Algebra structure for L^p of a vector measure

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ABSTRACT

In this paper, we present new results about the space $L^p(\nu)$ for ν being a vector measure defined in the Borel σ-algebra of a compact abelian group G and satisfying certain property concerning translation of simple functions. Namely, we show that $L^p(\nu)$ is a translation invariant space which can be endowed with an algebra structure via usual convolution product. We apply these results to the optimal domain of the Fourier transform and convolution operators.

Keywords:
Algebra group
Vector measure
Space of p-integrable functions
Optimal domain

1. Introduction

Given a compact abelian group G, the space $L^p(G)$, where $1 \leqslant p < \infty$ and the integration is with respect to the Haar measure m of G, is homogeneous, an $L^1(G)$-module and so a commutative Banach algebra for the convolution product. These are classical topics in harmonic analysis, see [12]. Our aim is to extend these facts to the setting of vector measures.

Vector measures turn out to be a powerful tool for the study of operators $T : E \to X$ between function spaces, for an overview see [2], [11, Chapter 4]. In fact, the *optimal domain* of T, that is the larger Banach function space to which T can be extended still with values in X, can be described as the space $L^1(\nu)$ of integrable functions with respect to the vector measure ν canonically associated to T via $\nu(A) = T(\chi_A)$. In this way, we obtain important information about the optimal domain of T, in particular, it is an order continuous Banach lattice with weak order unit. Now, using the results of this paper, we can obtain more useful information as it is homogeneous and can be endowed with an algebra structure. Note that $L^1(\nu)$ is not in general a classical Lebesgue space, indeed every order continuous Banach lattice with weak unit can be described as a space $L^1(\nu)$ for some vector measure ν, see [1, Theorem 8].

So, we consider a vector measure $\nu : \mathcal{B}(G) \to X$, where $\mathcal{B}(G)$ is the Borel σ-algebra of G and X is a Banach space. Of course, as in the case of $L^p(G)$ in which the properties of the Haar measure m are crucial, ν will have to satisfy some special conditions. Namely, ν is absolutely continuous with respect to m and the norms of the integrals with respect to ν of a simple function φ and any of its translations $\tau_a\varphi$ coincide. We show that the space $L^p(\nu)$ of functions whose pth power is integrable with respect to ν, is homogeneous (Theorem 3.8) and an $L^1(G)$-module (Theorem 4.6). In particular, $L^p(\nu)$ is closed for the convolution product and so can be endowed with a Banach algebra structure. Similar results are presented for the space $L^p_w(\nu)$ of functions whose pth power is weakly integrable with respect to ν. The proof of these facts relies on a result, interesting by itself and based on the Markoff–Kakutani fixed point theorem, establishing that actually the Haar

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measure m is a Rybakov control measure with a density h_0 belonging to the Köthe dual of $L^1(\nu)$, i.e. $dm = h_0\, d|x_0^*\nu|$ for some x_0^* in the topological dual space of X (Theorem 4.1).

In Section 5 we apply our results to the optimal domain of some classical operators. The first one is the Fourier transform $\mathcal{F}: L^p(G) \to \ell^q(\Gamma)$, where $1 \leqslant p \leqslant 2$, $1/p + 1/q = 1$ and Γ is the dual group of G. This operator has been recently investigated in [8] for the case $G = \mathbb{T}$ and in [11, Chapter 7.1] for the general case, where different descriptions of its optimal domain (i.e. the optimal domain of the Hausdorff–Young inequality) are given, providing an exact answer to a question posed by R.E. Edwards in [4] some forty years ago. Also, the authors establish another important facts for classical harmonic analysis which can be now deduced from our general setting. The second one is any linear continuous operator $T: E \to E$ commuting with the translation operator (i.e. $\tau_a T = T\tau_a$ for all $a \in G$), where E is a translation invariant, order continuous, Banach function space. In particular, we consider $E = L^p(G)$ for $1 \leqslant p < \infty$ and $Tf = f * \mu$ for μ any complex regular measure defined on $\mathcal{B}(G)$. This last example has been studied in [9,10] and [11, Chapter 7.3].

2. Preliminaries

Let G be a *compact abelian group*, that is, an abelian group with a structure of compact Hausdorff topological space such that the group operations are continuous. Denote by $\mathcal{B}(G)$ the Borel σ-algebra of G. There exists a unique translation invariant regular measure $m: \mathcal{B}(G) \to [0, \infty)$ with $m(G) = 1$, called the *Haar measure* of G. Denote by $L^0(G)$ the space of all complex measurable functions defined on G, where functions which are equal m-a.e. are identified. As usual, we denote by $L^p(G)$ $(1 \leqslant p < \infty)$ the space of functions in $L^0(G)$ whose pth power is m-integrable and by $L^\infty(G)$ the space of m-a.e. bounded functions, with standard norms. Note that $L^\infty(G) \subset L^p(G) \subset L^1(G)$ continuously with continuity constant 1. For each $a \in G$, the *translation operator* τ_a is given by $\tau_a f(s) = f(s - a)$ for $f \in L^0(G)$ and $s \in G$. A Banach space $Y \subset L^1(G)$ is *homogeneous* if

(i) $\tau_a f \in Y$ for every $f \in Y$ and $a \in G$ (i.e. Y is *translation invariant*) and $\|\tau_a f\|_Y = \|f\|_Y$,
(ii) for each $f \in Y$, the map $a \mapsto \tau_a f$ is continuous from G into Y.

