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Extremal bounded complete trajectories for nonautonomous reaction-diffusion equations with discontinuous forcing term

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Abstract In this paper we establish a strong comparison principle for a nonautonomous differential inclusion with a forcing term of Heaviside type. Using this principle, we study the structure of the global attractor in both the autonomous and nonautonomous cases. In particular, in the last case we prove that the pullback attractor is confined between two special bounded complete trajectories, which play the role of nonautonomous equilibria.

Keywords: differential inclusions, reaction-diffusion equations, pullback attractors, nonautonomous dynamical systems, multivalued dynamical systems, structure, comparison of solutions.

AMS Subject Classification (2010): 35B40, 35B41, 35B51, 35K55, 35K57

1 Introduction

Comparison of solutions for reaction-diffusion equations is a powerful tool in order to study the structure of global attractors. In particular, in the autonomous case it allows us to establish that the global attractor is confined between two stationary solutions, which are the maximal and minimal elements of the attractor. For a class of autonomous reaction-diffusion equations, such result was proved in [5], [31]. It is worth noticing that a general theory of monotone random dynamical systems was

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first studied in [1], [2], [17]. These results were extended to multivalued autonomous dynamical systems in [11].

In nonautonomous problems the situation is more complicated because stationary solutions do not exist in general but only in rather particular cases, at least not in the classical sense. For this reason, we need to replace them by a special type of bounded complete trajectories, which play the role of "nonautonomous equilibria". The general theory of order-preserving nonautonomous dynamical systems was studied in [16], [25]. In this sense, a result proved in [30] (see also [13], [25] and [29]) is remarkable because a complete bounded positive non-degenerate solution was constructed for a nonautonomous reaction-diffusion equation. Using this solution, a nonautonomous interval containing the pullback attractor is provided. In the multivalued nonautonomous framework, similar results were established in [12], where an ordinary nonautonomus differential inclusion was studied.

We aim to study the structure of attractors for the following nonautonomous differential inclusion

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} \in b(t)H_0(u) + \omega(t)u, \text{ on } (0,1) \times (\tau, \infty), \\ u(0,t) = u(1,t) = 0, \\ u(x,\tau) = u_\tau(x), \end{cases} \tag{1}$$

where H_0 is a Heaviside function. Problems of this type appear when we have a differential equation driven by a nonlinear function having a discontinuity, which can be rewritten as a differential inclusion by means of a Heaviside function. Well known applications like combustion in porous media [21], the conduction of electrical impulses in nerve axons (see [33], [34]) or the surface temperature on Earth (see [10], [20]) are modeled by inclusions of similar type.

The structure of the global attractor for problem (1) in the autonomous case has been studied in detail in [4]. Nevertheless, several challenging problems still remain open. For models concerning the climate on Earth, some results about bifurcations of steady states were proved in [8], [9].

In the multivalued framework, that is, when more than one solution can exist for the Cauchy problem of a differential equation, it is not possible to compare solutions with ordered initial data in the same way as in the single-valued case. Instead, we need to establish some sort of order relationship between the set of solutions corresponding to the ordered initial conditions. In this sense, different definitions have been given in the literature. A strong comparison principle was defined and applied to ordinary differential equations with delays in [11]. A weak comparison principle was established in [38] for reaction-diffusion equations without uniqueness. Also, an intermediate comparison principle was given in [14] for differential inclusions governed by subdifferential maps.

In this paper we firstly prove in the second section a strong comparison principle for the solutions of problem (1). Moreover, we obtain also strong comparison between positive solutions of (1) and its corresponding autonomous equation (that is, for $b(t)$, $\omega(t)$ identically equal to constants). After that, in the third section, we use this comparison principle and the abstract results from [11] in order to establish that the global attractor of the autonomous problem (1) is confined between maximal and minimal stationary points. Moreover, the expressions of these fixed points are known from [4].

The main results of this paper, concerning the structure of the pullback attractor for the nonautonomous inclusion (1), are given in the fourth section. We prove the existence of two bounded complete trajectories that generate a time-depending interval containing the pullback attractor. These solutions are strictly positive (negative) for any time and any $x \in (0,1)$, that is, they are non-degenerate. Moreover, they are the unique non-degenerate bounded complete trajectories of the problem and play the role of nonautonomous positive equilibria.

2 Comparison of solutions

In this section we establish a strong comparison principle for the strong solutions of a nonautonomous differential inclusion in a bounded n-dimensional domain.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with smooth boundary. We consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u \in b(t)H_0(u) + \omega(t)u, \text{ on } \Omega \times (\tau, \infty), \\ u|_{\partial\Omega} = 0, \\ u(\tau, x) = u_\tau(x), \end{cases} \tag{2}$$

where $b : \mathbb{R} \to \mathbb{R}^+$, $\omega : \mathbb{R} \to \mathbb{R}^+$ are continuous functions such that

$$0 < b_0 \le b(t) \le b_1, 0 \le \omega_0 \le \omega(t) \le \omega_1,$$

and

$$H_0(u) = \begin{cases} -1, & \text{if } u < 0, \\ [-1,1], & \text{if } u = 0, \\ 1, & \text{if } u > 0, \end{cases}$$

is the Heaviside function.

We rewrite (2) in the abstract form

$$\begin{cases} \dfrac{\partial u}{\partial t} + \partial\psi(u) - R(t, u(t)) \ni 0, \\ u(\tau) = u_\tau, \end{cases}$$

where $\partial\psi$ is the subdifferential of the proper, convex, lower semicontinuous function $\psi\colon L^2(\Omega) \to (-\infty, +\infty]$ given by

$$\psi(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx, \text{ if } u \in H_0^1(\Omega), \\ \qquad +\infty, \text{ otherwise}, \end{cases}$$

$$\partial\psi(u) = \left\{ y \in L^2(\Omega) : y(x) = -\Delta u(x), \text{ a.e. on } \Omega \right\},$$

$D(\partial\psi) = H^2(\Omega) \cap H_0^1(\Omega)$ and for any $t \in \mathbb{R}$,

$$R(t, u) = \left\{ y \in L^2(\Omega) : \ y(x) \in b(t)H_0(u(x)) + \omega(t)u(x), \text{ a.e. on } \Omega \right\}.$$

We note that in our particular case the operator $\partial\psi : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ is single-valued and linear. In the sequel, as $\partial\psi$ is also the generator of a C_0-semigroup, for convenience we shall use the notation $A = \partial\psi$. Also, we observe that $\overline{D(\psi)} = L^2(\Omega)$.

Let us introduce some notation. Throughout this paper we denote by $\|\cdot\|_X$ the norm in the Banach space X, whereas $\|\cdot\|, (\cdot,\cdot)$ will be used for the norm and scalar product in the space $L^2(\Omega)$ (and with some abuse of notation also in $(L^2(\Omega))^d$, $d \in \mathbb{N}$). Also, $P(X)$ will be the set of all non-empty subsets of X and $2^X = P(X) \cup \varnothing$. The Hausdorff semidistance from the set C to the set B is given by

$$dist(C,B) = \sup_{y\in C} \inf_{z\in B} \|y - z\|,$$

whereas the Hausdorff distance is defined by

$$dist_H(C,B) = \max\{dist(C,B), dist(B,C)\}.$$

For $C \subset X$ an ε-neighborhood is the set $O_\varepsilon(C) = \{z \in X : dist(z,C) < \varepsilon\}$.

For a multivalued map $G : X \to 2^X$ we denote $D(G) = \{u \in X : G(u) \in P(X)\}$. The map G is called upper semicontinuous if for any $u \in D(G)$ and any neighborhood O of $G(u)$ there exists $\delta > 0$ such that $G(v) \subset O$ as soon as $\|u - v\| < \delta$. it is said to be w-upper semicontinuous if for all $\epsilon > 0$, $u \in D(G)$, there is $\delta > 0$ such that $G(v) \subset O_\epsilon(G(u))$ if $\|u - v\| < \delta$. Any upper semicontinuous map is w-upper semicontinous, the converse being true if G has compact values [3, p.45].

We recall the concept of strong solution for problem (2).

Definition 1 We say that the function $u \in C([\tau,+\infty), L^2(\Omega))$ is a strong solution of (2) if:

1. $u(\tau) = u_\tau$;
2. For any $\delta > 0$, $\tau + \delta < T$, $u(\cdot)$ is absolutely continuous on $[\tau + \delta, T]$ and $u(t) \in D(A)$ for a.a. $t \in (\tau, T)$;
3. There exists a function $r : [\tau,+\infty) \to L^2(\Omega)$ such that $r(t) \in R(t, u(t))$, $r \in L^2\left(\tau, T; L^2(\Omega)\right)$ for any $T > \tau$, and

$$\frac{du}{dt} + Au(t) = r(t), \text{ for a.a. } t \in (\tau,+\infty), \tag{3}$$

where the equality is understood in the sense of the space $L^2(\Omega)$.

Lemma 1 *For every strong solution of (2) the function $r(\cdot)$ belongs to $L^\infty\left(\tau, T; L^2(\Omega)\right)$ for any $T > \tau$.*

Proof The statement follows readily from $u \in C([\tau,+\infty), L^2(\Omega))$, $r \in L^2\left(\tau, T; L^2(\Omega)\right)$ and the inequality

$$|r(t,x)| \le b(t) + \omega(t)|u(t,x)| \text{ for a.a. } (t,x).$$

We denote by $f : \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ the multivalued function given by $f(t,u) = b(t)H_0(u)$. Then f possesses nonempty, closed, bounded and convex values, and for all $t \in \mathbb{R}$ the map $f(t,\cdot) : \mathbb{R} \to P(\mathbb{R})$ is upper semicontinuous. Moreover, for any $t, s \in \mathbb{R}^+$, $u \in \mathbb{R}$,

$$dist_H(f(t,u), f(s,u)) = |b(t) - b(s)|,$$

and

$$\sup_{y\in f(t,u)} |y| = b(t).$$

The last equality implies that, for all $y \in L^2(\Omega)$ and for a.a. $t \in \mathbb{R}^+$,

$$\sup_{\xi \in R(t,y)} \|\xi\| \leq |\Omega|^{1/2} b(t).$$

The following result follows from Lemma 6.28 in [24].

Lemma 2 *The map R satisfies the following properties:*

1. *$R : \mathbb{R} \times L^2(\Omega) \to 2^{L^2(\Omega)}$ has nonempty, closed, bounded and convex values;*
2. *For any $t \in \mathbb{R}$, the map $R(t, \cdot) : L^2(\Omega) \to P(L^2(\Omega))$ is w-upper semicontinuous;*
3. *For all $y \in L^2(\Omega)$, $\tau \in \mathbb{R}$, the map $R(\cdot, y) : [\tau, +\infty) \to P(L^2(\Omega))$ possesses a measurable selection, that is, there exists a measurable function $h : [\tau, +\infty) \to L^2(\Omega)$ such that $h(t) \in R(t, y)$ for a.a. $t > \tau$.*

Theorem 1 *For any $u_\tau \in L^2(\Omega)$, problem (2) has at least one strong solution.*

Proof If we fix an interval $[\tau, T]$, the existence of a strong solution follows from Theorem 6.11 and Lemma 6.16 in [24]. Also, adapting Lemma 6.31 in [24] to the nonautonomous case we obtain that the concatenation of two strong solutions is again a strong solution, so every solution in an interval $[\tau, T]$ can be extended to a global one, that is, defined for $t \in [\tau, +\infty)$.

Let us consider the auxiliary problem

$$\begin{cases} \dfrac{du}{dt} + Au(t) = g(t), \ t \in (\tau, T), \\ \qquad\qquad u(\tau) = u_\tau, \end{cases} \tag{4}$$

where $g \in L^1(\tau, T; L^2(\Omega))$.

The continuous function $u : [\tau, T] \to L^2(\Omega)$ is said to be a strong solution of (4) on $[\tau, T]$, if $u(\cdot)$ is absolutely continuous on any compact subset of (τ, T), $u(t) \in D(A)$ for a.a. $t \in (\tau, T)$ and

$$\frac{du}{dt} + Au(t) = g(t) \text{ for a.a. } t \in (\tau, T).$$

The continuous function $u : [\tau, +\infty) \to L^2(\Omega)$ is called in general a strong solution if it is a strong solution on every interval $[\tau, T]$.

Proposition 1 *([7, Theorem 3.6] or [6, p.189]) For any $g(\cdot) \in L^2(\tau, T; L^2(\Omega))$, $u_\tau \in L^2(\Omega)$, there exists a unique strong solution of inclusion (2) on $[\tau, T]$ satisfying*

$$\sqrt{t}\frac{du}{dt} \in L^2(\tau, T; L^2(\Omega)), \ \psi(u(\cdot)) \in L^1(\tau, T). \tag{5}$$

Also, the map $t \mapsto \psi(u(t))$ is absolutely continuous on $[\tau + \delta, T]$, for all $0 < \delta < T - \tau$.

If, moreover, $u_\tau \in \mathcal{D}(\psi)$, then $\dfrac{du}{dt} \in L^2(\tau, T; L^2(\Omega))$ and $t \mapsto \psi(u(t))$ is absolutely continuous on $[\tau, T]$.

