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VALUES FOR GAMES WITH AUTHORIZATION STRUCTURE

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Tesis Doctoral

Values for games with authorization structure

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A mis padres,

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Chapter 1

Introduction

Although there is some earlier research on two-person games, it is generally considered that game theory was born as a field of mathematics with the paper published by John von Neumann [66] in 1928. Some years later, in 1944, the bases of this theory were established with the publication of *Theory of Games and Economic Behavior,* written by Oskar Morgenstern and John von Neumann [67]. In this book, it was shown that many social and economic situations can be described through strategic games, and these games can be studied by using tools of mathematical analysis.

In a general way, it can be said that game theory studies cooperation and conflict situations, using mathematical methods. This leads to the development of cooperative and noncooperative models. This research is focused on cooperative models.

1.1 Preliminaries and notation

Cooperative game theory deals with groups of players that aim to share the benefits derived from their cooperation. The cooperative model abstracts away from some details of the interaction among the players and describes only the outcomes that result when players cooperate within different coalitions. At this point we can consider two different approaches. In some situations, the outcome of each coalition is described by a real number. The games used in these cases are called transferable utility (TU) games. The adjective *transferable* refers to the assumption that a player can losslessly transfer any part of his utility to another player, usually through money, and that the players' utilities are linear in money with the same scale for all players. So we can think that the number assigned to a coalition in a TU game expresses an amount of money. If there is no possibility of transferring the utility between players by using money or, if there is, the utilities are not linear with the same scale with respect money, then we use nontransferable utility (NTU) games. In NTU games the possibilities of each coalition are represented by a set of utility vectors indexed by the members of the coalition.

1.1.1 Transferable utility games

A *transferable utility game* or TU game is a pair (N, v), where N is a set of cardinality n with $n \in \mathbb{N}$ and $v : 2^N \to \mathbb{R}$ is a function satisfying that $v(\emptyset) = 0$. The elements of N are called *players*, and the subsets of N are called *coalitions*. For each coalition E, the number $v(E)$, that is called the *worth* of E, can be interpreted as the maximal gain or minimal cost that the players in E can achieve when they decide to cooperate. Throughout this work, we interpret $v(E)$ as the maximal gain of each coalition. The function v is called *characteristic function* of the game. Usually, the TU game (N, v) is identified with the characteristic function v.

Depending on the properties of the characteristic function we can distinguish different types of games. A detailed description of these can be seen in Driessen [34]. Here, only those which are used throughout this work are defined.

If $v(E) \leq v(F)$ for all $E \subseteq F \subseteq N$ then the game v is said to be *monotonic*. If, in addition, v only takes values in the set $\{0, 1\}$, then the game is called *simple*.

A game v is *convex* if for any coalitions $E, F \subseteq N$ the following inequality is satisfied

$$v(E \cup F) + v(E \cap F) \geq v(E) + v(F).$$

Equivalently, a game v is convex if and only if for any coalitions $E, F \subseteq N$, such that $E \subseteq F$ and for all $i \in N \setminus F$, it holds that

$$v(E \cup \{i\}) - v(E) \leq v(F \cup \{i\}) - v(F).$$

Convex games were introduced by Shapley [64] and they are useful in several situations in economic sciences.

The set of all TU games on N is denoted by $\mathcal{G}^N$. Given $v, w \in \mathcal{G}^N$ the game $v + w$ is defined by $(v + w)(E) = v(E) + w(E)$ for all $E \subseteq N$. If $c \in \mathbb{R}$ the game cv is defined by $(cv)(E) = c\,v(E)$ for all $E \subseteq N$. With respect to these operations, $\mathcal{G}^N$ is a $2^n - 1$-dimensional real vector space. A well-known basis of this vector space is given by the set

$$\left\{u_F : F \in 2^N \setminus \{\emptyset\}\right\},$$

where

$$u_F(E) = \begin{cases} 1 & \text{if } F \subseteq E, \\ 0 & \text{otherwise.} \end{cases}$$

For each nonempty $F \subseteq N$ the game u_F is called the *unanimity game* of F. Every game $v \in \mathcal{G}^N$ can be written as a linear combination of unanimity games. Thus,

$$v = \sum_{F \in 2^N \setminus \{\emptyset\}} \triangle_v(F)\, u_F$$

where each coordinate $\triangle_v(F)$ of the game v with respect to the basis of the unanimity games is called *dividend of the coalition* F in the game v. For all $E \in 2^N \setminus \{\emptyset\}$ we have that

$$v(E) = \sum_{\{F \in 2^N \setminus \{\emptyset\} : F \subseteq E\}} \triangle_v(F).$$

Therefore, for each $E \in 2^N \setminus \{\emptyset\}$ it holds that

$$\triangle_v(E) = v(E) - \sum_{\{F \in 2^N \setminus \{\emptyset\} : F \subsetneq E\}} \triangle_v(F).$$

Solutions for TU games

Given a TU game (N, v), a problem that arises is how to assign a payoff to each player in a fair way. An *allocation rule* or *value* assigns to each game (N, v) a payoff vector $\psi(v) \in \mathbb{R}^N$. Many allocation rules have been defined in literature. The best known of them is the *Shapley value*, introduced by Shapley [62] in 1953. Given $v \in \mathcal{G}^N$, the Shapley value of v, denoted by $\phi(v)$, is defined by

$$\phi_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E \left[v(E) - v(E \setminus \{i\})\right] \quad \text{for all } i \in N,$$

where

$$p_E = \frac{(n - |E|)!\,(|E| - 1)!}{n!}$$

and $|E|$ denotes the cardinality of E.

The *Banzhaf value*, introduced by Owen [58] in 1975, arises from considering that each player is equally likely to join any coalition. Given $v \in \mathcal{G}^N$, the Banzhaf value of v, denoted by $\beta(v)$, is defined by

$$\beta_i(v) = \frac{1}{2^{n-1}} \sum_{\{E \subseteq N : i \in E\}} [v(E) - v(E \setminus \{i\})] \quad \text{for all } i \in N.$$

Some desirable properties for a value $\psi : \mathcal{G}^N \to \mathbb{R}^N$ are the following.

EFFICIENCY. For all $v \in \mathcal{G}^N$ it holds that

$$\sum_{k \in N} \psi_k(v) = v(N).$$

ADDITIVITY. For all $v_1, v_2 \in \mathcal{G}^N$ it holds that

$$\psi(v_1 + v_2) = \psi(v_1) + \psi(v_2).$$

A player $i \in N$ is said to be *a null player in* $v \in \mathcal{G}^N$ if $v(E) = v(E \setminus \{i\})$ for all $E \subseteq N$.

NULL PLAYER PROPERTY. If $i \in N$ is null player in $v \in \mathcal{G}^N$ it holds that

$$\psi_i(v) = 0.$$

A player i is said to be *a necessary player in* $v \in \mathcal{G}^N$ if $v(E) = 0$ for $E \subseteq N \setminus \{i\}$.

NECESSARY PLAYER PROPERTY. If i is a necessary player in a monotonic game $v \in \mathcal{G}^N$ it holds that

$$\psi_i(v) \geq \psi_j(v) \quad \text{for all } j \in N.$$

Let (N, v) be a TU game. Given two different players i and j, the amalgamation of i and j defines a new player denoted by $\widehat{ij}$. Let $N^{ij} = (N \setminus \{i,j\}) \cup \{\widehat{ij}\}$ and $v^{ij} : 2^{N^{ij}} \to \mathbb{R}$ defined by

$$v^{ij}(E) = \begin{cases} v(E) & \text{if } \widehat{ij} \notin E, \\ v\left(\left(E \setminus \{\widehat{ij}\}\right) \cup \{i,j\}\right) & \text{if } \widehat{ij} \in E. \end{cases}$$

AMALGAMATION. For all $v \in \mathcal{G}^N$ it holds that

$$\psi_i(v) + \psi_j(v) = \psi_{\widehat{ij}}\left(v^{ij}\right) \qquad \text{for all } i, j \in N.$$

The Shapley value satisfies all these properties except amalgamation whereas the Banzhaf value satisfies all of them except efficiency.

