Opuscula Math. 40, no. 3 (2020), 341–360
https://doi.org/10.7494/OpMath.2020.40.3.341 Opuscula Mathematica

EXISTENCE OF PERIODIC POSITIVE SOLUTIONS TO NONLINEAR LOTKA–VOLTERRA COMPETITION SYSTEMS

Mimia Benhadri, Tomás Caraballo, and Halim Zeghdoudi

Communicated by Josef Diblík

Abstract. We investigate the existence of positive periodic solutions of a nonlinear Lotka–Volterra competition system with deviating arguments. The main tool we use to obtain our result is the Krasnoselskii fixed point theorem. In particular, this paper improves important and interesting work [X.H. Tang, X. Zhou, *On positive periodic solution of Lotka–Volterra competition systems with deviating arguments*, Proc. Amer. Math. Soc. 134 (2006), 2967–2974]. Moreover, as an application, we also exhibit some special cases of the system, which have been studied extensively in the literature.

Keywords: Krasnoselskii's fixed point theorem, positive periodic solutions, Lotka–Volterra competition systems, variable delays.

Mathematics Subject Classification: 34K20, 34K13, 92B20.

1. INTRODUCTION

It is well known that the application of theories of functional differential equations in mathematical ecology or biology has developed rapidly and effectively. Lotka–Volterra system is one of the most celebrated models in mathematical biology and population dynamics. Recently, it has also been successfully applied to interesting applications in epidemiology, physics, chemistry, economics, biological science, and other areas (see [7, 11, 12, 24]). Lotka–Volterra model has been an active field of research, both in the deterministic and stochastic cases, since it was originally introduced in 1920 by Lotka [21], and later applied by Volterra [28] to a predator-prey interaction. This system can model the dynamics of ecological systems with predator-prey interactions, mutualism, disease and competition. Many important and influential results have been established and can be found in many articles and books. Particularly, the existence of positive periodic solutions for various Lotka–Volterra-type population dynamical systems has been extensively studied in [6,10,16,22,23,27] and the references cited therein. Motivated by this, in this paper we use a fixed point theorem due to Krasnoselskii to study the existence of positive periodic solutions of nonlinear Lotka–Volterra competition systems with deviating arguments as follows:

$$u_i'(t) = u_i(t)\left\{ r_i(t) - \sum_{j=1}^{n} a_{ij}(t)u_j(t) - \sum_{j=1}^{n} b_{ij}(t)f_j\left(u_j(t)\right) - \sum_{j=1}^{n} c_{ij}(t)g_j\left(u_j(t-\delta_j(t))\right) \right\} \tag{1.1}$$

for $j = 1, 2, \ldots, n$, where $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^T \in \mathbb{R}^n$, and since we are searching for the existence of periodic solutions for equation (1.1), we assume $r_i, a_{ij}, b_{ij}, c_{ij} \in C(\mathbb{R}^+, \mathbb{R}^+)$ are all ω-periodic functions ($\omega > 0$) with respect to time t,

$$\begin{aligned} a_{ij}(t+\omega) &= a_{ij}(t), \quad \delta_j(t+\omega) = \delta_j(t), \\ b_{ij}(t+\omega) &= b_{ij}(t), \quad c_{ij}(t+\omega) = c_{ij}(t) \end{aligned} \tag{1.2}$$

for $i, j = 1, 2, \ldots, n$, with δ_j being scalar continuous functions, and $\delta_j(t) \geq \delta_j^* > 0$ with

$$\begin{aligned} \overline{r}_i &= \frac{1}{\omega}\int_0^{\omega} r_i(s)ds > 0, \quad \overline{a}_{ij} = \frac{1}{\omega}\int_0^{\omega} a_{ij}(s)ds \geq 0, \\ \overline{b}_{ij} &= \frac{R_j}{\omega}\int_0^{\omega} b_{ij}(s)ds \geq 0, \quad \overline{c}_{ij} = \frac{T_j}{\omega}\int_0^{\omega} c_{ij}(s)ds \geq 0, \end{aligned} \tag{1.3}$$

for $i, j = 1, 2, \ldots, n$, where R_j and T_j are given in (A1) and (A2), respectively. We also assume that the functions, $f_i, g_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, \ldots, n$ are continuous,

$$\begin{aligned} f(u(t)) &= \left[f_1(u_1(t)), f_2(u_2(t)), \ldots, f_n(u_n(t))\right]^T \in \mathbb{R}_+^n, \\ g(u(t)) &= \left[g_1(u_1(t)), g_2(u_2(t)), \ldots, g_n(u_n(t))\right]^T \in \mathbb{R}_+^n \end{aligned}$$

are positive continuous in their respective arguments.

Now we will list the assumptions we will impose along our paper:

(A1) there exist nonnegative constants $\overline{T}_j, T_j$ such that for all $u \in \mathbb{R}^+$,

$$\overline{T}_j u \leq f_j(u) \leq T_j u, \quad j = 1, 2, \ldots, n. \tag{1.4}$$

(A2) there exist nonnegative constants $\overline{R}_j, R_j$ such that for all $u \in \mathbb{R}^+$,

$$\overline{R}_j u \leq g_j(u) \leq R_j u, \quad j = 1, 2, \ldots, n. \tag{1.5}$$

(A3) The linear system

$$\sum_{j=1}^{n} \left(\overline{a}_{ij} + \overline{b}_{ij} + \overline{c}_{ij}\right) u_j = \overline{r}_i, \quad i = 1, 2, \ldots, n, \tag{1.6}$$

possesses a positive solution.

Throughout this paper, a vector $u = (u_1, u_2, \ldots, u_n)^T$ is said to be positive if $u_i > 0$ $(i = 1, 2, \ldots, n)$.

It is worth noting that in 1998, Li [18] considered the following delayed periodic logistic equation

$$u'(t) = u(t)\left[r(t) - c(t)u(t - \delta(t))\right], \tag{1.7}$$

and proved that equation (1.7) always has a positive ω-periodic solution if $r, c, \delta \in C(\mathbb{R}, [0, \infty))$ are ω-periodic functions such that $\int_0^\omega r(s)ds > 0$ and $\int_0^\omega c(s)ds > 0$.

