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MORSE DECOMPOSITION FOR GRADIENT-LIKE MULTI-VALUED AUTONOMOUS AND NONAUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. In this paper, we first prove that the property of being a gradient-like general dynamical system and the existence of a Morse decomposition are equivalent. Next, the stability of gradient-like general dynamical systems is analyzed. In particular, we show that a gradient-like general dynamical system is stable under perturbations, and that Morse sets are upper semi-continuous with respect to perturbations. Moreover, we prove that any solution of perturbed general dynamical systems should be close to some Morse set of the unperturbed gradient-like general dynamical system. We do not assume local compactness for the metric phase space X, unlike previous results in the literature. Finally, we extend the Morse decomposition theory of single-valued nonautonomous dynamical systems to the multi-valued case, without imposing any compactness of the parameter spaces.

1. **Introduction.** The theory of Morse decompositions for autonomous and nonautonomous dynamical systems plays an important role in the theory of dynamical systems; see, for instance [2, 4, 6, 15, 12] and the references therein. Morse decompositions in the set-valued context have first been introduced by R. McGehee in the 1990's [14]. In recent years, there is an increasing interest in the study of Morse decompositions for set-valued dynamical systems. The Morse decomposition theory and the upper semi-continuity of Morse decompositions of attractors for general dynamical systems were established in [12]. Then the results in [12] were extended to periodic and nonautonomous general dynamical systems [16, 17]. For gradient multi-valued semiflows, the existence of a Lyapunov function, the property

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of being a dynamically gradient multi-valued semiflow and the existence of a Morse decomposition have been proved to be equivalent in [7, 8]. In this paper our aims here are the following ones.

First, we prove that the property of being a gradient-like general dynamical system and the existence of a Morse decomposition are equivalent. The results are similar to the corresponding ones in [7]. However, our definition of gradient-like general dynamical system imposes weaker conditions than the ones previously used in the literature for set-valued dynamical systems. For this reason and although the proofs follow the same lines of the single-valued case, we prefer to include the proofs or our results stated in this paper just for completeness.

Secondly, we are interested in the stability of gradient-like general dynamical systems. In particular, we show that a gradient-like general dynamical system is stable under perturbations, and that Morse sets are upper semi-continuous with respect to perturbations. Moreover, we prove that any solution of perturbed general dynamical systems should be close to some Morse set of the unperturbed gradient-like general dynamical system. The stability of Morse decompositions of attractors in a complete locally compact metric space has been studied in [12]. It is worth mentioning that here we do not assume the local compactness for the metric space X, and the proofs here are much more simpler and direct.

Finally, we extend the Morse decomposition theory of single-valued nonautonomous dynamical systems to the multi-valued case. We borrow some ideas from [2] for nonautonomous Morse decompositions of single-valued gradient-like processes. For multi-valued nonautonomous general dynamical systems with compact parameter spaces, the Morse decomposition theory was established in [17]. Here we use the concept of multi-valued processes to describe multi-valued nonautonomous dynamical systems, so the compactness of parameter spaces is not assumed.

2. Morse decomposition for gradient-like GDSs.

Let X be a metric space with metric $d(\cdot,\cdot)$. For any nonempty subsets A and B of X, define the Hausdorff semi-distance and Hausdorff distance, respectively, as

$$H_X^*(A,\,B)=\sup_{a\in A}d(a,\,B),$$

and

$$H_X(A,\,B)=\max\{H_X^*(A,\,B),H_X^*(B,A)\},$$

where $d(a,\,B)=\inf_{b\in B}d(a,\,b)$. Given a subset $A\subset X$, the ε-neighborhood of A is the set $\mathcal{O}_\varepsilon(A):=\{x\in X: d(x,\,A)<\varepsilon\}$.

Definition 2.1. [10] A set-valued mapping $G:\mathbb{R}^+\times X\to 2^X$ with nonempty closed images is said to be a general dynamical system (GDS in short), if the following axioms hold:

(1) Semigroup property:

$$G(0,x)=\{x\},\quad G(t,G(s,x))=G(t+s,x),\quad \forall x\in X,\ s,t\in\mathbb{R}^+;$$

(2) $G(t,x)$ is continuous in t for each fixed x in the sense of Hausdorff distance;
(3) $G(t,x)$ is upper semi-continuous in x uniformly in t on any compact interval.

For a GDS $G(t,x)$, we will simply write $G(t,x)$ as $G(t)x$. As usual, we may also drop x from $G(t)x$ and use the notation $G(t)$ or $\{G(t)\}$ to denote $G(t,x)$.

Definition 2.2. A subset A of X is said to be positively invariant (resp. negatively invariant, invariant), if

$$G(t)A \subset A \quad (\text{resp. } G(t)A \supset A, \quad G(t)A = A) \quad \text{ for all } t \geqslant 0.$$

In this section we will always assume that $\{G(t)\}$ is a GDS in X. Moreover, we assume that $\{G(t)\}$ is asymptotically upper-semicompact, that is, $\{G(t)\}$ satisfies the following assumption:

(AC) Any sequence $y_n \in G(t_n)x_n$ possesses a convergent subsequence in X whenever $t_n \xrightarrow{n\to\infty} \infty$, $x_n \in X$ and $\cup_{n\geq 1} G([0, t_n])x_n$ is bounded.

Definition 2.3. Given two subsets A, B of X, we say that A attracts B under the action of the GDS $\{G(t)\}$ if $\lim_{t\to\infty} H_X^*(G(t)B,\, A) = 0$, and we say that A absorbs B under the action of $\{G(t)\}$ if there is a $t_B \geqslant 0$ such that $G(t)B \subset A$ for all $t \geqslant t_B$.

Let A be a subset of X. The uniform attraction region $\Omega_u(A)$ of A is defined as: $x \in \Omega_u(A)$ if and only if A attracts a neighborhood U of x.

Definition 2.4. A subset $\mathcal{A}$ of X is said to be a global attractor for a GDS $\{G(t)\}$ if it is compact, invariant under the action of $\{G(t)\}$, and for every bounded subset B of X we have that $\mathcal{A}$ attracts B under the action of $\{G(t)\}$.

Definition 2.5. A global solution for a GDS $\{G(t)\}$ is a continuous function $\xi : \mathbb{R} \to X$ with the property that $\xi(t+s) \in G(t)\xi(s)$ for all $s \in \mathbb{R}$ and $t \in \mathbb{R}^+$. We say that $\xi : \mathbb{R} \to X$ is a global solution through $x \in X$ if it is a global solution with $\xi(0) = x$.

Proposition 2.6. *Let x_n be a bounded sequence of X, and for any sequence $\tau_n \xrightarrow{n\to\infty} \infty$, let $\xi_n(\cdot) : \mathbb{R} \to X$ be a sequence of global solutions with*

$$\xi_n(t) \in G(\tau_n + t)x_n, \quad t \in (-\tau_n, \infty).$$

Then there is a subsequence of ξ_n that converges to a global solution ξ of $\{G(t)\}$ uniformly on any compact interval.

Proof. Since $\{G(t)\}$ is asymptotically upper-semicompact, then for any fixed $t \geqslant 0$, there is a subsequence which we again denote by $\{\xi_n(t)\}$ and $y_t \in X$ such that $\xi_n(t) \xrightarrow{n\to\infty} y_t$. Write $\xi(t) = y_t$ for all $t \geqslant 0$. Note that

$$d(\xi(t),\, G(t)\xi(0)) \leqslant d(\xi(t),\, \xi_n(t)) + d(\xi_n(t),\, G(t)\xi_n(0)) + H_X^*(G(t)\xi_n(0),\, G(t)\xi(0)),$$

and $\xi_n(t) \in G(t)\xi_n(0)$. Then it follows from the upper semi-continuity of $G(t)x$ that $\xi(t) \in G(t)\xi(0)$ for all $t \geqslant 0$. Proceeding similarly, for any fixed $t \in [-1, 0]$, there is a subsequence of $\{\xi_n(t)\}$ (still denoted by $\{\xi_n(t)\}$) that converges to y_t. Let $\xi(t) = y_t$ for all $t \in [-1, 0]$. Then we obtain that $\xi(0) \in G(1)\xi(-1)$ and $\xi(t) \in G(t+1)\xi(-1)$ for all $t \geqslant -1$. Applying the above procedure several times we can construct $\xi(t) \in G(t+m)\xi(-m)$ for all $t \geqslant -m$. Letting $m \to \infty$, we have a global solution $\xi(t)$ of the GDS $\{G(t)\}$, and for any fixed $t \in \mathbb{R}$, we can find a subsequence, which we again denote by $\{\xi_n(t)\}$, such that $\xi_n(t) \xrightarrow{n\to\infty} \xi(t)$. Furthermore, using a diagonal method one can choose a subsequence of $\{\xi_n(\cdot)\}$ (again denoted by $\{\xi_n(\cdot)\}$) such that $\xi_n(t) \xrightarrow{n\to\infty} \xi(t)$ for all rational numbers $t \in \mathbb{Q}$.

It remains to prove that $\xi_n(t) \xrightarrow{n\to\infty} \xi(t)$ uniformly on any compact interval $[T_0, T_1]$. Without loss of generality, we assume that $T_1 \in \mathbb{Q}$. Assuming the opposite, there would exist $\varepsilon_0 > 0$, some subsequence $\{\xi_{n_j}(\cdot)\}$ and corresponding values $t_j \in [T_0, T_1]$ $(j = 1, 2, 3, \cdots)$ such that

$$|\xi_{n_j}(t_j) - \xi(t_j)| > \varepsilon_0, \quad \forall j \in \mathbb{N}. \tag{1}$$

By the asymptotically upper-semicompactness of $\{G(t)\}$, there is a subsequence of $\{\xi_{n_j}(t_j)\}$ (relabeled by $\{\xi_{n_j}(t_j)\}$) such that

$$\xi_{n_j}(t_j) \xrightarrow{j\to\infty} y. \tag{2}$$

Without loss of generality, we assume that $t_j \xrightarrow{j\to\infty} t_0 \in [T_0, T_1]$. Then (1)-(2) and the continuity of the global solution $t \mapsto \xi(t)$ imply that

$$|y - \xi(t_0)| \geqslant \varepsilon_0. \tag{3}$$

If $t_0 < T_1$, take any fixed rational value τ, $t_0 < \tau \leqslant T_1$; if $t_0 = T_1$, take $\tau = T_1$. Disregarding a finite number of terms, it may be assumed that $t_j < \tau$. Since $\xi_{n_j}(\tau) \in G(\tau - t_j)\xi_{n_j}(t_j)$ and $G(t)x$ is upper semi-continuous in (t, x), we have that

$$\begin{aligned} d(\xi(\tau), G(\tau - t_0)y) \quad &\leqslant \quad d(\xi(\tau), \xi_{n_j}(\tau)) + d(\xi_{n_j}(\tau), G(\tau - t_j)\xi_{n_j}(t_j)) \\ &\quad + H^*_X(G(\tau - t_j)\xi_{n_j}(t_j), G(\tau - t_0)y) \xrightarrow{j\to\infty} 0, \end{aligned}$$

and thus $\xi(\tau) \in G(\tau - t_0)y$. Therefore,

$$\xi(t_0) = \lim_{\tau\to t_0} \xi(\tau) = \lim_{\tau\to t_0} G(\tau - t_0)y = y,$$

which contradicts (3). The proof is therefore complete. □

Let $A \subset X$. The ω-limit set $\omega(A)$ of A is defined as

$$\omega(A) := \{y \in X : \exists t_n \to \infty \text{ and } y_n \in G(t_n)A \text{ such that } y_n \to y\}.$$

For a global solution ξ of $\{G(t)\}$, the ω-limit set $\omega(\gamma)$ (resp. α-limit set $\alpha(\gamma)$) of γ is defined by

$$\omega(\xi)(\text{resp. } \alpha(\xi)) = \{x \in X : \exists t_n \to +\infty \text{ (resp. } -\infty) \text{ such that } \xi(t_n) \to x\}.$$

A set M is said to be weakly invariant, if for any $x \in M$, there is a global solution ξ through x with $\xi(\mathbb{R}) \subset M$.

Note that if M is weakly invariant, then M is negatively invariant.

Proposition 2.7. [11] *Let A be a nonempty subset of X, and let ξ be a global solution. Then*

(1) *$\omega(A)$, $\omega(\xi)$ (resp. $\alpha(\xi)$) are nonempty compact and weakly invariant sets;*
(2) *$\omega(A)$ attracts A, and*

$$\lim_{t\to\infty} d(\xi(t), \omega(\xi)) = 0 \ (\textit{resp.} \ \lim_{t\to-\infty} d(\xi(t), \alpha(\xi)) = 0).$$

Next we introduce the concept of gradient-like GDSs as an extension of the corresponding one in the single-valued context (see [1] for the definition of gradient-like semigroups). It is worth mentioning that the previous definitions for gradient-like systems in the set-valued framework assumed a stronger condition (see, e.g. Definition 1 in [7]). First, we need the definition of isolated weakly invariant set.

Definition 2.8. Let $\{G(t)\}$ be a GDS in X. We say that a weakly invariant set $\Xi \subset X$ for the GDS $\{G(t)\}$ is an isolated weakly invariant set if there is an ε-neighborhood $\mathcal{O}_\varepsilon(\Xi)$ of Ξ such that if ξ is a global solution contained in $\mathcal{O}_\varepsilon(\Xi)$, then $\xi(\mathbb{R}) \subset \Xi$.

A disjoint family of isolated weakly invariant sets is a family $\{\Xi_1, \ldots, \Xi_n\}$ of isolated weakly invariant sets with the property that, for some $\varepsilon > 0$,

$$\mathcal{O}_\varepsilon(\Xi_i) \cap \mathcal{O}_\varepsilon(\Xi_j) = \emptyset, \quad 1 \leqslant i < j \leqslant n.$$

Definition 2.9. Let $\{G(t)\}$ be a GDS which possesses a disjoint family of isolated weakly invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \ldots, \Xi_n\}$. A homoclinic structure associated with $\boldsymbol{\Xi}$ is a subset $\{\Xi_{k_1}, \ldots, \Xi_{k_p}\}$ of $\boldsymbol{\Xi}$ ($p \leqslant n$) together with a set of global solutions $\{\xi_1, \ldots, \xi_p\}$ such that

$$\Xi_{k_j} \xleftarrow{t \to -\infty} \xi_j(t) \xrightarrow{t \to \infty} \Xi_{k_{j+1}}, \quad 1 \leqslant j \leqslant p$$

where $\Xi_{k_{p+1}} := \Xi_{k_1}$ and, if $p = 1$, $\xi_1(\mathbb{R}) \subsetneq \Xi_{k_1}$, the notations $\Xi_{k_j} \xleftarrow{t \to -\infty} \xi_j(t)$ and $\xi_j(t) \xrightarrow{t \to \infty} \Xi_{k_{j+1}}$ mean that $\lim_{t \to -\infty} d(\xi_j(t), \Xi_{k_j}) = 0$ and $\lim_{t \to \infty} d(\xi_j(t), \Xi_{k_{j+1}}) = 0$, respectively.