Corollary 1 *Every strong solution $u(\cdot)$ to problem (2) satisfies*

$$
\begin{aligned}
u &\in L^2\left(\tau, T; H_0^1(\Omega)\right), \qquad (6)\\
\sqrt{t}\frac{du}{dt} &\in L^2(\tau, T; L^2(\Omega)),\\
\frac{du}{dt} &\in L^2(\tau, T; H^{-1}(\Omega)),
\end{aligned}
$$

for all $T > \tau$. Moreover:

1. *The map $t \mapsto \|u(t)\|^2$ is absolutely continuous on every interval $[\tau, T]$ and*

$$\frac{d}{dt}\|u(t)\|^2 = 2\left(\frac{du}{dt}, u(t)\right) \text{ for a.a. } t \in (\tau, T). \qquad (7)$$

2. *The map $t \mapsto \|\nabla u(t)\|^2$ is absolutely continuous on every interval $[\tau+\delta, T]$ with $0 < \delta < T - \tau$,*

$$\frac{d}{dt}\|\nabla u(t)\|^2 = 2\left(\frac{du}{dt}, -\Delta u(t)\right), \text{ for a.a. } t \in (\tau, T), \qquad (8)$$

and

$$u \in C([\tau+\delta, +\infty), H_0^1(\Omega)). \qquad (9)$$

3. *If $u_\tau \in H_0^1(\Omega)$, then $\frac{du}{dt} \in L^2\left(\tau, T; L^2(\Omega)\right)$ and $t \mapsto \|\nabla u(t)\|^2$ is absolutely continuous on every interval $[\tau, T]$. Also,*

$$u \in C([\tau, +\infty), H_0^1(\Omega)). \qquad (10)$$

Proof Equality (7) follows from [15, p.285] and the rest of properties, except (8)-(10), are a consequence of Proposition 1. For the equality (8) see [6, p.189]. If $u_\tau \in H_0^1(\Omega)$, as $t \mapsto u(t)$ is weakly continuous with respect to $H_0^1(\Omega)$, $t \mapsto \|u(t)\|_{H_0^1(\Omega)}$ is continuous and $H_0^1(\Omega)$ is a Hilbert space, property (10) follows. The proof for (9) is analogous.

Our aim is to prove the following comparison principle.

Definition 2 The strong solutions of problem (2) satisfy a strong comparison principle if for any initial data $u_\tau \leq v_\tau$ there exist strong solutions $\underline{u}(\cdot)$, $\overline{v}(\cdot)$ such that $\underline{u}(\tau) = u_\tau$, $\overline{v}(\tau) = v_\tau$ and

$$
\begin{aligned}
&u(t) \leq \overline{v}(t),\\
&\underline{u}(t) \leq v(t), \ \forall t \geq \tau,
\end{aligned}
$$

where $u(\cdot)$, $v(\cdot)$ are arbitrary strong solutions of problem (2) such that $u(\tau) = u_\tau$, $v(\tau) = v_\tau$.

In particular, this definition implies that for every initial data there exist a maximal and a minimal strong solution.

Let us consider the following parabolic problems

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = b(t) f_\varepsilon(u) + \omega(t) u, \text{ on } \Omega \times (\tau, +\infty), \\ u|_{\partial\Omega} = 0, \\ u(\tau, x) = u_\tau(x), \end{cases} \tag{11}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = b(t) \overline{f}_\varepsilon(u) + \omega(t) u, \text{ on } \Omega \times (\tau, +\infty), \\ u|_{\partial\Omega} = 0, \\ u(\tau, x) = u_\tau(x), \end{cases} \tag{12}$$

where $f_\varepsilon, \overline{f}_\varepsilon \in C^1(\mathbb{R})$, $f'_\varepsilon(u), \overline{f}'_\varepsilon \geq 0$, $\left|f'_\varepsilon(u)\right|, \left|\overline{f}'_\varepsilon(u)\right| \leq C_\varepsilon$ for all u, and

$$f_\varepsilon(u) = \begin{cases} -1, & \text{if } u \leq -\varepsilon, \\ -1 \leq f_\varepsilon(u) \leq 1, & \text{if } -\varepsilon < u < 0, \\ 1, & \text{if } u \geq 0, \end{cases}$$

$$\overline{f}_\varepsilon(u) = \begin{cases} -1, & \text{if } u \leq 0, \\ -1 \leq f_\varepsilon(u) \leq 1, & \text{if } 0 < u < \varepsilon, \\ 1, & \text{if } u \geq \varepsilon. \end{cases}$$

It is straightforward that

$$\begin{aligned} f_\varepsilon(u) &\geq \sup_{y \in H_0(v)} y \text{ if } u \geq v, \\ \overline{f}_\varepsilon(u) &\leq \inf_{y \in H_0(v)} y \text{ if } u \leq v. \end{aligned} \tag{13}$$

With obvious little changes, we can extend the definition of strong solutions to problems (11)-(12). Let us show that these problems have a unique strong solution. Let us just consider problem (11).

From now on, for $v \in L^2(\Omega)$, we denote by $f_\varepsilon(v)$ ($\overline{f}_\varepsilon(v)$) the element $y \in L^2(\Omega))$ such that $y(x) = f_\varepsilon(v(x))$ ($= \overline{f}_\varepsilon(v(x))$) for a.a. x. In the same way, for $h \in L^1(\tau, T; L^2(\Omega))$ we denote $h(t) := h(t,\cdot) \in L^2(\Omega))$.

We know from [15, p.283] that for any $u_\tau \in L^2(\Omega)$ there exists a unique weak solution of problem (11), which means that $u \in C([0, +\infty), L^2(\Omega))$, $u \in L^2(\tau, T; H_0^1(\Omega))$, for all $T > \tau$, and

$$\frac{d}{dt}(u(t), v) - \langle \Delta u, v \rangle = (f_\varepsilon(u(t)), v), \ \forall v \in H_0^1(\Omega),$$

where the equality is understood in the sense of distributions on every interval (τ, T) and $\langle \cdot, \cdot \rangle$ is pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Lemma 3 *For any $u_\tau \in L^2(\Omega)$ there exists a unique strong solution $u_\varepsilon(\cdot)$ of problem (11), which coincides with the unique weak solution of (11).*

Proof Let $u_\varepsilon(\cdot)$ be the unique weak solution of problem (11). If we put $g_\varepsilon(t) = f_\varepsilon(u_\varepsilon(t)) + \omega(t)u_\varepsilon(t)$, then $g_\varepsilon \in L^2(\tau, T; L^2(\Omega))$, for all $T > \tau$, and we can consider the linear problem

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z = g_\varepsilon(t), \text{ on } \Omega \times (\tau, +\infty), \\ z|_{\partial\Omega} = 0, \\ z(\tau, x) = z_\tau(x). \end{cases} \tag{14}$$

On the one hand, $u_\varepsilon(\cdot)$ is the unique weak solution of problem (14). On the other hand, it follows from Proposition 1 that problem (14) possesses a unique strong solution $z(\cdot)$. If we were able to show that $z(\cdot)$ is also a weak solution of (14), then we would obtain that $z = u_\varepsilon$, so $u_\varepsilon(\cdot)$ would be a strong solution of (11). Indeed. In view of Proposition 1 we obtain that $z \in L^2(\tau, T; H_0^1(\Omega))$ for all $T > \tau$. From the equality in (14) we infer then that $\dfrac{dz}{dt} \in L^2(\tau, T; H^{-1}(\Omega))$, which implies by [28, Lemma 7.4] that

$$\left\langle \frac{dz}{dt}, v \right\rangle - \langle \Delta z, v\rangle = (g_\varepsilon(t), v), \ \forall v \in H_0^1(\Omega).$$

Hence, by [32, p. 250] we have

$$\frac{d}{dt}(z, v) - \langle \Delta z, v\rangle = (g_\varepsilon(t), v),$$

so $z(\cdot)$ is a weak solution of (14). Thus, $z = u_\varepsilon$ is a strong solution of problem (11).

It remains to check uniqueness. Let $v(\cdot)$ be an arbitrary strong solution of (11). Then, it is a strong solution of problem (14) with $g_\varepsilon(t) = f_\varepsilon(v(t)) + \omega(t)v(t)$. By the previous argument $v(\cdot)$ is a weak solution of (14) and then a weak solution of (11) as well. Therefore, v is equal to u_ε, the unique weak solution of (11). Thus, $v = z$.

Remark 1 The function $h_\varepsilon(t) = b(t)f_\varepsilon(u_\varepsilon(t)) + \omega(t)u_\varepsilon(t)$ belongs to $L^\infty(\tau, T; L^2(\Omega))$ for any $T > \tau$.

Corollary 2 *For any $u_\tau \in L^2(\Omega)$ the function $u_\varepsilon \in C([\tau, +\infty), L^2(\Omega))$ is a strong solution to problem (11) if and only if it is a weak solution.*

It is well known [27, Chapter 7] that operator $-A = \Delta u : D(A) = H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ is the generator of a strongly continuous semigroup of bounded linear operators $S(t) : L^2(\Omega) \to L^2(\Omega)$, $t \geq 0$, which will be denoted in the sequel by $S(t) = e^{-At}$. Moreover, it is a semigroup of contractions, that is, $\left\| e^{-At} \right\| \leq 1$. For every $x \in D(A)$ the function $u(t) = e^{-At}x$ is the unique classical solution (see the definition below) to the problem

$$\begin{cases} \dfrac{du}{dt} + Au(t) = 0, \ t > 0, \\ \qquad u(0) = x. \end{cases} \tag{15}$$

Also, the semigroup e^{-At} is positive for all $t \geq 0$ [13, Chapter 12].

Let us defined the concept of mild solution for the inhomogeneous problem (4).

Definition 3 Let $u_\tau \in L^2(\Omega)$ and $g \in L^1_{loc}(\tau, +\infty; L^2(\Omega))$. Then the function $u \in C([\tau, +\infty), L^2(\Omega))$ is called a mild solution to problem (4) if

$$u(t) = e^{-A(t-\tau)} u_\tau + \int_\tau^t e^{-A(t-s)} g(s) ds, \ \tau \leq t < \infty. \tag{16}$$

It is called a classical solution if $u(\cdot)$ is continuously differentiable on $(\tau, +\infty)$, $u(t) \in D(A)$ for any $t \in (\tau, +\infty)$, $u(\tau) = x$ and the equality in (4) is satisfied for every $t \in (\tau, \infty)$.

For every $u_\tau \in L^2(\Omega)$ and $g \in L^1_{loc}(\tau, +\infty; L^2(\Omega))$ there exists a unique mild solution to problem (4). Moreover, if $u_\tau \in D(A)$ and g is continuously differentiable on $[\tau, +\infty)$, then the mild solution is the unique classical solution [27, p.107].

We can also define mild solutions for problems (11) and (2).

Definition 4 Let $u_\tau \in L^2(\Omega)$. Then the function $u_\varepsilon \in C([\tau, +\infty), L^2(\Omega))$ is called a mild solution to problem (11) if

$$u_\varepsilon(t) = e^{-A(t-\tau)} u_\tau + \int_\tau^t e^{-A(t-s)} (b(t) f_\varepsilon(u_\varepsilon(t)) + \omega(t) u_\varepsilon(t)) ds, \ \tau \leq t < \infty.$$

We note that for any $u_\varepsilon \in C([\tau, \infty), L^2(\Omega))$ the map $f_\varepsilon(u_\varepsilon(t, \cdot))$ belongs to $L^\infty_{loc}\left(0, +\infty; L^2(\Omega)\right) \subset L^2_{loc}\left(0, +\infty; L^2(\Omega)\right)$.

Definition 5 Let $u_\tau \in L^2(\Omega)$. Then the function $u \in C([\tau, +\infty), L^2(\Omega))$ is called a mild solution to problem (2) if there exists h such that $h \in L^2_{loc}\left(0, +\infty; L^2(\Omega)\right)$, for any $T > \tau$, $h(t, x) \in H_0(u(t, x))$, for a.a. (t, x), and

$$u(t) = e^{-A(t-\tau)} u_\tau + \int_\tau^t e^{-A(t-s)} (b(t) h(t) + \omega(t) u(t)) ds, \ \tau \leq t < \infty.$$

Lemma 4 *For any $u_\tau \in L^2(\Omega)$ and $g \in L^2_{loc}\left(0, +\infty; L^2(\Omega)\right)$ the function $u \in C([\tau, +\infty), L^2(\Omega))$ is a strong solution to problem (4) if and only if it is a mild solution.*

Proof The proof follows the same lines of [39, Lemma 2], but we provide it in detail for the sake of completeness.