The space Y is an $L^1(G)$*-module* if for every $f \in L^1(G)$ and $g \in Y$ we have that $f * g \in Y$ with $\|f * g\|_Y \leqslant \|f\|_{L^1(G)}\|g\|_Y$, where $f * g$ denotes the *convolution product* of f and g, i.e.

$$f * g(t) = \int_G f(s)g(t - s)\, dm(s), \quad \text{for all } t \in G.$$

Let $\nu: \mathcal{B}(G) \to X$ be a *vector measure*, that is, a countably additive set function, where X is a complex (or real) Banach space. A set $A \in \mathcal{B}(G)$ is ν*-null* if $\nu(B) = 0$ for every $B \in \mathcal{B}(G)$ with $B \subset A$. The *semivariation* of ν is the set function $\|\nu\|: \mathcal{B}(G) \to [0, \infty)$ defined by

$$\|\nu\|(A) = \sup_{x^* \in B_{X^*}} \left|x^*\nu\right|(A), \quad \text{for all } A \in \mathcal{B}(G),$$

where B_{X^*} is the closed unit ball of the topological dual space X^* of X and $|x^*\nu|$ is the variation of the complex measure $x^*\nu$. A set $A \in \mathcal{B}(G)$ is ν-null if and only if $\|\nu\|(A) = 0$. The vector measure ν is *absolutely continuous* with respect to a positive measure λ on $\mathcal{B}(G)$ (written as $\nu \ll \lambda$) if $\|\nu\|(A) \to 0$ as $\lambda(A) \to 0$. In the case when λ is a finite measure, $\nu \ll \lambda$ if and only if every λ-null set is ν-null. A *Rybakov control measure* for ν is a finite positive measure $\mu = |x_0^*\nu|$ for some $x_0^* \in X^*$ such that $\nu \ll \mu$. Note that such a measure always exists (see [3, Theorem IX.2.2]) and has the same null sets as ν, since $\mu(A) \leqslant \|x_0^*\|_{X^*}\|\nu\|(A)$ for all $A \in \mathcal{B}(G)$.

A function $f \in L^0(G)$ is *integrable with respect to* ν if it satisfies:

(i) $\int_G |f|\, d|x^*\nu| < \infty$ for all $x^* \in X^*$.
(ii) For each $A \in \mathcal{B}(G)$, there exists $x_A \in X$ such that

$$x^*(x_A) = \int_A f\, dx^*\nu, \quad \text{for all } x^* \in X^*.$$

The element x_A will be written as $\int_A f\, d\nu$. Denote by $L^1(\nu)$ the space of integrable functions with respect to ν and by $L^1_w(\nu)$ the space of functions satisfying only condition (i), where functions which are equal ν-a.e. (i.e. except on a ν-null set) are identified. Note that if μ is a Rybakov control measure for ν, the ν-a.e. and μ-a.e. classes of functions coincide. The spaces $L^1(\nu)$ and $L^1_w(\nu)$ are Banach spaces endowed with the norm

$$\|f\|_\nu = \sup_{x^* \in B_{X^*}} \int_\Omega |f|\, d\left|x^*\nu\right|, \quad \text{for all } f \in L^1_w(\nu).$$

The space of all simple functions is dense in $L^1(\nu)$ and for a simple function $\varphi = \sum \alpha_j \chi_{A_j}$, we have that $\int_A \varphi\, d\nu = \sum \alpha_j \nu(A_j \cap A)$. Note that $L^1(\nu)$ is a closed subspace of $L^1_w(\nu)$. The *integration operator* $I_\nu : L^1(\nu) \to X$ defined by $I_\nu(f) = \int_G f\, d\nu$ for all $f \in L^1(\nu)$, is linear and continuous with $\|I_\nu\| \leqslant 1$.

Let μ be a Rybakov control measure for ν. Then $L^1(\nu)$ and $L^1_w(\nu)$ are *Banach function spaces* related to μ, in the sense of Lindenstrauss and Tzafriri [7, Definition 1.b.17], i.e. a Banach space E with $L^\infty(\mu) \subset E \subset L^1(\mu)$ and satisfying that if $f \in E$, $g \in L^0(\mu)$ and $|g| \leqslant |f|$ μ-a.e. then $g \in E$ and $\|g\|_E \leqslant \|f\|_E$. Moreover, the space $L^1(\nu)$ is *order continuous*, i.e. if $f, f_n \in L^1(\nu)$ are such that $0 \leqslant f_n \uparrow f$ ν-a.e. then $f_n \to f$ in $L^1(\nu)$. We will consider the *Köthe dual space* of $L^1(\nu)$, i.e.

$$L^1(\nu)' = \{g \in L^0(\mu):\ fg \in L^1(\mu),\ \text{for all } f \in L^1(\nu)\},$$

which is a Banach function space related to μ, endowed with the norm

$$\|g\|_{L^1(\nu)'} = \sup_{f \in B_{L^1(\nu)}} \int_G |fg|\, d\mu.$$

Since $L^1(\nu)$ is order continuous, then $L^1(\nu)^*$ can be identified with $L^1(\nu)'$, more precisely, each $x^* \in L^1(\nu)^*$ is identified with a function $g \in L^1(\nu)'$ via $x^*(f) = \int_G fg\, d\mu$ for all $f \in L^1(\nu)$ (see [7, p. 29]).

For $1 \leqslant p < \infty$, let $L^p_w(\nu)$ be the *p-power* of $L^1_w(\nu)$, that is, the space of functions $f \in L^0(\mu)$ such that $|f|^p \in L^1_w(\nu)$. Similarly, $L^p(\nu)$ will denote the p-power of $L^1(\nu)$. Both spaces are Banach function spaces with norm

$$\|f\|_{\nu,p} := \big\||f|^p\big\|_\nu^{1/p}, \quad \text{for all } f \in L^p_w(\nu),$$

and $L^p(\nu) \subset L^p_w(\nu) \subset L^1(\nu)$, see [5,13]. Note that for $p = \infty$, the space $L^\infty(\nu)$ of functions which are bounded ν-a.e. coincide with $L^\infty(\mu)$, so this case will be not considered.

