Cómo construir familias biespectrales de polinomios ortogonales a partir de familias clásicas\textsuperscript{1}

Manuel Domínguez de la Iglesia

Instituto de Matemáticas, C.U., UNAM

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\textsuperscript{1}trabajo conjunto con Antonio J. Durán
**OUTLINE**

1. **INTRODUCTION**
   - Classical orthogonal polynomials
   - Krall orthogonal polynomials

2. **METHODOLOGY**
   - $D$-operators
   - Choice of arbitrary polynomials
   - Identifying the measure

3. **EXAMPLES**
   - Charlier, Meixner and Krawtchouk polynomials
   - Laguerre polynomials
Introduction

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3. Examples
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THE SPACE $L^2_{\omega}(S)$

Let $\omega$ be a positive measure on $S \subset \mathbb{R}$ and consider the space of functions $L^2_{\omega}(S)$ with the inner product

$$\langle f, g \rangle_\omega = \int_S f(x)g(x)d\omega(x)$$

We say that $f \in L^2_{\omega}(S)$ if $\langle f, f \rangle_\omega = \|f\|_{\omega}^2 < \infty$.

$S$ can be a continuous interval, a discrete set of points or a combination of both. The discrete component of the measure is usually written as

$$\omega_d(x) = \sum_{x=0}^{N} a_x \delta_{t_x}, \quad t_{x_0}, \ldots, t_{x_N} \in \mathbb{R}$$

In that case the inner product can be thought of as

$$\langle f, g \rangle_{\omega_d} = \sum_{x=0}^{N} a_x \int f(x)g(x)\delta_{t_x} = \sum_{x=0}^{N} a_x f(t_x)g(t_x)$$
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**ORTHOGONAL POLYNOMIALS**

A system of polynomials $(p_n)_n = \{p_0(x), p_1(x), \ldots\}$ with $\text{deg}(p_n) = n$ is orthogonal in $L^2_\omega(S)$ if (Gramm-Schmidt)

$$\langle p_n, p_m \rangle_\omega = \int_S p_n(x)p_m(x)d\omega(x) = \|p_n\|_\omega^2 \delta_{nm}, \quad n, m \geq 0$$

Every family of OP's $(p_n)_n$ satisfy a three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + c_np_{n-1}(x), \quad n \geq 1$$

where $a_n, c_n \neq 0$, $b_n \in \mathbb{R}$ and $p_0(x) = 1$, $p_{-1}(x) = 0$.

Jacobi operator (tridiagonal):

$$Jp = \begin{pmatrix} b_0 & a_1 & & & \\ c_1 & b_1 & a_2 & & \\ & c_2 & b_2 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = xp, \quad x \in S$$

The converse result is also true (Favard's or spectral theorem).
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Classical Families (Continuous Case)

Bochner problem, 1929

\[ \sigma(x) \frac{d^2}{dx^2} p_n(x) + \tau(x) \frac{d}{dx} p_n(x) + \lambda_n p_n(x) = 0, \quad x \in S \subset \mathbb{R} \]
\[ \deg \sigma \leq 2, \quad \deg \tau = 1 \]

- Hermite (Normal, Gaussian): \( \omega(x) = e^{-x^2}, \quad x \in \mathbb{R} \)
  \[ H_n(x)'' - 2xH_n(x)' = -2nH_n(x) \]

- Laguerre (Gamma, Exponential): \( \omega(x) = x^\alpha e^{-x}, \quad x > 0, \quad \alpha > -1 \)
  \[ xL_n^\alpha(x)'' + (\alpha + 1 - x)L_n^\alpha(x)' = -nL_n^\alpha(x) \]

- Jacobi (Beta, Uniform): \( \omega(x) = x^\alpha (1-x)^\beta, \quad x \in (0,1), \quad \alpha, \beta > -1 \)
  \[ x(1-x)P_n^{(\alpha,\beta)}(x)'' + (\alpha + 1 - (\alpha + \beta + 2)x)P_n^{(\alpha,\beta)}(x)' = -n(n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}(x) \]
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**Classical families (discrete case)**

If we set

\[ \Delta f(x) = f(x + 1) - f(x), \quad \nabla f(x) = f(x) - f(x - 1) \]

the classification problem is to find discrete OP’s \((p_n)_n\)

\[ \sigma(x) \Delta \nabla p_n(x) + \tau(x) \Delta p_n(x) + \lambda_n p_n(x) = 0, \quad x \in S \subset \mathbb{N} \]

\[ \deg \sigma \leq 2, \quad \deg \tau = 1 \]

In other words, if we call the shift operator

\[ \mathcal{S}_j f(x) = f(x + j) \]

the difference equation reads

\[ [\sigma(x) + \tau(x)]\mathcal{S}_1 p_n(x) - [2\sigma(x) + \tau(x)]\mathcal{S}_0 p_n(x) + \sigma(x)\mathcal{S}_{-1} p_n(x) + \lambda_n p_n(x) = 0, \quad x \in S \subset \mathbb{N} \]
Classical families (discrete case)

- Charlier (Poisson):
  \[ \omega_a(x) = \sum_{x=0}^{\infty} \frac{a^x}{x!} \delta_x, \quad a > 0 \]

- Meixner (Pascal, Geometric):
  \[ ac^n_a(x + 1) - (x + a)c^n_a(x) + xc^n_a(x - 1) = -nc^n_a(x) \]

\[(i)_j = i(i + 1) \cdots (i + j - 1) \text{ is the Pochhammer symbol} \]

\[ a(x + c)m^a,c_n(x + 1) - (x + a(x + c))m^a,c_n(x) \\
+ xm^a,c_n(x - 1) = n(a - 1)m^a,c_n(x) \]
CLASSICAL FAMILIES (DISCRETE CASE)

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\[ \omega_a(x) = \sum_{x=0}^{\infty} \frac{a^x}{x!} \delta_x, \quad a > 0 \]