In [27], Tang and Zou studied the following n-species Lotka–Volterra competitive systems with several deviating arguments:

$$u_i'(t) = u_i(t)\left(r_i(t) - \sum_{j=1}^{n} c_{ij}(t)u_j(t - \delta_j(t))\right), \quad i = 1, 2, \ldots, n, \tag{1.8}$$

and, by using the Krasnoselskii fixed point theorem method, the authors proved that (1.8) has at least one positive ω-periodic solution provided that the corresponding system of linear equations

$$\sum_{j=1}^{n} \overline{c}_{ij} x_j = \overline{r}_i, \quad i = 1, 2, \ldots, n,$$

possesses a positive solution with

$$\overline{r}_i = \frac{1}{\omega}\int_0^\omega r_i(s)ds > 0, \quad \overline{c}_{ij} = \frac{1}{\omega}\int_0^\omega c_{ij}(s)ds \geq 0, \quad i, j = 1, 2, \ldots, n.$$

On the other hand, Fan *et al.* [10] and Li [19] established a set of easily verifiable sufficient conditions for the existence and global attractiveness of positive periodic solutions for equation (1.8) by using the method of coincidence degree and Lyapunov functional. Other competition models have been studied in [1–5, 13–15, 17, 23, 26, 29, 30].

The method used in [27] was also used in [23] where the authors investigated the existence and global attractiveness of positive periodic solutions of a 3-species Lotka–Volterra predator-prey system with several deviating arguments:

$$\begin{cases} u_1'(t) = u_1(t)\left(r_1(t) - c_{11}(t)u_1(t - \delta_1(t)) - c_{12}(t)u_2(t - \delta_2(t))\right. \\ \qquad\quad \left. - \, c_{13}(t)u_3(t - \delta_3(t))\right), \\ u_2'(t) = u_2(t)\left(-r_2(t) + c_{21}(t)u_1(t - \delta_1(t)) - c_{22}(t)u_2(t - \delta_2(t))\right. \\ \qquad\quad \left. - \, c_{23}(t)u_3(t - \delta_3(t))\right), \\ u_3'(t) = u_3(t)\left(-r_3(t) + c_{31}(t)u_1(t - \delta_1(t)) - c_{32}(t)u_2(t - \delta_2(t))\right. \\ \qquad\quad \left. - \, c_{33}(t)u_3(t - \delta_3(t))\right). \end{cases} \tag{1.9}$$

In the current paper we extend, in particular, the results in [27] to the nonlinear Lotka–Volterra system of equations (1.1). Notice that when $a_{ij} = 0$ in the second term on the right hand side of (1.1), $f_j(u_j) = 0$, and $g_j(u_j) = u_j$, then (1.1) reduces to (1.8). Thus, our results are more general than those obtained in [27].

The content of this paper is as follows. In Section 2, we recall some results which are necessary for our analysis. The existence of positive periodic solutions of system (1.1) by using the Krasnoselskii fixed point theorem is proved in Section 3. Finally, in Section 4, we analyse an example to illustrate how our result can be easily applied to interesting models.

2. PRELIMINARIES

For the reader convenience, we recall the definition of cone as well as the celebrated Krasnoselskii fixed point theorem.

Let X be a Banach space and let Ω be a closed, nonempty subset of X. We say that Ω is a cone if

(i) $\alpha u + \beta v \in \Omega$ for all $u, v \in \Omega$ and all $\alpha, \beta \geq 0$,
(ii) $u, -u \in \Omega$ imply $u = 0$.

The proof of Krasnoselskii's fixed point theorem stated below can be found in [17].

Theorem 2.1 ([17]). *Let X be a Banach space, and let $\Omega \subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let*

$$\mathcal{P} : \Omega \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \Omega$$

be a completely continuous operator such that either

$$\|\mathcal{P}u\| \leq \|u\| \ \text{for } u \in \Omega \cap \partial\Omega_1 \ \text{and} \ \|\mathcal{P}u\| \geq \|u\| \ \text{for } u \in \Omega \cap \partial\Omega_2$$

or

$$\|\mathcal{P}u\| \geq \|u\| \ \text{for } u \in \Omega \cap \partial\Omega_1 \ \text{and} \ \|\mathcal{P}u\| \leq \|u\| \ \text{for } u \in \Omega \cap \partial\Omega_2.$$

Then $\mathcal{P}$ has a fixed point in $\Omega \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

3. EXISTENCE OF POSITIVE PERIODIC SOLUTIONS

To apply Theorem 2.1, we need to define a Banach space C_ω, a closed S of C_ω and construct one mapping. Thus, we let $(C_\omega, \|\cdot\|) = (X, \|\cdot\|)$, where

$$C_\omega = \{u : u \in C(\mathbb{R}, \mathbb{R}^n), u(t+\omega) = u(t)\}, \tag{3.1}$$

with the norm

$$\forall u \in C_\omega : \|u\| = \sum_{i=1}^{n} |u_i|_0, \quad |u_i|_0 = \max_{t \in [0,\omega]} |u_i(t)|, \ i = 1, 2, \ldots, n. \tag{3.2}$$

Then C_ω is a real Banach space endowed with the above norm $\|\cdot\|$.

The following lemma is fundamental to our results.

Lemma 3.1. *The function u is an ω-periodic solution of equation* (1.1) *if and only if u is an ω-periodic solution of the following equation*

$$u_i(t) = \int_t^{t+\omega} G_i(t,s)u_i(s)\Bigg\{\sum_{j=1}^n a_{ij}(s)u_j(s) + \sum_{j=1}^n b_{ij}(s)f_j(u_j(s)) + \sum_{j=1}^n c_{ij}(s)g_j(u_j(s-\delta_j(s)))\Bigg\}ds, \tag{3.3}$$

where

$$G_i(t,s) = \frac{1}{1-e^{-\overline{r_i}\omega}} \exp\left(-\int_t^s r_i(\xi)\,d\xi\right), \quad i = 1,2,\dots,n, \tag{3.4}$$

and we assume

$$e^{-\overline{r_i}\omega} \neq 1.$$

Proof. Let u be an ω-periodic solution of equation (1.1), then

$$\begin{aligned}&\left[u_i(t)\exp\left(-\int_0^t r_i(s)\,ds\right)\right]' \\ &= -\exp\left(-\int_0^t r_i(s)\,ds\right)u_i(t)\Bigg\{\sum_{j=1}^n a_{ij}(t)u_j(t) + \sum_{j=1}^n b_{ij}(t)f_j(u_j(t)) \\ &\quad + \sum_{j=1}^n c_{ij}(t)g_j(u_j(t-\delta_j(t)))\Bigg\}.\end{aligned} \tag{3.5}$$