Let $u(\cdot)$ be a strong solution. We take sequences $u^n_\tau \in D(A)$, $g^n(\cdot) \in C^1\left([\tau, +\infty), L^2(\Omega)\right)$ such that

$$u^n_\tau \to u_\tau \text{ in } L^2(\Omega),$$
$$g^n \to g \text{ in } L^2\left(\tau, T; L^2(\Omega)\right) \ \forall T > \tau.$$

Denote by $u^n(\cdot)$ the unique classical solution to the problem

$$\begin{cases} \dfrac{du^n}{dt} = Au^n(t) + g^n(t), \ t > \tau, \\ u^n(\tau) = u^n_\tau. \end{cases}$$

Let $T > \tau$ and $0 < \varepsilon < T - \tau$ be arbitrary. We note that w^n is the unique strong solution to problem (4) with $\widetilde{g}(t) = g^n(t) - g(t)$ and $w(\tau) = u^n_\tau - u_\tau$, so by (7) we have

$$2\left(\frac{d}{dt}(u^n - u), u^n - u\right) = \frac{d}{dt}\left\|u^n(t) - u(t)\right\|^2 \text{ for a.a. } t \in (\tau + \varepsilon, T).$$

Hence, the difference $w^n = u^n - u$ satisfies

$$\frac{1}{2}\frac{d}{dt}\left\|w^n\right\|^2 + \left\|w^n(t)\right\|_{H_0^1}^2 \le \frac{1}{2}\left\|g^n(t) - g(t)\right\|^2 + \frac{1}{2}\left\|w^n(t)\right\|^2 \text{ for a.a. } t \in (\tau+\varepsilon, T).$$

By means of the Gronwall lemma we obtain easily that

$$u^n \to u \text{ in } C\Big([\tau+\varepsilon, T], L^2(\Omega)\Big).$$

Since u^n is a mild solution, we get

$$u^n(t) = e^{-A(t-\tau-\varepsilon)}u^n(\tau+\varepsilon) + \int_{\tau+\varepsilon}^{t} e^{-A(t-s)}g^n(s)\,ds \text{ for any } \tau+\varepsilon \le t \le T.$$

By continuity of the maps $e^{-At} : L^2(\Omega) \to L^2(\Omega)$ and Lebesgue's theorem, we have

$$e^{-A(t-\tau-\varepsilon)}u^n(\tau+\varepsilon) \to e^{-A(t-\tau-\varepsilon)}u(\tau+\varepsilon),$$
$$\int_{\tau+\varepsilon}^{t} e^{-A(t-s)}g^n(s)\,ds \to \int_{\tau+\varepsilon}^{t} e^{-A(t-s)}g(s)\,ds, \text{ as } n \to \infty,$$

which implies

$$u(t) = e^{-A(t-\tau-\varepsilon)}u(\tau+\varepsilon) + \int_{\tau+\varepsilon}^{t} e^{-A(t-s)}g(s)\,ds \text{ for any } \tau+\varepsilon \le t \le T.$$

Finally, passing to the limit as $\varepsilon \to 0$ and taking into account that $e^{-A(t-\tau-\varepsilon)}u(\tau+\varepsilon) \to e^{-A(t-\tau)}u(\tau)$, we obtain that u is a mild solution.

By uniqueness of the mild solution of (4) the converse statement follows immediately.

Corollary 3 *For any $u_\tau \in L^2(\Omega)$ the function $u_\varepsilon \in C([\tau,+\infty), L^2(\Omega))$ is a strong solution to problem (11) if and only if it is a mild solution.*

Proof Let u_ε be a strong (mild) solution to problem (11). We define the function $g(t) = b(t)f_\varepsilon(u_\varepsilon(t)) + \omega(t)u_\varepsilon(t)$, which belongs to $L^\infty_{loc}\left(\tau,+\infty; L^2(\Omega)\right) \subset L^2_{loc}\left(\tau,+\infty; L^2(\Omega)\right)$. Since u_ε is also the unique strong (mild) solution to problem (4), Lemma 4 implies that it is a mild (strong) solution to problem (11).

Corollary 4 *For any $u_\tau \in L^2(\Omega)$ the function $u \in C([\tau,+\infty), L^2(\Omega))$ is a strong solution to problem (2) if and only if it is a mild solution.*

Proof Let u be a strong solution to problem (2). As u is the unique strong solution to problem (4) with $g(t) = r(t)$, Lemma 4 implies that it is a mild solution to problem (2) with $h(t) = (r(t) - \omega(t)u(t))/b(t)$.

Conversely, let u be a mild solution to problem (2). As u is the unique mild solution to problem (4) with $g(t) = b(t)h(t) + \omega(t)u(t)$, Lemma 4 implies that it is a strong solution to problem (2) with $r(t) = g(t)$.

Now we are ready to prove the comparison principle.

Theorem 2 *The strong solutions of problem (2) satisfy the strong comparison principle.*

Proof We define the set

$$V=\{u\in C([\tau,\tau+t_0],L^2(\Omega)):u(\tau)=u_\tau,\ \|u(t)-u_\tau\|\leq 1,\ \forall t\in[\tau,\tau+t_0]\},$$

where $t_0>0$ satisfies

$$\begin{aligned}\left\|e^{-A(t-\tau)}u_\tau-u_\tau\right\|&\leq\frac{1}{2},\\ t_0\left(b_1+\omega_1\|u_\tau\|+\omega_1\right)&\leq\frac{1}{2},\\ t_0(b_1C_\varepsilon+\omega_1)&<1,\text{ for all }t\in[\tau,\tau+t_0],\end{aligned}\tag{17}$$

where C_ε is the constant for which $\left|f_\varepsilon'(u)\right|,\left|\overline{f}_\varepsilon'(u)\right|\leq C_\varepsilon$. The map $\mathcal{F}:V\to C([\tau,\tau+t_0],L^2(\Omega))$ is defined by

$$\mathcal{F}(u)(t)=e^{-A(t-\tau)}u_\tau+\int_\tau^t e^{-A(t-s)}\left(b\left(s\right)f_\varepsilon(u(s))+\omega(s)u(s)\right)ds.\tag{18}$$

For any $u\in V$ and $t\in[\tau,\tau+t_0]$ we have

$$\begin{aligned}\|\mathcal{F}(u)(t)-u_\tau\|&\leq\left\|e^{-A(t-\tau)}u_\tau-u_\tau\right\|+\int_\tau^t(b\left(s\right)\|f_\varepsilon(u(s))\|+\omega(s)\|u(s)\|)ds\\&\leq\frac{1}{2}+t_0\left(b_1+\omega_1\|u_\tau\|+\omega_1\right)\leq 1.\end{aligned}$$

Also, $\mathcal{F}(u)(\tau)=u_\tau$. Thus, $\mathcal{F}\left(u\right)\in V$ and then $\mathcal{F}:V\to V$.

We check further that $\mathcal{F}$ is a contraction mapping. Indeed,

$$\begin{aligned}\|\mathcal{F}(u)(t)-\mathcal{F}(v)(t)\|&\leq\int_\tau^t(b\left(s\right)\|f_\varepsilon(u(s))-f_\varepsilon(v(s))\|+\omega(s)\|u(s)-v(s)\|)ds\\&\leq\int_\tau^t(b\left(s\right)C_\varepsilon\|u(s)-v(s)\|+\omega(s)\|u(s)-v(s)\|)ds\\&\leq t_0(b_1C_\varepsilon+\omega_1)\|u-v\|_{\mathcal{C}([\tau,\tau+t_0],L^2(\Omega))}\\&=\beta_0\|u-v\|_{\mathcal{C}([\tau,\tau+t_0],L^2(\Omega))},\ \forall t\in[\tau,\tau+t_0],\end{aligned}$$

where $0<\beta_0<1$.

Therefore, the contraction fixed point theorem implies that $\mathcal{F}$ possesses a unique fixed point $u^*\in V$, which is a mild solution and coincides with the unique strong solution u_ε to problem (11), as by Lemma 4 u^* is also a strong solution.

In view of Definition 1 for any strong solution $v\left(\cdot\right)$ of (2) there exists a function h satisfying $h\in L^\infty\left(\tau,T;L^2(\Omega)\right)$, for any $T>\tau$, $h(s,x)\in H_0(v(s,x))$ for a.a. (s,x), and such that $v\left(\cdot\right)$ is the unique strong solution to the following problem

$$\begin{cases}\dfrac{\partial v}{\partial t}-\Delta v=b(t)h(t)+\omega(t)v,\ \text{ on }\Omega\times(\tau,\infty),\\ v|_{\partial\Omega}=0,\\ v(\tau,x)=v_\tau(x).\end{cases}\tag{19}$$

In view of Corollary 4, v is a mild solution to (2) as well, so it satisfies the variation of constants formula:

$$v(t) = e^{-A(t-\tau)}v_\tau + \int_\tau^t e^{-A(t-s)}(b(t)h(t) + \omega(t)v(t))ds. \tag{20}$$

We define the set

$$V_1 = \{v \in C([\tau, \tau + t_0], L^2(\Omega)) : v(\tau) = v_\tau,\ \|v(t) - v_\tau\| \leq 1,\ \forall t \in [\tau, \tau + t_0]\},$$

where $t_0 > 0$ satisfies (17) and also

$$\left\| e^{-A(t-\tau)}v_\tau - v_\tau \right\| \leq \frac{1}{2},$$
$$t_0\,(b_1 + \omega_1 \|v_\tau\| + \omega_1) \leq \frac{1}{2}, \text{ for all } t \in [\tau, \tau + t_0].$$

Arguing as before, $v(\cdot)$ is the unique fixed point of the map $F_1 : V_1 \to V_1$ given by

$$\mathcal{F}_1(u)(t) = e^{-A(t-\tau)}v_\tau + \int_\tau^t e^{-A(t-s)}\,(b(s)\,h(s) + \omega(s)u(s))\,ds. \tag{21}$$

Assume that $u_\tau \geq v_\tau$. Let us define the set

$$\widehat{V} = \{u \in V : u(t) \geq v(t),\ \forall t \in [\tau, \tau + t_0]\}.$$

This set is non-empty since $u(t) = v(t) + u_\tau - v_\tau$ belongs to it.

For any $u \in \widehat{V}$ using (13) and $e^{-At} \geq 0$ we have

$$\begin{aligned}\mathcal{F}(u)(t) - v(t) &= e^{-A(t-\tau)}(u_\tau - v_\tau) \\ &\quad + \int_\tau^t e^{-A(t-s)}\,(b(s)(f_\varepsilon(u(s)) - h(s)) + \omega(s)(u(s) - v(s)))\,ds \\ &\geq 0, \text{ for any } t \in [\tau, \tau + t_0].\end{aligned}$$

Therefore, $\mathcal{F}(\widehat{V}) \subset \widehat{V}$. Since $\mathcal{F}$ is a contraction in V, so does it in $\widehat{V}$. We deduce that $\mathcal{F}$ possesses a unique fixed point $u^* \in \widehat{V}$, which is equal to the solution u_ε to problem (11). It follows that

$$u_\varepsilon(t) \geq v(t) \text{ for any } t \in [\tau, \tau + t_0].$$

Using a standard continuation argument it is proved that

$$u_\varepsilon(t) \geq v(t) \text{ for any } t \geq \tau. \tag{22}$$

The last property is true for every strong solution $v(\cdot)$ to problem (2) such that $u_\tau \geq v(\tau) = v_\tau$.

Further, we will pass to the limit as $\varepsilon \to 0$.

Mutiplying the equation in (11) by u_ε we obtain easily that

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon\|^2 + \|\nabla u_\varepsilon\|^2 \leq C_1 + C_2\|u_\varepsilon\|^2,$$

for some constants $C_1, C_2 > 0$. By Gronwall's lemma for any $T > \tau$ there exists $D_1 = D_1(T, \|u_{\varepsilon\tau}\|)$ such that

$$\|u_\varepsilon(t)\|^2 \leq D_1 \text{ for all } t \in [\tau, T]. \tag{23}$$

Hence, there exists $D_2 = D_2(T, \|u_{\varepsilon\tau}\|)$ such that

$$\int_\tau^t \|\nabla u_\varepsilon(s)\|^2 \, ds \leq D_2 \text{ for all } t \in [\tau, T]. \tag{24}$$

Let $0 < \delta < T - \tau$ be arbitrary. Since $u_\varepsilon \in L^2(\tau + \delta, T; D(A))$ and $\dfrac{du_\varepsilon}{dt} \in L^2\left(\tau + \delta, T; L^2(\Omega)\right)$, by Corollary 1 we obtain that $u_\varepsilon \in C([\tau + \delta, T], H_0^1(\Omega))$ and

$$\frac{d}{dt}\|\nabla u_\varepsilon\|^2 = 2\left(Au_\varepsilon, \frac{du_\varepsilon}{dt}\right) \text{ for a.a. } t \in (\tau + \delta, T). \tag{25}$$

Multiplying now the equation in (11) by $\dfrac{du_\varepsilon}{dt}$ we have

$$\left\|\frac{du_\varepsilon}{dt}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_\varepsilon\|^2 \leq C_3 + \frac{1}{2}\left\|\frac{du_\varepsilon}{dt}\right\|^2 + C_4\|u_\varepsilon\|^2.$$

Using (23) and integrating over (s, t) with $t > s > \tau$ we deduce the existence of $D_3 = D_3(T, \|u_{\varepsilon\tau}\|)$ such that

$$\|\nabla u_\varepsilon(t)\|^2 \leq \|\nabla u_\varepsilon(s)\|^2 + D_3.$$

Integrating with respect to the variable s over (τ, t) and using (24) we obtain

$$\|\nabla u_\varepsilon(t)\|^2 \leq \frac{D_2}{t - \tau} + D_3, \text{ for any } \tau < t \leq T. \tag{26}$$

For any $0 < \delta < T - \tau$ we then have

$$\begin{aligned} \int_{\tau+\delta}^T \left\|\frac{du_\varepsilon}{dt}\right\|^2 dt &\leq \|\nabla u_\varepsilon(\tau + \delta)\|^2 + D_3 \\ &\leq \frac{D_2}{\delta} + 2D_3. \end{aligned} \tag{27}$$

Thanks to these estimates, there is a subsequence u_{ε_n} and a function $\overline{u}$ such that

$$u_{\varepsilon_n} \to \overline{u} \text{ weakly in } L^2\left(\tau, T; L^2(\Omega)\right) \text{ and weakly star in } L^\infty\left(\tau, T; L^2(\Omega)\right),$$
$$\frac{du_{\varepsilon_n}}{dt} \to \frac{d\overline{u}}{dt} \text{ weakly in } L^2\left(\tau + \delta, T; L^2(\Omega)\right) \text{ for any } 0 < \delta < T - \tau.$$

Hence, for any $N > 0$ such that $\frac{1}{N} < T - \tau$ we deduce that the maps $u_{\varepsilon_n} : [\frac{1}{N}, T] \to L^2(\Omega)$ are equicontinuous, so (26), the compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$ and Ascoli-Arzelà theorem imply that

$$u_{\varepsilon_n} \to \overline{u} \text{ in } C\left(\left[\frac{1}{N}, T\right], L^2(\Omega)\right).$$

By a diagonal argument this is true for any interval $\left[\frac{1}{N}, T\right]$.