Definition 2.10. Let $\{G(t)\}$ be a GDS with a global attractor $\mathcal{A}$ and a disjoint family of isolated weakly invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \ldots, \Xi_n\}$. The GDS $\{G(t)\}$ is said to be gradient-like relative to $\boldsymbol{\Xi}$ if

(1) for any global solution $\xi : \mathbb{R} \to \mathcal{A}$ there are $1 \leqslant i, j \leqslant n$ such that

$$\Xi_i \xleftarrow{t \to -\infty} \xi(t) \xrightarrow{t \to \infty} \Xi_j,$$

(2) there is no homoclinic structure associated with $\boldsymbol{\Xi}$.

Now we will introduce the notion of a Morse decomposition for an attractor $\mathcal{A}$ of a GDS $\{G(t)\}$. We start with the notion of attractor-repeller pairs.

Definition 2.11. Let $\{G(t)\}$ be a GDS with a global attractor $\mathcal{A}$. We say that a nonempty compact subset A of $\mathcal{A}$ is a local attractor for $\{G(t)\}$ in $\mathcal{A}$ if there is an $\varepsilon > 0$ such that $\omega(\mathcal{O}_\varepsilon(A) \cap \mathcal{A}) = A$. It is a local attractor in X if there is an $\varepsilon > 0$ such that $\omega(\mathcal{O}_\varepsilon(A)) = A$.

The repeller A^* associated with a local attractor A in $\mathcal{A}$ is the set defined by

$$A^* := \{x \in \mathcal{A} : H_X^*(G(t)x,\, A) \nrightarrow 0 \text{ as } t \to \infty\}.$$

The pair (A, A^*) is called an attractor-repeller pair for $\{G(t)\}$ on $\mathcal{A}$.

Proposition 2.12. [7, 12] *Let A be a local attractor of $\{G(t)\}$ in $\mathcal{A}$. Then*

(1) *A is compact and invariant;*
(2) *A^* is compact and weakly invariant.*

Remark 2.13. For single-valued dynamical systems (see [6, 15]), the repeller A^* is defined by

$$A^* := \{x \in \mathcal{A} : \omega(x) \cap A = \emptyset\}.$$

Note that if A is a local attractor, then A^* is compact and invariant.

For the multi-valued case, the repeller A^* is defined (see [12]) by

$$A^* := \{x \in \mathcal{A} : \omega(x) \backslash A \neq \emptyset\}.$$

It is worth noticing that

$$\omega(x) \backslash A = \emptyset \Longleftrightarrow \omega(x) \subset A \Longleftrightarrow H_X^*(G(t)x,\, A) \xrightarrow{t \to \infty} 0.$$

Then, it is easy to see that the two definitions of repeller are equivalent, i.e.,

$$A^* := \{x \in \mathcal{A} : H_X^*(G(t)x,\, A) \nrightarrow 0 \text{ as } t \to \infty\} = \{x \in \mathcal{A} : \omega(x) \backslash A \neq \emptyset\}.$$

Definition 2.14. Given an increasing family $\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = \mathcal{A}$, of $n+1$ local attractors, for $j = 1, \dots, n$, define $\Xi_j := A_j \cap A^*_{j-1}$. The ordered n-upla $\mathbf{\Xi} := \{\Xi_1, \Xi_2, \dots, \Xi_n\}$ is called a Morse decomposition for $\mathcal{A}$.

We will show that local attraction inside $\mathcal{A}$ is equivalent to local attraction in the whole space X. First, we prove the following result which plays a crucial role in the proof of Lemma 2.16. The similar result in the context of complete locally compact metric spaces can be found in [12], and also in [7] for multivalued semiflows.

Lemma 2.15. *Let $\{G(t)\}$ be a GDS in X with a global attractor $\mathcal{A}$. If $A \subset \mathcal{A}$ is a compact invariant set for $\{G(t)\}$ and there is an $\varepsilon > 0$ such that A attracts $\mathcal{O}_\varepsilon(A) \cap \mathcal{A}$ then, for each $\delta \in (0, \varepsilon)$ there is a $\delta' \in (0, \delta)$ such that $\gamma^+(\mathcal{O}_{\delta'}(A)) \subset \mathcal{O}_\delta(A)$, where $\gamma^+(\mathcal{O}_{\delta'}(A)) = \bigcup_{x \in \mathcal{O}_{\delta'}(A)} \bigcup_{t \geqslant 0} G(t)x$.*

Proof. Arguing by contradiction, let us assume the opposite. Then, there exists $0 < \delta < \varepsilon$ such that for any $\delta' \in (0, \delta)$, there exist an $x_{\delta'} \in \mathcal{O}_{\delta'}(A)$ and a $t_{\delta'} > 0$ such that $G(t_{\delta'})x_{\delta'} \backslash \mathcal{O}_\delta(A) \neq \emptyset$. Thus there are $x \in A$, $X \ni x_n \xrightarrow{n \to \infty} x$ and $t_n > 0$ such that $G(t_n)x_n \backslash \mathcal{O}_\delta(A) \neq \emptyset$.

Note that there are solutions ξ_n defined on $[0, t_n]$ such that $\xi_n(0) = x_n$ and $d(\xi_n(t_n), A) \geqslant \delta$. We can assume that

$$\xi_n([0, t_n]) \subset \overline{\mathcal{O}_\delta(A)},$$

otherwise we can choose t_n as $t_n = \inf\{t > 0 : d(\xi_n(t), A) \geqslant \delta\}$. We claim that $t_n \to \infty$ as $n \to \infty$. Indeed, if this is not the case, then the sequence t_n is bounded. By $x \in A$ and Definition 2.1, we have

$$d(\xi_n(t_n), A) = d(\xi_n(t_n), G(t_n)A) \leqslant H^*_X(G(t_n)x_n, G(t_n)x) \to 0$$

as $n \to \infty$, which leads to a contradiction and proves our claim. Now let

$$\eta_n(t) = \xi_n(t_n + t) \quad t \in [-t_n, 0].$$

By Proposition 2.6, we can extract a subsequence η_{n_k} such that η_{n_k} converges to some solution $\eta : (-\infty, 0] \to \overline{\mathcal{O}_\delta(A)}$ uniformly on any compact interval $[t, 0]$. Clearly $\eta(t) \in \overline{\mathcal{O}_\delta(A)} \cap \mathcal{A} \subset \mathcal{O}_\varepsilon(A) \cap \mathcal{A}$ for all $t \leqslant 0$ and $d(\eta(0), A) = \delta$, and consequently A cannot attract $\mathcal{O}_\varepsilon(A) \cap \mathcal{A}$. A contradiction which completes our proof. □

Lemma 2.16. *Let $\{G(t)\}$ be a GDS in X with a global attractor $\mathcal{A}$ and $S(t) := G(t)|_{\mathcal{A}}$, $\forall t \geqslant 0$. Then, $\{S(t)\}$ is a GDS in $\mathcal{A}$. If A is a local attractor for $\{S(t)\}$ in $\mathcal{A}$ (that is, there is an $\varepsilon > 0$ with $\omega(\mathcal{O}_\varepsilon(A) \cap \mathcal{A}) = A$) and K is a compact subset of $\mathcal{A}$ such that $K \cap A^* = \emptyset$, then A attracts K. Furthermore A is a local attractor for $\{G(t)\}$ in X.*

Proof. The proof is a slight modification of the corresponding one for single-valued systems in [1]. It is thus omitted. □

Remark 2.17. Let $\mathcal{A}$ be a global attractor of $\{G(t)\}$, and $A \subset \mathcal{A}$ be a local attractor of $\{G(t)\}$ in X. Then A is also a local attractor of $\{G(t)\}$ in $\mathcal{A}$.

Indeed, it is clear that $\omega(\mathcal{O}_\varepsilon(A) \cap \mathcal{A}) \subset \omega(\mathcal{O}_\varepsilon(A)) = A$, and from the invariance of A and $A \subset \mathcal{A}$, we have the converse inclusion $A \subset \omega(\mathcal{O}_\varepsilon(A) \cap \mathcal{A})$.

Lemma 2.18. *Let $\{G(t)\}$ be a GDS in X with a global attractor $\mathcal{A}$ and (A, A^*) an attractor-repeller pair for $\{G(t)\}$. Then*

(1) *If* $\xi : \mathbb{R} \to X$ *is a global bounded solution for* $\{G(t)\}$ *through* $x \notin A \cup A^*$, *then* $\xi(t) \xrightarrow{t\to\infty} A$ *and* $\xi(t) \xrightarrow{t\to-\infty} A^*$.
(2) *A global solution* $\xi : \mathbb{R} \to X$ *of* $\{G(t)\}$ *with the property that* $\xi(t) \in \mathcal{O}_\delta(A^*)$ *for all* $t \leqslant 0$ *for some* $\delta > 0$ *such that* $\mathcal{O}_\delta(A^*) \cap A = \emptyset$ *must satisfy* $d(\xi(t),\, A^*) \xrightarrow{t\to-\infty} 0$.

Proof. Part (1) follows from Proposition 3.5 in [12]. Part (2) can be proved in a similar way of the single-valued case; see [1], Lemma 2.13. We omit the details.

It is worthy mentioning that invariance implies weak invariance for compact sets, and that if a set Ξ is compact isolated weakly invariant and also invariant, then it is isolated invariant; that is, there is a $\delta > 0$ such that Ξ is the maximal invariant set in $\mathcal{O}_\delta(\Xi)$; see [12], Proposition 2.5.

Lemma 2.19. *Let* $\{G(t)\}$ *be a GDS with a global attractor* $\mathcal{A}$ *and let* $\Xi \subset \mathcal{A}$ *be a closed isolated weakly invariant set such that* $W^u(\Xi) = \Xi$. *Then* Ξ *is a local attractor, where*

$$W^u(\Xi) \;:=\; \{x \in X:\ \text{there is a global solution } \xi : \mathbb{R} \to X \text{ such that } \xi(0) = x \text{ and } \lim_{t\to-\infty} d(\xi(t),\, \Xi) = 0\}.$$

Proof. Since Ξ is weakly invariant, it is negatively invariant. In order to show that Ξ is invariant, we check that the converse inclusion $G(t)\Xi \subset \Xi$ holds true. Let $y \in G(t)\Xi$ be given. Then $y \in G(t)x$ for some $x \in \Xi \subset \mathcal{A}$. Recall that $\mathcal{A}$ is invariant and compact, so it is weakly invariant (see [12], Proposition 2.5), and consequently there is a global solution $\xi_3 : \mathbb{R} \to \mathcal{A}$ such that $\xi_3(t) = y$. Note that $x \in \Xi$, hence there is also a solution ξ_1 such that $\xi_1(t) \in \Xi\ \forall t \leqslant 0$, $\xi_1(0) = x$ and $\lim_{s\to-\infty} d(\xi_1(s),\, \Xi) = 0$. On the other hand, we can find a solution ξ_2 on $[0, t]$ such that $\xi_2(0) = x$ and $\xi_2(t) = y$, and it is clear that $\xi_2([0, t]) \subset \mathcal{A}$. We define a global solution $\xi : \mathbb{R} \to \mathcal{A}$ such that

$$\xi(s) = \begin{cases} \xi_3(s + t), & \forall s \geqslant 0; \\ \xi_2(s + t), & \forall -t \leqslant s \leqslant 0; \\ \xi_1(s + t), & \forall s \leqslant -t. \end{cases}$$

Then $\xi(0) = y$ and $\lim_{s\to-\infty} d(\xi(s),\, \Xi) = 0$, which implies $y \in W^u(\Xi) = \Xi$ and therefore the positive invariance of Ξ follows.

Now it suffices to prove that Ξ is a local attractor, that is, there is a $\delta > 0$ such that $\omega(\mathcal{O}_\delta(\Xi)) = \Xi$.

Let $\delta_0 > 0$ be such that Ξ is the maximal weakly invariant set in $\mathcal{O}_{\delta_0}(\Xi)$. Let us prove that, given $\delta \in (0, \delta_0)$, there exists $\delta' \in (0, \delta)$ such that $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \subset \mathcal{O}_\delta(\Xi)$. In fact, if the result were not true, there would exist a $\delta \in (0, \delta_0)$, a sequence $\{x_l\}$ in X with $d(x_l,\, \Xi) \xrightarrow{l\to\infty} 0$ and a sequence $\{t_l\}$ in $(0, \infty)$ such that $G(t_l)x_l \backslash \mathcal{O}_\delta(\Xi) \neq \emptyset$. Hence, there is a solution ξ_l on $[0, t_l]$ with $\xi_l(0) = x_l$ and $d(\xi_l(t_l),\, \Xi) \geqslant \delta$. We can assume that $\xi_l([0, t_l]) \subset \mathcal{O}_\delta(\Xi)$, otherwise we can choose t_l as $t_l = \inf\{t > 0 : d(\xi_l(t),\, \Xi) \geqslant \delta\}$. We claim that $t_l \to \infty$ as $l \to \infty$. Indeed, if this were not the case, then the compactness of Ξ would imply that there would exist a sequence $x_n := x_{l_n} \to x_0 \in \Xi$ such that the sequence $t_n := t_{l_n}$ is bounded. By the invariance of Ξ and Axiom (3) in Definition 2.1,

$$d(\xi_{l_n}(t_n),\, \Xi) = d(\xi_{l_n}(t_n),\, G(t_n)\Xi) \leqslant H_X^*(G(t_n)x_n,\, G(t_n)x_0) \to 0$$

as $n \to \infty$, which leads to a contradiction and proves our claim. Now, let

$$\sigma_l(t) = \xi_l(t_l + t), \qquad \text{for } t \in [-t_l, 0].$$

Then, σ_l is a solution of G on $[-t_l, 0]$. By Proposition 2.6, we can extract a subsequence $\sigma_n := \sigma_{l_n}$ with $l_n \to \infty$ such that σ_n converges to some solution $\sigma : (-\infty, 0] \to \overline{\mathcal{O}_\delta(\Xi)}$ uniformly on any compact interval $[t, 0]$. Since $d(\sigma_n(0), \Xi) = \delta$, we necessarily have

$$d(\sigma(0), \Xi) = \delta. \tag{4}$$

Now we extend σ to a global solution, still denoted by σ. Since $\alpha(\sigma)$ is weakly invariant and Ξ is maximal weakly invariant in $\mathcal{O}_{\delta_0}(\Xi)$, we have $\alpha(\sigma) \subset \Xi$. Clearly $\sigma(0) \in W^u(\Xi) = \Xi$, which contradicts (4).