For a complete overview about integration with respect to vector measures we refer to [2], [11, Chapter 3] and the references therein.

3. Homogeneity for L^p of a vector measure

Let G be a compact abelian group, X a Banach space and $\nu : \mathcal{B}(G) \to X$ a non-null vector measure.

Definition 3.1. We say that ν is *norm integral translation invariant* if

$$\big\|I_\nu(\tau_a \varphi)\big\|_X \leqslant \big\|I_\nu(\varphi)\big\|_X, \tag{1}$$

for every simple function φ and $a \in G$.

Remark 3.2. For every $a \in G$ and every simple function $\varphi = \sum \alpha_j \chi_{A_j}$ we have that $\tau_a \varphi = \sum \alpha_j \chi_{A_j + a}$ is also a simple function and so both are integrable with respect to ν. Actually, the inequality in (1) is an equality, since $\varphi = \tau_{-a}(\tau_a \varphi)$. In particular, taking $\varphi = \chi_A$ with $A \in \mathcal{B}(G)$,

$$\big\|\nu(A + a)\big\|_X = \big\|I_\nu(\tau_a \chi_A)\big\|_X = \big\|I_\nu(\chi_A)\big\|_X = \big\|\nu(A)\big\|_X.$$

Note that ν *translation invariant* (i.e. $\nu(A + a) = \nu(A)$ for all $A \in \mathcal{B}(G)$ and $a \in G$) implies ν norm integral translation invariant.

From now and on, the vector measure $\nu : \mathcal{B}(G) \to X$ will always be norm integral translation invariant. Let X_ν denote the subspace of X given by the image of all simple functions by I_ν. For every $a \in G$, we consider the correspondence $S_a : X_\nu \to X_\nu$ defined by

$$S_a\big(I_\nu(\varphi)\big) := I_\nu(\tau_a \varphi), \quad \text{for all simple functions } \varphi. \tag{2}$$

Then, (1) is equivalent to S_a being a well-defined linear continuous map with $\|S_a\| \leqslant 1$. Note that

$$S_a\big(\nu(A)\big) = \nu(A + a), \quad \text{for all } A \in \mathcal{B}(G) \tag{3}$$

(take $\varphi = \chi_A$ in (2)) and so $S_a \circ \nu : \mathcal{B}(G) \to X$ is a vector measure.

Lemma 3.3. *The following claims hold*:

(i) $\{S_a\}_{a \in G}$ *is a commutative group under composition of operators.*
(ii) S_a *is an isometric isomorphism for each* $a \in G$, *and so* $\|S_a\| = 1$.

Proof. Given $a, b \in G$, for every simple function φ we have that

$$S_a \circ S_b\big(I_\nu(\varphi)\big) = S_a\big(I_\nu(\tau_b\varphi)\big) = I_\nu\big(\tau_a(\tau_b\varphi)\big) = I_\nu(\tau_{a+b}\varphi) = S_{a+b}\big(I_\nu(\varphi)\big).$$

Then $S_a \circ S_b = S_{a+b}$ and so the family $\{S_a\}_{a\in G}$ is commutative and closed under composition of operators. Note that $S_a \circ S_{-a} = S_{-a} \circ S_a = S_0$ where S_0 is the identity map. Thus, (i) holds. Moreover, S_a is bijective and, since $\|S_a\| \leqslant 1$ for arbitrary $a \in G$, for every $x \in X_\nu$ it follows that

$$\big\|S_a(x)\big\|_X \leqslant \|x\|_X = \big\|S_{-a}\big(S_a(x)\big)\big\|_X \leqslant \big\|S_a(x)\big\|_X.$$

Then, S_a is an isometry and so (ii) holds. □

The group $\{S_a\}_{a\in G}$ given in Lemma 3.3 is the tool for proving the following technical result, which will be the key for showing that the spaces $L^1_w(\nu)$ and $L^1(\nu)$ are translation invariant.

Lemma 3.4. *For every $x^* \in X^*$ and $a \in G$, there exists $x^*_a \in X^*$ such that $\|x^*_a\|_{X^*} \leqslant \|x^*\|_{X^*}$ and*

$$x^*\nu(A + a) = x^*_a\nu(A), \quad \textit{for all } A \in \mathcal{B}(G). \tag{4}$$

Proof. Fix $x^* \in X^*$ and $a \in G$. Let S_a be the operator given in (2) and take $y^*_a := x^* \circ S_a \in X^*_\nu$ for which $\|y^*_a\|_{X^*_\nu} \leqslant \|x^*\|_{X^*}$. Then, from the Hahn Banach extension theorem, there exists $x^*_a \in X^*$ such that $x^*_a = y^*_a$ on X_ν and $\|x^*_a\|_{X^*} = \|y^*_a\|_{X^*_\nu}$. Moreover, from (3) it follows

$$x^*_a\nu(A) = y^*_a\nu(A) = \big(x^* \circ S_a\big)\nu(A) = x^*\nu(A + a)$$

for all $A \in \mathcal{B}(G)$. □

Proposition 3.5. *The spaces $L^1(\nu)$, $L^1_w(\nu)$ are translation invariant with*

(i) $\|\tau_a f\|_\nu = \|f\|_\nu$ *for every* $f \in L^1_w(\nu)$ *and* $a \in G$,
(ii) $\|I_\nu(\tau_a f)\|_X = \|I_\nu(f)\|_X$ *for every* $f \in L^1(\nu)$ *and* $a \in G$.