1.1.2 Non-transferable utility games

A *cooperative game with nontransferable utility* or NTU game is a pair (N, V) where N is a set of cardinality $n \in \mathbb{N}$ and V is a correspondence that assigns to each nonempty $E \subseteq N$ a nonempty subset $V(E) \subseteq \mathbb{R}^E$. The set valued function V is called the *characteristic function* of the NTU game (N, V). Usually, the NTU game (N, V) is identified with the characteristic function V.

If V and W are NTU games, the NTU game $V+W$ is defined by $(V+W)(E) = V(E) + W(E)$ for every nonempty $E \subseteq N$. If $\alpha \in \mathbb{R}^N$ the NTU game $\alpha * V$ is defined by $(\alpha * V)(E) = \alpha^E * V(E)$ for every nonempty $E \subseteq N$ where α^E is the restriction of α to E and $*$ denotes the Hadamard product. Given $v \in \mathcal{G}^N$ the NTU game *corresponding* to v is defined by

$$V_v(E) = \left\{ x \in \mathbb{R}^E : \sum_{k \in E} x_k \leqslant v(E) \right\} \qquad \text{for every nonempty } E \subseteq N.$$

Solutions for NTU games

The *Shapley NTU correspondence* was introduced by Shapley [63] and it was characterized by Aumann [7]. Aumann considered the NTU games V satisfying the following conditions:

(i) $V(E)$ is a convex comprehensive proper subset of $\mathbb{R}^E$ for all nonempty $E \subseteq N$.

(ii) $V(N)$ is smooth, i.e., it has a unique supporting hyperplane at each point of its boundary.

(iii) For every $x \in \partial(V(N))$ it holds $\{y \in \mathbb{R}^N : y \geqslant x\} \cap V(N) = \{x\}$.

(iv) There exists $x \in \mathbb{R}^N$ such than $V(E) \times \{0\}^{N \setminus E} \subseteq x + V(N)$ for every nonempty $E \subseteq N$.

We denote by $\hat{\Gamma}^N$ the set of NTU games satisfying (i), (ii), (iii) and (iv).

Let $V \in \hat{\Gamma}^N$. A vector $x \in \mathbb{R}^N$ is a *Shapley NTU payoff vector* of V if there exists $\lambda \in \mathbb{R}^N_{++}$ such that

1. $x \in \overline{V(N)}$,
2. The set $\left\{\lambda^E \cdot z : z \in V(E)\right\}$ is bounded above for every nonempty $E \subseteq N$,
3. $\lambda * x = \phi(v_\lambda)$ where v_λ is the TU game defined by

$$v_\lambda(E) = \sup\left\{\lambda^E \cdot z : z \in V(E)\right\} \quad \text{for every nonempty } E \subseteq N.$$

Let $SH : \hat{\Gamma}^N \to 2^{\mathbb{R}^N}$ be the mapping that assigns to each $V \in \hat{\Gamma}^N$ the set of Shapley NTU payoff vectors of V. The correspondence SH is called the *Shapley NTU correspondence* (*on* N). Aumann [7] proved that the Shapley NTU correspondence satisfies the following properties.

EFFICIENCY. For every $V \in \hat{\Gamma}^N$ it holds that

$$SH(V) \subseteq \partial(V(N)).$$

CONDITIONAL ADDITIVITY. For every $V, W \in \hat{\Gamma}^N$ such that $V + W = U \in \hat{\Gamma}^N$ it holds that

$$(SH(V) + SH(W)) \cap \partial(U(N)) \subseteq SH(U).$$

SCALE COVARIANCE. For every $V \in \hat{\Gamma}^N$ and $\alpha \in \mathbb{R}^N_{++}$ it holds that

$$SH(\alpha * V) = \alpha * SH(V).$$

INDEPENDENCE OF IRRELEVANT ALTERNATIVES. For every $V, W \in \hat{\Gamma}^N$ such that $V(N) \subseteq W(N)$ and $V(E) = W(E)$ for every nonempty $E \subsetneq N$, it holds that

$$SH(W) \cap V(N) \subseteq SH(V).$$

The *Harsanyi configuration correspondence for NTU games* was introduced by Harsanyi [42] and it was characterized by Hart [43]. Hart considered the NTU games V satisfying the following conditions:

(i) $V(E)$ is closed, convex and comprehensive for all nonempty $E \subseteq N$.

(ii) $V(N)$ is smooth.

(iii) For every $x \in \partial(V(N))$ it holds $\{y \in \mathbb{R}^N : y \geqslant x\} \cap V(N) = \{x\}$.

We denote by Ω^N the set of NTU games satisfying (i), (ii) and (iii).

A *payoff configuration* for N is an element $(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \prod_{E \in 2^N \setminus \{\emptyset\}} \mathbb{R}^E$. Let $V \in \Omega^N$. A payoff configuration $(x_E)_{E \in 2^N \setminus \{\emptyset\}}$ is a *Harsanyi solution* of V if there exists $\lambda \in \mathbb{R}^N_{++}$ such that

1. $x_E \in \partial (V(E))$ for every nonempty $E \subseteq N$,
2. $\lambda \cdot x_N = \max \{\lambda \cdot z : z \in V(N)\}$,
3. $\lambda^E * x_E = \phi(w_{|E})$ for every nonempty $E \subseteq N$, where w is the TU game given by $w(F) = \lambda^F \cdot x_F$ for every nonempty $F \subseteq N$.

The mapping that assigns to each $V \in \Omega^N$ the set of Harsanyi solutions of V is called the *Harsanyi configuration correspondence for NTU games* (*on* N) and is denoted by H. Hart [43] proved that H satisfies the following properties.

CONDITIONAL ADDITIVITY. For every $V, W \in \Omega^N$ such that $V + W = U \in \Omega^N$ and $(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in H(V) + H(W)$ such that $x_E \in \partial (U(E))$ for every nonempty $E \subseteq N$, it holds that

$$(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in H(U).$$

SCALE COVARIANCE. For every $V \in \Omega^N$ and $\alpha \in \mathbb{R}^N_{++}$ it holds that

$$H(\alpha * V) = \left\{ \left(\alpha^E * x_E\right)_{E \in 2^N \setminus \{\emptyset\}} : (x_E)_{E \in 2^N \setminus \{\emptyset\}} \in H(V) \right\}.$$

INDEPENDENCE OF IRRELEVANT ALTERNATIVES. Given $V, W \in \Omega^N$ such that $V(E) \subseteq W(E)$ for every nonempty $E \subseteq N$ and $(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in H(W)$ such that $x_E \in V(E)$ for every nonempty $E \subseteq N$, it holds that

$$(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in H(V).$$

1.1.3 Permission structures and restrictions

Permission structures were introduced by Gilles, Owen and van den Brink [40] to model hierarchical organizations. A *permission structure* on N is a mapping $S : N \to 2^N$. Given $i \in N$ the players in $S(i)$ are called the *successors of* i *in* S. The players in $P_S(i) = \{j \in N : i \in S(j)\}$ are called the *predecessors of* i *in* S. Let $\hat{S} : N \to 2^N$ denote the *transitive closure of* S, i.e., $j \in \hat{S}(i)$ if and only if there exists a sequence $\{i_p\}_{p=0}^{q}$ such that $i_0 = i$, $i_q = j$ and $i_p \in S(i_{p-1})$ for all $1 \leq p \leq q$. The players in $\hat{S}(i)$ are called the *subordinates of* i in S. We denote $\hat{P}_S(i) = \left\{j \in N : i \in \hat{S}(j)\right\}$. The players in $\hat{P}_S(i)$ are called the *superiors of* i in S. The collection of all permission structures on N is denoted by $\mathcal{S}^N$. A permission structure S on N can be identified with a directed graph (digraph) on N. The vertex set is N and the pair (i, j) is a link if $j \in S(i)$.

