frac{1}{p}}(m_T) \arrow[ur, dotted, "I_{m_T}"'] & \end{tikzcd}$$

with I_{m_T} being $\frac{1}{p}$th power factorable with a continuous extension.

(b) *If T is $\frac{1}{p}$-power dominated, then T factors optimally as*

$$\begin{tikzcd} \ell^1 \arrow[rr, "T"] \arrow[dr, dotted, "i"'] & & E \\ & \ell^{1}(m_T) \arrow[ur, dotted, "I_{m_T}"'] & \end{tikzcd}$$

with I_{m_T} being $\frac{1}{p}$th power factorable with a continuous extension.

Consider now the case when $E = \ell(c)$ is a B.f.s. related to c such that $\ell^1 \subset \ell(c) \subset \ell^0$. Then $\ell(c)$ is a *Köthe function space* in the sense of Lindenstrauss and Tzafriri, see [16, pp. 28–30]. For instance, $\ell(c)$ could be an ℓ^q space with $1 \le q \le \infty$, or a Lorentz sequence space $\ell^{q,r}$ with $1 \le r \le q \le \infty$ or an Orlicz sequence space ℓ_φ with φ being an Orlicz function.

Let us recall some facts about the Köthe dual of an space $\ell(c)$. Denote the *scalar product* of two sequences $x = (x_n), y = (y_n) \in \ell^0$ by

$$(x, y) = \sum x_n y_n$$

provided the sum exists. The *Köthe dual* of $\ell(c)$ is given by

$$\ell(c)' = \Big\{ y \in \ell^0 : \big(|x|, |y|\big) < \infty \quad \text{for all } x \in \ell(c) \Big\}.$$

Note that $\chi_A \in \ell(c)'$ for all $A \in \mathcal{P}_F(\mathbb{N})$. The space $\ell(c)'$ endowed with the norm

$$\|y\|_{\ell(c)'} = \sup_{x \in B_{\ell(c)}} \big(|x|, |y|\big)$$

is a B.f.s. in the sense of Lindenstrauss and Tzafriri. The map $j \colon \ell(c)' \to \ell(c)^*$ defined by $\langle j(y), x\rangle = \big(x, y\big)$ for all $y \in \ell(c)'$ and $x \in \ell(c)$, is a linear isometry. In particular, convergence in norm of $\ell(c)$ implies pointwise convergence, as $e_n \in \ell(c)'$ for all n. Note that $\ell(c) \subset \ell(c)''$. The equality $\ell(c) = \ell(c)''$ holds with equal norms if and only if $\ell(c)$ has the *Fatou property*, that is, if $(x^k) \subset \ell(c)$ is such that $0 \le x^k \uparrow x$ pointwise and $\sup \|x^k\|_{\ell(c)} < \infty$ then $x \in \ell(c)$ and $\|x^k\|_{\ell(c)} \uparrow \|x\|_{\ell(c)}$.

Let $M = (a_{ij})$ be an infinite matrix of real numbers and denote by C_j the jth column of M. Assume $C_j \neq 0$ for all j. Note that

$$Mx = \left(\sum_j a_{ij} x_j \right)_i$$

for any $x \in \ell^0$ for which it is meaningful to do so.

Proposition 7.7. *Suppose that $\ell(c)$ has the Fatou property. Then, the following statements are equivalent:*

(a) *M defines a continuous linear operator $M \colon \ell^1 \to \ell(c)$.*
(b) *$C_j \in \ell(c)$ for all j and* $\sup_j \|C_j\|_{\ell(c)} < \infty$.

