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The uniformly accelerated motion in General Relativity from a geometric point of view

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Abstract. The notion of a uniformly accelerated rectilinear motion of an observer in a general spacetime is analysed in detail. From a geometric viewpoint, a uniformly accelerated observer may be seen as a Lorentzian circle. Finally, we find geometric assumptions to ensure that an inextensible uniformly accelerated rectilinear observer does not disappear in a finite proper time.

1. Introduction

The definition of uniformly accelerated motion in General Relativity has been discussed many times over the last 50 years. In the pioneering work by Rindler [4], it was partially motivated by some aspects of intergalactic rocket travel by use of the special relativistic formulas for hyperbolic motion.

The relation between uniformly accelerated motion and Lorentzian circles in Lorentz-Minkowski spacetime was used by Rindler in [4] to define what he named hyperbolic motion in General Relativity, extending uniformly accelerated motion in Lorentz-Minkowski spacetime.

First, we present an approach to the study of uniformly accelerated motion in General Relativity in the realm of modern Lorentzian geometry. In order to do that, let $\gamma : I \longrightarrow M$ be an observer in the spacetime M. Its (proper) acceleration is given by the covariant derivative of its velocity $\gamma\,'$, i.e., $\frac{D\gamma\,'}{dt}$. Intuitively, the particle obeys a uniformly accelerated motion if its acceleration remains to be unchanged. Mathematically, we need a *connection along* γ which permits to compare spatial directions at different instants of the life of γ. In General Relativity this connection is known as the *Fermi-Walker connection* of γ. Thus, using the Fermi-Walker covariant derivative $\frac{\widehat{D}}{dt}$, we will say that a particle obeys a *uniformly accelerated* (UA) motion if,

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma\,'}{dt}\right) = 0. \tag{1}$$

The family of UA observers in the Lorentz-Minkowski spacetime $\mathbb{L}^n$ was

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completely determined long time ago [4]. It consists of timelike geodesics and Lorentzian circles. For instance, in $\mathbb{L}^2$, using the usual coordinates (x,t), the UA observer $\gamma(\tau) = \big(x(\tau), t(\tau)\big)$ throughout $(0,0)$ with zero velocity relative to certain family of inertial observers (the integral curves of vector field ∂_t) and proper acceleration a is given by,

$$x(\tau) = \tfrac{c^2}{a}\left[\cosh\left(\tfrac{a\tau}{c}\right) - 1\right], \qquad t(\tau) = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right),$$

where $\tau \in \mathbb{R}$ is the proper time of γ, and c is the light speed in vacuum.

In this paper we expose how UA observers can be seen as Lorentzian circles in any general spacetime. After that, we characterize UA observers as the projection on the spacetime of the integral curves of a vector field defined on a certain fiber bundle over the spacetime. Using this vector field, the completeness of inextensible UA motions is analysed in the search of geometric assumptions which assure that inextensible UA observers do not disappear in a finite proper time (in particular, the absence of timelike singularities). In particular, any inextensible UA observer is complete under the assumption of compactness of the spacetime and that it admits a conformal and closed timelike vector field.

2. Uniformly accelerated observer as a Lorentzian circle

A spacetime is a time orientable $n(\geq 2)$ – dimensional Lorentzian manifold $(M, \langle\ ,\ \rangle)$, endowed with a fixed time orientation. As usual, we will consider an observer in M as a (smooth) curve $\gamma : I \longrightarrow M$, I an open interval of $\mathbb{R}$, such that $\langle \gamma'(t), \gamma'(t)\rangle = -1$ and $\gamma'(t)$ is future pointing for any proper time t of γ. At each event $\gamma(t)$ the tangent space $T_{\gamma(t)}M$ splits as

$$T_{\gamma(t)}M = T_t \oplus R_t,$$

where $T_t = \mathrm{Span}\{\gamma'(t)\}$ and $R_t = T_t^{\perp}$. R_t is interpreted as the instantaneous physical space observed by γ at t. Clearly, the observer γ is able to compare spatial directions at t, but in order to compare $v_1 \in R_{t_1}$ with $v_2 \in R_{t_2}$, the observer γ must use a suitable connection. We proceed to describe it.

For each $Y \in \mathfrak{X}(\gamma)$ put Y_t^T, Y_t^R the orthogonal projections of Y_t on T_t and R_t, respectively, i.e., $Y_t^T = -\langle Y_t, \gamma'(t)\rangle\, \gamma'(t)$ and $Y_t^R = Y_t - Y_t^T$. In this way, if ∇ denotes the Levi-Civita connection, we have, [5, Prop. 2.2.1],

Along this paper the signature of a Lorentzian metric is $(-, +, ..., +)$.