We can also see that the functions $f_{\varepsilon_n}(u_{\varepsilon_n}(t))$, $g_{\varepsilon_n}(t) = b(t)f_{\varepsilon_n}(u_{\varepsilon_n}(t)) + \omega(t)u_{\varepsilon_n}(t)$ are bounded in $L^\infty(\tau, T; L^2(\Omega))$ and then up to a subsequence

$$\begin{aligned} &f_{\varepsilon_n}(u_{\varepsilon_n}) \to h, \\ &\quad g_{\varepsilon_n} \to g, \text{ weakly star in } L^\infty\left(\tau, T; L^2(\Omega)\right), \end{aligned}$$

where $h \in L^{\infty}(\tau, T; L^2(\Omega))$ and $g(t) = b(t)h(t) + \omega(t)\overline{u}(t)$. Therefore, the function $\overline{u}$ satisfies the equality

$$\frac{du}{dt} + A\overline{u}(t) = b(t)h(t) + \omega(t)\overline{u}(t), \text{ a.e. on } (\tau, T).$$

Let us prove that $h(t,x) \in H_0(\overline{u}(t,x))$ for a.a. $(t,x) \in (\tau, T) \times \Omega$. Let $A_0 = \{(t,x) : \overline{u}(t,x) = 0\}$. It is obvious that

$$f_{\varepsilon_n}(u_{\varepsilon_n}(t,x)) \in [-1,1] = H_0(\overline{u}(t,x)) \text{ for } (t,x) \in A_0.$$

If $(t,x) \in A_- = \{(t,x) : \overline{u}(t,x) < 0\}$, then

$$dist(f_{\varepsilon_n}(u_{\varepsilon_n}(t,x)), H_0(\overline{u}(t,x))) = |f_{\varepsilon_n}(u_{\varepsilon_n}(t,x)) + 1| \to 0, \text{ as } n \to \infty,$$

as for any $t \in (\tau, T]$ it is true that $u_{\varepsilon_n}(t,x) \to \overline{u}(t,x)$, for a.a. x, and by the definition of f_ε we can see that $f_{\varepsilon_n}(u_{\varepsilon_n}(t,x)) = -1$ for all $n \geq n_0(t,x)$. In the same way, if $(t,x) \in A_+ = \{(t,x) : \overline{u}(t,x) > 0\}$, then

$$dist(f_{\varepsilon_n}(u_{\varepsilon_n}(t,x)), H_0(\overline{u}(t,x))) = |f_{\varepsilon_n}(u_{\varepsilon_n}(t,x)) - 1| \to 0, \text{ as } n \to \infty.$$

Hence, for any $t \in (\tau, T]$ we have

$$dist(f_{\varepsilon_n}(u_{\varepsilon_n}(t,x)), H_0(\overline{u}(t,x))) \to 0 \text{ for a.a. } x. \tag{28}$$

Denote $h_n(\cdot) = f_{\varepsilon_n}(u_{\varepsilon_n}(\cdot)) : [\tau, T] \to L^2(\Omega)$. By Proposition 1.1 in [35] we have

$$h(t) \in \cap_{n \geq 0} \overline{co} \cup_{k \geq n} h_k(t) \text{ for a.a. } t.$$

Hence, if we fix t, there exists a sequence $g_n(t) = \sum_{i=1}^{M} \lambda_i h_{k_i}(t)$, where $\sum_{i=1}^{M} \lambda_i = 1$ and $k_i \geq n$, such that $g_n(t) \to h(t)$ strongly in $L^2(\Omega)$. In view of (28),

$$\begin{aligned} & dist(g_n(t,x), H_0(\overline{u}(t,x))) \\ & \leq \sum_{i=1}^{M} dist(h_{k_i}(t,x), H_0(\overline{u}(t,x))) \to 0 \text{ for a.a. } x. \end{aligned}$$

But passing to a subsequence $g_n(t,x) \to h(t,x)$ for a.a. x. Thus, $h(t,x) \in H_0(\overline{u}(t,x))$.

Then equality (3) holds and the second condition in Definition 1 is clearly satisfied. In order to prove that $u(\cdot)$ is a strong solution to (2) it remains to check that $\overline{u} \in C([\tau, +\infty), L^2(\Omega))$ and $\overline{u}(\tau) = u_\tau$.

Let z be the unique strong solution of problem

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z = \omega(t) z, \text{ on } \Omega \times (\tau, \infty), \\ z|_{\partial\Omega} = 0, \\ z(x,\tau) = u_\tau(x). \end{cases}$$

We put $w_n(t) = u_{\varepsilon_n}(t) - z(t)$. It is easy to obtain that

$$\begin{aligned} \frac{d}{dt}\|w_n\|^2 &\leq \omega(t)\|w_n(t)\|^2 + b(t)\|h_n(t)\|\|u_{\varepsilon_n}(t)\| \\ &\leq R_1 + R_2\|w_n(t)\|^2 + R_3\|u_{\varepsilon_n}(t)\|^2 \\ &\leq R_4. \end{aligned}$$

Since $w_n(\tau) = 0$, we get

$$\|w_n(t)\|^2 \leq R_4 t,$$

so for any $t > \tau$,

$$\|\overline{u}(t) - z(t)\| = \lim_{n\to\infty} \|w_n(t)\|^2 \leq R_4 t.$$

Therefore,

$$\|\overline{u}(t) - u_\tau\| \leq R_4 t + \|z(t) - u_\tau\| \to 0 \text{ as } t \to 0^+.$$

Finally, $u_{\varepsilon_n}(t) \to \overline{u}(t)$, for any $t > 0$, and (22) imply that

$$\overline{u}(t) \geq v(t) \text{ for any } t \geq \tau.$$

In conclusion, we have established that $\overline{u}(t) \geq v(t)$ for all $t \geq \tau$, where $v(\cdot)$ is any strong solution $v(\cdot)$ to problem (29) such that $v(\tau) = v_\tau \leq u_\tau$.

For the second part, we define the map $\mathcal{F}_2 : V \to C([\tau, \tau + t_0], L^2(\Omega))$ by

$$\mathcal{F}_2(u)(t) = e^{-A(t-\tau)}u_\tau + \int_\tau^t e^{-A(t-s)} \left(b(s)\overline{f}_\varepsilon(u(s)) + \omega(s)u(s)\right) ds,$$

which satisfies $\mathcal{F}_1(V) \subset V$ and is contractive. Then the unique strong solution u_ε to problem (12) is the unique fixed point of $\mathcal{F}_2$ in V.

Any strong solution $v(\cdot)$ of (2) with initial condition v_τ is the unique fixed point of the map $\mathcal{F}_1 : V_1 \to V_1$.

Let $u_\tau \leq v_\tau$ and define the set $\widehat{V}_1 = \{u \in V : u(t) \leq v(t)\}$. This set is non-empty because $u(t) = v(t) + u_\tau - v_\tau$ belongs to it.

For any $u \in \widehat{V}_1$, (13) and $e^{-At} \geq 0$ imply

$$\begin{aligned}\mathcal{F}(u)(t) - v(t) &= e^{-A(t-\tau)}(u_\tau - v_\tau) \\ &\quad + \int_\tau^t e^{-A(t-s)} \left(b(s)(\overline{f}_\varepsilon(u(s)) - h(s)) + \omega(s)(u(s) - v(s))\right) ds \\ &\leq 0, \text{ for any } t \in [\tau, \tau + t_0].\end{aligned}$$

It follows that $\mathcal{F}_2(\widehat{V}_1) \subset \widehat{V}_1$ and $\mathcal{F}_2$ is a contraction in $\widehat{V}_1$. Thus, $\mathcal{F}_2$ possesses a unique fixed point $u^* \in \widehat{V}_1$, which is equal to the solution u_ε to problem (12). It follows that

$$u_\varepsilon(t) \leq v(t) \text{ for any } t \geq \tau.$$

The last property is true for every strong solution $v(\cdot)$ to problem (2) such that $u_\tau \leq v(\tau) = v_\tau$.

Passing to the limit as $\varepsilon \to 0$ we obtain the existence of a strong solution $\underline{u}$ to problem (2) such that $\underline{u}(t) \leq v(t)$ for all $t \geq \tau$, where $v(\cdot)$ is any strong solution to problem (2) such that $u_\tau \leq v(\tau) = v_\tau$.

It follows that the solutions to problem (2) satisfy the strong comparison principle.

Corollary 5 *If $u_\tau \geq 0$, there exists at least one strong solution $u(\cdot)$ to problem (2) such that $u(t) \geq 0$ for all $t \geq \tau$.*

Proof It follows from the strong comparison principle and the fact that $v(t) \equiv 0$ is a strong solution for $v_\tau = 0$.

We now study a comparison principle between the solutions in the nonautonomous and autonomous cases.

Let us consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u \in bH_0(u) + \omega u, \text{ on } \Omega \times (\tau, \infty), \\ u|_{\partial\Omega} = 0, \\ u(\tau, x) = u_\tau(x), \end{cases} \tag{29}$$

where $0 < b,\ 0 \leq \omega$. Let us denote a strong solution to problem (29) by $u_{b,\omega}(\cdot)$ (and we do not take into account in this notation the initial time τ as the solution is the same whatever the value of τ).

Theorem 3 *For any initial datum $u_\tau \geq 0$ there exists a non-negative solution $\overline{u}_{b_1,\omega_1}(\cdot)$ to problem (29) with $u(\tau) = u_\tau,\ b = b_1,\ \omega = \omega_1$ such that*

$$v(t) \leq \overline{u}_{b_1,\omega_1}(t),\ \forall t \geq \tau, \tag{30}$$

where $v(\cdot)$ is an arbitrary strong non-negative solution to (2) with $u(\tau) = u_\tau$.

On the other hand, there exist a non-negative solution $\overline{u}(\cdot)$ to (2) with $u(\tau) = u_\tau$ such that

$$\overline{u}(t) \geq u_{b_0,\omega_0}(t),\ \forall t \geq \tau, \tag{31}$$

where $u_{b_0,\omega_0}(\cdot)$ is an arbitrary strong non-negative solution to (29) with $u(\tau) = u_\tau,\ b = b_0,\ \omega = \omega_0$.

Proof Let $\mathcal{F} : V \to C([\tau, \tau + t_0], L^2(\Omega))$ be defined by

$$\mathcal{F}(u)(t) = e^{-A(t-\tau)}u_\tau + \int_\tau^t e^{-A(t-s)}\left(b_1 f_\varepsilon(u(s)) + \omega_1 u(s)\right) ds, \tag{32}$$

where

$$V = \{u \in C([\tau, \tau+t_0], L^2(\Omega)) : u(\tau) = u_\tau,\ u(t) \geq 0,\ \|u(t) - u_\tau\| \leq 1,\ \forall t \in [\tau, \tau+t_0]\},$$

and $t_0 > 0$ satisfies (17). Let $z_0 \in L^2(\Omega)$ be such that $z_0(x) = 1$ for a.a. x. Since $f_\varepsilon(u(s)) = z_0$, for any $u \in V$, and $e^{-At} \geq 0$, we have that $\mathcal{F}(u)(t) \geq 0$. Then, arguing in the same way as in Theorem 2 we obtain that $\mathcal{F} : V \to V$ is a contracting map, so $\mathcal{F}$ possesses a unique fixed point $\overline{u} \in V$, which is a mild solution and, by Lemma 4, coincides with the unique strong solution u_ε to problem (11) with $b(t) = b_1,\ \omega(t) = \omega_1$.

Let $v(\cdot)$ be a non-negative strong solution of (2) such that $v(\tau) = u_\tau$. Then there exists h such that $h \in L^\infty\left(\tau, T; L^2(\Omega)\right)$, for any $T > \tau$, $h(s,x) \in H_0(v(s,x))$ for a.a. (s,x), and $v(\cdot)$ is the unique strong solution to problem (19). Moreover, $v(\cdot)$ satisfies (20) and it is the unique fixed point of the contractive map $\mathcal{F}_1 : V_1 \to V_1$ given by

$$\mathcal{F}_1(u)(t) = e^{-A(t-\tau)}u_\tau + \int_\tau^t e^{-A(t-s)}\left(b(s)h(s) + \omega(s)u(s)\right) ds,$$

where

$$V_1 = \{u \in C([\tau, \tau + t_0], L^2(\Omega)) : u(\tau) = u_\tau,\ \|u(t) - u_\tau\| \leq 1,\ \forall t \in [\tau, \tau + t_0]\}.$$

Let us define the set $\widehat{V} = \{u \in V : u(t) \geq v(t)\}$, which is obviously non-empty as $v \in \widehat{V}$.