Proof. (a) $\Rightarrow$ (b) Let $K > 0$ be such that $\|Mx\|_{\ell(c)} \le K\|x\|_{\ell^1}$ for all $x \in \ell^1$. For every j we have that $C_j = Me_j \in \ell(c)$. Moreover,

$$\sup_j \|C_j\|_{\ell(c)} = \sup_j \|Me_j\|_{\ell(c)} \le K \sup_j \|e_j\|_{\ell^1} = K.$$

(b) $\Rightarrow$ (c) Since $\ell(c)$ has the Fatou property then $\ell(c) = \ell(c)''$ with equal norms. Let $x \in \ell^1$. First note that for every i we have that

$$\begin{aligned}\sum_j |a_{ij} x_j| = \sum_j \big(|C_j|, e_i\big)|x_j| &\le \sum_j \|C_j\|_{\ell(c)} \|e_i\|_{\ell(c)'} |x_j| \\ &\le \|e_i\|_{\ell(c)'} \|x\|_{\ell^1} \sup_j \|C_j\|_{\ell(c)}\end{aligned}$$

and so $Mx \in \ell^0$. Given $y \in \ell(c)'$ it follows that

$$\begin{aligned}\big(|y|, |Mx|\big) &= \sum_i |y_i| \left| \sum_j a_{ij} x_j \right| \le \sum_i \sum_j |a_{ij} x_j y_i| = \sum_j |x_j| \sum_i |a_{ij} y_i| \\ &= \sum_j |x_j| \big(|C_j|, |y|\big) \le \sum_j |x_j| \, \|C_j\|_{\ell(c)} \|y\|_{\ell(c)'} \\ &\le \|y\|_{\ell(c)'} \|x\|_{\ell^1} \sup_j \|C_j\|_{\ell(c)}.\end{aligned}$$

Then $Mx \in \ell(c)'' = \ell(c)$ and

$$\|Mx\|_{\ell(c)} = \sup_{y \in B_{\ell(c)'}} \big(|y|, |Mx|\big) \leq \|x\|_{\ell^1} \sup_j \|C_j\|_{\ell(c)}.$$

$\square$

In what follows, assume that $\ell(c)$ has the Fatou property, $C_j \in \ell(c)$ for all j and $\sup_j \|C_j\|_{\ell(c)} < \infty$. Then, M defines a continuous linear operator $M\colon \ell^1 \to \ell(c)$ and so, by Remark 7.1 we have that M can be extended as

$$\begin{tikzcd}
\ell^1 \arrow[rr, "M"] \arrow[dr, dotted, "i"'] & & \ell(c) \\
& \ell^1(m_M) \arrow[ur, dotted, "I_{m_M}"'] &
\end{tikzcd}$$

and $\ell^1(m_M)$ is the largest σ-order continuous quasi-B.f.s. to which M can be extended as a continuous operator.

Remark 7.8. For every $x \in \ell^1(m_M)$ it follows that $I_{m_M}(x) = Mx$ and so M defines a continuous linear operator $M\colon \ell^1(m_M) \to \ell(c)$. Indeed, take $0 \leq x = (x_n) \in \ell^1(m_M)$ and $x^k = \sum_{j=1}^k x_j e_j \in \ell^1$. Since $x^k \uparrow x$ pointwise and $\ell^1(m_M)$ is σ-order continuous it follows that $x^k \to x$ in $\ell^1(m_M)$. Then, since $M = I_{m_M}$ on ℓ^1, we have that $Mx^k = I_{m_M}(x^k) \to I_{m_M}(x)$ in $\ell(c)$ and so pointwise. Hence, the ith coordinate $\sum_{j=1}^k a_{ij}x_j$ of Mx^k converges to the ith coordinate of $I_{m_M}(x)$ and thus $Mx = I_{m_M}(x) \in \ell(c)$. For a general $x \in \ell^1(m_M)$, we only have to take the positive and negative parts of x.

From Proposition 7.2 applied to $M\colon \ell^1 \to \ell(c)$ and Remark 7.8 we obtain the following conclusion.

Proposition 7.9. *The following statements are equivalent:*

(a) *M defines a continuous linear operator $M\colon \ell^p \to \ell(c)$.*
(b) *M is $\frac{1}{p}$th power factorable with a continuous extension.*
(c) *$\ell^p \subset \ell^1(m_M)$.*
(d) *$\ell^1 \subset \ell^{\frac{1}{p}}(m_M) \cap \ell^1(m_M)$.*
(e) *There exists $C > 0$ such that*

$$\left\| \sum_{j \in M} x_j C_j \right\|_{\ell(c)} \leq C \left(\sum_{j \in M} x_j^p \right)^{\frac{1}{p}}$$

for all $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \subset [0, \infty)$.