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\[ ac_n^a(x + 1) - (x + a)c_n^a(x) + xc_n^a(x - 1) = -nc_n^a(x) \]
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- **Krawtchuok** (Binomial, Bernoulli):

\[
\omega_{p,N}(x) = \sum_{x=0}^{N} \binom{N}{x} p^x (1-p)^{N-x} \delta_x, \quad 0 < p < 1
\]

\[
p(N - x) k_n^{p,N}(x + 1) - [p(N - x) + x(1-p)] k_n^{p,N}(x) \\
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- **Hahn** (Hypergeometric):

\[
\omega_{\alpha,\beta,N}(x) = \sum_{x=0}^{N} \binom{\alpha + x}{x} \binom{\beta + N - x}{N - x} \delta_x, \quad \alpha, \beta > -1, \alpha, \beta < -N
\]

\[
B(x) Q_n^{\alpha,\beta,N}(x + 1) - [B(x) + D(x)] Q_n^{\alpha,\beta,N}(x) \\
+ D(x) Q_n^{\alpha,\beta,N}(x - 1) = n(n + \alpha + \beta + 1) Q_n^{\alpha,\beta,N}(x)
\]

where \(B(x) = (x + \alpha + 1)(x - N)\) and \(D(x) = x(x - \beta - N - 1)\).
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**Krall polynomials (continuous case)**

**GOAL** (Krall, 1939): find families of OP’s \((q_n)_n\) which are also eigenfunctions of a higher-order **differential** operator of the form

\[
D_c = \sum_{j=0}^{2m} h_j(x) \frac{d^j}{dx^j}, \quad \text{deg}(h_j) \leq j \quad \Rightarrow \quad D_c(q_n) = \lambda_n q_n
\]

Littlejohn, Grünbaum, Heine, Iliev, Koekoek’s, Lesky, Bavinck, van Haeringen, Horozov, Koornwinder, etc (80’s, 90’s, 00’s).

**Common techniques:** ad-conditions, Darboux process, etc.

\((q_n)_n\) are typically orthogonal with respect to the measure

\[
\omega(x) + \sum_{j=0}^{m-1} a_j \delta_{x_0}^{(j)} , \quad a_j \in \mathbb{R}
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where \(\omega\) is a (modified) classical weight and \(x_0\) is an endpoint of the support of orthogonality of \(\omega\).
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**KRALL POLYNOMIALS (DISCRETE CASE)**

The same question arise in the discrete setting, i.e. find families of OP's \((q_n)_n\) which are also eigenfunctions of a higher order difference operator

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D_d = \sum_{j=-s}^{s} h_j(x)\delta_j, \quad h_s, h_{-s} \neq 0, \quad \Rightarrow \quad D_d(q_n) = \lambda_n q_n
\]

The same techniques of adding deltas does not work for the discrete case.

Surprisingly, it has not been until very recently (Durán, 2012) when the first examples appeared (\(D\)-operators).

\((q_n)_n\) are typically orthogonal with respect to the measure

\[
\omega^F(x) = \prod_{f \in F} (x - f) \omega(x)
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where \(\omega\) is a discrete classical weight and \(F\) is a finite set of numbers. This is also called a Christoffel transform of \(\omega\).
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3 **Examples**
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**D-OPERATORS**

Let $\mathcal{A}$ be an algebra of (differential or difference) operators and $(p_n)_n$ a family of polynomials such that there exists $D_p \in \mathcal{A}$ with $D_p(p_n) = np_n$. Given a sequence of numbers $(\varepsilon_n)_n$, let us consider the operator

$$D(p_n) = \sum_{j=1}^{n} (-1)^{j+1} \varepsilon_n \cdots \varepsilon_{n-j} p_{n-j} = \varepsilon_n p_{n-1} - \varepsilon_n \varepsilon_{n-1} p_{n-2} + \cdots$$

We say that $D$ is an **D-operator** associated with $\mathcal{A}$ and $(p_n)_n$ if $D \in \mathcal{A}$.

- **Laguerre**: $\varepsilon_n = -1 \Rightarrow D = \frac{d}{dx}$.
- **Charlier**: $\varepsilon_n = 1 \Rightarrow D = \nabla$.
- **Meixner**:
  
  $\varepsilon_n^1 = \frac{a}{1-a} \Rightarrow D_1 = \frac{a}{1-a} \Delta, \quad \varepsilon_n^2 = \frac{1}{1-a} \Rightarrow D_2 = \frac{1}{1-a} \nabla$.
- **Krawtchouk**:
  
  $\varepsilon_n^1 = \frac{1}{1-a} \Rightarrow D_1 = \frac{1}{1-a} \nabla, \quad \varepsilon_n^2 = -\frac{a}{1-a} \Rightarrow D_2 = -\frac{a}{1-a} \Delta$. 
**D-operators**

Let $\mathcal{A}$ be an algebra of (differential or difference) operators and $(p_n)_n$ a family of polynomials such that there exists $D_p \in \mathcal{A}$ with $D_p(p_n) = np_n$. Given a sequence of numbers $(\varepsilon_n)_n$, let us consider the operator

$$D(p_n) = \sum_{j=1}^{n} (-1)^{j+1} \varepsilon_n \cdots \varepsilon_{n-j} p_{n-j} = \varepsilon_n p_{n-1} - \varepsilon_n \varepsilon_{n-1} p_{n-2} + \cdots$$

We say that $D$ is an **D-operator** associated with $\mathcal{A}$ and $(p_n)_n$ if $D \in \mathcal{A}$.