For any $u \in \widehat{V}$, since $f_\varepsilon(u(\cdot, s)) = z_0$ and $u(s) \geq v(s) \geq 0$, we have

$$\begin{aligned} b_1 f_\varepsilon(u(s)) - b(s) h(s) &= b_1 z_0 - b(s) h(s) \geq (b_1 - b(s)) z_0 \geq 0, \\ \omega_1 u(s) - \omega(s) v(s) &\geq 0. \end{aligned}$$

Then, by $e^{-At} \geq 0$, we obtain

$$\begin{aligned} \mathcal{F}(u)(t) - v(t) &= \int_\tau^t e^{-A(t-s)} \left(b_1 f_\varepsilon(u(s)) - b(s) h(s) + \omega_1 u(s) - \omega(s) v(s)\right) ds \\ &\geq 0, \text{ for any } t \in [\tau, \tau + t_0]. \end{aligned}$$

Therefore, $\mathcal{F}(\widehat{V}) \subset \widehat{V}$. Since $\mathcal{F}$ is a contraction in V, it is a contraction in $\widehat{V}$ as well. We deduce that $\mathcal{F}$ possesses a unique fixed point $\overline{u} \in \widehat{V}$, which is equal to the solution u_ε to problem (11) with $b(t) = b_1,\ \omega(t) = \omega_1$. As before, by a standard continuation argument, it follows that

$$u_\varepsilon(t) \geq v(t) \text{ for any } t \geq \tau.$$

Passing to the limit in exactly the same way as in Theorem 2 we obtain the existence of a solution $\overline{u}_{b_1,\omega_1}(\cdot)$ to problem (29) such that (30) holds.

Let $\mathcal{F}_2 : V \to C([\tau, \tau + t_0], L^2(\Omega))$ be defined by

$$\mathcal{F}_2(u)(t) = e^{-A(t-\tau)} u_\tau + \int_\tau^t e^{-A(t-s)} \left(b(s) f_\varepsilon(u(s)) + \omega(s) u(s)\right) ds. \tag{33}$$

Let $z_0 \in L^2(\Omega)$ be such that $z_0(x) = 1$ for a.a. x. Since $f_\varepsilon(u(s)) = z_0$, for any $u \in V$, and $e^{-At} \geq 0$, we have $\mathcal{F}(u)(t) \geq 0$. Then, arguing in the same way as in Theorem 2 we obtain that $\mathcal{F}_2 : V \to V$ is a contracting map, so $\mathcal{F}_2$ possesses a unique fixed point $\overline{u} \in V$, which is a mild solution and, by Lemma 4, coincides with the unique strong solution u_ε to problem (11).

Let $u_{b_0,\omega_0}(\cdot)$ be a non-negative strong solution of (29) such that $u_{b_0,\omega_0}(\tau) = u_\tau$, $b = b_0,\ \omega = \omega_0$. Then there exists h such that $h \in L^\infty\left(\tau, T; L^2(\Omega)\right)$, for any $T > \tau$, $h(s,x) \in H_0(u_{b_0,\omega_0}(s,x))$ for a.a. (s,x), and $u_{b_0,\omega_0}(\cdot)$ is the unique strong solution to problem (19) with $b(t) = b_0,\ \omega(t) = \omega_0$. Moreover, $u_{b_0,\omega_0}(\cdot)$ satisfies (20) and it is the unique fixed point of the contractive map $\mathcal{F}_3 : V_1 \to V_1$ given by

$$\mathcal{F}_3(u)(t) = e^{-A(t-\tau)} u_\tau + \int_\tau^t e^{-A(t-s)} \left(b_0 h(s) + \omega_0 u(s)\right) ds.$$

We define the set $\widehat{V}_1 = \{u \in V : u(t) \geq u_{b_0,\omega_0}(t)\}$, which is non-empty as $u_{b_0,\omega_0} \in \widehat{V}$.

As before, for any $u \in \widehat{V}_1$ we have

$$\begin{aligned} b(s) f_\varepsilon(u(s)) - b_0 h(s) &= b(s) z_0 - b_0 h(s) \geq (b(s) - b_0) z_0 \geq 0, \\ \omega(s) u(s) - \omega_0 u_{b_0,\omega_0}(s) &\geq 0, \end{aligned}$$

and

$$\mathcal{F}_2(u)(t) - u_{b_0,\omega_0}(t) = \int_{\tau}^{t} e^{-A(t-s)} \left(b(s) f_{\varepsilon}(u(s)) - b_0 h(s) + \omega(s) u(s) - \omega_0 u_{b_0,\omega_0}(s)\right) ds$$
$$\geq 0, \text{ for any } t \in [\tau, \tau + t_0].$$

Thus, $\mathcal{F}_2(\widehat{V}) \subset \widehat{V}$. Since $\mathcal{F}_2$ is a contraction in V, it is a contraction in $\widehat{V}$ as well, so $\mathcal{F}_2$ possesses a unique fixed point $u^* \in \widehat{V}$, which is equal to the solution u_{ε} to problem (11). As before, by a standard continuation argument, it follows that

$$u_{\varepsilon}(t) \geq u_{b_0,\omega_0}(t) \text{ for any } t \geq \tau.$$

Again, passing to the limit we obtain the existence of a solution $\overline{u}(\cdot)$ to problem (29) such that (31) holds.

3 Characterization of the global attractor in the autonomous case

In this section we will study the autonomous differential inclusion (2) in the scalar case and will deduce from the strong comparison principle some properties concerning the structure of the global attractor.

Hence, we consider the autonomous problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} \in bH_0(u) + \omega u, \text{ on } (0,1) \times (\tau, \infty), \\ u|_{\partial\Omega} = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{34}$$

where $0 < b$, $0 \leq \omega$. We assume also throughout this section that

$$0 \leq \omega < \pi^2,$$

where π^2 is the first eigenvalue of the operator $-\dfrac{\partial^2}{\partial x^2}$ in $H_0^1(0,1)$. This restriction is necessary in order to guarantee the existence of a global attractor.

Let $\mathcal{D}(u_0)$ denote the set of all strong solutions of (34) for $u_0 \in L^2(\Omega)$ and $\tau = 0$. We define the multivalued family of operators $G : \mathbb{R}^+ \times L^2(\Omega) \to P(L^2(\Omega))$, where $P(X)$ stands for the set of all non-empty subsets of the space X, by

$$G(t, u_0) = \{u(t) : u(\cdot) \in \mathcal{D}(u_0)\}.$$

It is well-known [36] that G is a strict multivalued semiflow, i.e., $G(t+s, u_0) = G(t, G(s, u_0))$, for all $t, s \geq 0$, $u_0 \in L^2(\Omega)$, possessing a global compact invariant attractor $\mathcal{A}$. This means that:

- $\mathcal{A} = G(t, \mathcal{A})$ for all $t \geq 0$ (strict invariance);
- $dist(G(t, B), \mathcal{A}) \to 0$ as $t \to +\infty$ for any bounded set B (attracting property).

Moreover, the attractor $\mathcal{A}$ is connected [37] and bounded in $H_0^1(\Omega)$ [36]. Also, G has compact values and the operator $u_0 \mapsto G(t, u_0)$ is upper semicontinuous for any $t \geq 0$ (see [36] again).

We recall the concept of order-preserving multivalued semiflow, which was introduced in [11].

Definition 6 The multivalued semiflow G is called order-preserving if for any $u_0 \leq v_0$ and $t \geq 0$ we have:

1. There exists $\underline{y} \in G(t, u_0)$ such that
$$\underline{y} \leq y \text{ for all } y \in G(t, v_0).$$
2. There exists $\overline{y} \in G(t, v_0)$ such that
$$y \leq \overline{y} \text{ for all } y \in G(t, u_0).$$

Theorem 2 implies that the semiflow G generated by the solutions of (2) is order-preserving.

We recall that x is called a fixed point (or equilibrium) of the multivalued semiflow G if $x \in G(t, x)$ for all $t \geq 0$.

The set of strong solutions $\mathcal{R} = \cup_{u_0 \in L^2(\Omega)} \mathcal{D}(u_0)$ satisfies the following properties [18]:

$(H1)$ For any $x \in L^2(\Omega)$ there exists $\varphi \in \mathcal{R}$ such that $\varphi(0) = x$.
$(H2)$ $\varphi_\tau(\cdot) = \varphi(\cdot + \tau) \in \mathcal{R}$ for any $\tau \geq 0$, $\varphi(\cdot) \in \mathcal{R}$ (translation property).
$(H3)$ Let $\varphi_1, \varphi_2 \in \mathcal{R}$ be such that $\varphi_2(0) = \varphi_1(s)$, where $s > 0$. Then the function $\varphi(\cdot)$, defined by
$$\varphi(t) = \begin{cases} \varphi_1(t) & \text{if } 0 \leq t \leq s, \\ \varphi_2(t-s) & \text{if } s \leq t, \end{cases}$$
belongs to $\mathcal{R}$ (concatenation property).
$(H4)$ For any sequence $\varphi^n(\cdot) \in \mathcal{R}$ such that $\varphi^n(0) \to \varphi_0$ in $L^2(\Omega)$, there exists a subsequence φ^{n_k} and $\varphi \in \mathcal{R}$ such that
$$\varphi^{n_k}(t) \to \varphi(t), \ \forall t \geq 0.$$

The element x is called a fixed point (or equilibrium) of $\mathcal{R}$ if $\varphi(t) \equiv x \in \mathcal{R}$. Since $(H1) - (H4)$ hold, it is well-known that x is a fixed point of G if and only if it is a fixed point of $\mathcal{R}$ [23, Lemma 7].

Applying Theorem 2 in [11] we obtain the following result.

Theorem 4 *There exist two equilibria x_*, $y^* \in \mathcal{A}$ such that:*

1. $x_* \leq z \leq y^*$ *for all* $z \in \mathcal{A}$.
2. *If the solutions corresponding to the initial conditions x_*, y^* are unique, then*
$$dist(G(t, u_0), x_*) \to 0, \text{ as } t \to +\infty, \text{ for any } u_0 \leq x_*, \tag{35}$$
$$dist(G(t, u_0), y^*) \to 0, \text{ as } t \to +\infty, \text{ for any } u_0 \geq y^*. \tag{36}$$

We observe that in [11] the following additional assumption concerning the order relation '$\leq$' was assumed: for any bounded set B there exists a, d such that
$$a \leq y \leq d \text{ for all } y \in B,$$
which means that B is contained in an interval $[a, d]$. Though this assumption is not true in the space $L^2(\Omega)$, in the proof of Theorem 2 in [11] it is only necessary to use this property for the global attractor $\mathcal{A}$, and not for an arbitrary bounded set B. Since $\mathcal{A}$ is bounded in $H_0^1(\Omega)$, which is continuously embedded in $C([0, 1])$,

the global attractor is in fact contained in an interval $[a, d]$, so Theorem 2 in [11] is applicable.

The fixed points of G were described explicitly in [4]. There exists an infinite but countable number of fixed points, denoted by $v_0 = 0, v_1^+(x),\ v_1^-(x), ..., v_n^+(x),\ v_n^-(x), ...,$ where $v_j^\pm(x)$ possess exactly $j+1$ zeroes in $[0,1]$ and $v_j^+(x) = -v_j^-(x)$. The exact expression for the point v_1^+ is given by

$$v_1^+(x) = \frac{b_0}{\omega_0}\cos(\sqrt{\omega_0}x) + \frac{b_0(1-\cos(\sqrt{\omega_0}))}{\omega_0\sin(\sqrt{\omega_0})}\sin(\sqrt{\omega_0}x) - \frac{b_0}{\omega_0},$$

which is the unique solution of the boundary-value problem

$$u'' + \omega_0 u = -b_0,\ u(0) = u(1) = 0.$$

From the analysis in [4] we infer that

$$v_1^-(x) \leq v_j^\pm(x) \leq v_1^+(x) \text{ for all } j \geq 1,$$

and then $x_* = v_1^-$, $y^* = v_1^+$ so Theorem 4 implies that

$$v_1^- \leq z \leq v_1^+ \text{ for all } z \in \mathcal{A}.$$

Also, since the points $v_1^\pm$ are stable [4, Theorem 6.3], the solutions corresponding to the initial conditions $v_1^\pm$ are unique. Hence, the convergences (35), (36) hold true.

Finally, we remark that in [4, Theorem 6.3] the structure of the global attractor was studied.