Then it follows from the previous arguments that, for any $\delta \in (0, \delta_0)$, there is a $\delta' \in (0, \delta)$ such that $\omega(\mathcal{O}_{\delta'}(\Xi)) \subset \overline{\gamma^+(\mathcal{O}_{\delta'}(\Xi))} \subset \overline{\mathcal{O}_\delta(\Xi)} \subset \mathcal{O}_{\delta_0}(\Xi)$. Note that Ξ is an isolated weakly invariant set and $\omega(\mathcal{O}_{\delta'}(\Xi))$ is weakly invariant. Hence we must have that $\omega(\mathcal{O}_{\delta'}(\Xi)) \subset \Xi$. On the other hand, the invariance of Ξ implies that $\Xi \subset \omega(\mathcal{O}_{\delta'}(\Xi))$, and thus $\Xi = \omega(\mathcal{O}_{\delta'}(\Xi))$. The proof of this lemma is finished. □

Corollary 2.20. *Let $\{G(t)\}$ be a GDS with a global attractor $\mathcal{A}$ and let $\Xi \subset \mathcal{A}$ be a closed isolated weakly invariant set. Then, Ξ is a local attractor for $\{G(t)\}$ if and only if $W^u(\Xi) = \Xi$.*

Proof. Thanks to Lemma 2.19, it only remains to show that if Ξ is a local attractor for $\{G(t)\}$, then we have $W^u(\Xi) = \Xi$. Let $z \in W^u(\Xi)$ be given. Since Ξ is a local attractor, there is an $\varepsilon > 0$ such that Ξ attracts $\mathcal{O}_\varepsilon(\Xi)$. By the definition of the unstable set of the weakly invariant set, we obtain that there exist a global solution $\xi : \mathbb{R} \to X$ and $T > 0$ such that $\xi(0) = z$ and $\xi(t) \in \mathcal{O}_\varepsilon(\Xi)$ for each $t \leqslant -T$. Then we deduce that

$$d(z, \Xi) \leqslant d(z, G(-t)\xi(t)) + H_X^*(G(-t)\xi(t), \Xi) = H_X^*(G(-t)\xi(t), \Xi) \to 0$$

as $t \to -\infty$. Clearly $z \in \Xi$ and thus $W^u(\Xi) \subset \Xi$. The converse inclusion $\Xi \subset W^u(\Xi)$ follows immediately from the fact that Ξ is weakly invariant. □

Lemma 2.21. *Let $\{G(t)\}$ be a gradient-like GDS with respect to the disjoint family of closed isolated weakly invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \dots, \Xi_n\}$ and let $\mathcal{A}$ be its global attractor. Then, there is a $k \in \{1, \dots, n\}$ such that Ξ_k is a local attractor for $\{G(t)\}$.*

Proof. From a similar proof to that of Lemma 2.16 in [1] for the single-valued case, we deduce that there is a $k \in \{1, \dots, n\}$ such that $W^u(\Xi_k) = \Xi_k$. Then the conclusion follows immediately from Lemma 2.19. □

Let $\{G(t)\}$ be a gradient-like GDS with respect to the disjoint family of closed isolated weakly invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \dots, \Xi_n\}$. If (after possible reordering) Ξ_1 is a local attractor for $\{G(t)\}$ let $\Xi_1^* = \{x \in \mathcal{A} : \omega(x) \backslash \Xi_1 \neq \emptyset\}$ be its associated repeller, so each Ξ_i, with $i \geqslant 2$, is contained in Ξ_1^*. Considering the restriction $\{G_1(t)\}$ of $\{G(t)\}$ to Ξ_1^* we have that $\{G_1(t)\}$ is a gradient-like GDS in the space Ξ_1^* with respect to the disjoint family of isolated weakly invariant sets $\{\Xi_2, \dots, \Xi_n\}$, and we can assume, thanks to Lemma 2.21, that Ξ_2 is a local attractor for the GDS $\{G_1(t)\}$ in Ξ_1^*. If $\Xi_{2,1}^*$ is the repeller associated with the local attractor Ξ_2 for $\{G_1(t)\}$ in Ξ_1^* we can proceed and consider the restriction $\{G_2(t)\}$ of the GDS $\{G_1(t)\}$ to $\Xi_{2,1}^*$

and then $\{G_2(t)\}$ is a gradient-like GDS in $\Xi^*_{2,1}$ with respect to the disjoint family of isolated weakly invariant sets $\{\Xi_3,\dots,\Xi_n\}$.

Proceeding in this way until all isolated weakly invariant sets are exhausted we obtain a reordering of $\{\Xi_1,\dots,\Xi_n\}$ in such a way that Ξ_1 is a local attractor for $\{G(t)\}$. Setting $\mathcal{A} =: \Xi^*_{0,-1}$ and $\Xi^*_{1,0} := \Xi^*_1$, for $j = 2,\dots,n$, we have that Ξ_j is a local attractor for the restriction of $\{G(t)\}$ to $\Xi^*_{j-1,j-2}$ whose repeller will be indicated by $\Xi^*_{j,j-1}$.

With the construction above, if a global solution $\xi : \mathbb{R} \to \mathcal{A}$ satisfies

$$\Xi_i \xleftarrow{t\to-\infty} \xi(t) \xrightarrow{t\to\infty} \Xi_j \tag{5}$$

then $i \geqslant j$. Thanks to Theorem 3.9 in [12], we deduce that the n-upla $\{\Xi_1,\dots,\Xi_n\}$, ordered in the way that we explained above, is a Morse decomposition for the attractor $\mathcal{A}$ of $\{G(t)\}$.

The proof of the following theorem will be given here (more closely related to the gradient-like GDSs and shorter) just for completeness.

Theorem 2.22. *Let $\{G(t)\}$ be a gradient-like GDS with respect to the disjoint family of closed isolated weakly invariant sets $\boldsymbol{\Xi} = \{\Xi_1,\dots,\Xi_n\}$ reordered in such a way that Ξ_j is a local attractor for the restriction of $\{G(t)\}$ to $\Xi^*_{j-1,j-2}$, as we have explained above, and let $\mathcal{A}$ be its global attractor. Then, $\boldsymbol{\Xi}$ defines a Morse decomposition on $\mathcal{A}$.*

Proof. Set $A_0 := \emptyset$, $A_1 := \Xi_1$ and for $j = 2, 3,\dots,n$

$$A_j := A_{j-1} \cup W^u(\Xi_j). \tag{6}$$

It is clear that $A_n = \mathcal{A}$. By Lemma 2.21, we see that $A_1 := \Xi_1$ is invariant. Then we proceed by induction, and by slightly modifying the proof of Lemma 2.19, the invariance of A_j follows immediately for each j. Recall that $A_1 := \Xi_1$ is closed. By using mathematical induction and Barbashin's compactness theorem, we can prove that A_j is closed for each j; see Theorem 3.8 in [12] for more details. Thus for each j, we choose $d > 0$ such that

$$\mathcal{O}_d(A_j) \cap \left(\bigcup_{i=j+1}^{n} \Xi_i \right) = \emptyset. \tag{7}$$

Step 1. We show that there are $\delta < d$ and $\delta' < \delta$ such that $\gamma^+(\mathcal{O}_{\delta'}(A_j) \cap \mathcal{A}) \subset \mathcal{O}_\delta(A_j)$. If that were not the case, there would exist a sequence $\{x_k\} \subset \mathcal{A}$ with $d(x_k,\, A_j) \xrightarrow{k\to\infty} 0$, a sequence $\{t_k\}$ in $(0,\infty)$ such that $G(t_k)x_k \backslash \mathcal{O}_\delta(A_j) \neq \emptyset$. Hence, for each x_k, there is a solution ξ_k on $[0, t_k]$ with $\xi_k(0) = x_k$ and $d(\xi_k(t_k),\, A_j) \geqslant \delta$. We can assume that $\xi_k([0, t_k]) \subset \mathcal{O}_\delta(A_j)$, otherwise we could choose t_k as $t_k = \inf\{t > 0 : d(\xi_k(t),\, A_j) \geqslant \delta\}$. We claim that $t_k \to \infty$ as $k \to \infty$. Indeed, suppose the opposite, then there would exist a sequence $x_n := x_{k_n} \to x_0 \in A_j$ such that the sequence $t_n := t_{k_n}$ is bounded. By the invariance of A_j and Axiom (3) in Definition 2.1,

$$d(\xi_{k_n}(t_n),\, A_j) = d(\xi_{k_n}(t_n),\, G(t_n)A_j) \leqslant H^*_X(G(t_n)x_n,\, G(t_n)x_0) \to 0,$$

as $n \to \infty$, which leads to a contradiction and proves our claim. Now, let

$$\sigma_k(t) = \xi_k(t_k + t), \qquad \text{for } t \in [-t_k, 0].$$

It is clear that σ_k is a solution of G on $[-t_k, 0]$. By Proposition 2.6, we can extract a subsequence $\sigma_n := \sigma_{k_n}$ with $k_n \to \infty$ such that σ_n converges to some solution $\sigma : (-\infty, 0] \to \overline{\mathcal{O}_\delta(A_j)} \cap \mathcal{A}$ uniformly on any compact interval $[t, 0]$ and $d(\sigma(0), A_j) \geqslant \delta$. From (7) and the properties of gradient-like GDSs we must have that $\sigma(t) \xrightarrow{t\to-\infty} \Xi_l$, for some $1 \leqslant l \leqslant j$ and, consequently, $\sigma(0) \in W^u(\Xi_l) \subset A_j$. This is a contradiction with $d(\sigma(0), A_j) \geqslant \delta$.

Step 2. Let us prove that A_j is a local attractor for each $1 \leqslant j \leqslant n$. By Proposition 2.7, we see that $\omega(\mathcal{O}_{\delta'}(A_j)\cap\mathcal{A})$ is weakly invariant. Let $y \in \omega(\mathcal{O}_{\delta'}(A_j)\cap \mathcal{A})$ be arbitrary. Then from the weakly invariance of $\omega(\mathcal{O}_{\delta'}(A_j)\cap\mathcal{A})$, there is a global solution $\xi : \mathbb{R} \to \omega(\mathcal{O}_{\delta'}(A_j) \cap \mathcal{A}) \subset \overline{\gamma^+(\mathcal{O}_{\delta'}(A_j) \cap \mathcal{A})} \subset \mathcal{O}_\delta(A_j)$ with $\xi(0) = y$. By (7) and the properties of gradient-like GDSs, we have $y = \xi(0) \in W^u(\Xi_l)$, for some $1 \leqslant l \leqslant j$ and, consequently, $y \in A_j$ and thus $\omega(\mathcal{O}_{\delta'}(A_j) \cap \mathcal{A}) \subset A_j$. On the other hand, the invariance of A_j implies that $A_j \subset \omega(\mathcal{O}_{\delta'}(A_j) \cap \mathcal{A})$. Therefore, $\omega(\mathcal{O}_{\delta'}(A_j) \cap \mathcal{A}) = A_j$, and consequently A_j is a local attractor.

Step 3. Finally, it suffices to prove $\Xi_j = A_j \cap A^*_{j-1}$ for all $1 \leqslant j \leqslant n$. Indeed, if $x \in A_j \cap A^*_{j-1}$, then from $A_j = \bigcup_{i=1}^{j} W^u(\Xi_i)$ we can choose a global solution ξ_1 such that $\xi_1(0) = x$ and $\xi_1(t) \to \bigcup_{i=1}^{j} \Xi_i$ as $t \to -\infty$. On the other hand, by Proposition 2.12, there is a global solution $\xi_2 : \mathbb{R} \to A^*_{j-1}$ through x. We define a global solution $\xi : \mathbb{R} \to \mathcal{A}$ such that

$$\xi(s) = \begin{cases} \xi_2(s), & \forall s \geqslant 0; \\ \xi_1(s), & \forall s \leqslant 0. \end{cases}$$

Note that $A^*_{j-1} = \{z \in \mathcal{A} : \omega(z)\backslash A_{j-1} \neq \emptyset\}$. By the weak invariance of A^*_{j-1} and the gradient-like property of $\{G(t)\}$, we have

$$\bigcup_{i=1}^{j} \Xi_i \xleftarrow{t\to-\infty} \xi(t) \xrightarrow{t\to\infty} \bigcup_{i=j}^{n} \Xi_i.$$

Since $\{G(t)\}$ is a gradient-like GDS with reordered isolated weakly invariant sets $\{\Xi_1, \ldots, \Xi_n\}$, any global solution $\xi : \mathbb{R} \to \mathcal{A}$ satisfies $\Xi_l \xleftarrow{t\to-\infty} \xi(t) \xrightarrow{t\to\infty} \Xi_k$ with $l \geqslant k$, and we obtain that $x \in \Xi_j$.

Conversely, if $x \in \Xi_j$, then there is a global solution $\xi : \mathbb{R} \to \Xi_j$ through x, and thus, by definition of A_j, we have $x \in A_j$. If $x \notin A^*_{j-1}$, then $\omega(x)\backslash A_{j-1} = \emptyset$, and consequently $\omega(x) \subset A_{j-1}$. Note that $\omega(\xi) \subset \omega(x)$, therefore $\omega(\xi) \subset \Xi_i$ for some $1 \leqslant i \leqslant j-1$, which contradicts $\omega(\xi) \subset \Xi_j$. Thus $x \in A^*_{j-1}$ and $\Xi_j \subset A_j \cap A^*_{j-1}$. The proof is now complete. $\square$

The following theorem shows the relationship between the notions of gradient-like GDSs and of Morse decomposition for a global attractor.