Proof. The equivalence among statements (b), (c), (d), (e) is given by Proposition 7.2. The statement (a) implies (b) obviously. From Remark 7.8 we have that M defines a continuous linear operator $M\colon \ell^1(m_M) \to \ell(c)$, so (c) implies (a). $\square$

Let us give two conditions guaranteeing that M defines a continuous linear operator $M\colon \ell^p \to \ell(c)$:

(I) If p' is the conjugate exponent of p and $\sum \|C_j\|_{\ell(c)}^{p'} < \infty$, then (e) in Proposition 7.9 holds. Indeed, for every $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j\in M} \subset [0,\infty)$ we have that

$$\left\|\sum_{j\in M} x_j C_j\right\|_{\ell(c)} \leq \sum_{j\in M} x_j \|C_j\|_{\ell(c)} \leq \left(\sum_{j\in M} x_j^p\right)^{\frac{1}{p}} \left(\sum_{j\in M} \|C_j\|_{\ell(c)}^{p'}\right)^{\frac{1}{p'}}$$
$$\leq \left(\sum \|C_j\|_{\ell(c)}^{p'}\right)^{\frac{1}{p'}} \left(\sum_{j\in M} x_j^p\right)^{\frac{1}{p}}.$$

(II) If M is $\frac{1}{p}$-power dominated, that is, there exists $C > 0$ such that

$$\left\|\sum_{j\in M} x_j^{\frac{1}{p}} C_j\right\|_{\ell(c)}^p \leq C \sup_{N\subset M} \left\|\sum_{j\in N} x_j C_j\right\|_{\ell(c)}$$

for every $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j\in M} \in [0,\infty)$, then (b) in Proposition 7.9 holds by Lemma 7.5.

For instance, in the case when $\ell(c) = \ell^q$ and $a_{ij} \geq 0$ for all i, j, condition (II) is satisfied if $F_i \in \ell^1$ for all i and $\sum \|F_i\|_{\ell^1}^q < \infty$, where F_i denotes the ith file of M. Indeed, for every $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j\in M} \in [0,\infty)$, applying Hölder's inequality twice for p and its conjugate exponent p', we have that

$$\left\|\sum_{j\in M} x_j^{\frac{1}{p}} C_j\right\|_{\ell^q}^p = \left(\sum_i \left(\sum_{j\in M} x_j^{\frac{1}{p}} a_{ij}\right)^q\right)^{\frac{p}{q}} = \left(\sum_i \left(\sum_{j\in M} x_j^{\frac{1}{p}} a_{ij}^{\frac{1}{p}} a_{ij}^{1-\frac{1}{p}}\right)^q\right)^{\frac{p}{q}}$$
$$\leq \left(\sum_i \left(\sum_{j\in M} x_j a_{ij}\right)^{\frac{q}{p}} \left(\sum_{j\in M} a_{ij}\right)^{\frac{q}{p'}}\right)^{\frac{p}{q}}$$
$$\leq \left(\sum_i \left(\sum_{j\in M} x_j a_{ij}\right)^q\right)^{\frac{1}{q}} \cdot \left(\sum_i \left(\sum_{j\in M} a_{ij}\right)^q\right)^{\frac{p}{qp'}}$$
$$\leq \left\|\sum_{j\in M} x_j C_j\right\|_{\ell^q} \left(\sum_i \|F_i\|_{\ell^1}^q\right)^{\frac{p}{qp'}}.$$

Note that $\sup_{N\subset M} \|\sum_{j\in N} x_j C_j\|_{\ell^q} = \|\sum_{j\in M} x_j C_j\|_{\ell^q}$ as $a_{ij} \geq 0$ for all i, j.

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