We recall that a map $\phi\colon \mathbb{R} \to L^2(\Omega)$ is a complete trajectory if

$$\phi(\cdot + h)|_{[0,\infty)} \in \mathcal{R},\ \text{for all } h \in \mathbb{R}.$$

The global attractor $\mathcal{A}$ consists of all bounded complete trajectories and it consists if fact of the fixed points and all complete bounded trajectories $\psi(\cdot)$ connecting two fixed points, that is,

$$\begin{aligned} \psi(t) &\to z_1 \text{ as } t \to +\infty, \\ \psi(t) &\to z_2 \text{ as } t \to -\infty, \end{aligned} \tag{37}$$

where z_j are fixed points. Partial results related to how the fixed points are connected were given in [4, Theorem 6.3]: if $v \rightsquigarrow z$ means that there is a connection from v to z, then:

1. $v_0 \rightsquigarrow v_j^\pm$, $\forall j \geq 1$;
2. $v_j^+ \rightsquigarrow v_{j-1}^\pm$, $v_j^- \rightsquigarrow v_{j-1}^\pm$, $\forall j \geq 2$;
3. $v_j^+ \rightsquigarrow v_1^\pm$, $v_j^- \rightsquigarrow e_1^\pm$, $\forall j \geq 2$;
4. If $v_k^\pm \rightsquigarrow v_j^\pm$ $(k, j \neq 0)$, then
$$v_{kn}^\pm \rightsquigarrow v_{jn}^\pm,\ \forall n \geq 1;$$
5. If $1 \leq k \leq j$, then $v_k^\pm \rightsquigarrow v_j^\pm$, $v_k^\pm \rightsquigarrow v_0$ are forbidden.

Now we have completed this description by showing that all these bounded complete trajectories lie inside the interval $[v_1^-, v_1^+]$.

4 Characterization of the pullback attractor in the nonautonomous case

In this section we will treat the nonautonomous differential inclusion (2) in the scalar case, that is, we consider the nonautonomous problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} \in b(t)H_0(u) + \omega(t)u, \text{ on } (0,1)\times(\tau,\infty), \\ u|_{\partial\Omega} = 0, \\ u(\tau,x) = u_0(x), \end{cases} \tag{38}$$

where $b\left(t\right)$, $\omega\left(t\right)$, $H_0(u)$ are as in Section 2. Additionally, we assume in the sequel that

$$\omega_1 < \pi^2.$$

We start by defining a multivalued process associated with the strong solutions of problem (38) and showing that it possesses a pullback attractor. After that we will give a characterization of this attractor in a similar manner as in Theorem 4 for the autonomous case. In particular, we will show the existence of a bounded complete non-degenerate trajectory at $-\infty$ which is unique in the class of non-degenerate bounded complete trajectories in the whole line.

Let $\mathcal{D}_\tau(u_\tau)$ be the set of all strong solutions of (38) with initial condition u_τ at time τ and let $\mathcal{R}_\tau = \cup_{x\in L^2(\Omega)}\mathcal{D}_\tau(x)$. In the same way as in the autonomous case, one can prove that the sets $\mathcal{R}_\tau$ satisfy the following properties:

$(K1)$ For any $\tau \in \mathbb{R}$ and $x \in L^2(\Omega)$ there exists $\varphi \in \mathcal{R}_\tau$ such that $\varphi\left(\tau\right) = x$.
$(K2)$ $\varphi_s = \varphi\mid_{[\tau+s,\infty)} \in \mathcal{R}_{\tau+s}$ for any $s \geq 0$, $\varphi \in \mathcal{R}_\tau$ (translation property).
$(K3)$ Let $\varphi, \psi \in \mathcal{R}$ be such that $\varphi \in \mathcal{R}_\tau$, $\psi \in \mathcal{R}_r$ and $\varphi(s) = \psi(s)$ for some $s \geq r \geq \tau$. Then the function θ defined by

$$\theta(t) := \begin{cases} \varphi(t), \ t \in [\tau, s], \\ \psi(t), \ t \in [s,\infty), \end{cases}$$

belongs to $\mathcal{R}_\tau$ (concatenation property).
$(K4)$ For any sequence $\varphi^n \in \mathcal{R}_\tau$ such that $\varphi^n\left(\tau\right) \to \varphi_0$ in $L^2(\Omega)$, there exists a subsequence φ^{n_k} and $\varphi \in \mathcal{R}_\tau$ such that

$$\varphi^{n_k}\left(t\right) \to \varphi\left(t\right), \, \forall t \geq \tau.$$

We define the multivalued family of operators $U : \mathbb{R}^2_{\geq} \times L^2(\Omega) \to P(L^2(\Omega))$, where $\mathbb{R}^2_{\geq} = \{(t,s) \in \mathbb{R}^2 : t \geq s\}$, by

$$U(t,s,x) = \{u(t) : u(\cdot) \in \mathcal{D}_s(x)\}.$$

It easily follows from $(K1) - (K3)$ that U is a strict multivalued process, that is, $U\left(t,t,\cdot\right) = Id$ is the identity map and $U\left(t,s,x\right) = U\left(t,\tau,U\left(\tau,s,x\right)\right)$ for all $s \leq \tau \leq t$, $x \in L^2(\Omega)$. Moreover, $(K4)$ implies that the graph of the map $x \mapsto U(t,s,x)$ is closed for all $(t,x) \in \mathbb{R}_d$.

We recall that the familiy of sets $\{K(t)\}_{t\in\mathbb{R}}$ is called pullback attracting for U if it attracts every bounded set B in the pullback sense, that is,

$$dist(U(t,s,B),K(t)) \to 0, \text{ as } s \to -\infty. \tag{39}$$

Lemma 5 *The process U has a pullback attracting family of compact sets $\{K(t)\}_{t\in\mathbb{R}}$.*

Proof For any strong solution multiplying equality (3) by u we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2+\left\|\frac{\partial u}{\partial x}(t)\right\|^2 &= \int_0^1 r\left(t,x\right)u(t,x)dx \\ &\leq b\left(t\right)\int_0^1 \left|u\left(t,x\right)\right|dx+\omega\left(t\right)\left\|u\left(t\right)\right\|^2 \\ &\leq b_1\left\|u\left(t\right)\right\|+\omega_1\left\|u\left(t\right)\right\|^2 \\ &\leq \frac{b_1^2}{4\varepsilon_0}+\left(\varepsilon_0+\omega_1\right)\left\|u\left(t\right)\right\|^2,\end{aligned}$$

where ε_0 is chosen such that $\varepsilon_0+\omega_1<\pi^2$. Then, as π^2 is the first eigenvalue of the operator $-\dfrac{\partial^2 u}{\partial x^2}$ in $H_0^1\left(\Omega\right)$, we obtain

$$\frac{d}{dt}\|u\|^2+\delta\left\|u\left(t\right)\right\|^2\leq\frac{d}{dt}\|u\|^2+\frac{\delta}{\pi^2}\left\|\frac{\partial u}{\partial x}(t)\right\|^2\leq\frac{b_1^2}{2\varepsilon_0}=C_1,$$

where $\delta=2(\pi^2-\omega_1-\varepsilon_0)>0$. By Gronwall's lemma we get

$$\left\|u\left(t\right)\right\|^2\leq e^{-\delta(t-s)}\left\|u\left(s\right)\right\|^2+\frac{C_1}{\delta}\text{ for all }t\geq s. \tag{40}$$

Also, integrating over $(t-\alpha,t)$, where $0<\alpha\leq 1$, we have

$$\int_{t-\alpha}^t\left\|\frac{\partial u}{\partial x}\right\|^2dr\leq\frac{\pi^2C_1}{\delta}+\frac{\pi^2}{\delta}\left\|u(t-\alpha)\right\|^2. \tag{41}$$

Further, we multiply (3) by $\dfrac{du}{dt}$ and use Corollary 1 to obtain that

$$\begin{aligned}\left\|\frac{du}{dt}\right\|^2+\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial x}\right\|^2 &\leq b\left(t\right)\int_0^1\left|\frac{\partial u}{\partial x}(t,x)\right|dx+\omega\left(t\right)\int_0^1 u\left(t,x\right)\frac{\partial u}{\partial x}(t,x)dx \\ &\leq b_1\left\|\frac{\partial u}{\partial x}\left(t\right)\right\|+\omega_1\left\|u\left(t\right)\right\|\left\|\frac{\partial u}{\partial x}\left(t\right)\right\| \\ &\leq\frac{b_1^2}{2}+\frac{\omega_1^2}{2}\left\|u\left(t\right)\right\|^2+\left\|\frac{\partial u}{\partial x}\left(t\right)\right\|^2.\end{aligned} \tag{42}$$

For $s\leq t-\alpha\leq r\leq t$ we integrate over the interval (r,t). Hence, by (40) and (41) we have

$$\begin{aligned}\left\|\frac{\partial u}{\partial x}(t)\right\|^2 &\leq\left\|\frac{\partial u}{\partial x}(r)\right\|^2+b_1^2+\omega_1^2\int_r^t\left\|u\left(\tau\right)\right\|^2d\tau+2\int_r^t\left\|\frac{\partial u}{\partial x}\left(\tau\right)\right\|^2d\tau \\ &\leq\left\|\frac{\partial u}{\partial x}(r)\right\|^2+b_1^2+(\omega_1^2+\frac{2\pi^2}{\delta})e^{-\delta(t-\alpha-s)}\left\|u\left(s\right)\right\|^2+\frac{C_1}{\delta}(\omega_1^2+2\pi^2+\frac{2\pi^2}{\delta}).\end{aligned}$$

Integrating now with respect to the variable r over the interval $(t-\alpha,t)$ and using again (40) and (41) we get

$$\begin{aligned}\alpha\left\|\frac{\partial u}{\partial x}(t)\right\|^2 &\leq\frac{\pi^2C_1}{\delta}+\frac{\pi^2}{\delta}\left\|u(t-\alpha)\right\|^2 \\ &\quad+b_1^2+(\omega_1^2+\frac{2\pi^2}{\delta})(e^{-\delta(t-\alpha-s)}\left\|u\left(s\right)\right\|^2)+\frac{C_1}{\delta}(\omega_1^2+2\pi^2+\frac{2\pi^2}{\delta}) \\ &\leq C_2+C_3e^{-\delta(t-\alpha-s)}\left\|u\left(s\right)\right\|^2,\end{aligned} \tag{43}$$

where $C_2, C_3 > 0$ are some constants.

We define the family $K(t)$ by

$$K(t) = \{y \in H_0^1(\Omega) : \|y\|_{H_0^1}^2 \leq \frac{2C_2}{\alpha}\}.$$

The compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$ implies that $K(t)$ are relatively compact in $L^2(\Omega)$. Also, as $K(t)$ is weakly closed in $H_0^1(\Omega)$, it is closed in $L^2(\Omega)$. Thus, $K(t)$ are compact in $L^2(\Omega)$. Finally, we obtain readily from (43) that for any bounded set B and any $t \in \mathbb{R}$ there exists $T(B,t)$ such that $U(t,s,B) \subset K(t)$ for all $s \leq T(B,t)$. Thus, (39) follows.

We recall the definition of pullback attractor.

Definition 7 The family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is called a pullback attractor if:

1. It is pullback attracting.
2. $\mathcal{A}(t) \subseteq U(t,s,\mathcal{A}(s))$, for all $t \geq s$ (negative semi-invariance);
3. $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is minimal in the sense that if $\{K(t)\}_{t\in\mathbb{R}}$ is a pullback attracting family of closed sets, then $\mathcal{A}(t) \subset K(t)$ for all $t \in \mathbb{R}$.

The pullback attractor is strictly invariant if $\mathcal{A}(t) = U(t,s,\mathcal{A}(s))$, for any $t \geq s$.

Theorem 5 *The process U possesses a strictly invariant pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$. Moreover, $\cup_{t\in\mathbb{R}}\mathcal{A}(t)$ is bounded in $H_0^1(\Omega)$ and $\overline{\cup_{t\in\mathbb{R}}\mathcal{A}(t)}$ is compact in $L^2(\Omega)$.*

Proof Since there exists a pullback attracting family of compact sets $\{K(t)\}_{t\in\mathbb{R}}$ and the map $x \mapsto U(t,\tau,x)$ has closed graph for all $t \geq \tau$ (by (K4)), there exists a compact pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, which satisfies $\mathcal{A}(t) \subset K(t)$ for all $t \in \mathbb{R}$ (see Theorem 1 in [22]). Moreover, since $\mathcal{A}(t) \subset K(t)$ and the definition of $K(t)$ we deduce that $\cup_{t\in\mathbb{R}}\mathcal{A}(t)$ is bounded in $H_0^1(\Omega)$ and $\overline{\cup_{t\in\mathbb{R}}\mathcal{A}(t)}$ is compact in $L^2(\Omega)$. Using this and the fact that U is a strict process we also obtain that the pullback attractor is strictly invariant (see Lemma 2.5 in [12] or Proposition 4.3 in [19]).

A map $\gamma : \mathbb{R} \to L^2(\Omega)$ is called a complete trajectory if

$$\varphi = \gamma|_{[\tau,+\infty)} \in \mathcal{R}_\tau, \text{ for all } \tau \in \mathbb{R}. \tag{44}$$

It is obvious that any complete trajectory satisfies

$$\gamma(t) \in U(t,s,\gamma(s)) \text{ for all } s \leq t. \tag{45}$$

The complete trajectory γ is said to be bounded if $\cup_{r\in\mathbb{R}}\gamma(r)$ is a bounded set. By the pullback attracting property and (45) it is easy to see that if $\gamma(\cdot)$ is a bounded complete trajectory, then $\gamma(t) \subset \mathcal{A}(t)$ for any $t \in \mathbb{R}$, where $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is the pullback attractor.

We have the following characterization of the pullback attractor.

Lemma 6 $\mathcal{A}(t) = \{\gamma(t) : \gamma$ *is a bounded complete trajectory*$\}$.