Theorem 2.23. *Let $\{G(t)\}$ be a GDS with global attractor $\mathcal{A}$. Then, $\{G(t)\}$ is a gradient-like GDS with respect to the disjoint family of closed isolated weakly invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \ldots, \Xi_n\}$ reordered in such a way that Ξ_j is a local attractor for the restriction of $\{G(t)\}$ to $\Xi^*_{j-1,j-2}$, if and only if $\boldsymbol{\Xi}$ is a Morse decomposition on $\mathcal{A}$.*

Proof. We only need to prove the necessity. Since $\boldsymbol{\Xi}$ is a Morse decomposition on $\mathcal{A}$, in a quite similar manner as in the situation of single-valued dynamical systems (see Chapter 3, Theorem 1.7 in [15] and Theorem 3.7 in [12] for details), we deduce that $\boldsymbol{\Xi}$ is a family of disjoint compact isolated weakly invariant sets, and for any global solution $\xi : \mathbb{R} \to \mathcal{A}$, either $\xi(\mathbb{R}) \subset \Xi_k$ for some Morse set Ξ_k, or else there are indices $i < j$ such that

$$\Xi_j \xleftarrow{t\to-\infty} \xi(t) \xrightarrow{t\to\infty} \Xi_i. \tag{8}$$

Consequently, $\{G(t)\}$ is a gradient-like GDS with respect to the disjoint family of isolated weakly invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \dots, \Xi_n\}$. In fact, if the result were not true, then there would exist a homoclinic structure associated with $\boldsymbol{\Xi}$. Hence, there is a subset $\{\Xi_{k_1}, \dots, \Xi_{k_p}\}$ of $\boldsymbol{\Xi}$, together with a set of global solutions $\{\xi_1, \dots, \xi_p\}$ such that

$$\Xi_{k_j} \xleftarrow{t\to-\infty} \xi_j(t) \xrightarrow{t\to\infty} \Xi_{k_{j+1}}, \quad 1 \leqslant j \leqslant p,$$

where $\Xi_{k_{p+1}} := \Xi_{k_1}$. It follows from (8) that $k_1 > k_2 > \cdots > k_p > k_{p+1} := k_1$, this is a contradiction. □

Remark 2.24. It is worth recalling that, from the arguments of Theorem 3.4 in [1] for single-valued dynamical systems, we conclude that if $\{G(t)\}$ is a semigroup (single-valued case) with global attractor $\mathcal{A}$ and a disjoint family of closed isolated weakly invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \dots, \Xi_n\}$, then the following assertions are equivalent:

1. $\{G(t)\}$ is a gradient semigroup with respect to $\boldsymbol{\Xi}$ in the sense of Definition 3.1 in [1].
2. $\{G(t)\}$ is a gradient-like semigroup with respect to $\boldsymbol{\Xi}$ in the sense of Definition 2.8 in [1].
3. $\boldsymbol{\Xi}$ is a Morse decomposition on $\mathcal{A}$.

2.1. **Stability of gradient-like GDSs under perturbations.** Let Λ be a metric space with metric $\rho(\cdot,\cdot)$, and let $\{G^\lambda(t)\}$ $(\lambda \in \Lambda)$ be a family of GDSs in X. We start by recalling the following basic result from [13].

Theorem 2.25. *Let $\lambda_0 \in \Lambda$. Assume the following continuity assumption holds:*

($C0$) *For any ε, $T > 0$ and bounded set $B \subset X$, there exists $\delta > 0$ such that whenever $\rho(\lambda, \lambda_0) < \delta$,*

$$H_X^*(G^\lambda(t)x,\, G^{\lambda_0}(t)\mathcal{O}_\varepsilon(x)) < \varepsilon, \quad \forall (t,x) \in [0,T] \times B.$$

Assume $\{G^{\lambda_0}(t)\}$ possesses a local attractor $\mathcal{A}$. Then

(1) *there is a neighborhood B of $\mathcal{A}$ with $B \subset \Omega_u(\mathcal{A})$ such that for any $\varepsilon > 0$, there exist a $T_0 > 0$ and a $\delta' > 0$ such that when $\rho(\lambda, \lambda_0) < \delta'$,*

$$H_X^*(G^\lambda(t)B,\, \mathcal{A}) < \varepsilon, \quad \forall t \geqslant T_0;$$

(2) *if $\{G^\lambda(t)\}$ is asymptotically upper-semicompact in X, then $\{G^\lambda(t)\}$ has a local attractor $\mathcal{A}^\lambda$ when $\rho(\lambda, \lambda_0)$ is sufficiently small, and*

$$H_X^*(\mathcal{A}^\lambda,\, \mathcal{A}) \to 0 \quad \text{as } \lambda \to \lambda_0; \tag{9}$$

(3) *B is a neighborhood of $\mathcal{A}^\lambda$ with $B \subset \Omega_u(\mathcal{A}^\lambda)$ when $\rho(\lambda, \lambda_0)$ is sufficiently small,*

where

$$\Omega_u(\mathcal{A}) = \{x \in X : \mathcal{A} \text{ attracts a neighborhood of } x \text{ under } G^{\lambda_0}\},$$

and

$$\Omega_u(\mathcal{A}^\lambda) = \{x \in X : \mathcal{A}^\lambda \text{ attracts a neighborhood of } x \text{ under } G^{\lambda}\}.$$

The main result in this subsection is the following theorem.

Theorem 2.26. *Let $\{G^\lambda(t)\}$ $(\lambda \in \Lambda)$ be a family of GDSs in X. Assume $\{G^\lambda(t)\}$ possesses a global attractor $\mathcal{A}^\lambda$ for each $\lambda \in \Lambda$, the continuity assumption $(C0)$ in Theorem 2.25 at $\lambda_0 \in \Lambda$ holds, and $G^\lambda(t)x$ is upper semi-continuous with respect to λ and x for each fixed $t \in \mathbb{R}^+$.*

Let $\{G(t)\} = \{G^{\lambda_0}(t)\}$ be a gradient-like GDS with respect to the disjoint family of closed isolated weakly invariant sets $\mathbf{\Xi} = \{\Xi_1, \dots, \Xi_n\}$ reordered in such a way that Ξ_j is a local attractor for the restriction of $\{G(t)\}$ to $\Xi^*_{j-1,j-2}$. Then, when $\rho(\lambda, \lambda_0)$ is sufficiently small, $\{G^\lambda(t)\}$ is a gradient-like GDS with respect to the disjoint family of isolated weakly invariant sets $\mathbf{\Xi}^\lambda = \{\Xi^\lambda_1, \dots, \Xi^\lambda_n\}$ and consequently $\mathcal{A}^\lambda = \bigcup_{j=1}^{n} W^u(\Xi^\lambda_j)$; moreover, for each $1 \leqslant j \leqslant n$, we have

$$\lim_{\lambda \to \lambda_0} H^*_X(\Xi^\lambda_j, \Xi_j) = 0.$$

Proof. Let $A_0 := \emptyset$, $A_1 := \Xi_1$ and for $j = 2, 3, \dots, n$

$$A_j := A_{j-1} \cup W^u(\Xi_j).$$

Then it follows from Theorem 2.22 that A_j is a local attractor for $\{G(t)\}$ in X, and

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = \mathcal{A},$$

where $\mathcal{A}$ is a global attractor for $\{G(t)\}$ on X. Thanks to Theorem 2.25, there exist a $\delta > 0$ and a $\eta > 0$ such that when $\rho(\lambda, \lambda_0) < \delta$, $\{G^\lambda(t)\}$ has a local attractor A^λ_j and $\mathcal{O}_\eta(A_j) \subset \Omega_u(A^\lambda_j)$ for each j. Note that we can assume $\delta > 0$ is sufficiently small so that by Theorem 2.25, we have for each fixed k,

$$A^\lambda_j \subset \mathcal{O}_\eta(A_j) \subset \mathcal{O}_\eta(A_k) \subset \Omega_u(A^\lambda_k), \qquad \text{for } j \leqslant k.$$

Combining this with the invariance and compactness of A^λ_j, we deduce that $A^\lambda_j \subset A^\lambda_k$ for all $j \leqslant k$. Therefore,

$$\emptyset = A^\lambda_0 \subset A^\lambda_1 \subset \cdots \subset A^\lambda_n = \mathcal{A}^\lambda$$

is an increasing sequence of local attractors of $\{G^\lambda(t)\}$.

Let $\Xi^\lambda_j = A^\lambda_j \cap A^{\lambda,*}_{j-1}$. Then $\mathbf{\Xi}^\lambda = \{\Xi^\lambda_1, \dots, \Xi^\lambda_n\}$ is a Morse decomposition of $\mathcal{A}^\lambda$. It follows from Theorem 2.23 that $\mathbf{\Xi}^\lambda$ is a family of disjoint isolated weakly invariant sets, $\mathcal{A}^\lambda = \bigcup_{j=1}^{n} W^u(\Xi^\lambda_j)$ and for any global solution $\xi : \mathbb{R} \to \mathcal{A}^\lambda$, either $\xi(\mathbb{R}) \subset \Xi^\lambda_k$ for some Morse set Ξ^λ_k, or else there are indices $i < j$ such that

$$\Xi^\lambda_j \xleftarrow{t \to -\infty} \xi(t) \xrightarrow{t \to \infty} \Xi^\lambda_i, \tag{10}$$

and consequently when $\rho(\lambda, \lambda_0) < \delta$, $\{G^\lambda(t)\}$ is a gradient-like GDS with respect to the disjoint family of isolated weakly invariant sets $\mathbf{\Xi}^\lambda = \{\Xi^\lambda_1, \dots, \Xi^\lambda_n\}$.

Finally, let us prove that for each $1 \leqslant j \leqslant n$, $\lim_{\lambda \to \lambda_0} H^*_X(\Xi^\lambda_j, \Xi_j) = 0$. We argue by contradiction. Assume that there exist a j with $1 \leqslant j \leqslant n$, a $\varepsilon_0 > 0$, sequences $\lambda_n \to \lambda_0$ and $x_n \in \Xi^{\lambda_n}_j$ such that

$$d(x_n, \Xi_j) \geqslant \varepsilon_0. \tag{11}$$

Recall that $x_n \in \Xi^{\lambda_n}_j = A^{\lambda_n}_j \cap A^{\lambda_n,*}_{j-1}$ and $\lim_{n\to\infty} H^*_X(A^{\lambda_n}_j, A_j) = 0$. Hence the compactness of A_j implies that there is a subsequence of $\{x_n\}$ (which we still denote by $\{x_n\}$) such that $x_n \to x_0 \in A_j$. Noting that $\Xi_j = A_j \cap A^*_{j-1}$, we need to show $x_0 \in A^*_{j-1}$. Suppose not. Then by the definition of the repeller, we have

$\omega(x_0) \subset A_{j-1}$. Therefore $G(T)x_0 \subset \mathcal{O}_{\frac{\eta}{3}}(A_{j-1})$ for sufficiently large T. It follows from the upper semi-continuity assumption that for all sufficiently large n,

$$H_X^*(G^{\lambda_n}(T)x_n,\, G^{\lambda_0}(T)x_0) < \frac{\eta}{3}.$$

Hence for sufficiently large n, $G^{\lambda_n}(T)x_n \subset \mathcal{O}_\eta(A_{j-1}) \subset \Omega_u(A_{j-1}^{\lambda_n})$. Since $x_n \in A_j^{\lambda_n}$ and $A_j^{\lambda_n}$ is invariant under the action of G^{λ_n}, we have that $G^{\lambda_n}(T)x_n \subset A_j^{\lambda_n}$ and thus $G^{\lambda_n}(T)x_n$ is relatively compact. From the definition of uniform attraction region, we see that $A_{j-1}^{\lambda_n}$ attracts $G^{\lambda_n}(T)x_n$, and consequently $\omega(x_n) \subset A_{j-1}^{\lambda_n}$. This implies that for sufficiently large n, $x_n \notin A_{j-1}^{\lambda_n,*}$, which contradicts $x_n \in \Xi_j^{\lambda_n} = A_j^{\lambda_n} \cap A_{j-1}^{\lambda_n,*}$ for each n and proves our claim. Therefore, $x_n \to x_0 \in A_j \cap A_{j-1}^* = \Xi_j$, which leads to a contradiction with (11) and thus the proof of this theorem is completed. □

Remark 2.27. Let us recall that for single-valued dynamical systems, similar to Theorem 2.26, we have the following result.

Let $\{G^\lambda(t)\}$ $(\lambda \in \Lambda)$ be a family of semigroups in X. Assume $\{G^\lambda(t)\}$ possesses a global attractor $\mathcal{A}^\lambda$ for each $\lambda \in \Lambda$, for any $\varepsilon,\, T > 0$ and bounded set $B \subset X$, there exists $\delta > 0$ such that when $\rho(\lambda,\, \lambda_0) < \delta$,

$$H_X^*(G^\lambda(t)x,\, G^{\lambda_0}(t)x) < \varepsilon, \quad \forall (t,x) \in [0,T] \times B,$$

and $G^\lambda(t)x$ is continuous with respect to λ and x for each fixed $t \in \mathbb{R}^+$.