Proposition 2.1. *There exists a unique connection $\widehat{\nabla}$ along γ such that*

$$\widehat{\nabla}_X Y = \left(\nabla_X Y^T\right)^T + \left(\nabla_X Y^R\right)^R ,$$

for any $X \in \mathfrak{X}(I)$ *and* $Y \in \mathfrak{X}(\gamma)$.

This connection $\widehat{\nabla}$ is called the *Fermi-Walker connection of* γ. It shows the suggestive property that if $Y \in \mathfrak{X}(\gamma)$ satisfies $Y = Y^R$ (i.e., Y_t may be observed by γ at any t) then $\left(\widehat{\nabla}_X Y\right)_t \in R_t$ for any t.

Denote by $\widehat{D}/dt$ the covariant derivative corresponding to $\widehat{\nabla}$. Then, we have [5, Prop. 2.2.2],

$$\frac{\widehat{D}Y}{dt} = \frac{DY}{dt} + \langle \gamma', Y\rangle \frac{D\gamma'}{dt} - \langle \frac{D\gamma'}{dt}, Y\rangle \gamma', \tag{2}$$

for any $Y \in \mathfrak{X}(\gamma)$. Note that $\frac{\widehat{D}}{dt} = \frac{D}{dt}$ if and only if γ is free falling.

The acceleration $\frac{D\gamma'}{dt}$ satisfies $\frac{D\gamma'}{dt}(t) \in R_t$, for any t. Therefore, it may be observed by γ whereas the velocity γ' is not observable by γ.

Now, we are in a position to give rigorously the notion of UA observer. An observer $\gamma : I \longrightarrow M$ is said to obey a uniformly accelerated motion if

$$\widehat{P}^{\gamma}_{t_1,t_2}\left(\frac{D\gamma'}{dt}(t_1)\right) = \frac{D\gamma'}{dt}(t_2), \tag{3}$$

for any $t_1, t_2 \in I$ with $t_1 < t_2$, equivalently, if the equation (1) holds everywhere on I, i.e., $\frac{D\gamma'}{dt}$ is Fermi-Walker parallel along γ. Clearly, if γ is free falling, then it is a UA observer.

Since we deal with a third-order ordinary differential equation, the following initial value problem has a unique local solution,

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right) = 0, \tag{4}$$
$$\gamma(0) = p, \quad \gamma'(0) = v, \quad \frac{D\gamma'}{dt}(0) = w,$$

where $p \in M$ and $v, w \in T_pM$ such that $|v|^2 = -1$, $\langle v, w\rangle = 0$, $|w|^2 = a^2$, and a is a positive constant. In addition, from definition it is clear that $\left|\frac{D\sigma'}{dt}\right|^2(t)$ is constant on I.

Taking into account formula (2), an observer γ satisfies equation (1) if and only if

$$\frac{D^2\gamma'}{dt^2} = \left\langle \frac{D\gamma'}{dt}, \frac{D\gamma'}{dt}\right\rangle \gamma', \tag{5}$$

which is a third order equation.

Consider a UA observer $\gamma : I \longrightarrow M$ with $a = \left|\frac{D\gamma'}{dt}\right| > 0$ and put $e_1(t) = \gamma'(t)$, $e_2(t) = \frac{1}{a}\frac{D\gamma'}{dt}(t)$. Then, from (5) we have

$$\frac{De_1}{dt} = ae_2(t) \qquad \text{and} \qquad \frac{De_2}{dt} = ae_1(t).$$

Conversely, assume this system holds true for an observer γ with $a > 0$ constant. Then, a (non free falling) UA observer may be seen as a *Lorentzian circle* of constant curvature a and identically zero torsion.

The previous results can be summarized as follows,

Proposition 2.2. *$\gamma : I \longrightarrow M$ is a UA observer iff one of the following assertions holds:*

(a) γ is a solution of third-order differential equation (5).

(b) γ is a Lorentzian circle or it is free falling.

(c) γ has constant curvature and the remaining curvatures equal to zero.

(d) γ, viewed as an isometric immersion from $(I, -dt^2)$ to M, is totally umbilical with parallel mean curvature vector.

3. Completeness of the inextensible UA trajectories

First of all, we are going to relate the solutions of equation (5) with the integral curves of a certain vector field on a Stiefel bundle type on M.

Given a Lorentzian linear space E and $a \in \mathbb{R}$, $a > 0$, denote by $V^a_{n,2}(E)$ the (n,2)-Stiefel manifold over E, defined by

$$V^a_{n,2}(E) = \big\{(v, w) \in E^2 : |v|^2 = -1, |w|^2 = a^2, \langle v, w\rangle = 0\big\}.$$

We will call (n,2)-Stiefel bundle, $V^a_{n,2}(M)$, to the bundle over M with fiber $V^a_{n,2}(T_pM)$.

A key tool in the study of completeness is contained in the following lemma, which is proved in detail in [2].