- **Laguerre**: $\varepsilon_n = -1 \Rightarrow D = \frac{d}{dx}$.
- **Charlier**: $\varepsilon_n = 1 \Rightarrow D = \nabla$.
- **Meixner**:
  - $\varepsilon_1^n = \frac{a}{1-a} \Rightarrow D_1 = \frac{a}{1-a} \Delta$, $\varepsilon_2^n = \frac{1}{1-a} \Rightarrow D_2 = \frac{1}{1-a} \nabla$.
- **Krawtchouk**:
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**D-OPERATORS**

Let $\mathcal{A}$ be an algebra of (differential or difference) operators and $(p_n)_n$ a family of polynomials such that there exists $D_p \in \mathcal{A}$ with $D_p(p_n) = np_n$. Given a sequence of numbers $(\varepsilon_n)_n$, let us consider the operator

$$\mathcal{D}(p_n) = \sum_{j=1}^{n} (-1)^{j+1} \varepsilon_n \cdots \varepsilon_{n-j} p_{n-j} = \varepsilon_n p_{n-1} - \varepsilon_n \varepsilon_{n-1} p_{n-2} + \cdots$$

We say that $\mathcal{D}$ is an $\mathcal{D}$-operator associated with $\mathcal{A}$ and $(p_n)_n$ if $\mathcal{D} \in \mathcal{A}$.

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**D-operators**

**Theorem (Durán, 2013)**

Let $\mathcal{A}$, $(p_n)_n$, $D_p(p_n) = np_n$, $(\varepsilon_n)_n$ and $D$. For an arbitrary polynomial $R$ such that $R(n) \neq 0$, $n \geq 0$, we define a new polynomial $P$ by

$$P(x) - P(x - 1) = R(x)$$

and a sequence of polynomials $(q_n)_n$ by $q_0 = 1$ and

$$q_n = p_n + \beta_n p_{n-1}, \quad n \geq 1$$

where the numbers $\beta_n$, $n \geq 0$, are given by

$$\beta_n = \varepsilon_n \frac{R(n)}{R(n-1)}, \quad n \geq 1$$

Then there exist $D_q \in \mathcal{A}$ such that $D_q(q_n) = P(n)q_n$ where

$$D_q = P(D_p) + DR(D_p)$$
**D-OPERATORS**

**GOAL:** Extend the previous Theorem for the case that we consider a linear combination of $m + 1$ consecutive $p_n$'s:

$$q_n = p_n + \beta_{n,1}p_{n-1} + \beta_{n,2}p_{n-2} + \cdots + \beta_{n,m}p_{n-m}$$

Let $R_1, R_2, \ldots, R_m$ be $m$ arbitrary polynomials and $m$ $\mathcal{D}$-operators $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m$ defined by the sequences $(\varepsilon^h_n)_n$, $h = 1, \ldots, m$. Define the auxiliary functions $\xi_{n,i}^h$ by

$$\xi_{n,i}^h = \varepsilon^h_n \varepsilon^h_{n-1} \cdots \varepsilon^h_{n-i+1}$$

and assume that the following **Casorati determinant** never vanish ($n \geq 0$)

$$\Omega(n) = \begin{vmatrix} \xi_{n-1,m-1}^1 R_1(n-1) & \xi_{n-2,m-2}^1 R_1(n-2) & \cdots & R_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n-1,m-1}^m R_m(n-1) & \xi_{n-2,m-2}^m R_m(n-2) & \cdots & R_m(n-m) \end{vmatrix} \neq 0$$
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Now consider the sequence of polynomials \((q_n)_n\) defined by

\[
q_n(x) = \begin{vmatrix}
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\end{vmatrix}
\]

Observation: \(q_n\) is a linear combination of \(m + 1\) consecutive \(p_n\)'s.

Define for \(h = 1, \ldots, m\), the following functions

\[
M_h(x) = \sum_{j=1}^{m} (-1)^{h+j} \xi_{x,m-j}^h \det (\xi_{x+j-r,m-r}^l R_l(x+j-r)) \quad \{ \begin{array}{l}
  l \neq h \\
  r \neq j
\end{array} \}
\]

Observation: \(M_h\) are linear combinations of adjoint determinants of \(\Omega(x)\).

If we assume that \(\Omega(x)\) and \(M_h(x)\) are polynomials in \(x\), then \(\exists D_q \in A\) with \(D_q(q_n) = P(n)q_n\) and \(P(x) - P(x-1) = \Omega(x)\), where

\[
D_q = P(D_p) + \sum_{h=1}^{m} M_h(D_p) D_h R_h(D_p)
\]
\( D\)-OPERATORS

Now consider the sequence of polynomials \((q_n)_n\) defined by

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q_n(x) = \begin{vmatrix}
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**Choice of $R_1, R_2, \ldots, R_m$**

**GOAL:** Make $(q_n)_n$ bispectral (we already have $D_q(q_n) = \lambda_n q_n$).

For that we have to make an appropriate choice of the arbitrary polynomials $R_1, R_2, \ldots, R_m$. This choice is based on the following recurrence formula ($h = 1, \ldots, m$):

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\varepsilon_{n+1}^h a_{n+1} R_j^h(n+1) - b_n R_j^h(n) + \frac{c_n}{\varepsilon_n^h} R_j^h(n-1) = (\eta_j^h + \kappa_j^h) R_j^h(n), \quad n \in \mathbb{Z}
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where $\eta_j^h$ and $\kappa_j^h$ are real numbers independent of $n$ and $j$, $(a_n)_n \in \mathbb{Z}$, $(b_n)_n \in \mathbb{Z}$, $(c_n)_n \in \mathbb{Z}$ are the coefficients in the TTRR for the OP’s $(p_n)_n$, and $(\varepsilon_n^h)_n$ defines a $\mathcal{D}$-operator for $(p_n)_n$.

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**CHOICE OF** $R_1, R_2, \ldots, R_m$

Given a set $G$ of $m$ positive integers, $G = \{g_1, \ldots, g_m\}$, call
\[ \tilde{G} = \{\tilde{g}_1, \ldots, \tilde{g}_m\} \] where $\tilde{g}_h = \eta_h g_h + \kappa_h$.