Let $\{G(t)\} = \{G^{\lambda_0}(t)\}$ be a gradient-like semigroup with respect to the disjoint family of closed isolated invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \ldots, \Xi_n\}$ reordered in such a way that Ξ_j is a local attractor for the restriction of $\{G(t)\}$ to $\Xi_{j-1,j-2}^*$. Then when $\rho(\lambda,\, \lambda_0)$ is sufficiently small, $\{G^\lambda(t)\}$ is gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\boldsymbol{\Xi}^\lambda = \{\Xi_1^\lambda, \ldots, \Xi_n^\lambda\}$ and consequently $\mathcal{A}^\lambda = \bigcup_{j=1}^{n} W^u(\Xi_j^\lambda)$; moreover, for each $1 \leqslant j \leqslant n$, we have

$$\lim_{\lambda \to \lambda_0} H_X^*(\Xi_j^\lambda,\, \Xi_j) = 0.$$

Theorem 2.28. *Assume the hypotheses in Theorem 2.26. Let $\{G(t)\} = \{G^{\lambda_0}(t)\}$ be a gradient-like GDS with respect to the disjoint family of closed isolated weakly invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \ldots, \Xi_n\}$ reordered in such a way that Ξ_j is a local attractor for the restriction of $\{G(t)\}$ to $\Xi_{j-1,j-2}^*$. Then, for any $\varepsilon > 0$, there is a $\delta_1 > 0$ such that when $\rho(\lambda,\, \lambda_0) < \delta_1$, for any solution ξ of $\{G^\lambda(t)\}$, there is a Ξ_j such that*

$$\lim_{t \to \infty} d(\xi(t),\, \Xi_j) \leqslant \varepsilon.$$

Proof. From the proof of Theorem 2.26, there is a $\delta_1 > 0$ such that when $\rho(\lambda,\, \lambda_0) < \delta_1$, $\{G^\lambda(t)\}$ has a global attractor $\mathcal{A}^\lambda$, and there is an increasing sequence

$$\emptyset = A_0^\lambda \subset A_1^\lambda \subset \cdots \subset A_n^\lambda = \mathcal{A}^\lambda$$

of local attractors A_j^λ of GDS $\{G^\lambda(t)\}$ in $\mathcal{A}^\lambda$ such that

$$\Xi_j^\lambda = A_j^\lambda \cap A_{j-1}^{\lambda,*}, \quad \forall 1 \leqslant j \leqslant n.$$

Hence when $\rho(\lambda,\, \lambda_0) < \delta_1$, for any solution ξ of $\{G^\lambda(t)\}$, there is a smallest j such that

$$\lim_{t \to \infty} d(\xi(t),\, A_j^\lambda) = 0.$$

Clearly $j > 0$. Since A^{λ}_{j-1} is a local attractor, there is a $\delta^* > 0$ such that $\omega(\mathcal{O}_{\delta^*}(A^{\lambda}_{j-1})) = A^{\lambda}_{j-1}$. By Lemma 2.15, we choose $\delta' \in (0, \delta^*)$ such that

$$\gamma^+(\mathcal{O}_{\delta'}(A^{\lambda}_{j-1})) := \bigcup_{x\in\mathcal{O}_{\delta'}(A^{\lambda}_{j-1})} \bigcup_{t\geqslant 0} G^{\lambda}(t)x \subset \mathcal{O}_{\delta^*}(A^{\lambda}_{j-1}).$$

Now two cases may occur:

(*i*) there is a $t_0 > 0$ such that $\xi(t_0) \in \mathcal{O}_{\delta'}(A^{\lambda}_{j-1})$. In this case we have that

$$\lim_{t\to\infty} d(\xi(t),\, A^{\lambda}_{j-1}) = 0,$$

which contradicts the choice of j.

(*ii*) $d(\xi(t),\, A^{\lambda}_{j-1}) \geqslant \delta'$ for all $t \geqslant 0$. When this takes place, we must have that $\omega(\xi) \subset A^{\lambda,*}_{j-1}$. Indeed, if there exist a $y \in \omega(\xi)$ and a sequence $s_n \to \infty$ such that $\xi(s_n) \to y$ and $y \notin A^{\lambda,*}_{j-1}$, then by the definition of repeller, we have $\omega(y) \subset A^{\lambda}_{j-1}$. Thus, there exists a sufficiently large $T > 0$ such that $G^{\lambda}(T)y \subset \mathcal{O}_{\frac{\delta'}{3}}(A^{\lambda}_{j-1})$, and consequently for all sufficiently large n, $G^{\lambda}(T)\xi(s_n) \subset \mathcal{O}_{\frac{2\delta'}{3}}(A^{\lambda}_{j-1})$. Therefore, $\xi(T + s_n) \in \mathcal{O}_{\frac{2\delta'}{3}}(A^{\lambda}_{j-1})$ for all sufficiently large n, what contradicts $d(\xi(t),\, A^{\lambda}_{j-1}) \geqslant \delta'$ for all $t \geqslant 0$. Recall that $\lim_{t\to\infty} d(\xi(t),\, A^{\lambda}_{j}) = 0$ and $\Xi^{\lambda}_j = A^{\lambda}_j \cap A^{\lambda,*}_{j-1}$. Hence

$$\lim_{t\to\infty} d(\xi(t),\, \Xi^{\lambda}_j) = 0. \tag{12}$$

It follows from Theorem 2.26 that for any $\varepsilon > 0$, we can choose δ_1, sufficiently small, such that

$$H^*_X(\Xi^{\lambda}_j,\, \Xi_j) < \varepsilon.$$

Combining this with (12), we deduce that when $\rho(\lambda,\, \lambda_0) < \delta_1$,

$$\lim_{t\to\infty} d(\xi(t),\, \Xi_j) \leqslant \varepsilon,$$

and thus the proof of this theorem is finished. □

3. **Nonautonomous Morse decomposition for gradient-like multi-valued processes.** In this section, we prove that all properties observed for gradient-like GDSs can be extended also to gradient-like multi-valued processes.

Definition 3.1. A multi-valued mapping $T : \mathbb{R} \times \mathbb{R} \times X \to X$ with nonempty closed images is said to be a multi-valued process, if the following axioms hold:

(1) $T(t,t)x = \{x\}$, $\forall (t,x) \in \mathbb{R} \times X$;
(2) $T(t,s)T(s,\tau)x = T(t,\tau)x$, $\forall t \geqslant s \geqslant \tau$, $x \in X$;
(3) $T(t,s)x$ is continuous in t for each fixed (s,x) in the sense of Hausdorff distance;
(4) $T(t,s)x$ is upper semi-continuous in (s,x) uniformly in t on any interval.

A family of subsets $D := \{D(t) : t \in \mathbb{R}\}$ of X is called a nonautonomous set. A nonautonomous set D is said to be closed (open), if $D(t)$ is closed (open) in X for each $t \in \mathbb{R}$.

Let $D := \{D(t) : t \in \mathbb{R}\}$ and $B := \{B(t) : t \in \mathbb{R}\}$ be two nonautonomous sets. If for each $t \in \mathbb{R}$, $B(t)$ is a neighborhood of $D(t)$ in X, then B is simply called a (nonautonomous) neighborhood of D.

Let $\mathcal{D}$ be a nonempty class of nonempty subsets family $D := \{D(t) : t \in \mathbb{R}\}$ of X.

Definition 3.2. A collection $\mathcal{D}$ of some families of nonempty subsets of X is said to be neighborhood closed if for each $D := \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, there exists a positive number ε depending on D such that the family

$$\{B(t) : B(t) \text{ is a nonempty subset of } \mathcal{O}_\varepsilon(D(t)), \forall t \in \mathbb{R}\} \tag{13}$$

also belongs to $\mathcal{D}$.

Note that the neighborhood closedness of $\mathcal{D}$ implies for each $D \in \mathcal{D}$,

$$\{\tilde{D}(t) : \tilde{D}(t) \text{ is a nonempty subset of } D(t), \forall t \in \mathbb{R}\} \in \mathcal{D}. \tag{14}$$

A collection $\mathcal{D}$ satisfying (14) is said to be inclusion-closed in the literature, see, e.g., [9].

Definition 3.3. A family of sets $\Xi := \{\Xi(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is said to be positively invariant (resp. negatively invariant, invariant), if for all $t \geqslant s$,

$$T(t,s)\Xi(s) \subset \Xi(t) \quad (\text{resp. } T(t,s)\Xi(s) \supset \Xi(t),\ T(t,s)\Xi(s) = \Xi(t)).$$

In this section we will always assume that $\{T(t,s)\}$ is a multi-valued process in X. Moreover, we assume that $\{T(t,s)\}$ is $\mathcal{D}$-pullback asymptotically upper-semicompact, that is, satisfies the following assumption:

(PAC) For all $t \in \mathbb{R}$, any sequence $y_n \in T(t, t - s_n)x_n$ has a convergent subsequence in X whenever $s_n \xrightarrow{n\to\infty} \infty$, $x_n \in B(t - s_n)$ with $B = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$.

Definition 3.4. An invariant family $\mathcal{A} := \{\mathcal{A}(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is said to be a (global) pullback attractor for a multi-valued process $\{T(t,s)\}$ if for each $t \in \mathbb{R}$, $\mathcal{A}(t)$ is compact, pullback attracts every member of $\mathcal{D}$; that is,

$$\lim_{s\to-\infty} H_X^*(T(t,s)B(s),\, \mathcal{A}(t)) = 0 \quad \text{for each } t \in \mathbb{R} \text{ and } B := \{B(t) : t \in \mathbb{R}\} \in \mathcal{D},$$

and $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ is minimal among all closed invariant families $\{e(t) : t \in \mathbb{R}\} \in \mathcal{D}$ with the property that $e(t)$ pullback attracts every member of $\mathcal{D}$ for each $t \in \mathbb{R}$.

Pullback and forward attractors are suitable concepts to describe the dynamics of nonautonomous dynamical systems. Simply said, a forward attractor is a family $\mathcal{A} := \{\mathcal{A}(t) : t \in \mathbb{R}\}$ of nonempty compact subsets of the state space, which is invariant and forward attracting. For nonautonomous dissipative systems, there are quite general existence results for pullback attractors, but in general, it is difficult to establish similar results for forward attractors. It is worth mentioning that due to the invariance property, a forward attractor, if it exists, necessarily coincides with the pullback one.

Definition 3.5. A continuous function $\xi : \mathbb{R} \to X$ is called a global solution for a multi-valued process $\{T(t,s)\}$ if it satisfies

$$\xi(t) \in T(t,s)\xi(s) \quad \text{for all } t \geqslant s,\ s \in \mathbb{R}.$$

A global solution ξ through $x \in X$ at $t \in \mathbb{R}$ means a global solution with $\xi(t) = x$.

If, in addition, there exists $D := \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$ such that $\xi(t) \in D(t)$ for every $t \in \mathbb{R}$, then ξ is called a $\mathcal{D}$-global solution for $\{T(t,s)\}$.

Definition 3.6. A family of sets $\Xi := \{\Xi(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is said to be weakly invariant, if for any $t \in \mathbb{R}$ and $x \in \Xi(t)$, there is a $\mathcal{D}$-global solution ξ through x at t with $\xi(s) \in \Xi(s)$ for all $s \in \mathbb{R}$.

Clearly, if $\Xi := \{\Xi(t) : t \in \mathbb{R}\}$ is a weakly invariant family, then Ξ is negatively invariant.

Let $B := \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$. The pullback ω-limit set $\omega(B) = \{\omega(B)(t) : t \in \mathbb{R}\}$ of B is defined, for all $t \in \mathbb{R}$, as follows:

$$\begin{aligned}\omega(B)(t) \quad := \quad & \{y \in X : \text{ there exist } s_n \xrightarrow{n\to\infty} \infty \text{ and } y_n \in T(t, t-s_n)B(t-s_n) \\ & \text{such that } y_n \xrightarrow{n\to\infty} y\}.\end{aligned}$$

Proposition 3.7. *Let $B := \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$. Then*

(1) *$\omega(B)(t)$ is nonempty compact and pullback attracts B for each $t \in \mathbb{R}$.*
(2) *$\omega(B) = \{\omega(B)(t) : t \in \mathbb{R}\}$ is weakly invariant.*

Proof. (1) Let $t \in \mathbb{R}$, $s_n \xrightarrow{n\to\infty} \infty$ and $x_n \in B(t - s_n)$ be given arbitrarily. Then, the $\mathcal{D}$-pullback asymptotically upper-semicompactness of $\{T(t, s)\}$ implies that for any sequence $y_n \in T(t, t-s_n)x_n$, there exists $y \in X$ such that, up to a subsequence, $y_n \xrightarrow{n\to\infty} y$. Hence $y \in \omega(B)(t)$ and $\omega(B)(t)$ is nonempty.

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\omega(B)(t)$. Then, there exist $s_n \xrightarrow{n\to\infty} \infty$ and $y'_n \in T(t, t - s_n)B(t - s_n)$ such that for all $n \in \mathbb{N}$,

$$d(z_n, y'_n) \leqslant \frac{1}{n}. \tag{15}$$

By the $\mathcal{D}$-pullback asymptotically upper-semicompactness of $\{T(t, s)\}$, we find that there exists $y' \in \omega(B)(t)$ such that, up to a subsequence, $y'_n \xrightarrow{n\to\infty} y'$, which together with (15) implies that $z_n \xrightarrow{n\to\infty} y' \in \omega(B)(t)$. Thus $\omega(B)(t)$ is compact.

We now prove that for all $t \in \mathbb{R}$,

$$\lim_{s\to\infty} d(T(t, t-s)B(t-s), \omega(B)(t)) = 0. \tag{16}$$

Suppose (16) is not true. Then there exist $t \in \mathbb{R}$, $\varepsilon' > 0$, sequences $s_n \xrightarrow{n\to\infty} \infty$, $x_n \in B(t - s_n)$ and $y_n \in T(t, t - s_n)x_n$ such that for all $n \in \mathbb{N}$,

$$d(y_n, \omega(B)(t)) \geqslant \varepsilon'. \tag{17}$$

By the $\mathcal{D}$-pullback asymptotically upper-semicompactness of $\{T(t, s)\}$, there exists $y \in \omega(B)(t)$ such that, up to a subsequence, $y_n \xrightarrow{n\to\infty} y$. This contradicts (17).

(2) We first prove that $\omega(B)$ is negatively invariant. Let $t \in \mathbb{R}$ and $y \in \omega(B)(t)$ be given arbitrarily. Then, there exist $\tau_n \xrightarrow{n\to\infty} \infty$ and $y_n \in T(t, t - \tau_n)B(t - \tau_n)$ such that $y_n \xrightarrow{n\to\infty} y$. Let $s \in \mathbb{R}$ with $s \leqslant t$ be given arbitrarily. Disregarding a finite number of terms, it can be assumed that $\tau_n > t - s$ for all $n \in \mathbb{N}$. Since $y_n \in T(t, t-\tau_n)B(t-\tau_n) = T(t, s)T(s, t-\tau_n)B(t-\tau_n)$, we have for every n, there exists $x_n \in T(s, t - \tau_n)B(t - \tau_n)$ such that $y_n \in T(t, s)x_n$. By the $\mathcal{D}$-pullback asymptotically upper-semicompactness of $\{T(t, s)\}$, taking a subsequence, if necessary, we can assume that $x_n \xrightarrow{n\to\infty} x \in \omega(B)(s)$. Then, it follows from Axiom (4) in Definition 3.1 that

$$d(y, T(t, s)x) \leqslant d(y, y_n) + d(y_n, T(t, s)x_n) + d(T(t, s)x_n, T(t, s)x) \xrightarrow{n\to\infty} 0.$$

It is clear that $y \in T(t, s)x \subset T(t, s)\omega(B)(s)$, and thus $\omega(B)(t) \subset T(t, s)\omega(B)(s)$.