If $v \in \mathcal{G}^N$ and $S \in \mathcal{S}^N$ the pair (v, S) is said to be a *game with permission structure*. Different assumptions can be made about how a permission structure restricts the formation of coalitions. In the *conjunctive approach*, Gilles, Owen and van den Brink [40] assumed that every player needs the permission from all his superiors. So, if a coalition E is formed, a player in E is allowed to play if and only if all his superiors belong to E. The set of players who are allowed to play within coalition E is called the *conjunctive sovereign part of* E and is denoted by $A_c^S(E)$, i.e.,

$$A_c^S(E) = \left\{i \in E : \hat{P}_S(i) \subseteq E\right\}.$$

In the *disjunctive approach*, van den Brink [24] assumed that each player only needs the permission from one of his predecessors (if he has any). In this case, a coalition is *autonomous* if for any player in the coalition either he does not have any predecessors or at least one of his predecessors is in the coalition. The *disjunctive sovereign part of a coalition* E, denoted by $A_d^S(E)$, is the largest autonomous subset of E.

In order to find reasonable payoff vectors for games with permission structure, van den Brink and Gilles [27] proposed to modify the characteristic function $v \in \mathcal{G}^N$ taking account of the limited possibilities of cooperation determined by the permission structure $S \in \mathcal{S}^N$. The *conjunctive* and *disjunctive restricted games* are defined, respectively, as $v_c^S(E) = v\left(A_c^S(E)\right)$ and $v_d^S(E) = v\left(A_d^S(E)\right)$ for every coalition $E \subseteq N$. A *value for games with permission structure on* N is a mapping $\psi : \mathcal{G}^N \times \mathcal{S}^N \to \mathbb{R}^N$. The *conjunctive permission value* and the *disjunctive permission value* are defined as $\phi^c(v, S) = \phi\left(v_c^S\right)$ and $\phi^d(v, S) = \phi\left(v_d^S\right)$

respectively. In a similar way, van den Brink [26] introduced *Banzhaf permission values*. A complete description of permission structures can be seen in van den Brink [23].

Derks and Peters [33] generalized the model proposed by van den Brink. They considered the concept of *restriction*, that is defined as a mapping $\rho : 2^N \to 2^N$ satisfying

1. $\rho(E) \subseteq E$ for any $E \subseteq N$,
2. If $E \subset F$ then $\rho(E) \subseteq \rho(F)$,
3. $\rho(\rho(E)) = \rho(E)$.

They interpreted the coalitions in the image of ρ as the only coalitions in which all the players can cooperate freely. We could also interpret $\rho(E)$ as the set of players that are allowed to play within coalition E. With each restriction ρ, they associated the so-called *restricted Shapley value* ψ^ρ, defined by $\psi^\rho(v) = \phi\,(v \circ \rho)$ for every $v \in \mathcal{G}^N$.

1.1.4 Fuzzy sets. The Choquet integral

Fuzzy sets were introduced by Zadeh [71]. A *fuzzy subset* of N is an element of $[0,1]^N$. Given e a fuzzy subset of N and $i \in N$, the number e_i is called the *grade of membership* of i in e. The *support* of e is the set $supp\,(e) = \{i \in N : e_i > 0\}$. For every $t \in [0,1]$ the *t-cut* of e is the set

$$[e]_t = \{i \in N : \, e_i \geqslant t\}.$$

Given $e, f \in [0,1]^N$, the fuzzy sets $e \cup f$ and $e \cap f$ are defined, respectively, as

$$(e \cap f)_i = \min\,(e_i, f_i) \qquad \text{for every } i \in N,$$

and

$$(e \cup f)_i = \max\,(e_i, f_i) \qquad \text{for every } i \in N.$$

The fuzzy subset e is contained in f, which is denoted by $e \subseteq f$, if $e_i \leqslant f_i$ for every $i \in N$.
A *fuzzy relation on* N is a fuzzy subset of $N \times N$.

In cooperative game theory, Aubin [6] defined a *fuzzy coalition* in N to be a fuzzy subset of N, where for all $i \in N$ the number $e_i \in [0,1]$ is regarded as the degree of participation of player i in e.

Every coalition $E \subseteq N$ can be identified with the fuzzy coalition $\mathbf{1}_E \in [0,1]^N$ defined by

$$(\mathbf{1}_E)_i = \begin{cases} 1 & \text{if } i \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The Choquet integral was introduced by Choquet [32] to deal with capacities. Later on it was extended to set functions by Schmeidler [61]. Given $v : 2^N \to \mathbb{R}$ and $e \in [0,1]^N$, the *Choquet integral of* e *with respect to* v is

$$\int e\, dv = \sum_{p=1}^{q} (s_p - s_{p-1})\, v\left([e]_{s_p}\right),$$

where $\{s_0, \ldots, s_q\} = \{e_i : i \in N\} \cup \{0\}$ with $0 = s_0 < \ldots < s_q$.

1.2 Historical background

1.2.1 Games with restricted cooperation

In a general model of cooperative games it is assumed that there are no restrictions in the cooperation among the players, and, therefore, every subset of players can form a coalition. In real life, however, political, social or economic circumstances may impose certain restraints on coalition formation. This idea has led several authors to develop models of cooperative games with partial cooperation.

One of the first approximations to partial cooperation is due to Aumann and Maschler [9]. They used coalition structures to define some solution notions for TU games. A *coalition structure* is a partition $\mathcal{B} = \{B_1, \ldots, B_k\}$ of the set N of players such that the cooperation is possible only among the players that belong to an element B_i. Later, in 1974, Aumann and Dréze [8] introduced the concept of value for games with coalition structure. Other authors who have developed this line of research are Owen [59], Hart and Kurz [44], Levy and McLean [48], Winter [68] [69] [70] and McLean [50].

In 1977, Myerson [53], in his seminal work *Graphs and Cooperation in Games*, presented a new class of games with partial cooperation structure. He considered a TU game (N, v) and a graph G on the set of players. The links of G represent the bilateral agreements among the players. The pair

(v, G) is called a *communication situation*. Given a coalition E, two players in E will be able to cooperate within the coalition only if they are connected in E, that is, if there is a path between the players that stays within E. In order to get together the information from the game and the graph, a new characteristic function v^G is defined

$$v^G(E) = \sum_{F \in E/G} v(F) \quad \text{for all } E \subseteq N,$$

where E/G is the set of connected components of the subgraph of G induced by E. The number $v^G(E)$ represents the worth of E when we consider the game v and the requirement that players can only communicate along links in G. By considering the Shapley value of this new game, Myerson obtains an allocation rule for communication situations, known as the *Myerson value*

$$\mu(v, G) = \phi(v^G).$$

Myerson characterized this allocation rule through certain properties of fairness and stability. The model of partial cooperation introduced by Myerson gave rise to a line of research that was followed, amongst others, by Owen [60], Hamiache [41], van den Nouweland [55], van den Nouweland and Borm [56], Carreras [30], van den Nouweland, Borm and Tijs [57], Borm, Owen and Tijs [22], Bergantiños, Carreras and García Jurado [10], Calvo and Lasaga [29] and Fernández, Algaba, Bilbao, Jiménez-Losada, Jiménez and López [37].