Proof. Let $a \in G$, $x^* \in X^*$ and consider the element $x^*_a \in X^*$ given in Lemma 3.4. It is direct to check that (4) implies $|x^*\nu|(A + a) = |x^*_a\nu|(A)$ for all $A \in \mathcal{B}(G)$. Then, for every simple function $\varphi = \sum \alpha_j \chi_{A_j}$ for which $\tau_a\varphi = \sum \alpha_j \chi_{A_j + a}$, it follows that

$$\int_G \tau_a\varphi\, d\big|x^*\nu\big| = \sum \alpha_j \big|x^*\nu\big|(A_j + a) = \sum \alpha_j \big|x^*_a\nu\big|(A_j) = \int_G \varphi\, d\big|x^*_a\nu\big|.$$

Given $f \in L^0(G)$ and a sequence (φ_n) of simple functions such that $0 \leqslant \varphi_n \uparrow |f|$ pointwise, since $0 \leqslant \tau_a\varphi_n \uparrow |\tau_a f|$ pointwise, applying the monotone convergence theorem we have that

$$\int_G |\tau_a f|\, d\big|x^*\nu\big| = \lim_n \int_G \tau_a\varphi_n\, d\big|x^*\nu\big| = \lim_n \int_G \varphi_n\, d\big|x^*_a\nu\big| = \int_G |f|\, d\big|x^*_a\nu\big|.$$

So, if $f \in L^1_w(\nu)$ then $\tau_a f \in L^1_w(\nu)$ and, since $\|x^*_a\|_{X^*} \leqslant \|x^*\|_{X^*}$, we have that $\|\tau_a f\|_\nu \leqslant \|f\|_\nu$. This is for arbitrary $a \in G$, so actually we have that $\|\tau_a f\|_\nu = \|f\|_\nu$, since $f = \tau_{-a}(\tau_a f)$.

For $f \in L^1(\nu)$, taking a sequence (φ_n) of simple functions converging to f in $L^1(\nu)$, we have that $\tau_a f \in L^1_w(\nu)$ and

$$\|\tau_a f - \tau_a\varphi_n\|_\nu = \big\|\tau_a(f - \varphi_n)\big\|_\nu = \|f - \varphi_n\|_\nu \to 0.$$

Since $L^1(\nu)$ is the closure of the simple functions in $L^1_w(\nu)$ and $(\tau_a\varphi_n)$ is a sequence of simple functions, then $\tau_a f \in L^1(\nu)$. Moreover, by the continuity of the integration operator I_ν and by (1), we have that

$$\big\|I_\nu(\tau_a f)\big\|_X = \lim_n \big\|I_\nu(\tau_a\varphi_n)\big\|_X = \lim_n \big\|I_\nu(\varphi_n)\big\|_X = \big\|I_\nu(f)\big\|_X. \qquad \square$$

Remark 3.6. Applying Proposition 3.5(i) to $f = \chi_A$ with $A \in \mathcal{B}(G)$ and noting that $\|\nu\|(A) = \|\chi_A\|_\nu$ for arbitrary A, it follows that $\|\nu\|(A + a) = \|\nu\|(A)$ for all $a \in G$. That is, the semivariation of ν is translation invariant. As a consequence of Proposition 3.5(ii), the operator S_a given in (2) can be extended to $I_\nu(L^1(\nu))$ in the way $S_a(I_\nu(f)) := I_\nu(\tau_a f)$ for all $f \in L^1(\nu)$ and the extension is an isometric isomorphism from $I_\nu(L^1(\nu))$ into itself.

Consider now the spaces $L^p(\nu)$ and $L^p_w(\nu)$ with $1 \leqslant p < \infty$. Since they are the p-powers of $L^1(\nu)$ and $L^1_w(\nu)$ respectively, which are translation invariant and satisfy Proposition 3.5(i), the following result holds. Note that $\tau_a(|f|^p) = |\tau_a f|^p$ for all $a \in G$ and $f \in L^0(G)$.

Corollary 3.7. *The spaces $L^p(\nu)$ and $L^p_w(\nu)$ are translation invariant with $\|\tau_a f\|_{\nu,p} = \|f\|_{\nu,p}$ for all $a \in G$ and $f \in L^p_w(\nu)$.*

Due to the density of the simple functions in $L^p(\nu)$, we get the next theorem whose proof is quite similar to that in [12, Theorem 1.1.5] for the spaces $L^p(G)$. We include it here for completeness.

Theorem 3.8. *Assume that $\nu \ll m$. For each fixed $f \in L^p(\nu)$, the map $a \mapsto \tau_a f$ is uniformly continuous from G into $L^p(\nu)$. Consequently, $L^p(\nu)$ is homogeneous.*