Lemma 3.1. *There exists a unique vector field G on $V^a_{n,2}(M)$ such that the curves $t \longmapsto \big(\gamma(t), \gamma'(t), \frac{D\gamma'}{dt}(t)\big)$ are the integral curves of G, for any solution γ of equation* (4).

Once defined G, we will look for assumptions which assert its completeness. The following result directly follows from Lemma 3.1.

Lemma 3.2. *Let $\gamma : [0, b) \longrightarrow M$ be a solution of equation (4) with $0 < b < \infty$. The curve γ can be extended to b as a solution of (4) if and only if there exists a sequence $\big\{\gamma(t_n), \gamma'(t_n), \frac{D\gamma'}{dt}(t_n)\big\}_n$ which is convergent in $V^a_{n,2}(M)$.*

Although we know that $|\gamma'(t)|^2 = -1$, this is not enough to apply Lemma 3.2 even in the geometrically relevant case of M compact.

Recall that a vector field K on M is said *closed* and *conformal* if satisfies

$$\nabla_X K = hX \qquad \text{for} \quad \text{all} \qquad X \in \mathfrak{X}(M). \tag{6}$$

Note that for any curve $\gamma : I \longrightarrow M$, if K is closed and conformal, we have

$$\frac{d}{dt}\langle K, \gamma'\rangle = \big\langle K, \frac{D\gamma'}{dt}\big\rangle + h(\gamma)|\,\gamma'|^2. \tag{7}$$

The following result, inspired from [1, Lemma 9], will be decisive to assure that the image of the curve in $V^a_{n,2}(M)$, associated to a UA observer γ, is contained in a compact subset.

Lemma 3.3. *Let M be a spacetime and let Q be a unitary timelike vector field. If $\gamma : I \longrightarrow M$ is a solution of (4) such that $\gamma(I)$ lies in a compact subset of M and $\langle Q, \gamma'\rangle$ is bounded on I, then the image of $t \longmapsto \big(\gamma(t), \gamma'(t), \frac{D\gamma'}{dt}\big)$ is contained in a compact subset of $V^a_{n,2}(M)$ where a is the constant $|\frac{D\gamma'}{dt}|$.*

Proof. Consider the 1-form Q^b metrically equivalent to Q and the associated Riemannian metric $g_R := \langle\, ,\, \rangle + 2\, Q^b \otimes Q^b$. We have,

$$g_R(\gamma', \gamma') = \langle \gamma', \gamma'\rangle + 2\,\langle Q, \gamma'\rangle^2,$$

which, by hypothesis, is bounded on I. Hence, there exists a constant $c > 0$ such that

$$\big(\gamma(I), \gamma'(I), \tfrac{D\gamma'}{dt}(I)\big) \subset C,$$
$$C := \big\{(p, v, w) \in V^a_{n,2}(M) \,:\, p \in C_1, \quad g_R(v, v) \leq c\big\},$$

where C_1 is a compact set on M such that $\gamma(I) \subset C_1$. Hence, C is a compact in $V^a_{n,2}(M)$. $\square$

Now, we are in a position to state the following completeness result.

Theorem 3.1. *Let M be a spacetime which admits a timelike conformal and closed vector field K. If* $\inf_M \sqrt{-\langle K, K\rangle} > 0$ *then, each solution* $\gamma : I \longrightarrow M$ *of* (4) *such that* $\gamma(I)$ *lies in a compact subset of M can be extended.*

Proof. Let $I = [0, b)$, $0 < b < +\infty$, be the domain of a solution γ of equation (4). Derivating (7), it follows

$$\frac{d^2}{dt^2}\langle K, \gamma'\rangle = \langle \frac{DK}{dt}, \frac{D\gamma'}{dt}\rangle + \langle K, \frac{D^2\gamma'}{dt^2}\rangle - \frac{d}{dt}(h \circ \gamma).$$

The first right term vanishes because K is conformal and closed. On the other hand, the second right term equals to $a^2\langle K, \gamma'\rangle$. Thus, the function $t \mapsto \langle K, \gamma'\rangle$ satisfies the following differential equation,

$$\frac{d^2}{dt^2}\langle K, \gamma'\rangle - a^2\langle K, \gamma'\rangle = (h \circ \gamma)'(t). \tag{8}$$

Using now that $\gamma(I)$ is contained in a compact of M, the function $h \circ \gamma$ is bounded on I. Moreover, since I is assumed bounded, using (8) there exists a constant $c_1 > 0$ such that

$$|\langle K, \gamma'\rangle| < c_1. \tag{9}$$

Now, if we put $Q := \frac{K}{|K|}$, where $|K|^2 = -\langle K, K\rangle > 0$, then Q is a unitary timelike vector field such that, by (9),

$$|\langle Q, \gamma'\rangle| \leq m\, c_1 \qquad \text{on} \quad I,$$

where $m = \sup_M |K|^{-1} < \infty$. The proof ends making use of Lemmas 3.2 and 3.3. □

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