Proof Since $(K1) - (K4)$ are satisfied and $\cup_{t\in\mathbb{R}}\mathcal{A}(t)$ is bounded, the result follows either from [12, Corollary 2.10] or [12, Corollary 2.12].

We will also prove that the sets $\mathcal{A}(t)$ are compact in $H_0^1(\Omega)$.

Lemma 7 *Let $u_\tau^n \to u_\tau$ in $L^2(\Omega)$ and let $u^n \in \mathcal{D}_\tau(u_\tau^n)$. Then there exists a subsequence $\{u^{n_k}\}$ and $u \in \mathcal{D}_\tau(u_\tau)$ such that*

$$u^{n_k} \to u \text{ in } C([\tau+r,T], H_0^1(\Omega)) \text{ for all } 0<r<T-\tau,\ T>\tau. \tag{46}$$

Proof We know by $(K4)$ that there exists $u \in \mathcal{D}_\tau(u_\tau)$ such that, up to a subsequence, $u^n(t) \to u(t)$ in $L^2(\Omega)$ for any $t \geq \tau$. We need to prove that (46) holds for this solution u.

We fix an interval $[\tau+r,T]$. Taking in (43) $s=\tau$, $t-\alpha=\tau$, we obtain a constant $D_1 = D_1(r)$ such that

$$\left\|u^n(t)\right\|_{H_0^1(\Omega)} \leq D_1,\ \forall t \in [\tau+r,T]. \tag{47}$$

By integrating (42) over $(\tau+r,T)$ and using (40) and (47) we have a constant $D_2 = D_2(\tau,r,T)$ satisfying

$$\int_{\tau+r}^T \left\|\frac{du^n}{dt}\right\|^2 ds \leq D_2. \tag{48}$$

Hence, Ascoli-Arzelá theorem implies, passing to a subsequence, that

$$u^n \to u \text{ in } C([\tau+r,T], L^2(\Omega)).$$

Also, from (48) and equality (3) we get a constant $D_3 = D_3(\tau,r,T)$ such that

$$\int_{\tau+r}^T \left\|\frac{\partial^2 u^n}{\partial x^2}\right\|^2 ds \leq D_3.$$

These inequalities and the Compactness Theorem [26, p.58] imply that

$$\begin{aligned}
u^n &\to u \text{ weakly star in } L^\infty(\tau+r,T;H_0^1(\Omega)),\\
u^n &\to u \text{ weakly in } L^2(\tau+r,T;H^2(\Omega)),\\
\frac{du^n}{dt} &\to \frac{du}{dt} \text{ weakly in } L^2(\tau+r,T;L^2(\Omega)),\\
u^n &\to u \text{ strongly in } L^2(\tau+r,T;H_0^1(\Omega)),\\
u^n(t) &\to u(t) \text{ in } H_0^1(\Omega) \text{ for a.a. } t \in (\tau+r,T).
\end{aligned}$$

In view of (9) we obtain also that $u^n, u \in C([\tau+r,T], H_0^1(\Omega))$.

Now, making use of (42) and (47) we deduce the existence of $D_4 = D_4(r)$ satisfying

$$\left\|u^n(t)\right\|_{H_0^1(\Omega)}^2 \leq \left\|u^n(s)\right\|_{H_0^1(\Omega)}^2 + D_4(t-s), \text{ for } \tau+r \leq s \leq t \leq T,$$

and the same inequality is true for the limit function u. Hence, the functions $J_n(t) = \|u^n(t)\|_{H_0^1(\Omega)}^2 + D_4 t$, $J(t) = \|u(t)\|_{H_0^1(\Omega)}^2 + D_4 t$ are continuous and non-increasing in the interval $[\tau+r,T]$. Moreover, $J_n(t) \to J(t)$ for a.a. $t \in (\tau+r,T)$.

Let $t_n \in [\tau + r, T]$ and $t_n \to t_0 \in (\tau + r, T]$. We choose $t_m \in (\tau + r, t_0)$ such that $t_m \to t_0$ and $J_n(t_m) \to J(t_m)$ for each m. It is important to observe that when we fix m the elements t_n are greater than t_m for n big enough. By the above properties we have

$$\begin{aligned} J_n(t_n) - J(t_0) &= J_n(t_n) - J_n(t_m) + J_n(t_m) - J(t_m) + J(t_m) - J(t_0) \\ &\leq J_n(t_m) - J(t_m) + J(t_m) - J(t_0) \leq \varepsilon, \end{aligned}$$

if $n \geq N(\varepsilon, m(\varepsilon))$, where $\varepsilon > 0$. We infer that $\limsup \|u(t_n)\|^2_{H^1_0(\Omega)} \leq \|u(t_0)\|^2_{H^1_0(\Omega)}$. Since $u(t_n) \to u(t_0)$ weakly in $H^1_0(\Omega)$, then $\liminf \|u(t_n)\|^2_{H^1_0(\Omega)} \geq \|u(t_0)\|^2_{H^1_0(\Omega)}$, so $\lim \|u(t_n)\|^2_{H^1_0(\Omega)} = \|u(t_0)\|^2_{H^1_0(\Omega)}$ and thus

$$u^n(t_n) \to u(t_0) \text{ in } H^1_0(\Omega).$$

As this argument is valid also in the interval $[\tau + \frac{r}{2}, T]$, this convergence holds for $t_n \to \tau + r$ as well.

Using a standard diagonal procedure we obtain that (46) is true.

Corollary 6 *The sets $\mathcal{A}(t)$ are compact in $H^1_0(\Omega)$.*

Proof Let $y_n \in \mathcal{A}(t)$, $t \in \mathbb{R}$. Since $\mathcal{A}(t)$ is compact in $L^2(\Omega)$, up to a subsequence $y_n \to y$ in $L^2(\Omega)$. The invariance of $\mathcal{A}(t)$ implies the existence of solutions $u_n(\cdot) \in \mathcal{R}_{t-1}$ such that $u_n(t) = y_n$ and $u_n(t-1) \in \mathcal{A}(t-1)$. Again, passing to a subsequence $u_n(t-1) \to \overline{u}$ in $L^2(\Omega)$. Hence, by Lemma 7 we obtain the existence of $u(\cdot) \in \mathcal{D}_{t-1}(\overline{u})$ such that $y_n = u_n(t) \to u(t)$ in $H^1_0(\Omega)$. This proves that the sets $\mathcal{A}(t)$ are relatively compact in $H^1_0(\Omega)$. As they are closed in $L^2(\Omega)$, so they are in $H^1_0(\Omega)$. Thus, $\mathcal{A}(t)$ are compact in $H^1_0(\Omega)$.

Further we are going to give a deeper characterization of the pullback attractor by showing that any bounded complete trajectory is contained in an interval defined by two special bounded complete trajectories.

Let $w^+_{b_i,\omega_i}$, $i = 0, 1$, denote the positive fixed point v^+_1 of problem (29) for the parameters $b = b_i$, $\omega = \omega_i$.

Theorem 6 *There exists a bounded complete trajectory ξ_M such that any complete bounded trajectory γ satisfies*

$$-\xi_M(t) \leq \gamma(t) \leq \xi_M(t) \text{ for all } t \in \mathbb{R}. \tag{49}$$

Moreover,

$$w^+_{b_0,\omega_0} \leq \xi_M(t) \leq w^+_{b_1,\omega_1}, \tag{50}$$

$$-\xi_M(t) \leq y \leq \xi_M(t) \text{ for all } y \in \mathcal{A}(t), \tag{51}$$

$$-\xi_M(t) \leq \lim \inf_{s \to -\infty} u(t) \leq \lim \sup_{s \to -\infty} u(t) \leq \xi_M(t), \tag{52}$$

uniformly for $u \in \mathcal{D}_s(u_\tau)$, $u_\tau \in B$, where B is bounded.

Proof Let $\mathcal{D}_\tau^+(u_\tau)$ be the set of all non-negative strong solutions of (2) with initial condition u_τ at time τ (which is non-empty for any $u_\tau \in L^2(\Omega)$ by Corollary 5) and let $\mathcal{R}_\tau^+ = \cup_{x\in L^2(\Omega)}\mathcal{D}_\tau^+(x)$, that is, $\mathcal{R}_\tau^+$ is the set of all non-negative strong solutions of problem (2).

We divide the proof into several steps.

Step 1. Theorem 3 and the fact that the solution of (29) corresponding to the initial condition w_{b_1,ω_1}^+ is unique imply that for any $u \in \mathcal{D}_\tau^+(w_{b_1,\omega_1}^+)$,

$$u(t) \le w_{b_1,\omega_1}^+, \text{ for all } t \ge \tau, \tag{53}$$

and the existence of $v \in \mathcal{D}_\tau^+(w_{b_0,\omega_0}^+)$ such that

$$w_{b_0,\omega_0}^+ \le v(t), \text{ for all } t \ge \tau. \tag{54}$$

Using the comparison principle given in Theorem 2 we can choose a maximal solution $u_{\max}^\tau$ of the set $\mathcal{D}_\tau(w_{b_1,\omega_1}^+)$. As (54) and $u_{\max}^\tau(t) \ge v(t)$ imply that

$$u_{\max}^\tau(t) \ge w_{b_0,\omega_0}^+, \text{ for all } t \ge \tau, \tag{55}$$

and $w_{b_0,\omega_0}^+(x) > 0$ for $x \in (0,1)$, $u_{\max}^\tau$ is the unique solution of the linear problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = b(t) + \omega(t)u, \text{ on } \Omega \times (\tau,\infty), \\ u|_{\partial\Omega} = 0, \\ u(x,\tau) = u_\tau(x). \end{cases} \tag{56}$$

Let $s_1 < s_2 \le t$ and let $u_i = u_{\max}^{s_i}$, $i = 1,2$, be the maximal solutions of the sets $\mathcal{D}_{s_i}(w_{b_1,\omega_1}^+)$. By (53), (55) and the strong comparison principle we get

$$\begin{gathered} u_1(s_2) \le w_{b_1,\omega_1}^+, \\ w_{b_0,\omega_0}^+ \le u_1(t) \le u_2(t). \end{gathered} \tag{57}$$

Thus, the function $s \mapsto u_{\max}^s(t)$ is non-increasing as $s \to -\infty$ and it is bounded from below. Then there exists a function $\xi_M(t)$ such that

$$\xi_M(t,x) = \lim_{s\to-\infty} u_{\max}^s(t,x) \text{ pointwise in } x.$$

However, by Lemma 5 and Lemma 7 we have that $u_{\max}^s(t) \to \xi_M(t)$ in $H_0^1(\Omega) \subset C([0,1)$, for any $t \in \mathbb{R}$, so

$$\xi_M(t,x) = \lim_{s\to-\infty} u_{\max}^s(t,x) \text{ uniformly in } x. \tag{58}$$

Step 2. $\xi_M(t)$ is a bounded complete trajectory.

We fix some $t_0 \in \mathbb{R}$ and put $x_n = u_{\max}^{s_n}(t_0)$, $\varphi_n(t) = u_{\max}^{s_n}(t)$, $t \ge t_0$. where $s_n \to -\infty$. Then $\varphi_n \in \mathcal{D}_{t_0}(x_n)$ and from $(K4)$ and $x_n \to \xi_M(t_0)$ we infer that $\varphi_n(t) \to \varphi(t)$, for all $t \ge t_0$, where $\varphi \in \mathcal{D}_{t_0}(\xi_M(t_0))$. By (58) we have that $\varphi(t) = \xi_M(t)$ for all $t \ge t_0$, so $\xi_M\mid_{[t_0,+\infty)} \in \mathcal{D}_{t_0}(\xi_M(t_0))$ for all $t_0 \in \mathbb{R}$. Therefore, ξ_M is a complete trajectory. Since $\xi_M(t) \in \mathcal{A}(t)$, ξ_M is bounded.

Also, from (53) and (57) we obtain (50).

Step 3. ξ_M is maximal.