Integrating both sides of (3.5) from t to $t+\omega$, we can obtain

$$\begin{aligned}&u_i(t+\omega)\exp\left(-\int_0^{t+\omega} r_i(s)\,ds\right) - u_i(t)\exp\left(-\int_0^t r_i(s)\,ds\right) \\ &= \int_t^{t+\omega} u_i(s)\exp\left(-\int_0^s r_i(\xi)\,d\xi\right)\Bigg\{\sum_{j=1}^n a_{ij}(s)u_j(s) + \sum_{j=1}^n b_{ij}(s)f_j(u_j(s)) \\ &\quad + \sum_{j=1}^n c_{ij}(s)g_j(u_j(s-\delta_j(s)))\Bigg\}ds.\end{aligned}$$

Since the functions $u, a_{ij}, b_{ij}, c_{ij}, \delta_j$ are ω-periodic with respect to t, we have

$$u_i(t) = \int\limits_t^{t+\omega} \frac{\exp\left(-\int_0^s r_i(\xi)\,d\xi\right)}{\exp\left(-\int_0^{t+\omega} r_i(s)\,ds\right) - \exp\left(-\int_0^t r_i(s)\,ds\right)} u_i(s) \times \left\{ \sum_{j=1}^n a_{ij}(s)u_j(s) + \sum_{j=1}^n b_{ij}(s) f_j\left(u_j(s)\right) + \sum_{j=1}^n c_{ij}(s) g_j\left(u_j(s - \delta_j(s))\right) \right\} ds,$$

and therefore

$$u_i(t) = \int\limits_t^{t+\omega} G_i(t,s)\, u_i(s) \left\{ \sum_{j=1}^n a_{ij}(s)u_j(s) + \sum_{j=1}^n b_{ij}(s) f_j\left(u_j(s)\right) + \sum_{j=1}^n c_{ij}(s) g_j\left(u_j(s - \delta_j(s))\right) \right\} ds.$$

To simplify notation, we denote

$$G_i(t,s) = \frac{1}{1 - e^{-\overline{r_i}\omega}} \exp\left(- \int\limits_t^s r_i(\xi)\,d\xi \right), \quad i = 1, 2, \ldots, n.$$

It is easy to see that for all $(t,s) \in [0,\omega] \times [0,\omega]$ we have

$$G_i(t+\omega, s+\omega) = G_i(t,s), \quad i = 1, 2, \ldots, n.$$

Thus, u is an ω-periodic function of equation (3.3).

On the other hand, if u is an ω-periodic solution of equation (3.3), then differentiating equation (3.3) with respect to t,

$$
\begin{aligned}
u_i'(t) &= G_i\left(t, t+\omega\right) u_i(t+\omega) \Bigg\{ \sum_{j=1}^{n} a_{ij}(t+\omega) u_j(t+\omega) + \sum_{j=1}^{n} b_{ij}(t+\omega) f_j\left(u_j(t+\omega)\right) \\
&\quad + \sum_{j=1}^{n} c_{ij}(t+\omega) g_j\left(u_j(t+\omega-\delta_j\left(t+\omega\right))\right) \Bigg\} \\
&\quad - G_i\left(t,t\right) u_i(t) \Bigg\{ \sum_{j=1}^{n} a_{ij}(t) u_j(t) + \sum_{j=1}^{n} b_{ij}(t) f_j\left(u_j(t)\right) \\
&\quad + \sum_{j=1}^{n} c_{ij}(t) g_j\left(u_j(t-\delta_j(t))\right) \Bigg\} \\
&\quad - \int_{t}^{t+\omega} \left(\frac{d}{dt} G_i\left(t,s\right) \right) u_i(s) \\
&\quad \times \Bigg\{ \sum_{j=1}^{n} a_{ij}(s) u_j(s) + \sum_{j=1}^{n} b_{ij}(s) f_j\left(u_j(s)\right) + \sum_{j=1}^{n} c_{ij}(s) g_j\left(u_j(s-\delta_j\left(s\right))\right) \Bigg\} ds \\
&= \left(G_i\left(t,t+\omega\right) - G_i\left(t,t\right)\right) u_i(t) \\
&\quad \times \Bigg\{ \sum_{j=1}^{n} a_{ij}(t) u_j(t) + \sum_{j=1}^{n} b_{ij}(t) f_j\left(u_j(t)\right) + \sum_{j=1}^{n} c_{ij}(t) g_j\left(u_j(t-\delta_j(t))\right) \Bigg\} \\
&\quad + r_i(t) \int_{t}^{t+\omega} G_i\left(t,s\right) u_i(s) \\
&\qquad \times \Bigg\{ \sum_{j=1}^{n} a_{ij}(s) u_j(s) + \sum_{j=1}^{n} b_{ij}(s) f_j\left(u_j(s)\right) + \sum_{j=1}^{n} c_{ij}(s) g_j\left(u_j(s-\delta_j\left(s\right))\right) \Bigg\} ds \\
&= -u_i(t) \Bigg\{ \sum_{j=1}^{n} a_{ij}(t) u_j(t) + \sum_{j=1}^{n} b_{ij}(t) f_j\left(u_j(t)\right) + \sum_{j=1}^{n} c_{ij}(t) g_j\left(u_j(t-\delta_j(t))\right) \Bigg\} \\
&\quad + r_i(t) \int_{t}^{t+\omega} G_i\left(t,s\right) u_i(s) \\
&\qquad \times \Bigg\{ \sum_{j=1}^{n} a_{ij}(s) u_j(s) + \sum_{j=1}^{n} b_{ij}(s) f_j\left(u_j(s)\right) + \sum_{j=1}^{n} c_{ij}(s) g_j\left(u_j(s-\delta_j\left(s\right))\right) \Bigg\} ds \\
&= u_i(t) \Bigg\{ r_i(t) - \sum_{j=1}^{n} a_{ij}(s) u_j(s) - \sum_{j=1}^{n} b_{ij}(s) f_j\left(u_j(s)\right) \\
&\qquad - \sum_{j=1}^{n} c_{ij}(s) g_j\left(u_j(s-\delta_j\left(s\right))\right) \Bigg\}.
\end{aligned}
$$

The proof is completed. □

Let

$$\sigma = \min\left\{e^{-\bar{r}_i \omega} : i = 1, 2, \ldots, n\right\}.$$

Now choose the cone Ω of C_ω defined by

$$\Omega = \{u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T \in C_\omega : u_i(t) \geq \sigma \left|u_i\right|_0, i = 1, 2, \ldots, n\}.$$

We use (3.3) to define the operator $\mathcal{P} : C_\omega \to C_\omega$ by

$$(\mathcal{P}u)(t) := \left[(\mathcal{P}_1 u_1)(t), (\mathcal{P}_2 u_2)(t), \ldots, (\mathcal{P}_n u_n)(t)\right]^T,$$

where

$$\begin{aligned}(\mathcal{P}_i u_i)(t) = \int_t^{t+\omega} G_i(t,s)\, u_i(s) \Bigg\{ \sum_{j=1}^n a_{ij}(s) u_j(s) + \sum_{j=1}^n b_{ij}(s) f_j(u_j(s)) \\ + \sum_{j=1}^n c_{ij}(s) g_j(u_j(s - \tau_j(s))) \Bigg\} ds.\end{aligned} \tag{3.6}$$

By (3.6), it is easy to check that $u \in C_\omega$ is an ω-periodic solution of equation (1.1) provided u is a fixed point of $\mathcal{P}$.