We then define the sequence of polynomials $(q^G_n)_n$ by
\[
q^G_n(x) = \begin{vmatrix}
 p_n(x) & -p_{n-1}(x) & \cdots & (-1)^m p_{n-m}(x) \\
 \xi_{n,m}^1 R_{g_1}^1(n) & \xi_{n-1,m-1}^1 R_{g_1}^1(n-1) & \cdots & R_{g_1}^1(n-m) \\
 \vdots & \vdots & \ddots & \vdots \\
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\end{vmatrix}
\]

Let $p_{\tilde{G}}(x) = \prod_{i=1}^m (x - \tilde{g}_i)$. $(q^G_n)_n$ are orthogonal w.r.t. a measure $\tilde{\omega}$ if
\[
\langle \tilde{\omega}, p_n \rangle = (-1)^n c_G \sum_{i=1}^m \frac{\xi_{n,n+1} R_{g_i}^i(n)}{p'_{\tilde{G}}(\tilde{g}_i) R_{g_i}^i(-1)}, \quad n \geq 0, \quad c_G \neq 0
\]
\[
0 = \sum_{i=1}^m \frac{R_{g_i}^i(n)}{p'_{\tilde{G}}(\tilde{g}_i) \xi_{-1,-n-1}^i R_{g_i}^i(-1)}, \quad 1 - m \leq n < 0
\]
\[
0 \neq \sum_{i=1}^m \frac{R_{g_i}^i(-m)}{p'_{\tilde{G}}(\tilde{g}_i) \xi_{-1,m-1} R_{g_i}^i(-1)}
\]


**Choice of \( R_1, R_2, \ldots, R_m \)**

Given a set \( G \) of \( m \) positive integers, \( G = \{ g_1, \ldots, g_m \} \), call \( \tilde{G} = \{ \tilde{g}_1, \ldots, \tilde{g}_m \} \) where \( \tilde{g}_h = \eta_h g_h + \kappa_h \).

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Let \( p_{\tilde{G}}(x) = \prod_{i=1}^m (x - \tilde{g}_i) \). \( (q^G_n) \) are **orthogonal** w.r.t. a measure \( \tilde{\omega} \) if

\[
\langle \tilde{\omega}, p_n \rangle = (-1)^n c_G \sum_{i=1}^m \frac{\xi^i_{n,n+1} R^i_{g_i}(n)}{p'_{\tilde{G}}(\tilde{g}_i) R^i_{g_i}(-1)}, \quad n \geq 0, \quad c_G \neq 0
\]

\[
0 = \sum_{i=1}^m \frac{R^i_{g_i}(n)}{p'_{\tilde{G}}(\tilde{g}_i) \xi^i_{-1,-n-1} R^i_{g_i}(-1)}, \quad 1 - m \leq n < 0
\]

\[
0 \neq \sum_{i=1}^m \frac{R^i_{g_i}(-m)}{p'_{\tilde{G}}(\tilde{g}_i) \xi^i_{-1,m-1} R^i_{g_i}(-1)}
\]
**Choice of** $R_1, R_2, \ldots, R_m$

Given a set $G$ of $m$ positive integers, $G = \{g_1, \ldots, g_m\}$, call $\tilde{G} = \{\tilde{g}_1, \ldots, \tilde{g}_m\}$ where $\tilde{g}_h = \eta_h g_h + \kappa_h$.

We then define the sequence of polynomials $(q_n^G)$ by

$$q_n^G(x) = \begin{vmatrix} p_n(x) & -p_{n-1}(x) & \cdots & (-1)^m p_{n-m}(x) \\ \xi_{n,m}^1 R_{g_1}^1(n) & \xi_{n-1,m-1}^1 R_{g_1}^1(n-1) & \cdots & R_{g_1}^1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n,m}^m R_{g_m}^m(n) & \xi_{n-1,m-1}^m R_{g_m}^m(n-1) & \cdots & R_{g_m}^m(n-m) \end{vmatrix}$$

Let $p_{\tilde{G}}(x) = \prod_{j=1}^m (x - \tilde{g}_i)$. $(q_n^G)$ are **orthogonal** w.r.t. a measure $\tilde{\omega}$ if

$$\langle \tilde{\omega}, p_n \rangle = (-1)^n c_G \sum_{i=1}^m \frac{\xi_{n,n+1}^i R_{g_i}^i(n)}{p_{\tilde{G}}'(\tilde{g}_i) R_{g_i}^i(-1)}, \quad n \geq 0, \quad c_G \neq 0$$

$$0 = \sum_{i=1}^m \frac{R_{g_i}^i(n)}{p_{\tilde{G}}'(\tilde{g}_i) \xi_{-1,-n-1}^i R_{g_i}^i(-1)}, \quad 1 - m \leq n < 0$$

$$0 \neq \sum_{i=1}^m \frac{R_{g_i}^i(-m)}{p_{\tilde{G}}'(\tilde{g}_i) \xi_{-1,m-1}^i R_{g_i}^i(-1)}$$
Identifying the measure $\tilde{\omega}$

$\tilde{\omega}$ will be identified by the Christoffel transform of $\omega$

$$\omega^F(x) = \prod_{f \in F} (x - f) \omega(x)$$

The set $G$ will be closely related with the set $F$.

In fact $G$ will be identified by one of the following sets:

$$I(F) = \{1, 2, \ldots, f_k\} \setminus \{f_k - f, f \in F\},$$
$$J_h(F) = \{0, 1, 2, \ldots, f_k + h - 1\} \setminus \{f - 1, f \in F\}, \quad h \geq 1$$

where $f_k = \max F$ and $k = \#(F)$.