Proof. Denote by $C(G)$ the space of all continuous complex functions on G. Let φ be a simple function and $\varepsilon > 0$. Since G is compact and m is finite and regular, for every $\delta > 0$, Lusin's theorem gives a function $g \in C(G)$ such that $g = \varphi$ except on a set S with $m(S) < \delta$ and $\|g\|_\infty \leqslant \|\varphi\|_\infty$. Then,

$$\|\varphi - g\|_{\nu,p} \leqslant \|\varphi - g\|_\infty \|\chi_S\|_{\nu,p} \leqslant 2\|\varphi\|_\infty \|\nu\|(S)^{1/p} < \varepsilon$$

taking δ small enough, since $\nu \ll m$. From this it follows that $C(G)$ is dense in $L^p(\nu)$, since the space of simple functions is so. Given $f \in L^p(\nu)$ and $\varepsilon > 0$, we take $g \in C(G)$ such that $\|f - g\|_{\nu,p} < \varepsilon/3$. Since g is uniformly continuous in G, there exists a neighborhood V of 0 such that $|g(a) - g(b)| < \varepsilon(3\|\nu\|(G)^{1/p})^{-1}$ for all $a, b \in G$ with $a - b \in V$. Then,

$$\begin{aligned}\|\tau_a f - \tau_b f\|_{\nu,p} &\leqslant \big\|\tau_a(f-g)\big\|_{\nu,p} + \|\tau_a g - \tau_b g\|_{\nu,p} + \big\|\tau_b(g-f)\big\|_{\nu,p} \\ &\leqslant 2\|f-g\|_{\nu,p} + \|\tau_a g - \tau_b g\|_\infty \|\nu\|(G)^{1/p} < \varepsilon\end{aligned}$$

for all $a, b \in G$ with $a - b \in V$. From this and Corollary 3.7 it follows that $L^p(\nu)$ is homogeneous. □

Since G is compact, the trigonometric polynomials on G (i.e. $\sum_{j=1}^n \alpha_j \gamma_j$ with $\gamma_j \in \Gamma$ and $\alpha_j \in \mathbb{C}$) are dense in $C(G)$, see [12, p. 24]. In the case when $\nu \ll m$, the space $C(G)$ is dense in $L^p(\nu)$, as proved in Proposition 3.5.

Corollary 3.9. *If $\nu \ll m$, then the trigonometric polynomials on G are dense in $L^p(\nu)$.*

Remark 3.10. The conclusions of Theorem 3.8 and Corollary 3.9 hold if there exists some positive finite regular measure λ on $\mathcal{B}(G)$ such that $\nu \ll \lambda$.

4. Convolution product in $L^p(\nu)$

Let $\nu : \mathcal{B}(G) \to X$ be a norm integral translation invariant vector measure such that $\nu \ll m$. We begin this section by showing that the Haar measure m is just a Rybakov control measure for ν with a certain density. This fact will be the key for proving that $L^1_w(\nu)$ is embedded in $L^1(G)$, which will allow us to consider the convolution product of functions in $L^p(\nu)$ and $L^p_w(\nu)$.

Theorem 4.1. *Let μ be a Rybakov control measure for ν. Then, there exists $0 \leqslant h_0 \in L^1(\nu)'$ with $\|h_0\|_{L^1(\nu)'} = \|\nu\|(G)^{-1}$ such that $m(A) = \int_A h_0\, d\mu$ for all $A \in \mathcal{B}(G)$.*

Proof. Consider the set

$$K = \big\{\xi^* \in B_{L^1(\nu)^*}\colon \xi^*(\chi_G) = \|\nu\|(G)\big\}.$$

Note that $0 \neq \chi_G \in L^1(\nu)$, since $\|\chi_G\|_\nu = \|\nu\|(G) \neq 0$ as ν is considered non-null. Then, from the Hahn–Banach theorem, there exists $\xi_0^* \in L^1(\nu)^*$ with $\|\xi_0^*\|_{L^1(\nu)^*} = 1$ such that $\xi_0^*(\chi_G) = \|\chi_G\|_\nu = \|\nu\|(G)$. That is, K is non-empty. It can be directly checked that K is convex. Consider the weak* topology in $L^1(\nu)^*$, for which $B_{L^1(\nu)^*}$ is compact. Since the functional $f : L^1(\nu)^* \to \mathbb{C}$, given by $f(\xi^*) = \xi^*(\chi_G)$ for all $\xi \in L^1(\nu)^*$, is continuous and $K = f^{-1}(\{\|\nu\|(G)\}) \cap B_{L^1(\nu)^*}$, we have that K is closed inside of a compact set. So, K is compact.

For every $a \in G$, we consider the linear operator $T_a : L^1(\nu)^* \to L^1(\nu)^*$ defined by $T_a(\xi^*) = \xi^* \circ \tau_a$ for all $\xi^* \in L^1(\nu)^*$. Since by Proposition 3.5, the translation operator $\tau_a : L^1(\nu) \to L^1(\nu)$ is an isometric isomorphism and T_a is just the transposed operator of τ_a, then T_a is weak*–weak* continuous. Moreover, the family of operators $\{T_a\}_{a\in G}$ is commuting as $\{\tau_a\}_{a\in G}$ is so. For every $\xi^* \in K$, we have that

$$\big\|T_a(\xi^*)\big\|_{L^1(\nu)^*} \leqslant \|T_a\| \big\|\xi^*\big\|_{L^1(\nu)^*} = \|\tau_a\| \big\|\xi^*\big\|_{L^1(\nu)^*} = \big\|\xi^*\big\|_{L^1(\nu)^*} \leqslant 1.$$

Moreover, noting that $\tau_a \chi_G = \chi_G$ for all $a \in G$, we have that

$$T_a(\xi^*)(\chi_G) = \xi^*(\tau_a \chi_G) = \xi^*(\chi_G) = \|\nu\|(G).$$

So, $T_a(\xi^*) \in K$. Therefore, $\{T_a\}_{a\in G}$ is a commuting family of weak*–weak* continuous affine maps from K into K, where K is non-empty, convex and weak*-compact. This is the hypothesis of the Markoff–Kakutani theorem (see for instance [6, Theorem 3.2]) which ensures the existence of a common fixed point for the family $\{T_a\}_{a\in G}$. Namely, there exists $\xi_0^* \in K$ such that $T_a(\xi_0^*) = \xi_0^*$ for every $a \in G$. Note that $\|\xi_0^*\|_{L^1(\nu)^*} = 1$, since

$$1 = \frac{1}{\|\nu\|(G)} \xi_0^*(\chi_G) = \frac{1}{\|\chi_G\|_\nu} \xi_0^*(\chi_G) \leqslant \|\xi_0^*\|_{L^1(\nu)^*} \leqslant 1.$$