Lemma 3.2. *The mapping $\mathcal{P}$ maps Ω into Ω, i.e. $\mathcal{P}\Omega \subset \Omega$.*

Proof. It is easy to see that for any $s \in [t, t+\omega]$, thanks to (3.4), we have

$$A_i := \frac{e^{-\overline{r_i}\omega}}{1 - e^{-\overline{r_i}\omega}} \leq G_i(t,s) \leq \frac{1}{1 - e^{-\overline{r_i}\omega}} =: B_i, \quad i = 1, 2, \ldots, n. \tag{3.7}$$

For any $u \in \Omega$, we can obtain

$$
\begin{aligned}
(\mathcal{P}_i u_i)(t+\omega) &= \int\limits_{t+\omega}^{t+2\omega} G_i(t+\omega, s)\, u_i(s) \Bigg\{ \sum_{j=1}^{n} a_{ij}(s) u_j(s) \\
&\qquad + \sum_{j=1}^{n} b_{ij}(s) f_j(u_j(s)) + \sum_{j=1}^{n} c_{ij}(s) g_j(u_j(s-\delta_j(s))) \Bigg\} ds \\
&= \int\limits_{t}^{t+\omega} G_i(t+\omega, s+\omega)\, u_i(s+\omega) \\
&\qquad \times \Bigg\{ \sum_{j=1}^{n} a_{ij}(s+\omega) u_j(s+\omega) + \sum_{j=1}^{n} b_{ij}(s+\omega) f_j(u_j(s+\omega)) \\
&\qquad + \sum_{j=1}^{n} c_{ij}(s+\omega) g_j(u_j(s+\omega-\delta_j(s+\omega))) \Bigg\} ds \\
&= \int\limits_{t}^{t+\omega} G_i(t, s)\, u_i(s) \Bigg\{ \sum_{j=1}^{n} a_{ij}(s) u_j(s) + \sum_{j=1}^{n} b_{ij}(s) f_j(u_j(s)) \\
&\qquad + \sum_{j=1}^{n} c_{ij}(s) g_j(u_j(s-\delta_j(s))) \Bigg\} ds \\
&= (\mathcal{P}_i u_i)(t).
\end{aligned}
$$

Thus $\mathcal{P}u \in C_\omega$. Moreover, from (3.6) and (3.7), we have for $u \in \Omega$

$$
\begin{aligned}
|(\mathcal{P}_i u_i)|_0 \le B_i \int\limits_{0}^{\omega} u_i(s) \Bigg\{ &\sum_{j=1}^{n} a_{ij}(s) u_j(s) + \sum_{j=1}^{n} b_{ij}(s) f_j(u_j(s)) \\
&+ \sum_{j=1}^{n} c_{ij}(s) g_j(u_j(s-\delta_j(s))) \Bigg\} ds
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathcal{P}_i u_i) &\ge A_i \int\limits_{0}^{\omega} u_i(s) \Bigg\{ \sum_{j=1}^{n} a_{ij}(s) u_j(s) + \sum_{j=1}^{n} b_{ij}(s) f_j(u_j(s)) \\
&\qquad + \sum_{j=1}^{n} c_{ij}(s) g_j(u_j(s-\delta_j(s))) \Bigg\} ds \\
&\ge \frac{A_i}{B_i} |(\mathcal{P}_i u_i)|_0 \ge \sigma |(\mathcal{P}_i u_i)|_0 .
\end{aligned}
$$

Hence, $\mathcal{P}\Omega \subset \Omega$. The proof is complete. □

Lemma 3.3. *The mapping $\mathcal{P} : \Omega \to \Omega$ is completely continuous.*

Proof. For $i = 1, 2, \dots, n$, we set

$$K_i(t, u(t)) = u_i(t) \sum_{j=1}^{n} a_{ij}(t) u_j(t),$$

$$F_i(t, u(t)) = u_i(t) \sum_{j=1}^{n} b_{ij}(t) f_j(u_j(t)),$$

and

$$H_i(t, u_t) = u_i(t) \sum_{j=1}^{n} c_{ij}(t) g_j(u_j(t - \delta_j(t))).$$

We first show that $\mathcal{P}$ is continuous. For any $L > 0$ and $\varepsilon > 0$, there exists $\eta_1 > 0$ such that for $\|u\| \leq L$, $\|v\| \leq L$, and $\|u - v\| < \eta_1$ imply

$$|K_i(s, u(s)) - K_i(s, v(s))| < \frac{\varepsilon}{3nB\omega}, \quad i = 1, 2, \dots, n. \tag{3.8}$$

For any $L > 0$ and $\varepsilon > 0$, there exists $\eta_2 > 0$ such that for $\|u\| \leq L$, $\|v\| \leq L$, and $\|u - v\| < \eta_2$ imply

$$|F_i(s, u(s)) - F_i(s, v(s))| < \frac{\varepsilon}{3nB\omega}, \quad i = 1, 2, \dots, n. \tag{3.9}$$

For any $L > 0$ and $\varepsilon > 0$, there exists $\eta_3 > 0$ such that for $\|u\| \leq L$, $\|v\| \leq L$, and $\|u - v\| < \eta_3$ imply

$$|H_i(s, u_s) - H_i(s, v_s)| < \frac{\varepsilon}{3nB\omega}, \quad i = 1, 2, \dots, n, \tag{3.10}$$

where $B = \max_{1 \leq i \leq n} B_i$.