In 1980, in his work *Conference Structures and Fair Allocation Rules*, Myerson [54] proposed a generalization of his model of partial cooperation. This generalization arises from the fact that the graph of a communication situation divides the set of coalitions into two classes. If the subgraph induced by a coalition E is connected, then E is a feasible coalition, in the sense that all the players in E can join together and cooperate regardless of the actions taken by the rest of players. If the subgraph induced by E is not connected, then E is a non-feasible coalition. Abstracting away from the graph, Myerson considered a structure that just indicates whether a coalition is feasible or not. In this way, he covered a wider range of games with partial cooperation. Following this idea, Bilbao [11] and López [49] developed a model of partial cooperation based on the so called *systems of feasible coalitions* and *partition systems*, which generalized the communication situations. A system of feasible coalitions is a collection $\mathcal{F}$ of subsets of N that contains the empty set and the singletons. A partition system is a system of feasible coalitions such that for every coalition E the maximal elements in the collection of feasible coalitions contained in E form a partition of E. If a pair $(v, \mathcal{F})$ is considered, where v is a TU game on N and $\mathcal{F}$ is a partition system on N, a new characteristic function that collects the information from both the game and the structure is introduced. This characteristic function, called the *restricted game*, is defined as

$$v^{\mathcal{F}}(E) = \sum_{F \in C_E^{\mathcal{F}}} v(F) \quad \text{for all } E \subseteq N,$$

where $C_E^{\mathcal{F}}$ denotes the partition of E formed by the maximal elements in the collection of feasible coalitions contained in E. Similarly as in the definition of the Myerson value, Bilbao and López considered the Shapley value of the restricted game, thus obtaining an allocation rule for games with partition system.

Notice that in the models proposed by Myerson and Bilbao the values of the game v on the non feasible coalitions are irrelevant. This is connected to another model of partial cooperation that was initiated by Faigle [36]. In this model, a game with restricted cooperation is a pair $(\mathcal{F}, v)$ where $\mathcal{F}$ is the set of feasible coalitions, without any a priori structure, and v is a real function defined on $\mathcal{F}$. Although Faigle extended v to a collection possibly larger than $\mathcal{F}$, he does not necessarily get a characteristic function on 2^N. This is an essential difference with respect to the previous models, which are based on obtaining a new game (from 2^N into $\mathbb{R}$) gathering the information from the original game and the structure.

Faigle's work gave rise to a new line of research, in which his model is assumed and, additionally, it is supposed that the family of feasible coalitions has a certain combinatorial structure. In this line of research different types of combinatorial structures have been considered, like *closure spaces* (see Bilbao [12], Jiménez [46], Bilbao, Lebrón and Jiménez [21]), *convex geometries* (see Bilbao and Edelman [15], Bilbao, Jiménez-Losada and López [18] and Bilbao, Lebrón and Jiménez [20], [19]) or *matroids* (see Jiménez-Losada [47], Bilbao, Jiménez-Losada, Lebrón and Tijs [17], Bilbao, Jiménez-Losada and Lebrón [16] and Bilbao, Driessen, Jiménez-Losada and Lebrón [14]).

In 1992, Gilles, Owen and van den Brink [40] presented another type of cooperation restriction. They considered situations in which some players have veto power over the actions undertaken by some other players. In order to model these situations, they introduced *games with permission structure*, that consist of a cooperative TU game and a mapping that assigns to each player a subset of direct subordinates. Depending on the interpretation of the superior-subordinate relationship, they considered two different approaches. In the conjunctive approach [40] it is assumed that every player needs the permission from all his superiors, whereas in the disjunctive approach [24] the permission from any of his superiors is sufficient. In each case, proceeding in a similar way as Myerson did with communication situations, they defined a new characteristic function, gathering the information given by the game and the structure, that allowed them to define a value for games with conjunctive or, respectively, disjunctive permission structures. They provided intuitive characterizations for each case, showing in this way that the values obtained are reasonable (see van den Brink and Gilles [27] and van den Brink [24], [26]).

Algaba, Bilbao, van den Brink and Jiménez-Losada [3], [4], [5] and Chacón [31] studied games on *antimatroids* as a generalization of the model of Gilles, Owen and van den Brink. In Bilbao [13], games on antimatroids are considered as a particular case of games on *augmenting systems*.

In order to calculate the values introduced by Gilles, Owen and van den Brink, what we really need to know about the corresponding permission structure is the so called sovereign part of each coalition, that is, the set of players that are allowed to play within each coalition. Considering this fact, Derks and Peters [33] abstracted away from the hierarchical nature of permission structures and introduced the more general concept of *restriction*, which is a monotonic projection $\rho : 2^N \rightarrow 2^N$ assigning to each coalition E the subcoalition $\rho(E)$ of players who can play when coalition E is formed. Given a TU game (N, v) and a restriction $\rho : 2^N \rightarrow 2^N$ they considered the restricted game $v \circ \rho$. By applying the Shapley value to this new game they obtained a value for games with restricted coalitions.

1.2.2 Fuzzy coalitions

In the previous models, the dependency relationships are complete in the sense that, when a coalition is formed, a player in the coalition either can fully cooperate within the coalition or he cannot cooperate at all. However, there are situations in which a player has a degree of freedom to cooperate within a coalition. The concept of fuzzy coalition, introduced by Aubin [6], is useful for modelling this kind of dependency relationships.

A critical issue arises when dealing with TU games and fuzzy coalitions: how to assign a worth to a fuzzy coalition. In his seminal paper, Aubin proposed an optimal value, also studied by Jiaquan and

Qiang [45]. Butnariu [28] assumed that different players should have the same membership grade in order to cooperate and provided a different way to assign a gain to a fuzzy coalition. Tsurumi, Tanino and Inuiguchi [65], by using the Choquet integral, came up with a reasonable method to extend a game to the set of fuzzy coalitions.

1.3 Synthesis of contents

Our aim in this work is to propose a new model of games with restricted cooperation.

In chapter 2 we present the so called authorization structures, which extend the concept of restriction introduced by Derks and Peters [33]. We define and characterize a Shapley value and a Banzhaf value for games with authorization structure.

In chapter 3 we aim to model situations in which the dependency relationships among the players are not complete. To that end we introduce fuzzy authorization structures. A Shapley value and a Banzhaf value for games with fuzzy authorization structure are obtained and characterized. To do this, we follow the approach described by Tsurumi, Tanino and Inuiguchi [65], using the Choquet integral to define a new auxiliary game that combines the information from the original game and from the fuzzy dependency relationships.

In chapter 4 the power in authorization structures is studied. Using the Shapley value and the Banzhaf value we measure how favorable the situation of each agent in an authorization structure is. In both cases, a fuzzy digraph is assigned to each authorization structure, measuring the dependence or the dominance relationship between any two agents. Moreover, a characterization of those measures is given. In a similar way, the power in fuzzy authorization structures is also discussed.

In chapter 5 a particular type of authorization structure, called interior operator structure, is analyzed. Interior operator structures are characterized in terms of transitivity of the veto relationships, and a Shapley value for games with interior operator structure is studied. Because of the transitivity of the veto relationships, this value turns out to satisfy a property of structural monotonicity. Finally, fuzzy interior operator structures are studied.