Let $\sigma \in \mathbb{R}$ with $\sigma \geqslant t$ be given arbitrarily. Now we check that $T(\sigma, t)y \cap \omega(B)(\sigma) \neq \emptyset$. For each n, take $z_n \in T(\sigma, t)y_n$. Since $y_n \in T(t, t - \tau_n)B(t - \tau_n)$, we see that $z_n \in T(\sigma, t)y_n \subset T(\sigma, t - \tau_n)B(t - \tau_n)$. By the $\mathcal{D}$-pullback asymptotically upper-semicompactness of $\{T(t, s)\}$, taking a subsequence if necessary, we can assume that

$z_n \xrightarrow{n\to\infty} z \in \omega(B)(\sigma)$. From Axiom (4) in Definition 3.1, we deduce that

$$d(z,\, T(\sigma,t)y) \leqslant d(z,\, z_n) + d(z_n,\, T(\sigma,t)y_n) + d(T(\sigma,t)y_n,\, T(\sigma,t)y) \xrightarrow{n\to\infty} 0,$$

and, consequently, $z \in T(\sigma,t)y \cap \omega(B)(\sigma)$.

Finally, similar to the proof of Proposition 2.9 in [11], by the compactness of $\omega(B)(t)$ for each $t \in \mathbb{R}$ and the generalized Barbashin's Theorem for set-valued processes (see Theorem 3 in [3] for details), we conclude that there is a global solution ξ through y at t with $\xi(s) \in \omega(B)(s)$ for all $s \in \mathbb{R}$. This shows that $\omega(B)$ is weakly invariant under $\{T(t,s)\}$. □

Now we introduce the notions of isolated weakly invariant sets and disjoint families of isolated weakly invariant sets.

Definition 3.8. Let $\mathcal{D}$ be a neighborhood closed collection of some families of nonempty subsets of X, and let $\Xi := \{\Xi(t) : t \in \mathbb{R}\} \in \mathcal{D}$ be a weakly invariant family for the multi-valued process $\{T(t,s)\}$. Ξ is called an isolated weakly invariant family if there exists a $\delta > 0$ with the property that any $\mathcal{D}$-global solution $\xi : \mathbb{R} \to X$ with $\xi(t) \in \mathcal{O}_\delta(\Xi(t))$ must satisfy that $\xi(t) \in \Xi(t)$ for all $t \in \mathbb{R}$.

A set $\boldsymbol{\Xi} = \{\Xi_1, \dots, \Xi_n\}$ is said to be a disjoint set of isolated weakly invariant families if each Ξ_i, $1 \leqslant i \leqslant n$, is an isolated weakly invariant family and there exists $\delta > 0$ such that $\mathcal{O}_\delta(\Xi_i(t)) \cap \mathcal{O}_\delta(\Xi_j(t)) = \emptyset$, for all $t \in \mathbb{R}$ and for $1 \leqslant i < j \leqslant n$.

Remark 3.9. For the single-valued case, let $\mathcal{D}$ be a neighborhood closed collection of some families of nonempty subsets of X, and let $\{T(t,s)\}$ be a process in X. Then, an invariant family $\Xi := \{\Xi(t) : t \in \mathbb{R}\}$ is an isolated invariant family for $\{T(t,s)\}$ if there exists a $\delta > 0$ such that for any invariant family $A := \{A(t) : t \in \mathbb{R}\}$ with $A(t) \subset \mathcal{O}_\delta(\Xi(t))$ for all $t \in \mathbb{R}$, we have $A(t) \subseteq \Xi(t)$ for all $t \in \mathbb{R}$. It is important to notice that if $\Xi := \{\Xi(t) : t \in \mathbb{R}\}$ is a compact isolated weakly invariant family, further we can show that Ξ is invariant, then Ξ is an isolated invariant family; see Remark 2.5 in [17] for details.

Definition 3.10. Let $\{T(t,s)\}$ be a multi-valued process which possesses a disjoint set of isolated weakly invariant families $\boldsymbol{\Xi} = \{\Xi_1, \dots, \Xi_n\} \subset \mathcal{D}$. A homoclinic structure associated with $\boldsymbol{\Xi}$ is a subset $\{\Xi_{k_1}, \dots, \Xi_{k_p}\}$ of $\boldsymbol{\Xi}$ ($p \leqslant n$) together with a set of $\mathcal{D}$-global solutions $\{\xi_1, \dots, \xi_p\}$ such that

$$\lim_{t\to-\infty} d(\xi_j(t),\, \Xi_{k_j}(t)) = 0 \quad \text{and} \quad \lim_{t\to\infty} d(\xi_j(t),\, \Xi_{k_{j+1}}(t)) = 0, \quad 1 \leqslant j \leqslant p,$$

where $\Xi_{k_{p+1}} := \Xi_{k_1}$.

Let us now define the notion of a gradient-like multi-valued process.

Definition 3.11. Let $\{T(t,s)\}$ be a multi-valued process in X with a pullback attractor $\mathcal{A} := \{\mathcal{A}(t) : t \in \mathbb{R}\} \in \mathcal{D}$ and a disjoint set of isolated weakly invariant families $\boldsymbol{\Xi} = \{\Xi_1, \dots, \Xi_n\}$ in $\{\mathcal{A}(t) : t \in \mathbb{R}\}$. We say that $\{T(t,s)\}$ is a gradient-like multi-valued process relative to $\boldsymbol{\Xi}$ if the following two conditions are satisfied:

(1) Any $\mathcal{D}$-global solution $\xi : \mathbb{R} \to X$ in $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ satisfies
$$\lim_{t\to-\infty} d(\xi(t),\, \Xi_i(t)) = 0 \quad \text{and} \quad \lim_{t\to\infty} d(\xi(t),\, \Xi_j(t)) = 0,$$
for some $1 \leqslant i, j \leqslant n$.

(2) There is no homoclinic structure associated with $\boldsymbol{\Xi}$.

Let $\{T(t,s)\}$ be a multi-valued process with a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ which contains a disjoint set of isolated weakly invariant families $\boldsymbol{\Xi} = \{\Xi_1, \dots, \Xi_n\}$, that is, $\Xi_i(t) \subset \mathcal{A}(t)$ for each i and t. Now we can define the concept of pinned-chain recurrence (see [2, 5, 6]) for similar concepts in the single-valued case).

Definition 3.12. Let δ be as in Definition 3.8 and fix $\varepsilon_0 \in (0, \delta)$. For $\Xi \in \boldsymbol{\Xi}$ and $\varepsilon \in (0, \varepsilon_0)$, an ε-pinned-chain from Ξ to Ξ is a sequence l_i, $1 \leqslant i \leqslant k$, in $\{1, \ldots, n\}$, a sequence of real numbers t_i, σ_i, τ_i with $\tau_i < \sigma_i < t_i$, a sequence of points $z_i \in X$, $1 \leqslant i \leqslant k$, and a sequence of solutions ξ_i for $\{T(t,s)\}$, $1 \leqslant i \leqslant k$, such that $\xi_i(\tau_i) = z_i \in \mathcal{O}_\varepsilon(\Xi_{l_i}(\tau_i))$, $\xi(\sigma_i) \notin \mathcal{O}_{\varepsilon_0}(\bigcup_{i=1}^n \Xi_i(\sigma_i))$ and $\xi_i(t_i) \in \mathcal{O}_\varepsilon(\Xi_{l_{i+1}}(t_i))$, $1 \leqslant i \leqslant k$, with $\Xi = \Xi_{l_{k+1}} = \Xi_{l_1}$. We say that $\Xi \in \boldsymbol{\Xi}$ is pinned-chain recurrent if there is an $\varepsilon_0 \in (0, \delta)$ and ε-pinned-chain from Ξ to Ξ for each $\varepsilon \in (0, \varepsilon_0)$.

Remark 3.13. It is clear that if $\{\mathcal{A}(t) : t \in \mathbb{R}\} \in \mathcal{D}$ has a homoclinic structure, then there is a pinned-chain recurrent isolated weakly invariant family $\Xi \in \boldsymbol{\Xi}$.

In particular, in the single-valued autonomous dynamical system case, the notion of pinned-chain recurrence is closely related to the existence of homoclinic structures (see Lemma 2.10 in [5] for details).

Similar to the single-valued nonautonomous dynamical system case in [2], we next define the notions of a local attractor and of an attractor-repeller pair for a multi-valued process $\{T(t,s)\}$. Before we define these concepts. Let us begin with the definition of an unstable set of a weakly invariant family.

Definition 3.14. Let $\{T(t,s)\}$ be a multi-valued process. The unstable set of a weakly invariant family $\Xi := \{\Xi(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is the set

$$W^u(\Xi) := \{(t,z) \in \mathbb{R} \times X : \text{ there is a } \mathcal{D}\text{-global solution } \xi : \mathbb{R} \to X \text{ such that } \xi(t) = z \text{ and } \lim_{s \to -\infty} d(\xi(s), \Xi(s)) = 0\}.$$

Also, $W^u(\Xi)(t) := \{z \in X : (t,z) \in W^u(\Xi)\}$ for each $t \in \mathbb{R}$.

Definition 3.15. Let $\{T(t,s)\}$ be a multi-valued process in X with a pullback attractor $\mathcal{A} := \{\mathcal{A}(t) : t \in \mathbb{R}\} \in \mathcal{D}$. We say that an isolated weakly invariant family $A := \{A(t) : t \in \mathbb{R}\}$, with $A(t) \subset \mathcal{A}(t)$ for all $t \in \mathbb{R}$, is a (nonautonomous) local attractor if $W^u(A)(t) = A(t)$ for all $t \in \mathbb{R}$.

If A is a local attractor, we define its associated repeller $A^* := \{A^*(t) : t \in \mathbb{R}\}$ by

$$A^*(t) := \{z \in \mathcal{A}(t) : H_X^*(T(r+t,t)z,\, A(r+t)) \nrightarrow 0 \text{ as } r \to \infty\}, \quad \text{for each } t \in \mathbb{R}.$$

The pair (A, A^*) is called an attractor-repeller pair.

Lemma 3.16. *Let $\mathcal{D}$ be a neighborhood closed collection of some families of nonempty subsets of X, $\{T(t,s)\}$ be a multi-valued process with a pullback attractor $\mathcal{A} := \{\mathcal{A}(t) : t \in \mathbb{R}\} \in \mathcal{D}$, and let $\Xi := \{\Xi(t) : t \in \mathbb{R}\}$ be such that for some $\varepsilon > 0$, $\omega(\mathcal{O}_\varepsilon(\Xi) \cap \mathcal{A})(t) = \Xi(t)$ for all $t \in \mathbb{R}$, where*

$$\omega(\mathcal{O}_\varepsilon(\Xi) \cap \mathcal{A})(t) := \{y \in X : \text{ there exist } s_n \xrightarrow{n\to\infty} \infty \text{ and } y_n \in T(t, t-s_n)(\mathcal{O}_\varepsilon(\Xi(t-s_n)) \cap \mathcal{A}(t-s_n)) \text{ such that } y_n \xrightarrow{n\to\infty} y\}.$$

Then $\Xi := \{\Xi(t) : t \in \mathbb{R}\}$ is a local attractor and Ξ is a compact isolated invariant family.

Proof. Since $\mathcal{D}$ is neighborhood closed and $\{\mathcal{A}(t) : t \in \mathbb{R}\} \in \mathcal{D}$, there exists a $\varepsilon_0 > 0$ with $\varepsilon_0 < \varepsilon$ such that $\{\mathcal{O}_{\varepsilon_0}(\Xi(t)) : t \in \mathbb{R}\} \in \mathcal{D}$. Thanks to Proposition 3.7, we see that $\Xi := \{\Xi(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is compact weakly invariant and $\Xi(t)$ pullback attracts $\{\mathcal{O}_{\varepsilon_0}(\Xi(t)) \cap \mathcal{A}(t) : t \in \mathbb{R}\}$ for each $t \in \mathbb{R}$. Let $\xi : \mathbb{R} \to X$ be a $\mathcal{D}$-global solution with $\xi(t) \in \mathcal{O}_{\varepsilon_0}(\Xi(t))$ for all $t \in \mathbb{R}$. Since every $\mathcal{D}$-global solution belongs to the pullback attractor, by assumption $\omega(\mathcal{O}_{\varepsilon_0}(\Xi) \cap \mathcal{A})(t) = \Xi(t)$, we deduce that, for any $t \in \mathbb{R}$,

$$\begin{aligned} d(\xi(t), \Xi(t)) \quad &\leqslant \quad d(\xi(t), T(t, t-s)\xi(t-s)) \\ &\quad + d(T(t, t-s)\xi(t-s), T(t, t-s)(\mathcal{O}_{\varepsilon_0}(\Xi(t-s)) \cap \mathcal{A}(t-s))) \\ &\quad + d(T(t, t-s)(\mathcal{O}_{\varepsilon_0}(\Xi(t-s)) \cap \mathcal{A}(t-s)), \Xi(t)) \to 0 \text{ as } s \to \infty, \end{aligned}$$

and consequently, $\xi(t) \in \Xi(t)$ for all $t \in \mathbb{R}$ and $\Xi := \{\Xi(t) : t \in \mathbb{R}\}$ is a compact isolated weakly invariant family.

Now we show that $W^u(\Xi)(t) = \Xi(t)$ for all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$ and $z \in W^u(\Xi)(t)$. Then by the definition of the unstable set of the weakly invariant set, we obtain that there exist a $\mathcal{D}$-global solution $\xi : \mathbb{R} \to X$ and $t_0' > 0$ such that $\xi(t) = z$ and $\xi(s) \in \mathcal{O}_{\varepsilon_0}(\Xi(s)) \cap \mathcal{A}(s)$ for each $s \leqslant -t_0'$. Note that for all $s \leqslant -t_0'$,

$$\begin{aligned} d(z, \Xi(t)) \quad &\leqslant \quad d(\xi(t), T(t, s)\xi(s)) + d(T(t, s)(\mathcal{O}_{\varepsilon_0}(\Xi(s)) \cap \mathcal{A}(s)), \Xi(t)) \\ &= \quad d(T(t, s)(\mathcal{O}_{\varepsilon_0}(\Xi(s)) \cap \mathcal{A}(s)), \Xi(t)), \end{aligned}$$

and that $\Xi(t)$ pullback attracts $\{\mathcal{O}_{\varepsilon_0}(\Xi(t)) : t \in \mathbb{R}\}$ for each $t \in \mathbb{R}$. Therefore $z \in \Xi(t)$ and $W^u(\Xi)(t) \subset \Xi(t)$. The converse inclusion $\Xi(t) \subset W^u(\Xi)(t)$ follows from the fact that $\Xi := \{\Xi(t) : t \in \mathbb{R}\}$ is weakly invariant.