In particular, $\xi_0^* \neq 0$. By the identification of $L^1(\nu)^*$ with $L^1(\nu)'$ (see Preliminaries), there exists $g_0 \in L^1(\nu)'$ such that $\|g_0\|_{L^1(\nu)'} = \|\xi_0\|_{L^1(\nu)^*} = 1$ and

$$\xi_0^*(f) = \int_G f g_0 \, d\mu, \quad \text{for all } f \in L^1(\nu).$$

Note that $g_0 \in L^1(\mu)$, since $\chi_G \in L^1(\nu)$. Let μ_0 denote the measure μ with density g_0. Then, for all $A \in \mathcal{B}(G)$ and $a \in G$, it follows

$$\mu_0(A+a) = \int_{A+a} g_0 \, d\mu = \xi_0^*(\chi_{A+a}) = \xi_0^*(\tau_a \chi_A) = T_a(\xi_0^*)(\chi_A) = \xi_0^*(\chi_A) = \int_A g_0 \, d\mu = \mu_0(A).$$

Hence, μ_0 is translation invariant and thus its variation $|\mu_0|$ is so. Of course, $|\mu_0|$ is the measure μ with density $|g_0|$ and so is non-null. For all $A \in \mathcal{B}(G)$,

$$|\mu_0|(A) = \int_A |g_0| \, d\mu \leqslant \|g_0\|_{L^1(\nu)'} \|\chi_A\|_\nu = \|\nu\|(A). \tag{5}$$

This, together with the fact that $\nu \ll m$, implies that $|\mu_0| \ll m$. By using the Radon–Nikodym theorem, it can be checked that $|\mu_0|$ is regular as m is so. Then, by uniqueness of the Haar measure, $|\mu_0| = |\mu_0|(G)m$. So, for all $A \in \mathcal{B}(G)$, we have that $m(A) = \int_A h_0 \, d\mu$ with $0 \leqslant h_0 = \frac{1}{|\mu_0|(G)} |g_0| \in L^1(\nu)'$. Moreover, since

$$\|\nu\|(G) = \xi_0^*(\chi_G) = \left| \int_G g_0 \, d\mu \right| \leqslant \int_G |g_0| \, d\mu = |\mu_0|(G),$$

from (5) it follows that $\|h_0\|_{L^1(\mu)} = |\mu_0|(G)^{-1} = \|\nu\|(G)^{-1}$. $\square$

As a consequence of Theorem 4.1, we have that actually ν and m are equivalent, that is, have the same null sets. This condition will be needed for the inclusion of $L_w^1(\nu)$ in $L^1(G)$ to be injective.

Remark 4.2. Again, the conclusion of Theorem 4.1 holds if there exists some positive finite regular measure λ on $\mathcal{B}(G)$ such that $\nu \ll \lambda$. But in this case it is not guaranteed that ν and m are equivalent. Also, the conclusion holds if μ is any finite positive measure equivalent to ν.

Remark 4.3. Theorem 4.1 can be improved in the case when $\nu(G)$ is a non-null element of X. Namely, in this case m is just a Rybakov control measure for ν, i.e. $m = |x_0^*\nu|$ for some $x_0^* \in X^*$. The proof is also based in the Markoff–Kakutani theorem but using different K and $\{T_a\}_{a\in G}$. More precisely, considering the set

$$K = \{ y^* \in B_{X_\nu^*} \colon y^*\nu(G) = \|\nu(G)\|_X \},$$

where X_ν is the subspace of X given by the image of all simple functions by I_ν, and the linear operators $T_a : X_\nu^* \to X_\nu^*$ defined by $T_a(x^*) = x^* \circ S_a$, where $\{S_a\}_{a\in G}$ is the family of operators given in (2), we have that there exists $y_0^* \in K$ such that $T_a(y_0^*) = y_0^*$ for every $a \in G$. Taking $y_1^* \in X^*$ such that $y_1^* = y_0^*$ on X_ν and $\|y_1^*\|_{X^*} = \|y_0^*\|_{X_\nu^*} = 1$, it can be proved that $|y_1^*\nu|$ is translation invariant and $|y_1^*\nu| \ll m$. Then, $|y_1^*\nu| = |y_1^*\nu|(G)m$.

Theorem 4.4. *The continuous inclusion*

$$L_w^1(\nu) \hookrightarrow L^1(G)$$

holds with $\|f\|_{L^1(G)} \leqslant \frac{1}{\|\nu\|(G)} \|f\|_\nu$ *for all* $f \in L_w^1(\nu)$.

Proof. Let μ be a Rybakov control measure for ν. From Theorem 4.1, there exists $0 \leqslant h_0 \in L^1(\nu)'$ with $\|h_0\|_{L^1(\nu)'} = \frac{1}{\|\nu\|(G)}$, such that m is the measure μ with density h_0. Let $f \in L^1_w(\nu)$ and (φ_n) a sequence of simple functions such that $0 \leqslant \varphi_n \uparrow |f|$ pointwise. Then,

$$\int_G |f|\,dm = \int_G |f| h_0\,d\mu = \lim_n \int_G \varphi_n h_0\,d\mu \leqslant \|h_0\|_{L^1(\nu)'} \lim_n \|\varphi_n\|_\nu \leqslant \frac{1}{\|\nu\|(G)} \|f\|_\nu. \qquad \square$$

From Theorem 4.4 we deduce the following corollary only by using the definition of the p-power space $L^p_w(\nu)$ and its norm.