If $u, v \in C_\omega$ with $\|u\| \le L, \|v\| \le L$, and $\|u - v\| \le \eta$, where $\eta = \min\{\eta_1, \eta_2, \eta_3\}$. Then, from (3.6), (3.7) and (3.8), (3.9), (3.10), we have

$$\begin{aligned}
|(\mathcal{P}_i u_i) - (\mathcal{P}_i v_i)|_0 \le & \int\limits_t^{t+\omega} G_i(t,s)\,|K_i(s,u(s)) - K_i(s,v(s))|\,ds \\
& + \int\limits_t^{t+\omega} G_i(t,s)\,|F_i(s,u(s)) - F_i(s,v(s))|\,ds \\
& + \int\limits_t^{t+\omega} G_i(t,s)\,|H_i(s,u_s) - H_i(s,v_s)|\,ds \\
\le & B \int\limits_0^{t+\omega} |K_i(s,u(s)) - K_i(s,v(s))|\,ds \\
& + B \int\limits_0^{t+\omega} |F_i(s,u(s)) - F_i(s,v(s))|\,ds \\
& + B \int\limits_0^{t+\omega} |H_i(s,u_s) - H_i(s,v_s)|\,ds \\
< & \frac{\varepsilon}{n}, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

This yields

$$\|\mathcal{P}u - \mathcal{P}v\| = \sum_{i=1}^{n} |(\mathcal{P}_i u_i) - (\mathcal{P}_i v_i)|_0 < \varepsilon.$$

Thus, $\mathcal{P}$ is continuous.

Next, we show that $\mathcal{P}$ is compact. Set

$$a = \max_{1\le i\le n} \sum_{j=1}^{n} \overline{a}_{ij}, \quad b = \max_{1\le i\le n} \sum_{j=1}^{n} \overline{b}_{ij}, \quad c = \max_{1\le i\le n} \sum_{j=1}^{n} \overline{c}_{ij}.$$

We let

$$S = \{u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T \in C_\omega : \|u\| \le M\},$$

where M is non-negative constant.

For any $u \in S$, it follows from (3.6) and (3.7) that

$$\begin{aligned}
|(\mathcal{P}_i u_i)|_0 &\leq B_i \int\limits_0^\omega |u_i|_0 \sum_{j=1}^n a_{ij}(s)\, |u_j|_0\, ds + B_i \int\limits_0^\omega |u_i|_0 \sum_{j=1}^n T_j b_{ij}(s)\, |u_j|_0\, ds \\
&\quad + B_i \int\limits_0^\omega |u_i|_0 \sum_{j=1}^n R_j c_{ij}(s)\, |u_j|_0\, ds \\
&= \omega B M^2 \sum_{j=1}^n \left(\overline{a}_{ij} + \overline{b}_{ij} + \overline{c}_{ij}\right) \\
&\leq B\omega M^2 (a+b+c),
\end{aligned}$$

and so

$$\|\mathcal{P}u\| = \sum_{i=1}^n |(\mathcal{P}_i u_i)|_0 \leq Bn\omega M^2 (a+b+c), \quad u \in S.$$

This shows that $\mathcal{P}(S)$ is uniformly bounded.

To show that $\mathcal{P}(S)$ is equicontinuous. Let $u \in S$, we calculate $\frac{d}{dt}(\mathcal{P}_i u_i)(t)$ and show that it is uniformly bounded, we obtain by taking the derivative in (3.6) that

$$\begin{aligned}
\left|(\mathcal{P}_i u_i)'(t)\right| &\leq r_i(t)\, |(\mathcal{P}_i u_i)(t)| + |u_i(t)| \sum_{j=1}^n a_{ij}(t)\, |u_j(t)| \\
&\quad + |u_i(t)| \sum_{j=1}^n b_{ij}(t)\, |f_j(u_j(t))| \\
&\quad + |u_i(t)| \sum_{j=1}^n c_{ij}(t)\, |g_j(u_j(t-\delta_j(t)))| \\
&\leq r_i^* B\omega M^2 (a+b+c) + M^2 \sum_{j=1}^n a_{ij}(s) \\
&\quad + M^2 \sum_{j=1}^n b_{ij}(s) T_j + M^2 \sum_{j=1}^n c_{ij}(s) R_j \\
&\leq r_i^* B M^2 \omega (a+b+c) + M^2 \sum_{j=1}^n a_{ij}^* \\
&\quad + \lambda_1 M^2 \sum_{j=1}^n b_{ij}^* + \lambda_2 M^2 \sum_{j=1}^n c_{ij}^* \leq DM^2, \quad i = 1, 2, \ldots, n,
\end{aligned}$$

where

$$D = \max_{1 \le i \le n} \left(r_i^* B\omega \left(a + b + c\right) + \lambda_1 \sum_{j=1}^{n} b_{ij}^* + \lambda_2 \sum_{j=1}^{n} c_{ij}^* \right),$$

$$\lambda_1 = \max \left\{ T_j, j = \overline{1,n} \right\}, \quad \lambda_2 = \max \left\{ R_j, j = \overline{1,n} \right\},$$

and

$$\begin{aligned} r_i^* &= \max_{t \in [0,\omega]} r_i\left(t\right), \\ b_{ij}^* &= \max_{t \in [0,\omega]} b_{ij}(t), \\ c_{ij}^* &= \max_{t \in [0,\omega]} c_{ij}(t), \end{aligned}$$

for $i, j = 1, 2, \ldots, n$.

Hence, $\mathcal{P}S \subset C_\omega$ is a family of uniformly bounded and equi-continuous functions. By the Ascoli–Arzelà Theorem (see [26, p. 169]), the operator $\mathcal{P}$ is compact, and therefore completely continuous. The proof is complete. □

We can now state and prove our main result of this paper.

Theorem 3.4. *Assume condition* (1.6) *holds. Then Eq.* (1.1) *possesses at least one positive ω-periodic solution.*

Proof. Let

$$u^* = \left(u_1^*, u_2^*, \ldots, u_n^*\right)^T$$

with $u_i^* > 0$, $i = 1, 2, \ldots, n$, be a positive solution of (1.6).

$$\begin{aligned} A &= \min \left\{ \overline{r}_i A_i : i = 1, 2, \ldots, n \right\}, \\ B &= \min \left\{ \overline{r}_i B_i : i = 1, 2, \ldots, n \right\}. \end{aligned}$$

Then $0 < A < B < +\infty$. Define

$$\Omega_1 = \left\{ u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T \in C_\omega : \left|u_i\right|_0 < \frac{u_i^*}{B\omega}, i = 1, 2, \ldots, n \right\}.$$