For the transformation $I$, the bigger the holes in $F$ (with respect to the set $\{1, 2, \ldots, f_k\}$), the bigger the set $I(F)$:

$$I(\{1, 2, 3, \ldots, k\}) = \{k\}, \quad I(\{1, k\}) = \{1, 2, \ldots, k - 2, k\}$$
IDENTIFYING THE MEASURE $\tilde{\omega}$

$\tilde{\omega}$ will be identified by the Christoffel transform of $\omega$

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IDENTIFYING THE MEASURE $\tilde{\omega}$: EXAMPLE

Imagine we have a discrete classical weight $\omega$ supported on \{0, 1, 2, ...\}

Let $F = \{1, 4, 6\}$ and consider the discrete weight $\omega^F$ given by

$$\omega^F(x) = \prod_{f \in F} (x - f) \omega(x) = (x - 1)(x - 4)(x - 6) \omega(x)$$

The new discrete weight $\omega^F$ will be supported on \{0, 2, 3, 5, 7 ...\}

The set of indexes $G$ we have to take to construct the orthogonal polynomials $(q_n^G)_n$ with respect to $\tilde{\omega} = \omega^F$ will be given by

$$G = I(F) = \{1, 2, 3, 4, 5, 6\} \setminus \{5, 2, 0\} = \{1, 3, 4, 6\}$$
Identifying the measure $\tilde{\omega}$: example

Imagine we have a discrete classical weight $\omega$ supported on $\{0, 1, 2, \ldots \}$

Let $F = \{1, 4, 6\}$ and consider the discrete weight $\omega^F$ given by

$$\omega^F(x) = \prod_{f \in F} (x - f) \omega(x) = (x - 1)(x - 4)(x - 6) \omega(x)$$

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The set of indexes $G$ we have to take to construct the orthogonal polynomials $(q_n^G)_n$ with respect to $\tilde{\omega} = \omega^F$ will be given by

$$G = I(F) = \{1, 2, 3, 4, 5, 6\} \setminus \{5, 2, 0\} = \{1, 3, 4, 6\}$$
1 Introduction
   - Classical orthogonal polynomials
   - Krall orthogonal polynomials

2 Methodology
   - $D$-operators
   - Choice of arbitrary polynomials
   - Identifying the measure

3 Examples
   - Charlier, Meixner and Krawtchouk polynomials
   - Laguerre polynomials
Charlier Polynomials

Let \( F \subset \mathbb{N} \) be finite and consider \( G = I(F) = \{g_1, \ldots, g_m\} \).
Let \( \omega_a \) be the Charlier measure and \((c^a_n)_n\) its sequence of OP’s. Assume that \( \Omega_G(n) = \det(c^{-a}_g(-n-j-1))_{i,j=1}^m \neq 0 \).

If we define \((q_n)_n\) by

\[
q_n(x) = \begin{vmatrix}
    c^a_n(x) & -c^a_{n-1}(x) & \cdots & (-1)^m c^a_{n-m}(x) \\
    c^{-a}_{g_1}(-n-1) & c^{-a}_{g_1}(-n) & \cdots & c^{-a}_{g_1}(-n+m-1) \\
    \vdots & \vdots & \ddots & \vdots \\
    c^{-a}_{g_m}(-n-1) & c^{-a}_{g_m}(-n) & \cdots & c^{-a}_{g_m}(-n+m-1)
\end{vmatrix}
\]

then the polynomials \((q_n)_n\) are orthogonal with respect to the measure

\[
\omega^F_a = \prod_{f \in F} (x - f) \omega_a
\]

and they are eigenfunctions of a higher order difference operator \(D_q\) with

\[
-s = r = \sum_{f \in F} f - \frac{k(k - 1)}{2} + 1, \quad k = \#(F)
\]
Charlier polynomials

Let $F \subset \mathbb{N}$ be finite and consider $G = l(F) = \{g_1, \ldots, g_m\}$. Let $\omega_a$ be the Charlier measure and $(c_a^n)_n$ its sequence of OP’s. Assume that $\Omega_G(n) = \det (c_g^a(-n - j - 1))_{l,j=1}^m \neq 0$.

If we define $(q_n)_n$ by

$$q_n(x) = \begin{vmatrix}
c_n^a(x) & -c_{n-1}^a(x) & \cdots & (-1)^m c_{n-m}^a(x) \\
-c_{g_1}^a(-n - 1) & c_{g_1}^a(-n) & \cdots & c_{g_1}^a(-n + m - 1) \\
\vdots & \vdots & \ddots & \vdots \\
-c_{g_m}^a(-n - 1) & c_{g_m}^a(-n) & \cdots & c_{g_m}^a(-n + m - 1)
\end{vmatrix}$$

then the polynomials $(q_n)_n$ are orthogonal with respect to the measure

$$\omega_a^F = \prod_{f \in F} (x - f) \omega_a$$

and they are eigenfunctions of a higher order difference operator $D_q$ with

$$-s = r = \sum_{f \in F} f - \frac{k(k - 1)}{2} + 1, \quad k = \#(F)$$
In this case have two different $\mathcal{D}$-operators. That means that we will have to consider two sets of positive integers $F_1, F_2 \subset \mathbb{N}$.

Consider $H = J_h(F_1) = \{h_1, \ldots, h_{m_1}\}$ and $K = I(F_2) = \{k_1, \ldots, k_{m_2}\}$.

Define $m = m_1 + m_2$ and consider the Meixner polynomials $(m_n^{a,c})_n$.

Assume that

$$
\Omega_{a,c}^{H,K}(n) = \begin{vmatrix}
\frac{m_1^{1/a,2-c}}{a^{m_1-1}} & \cdots & \frac{m_{h_1}^{1/a,2-c}}{a^{m_1-1}} & \cdots \\
\frac{m_{h_1}^{1/a,2-c}}{a^{m_1-1}} & \cdots & \frac{m_{h_1}^{1/a,2-c}}{a^{m_1-1}} & \cdots \\
\frac{m^{a,2-c}}{a^{m_1-1}} & \cdots & \frac{m^{a,2-c}}{a^{m_1-1}} & \cdots \\
\frac{m_{k_1}^{a,2-c}}{a^{m_1-1}} & \cdots & \frac{m_{k_1}^{a,2-c}}{a^{m_1-1}} & \cdots \\
\frac{m_{k_1}^{a,2-c}}{a^{m_1-1}} & \cdots & \frac{m_{k_1}^{a,2-c}}{a^{m_1-1}} & \cdots \\
\frac{m^{a,2-c}}{a^{m_1-1}} & \cdots & \frac{m^{a,2-c}}{a^{m_1-1}} & \cdots \\
\frac{m_{k_{m_2}}^{a,2-c}}{a^{m_1-1}} & \cdots & \frac{m_{k_{m_2}}^{a,2-c}}{a^{m_1-1}} & \cdots \\
\frac{m_{k_{m_2}}^{a,2-c}}{a^{m_1-1}} & \cdots & \frac{m_{k_{m_2}}^{a,2-c}}{a^{m_1-1}} & \cdots
\end{vmatrix} \neq 0
$$
In this case have two different $D$-operators. That means that we will have to consider two sets of positive integers $F_1, F_2 \subset \mathbb{N}$.