Corollary 4.5. *The continuous inclusions*

$$L^p(\nu) \hookrightarrow L^p_w(\nu) \hookrightarrow L^p(G) \hookrightarrow L^1(G)$$

hold with $\|f\|_{L^p(G)} \leqslant \frac{1}{\|\nu\|(G)^{1/p}} \|f\|_{\nu,p}$ *for all* $f \in L^p_w(\nu)$.

Now, let us show that the spaces $L^p(\nu)$ and $L^p_w(\nu)$ are $L^1(G)$-modules.

Theorem 4.6. *If* $f \in L^1(G)$ *and* $g \in L^p_w(\nu)$*, then* $f * g \in L^p_w(\nu)$ *and*

$$\|f * g\|_{\nu,p} \leqslant \|f\|_{L^1(G)} \|g\|_{\nu,p}.$$

Moreover, if $g \in L^p(\nu)$*, then* $f * g \in L^p(\nu)$.

Proof. Take $f \in L^1(G)$ and $g \in L^p_w(\nu)$. From Corollary 4.5, $g \in L^1(G)$ and so $f * g \in L^1(G)$. Note that by Corollary 3.7, $\tau_s g \in L^p_w(\nu)$ and $\|\tau_s g\|_{\nu,p} = \|g\|_{\nu,p}$ for all $s \in G$. Then, by using the Minkowsky inequality, for every $x^* \in X^*$ with $\|x^*\|_{X^*} \leqslant 1$, it follows that

$$\begin{aligned}
\left(\int_G \left|f * g(t)\right|^p d\left|x^*\nu\right|(t) \right)^{\frac{1}{p}} &\leqslant \left(\int_G \left(\int_G \left|f(s)\tau_s g(t)\right| dm(s) \right)^p d\left|x^*\nu\right|(t) \right)^{\frac{1}{p}} \\
&\leqslant \int_G \left|f(s)\right| \left(\int_G \left|\tau_s g(t)\right|^p d\left|x^*\nu\right|(t) \right)^{\frac{1}{p}} dm(s) \\
&\leqslant \int_G \left|f(s)\right| \|\tau_s g\|_{\nu,p}\, dm(s) \\
&= \|g\|_{\nu,p} \|f\|_{L^1(G)}.
\end{aligned}$$

Thus, $f * g \in L^p_w(\nu)$ and $\|f * g\|_{\nu,p} \leqslant \|f\|_{L^1(G)} \|g\|_{\nu,p}$.

Suppose now that $g \in L^p(\nu)$. We have seen that $f * g \in L^p_w(\nu)$. Consider two sequences (φ_n) and (ψ_n) of simple functions converging to f and g in $L^1(G)$ and $L^p(\nu)$ respectively. Note that $\varphi_n * \psi_n \in L^p(\nu)$, since $\varphi_n * \psi_n$ is bounded. Moreover,

$$\begin{aligned}
\|f * g - \varphi_n * \psi_n\|_{\nu,p} &\leqslant \left\|f * (g - \psi_n)\right\|_{\nu,p} + \left\|(f - \varphi_n) * \psi_n\right\|_{\nu,p} \\
&\leqslant \|f\|_{L^1(G)} \|g - \psi_n\|_{\nu,p} + \|f - \varphi_n\|_{L^1(G)} \|\psi_n\|_{\nu,p} \to 0.
\end{aligned}$$

Then $f * g \in L^p(\nu)$, since $L^p(\nu)$ is a closed subspace of $L^p_w(\nu)$. $\square$

As a consequence of Corollary 4.5 and Theorem 4.6, we deduce that $L^p_w(\nu)$ and $L^p(\nu)$ are closed under convolution product.

Corollary 4.7. *If* $f, g \in L^p_w(\nu)$*, then* $f * g \in L^p_w(\nu)$ *and*

$$\|f * g\|_{\nu,p} \leqslant \frac{1}{\|\nu\|(G)^{\frac{1}{p}}} \|f\|_{\nu,p} \|g\|_{\nu,p}.$$

Moreover, if $f, g \in L^p(\nu)$*, then* $f * g \in L^p(\nu)$.

Note that if $\|\nu\|(G) \geqslant 1$, then $L^p(\nu)$ and $L^p_w(\nu)$ are Banach algebras for the convolution product. Also, as an extension of Theorem 4.6, we obtain the following result.

Corollary 4.8. *Consider $1 \leqslant q, p \leqslant r < \infty$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then, for $f \in L^q(G)$ and $g \in L^p_w(\nu)$, we have that $f * g \in L^r_w(\nu)$ and*

$$\|f * g\|_{\nu,r} \leqslant \frac{1}{\|\nu\|(G)^{\frac{1}{p}-\frac{1}{r}}} \|f\|_{L^q(G)} \|g\|_{\nu,p}.$$

Moreover, if $g \in L^p(\nu)$, then $f * g \in L^r(\nu)$.