If $u \in \Omega \cap \partial\Omega_1$, then

$$\sigma \left|u_i\right|_0 \le u_i(t) \le \left|u_i\right|_0 = \left(B\omega\right)^{-1} u_i^*, \quad i = 1, 2, \ldots, n,$$

and

$$
\begin{aligned}
(\mathcal{P}_i u_i)(t) &\leq B_i \int_0^\omega \Bigg\{ u_j(s) \sum_{j=1}^n a_{ij}(s) u_j(s) + u_j(s) \sum_{j=1}^n b_{ij}(s) f_j(u_j(s)) \\
&\qquad + u_j(s) \sum_{j=1}^n c_{ij}(s) g_j(u_j(s-\delta_j(s))) \Bigg\} ds \\
&\leq B_i \int_0^\omega |u_i|_0 \sum_{j=1}^n a_{ij}(s) |u_j|_0 \, ds + B_i \int_0^\omega |u_i|_0 \sum_{j=1}^n T_j b_{ij}(s) |u_j|_0 \, ds \\
&\quad + B_i \int_0^\omega |u_i|_0 \sum_{j=1}^n R_j c_{ij}(s) |u_j|_0 \, ds \\
&= B_i \omega |u_i|_0 \sum_{j=1}^n \overline{a}_{ij} |u_j|_0 + B_i \omega |u_i|_0 \sum_{j=1}^n \overline{b}_{ij} |u_j|_0 + B_i \omega |u_i|_0 \sum_{j=1}^n \overline{c}_{ij} |u_j|_0 \\
&= B_i \omega (B\omega)^{-1} |u_i|_0 \sum_{j=1}^n \overline{a}_{ij} u_j^* + B_i \omega (B\omega)^{-1} |u_i|_0 \sum_{j=1}^n \overline{b}_{ij} u_j^* \\
&\quad + B_i \omega (B\omega)^{-1} |u_i|_0 \sum_{j=1}^n \overline{c}_{ij} u_j^* \\
&= B_i \omega (B\omega)^{-1} |u_i|_0 \left(\sum_{j=1}^n \overline{a}_{ij} u_j^* + \sum_{j=1}^n \overline{b}_{ij} u_j^* + \sum_{j=1}^n \overline{c}_{ij} u_j^* \right) \\
&= B_i \omega (B\omega)^{-1} |u_i|_0 \left[\sum_{j=1}^n \left(\overline{a}_{ij} + \overline{b}_{ij} + \overline{c}_{ij} \right) u_j^* \right] \\
&= B_i \overline{r}_i \omega (B\omega)^{-1} |u_i|_0 \\
&\leq |u_i|_0, \quad i = 1, 2, \ldots, n,
\end{aligned}
$$

and therefore

$$
\|\mathcal{P}u\| = \sum_{i=1}^n |(\mathcal{P}_i u_i)|_0 \leq \sum_{i=1}^n |u_i|_0 = \|u\|, \quad u \in \Omega \cap \partial\Omega_1.
$$

Let $\overline{\theta} = \min\{1, \theta_1, \theta_2\}$, where

$$
\theta_1 = \min\left\{ \frac{\overline{T}_j}{T_j}, j = \overline{1, n} \right\} \text{ and } \theta_2 = \min\left\{ \frac{\overline{R}_j}{R_j}, j = \overline{1, n} \right\}.
$$

Next, we define

$$
\Omega_2 = \left\{ u \in C_\omega : |u_i|_0 < \frac{u_i^*}{\overline{\theta} \sigma^2 A \omega}, i = 1, 2, \ldots, n \right\}.
$$

If $u \in \Omega \cap \partial\Omega_2$, then

$$\sigma |u_i|_0 \leq u_i(t) \leq |u_i|_0 = \left(\overline{\theta}\sigma^2 A\omega\right)^{-1} u_i^*, \quad i = 1, 2, \ldots, n,$$

and consequently

$$\begin{aligned}
\left(\mathcal{P}_i u_i\right)(t) \geq{}& A_i \int_0^\omega u_i(s) \Bigg\{ \sum_{j=1}^n a_{ij}(s) u_j(s) + \sum_{j=1}^n b_{ij}(s) f_j\left(u_j(s)\right) \\
&\quad + \sum_{j=1}^n c_{ij}(s) g_j\left(u_j(s - \tau_j(s))\right) \Bigg\} ds \\
\geq{}& \sigma^2 A_i |u_i|_0 \sum_{j=1}^n \int_0^\omega a_{ij}(s) |u_j|_0 \, ds + \sigma^2 A_i |u_i|_0 \sum_{j=1}^n \int_0^\omega T_j \frac{\overline{T}_j}{T_j} b_{ij}(s) |u_j|_0 \, ds \\
&+ \sigma^2 A_i |u_i|_0 \sum_{j=1}^n \int_0^\omega R_j \frac{\overline{R}_j}{R_j} c_{ij}(s) |u_j|_0 \, ds \\
\geq{}& 1 \times \sigma^2 A_i \omega |u_i|_0 \sum_{j=1}^n \overline{a}_{ij} |u_j|_0 + \theta_1 \times \sigma^2 A_i \omega |u_i|_0 \sum_{j=1}^n \overline{b}_{ij} |u_j|_0 \\
&+ \theta_2 \times \sigma^2 A_i \omega |u_i|_0 \sum_{j=1}^n \overline{c}_{ij} |u_j|_0 \\
\geq{}& \overline{\theta}\sigma^2 A_i \omega |u_i|_0 \sum_{j=1}^n \overline{a}_{ij} |u_j|_0 + \overline{\theta}\sigma^2 A_i \omega |u_i|_0 \sum_{j=1}^n \overline{b}_{ij} |u_j|_0 \\
&+ \overline{\theta}\sigma^2 A_i \omega |u_i|_0 \sum_{j=1}^n \overline{c}_{ij} |u_j|_0 \\
={}& \overline{\theta}\sigma^2 A_i \omega |u_i|_0 \sum_{j=1}^n \overline{a}_{ij} \left(\overline{\theta}\sigma^2 A\omega\right)^{-1} u_i^* + \overline{\theta}\sigma^2 A_i \omega |u_i|_0 \sum_{j=1}^n \overline{b}_{ij} \left(\overline{\theta}\sigma^2 A\omega\right)^{-1} u_i^* \\
&+ \overline{\theta}\sigma^2 A_i \omega |u_i|_0 \sum_{j=1}^n \overline{c}_{ij} \left(\overline{\theta}\sigma^2 A\omega\right)^{-1} u_i^* \\
={}& A_i \omega \left(A\omega\right)^{-1} |u_i|_0 \left[\sum_{j=1}^n \left(\overline{a}_{ij} + \overline{b}_{ij} + \overline{c}_{ij}\right) u_j^* \right] \\
={}& A_i \overline{r}_i \omega \left(A\omega\right)^{-1} |u_i|_0 \\
\geq{}& |u_i|_0, \quad i = 1, 2, \ldots, n,
\end{aligned}$$

and thus

$$\|\mathcal{P}u\| = \sum_{i=1}^n |(\mathcal{P}_i u_i)|_0 \geq \sum_{i=1}^n |u_i|_0 = \|u\|, \quad u \in \Omega \cap \partial\Omega_2.$$