Finally, we prove that $\Xi := \{\Xi(t) : t \in \mathbb{R}\}$ is invariant. Since $\Xi := \{\Xi(t) : t \in \mathbb{R}\}$ is weakly invariant, clearly Ξ is negatively invariant. Now it only remains to check that $T(t, s)\Xi(s) \subset \Xi(t)$ for all $t \geqslant s$ and $s \in \mathbb{R}$. Let $s \in \mathbb{R}$, $t \geqslant s$ and $y \in T(t, s)\Xi(s)$ be given arbitrarily. Then $y \in T(t, s)x$ for some $x \in \Xi(s) \subset \mathcal{A}(s)$. Since $\mathcal{A}$ is a compact invariant family, in view of Remark 2.5 in [17], there is a $\mathcal{D}$-global solution $\xi_3 : \mathbb{R} \to \mathcal{A}$ such that $\xi_3(t) = y$. Recall that $x \in \Xi(s)$, hence there is also a $\mathcal{D}$-global solution $\xi_1 : \mathbb{R} \to \mathcal{A}$ such that $\xi_1(s) = x$ and $\lim_{\tau \to -\infty} d(\xi_1(\tau), \Xi(\tau)) = 0$. On the other hand, we can find a solution ξ_2 on $[s, t]$ such that $\xi_2(s) = x$, $\xi_2(t) = y$ and clearly $\xi_2(\tau) \in \mathcal{A}(\tau)$ for each $\tau \in [s, t]$. We define a $\mathcal{D}$-global solution $\xi : \mathbb{R} \to \mathcal{A}$ such that

$$\xi(\tau) = \begin{cases} \xi_3(\tau), & \forall \tau \geqslant t; \\ \xi_2(\tau), & \forall s \leqslant \tau \leqslant t; \\ \xi_1(\tau), & \forall \tau \leqslant s. \end{cases}$$

Then we have $\xi(t) = y$ and $\lim_{\tau \to -\infty} d(\xi(\tau), \Xi(\tau)) = 0$. This implies $y \in W^u(\Xi)(t) = \Xi(t)$, and thus $T(t, s)\Xi(s) \subset \Xi(t)$.

Thanks to Remark 3.9, since $\Xi := \{\Xi(t) : t \in \mathbb{R}\}$ is a compact isolated weakly invariant family and Ξ is invariant, we obtain that Ξ is a compact isolated invariant family. The proof of this lemma is therefore finished. □

Indeed, we can also define the (nonautonomous) local attractor in the sense of pullback attraction, i.e., for some $\varepsilon > 0$, $\omega(\mathcal{O}_\varepsilon(\Xi))(t) = \Xi(t)$ for all $t \in \mathbb{R}$. This implies the (nonautonomous) local attractor in the sense of Definition 3.15 due to Lemma 3.16.

Lemma 3.17. *Let $\{T(t, s)\}$ be a multi-valued process with a pullback attractor $\mathcal{A} := \{\mathcal{A}(t) : t \in \mathbb{R}\} \in \mathcal{D}$ and let $A := \{A(t) : t \in \mathbb{R}\}$ be a local attractor with $A(t) \subset \mathcal{A}(t)$ for each $t \in \mathbb{R}$. Suppose that there exists $\varepsilon > 0$ with $\mathcal{A}(t) \cap \mathcal{O}_\varepsilon(A(t)) \cap \mathcal{O}_\varepsilon(A^*(t)) = \emptyset$ for all $t \in \mathbb{R}$. Then, the repeller A^* of A is an isolated weakly invariant family and $A^*(t)$ is compact for each $t \in \mathbb{R}$.*

Proof. If $A^*(t_0)$ is empty for some $t_0 \in \mathbb{R}$, then it is empty for all $t \in \mathbb{R}$ and the proof is obvious. Assume that $A^*(t)$ is nonempty for all $t \in \mathbb{R}$.

Let us first show that $A^*(t)$ is compact for each $t \in \mathbb{R}$. Observe that $A^*(t) \subset \mathcal{A}(t)$ and $\mathcal{A}(t)$ is compact, it is sufficient to prove that $A^*(t)$ is closed. Let $y_n \xrightarrow{n\to\infty} y$, where $y_n \in A^*(t)$. If $y \notin A^*(t)$, then we have that

$$H_X^*(T(r+t,t)y,\ A(r+t)) \to 0 \text{ as } r \to \infty.$$

By the assumption, there exists $\varepsilon > 0$ such that $\mathcal{A}(t) \cap \mathcal{O}_\varepsilon(A(t)) \cap A^*(t) = \emptyset$ for all $t \in \mathbb{R}$. Hence there is a t_0 such that

$$H_X^*(T(t_0+t,t)y,\ A(t_0+t)) < \frac{\varepsilon}{2}.$$

By Axiom (4) in Definition 3.1, in view of $y_n \to y$, we obtain that there exists n_0 such that for all $n \geqslant n_0$,

$$\begin{aligned} H_X^*(T(t_0+t,t)y_n,\ A(t_0+t)) \quad &\leqslant \quad H_X^*(T(t_0+t,t)y_n,\ T(t_0+t,t)y) \\ &\quad +H_X^*(T(t_0+t,t)y,\ A(t_0+t)) < \varepsilon. \end{aligned}$$

This implies that for all $n \geqslant n_0$, $T(t_0+t,t)y_n \subset \mathcal{O}_\varepsilon(A(t_0+t))$, and thus we have $x \notin A^*(t_0+t)$ for all $x \in T(t_0+t,t)y_n$. From the definition of the repeller, the compactness of $T(t_0+t,t)y_n$ and Axiom (4) in Definition 3.1, we deduce that for all $n \geqslant n_0$,

$$H_X^*(T(r{+}t,t)y_n,\ A(r{+}t)) = H_X^*(T(r{+}t,t_0{+}t)T(t_0{+}t,t)y_n,\ A(r{+}t)) \to 0 \text{ as } r \to \infty,$$

which contradicts the fact that $y_n \in A^*(t)$ for all $n \in \mathbb{N}$.

We will prove now that A^* is negatively invariant. Let $t \geqslant s$ and $w \in A^*(t) \subset \mathcal{A}(t)$, then the invariance of $\mathcal{A}$ implies that there is $v \in \mathcal{A}(s)$ such that $w \in T(t,s)v$. In fact, v belongs to $A^*(s)$. If this were not true, then by the definition of repeller,

$$\lim_{\tau\to\infty} H_X^*(T(\tau,s)v,\ A(\tau)) = 0.$$

Note that $T(\tau,t)w \subset T(\tau,t)T(t,s)v = T(\tau,s)v$. Hence,

$$\lim_{\tau\to\infty} H_X^*(T(\tau,t)w,\ A(\tau)) = 0,$$

which contradicts the fact that $w \in A^*(t)$ and proves that $A^*(t) \subset T(t,s)A^*(s)$.

We assume that $w \in A^*(t)$. Then by the compactness and negative invariance of A^*, in view of Theorems 2.2-2.3 in [17], we deduce from the similar proof of Proposition 2.9 in [11] that there is a solution ξ^- on $(-\infty,t]$ such that $\xi^-(s) \in T(s,\tau)\xi^-(\tau)$ for all $s \leqslant t$, $\tau \leqslant s$, $\xi^-(s) \in A^*(s)$ for all $s \leqslant t$ and $\xi^-(t) = w$.

To prove that A^* is weakly invariant, according to Remark 2.5 in [17], it suffices to verify that for any $t \geqslant s$, $s \in \mathbb{R}$ and $x \in A^*(s)$, $T(t,s)x \cap A^*(t) \neq \emptyset$. Assume on the contrary that this is not the case. Then there exist $s_0 \in \mathbb{R}$, $t_0 \geqslant s_0$ and $x_0 \in A^*(s_0)$ such that $T(t_0,s_0)x_0 \cap A^*(t_0) = \emptyset$. This implies that $y \notin A^*(t_0)$ for all $y \in T(t_0,s_0)x_0$. Hence, by the definition of the repeller, the compactness of $T(t_0,s_0)x_0$ and Axiom (4) in Definition 3.1, we obtain that

$$H_X^*(T(t{+}s_0,s_0)x_0,\ A(t{+}s_0)) = H_X^*(T(t{+}s_0,t_0)T(t_0,s_0)x_0,\ A(t{+}s_0)) \to 0 \text{ as } t \to \infty,$$

which contradicts the fact that $x_0 \in A^*(s_0)$.

Finally, we prove that A^* is an isolated weakly invariant family. Let $\xi : \mathbb{R} \to X$ be a $\mathcal{D}$-global solution with $\xi(t) \in \mathcal{O}_\varepsilon(A^*(t)) \cap \mathcal{A}(t)$ for all $t \in \mathbb{R}$. To complete the proof, it suffices to show that $\xi(t) \in A^*(t)$ for all $t \in \mathbb{R}$. If that were not the case, then there would exist a $t_0 \in \mathbb{R}$ such that $\xi(t_0) \notin A^*(t_0)$. By the definition of repeller, we have

$$\lim_{t\to\infty} H_X^*(T(t,t_0)\xi(t_0),\ A(t)) = 0,$$

and consequently, $\xi(t) \in \mathcal{O}_\varepsilon(A(t))$ for sufficiently large t, which is in contradiction with $\mathcal{A}(t) \cap \mathcal{O}_\varepsilon(A(t)) \cap \mathcal{O}_\varepsilon(A^*(t)) = \emptyset$ for all $t \in \mathbb{R}$. The proof is complete. □

Remark 3.18. Notice that for single-valued evolution processes, a repeller $A^* := \{A^*(t) : t \in \mathbb{R}\}$ associated with a local attractor A is invariant, as shown in [2, Proposition 2.7].

Definition 3.19. Let $\{T(t,s)\}$ be a multi-valued process in X with a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\} \in \mathcal{D}$ and let $A_0 = \{A_0(t) : t \in \mathbb{R}\}$, $A_1 = \{A_1(t) : t \in \mathbb{R}\}$, $\dots$, $A_n = \{A_n(t) : t \in \mathbb{R}\}$ be $n+1$ local attractors with $\emptyset = A_0(t) \subset A_1(t) \subset \cdots \subset A_n(t) = \mathcal{A}(t)$ for each $t \in \mathbb{R}$.

Define $\Xi_j(t) := A_j(t) \cap A^*_{j-1}(t)$ for each $t \in \mathbb{R}$ and $j = 1, \dots, n$. The ordered set of weakly invariant families $\mathbf{\Xi} := \{\Xi_1, \Xi_2, \dots, \Xi_n\}$ is called a Morse-decomposition for the pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$.

The following result extends Lemma 2.16 to the nonautonomous case.

Lemma 3.20. *Let $\mathcal{D}$ be a neighborhood closed collection of some families of nonempty subsets of X, $\{T(t,s)\}$ be a multi-valued process in X with a pullback attractor $\mathcal{A} := \{\mathcal{A}(t) : t \in \mathbb{R}\} \in \mathcal{D}$, and let $A := \{A(t) : t \in \mathbb{R}\}$ be a local attractor for $\{T(t,s)\}$ in $\mathcal{A}$, that is, $A := \{A(t) : t \in \mathbb{R}\}$ is an isolated weakly invariant family with $W^u_{\mathcal{A}}(A)(t) = A(t)$ for all $t \in \mathbb{R}$, where*

$$W^u_{\mathcal{A}}(A)(t) = \{z \in \mathcal{A}(t) : \text{ there is a } \mathcal{D}\text{-global solution } \xi : \mathbb{R} \to \mathcal{A} \text{ such that } \xi(t) = z \text{ and } \lim_{s \to -\infty} d(\xi(s), A(s)) = 0\}.$$

Then A is a local attractor for $\{T(t,s)\}$ in X.

Proof. We first show that $W^u(A)(\tau) \subset A(\tau)$ for each $\tau \in \mathbb{R}$. Let $t \in \mathbb{R}$ and $z \in W^u(A)(t)$ be given arbitrarily. Then there is a $\mathcal{D}$-global solution $\xi : \mathbb{R} \to X$ such that $\xi(t) = z$ and

$$\lim_{s \to -\infty} d(\xi(s), A(s)) = 0. \tag{18}$$

Since $A \in \mathcal{D}$ and $\mathcal{D}$ is neighborhood closed, there exists $\varepsilon > 0$ such that

$$\{B(t) : B(t) \text{ is a nonempty subset of } \mathcal{O}_\varepsilon(A(t)), \forall t \in \mathbb{R}\} \in \mathcal{D}.$$

Choose an arbitrary number $\varepsilon_1 \in (0, \varepsilon)$. By (18), there is a $T > 0$ such that for all $s \leqslant -T$,

$$d(\xi(s), A(s)) < \varepsilon_1,$$

and consequently, $\xi(s) \in \mathcal{O}_\varepsilon(A(s))$ for each $s \leqslant -T$ and $\{\mathcal{O}_{\varepsilon_1}(A(t)) : t \in \mathbb{R}\} \in \mathcal{D}$. Then by the definition of pullback attractor, we have

$$d(\xi(t), \mathcal{A}(t)) \leqslant \lim_{s \to -\infty} H^*_X(T(t,s)\xi(s), \mathcal{A}(t)) = 0.$$

This implies that $\xi(t) \in \mathcal{A}(t)$, thus ξ is a $\mathcal{D}$-global solution defined in $\mathcal{A}$ and $z = \xi(t) \in W^u_{\mathcal{A}}(A)(t)$. Noting that $W^u_{\mathcal{A}}(A)(\tau) = A(\tau)$ for all $\tau \in \mathbb{R}$, hence $z \in A(t)$ and this proves $W^u(A)(t) \subset A(t)$.