Proof. Using the Hölder inequality, it can be checked that

$$\|FGH\|_{L^1(G)} \leqslant \|F\|_{L^\alpha(G)} \|G\|_{L^\beta(G)} \|H\|_{L^\rho(G)}$$

for all $F \in L^\alpha(G)$, $G \in L^\beta(G)$, $H \in L^\rho(G)$ with $1 = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\rho}$. Taking $\alpha = \frac{rq}{r-q}$, $\beta = \frac{rp}{r-p}$ and $\rho = r$, we get the inequality

$$|f * g|(t) \leqslant \int_G |f(t-s)|^{1-\frac{q}{r}} |g(s)|^{1-\frac{p}{r}} |f(t-s)|^{\frac{q}{r}} |g(s)|^{\frac{p}{r}} \, dm(s) \leqslant \|f\|_{L^q(G)}^{1-\frac{q}{r}} \|g\|_{L^p(G)}^{1-\frac{p}{r}} \big(|f|^q * |g|^p(t)\big)^{\frac{1}{r}},$$

for $f \in L^q(G)$ and $g \in L^p_w(\nu)$. Note that by Theorem 4.6, we have that $|f|^q * |g|^p$ belongs to $L^1_w(\nu)$ (or $L^1(\nu)$ if $g \in L^p(\nu)$), so $f * g \in L^r_w(\nu)$ (or $L^r(\nu)$ if $g \in L^p(\nu)$), and

$$\|f * g\|_{\nu,r} \leqslant \|f\|_{L^q(G)} \|g\|_{L^p(G)}^{1-\frac{p}{r}} \|g\|_{\nu,p}^{\frac{p}{r}}.$$

Now applying Corollary 4.5 we obtain the desired result. □

5. Applications

In this section, we consider two classical operators, the Fourier transform and the convolution operator. We will see that each of these operators yields an associated vector measure ν which is included in our general framework, that is, ν is norm integral translation invariant and $\nu \ll m$. Therefore, all the results of this paper hold for the space $L^1(\nu)$ and so for the optimal domain of these classical operators.

5.1. Fourier transform

Let Γ be the *dual group* of G. The *Fourier transform* $\mathcal{F} : L^1(G) \to \ell^\infty(\Gamma)$ is defined by $\mathcal{F}(f) = \hat{f}$ for all $f \in L^1(G)$, where

$$\hat{f}(\gamma) = \int_G f(s)\overline{\gamma(s)} \, dm(s), \quad \text{for all } \gamma \in \Gamma.$$

For $1 \leqslant p \leqslant 2$ and $1/p + 1/q = 1$, the *Hausdorff–Young inequality*,

$$\|\hat{f}\|_{\ell^q(\Gamma)} \leqslant \|f\|_{L^p(G)}, \quad \text{for all } f \in L^p(G),$$

establishes that $\mathcal{F} : L^p(G) \to \ell^q(\Gamma)$ is a well-defined continuous operator. Consider its associated vector measure, that is, $\nu : \mathcal{B}(G) \to \ell^q(\Gamma)$ defined by

$$\nu(A) := \mathcal{F}(\chi_A),$$

which obviously satisfies that $\nu \ll m$. Since $I_\nu(\varphi) = \widehat{\varphi}$ for every simple function φ, we have that

$$I_\nu(\tau_a\varphi) = \widehat{\tau_a\varphi} = \overline{\gamma(a)}\widehat{\varphi} = \overline{\gamma(a)} I_\nu(\varphi)$$

and so $\|I_\nu(\tau_a\varphi)\|_{\ell^q(\Gamma)} = \|I_\nu(\varphi)\|_{\ell^q(\Gamma)}$. Then, ν is norm integral translation invariant. Note that $\nu(G) = \widehat{\chi_G} \neq 0$ and so, by Remark 4.3, the Haar measure is a Rybakov control measure for ν. Namely, $m = x_0^*\nu$ for $x_0^* = (e_\gamma)_{\gamma\in\Gamma}$ with $e_\gamma = 1$ if $\gamma = \chi_G$ and $e_\gamma = 0$ in other case.

5.2. Convolution operator

Let E be a Banach function space related to m [7, Definition 1.b.17], which is order continuous, translation invariant and satisfies that $\|\tau_a f\|_E = \|f\|_E$ for all $a \in G$ and $f \in E$. Consider a continuous linear operator $T : E \to E$ commuting with the translation operator (i.e. $\tau_a T = T\tau_a$ for all $a \in G$) and its associated vector measure $\nu : \mathcal{B}(G) \to E$ given by $\nu(E) := T(\chi_E)$ for all $E \in \mathcal{B}(G)$. Note that $\nu \ll m$. Moreover, since $I_\nu(\varphi) = T(\varphi)$ for every simple function φ, it follows that

$$I_\nu(\tau_a\varphi) = T(\tau_a\varphi) = \tau_a T(\varphi) = \tau_a I_\nu(\varphi)$$

and so $\|I_\nu(\tau_a\varphi)\|_E = \|\tau_a I_\nu(\varphi)\|_E = \|I_\nu(\varphi)\|_E$. Therefore, ν is norm integral translation invariant.

For instance, taking $E = L^p(G)$ for $1 \leqslant p < \infty$ and $Tf = f * \mu$ with μ being a non-null complex regular measure defined on $\mathcal{B}(G)$, where

$$f * \mu(t) = \int_G f(t-s)\,d\mu(s), \quad \text{for all } t \in G,$$

all the above conditions are satisfied. Hence, the convolution operator T has an associated vector measure $\nu : \mathcal{B}(G) \to L^p(G)$, given by

$$\nu(A) := \chi_A * \mu, \quad \text{for all } A \in \mathcal{B}(G),$$

which is norm integral translation invariant. Since $\nu(G) = \chi_G * \mu \neq 0$, by Remark 4.3, we have that the Haar measure m is a Rybakov control measure for ν. Namely, $m = |x_0^* \nu|$ for $x_0^* \in L^p(G)^*$ identified with $\frac{1}{|\mu(G)|}\chi_G \in L^q(G)$.

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