Obviously, Ω_1 and Ω_2 are open bounded subsets of C_ω with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Hence, $\mathcal{P} : \Omega \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \Omega$ is a completely continuous operator and satisfies condition (i) in Theorem 2.1. By Krasnoselskii's Theorem, there exists a fixed point $u \in \Omega \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $u(t) = (\mathcal{P}u)(t)$, i.e. u is a positive ω-periodic solution of Eq. (1.1). □

4. AN EXAMPLE

In this section, we analyze an example to illustrate the application of our main result.

Example 4.1. Let us consider the following system:

$$u_i'(t) = u_i(t) \left\{ r_i(t) - \sum_{j=1}^{2} b_{ij}(t) f_j\left(u_j(t)\right) - \sum_{j=1}^{2} c_{ij}(t) g_j\left(u_j(t - \delta_j(t))\right) \right\}, \tag{4.1}$$

for $i = 1, 2$. This model corresponds to system (1.1) when $n = 2, \omega = \pi$. Let

$$r_1(t) = \frac{1}{2}\left(1 + \sin 2t\right), \quad r_2(t) = \frac{3}{4}\left(1 + \cos 4t\right),$$

and $\delta_j \in (\mathbb{R}^+, \mathbb{R}^+)$ be arbitrary continuous functions which satisfy $\delta_j(t + \omega) = \delta_j(t)$, $i = 1, 2$. We then have

$$\overline{r}_1 = \frac{1}{\omega} \int_0^{\omega} r_1(t)dt = \frac{1}{2\pi} \int_0^{\pi} \left(1 + \sin 2t\right) dt = \frac{1}{2},$$

$$\overline{r}_2 = \frac{1}{\omega} \int_0^{\omega} r_1(t)dt = \frac{3}{4\pi} \int_0^{\pi} \left(1 + \cos 4t\right) dt = \frac{3}{4},$$

and it is straightforward to check that $A_i \leq G_i\,(t, s) \leq B_i$, for $i = 1, 2$, where

$$G_i\,(t, s) = \frac{1}{1 - e^{-\overline{r_i}\omega}} \exp\left(- \int_t^s r_i\,(\xi)\, d\xi \right), \quad i = 1, 2,$$

and

$$A_1 := \frac{e^{-\overline{r}_1\omega}}{1 - e^{-\overline{r_1}\omega}} = \frac{e^{-\frac{\pi}{2}}}{1 - e^{-\frac{\pi}{2}}}, \qquad A_2 := \frac{e^{-\overline{r_2}\omega}}{1 - e^{-\overline{r_1}\omega}} = \frac{e^{-\frac{3\pi}{4}}}{1 - e^{-\frac{3\pi}{4}}},$$

$$B_1 := \frac{1}{1 - e^{-\overline{r_1}\omega}} = \frac{1}{1 - e^{-\frac{\pi}{2}}}, \qquad B_2 := \frac{1}{1 - e^{-\overline{r_2}\omega}} = \frac{1}{1 - e^{-\frac{3\pi}{4}}}.$$

Let

$$f_j\,(u) = \sqrt{\frac{u^2}{2} e^{\sin u}}, \qquad g_j\,(u) = \sqrt{\frac{u^2}{2}\left(e^{\cos u} + 1\right)}, \qquad j = 1, 2.$$

Since $|\sin u| \leq 1$ and $|\cos u| \leq 1$ for $u \in \mathbb{R}$,

$$\overline{T}_j u \leq f_j(u) \leq T_j u, \text{ for } u \geq 0,$$

$$\overline{R}_j u \leq g_j(u) \leq R_j u, \text{ for } u \geq 0,$$

where $T_j = \frac{e+1}{2}, \overline{T}_j = \frac{e^{-1}+1}{2}$ and $R_j = \frac{e}{2}, \overline{R}_j = \frac{e^{-1}}{2}$, $j = 1, 2$.

We can choose

$$b_{11}(t) = \frac{(1+\cos 2t)}{3T_1}, \quad b_{12}(t) = \frac{(1+\sin 2t)}{2T_2}, \quad b_{21}(t) = 0, \quad b_{22}(t) = \cos(4t),$$

which implies

$$\bar{b}_{11} = \frac{T_1}{\omega}\int_0^\omega b_{11}(s)ds = \frac{1}{3}, \quad \bar{b}_{12} = \frac{T_2}{\omega}\int_0^\omega b_{12}(s)ds = \frac{1}{2},$$

$$\bar{b}_{21} = \frac{T_1}{\omega}\int_0^\omega b_{21}(s)ds = 0, \quad \bar{b}_{22} = \frac{T_2}{\omega}\int_0^\omega b_{22}(s)ds = 0,$$

and also choose

$$c_{11}(t) = 0, \quad c_{12}(t) = \frac{2(1+\cos 4t)}{R_1}, \quad c_{21}(t) = \frac{(1+\sin 2t)}{R_2}, \quad c_{22}(t) = \frac{(1+\sin 2t)}{2R_2},$$

obtaining

$$\bar{c}_{11} = \frac{R_1}{\omega}\int_0^\omega c_{11}(s)ds = 0, \quad \bar{c}_{12} = \frac{R_2}{\omega}\int_0^\omega c_{12}(s)ds = 2,$$

$$\bar{c}_{21} = \frac{R_1}{\omega}\int_0^\omega c_{12}(s)ds = 1, \quad \bar{c}_{22} = \frac{R_2}{\omega}\int_0^\omega c_{22}(s)ds = \frac{1}{2}.$$

Moreover, it is easy to verify that the corresponding system of nonlinear equation (4.1)

$$\begin{cases} \sum_{j=1}^{2} \bar{b}_{1j}u_j + \sum_{j=1}^{2} \bar{c}_{1j}u_j = \bar{r}_1, \\ \sum_{j=1}^{2} \bar{b}_{2j}u_j + \sum_{j=1}^{2} \bar{c}_{2j}u_j = \bar{r}_2, \end{cases}$$

has a unique positive solution $u = (u_1, u_2) = \left(\frac{39}{56}, \frac{3}{28}\right)$. The conditions of Theorem 3.4 are fulfilled and system (4.1) possesses at least one positive π-periodic solution.