In the similar way, the isolation of A for $\{T(t,s)\}$ in X follows from the one of A for $\{T(t,s)\}$ in $\mathcal{A}$. It is clear that A is a weakly invariant family for $\{T(t,s)\}$ in X and $A(\tau) \subset W^u(A)(\tau)$ for all $\tau \in \mathbb{R}$, and thus the proof of this theorem is complete. □

Remark 3.21. Let $\{G(t)\}$ be a GDS in X with a global attractor $\mathcal{A}$. By slightly modifying the proof of Lemma 2.19 and Corollary 2.20, we obtain that an isolated invariant set $A \subset \mathcal{A}$ is a local attractor for $\{G(t)\}$ in $\mathcal{A}$ (that is, there is an $\varepsilon > 0$ with $\omega(\mathcal{O}_\varepsilon(A) \cap \mathcal{A}) = A$) if and only if $W^u_{\mathcal{A}}(A) = A$ where

$$W^u_{\mathcal{A}}(A) \quad = \quad \{z \in \mathcal{A} : \text{ there is a global solution } \xi : \mathbb{R} \to \mathcal{A} \text{ such that } \xi(0) = z \text{ and } \lim_{t\to-\infty} d(\xi(t), A) = 0\}.$$

Next we describe the construction of a Morse decomposition for the pullback attractor of a gradient-like multi-valued process relative to the disjoint set of isolated weakly invariant families $\{\Xi_1, \dots, \Xi_n\}$. First, we need the following fundamental result.

Lemma 3.22. *Let $\{T(t, s)\}$ be a gradient-like multi-valued process with an associated disjoint set of isolated weakly invariant families $\mathbf{\Xi} = \{\Xi_1, \dots, \Xi_n\}$. Then, there exists $i \in \{1, \dots, n\}$ such that Ξ_i is a local attractor for $\{T(t, s)\}$.*

Proof. First, we show that there is some $i \in \{1, \dots, n\}$ such that $W^u(\Xi_i)(t) = \Xi_i(t)$ for each $t \in \mathbb{R}$. Assume, by contradiction, that this were not the case. Then, for each $1 \leqslant i \leqslant n$, there would exist a $\mathcal{D}$-global solution $\xi_i(t) \in \mathcal{A}(t)$ (with $\xi_i(s) \notin \Xi_i(s)$ for some $s \in \mathbb{R}$) such that $\lim_{t\to-\infty} d(\xi_i(t), \Xi_i(t)) = 0$. From the fact that $\{T(t, s)\}$ is gradient-like, $\xi_i(t)$ converges to some element of $\mathbf{\Xi}$ as $t \to \infty$, this necessarily would produce a homoclinic structure which would be a contradiction.

Note that $\mathbf{\Xi} = \{\Xi_1, \dots, \Xi_n\}$ is a disjoint set of isolated weakly invariant families, and thus the proof of Lemma 3.22 is finished. □

Let $\{T(t, s)\}$ be a gradient-like multi-valued process with the associated disjoint set of isolated weakly invariant families $\mathbf{\Xi} = \{\Xi_1, \dots, \Xi_n\}$. If (after possible reordering) Ξ_1 is a local attractor for $\{T(t, s)\}$, let Ξ_1^* as in Definition 3.15 be its associated repeller, then we have that each $\Xi_i(s)$, for $i \geqslant 2$ and $s \in \mathbb{R}$, is contained in $\Xi_1^*(s)$. We can repeat the reasoning in Lemma 3.22 to deduce that there is $i \geqslant 2$ such that

$$W^u(\Xi_i)(t) \cap \Xi_1^*(t) = \Xi_i(t) \quad \text{ for all } t \in \mathbb{R}.$$

We relabel this isolated weakly invariant family as Ξ_2 and define for each $t \in \mathbb{R}$,

$$\Xi_{2,1}^*(t) := \{z \in \Xi_1^*(t) : H_X^*(T(r + t, t)z, \Xi_2(r + t)) \nrightarrow 0 \text{ as } r \to \infty\}.$$

Then we have that, for each $t \in \mathbb{R}$ and $i = 3, \dots, n$, $\Xi_i(t) \subset \Xi_{2,1}^*(t)$.

Proceeding in this way untill all isolated weakly invariant families are exhausted, we obtain a reordering of $\mathbf{\Xi} = \{\Xi_1, \dots, \Xi_n\}$ such that Ξ_1 is a local attractor for $\{T(t, s)\}$. Setting $\Xi_{1,0}^* := \Xi_1^*$, and

$$W^u(\Xi_i)(t) \cap \Xi_{i-1,i-2}^*(t) = \Xi_i(t) \quad \text{ for all } t \in \mathbb{R} \text{ and } i = 2, \dots, n,$$

where, for $i = 2, \dots, n$,

$$\Xi_{i,i-1}^*(t) := \{z \in \Xi_{i-1,i-2}^*(t) : H_X^*(T(r + t, t)z, \Xi_i(r + t)) \nrightarrow 0 \text{ as } r \to \infty\}.$$

Remark 3.23. Similar to the proof of Lemma 2.9 in [2] for single-valued systems, we have the following result:

Let $\{T(t, s)\}$ be a gradient-like multi-valued process with a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ and an associated (reordered) disjoint set of isolated weakly invariant families $\mathbf{\Xi} = \{\Xi_1, \dots, \Xi_n\}$. Then, any $\mathcal{D}$-global solution $\xi : \mathbb{R} \to X$ in $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ satisfies

$$\lim_{t\to-\infty} d(\xi(t), \Xi_i(t)) = 0 \quad \text{ and } \quad \lim_{t\to\infty} d(\xi(t), \Xi_j(t)) = 0,$$

with $i \geqslant j$.

Theorem 3.24. *Let $\{T(t,s)\}$ be a gradient-like multi-valued process with a pullback attractor $\mathcal{A} := \{\mathcal{A}(t) : t \in \mathbb{R}\}$ and an associated disjoint set of isolated weakly invariant families $\boldsymbol{\Xi} = \{\Xi_1, \dots, \Xi_n\}$ reordered as explained above. Assume that there exists $\delta > 0$ such that, for $j = 1, 2, \dots, n-1$ and $t \in \mathbb{R}$, it holds that*

$$\mathcal{O}_\delta(A_j(t)) \cap \left(\bigcup_{i=j+1}^{n} \Xi_i(t) \right) = \emptyset. \tag{19}$$

Then $\boldsymbol{\Xi}$ defines a Morse decomposition for $\{\mathcal{A}(t) : t \in \mathbb{R}\}$.

Proof. We divide the proof into four steps.

Step 1. For each $t \in \mathbb{R}$, set $A_0(t) := \emptyset$, $A_1(t) := \Xi_1(t)$, and for $j = 2, 3, \dots, n$,

$$A_j(t) := A_{j-1}(t) \cup W^u(\Xi_j)(t) = \bigcup_{i=1}^{j} W^u(\Xi_i)(t).$$

Clearly $A_n(t) = \mathcal{A}(t)$. From Lemma 3.22, it is easy to see that $A_1 := \Xi_1$ is a weakly invariant family. We prove that $\{W^u(\Xi_2)(t) : t \in \mathbb{R}\}$ is a weakly invariant family. Let $t \geqslant s$, $s \in \mathbb{R}$ and $y \in W^u(\Xi_2)(t)$ be given. By the definition of the unstable set of a weakly invariant family, we find that there is a $\mathcal{D}$-global solution $\xi : \mathbb{R} \to X$ such that $\xi(t) = y$ and $\lim_{\tau \to -\infty} d(\xi(\tau), \Xi_2(\tau)) = 0$. Hence $\xi(s) \in W^u(\Xi_2)(s)$ for all $s \in \mathbb{R}$. Consequently, we obtain that $A_2 := \{A_2(t) : t \in \mathbb{R}\}$ is a weakly invariant family. Proceeding in this way, we obtain that for $j = 3, \dots, n$, $A_j := \{A_j(t) : t \in \mathbb{R}\}$ is a weakly invariant family.

Step 2. We prove that $W^u(A_j)(t) = A_j(t)$ for each $1 \leqslant j \leqslant n$ and $t \in \mathbb{R}$. Noting that $A_j(t) \subset W^u(A_j)(t)$ for each $t \in \mathbb{R}$ due to the weak invariance of A_j. On the other hand, if $z \in W^u(A_j)(t)$, then there is a $\mathcal{D}$-global solution $\xi : \mathbb{R} \to X$ with $\xi(t) = z$ and $\lim_{s \to -\infty} d(\xi(s), A_j(s)) = 0$. Since $\{T(t,s)\}$ is gradient-like and from (19), we must have that there exists $k \in \{1, 2, \dots, j\}$ such that $\lim_{s \to -\infty} d(\xi(s), \Xi_k(s)) = 0$. This implies that $z \in W^u(\Xi_k)(t) \subset A_j(t)$ and thus $W^u(A_j)(t) \subset A_j(t)$.

Step 3. We show that $\{A_j(t) : t \in \mathbb{R}\}$ is a local attractor for each j. From the previous results and the definition of local attractors, now we need to prove that $\{A_j(t) : t \in \mathbb{R}\}$ is an isolated weakly invariant family. Indeed, let $\delta > 0$ be as in (19) and $\xi : \mathbb{R} \to X$ be a $\mathcal{D}$-global solution with $\xi(t) \in \mathcal{O}_\delta(A_j(t))$ for all $t \in \mathbb{R}$. Since $\{T(t,s)\}$ is gradient-like, there is a $k \in \{1, \dots, j\}$ such that $d(\xi(t), \Xi_k(t)) \xrightarrow{t \to -\infty} 0$. Then $\xi(t) \in W^u(\Xi_k)(t) \subset A_j(t)$ for all $t \in \mathbb{R}$. Hence, $\{A_j(t) : t \in \mathbb{R}\}$ is a local attractor for each j.

Step 4. Let us prove that $\Xi_j(t) = A_j(t) \cap A^*_{j-1}(t)$ for each j. By a similar proof of Theorem 2.10 in [2], we can show that $A_j(t) \cap A^*_{j-1}(t) \subset \Xi_j(t)$ for each j. Now it only remains to prove the reverse inclusion. Let $z \in \Xi_j(t)$, then there is a $\mathcal{D}$-global solution $\xi : \mathbb{R} \to X$ with $\xi(t) = z$ and $\xi(s) \in \Xi_j(s)$ for each $s \in \mathbb{R}$. Clearly $z \in A_j(t)$. Suppose $z \notin A^*_{j-1}(t)$. Then we have

$$\lim_{s \to \infty} H^*_X(T(s,t)z, A_{j-1}(s)) = 0,$$

in particular, $\lim_{s\to\infty} d(\xi(s), A_{j-1}(s)) = 0$. By assumption (19), we deduce that

$$d\left(\xi(s), \bigcup_{i=j}^{n} \Xi_i(s)\right) \nrightarrow 0 \quad \text{as } s \to \infty.$$

Note that $\{T(t,s)\}$ is gradient-like, hence there is $i \in \{1, 2, \ldots, j-1\}$ such that

$$\lim_{s\to\infty} d(\xi(s), \Xi_i(s)) = 0,$$

which contradicts the fact that $\xi(s) \in \Xi_j(s)$ for each $s \in \mathbb{R}$ and $\mathbf{\Xi} = \{\Xi_1, \ldots, \Xi_n\}$ is a disjoint set of isolated weakly invariant families. Therefore $z \in A_j(t) \cap A^*_{j-1}(t)$ and $\Xi_j(t) \subset A_j(t) \cap A^*_{j-1}(t)$. The proof of Theorem 3.24 is therefore complete. □

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REFERENCES

[1] E. R. Aragão-Costa, T. Caraballo, A. N. Carvalho, J. A. Langa, Stability of gradient semigroups under perturbations, *Nonlinearity*, 24 (2011), 2099-2117.

[2] E. R. Aragão-Costa, T. Caraballo, A. N. Carvalho, J. A. Langa, Non-autonomous Morse-decomposition and lyapunov functions for gradient-like processes, *Trans. Amer. Math. Soc.*, 365 (2013) 5277-5312.

[3] T. Caraballo, P. E. Kloeden, P. Marín-Rubio, Weak pullback attractors of setvalued processes, *J. Math. Anal. Appl.*, 288 (2003) 692-707.

[4] T. Caraballo, J. C. Jara, J. A. Langa, J. Valero, Morse decomposition of global attractors with infinite components, *Discrete Contin. Dyn. Syst. Ser. A*, 25 (2015), 2845–2861.

[5] A. N. Carvalho, J. A. Langa, An extension of the concept of gradient semigroups which is stable under perturbation, *J. Differ. Equ.*, 246 (2009), 2646-2668.

[6] C. Conley, *Isolated Invariant Sets and the Morse Index*, In: Regional Conference Series in Mathematics 38, American Matematical Society, Providence, 1978.

[7] H. B. da Costa, J. Valero, Morse decompositions and Lyapunov functions for dynamically gradient multivalued semiflows, *Nonlinear Dyn.*, 84 (2016), 19-34.

[8] H. B. da Costa, J. Valero, Morse decompositions with infinite components for multivalued semiflows, *Set-Valued Var. Anal.*, 25 (2017), 25-41.

[9] F. Flandoli, B. Schmalfuß, Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise, *Stoch. Stoch. Rep.*, 59 (1996), 21-45.

[10] P. E. Kloeden, Asymptotic invariance and limit sets of general control systems, *J. Diff. Eqns.*, 19 (1975), 91-105.

[11] D. S. Li, On dynamical stability in general dynamical systems, *J. Math. Anal. Appl.*, 263 (2001), 455-478.

[12] D. S. Li, Morse decompositions for general dynamical systems and differential inclusions with applications to control systems, *SIAM J. Control Optim.*, 46 (2007), 35-60.

[13] D. S. Li, Y. J. Wang, S. Y. Wang, On the dynamics of nonautonomous general dynamical systems and differential inclusions, *Set-Valued Anal.*, 16 (2008), 651-671.

[14] R. McGehee, Attractors for closed relations on compact Hausdorff spaces, *Indiana Univ. Math. J.*, 41 (1992), 1165-1209.

[15] K.P. Rybakowski, *The Homotopy Index and Partial Differential Equations*, Springer, Berlin, 1987.

[16] Y. J. Wang, D. S. Li, Morse decompositions for periodic general dynamical systems and differential inclusions, *Set-Valued Var. Anal.*, 20 (2012), 519-549.

[17] Y. J. Wang, D. S. Li, Morse decompositions for nonautonomous general dynamical systems, *Set-Valued Var. Anal.*, 22 (2014), 117-154.

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