Consider $H = J_h(F_1) = \{ h_1, \ldots, h_{m_1} \}$ and $K = I(F_2) = \{ k_1, \ldots, k_{m_2} \}$.

Define $m = m_1 + m_2$ and consider the Meixner polynomials $(m_n^{a,c})$. Assume that

$$\Omega_{a,c}^{H,K}(n) = \begin{vmatrix} m_{h_1}^{1/a,2-c}(-n) & \cdots & m_{h_1}^{1/a,2-c}(-n+m-1) \\ \vdots & \ddots & \vdots \\ m_{h_{m_1}}^{1/a,2-c}(-n) & \cdots & m_{h_{m_1}}^{1/a,2-c}(-n+m-1) \\ \frac{m_{k_1}^{a,2-c}(-n)}{am^{-1}} & \cdots & \frac{m_{k_1}^{a,2-c}(-n+m-1)}{am^{-1}} \\ \vdots & \ddots & \vdots \\ \frac{m_{k_{m_2}}^{a,2-c}(-n)}{am^{-1}} & \cdots & \frac{m_{k_{m_2}}^{a,2-c}(-n+m-1)}{am^{-1}} \end{vmatrix} \neq 0$$
### Meixner Polynomials

If we define \((q_n)_n\) by

\[
q_n(x) = \begin{vmatrix}
(1 - a)^m m_n^{a,c}(x) & -(1 - a)^{m-1} m_{n-1}^{a,c}(x) & \ldots & (-1)^m m_{n-m}^{a,c}(x) \\
\frac{a^m}{m_{h_1}^{1/a,2-c}}(-n-1) & \frac{a^{m-1}}{m_{h_1}^{1/a,2-c}}(-n) & \ldots & \frac{a^{m-1}}{m_{h_1}^{1/a,2-c}}(-n+m-1) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{a^m}{m_{h_{m_1}}^{1/a,2-c}}(-n-1) & \frac{a^{m-1}}{m_{h_{m_1}}^{1/a,2-c}}(-n) & \ldots & \frac{a^{m-1}}{m_{h_{m_1}}^{1/a,2-c}}(-n+m-1) \\
\frac{a^m}{m_{k_1}^{a,2-c}}(-n-1) & \frac{a^{m-1}}{m_{k_1}^{a,2-c}}(-n) & \ldots & \frac{a^{m-1}}{m_{k_1}^{a,2-c}}(-n+m-1) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{a^m}{m_{k_{m_2}}^{a,2-c}}(-n-1) & \frac{a^{m-1}}{m_{k_{m_2}}^{a,2-c}}(-n) & \ldots & \frac{a^{m-1}}{m_{k_{m_2}}^{a,2-c}}(-n+m-1)
\end{vmatrix}
\]

then the polynomials \((q_n)_n\) are **eigenfunctions** of a higher order difference operator \(D_q\) and they are **orthogonal** with respect to the measure

\[
\omega_{a,c}^{F_1,F_2} = \prod_{f \in F_1} (x + c + f) \prod_{f \in F_2} (x - f) \omega_{a,c}
\]


**Krawtchouk polynomials**

Again, for $F_1, F_2 \subset \mathbb{N}$ consider $K = I(F_1) = \{k_1, \ldots, k_{m_2}\}$ and $H = J_h(F_2) = \{h_1, \ldots, h_{m_1}\}$ with $m = m_1 + m_2$.

If we define $(q_n)_n$ by

$$q_n(x) = \begin{pmatrix}
(1 + a)^m k_n^{a,N}(x) & -(1 + a)^{m-1} k_n^{a,N}(x) & \cdots & (-1)^m k_n^{a,N}(x) \\
(1 + a)^m k_{n-1}^{a,-N}(x) & -(1 + a)^{m-1} k_{n-1}^{a,-N}(x) & \cdots & (-1)^m k_{n-1}^{a,-N}(x) \\
\vdots & \vdots & \ddots & \vdots \\
(1 + a)^m k_{n-m}^{a,-N}(x) & -(1 + a)^{m-1} k_{n-m}^{a,-N}(x) & \cdots & (-1)^m k_{n-m}^{a,-N}(x)
\end{pmatrix}$$

then the polynomials $(q_n)_n$ are **eigenfunctions** of a higher order difference operator $D_q$ and **orthogonal** with respect to the measure

$$\omega_{a,N}^{F_1,F_2} = \prod_{f \in F_1} (x - f) \prod_{f \in F_2} (N - 1 - f - x) \omega_{a,N}$$
Again, for $F_1, F_2 \subseteq \mathbb{N}$ consider $K = I(F_1) = \{k_1, \ldots, k_{m_2}\}$ and $H = J_h(F_2) = \{h_1, \ldots, h_{m_1}\}$ with $m = m_1 + m_2$. If we define $(q_n)_n$ by

$$q_n(x) = \begin{bmatrix}
(1 + a)^m k_n^{a,N}(x) & -(1 + a)^{m-1} k_n^{a,N}(x) & \cdots & (-1)^m k_{n-m}^{a,N}(x) \\
k_k^{a,-N}(-n-1) & k_k^{a,-N}(-n) & \cdots & k_k^{a,-N}(-n + m - 1) \\
\vdots & \vdots & \ddots & \vdots \\
k_{k_{m_1}}^{a,-N}(-n-1) & k_{k_{m_1}}^{a,-N}(-n) & \cdots & k_{k_{m_1}}^{a,-N}(-n + m - 1) \\
(-a)^m k_{h_1}^{1/a,-N}(-n-1) & (-a)^{m-1} k_{h_1}^{1/a,-N}(-n) & \cdots & k_{h_1}^{1/a,-N}(-n + m - 1) \\
\vdots & \vdots & \ddots & \vdots \\
(-a)^m k_{h_{m_2}}^{1/a,-N}(-n-1) & (-a)^{m-1} k_{h_{m_2}}^{1/a,-N}(-n) & \cdots & k_{h_{m_2}}^{1/a,-N}(-n + m - 1)
\end{bmatrix}$$

then the polynomials $(q_n)_n$ are eigenfunctions of a higher order difference operator $D_q$ and orthogonal with respect to the measure

$$\omega_{a,N}^{F_1,F_2} = \prod_{f \in F_1} (x - f) \prod_{f \in F_2} (N - 1 - f - x) \omega_{a,N}$$
**LAGUERRE POLYNOMIALS**