In chapter 6 we focus on the most simple authorization structures: conjunctive authorization structures. We characterize them by proving that an authorization structure is conjunctive if and only if all the dependency relationships induced are bilateral. A simplified expression of the Shapley power measures defined in chapter 4 is given for these structures. We give a Shapley value and a Banzhaf value for games with conjunctive authorization structure. We also study fuzzy conjunctive authorization structures.

In chapter 7 we deal with NTU games. NTU games with authorization structure are introduced, and a Shapley solution is obtained. We analyze the fuzzy case as well. Lastly, a Harsanyi solution for NTU games with interior operator structure is presented.

Chapter 2

Games with authorization structure

Different approaches have been developed to model games with permission relationships among the agents. In 1992, Gilles, Owen and van den Brink [40] introduced conjunctive permission structures. Later, van den Brink [24] introduced disjunctive permission structures. Subsequently, van den Brink and Gilles [27] and van den Brink [24] provided and characterized a Shapley value for, respectively, conjunctive and disjunctive permission structures. Derks and Peters [33] generalized those approaches by abstracting away from hierarchies and considering the so called restrictions. On the one hand, the model considered by Derks and Peters is more general, but on the other hand the axiomatizations given in [27] and [24] are somewhat more intuitive than that given in [33]. Our aim in this chapter will be to provide a new model for games with permission relationships among the players. This new model will fulfill two requirements. Firstly, it will more general than the one given in [33]. And, secondly, it will allow us to define and characterize a sharing value in a similar way as in [27] and [24].

2.1 Authorization structures

Let $n \in \mathbb{N}$ and let N be a set of cardinality n. We think of N as a set of players in a cooperative game.

Definition 2.1 *An authorization operator on N is a mapping $A : 2^N \to 2^N$ that satisfies the following conditions:*

1. *$A(E) \subseteq E$ for any $E \subseteq N$,*
2. *If $E \subset F$ then $A(E) \subseteq A(F)$.*

The pair (N, A) is called an authorization structure. The set of all authorization operators on N is denoted by $\mathcal{A}^N$.

If $A \in \mathcal{A}^N$ and $E \subseteq N$ we interpret $A(E)$ as the set of players that will be allowed to play when coalition E is formed. Bearing this in mind, the two conditions considered in the definition of authorization operator seem to be reasonable.

Definition 2.2 *An authorization operator A is said to be normal if $A(N) = N$. The set of normal authorization operators is denoted by $\widetilde{\mathcal{A}}^N$.*

Note that the concept of authorization operator is more general than that of restriction introduced by Derks and Peters [33]. If A is a restriction on N then the coalitions $A(E)$ are actually autonomous, that is, $A(A(E)) = A(E)$ for any $E \subseteq N$, which, in general, is not true for authorization operators. Therefore, restrictions are special cases of authorization operators.

Example 2.3 Imagine the following situation. A consumer electronics company wants to make a new product. To do this, the company needs several components from various suppliers. We will focus on three of those suppliers. For $i = 1, 2, 3$, supplier i produces component i. The company has signed an agreement with the three suppliers that establishes the following:

- The company will pay i dollars to supplier i for every unit of component i delivered before the deadline.
- The company will pay a total of $2(i + j)$ dollars to suppliers i and j for every pair made up of a unit of component i and a unit of component j delivered before the deadline.
- The company will pay a total of 20 dollars to the three suppliers for every set made up of a unit of each component delivered before the deadline.

Each supplier has calculated that it would be able to produce one million units of the corresponding component before the deadline. This situation can be modeled with a cooperative game (N, v), where $N = \{1, 2, 3\}$ and, for every $E \subseteq N$, $v(E)$ is the revenue (in millions) obtained by coalition E,

$$v(\{1\}) = 1, \quad v(\{2\}) = 2, \quad v(\{3\}) = 3,$$
$$v(\{1,2\}) = 6, \quad v(\{1,3\}) = 8, \quad v(\{2,3\}) = 10,$$
$$v(\{1,2,3\}) = 20.$$

Imagine now the following. Supplier 2 finds out that supplier 1 makes use, in the production of component 1, of a technology developed and patented by 2. So 2 sues 1 for violation of patent rights. Supplier 3 preferred to avoid a patent war, but, in view of the maneuvers of 2, decides to sue 2. The final scenario is this: supplier 1 cannot use or sell the component they make without the authorization from supplier 2, and the latter cannot use or sell their component without the authorization from supplier 3. Let us consider an operator A that assigns to each coalition $E \subseteq N$ the set of players in E that don't need the authorization from a player in $N \setminus E$. We can represent A with the following table

E	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$A(E)$	$\emptyset$	$\emptyset$	$\{3\}$	$\{1\}$	$\{3\}$	$\{2,3\}$	$\{1,2,3\}$

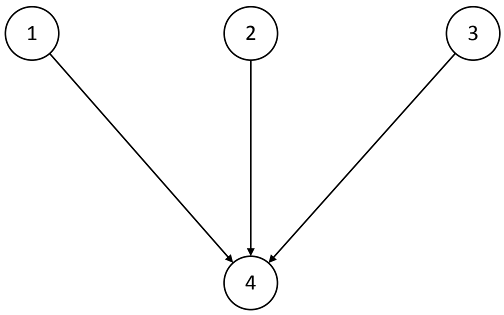
It is clear that A is a normal authorization operator and is not a restriction. Notice that only player 1 would be allowed to play within coalition $\{1,2\}$. But observe that player 1 needs the permission from player 2. In other words, the key point is that within coalition $\{1,2\}$ player 2 cannot play but gives authorization to play. That role would not be possible in the case of a restriction.

We know that both conjunctive and disjunctive permission structures are particular cases of restrictions, and hence authorization structures. Look at the following example.

Example 2.4 On the table below we have defined three different authorization operators A, B and C on $N = \{1,2,3,4\}$.

E	$A(E)$	$B(E)$	$C(E)$
$E : 4 \notin E$	E	E	E
$\{4\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1,4\}$	$\{1\}$	$\{1,4\}$	$\{1\}$
$\{2,4\}$	$\{2\}$	$\{2,4\}$	$\{2\}$
$\{3,4\}$	$\{3\}$	$\{3,4\}$	$\{3\}$
$\{1,2,4\}$	$\{1,2\}$	$\{1,2,4\}$	$\{1,2,4\}$
$\{1,3,4\}$	$\{1,3\}$	$\{1,3,4\}$	$\{1,3,4\}$
$\{2,3,4\}$	$\{2,3\}$	$\{2,3,4\}$	$\{2,3,4\}$
$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$

Let us consider the following digraph



It is plain to see that, for any $E \subseteq N$, $A(E)$ is equal to the sovereign part of E in the conjunctive hierarchy represented by the digraph above. Similarly, $B(E)$ is equal to the sovereign part of E in the disjunctive hierarchy represented by the same digraph. Finally, the structure determined by C is also hierarchical (and induced by the digraph above), but neither conjunctive nor disjunctive. In this structure, a coalition is autonomous (note that C is, as well as A and B, a restriction) if for any element in the coalition, the majority of his predecessors are in the coalition too.

Definition 2.5 *A game with authorization structure on N is a pair (v, A) where $v \in \mathcal{G}^N$ and $A \in \mathcal{A}^N$.*

Given a game with authorization structure, we can define a characteristic function that gathers the information from the game and the structure in a reasonable way.

Definition 2.6 *Let $v \in \mathcal{G}^N$ and $A \in \mathcal{A}^N$. The restricted game of (v, A) is the game $v^A \in \mathcal{G}^N$ given by*

$$v^A(E) = v(A(E)) \quad \textit{for all } E \subseteq N.$$

The number $v^A(E)$ is the worth of E in the game with authorization structure (v, A).