Acknowledgements

We thank the referee for the helpful suggestions which allowed us to improve the presentation of our paper. The research of T.C. has been partially supported by FEDER and Ministerio de Ciencia, Innovación y Universidades (Spain) under grant PGC2018-096540-B-I00.

REFERENCES

[1] S. Ahmad, *On the nonautonomous Lotka–Volterra competition equations*, Proc. Amer. Math. Soc. **117** (1993), 199–204.

[2] C. Alvarez, A.C. Lazer, *An application of topological degree to the periodic competing species model*, J. Aust. Math. Soc. Ser. B **28** (1986), 202–219.

[3] A. Battaaz, F. Zanolin, *Coexistence states for periodic competition Kolmogorov systems*, J. Math. Anal. Appl. **219** (1998), 179–199.

[4] F.D. Chen, *Periodic solution and almost periodic solution for a delay multispecies logarithmic population model*, Appl. Math. Comput. **171** (2005), 760–770.

[5] F.D. Chen, *Periodic solutions and almost periodic solutions of a neutral multispecies logarithmic population model*, Appl. Math. Comput. **176** (2006), 431–441.

[6] S. Chen, T. Wang, J. Zhang, *Positive periodic solution for non-autonomous competition Lotka–Volterra patch system with time delay*, Nonlinear Anal. Real World Appl. **5** (2004), 409–419.

[7] T. Cheon, *Evolutionary stability of ecological hierarchy*, Physical Review Letters **90** (2003) 25, Article ID 258105, 4 pages.

[8] A. Dénes, L. Hatvani, *On the asymptotic behaviour of solutions of an asymptotically Lotka–Volterra model*, Electron. J. Qual. Theory Differ. Equ. **67** (2016), 1–10.

[9] M. Fan, K. Wang, *Global periodic solutions of a generalized n-species Gilpn Ayala competition model*, Comput. Math. Appl. **40** (2000), 1141–1151.

[10] M. Fan, K. Wang, D.Q. Jiang, *Existence and global attractivity of positive peridic solutions of n-species Lotka–Volterra competition systems with several deviating arguments*, Math. Biosci. **160** (1999), 47–61.

[11] P. Gao, *Hamiltonian structure and first integrals for the Lotka–Volterra systems*, Physics Letters A **273** (2000) 1–2, 85–96.

[12] K. Geisshirt, E. Praestgaard, S. Toxvaerd, *Oscillating chemical reactions and phase separation simulated by molecular dynamics*, J. Chem. Phys. **107** (1997) 22, 9406–9412.

[13] S.A.H. Geritz, M. Gyllenberg, *Seven answers from adaptive dynamics*, J. Evol. Biol. **18** (2005), 1174–1177.

[14] K. Gopalsamy, *Global asymptotical stability in a periodic Lotka–Volterra system*, J. Aust. Math. Soc. Ser. B **29** (1985), 66–72.

[15] M. Gyllenberg, Y. Wang, *Dynamics of the periodic type-K competitive Kolmogorov systems*, J. Differ. Equ. **205** (2004), 50–76.

[16] D. Hu, Z. Zhang, *Four positive periodic solutions to a Lotka–Volterra cooperative system with harvesting terms*, Nonlinear Anal. Real World Appl. **11** (2010), 1115–1121.

[17] M.A. Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[18] Y.K. Li, *On a periodic delay logistic type population model*, Ann. Differential Equations **14** (1998), 29–36.

[19] Y.K. Li, *Periodic solutions for delay Lotka–Volterra competition systems*, J. Math. Anal. Appl. **246** (2000), 230–244.

[20] G. Lin, Y. Hong, *Periodic solution in nonautonomous predator-prey system with delays*, Nonlinear Anal. Real World Appl. **10** (2009), 1589–1600.

[21] A.J. Lotka, *Undamped oscillations derived from the law of mass action*, J. Am. Chem. Soc. **42** (1920), 1595–1599.

[22] S. Lu, *On the existence of positive periodic solutions to a Lotka–Volterra cooperative population model with multiple delays*, Nonlinear Anal. **68** (2008), 1746–1753.

[23] X. Lv, S.P. Lu, P. Yan, *Existence and global attractivity of positive periodic solutions of Lotka–Volterra predator-prey systems with deviating arguments*, Nonlinear Anal. Real World Appl. **11** (2010), 574–583.

[24] A. Provata, G.A. Tsekouras, *Spontaneous formation of dynamical patterns with fractal fronts in the cyclic lattice Lotka–Volterra model*, Physical Review E **67** (2003) 5, part 2, Article ID 056602.

[25] Y.R. Raffoul, *Positive periodic solutions in neutral nonlinear differential equations*, Electronic Journal of Qualitative Theory of Differential Equations **16** (2007), 1–10.

[26] H.L. Royden, *Real Analysis*, MacMillan Publishing Company, New York, 1998.

[27] X.H. Tang, X. Zhou, *On positive periodic solution of Lotka–Volterra competition systems with deviating arguments*, Proc. Amer. Math. Soc. **134** (2006), 2967–2974.

[28] V. Volterra, *Variations and fluctuations of the number of individuals in animal species living together*, [in:] R.N. Chapman (ed.), *Animal Ecology*, McGraw-Hill, New York, 1926.

[29] G. Zhang, S.S. Cheng, *Positive periodic solutions of coupled delay differential systems depending on two parameters*, Taiwan. Math. J. **8** (2004), 639–652.

[30] H.Y. Zhao, L. Sun, *Periodic oscillatory and global attractivity for chemostat model involving distributed delays*, Nonlinear Anal. Real World Appl. **7** (2006), 385–394.

Mimia Benhadri
mbenhadri@yahoo.com

LAMAHIS Lab
Faculty of Sciences
Department of Mathematics
University of Skikda
P.O. Box 26, Skikda 21000, Algeria

Tomás Caraballo (corresponding author)
caraball@us.es

Departamento de Ecuaciones Difererenciales y Análisis Numérico
Universidad de Sevilla
c/ Tarfia s/n, 41012 – Sevilla, Spain

Halim Zeghdoudi
halim.zeghdoudi@univ-annaba.dz

LaPS Laboratory
Badji–Mokhtar University
P.O. Box 12, Annaba 23000, Algeria

Received: December 23, 2018.
Revised: February 23, 2020.
Accepted: February 29, 2020.