For $m \geq 1$, let $M = (M_{i,j})_{i,j=0}^{m-1}$ be any $m \times m$ matrix. For $\alpha \neq m - 1, m - 2, \ldots$, consider the discrete Laguerre-Sobolev bilinear form defined by

$$\langle p, q \rangle = \int_0^{\infty} p(x)q(x)x^{\alpha-m}e^{-x}dx + (p(0), \ldots, p^{(m-1)}(0))M \begin{pmatrix} q(0) \\ \vdots \\ q^{(m-1)}(0) \end{pmatrix}$$

Then the family $(q_n)_n$ defined by

$$q_n(x) = \begin{pmatrix} \frac{L_n^\alpha(x)}{\Omega(n)} \\ \frac{L_{n-1}^\alpha(x)}{\Omega(n)} \\ \vdots \\ \frac{L_{n-m}^\alpha(x)}{\Omega(n)} \end{pmatrix} = \begin{pmatrix} L_n^\alpha(x) \\ L_{n-1}^\alpha(x) \\ \vdots \\ L_{n-m}^\alpha(x) \end{pmatrix} + (p(0), \ldots, p^{(m-1)}(0))M \begin{pmatrix} q(0) \\ \vdots \\ q^{(m-1)}(0) \end{pmatrix}$$

is orthogonal with respect to the discrete Laguerre-Sobolev bilinear form, as long as $\Omega(n) = \det(\mathcal{R}_i(n-j))_{i,j=1}^{m} \neq 0, n \geq 0$, where

$$\mathcal{R}_i(x) = \frac{\Gamma(\alpha - m + l)}{(m - l)!} (x+1)_{m-l} + (l-1)! \frac{\Gamma(\alpha + 1 + x)}{\Gamma(1 + x)} \sum_{i=0}^{m-1} \frac{(-1)^i M_{i-1,i}}{\Gamma(\alpha + i + 1)} (x-i+1)_i$$

Observation: $\mathcal{R}_1(x), \ldots, \mathcal{R}_m(x)$ are not polynomials in general.
LAGUERRE POLYNOMIALS

For \( m \geq 1 \), let \( M = (M_{i,j})_{i,j=1}^{m-1} \) be any \( m \times m \) matrix. For \( \alpha \neq m - 1, m - 2, \ldots \), consider the discrete Laguerre-Sobolev bilinear form defined by

\[
\langle p, q \rangle = \int_0^\infty p(x)q(x)x^{\alpha-m}e^{-x}dx + (p(0), \ldots, p^{(m-1)}(0))M \begin{pmatrix} q(0) \\ \vdots \\ q^{(m-1)}(0) \end{pmatrix}
\]

Then the family \( (q_n)_n \) defined by

\[
q_n(x) = \frac{L_n^\alpha(x) - L_{n-1}^\alpha(x) - \cdots - L_{n-m}^\alpha(x)}{R_1(n) - R_1(n-1) - \cdots - R_1(n-m)}
\]

is orthogonal with respect to the discrete Laguerre-Sobolev bilinear form, as long as \( \Omega(n) = \det(R_i(n-j))_{i,j=1}^{m} \neq 0, n \geq 0 \), where

\[
R_l(x) = \frac{\Gamma(\alpha - m + l)}{(m - l)!} (x+1)_{m-l} + (l-1)! \frac{\Gamma(\alpha + 1 + x)}{\Gamma(1 + x)} \sum_{i=0}^{m-1} \frac{(-1)^i M_{l-1,i}}{\Gamma(\alpha + i + 1)} (x-i+1)_i
\]

Observation: \( R_1(x), \ldots, R_m(x) \) are not polynomials in general.
Laguerre Polynomials

Let \((L_n^\alpha)_n\) be the family of Laguerre polynomials and \(D_p\) the corresponding second-order differential equation such that \(D_p(L_n^\alpha) = nL_n^\alpha\).

Assume that \(\alpha\) is a positive integer with \(\alpha \geq m\).

Then there exists a differential operator \(D_q\) of the form

\[
D_q = P(D_p) + \sum_{h=1}^{m} M_h(D_p) \frac{d}{dx} \mathcal{R}_h(D_p),
\]

such that \(D_q(q_n) = P(n)q_n\) where

\[
P(x) - P(x - 1) = \Omega(x)
\]

and the polynomials \(M_h(x), h = 1, \ldots, m\) are defined by

\[
M_h(x) = \sum_{j=1}^{m} (-1)^{h+j} \det(\mathcal{R}_l(x + j - r)) \begin{cases} i \neq h \quad \{ \quad \} \\ r \neq j \end{cases}
\]

\[
\mathcal{R}_l(x) = \frac{(\alpha - m + l - 1)!}{(m - l)!} (x+1)^{m-l} + (l-1)! (x+1)^\alpha \sum_{i=0}^{m-1} \frac{(-1)^i M_{l-1,i}}{(\alpha + i)!} (x-i+1)^i
\]
LAGUERRE POLYNOMIALS