Example 2.7 Let us calculate the restricted game of the game with authorization structure (v, A) given in Example 2.3.

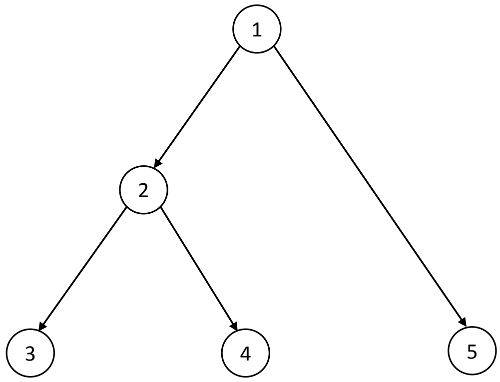
$$\begin{aligned}
v^A(\{1\}) &= v(\emptyset) = 0, \\
v^A(\{2\}) &= v(\emptyset) = 0, \\
v^A(\{3\}) &= v(\{3\}) = 3, \\
v^A(\{1,2\}) &= v(\{1\}) = 1, \\
v^A(\{1,3\}) &= v(\{3\}) = 3, \\
v^A(\{2,3\}) &= v(\{2,3\}) = 10, \\
v^A(\{1,2,3\}) &= v(\{1,2,3\}) = 20.
\end{aligned}$$

Let us finish this section with another example. We study the following situation proposed by van den Brink [25].

Example 2.8 Let $N = \{1, 2, 3, 4, 5\}$ be the set of employees of a firm. This firm is hierarchically structured. The firm structure $S : N \to 2^N$ describes the hierarchical structure of the firm,

$$S(1) = \{2, 5\}, \;\; S(2) = \{3, 4\}, \;\; S(3) = S(4) = S(5) = \emptyset.$$

This firm structure is illustrated in the following digraph



The employees $3, 4$ and 5 are referred to as the workers. We assume that they operate the production process in the firm. The other employees are the managers or coordinators, who do not actively produce but coordinate. In order to produce, a worker must be coordinated by all his superiors in the structure. The set of workers is denoted by W_S, and the set of managers is denoted by M_S. So we have that

$$W_S = \{3, 4, 5\}, \;\; M_S = \{1, 2\}.$$

Finally we have a production game $w : 2^{W_S} \to \mathbb{R}_+$ that describes the potential production possibilities of the workers in the firm. The value $w(E)$ is the nonnegative production output value that can be generated if exactly the workers in $E \subseteq W_S$ are active in the production process. In this example we consider the production game given by

$$w(E) = |E|^2 \quad \text{for all } E \subseteq W_S.$$

We aim to give a fair payoff vector, that will be used to determine the wages of the employees of the firm. The situation described can be modeled with a game with authorization structure (v, A), where $v : 2^N \to \mathbb{R}$ is given by

$$v(E) = w(E \cap W_S) \quad \text{for all } E \subseteq N,$$

and the authorization structure A is given by

$$A(E) = \left\{ i \in W : \hat{S}^{-1}(i) \subseteq E \right\} \quad \text{for all } E \subseteq N,$$

where, for each $i \in N$, $\hat{S}^{-1}(i)$ denotes the set of superiors of i in the firm structure.

Notice that the restricted game v^A is given by

$$v^A(E) = w(A(E)) \quad \text{for all } E \subseteq N.$$

So we have that

$$\begin{aligned}
&v^A(\{1,5\}) = w(\{5\}) = 1, \\
&v^A(\{1,2,3\}) = w(\{3\}) = 1, \\
&v^A(\{1,2,4\}) = w(\{4\}) = 1, \\
&v^A(\{1,2,5\}) = v^A(\{1,3,5\}) = v^A(\{1,4,5\}) = w(\{5\}) = 1, \\
&v^A(\{1,2,3,4\}) = w(\{3,4\}) = 4, \\
&v^A(\{1,2,3,5\}) = w(\{3,5\}) = 4, \\
&v^A(\{1,2,4,5\}) = w(\{4,5\}) = 4, \\
&v^A(\{1,3,4,5\}) = w(\{5\}) = 1, \\
&v^A(\{1,2,3,4,5\}) = w(\{3,4,5\}) = 9, \\
&v^A(E) = w(\emptyset) = 0 \quad \text{for any other } E \subseteq N.
\end{aligned}$$

2.2 The Shapley authorization value

An allocation rule for games with authorization structure assigns to every game with authorization structure a payoff vector. In this section we define and characterize an allocation rule for games with authorization structure.

Definition 2.9 *The Shapley authorization value, denoted by Φ, assigns to each game with authorization structure (v, A) the Shapley value of v^A,*

$$\Phi(v, A) = \phi\left(v^A\right) \quad \textit{for all } v \in \mathcal{G}^N \textit{ and } A \in \mathcal{A}^N.$$

We aim to characterize the Shapley authorization value. To that end, we consider the following properties.

- **EFFICIENCY**. For every $v \in \mathcal{G}^N$ and $A \in \mathcal{A}^N$ it holds that

$$\sum_{k \in N} \Psi_k(v, A) = v(A(N)).$$

- **ADDITIVITY**. For every $v, w \in \mathcal{G}^N$ and $A \in \mathcal{A}^N$ it holds that

$$\Psi(v + w, A) = \Psi(v, A) + \Psi(w, A).$$

It is well known that the Shapley value satisfies the so-called null-player property. That is, for every $v \in \mathcal{G}^N$ and $i \in N$ such that i is a null player in v it holds that $\phi_i(v) = 0$. If we want to determine an "adequate" allocation rule for games with authorization structure, it would not be a good idea to look for an allocation rule satisfying that property. That is due to the fact that we have to take the structure as well into consideration, since a player could make profit not only by playing, but also by giving authorization to play. Suppose that we have $v \in \mathcal{G}^N$, an authorization structure on N and $i \in N$ a null player in v. There might be other players depending on the authorization from i, in which case player i could still reasonably expect to be rewarded. But if all players depending on i are also null players in v, i should not expect anything but a zero-payoff. That is the irrelevant player property, that we state next. We need some previous definitions.

Definition 2.10 *Let $A \in \mathcal{A}^N$ and $i, j \in N$. A player j depends partially on i in (N, A) if there exists $E \subseteq N$ such that $j \in A(E) \setminus A(E \setminus \{i\})$.*

Definition 2.11 *Let $v \in \mathcal{G}^N$, $A \in \mathcal{A}^N$ and $i \in N$. A player i is irrelevant in (v, A) if for every $j \in N$ such that j depends partially on i in (N, A) it holds that j is a null player in v.*

- **IRRELEVANT PLAYER PROPERTY**. For every $v \in \mathcal{G}^N$, $A \in \mathcal{A}^N$ and $i \in N$ such that i is an irrelevant player in (v, A), it holds that

$$\Psi_i(v, A) = 0.$$

Let $v \in \mathcal{G}^N$ and $j \in N$. Recall that the player j is a necessary player in v if $v(E) = 0$ for every $E \subseteq N \setminus \{j\}$. We know that for every monotonic game $v \in \mathcal{G}^N$ and j a necessary player in v it holds that $\phi_j(v) \geqslant \phi_k(v)$ for all $k \in N$. The allocation rule we are looking for should satisfy this "natural" property too. In fact, it should satisfy something more. Suppose that we have $v \in \mathcal{G}^N$, an authorization structure on N, $j \in N$ a necessary player in v and $i \in N$ such that player j cannot play without the authorization from player i. It seems logical to expect that no player will receive more than player i. That is the property we state below.