Let ψ be any bounded global trajectory. Since the pullback attractor is bounded in $H_0^1(\Omega) \subset C([0,1])$, there is $\Phi \in L^2(\Omega)$ satisfying

$$\Phi \geq y \text{ for all } y \in \cup_{t\in\mathbb{R}}\mathcal{A}(t).$$

Thus, $\Phi \geq \psi(t)$ for any $t \in \mathbb{R}$. Take a sequence $s_n \to -\infty$. Denote by $v_{\max}^n$ the maximal solution of the set $\mathcal{D}_{s_n}(\Phi)$. Then as $\psi(t) = u^n(t)$, where $u^n = \psi \mid_{[s_n,+\infty)} \in \mathcal{D}_{s_n}(\psi(s_n))$, Theorem 2 implies

$$\psi(t) \leq v_{\max}^n(t).$$

Let $s_n < r \leq t$ be arbitrary. Taking into account that

$$\overline{v}^n = v_{\max}^n \mid_{[r,+\infty)} \in \mathcal{D}_r^+(v_{\max}^n(r)),$$

and being $w_{\max}^n$ the maximal solution of the autonomous problem (29) at $\tau = s_n$ with initial condition $w_{\max}^n(s_n) = \Phi$ and $b = b_1,\ \omega = \omega_1$, which is the unique solution of the linear problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = b_1 + \omega_1 u, \text{ on } \Omega \times (\tau, \infty), \\ u|_{\partial\Omega} = 0, \\ u(x, s_n) = \Phi(x), \end{cases} \tag{59}$$

Theorem 3 implies that

$$v_{\max}^n(r) \leq w_{\max}^n(r).$$

Now let $p_{\max}^n$ be the maximal solution of the set $\mathcal{D}_r(w_{\max}^n(r))$. Then, again by the comparison principle,

$$\psi(t) \leq \overline{v}^n(t) \leq p_{\max}^n(t).$$

Since $w_{\max}^n$ is the unique solution to problem (59), $w_{\max}^n(r) \to w_{b_1,\omega_1}^+$ as $n \to \infty$. Hence, it follows from $(K4)$ the existence of $p \in \mathcal{D}_r^+(w_{b_1,\omega_1}^+)$ such that up to a subsequence

$$p_{\max}^n(t) \to p\,(t)\,.$$

But $u_{\max}^r$ is the maximal solution of $\mathcal{D}_r(w_{b_1,\omega_1}^+)$, so

$$\psi(t) \leq p\,(t) \leq u_{\max}^r(t).$$

Using the definition of ξ_M,

$$\psi(t) \leq \lim_{r\to-\infty} u_{\max}^r(t) = \xi_M(t),$$

and by symmetry of solutions,

$$-\xi_M(t) \leq \psi(t) \leq \xi_M(t),$$

proving (49). Therefore, Lemma 6 and the pullback attracting property of $\mathcal{A}(t)$ imply (51)-(52).

Let

$$\Phi(0,1)=\{u\in C([0,1]):u(x)>0,\,\forall x\in(0,1)\,,\ u\,(0)=u\,(1)=0\}.$$

In other words, $\Phi(0,1)$ is the subset of continuous functions on $[0,1]$ satisfying zero Dirichlet boundary conditions and being strictly positive on $(0,1)$.

Lemma 8 *Let ξ_1,ξ_2 be two bounded complete trajectories of (2) such that $\xi_i(t)\in\Phi(0,1)$ for all $t\in\mathbb{R}$. Then $\xi_1(t)=\xi_2(t)$ for all $t\in\mathbb{R}$.*

Proof Since $\xi_i(t)\in\Phi(0,1)$, $u_i=\xi_i\mid_{[s,\infty)}$ are solutions to the linear problem (56) with initial conditions $u_i(s)=\xi_i(s)$, respectively. Then, taking the difference of the two equations and multiplying by $v=u_1-u_2$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|v\right\|^2+\left\|\frac{\partial v}{\partial x}\right\|^2=\omega\left(t\right)\left\|v\right\|^2.$$

Then $\left\|\frac{\partial v}{\partial x}\right\|^2\geq\pi^2\left\|v\right\|^2$, $\omega\left(t\right)\leq\omega_1<\pi^2$ and Gronwall's lemma imply

$$\left\|v(t)\right\|^2\leq\left\|v(s)\right\|^2e^{-\beta(t-s)},$$

where $\beta=2(\pi^2-\omega_1)>0$. Passing to the limit as $s\to-\infty$ the statement follows.

Since (50) implies that $\xi_M(t)\in\Phi(0,1)$ for all $t\in\mathbb{R}$, we obtain the following straightforward consequence of Lemma 8.

Corollary 7 *ξ_M is the unique bounded complete trajectory such that $\xi_M(t)\in\Phi(0,1)$ for all $t\in\mathbb{R}$.*

The next lemma is proved exactly in the same way as Lemma 8.

Lemma 9 *Let ξ_1,ξ_2 be two bounded complete trajectories of (2) such that for some t_0 we have $\xi_i(t)\in\Phi(0,1)$ for all $t\leq t_0$, that is, they are non-degenerate at $-\infty$. Then $\xi_1(t)=\xi_2(t)$ for all $t\leq t_0$.*

Corollary 8 *If ψ is a bounded complete trajectory such that for some t_0 we have $\psi(t)\in\Phi(0,1)$ for all $t\leq t_0$, then $\psi(t)=\xi_M(t)$ for all $t\leq t_0$.*

Remark 2 If we could prove that the solutions are unique for the initial conditions $\xi_M(s)$, $s\in\mathbb{R}$, then ξ_M would be the unique bounded complete trajectory such that for some t_0 we have $\xi_M(t)\in\Phi(0,1)$ for all $t\leq t_0$, that is, the only non-degenerate bounded complete trajectory at $-\infty$.

Lemma 10 *If $u_1\left(\cdot\right),\ u_2\left(\cdot\right)\in\mathcal{R}^+_{t_0}$ are two solutions such that $u_i(t)\in\Phi(0,1)$ for all $t\geq t_0$, that is, they are non-degenerate at $+\infty$, then*

$$\left\|u_1(t)-u_2(t)\right\|\to0\ \text{as}\ t\to+\infty.$$

Proof Arguing in the same way as in Lemma 8 we obtain that

$$\left\|u_1(t)-u_2(t)\right\|^2\leq\left\|u_1(t_0)-u_2(t_0)\right\|^2e^{-\beta(t-t_0)}\to0\text{ as }t\to+\infty.$$

Corollary 9 *If* $u(\cdot) \in \mathcal{R}_{t_0}^{+}$ *is non-degenerate at* $+\infty$*, then*

$$\|u(t) - \xi_M(t)\| \to 0 \text{ as } t \to +\infty.$$

In fact, we can prove that for every non-negative initial datum at least one non-degenerate solution at $+\infty$ exists.

Lemma 11 *Let* $u_s \geq 0$*,* $s \in \mathbb{R}$*. Then there exists* $u(\cdot) \in \mathcal{R}_{t_0}^{+}$ *with* $u(s) = u_s$ *such that*

$$\|u(t) - \xi_M(t)\| \to 0 \text{ as } t \to +\infty.$$

Proof Let $u(\cdot)$ be the unique solution to the linear problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = b(t) + \omega(t)u, \text{ on } \Omega \times (\tau, \infty), \\ u|_{\partial\Omega} = 0, \\ u(x,s) = u_s(x), \end{cases}$$

which clearly belongs to $\mathcal{R}_s$ and, moreover, is the maximal solution for the initial condition u_s. Since $\xi_M(\cdot)$ is a solution of the same problem but with initial condition $\xi_M(s)$, arguing as before we have

$$\|u(t) - \xi_M(t)\|^2 \leq \|u_s - \xi_M(s)\|^2 e^{-\beta(t-s)} \to 0 \text{ as } t \to +\infty.$$

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References

1. L. Arnold, I. Chueshov, Order-preserving random dynamical systems: equilibria, attractors, equilibria, *Dynamics and Stability of Systems,* **14** (1998), 265-272.
2. L. Arnold, I.Chueshov, A limit set trichotomy for order-preserving random systems, *Positivity,* **5** (2001), 95-114.
3. J.P. Aubin, A. Cellina, "Differential Inclusions", Springer-Verlag, Berlin, 1984.
4. J.M. Arrieta, A. Rodríguez-Bernal, J. Valero, Dynamics of a reaction-diffusion equation with a discontinuous nonlinearity, *International Journal of Bifurcations and Chaos*, **16** (2006), 2965-2984.
5. J.M. Arrieta, A. Rodríguez-Bernal,, A. Vidal-López, Regiones atractoras delimitadas por puntos de equilibrio extremales para ecuaciones de reacción-difusión, Proceedings in CDRom of the XVIII CEDYA-VIII CMA, Tarragona (Spain), 2003.
6. V. Barbu, "Nonlinear Semigroups and Differential Equations in Banach Spaces", Editura Academiei, Bucuresti, 1976.
7. H. Brézis, "Operateurs maximaux monotones et semi-groups de contractions dans les espaces de Hilbert", North-Holland, Amsterdam, 1973.
8. S. Bensid, J.I. Díaz, Stability results for discontinuous nonlinear elliptic and parabolic problems with a s-shaped bifurcation branch of stationary solutions, *Discrete Contin. Dyn. Syst., Ser. B*, **22** *(*2017), 1757-1778.
9. S. Bensid, J.I. Díaz, On the exact number of monotone solutions of a simplified Budyko climate model and their different stability, *Discrete Contin. Dyn. Syst., Ser. B* (to appear).
10. M.I. Budyko, The effects of solar radiation variations on the climate of the Earth, *Tellus*, **21** (1969), 611-619.
11. T. Caraballo, J.A. Langa, J. Valero, Asymptotic behaviour of monotone multi-valued dynamical systems, *Dynam. Syst.*, **20** (2005), 301-321.

12. T. Caraballo, J.A. Langa, J. Valero, Structure of the pullback attractor for a non-autonomous scalar differential inclusion, *Discrete Cont. Dyn. Systems, Series S*, **9** (2016), 979-994.
13. A.N. Carvalho, J.A. Langa, J. Robinson, "Attractors for infinite-dimensional non-autonomous dynamical systems", Springer, New-York, 2013.
14. A.N. Carvalho, C.B. Gentile, Comparison results for nonlinear parabolic equations with monotone principal part, *J. Math. Anal. Appl.*, **259** (2001), 319-337.
15. V.V. Chepyzhov, M.I.Vishik, "Attractors for equations of mathematical physics", American Mathematical Society, Providence, Rhode Island, 2002.
16. I.D. Chueshov, Order-preserving skew-product flows and nonautonomous parabolic systems, *Acta Appl. Math.*, 65 (2001), 185-205.
17. I.D. Chueshov, "Monotone Random Systems: Theory and Applications", Lecture Notes in Mathematics 1779, Springer-Verlag, Berlin, 2002.
18. H.B. da Costa, J. Valero, Morse decompositions and Lyapunov functions for dynamically gradient multivalued semiflows, *Nonlinear Dyn.*, **84** (2016), 19-34.
19. M. Coti Zelati, P. Kalita, Minimality properties of set-valued processes and their pullback attractors, *SIAM J. Math. Anal.*, **47** (2015), 1530-1561.
20. J.I. Díaz, J. Hernández, L.Tello, Some results about multiplicity and bifurcation of stationary solutions of a reaction diffusion climatological model, *Rev. R. Acad. Cien. Serie A. Mat.,* **96** (2002), 357-366.
21. E. Feireisl, J. Norbury, Some existence and nonuniqueness theorems for solutions of parabolic equations with discontinuous nonlinearities, *Proc. Roy. Soc. Edinburgh*, **119A (**1991**)**, 1-17.
22. O.V. Kapustyan, P.O. Kasyanov, J. Valero, Pullback attractors for a class of extremal solutions of the 3D Navier-Stokes system, *J. Math. Anal. Appl.*, **373** (2011), 535-547.
23. O.V. Kapustyan, P.O. Kasyanov, J. Valero, Structure and regularity of the global attractor of a reaction-diffusion equation with non-smooth nonlinear term, *Discrete Cont. Dyn. Systems*, **34** (2014), 4155-4182.
24. O.V. Kapustyan, V.S. Melnik, J. Valero, V.V. Yasinski, "Global Attractors of Multi-Valued Dynamical Systems and Evolution Equations Without Uniqueness", National Academy of Sciences of Ukraine, Naukova Dumka, 2008.
25. J.A. Langa, A. Suárez, Pullback permanence for non-autonomous partial differential equations, *Electron. J. Differential Equations*, **2002** (2002), No. 72, pp. 1-20.
26. J.L. Lions, "Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires", Gauthier-Villar, Paris, 1969.
27. A. Pazy, "Semigroups of linear operators and applications to partial differential equations", Springer, New-York, 1983.
28. J. Robinson, "Infinite-dimensional dynamical systems", Cambridge University Press, Cambridge, 2001.
29. J. Robinson, A. Rodríguez-Bernal, A. Vidal-López, Pullback attractors and extremal complete trajectories for non-autonomous reaction-diffusion problems, *J. Differential Equations*, **238** (2007), 289–337.
30. A. Rodríguez-Bernal, A. Vidal-Lopez, Existence, uniqueness and attractivity properties of positive complete trajectories for non-autonomous reaction-diffusion problems, *Discrete Continuous Dynamical Systems*, **18** (2007), 537-567.
31. A. Rodríguez-Bernal, A. Vidal-López, Extremal equilibria for reaction-diffusion equationsin bounded domains and applications, *J. Differential Equations*, **244** (2008), 2983-3030.
32. R. Temam, "Navier-Stokes Equations", North-Holland, Amsterdam-New York, 1977.
33. D. Terman, A free boundary problem arising from a bistable reaction-diffusion equation*, Siam J. Math. Anal.*, **14** (1983), 1107-1129.
34. D. Terman, A free boundary arising from a model for nerve conduction*, J. Differential Equations*, **58** (1985), 345-363.
35. A.A. Tolstonogov, On solutions of evolution inclusions I, *Siberian Math. J.*, **33** (1992), 500-511.
36. J. Valero, Attractors of parabolic equations without uniqueness, *J. Dynamics Differential Equations*, **13** (2001), 711-744.
37. J. Valero, On the Kneser property for some parabolic problems, *Topology Appl.*, **153** (2005), 975-989.
38. J. Valero, A weak comparison principle for reaction-diffusion systems, *J. Function Spaces and Applications*, 2012, V.2012, 30pp.
39. J. Valero, On L^r-regularity of global attractors generated by strong solutions of reaction-diffusion equations, *Applied Mathematics and Nonlinear Sciences*, **1** (2016), 375-390.