Let \((L_n^\alpha)_n\) be the family of Laguerre polynomials and \(D_p\) the corresponding second-order differential equation such that \(D_p(L_n^\alpha) = nL_n^\alpha\). Assume that \(\alpha\) is a positive integer with \(\alpha \geq m\). Then there exists a differential operator \(D_q\) of the form

\[
D_q = P(D_p) + \sum_{h=1}^{m} M_h(D_p) \frac{d}{dx} \mathcal{R}_h(D_p),
\]

such that \(D_q(q_n) = P(n)q_n\) where

\[
P(x) - P(x - 1) = \Omega(x)
\]

and the polynomials \(M_n(x), h = 1, \ldots, m\) are defined by

\[
M_h(x) = \sum_{j=1}^{m} (-1)^{h+j} \det (\mathcal{R}_l(x + j - r)) \{\begin{array}{cc}
  l \neq h \\
  r \neq j
\end{array}\}
\]

\[
\mathcal{R}_l(x) = \frac{\alpha - m + l - 1}{(m-l)}((x+1)^{m-l} + (l-1)!)(x+1)\alpha \sum_{i=0}^{m-1} \frac{(-1)^i M_{l-1,i}}{(\alpha + i)!} (x-i+1)i
\]
Moreover, the minimal order of the differential operator $D_q$ having the orthogonal polynomials $(q_n)_n$ as eigenfunctions is at most $2(\alpha\text{-wr}(M) + 1)$ where $\alpha\text{-wr}(M)$ is the $\alpha$-weighted rank of the matrix $M$, given by

$$
\alpha\text{-wr}(M) = \sum_{j=1}^{m} n_j + \sum_{j=1}^{m-1} m_j - \frac{m(m-1)}{2}
$$

The indexes $n_j$ and $m_j$ are related with how singular are the columns and the rows of the matrix $M$.

- When $M = (M_{i,j})_{i,j=0}^{m-1}$ is the symmetric matrix with entries $M_{i,j} = a_{i+j}$ for $i + j \leq m - 1$ and $M_{i,j} = 0$ for $i + j > m - 1$, the discrete Laguerre Sobolev inner product reduces

$$
x^{\alpha-m}e^{-x} + \sum_{i=0}^{m-1} a_i \delta_0^{(i)} , \quad \alpha\text{-wr}(M) = m\alpha
$$

- When $M$ is diagonal, $M = \text{diag}(M_0, \ldots, M_{m-1})$, $M_{m-1} \neq 0$, we have

$$
\alpha\text{-wr}(M) = s\alpha + (m-s)(m+1) - \sum_{1 \leq j \leq m, M_j \neq 0} j , \quad s = |\{j : 1 \leq j \leq m, M_j \neq 0\}|
$$
Moreover, the minimal order of the differential operator $D_q$ having the orthogonal polynomials $(q_n)_n$ as eigenfunctions is at most $2(\alpha\text{-wr}(M) + 1)$ where $\alpha\text{-wr}(M)$ is the $\alpha$-weighted rank of the matrix $M$, given by

$$\alpha\text{-wr}(M) = \sum_{j=1}^{m} n_j + \sum_{j=1}^{m-1} m_j - \frac{m(m-1)}{2}$$

The indexes $n_j$ and $m_j$ are related with how singular are the columns and the rows of the matrix $M$.

- When $M = (M_{i,j})_{i,j=0}^{m-1}$ is the symmetric matrix with entries $M_{i,j} = a_{i+j}$ for $i + j \leq m - 1$ and $M_{i,j} = 0$ for $i + j > m - 1$, the discrete Laguerre Sobolev inner product reduces

$$x^{\alpha-m} e^{-x} + \sum_{i=0}^{m-1} a_i \delta_0^{(i)}, \quad \alpha\text{-wr}(M) = ma$$

- When $M$ is diagonal, $M = \text{diag}(M_0, \ldots, M_{m-1})$, $M_{m-1} \neq 0$, we have

$$\alpha\text{-wr}(M) = s\alpha + (m-s)(m+1)-2 \sum_{1\leq j\leq m, M_j \neq 0} j, \ s = |\{j : 1 \leq j \leq m, M_j \neq 0\}|$$
Moreover, the minimal order of the differential operator $D_q$ having the orthogonal polynomials $(q_n)_n$ as eigenfunctions is at most $2(\alpha\text{-wr}(M) + 1)$ where $\alpha\text{-wr}(M)$ is the $\alpha$-weighted rank of the matrix $M$, given by

$$\alpha\text{-wr}(M) = \sum_{j=1}^{m} n_j + \sum_{j=1}^{m-1} m_j - \frac{m(m-1)}{2}$$

The indexes $n_j$ and $m_j$ are related with how singular are the columns and the rows of the matrix $M$.

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LAGUERRE POLYNOMIALS: EXPPLICIT EXAMPLE

Let $\alpha = 3$, $m = 3$ and $M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $R_1(x), R_2(x), R_3(x)$ are given by

$R_1(x) = -\frac{(x + 1)(x + 2)(x^2 - x - 24)}{24}$

$R_2(x) = -\frac{(x + 1)(x^3 + x^2 - 14x - 48)}{24}$

$R_3(x) = \frac{(x + 4)(x^4 + x^3 + x^2 - 9x + 30)}{60}$

The differential operator (of order 18) satisfying $D_q(q_n) = P(n)q_n$ is

$D_q = P(D_p) + \sum_{h=1}^{3} M_h(D_p) \frac{d}{dx} R_h(D_p)$

where

$P(x) = -\frac{x^9}{4320} + \frac{x^8}{480} - \frac{x^7}{144} - \frac{17x^6}{720} + \frac{47x^5}{480} - \frac{253x^4}{1440} + \frac{55x^3}{108} - \frac{289x^2}{360} - \frac{18x}{5}$
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**Conclusions and extensions**

**Conclusions**: Given a classical family of OP’s \((p_n)_n\) with a second-order difference or differential operator \(D_p\) such that \(D_p(p_n) = np_n\) (Charlier, Meixner, Krawtchouk and Laguerre) we can construct a new bispectral family of OP’s satisfying higher-order difference or differential operators.


**Future work**: Examples of the form \(D_p\) with \(D_p(p_n) = \theta_n p_n\), where \(\theta_n\) is any function of \(n\). The classical families to study in this case are the Jacobi (continuous) and Hahn (discrete).
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