Definition 2.12 *Let $A \in \mathcal{A}^N$ and $i, j \in N$. A player i has veto power over j in (N, A) if $j \notin A(N \setminus \{i\})$.*

- PROPERTY OF VETO POWER OVER A NECESSARY PLAYER. For every monotonic $v \in \mathcal{G}^N$, $A \in \mathcal{A}^N$ and $i, j \in N$ such that j is a necessary player in v and i has veto power over j in (N, A), it holds that

$$\Psi_i(v, A) \geqslant \Psi_k(v, A) \quad \text{for all } k \in N.$$

Although at first sight the following fairness property might seem a little contrived, it is actually quite natural and intuitive. Suppose that we have a game and an authorization structure on N, $T \subseteq N$ and $i \in T$ such that in case coalition T were formed i could not play. Imagine now that somehow coalition T acquires the power to authorize i to play. It seems reasonable to think that all the players in T will benefit equally from that fact. That's what the fairness property states.
Let $A \in \mathcal{A}^N$, $T \in 2^N \setminus \{\emptyset\}$ and $i \in T$. We define

$$\begin{array}{rccl} A^{T,i}: & 2^N & \to & 2^N \\ & E & \to & A^{T,i}(E) = \begin{cases} A(E) & \text{if } T \not\subseteq E, \\ A(E) \cup \{i\} & \text{if } T \subseteq E. \end{cases} \end{array}$$

It is clear that $A^{T,i} \in \mathcal{A}^N$.

- FAIRNESS. For every $v \in \mathcal{G}^N$, $A \in \mathcal{A}^N$, $T \in 2^N \setminus \{\emptyset\}$ and $i \in T$ it holds that

$$\Psi_j\left(v, A^{T,i}\right) - \Psi_j(v, A) = \Psi_i\left(v, A^{T,i}\right) - \Psi_i(v, A) \quad \text{for all } j \in T.$$

Notice that if $i \in A(T)$ then $A^{T,i} = A$. Therefore, the expression above is non trivial only if $i \notin A(T)$.

In the following results we show that the five properties seen before uniquely determine the Shapley authorization value.

Theorem 2.13 *The Shapley authorization value satisfies the properties of efficiency, additivity, irrelevant player, veto power over a necessary player and fairness.*

Proof. We are going to show that the Shapley authorization value satisfies the five properties.

EFFICIENCY. Let $v \in \mathcal{G}^N$ and $A \in \mathcal{A}^N$. It holds that

$$\sum_{k \in N} \Phi_k(v, A) = \sum_{k \in N} \phi_k\left(v^A\right) = v^A(N) = v(A(N)).$$

ADDITIVITY. Let $v, w \in \mathcal{G}^N$ and $A \in \mathcal{A}^N$. It is easy to check that $(v+w)^A = v^A + w^A$. We can derive that

$$\Phi(v+w, A) = \phi\left((v+w)^A\right) = \phi(v^A + w^A) = \phi(v^A) + \phi(w^A) = \Phi(v, A) + \Phi(w, A).$$

IRRELEVANT PLAYER PROPERTY. Let $v \in \mathcal{G}^N$, $A \in \mathcal{A}^N$ and $i \in N$ an irrelevant player in (v, A). We must prove that $\Phi_i(v, A) = 0$. Taking into consideration that $\Phi_i(v, A) = \phi_i\left(v^A\right)$ and that ϕ satisfies the null-player property, it is enough to show that i is a null player in v^A. For that, take $E \subseteq N$. We have to show that $v^A(E) = v^A\left(E \setminus \{i\}\right)$.

First, notice that

$$A(E) \setminus A(E \setminus \{i\}) \subseteq \{j \in N : j \text{ depends partially on } i \text{ in } (N, A)\}.$$

Since i is an irrelevant player in (v, A), every player that depends partially on i in (N, A) is a null player in v. So we can derive that

$$A(E) \setminus A(E \setminus \{i\}) \subseteq \{j \in N : j \text{ is a null player in } v\}$$

and, hence

$$v(A(E)) = v\left(A\left(E \setminus \{i\}\right)\right)$$

or, equivalently

$$v^A(E) = v^A\left(E \setminus \{i\}\right).$$

PROPERTY OF VETO POWER OVER A NECESSARY PLAYER. Let $v \in \mathcal{G}^N$ be a monotonic game, $A \in \mathcal{A}^N$ and $i, j \in N$ such that j is a necessary player in v and i has veto power over j in (N, A). We must prove that $\Phi_i(v, A) \geqslant \Phi_k(v, A)$ for all $k \in N$. Recall that given a monotonic $w \in \mathcal{G}^N$ and $i \in N$ a necessary player in w it holds that $\phi_i(w) \geqslant \phi_k(w)$ for all $k \in N$. Keeping that in mind and the fact that $\Phi(v, A) = \phi\left(v^A\right)$, it is enough to prove that $v^A \in \mathcal{G}^N$ is a monotonic game and i is a necessary player in v^A. Firstly, the monotonicity of v^A derives directly from the monotonicity of v and the definition of authorization operator. It only remains to prove that i is a necessary player in v^A. Since v^A is monotonic it is enough to prove that $v^A\left(N \setminus \{i\}\right) = 0$. As i has veto power over j in (N, A), it holds that $j \notin A\left(N \setminus \{i\}\right)$. From this and the fact that j is necessary in v we obtain that $v\left(A\left(N \setminus \{i\}\right)\right) = 0$, or, equivalently, $v^A\left(N \setminus \{i\}\right) = 0$.

FAIRNESS. Let $v \in \mathcal{G}^N$, $A \in \mathcal{A}^N$, $T \in 2^N \setminus \{\emptyset\}$ and $i, j \in T$. We must prove that

$$\Phi_j\left(v, A^{T,i}\right) - \Phi_j(v, A) = \Phi_i\left(v, A^{T,i}\right) - \Phi_i(v, A). \tag{2.1}$$

Let's focus on the left-hand side.

$$\Phi_j\left(v, A^{T,i}\right) - \Phi_j(v, A) = \phi_j\left(v^{A^{T,i}}\right) - \phi_j\left(v^A\right)$$

$$= \sum_{\{E \subseteq N:\, j \in E\}} p_E\left[v\left(A^{T,i}(E)\right) - v\left(A^{T,i}(E \setminus \{j\})\right) - v(A(E)) + v(A(E \setminus \{j\}))\right],$$

where the numbers p_E are the coefficients of the Shapley value. As $j \in T$, it is clear, from the definition of $A^{T,i}$ that $A^{T,i}(E \setminus \{j\}) = A(E \setminus \{j\})$ for all $E \subseteq N$. So the sum above is equal to

$$\sum_{\{E \subseteq N:\, j \in E\}} p_E\left[v\left(A^{T,i}(E)\right) - v(A(E))\right].$$

Notice that if we take $E \subseteq N$ such that $T \nsubseteq E$ it holds that $A^{T,i}(E) = A(E)$. So we can write the expression above as

$$\sum_{\{E \subseteq N:\, T \subseteq E\}} p_E \left[v\left(A^{T,i}(E)\right) - v(A(E)) \right].$$

We have shown that

$$\Phi_j\left(v, A^{T,i}\right) - \Phi_j(v, A) = \sum_{\{E \subseteq N:\, T \subseteq E\}} p_E \left[v\left(A^{T,i}(E)\right) - v(A(E)) \right]. \tag{2.2}$$

Following a similar reasoning as above, we can also prove that

$$\Phi_i\left(v, A^{T,i}\right) - \Phi_i(v, A) = \sum_{\{E \subseteq N:\, T \subseteq E\}} p_E \left[v\left(A^{T,i}(E)\right) - v(A(E)) \right]. \tag{2.3}$$

From (2.2) and (2.3) we conclude (2.1). □

We have already seen that the Shapley authorization value satisfies the five properties. Now we see that such properties uniquely determine the Shapley authorization value.

Theorem 2.14 *An allocation rule for games with authorization structure is equal to the Shapley authorization value if it satisfies the properties of efficiency, additivity, irrelevant player, veto power over a necessary player and fairness.*

Proof. Let Ψ be an allocation rule for games with authorization structure satisfying the properties of efficiency, additivity, irrelevant player, veto power over a necessary player and fairness. We must prove that $\Psi = \Phi$.

Let $n \in \mathbb{N}$ and let N be a set of cardinality n. Our first goal will be to show that $\Psi(cu_E, A) = \Phi(cu_E, A)$ for all $c > 0$, $E \in 2^N \setminus \{\emptyset\}$ and $A \in \mathcal{A}^N$. So, take $c > 0$ and $E \in 2^N \setminus \{\emptyset\}$. We want to see that

$$\Psi(cu_E, A) = \Phi(cu_E, A) \quad \text{for all } A \in \mathcal{A}^N. \tag{2.4}$$

Firstly, we define

$$m(A) = \sum_{F \subseteq N} |A(F)| \quad \text{for every } A \in \mathcal{A}^N.$$

We prove (2.4) by induction on $m(A)$.

1. BASE CASE. Let $A \in \mathcal{A}^N$ be such that $m(A) = 0$. In this case all players are irrelevant. Since Ψ and Φ satisfy the irrelevant player property, it holds that $\Phi_i(cu_E, A) = \Psi_i(cu_E, A)$ for every $i \in N$.

2. INDUCTIVE STEP. Let $A \in \mathcal{A}^N$. We consider the three following sets:

$$H_1 = \{i \in N : i \text{ is an irrelevant player in } (cu_E, A)\},$$
$$H_2 = \{i \in N : \text{there exists } j \in E \text{ such that } i \text{ has veto power over } j \text{ in } (N, A)\},$$
$$H_3 = N \setminus (H_1 \cup H_2).$$

Since Φ and Ψ satisfy the irrelevant player property it holds that

$$\Phi_i(cu_E, A) = 0 \quad \text{for all } i \in H_1, \tag{2.5}$$
$$\Psi_i(cu_E, A) = 0 \quad \text{for all } i \in H_1. \tag{2.6}$$

And from the property of veto power over a necessary player we can derive that there exist $b, b' \in \mathbb{R}$ such that

$$\Phi_i(cu_E, A) = b \quad \text{for all } i \in H_2, \tag{2.7}$$
$$\Psi_i(cu_E, A) = b' \quad \text{for all } i \in H_2. \tag{2.8}$$

Now suppose that $i \in H_3$. Since $i \notin H_1$ there must exist $j \in E$ such that j depends partially on i in (N, A). This means that there exists $F \subseteq N$ such that $j \in A(F) \setminus A(F \setminus \{i\})$. Notice that $F \neq N$, since otherwise i would have veto power over j and this would contradict $i \notin H_2$. Take K minimal such that $K \subseteq F$ and $j \in A(K)$. It is clear that $i \in K$. We define

$$\begin{array}{rccl} B: & 2^N & \to & 2^N \\ & T & \to & B(T) = \begin{cases} A(T) & \text{if } T \neq K, \\ A(K) \setminus \{j\} & \text{if } T = K. \end{cases} \end{array}$$

It is straightforward to check that $B \in \mathcal{A}^N$ and $B^{K,j} = A$. By using the fairness property we obtain

$$\begin{aligned}\Phi_i\,(cu_E,A)-\Phi_i(cu_E,B) &= \Phi_j\,(cu_E,A)-\Phi_j(cu_E,B),\\ \Psi_i\,(cu_E,A)-\Psi_i(cu_E,B) &= \Psi_j\,(cu_E,A)-\Psi_j(cu_E,B).\end{aligned}$$

Since $j\in E\subseteq H_2$ we know that $\Phi_j(cu_E,A)=b$ and $\Psi_j(cu_E,A)=b'$. If we substitute those values into the equalities above we have

$$\begin{aligned}\Phi_i\,(cu_E,A) &= b+\Phi_i(cu_E,B)-\Phi_j(cu_E,B),\\ \Psi_i\,(cu_E,A) &= b'+\Psi_i(cu_E,B)-\Psi_j(cu_E,B).\end{aligned}$$

As $m(B)=m(A)-1$, it follows by induction hypothesis that $\Psi(cu_E,B)=\Phi(cu_E,B)$. Therefore,

$$\begin{aligned}\Phi_i\,(cu_E,A) &= b+\Phi_i(cu_E,B)-\Phi_j(cu_E,B),\\ \Psi_i\,(cu_E,A) &= b'+\Phi_i(cu_E,B)-\Phi_j(cu_E,B),\end{aligned}$$

and hence

$$\Phi_i\,(cu_E,A)-\Psi_i\,(cu_E,A)=b-b'.$$

So we have proved that

$$\Phi_i\,(cu_E,A)-\Psi_i\,(cu_E,A)=b-b' \quad \text{for all } i\in H_3. \tag{2.9}$$

Now, on the one hand, from (2.5), (2.6), (2.7), (2.8) and (2.9), we can obtain

$$\sum_{i\in N}\Phi_i(cu_E,A)-\sum_{i\in N}\Psi_i(cu_E,A)=(b-b')|H_2\cup H_3|,$$

and, on the other hand, as Φ and Ψ are efficient we know that

$$\sum_{i\in N}\Phi_i(cu_E,A)=\sum_{i\in N}\Psi_i(cu_E,A).$$

Therefore, it follows that $(b-b')|H_2\cup H_3|=0$. Since $E\subseteq H_2, E\neq\emptyset$, it holds that $b=b'$, what leads to $\Psi(cu_E, A) = \Phi(cu_E, A)$.

So we have seen that $\Psi(cu_E, A) = \Phi(cu_E, A)$ for all $c > 0$, $E \in 2^N \setminus \{\emptyset\}$ and $A \in \mathcal{A}^N$.

Let $c > 0$, $E \in 2^N \setminus \{\emptyset\}$ and $A \in \mathcal{A}^N$. It holds that

$$\Psi(cu_E, A) + \Psi(-cu_E, A) = \Psi((c-c)u_E, A) = 0,$$

where we have respectively used additivity and the irrelevant player property. We conclude that

$$\Psi(-cu_E, A) = -\Psi(cu_E, A).$$

We can write

$$\Psi(-cu_E, A) = -\Psi(cu_E, A) = -\Phi(cu_E, A) = \Phi(-cu_E, A).$$

So we already know that $\Psi(cu_E, A) = \Phi(cu_E, A)$ for all $c \in \mathbb{R}$, $E \in 2^N \setminus \{\emptyset\}$ and $A \in \mathcal{A}^N$. Finally, take $v \in \mathcal{G}^N$ and $A \in \mathcal{A}^N$. It holds that

$$\begin{aligned}\Psi(v, A) &= \Psi\left(\sum_{E \in 2^N \setminus \{\emptyset\}} \Delta_v(E) u_E, A\right) = \sum_{E \in 2^N \setminus \{\emptyset\}} \Psi\left(\Delta_v(E) u_E, A\right) \\ &= \sum_{E \in 2^N \setminus \{\emptyset\}} \Phi\left(\Delta_v(E) u_E, A\right) = \Phi\left(\sum_{E \in 2^N \setminus \{\emptyset\}} \Delta_v(E) u_E, A\right) = \Phi(v, A).\end{aligned}$$

□

Example 2.15 Let us calculate $\Phi(v, A)$ where v and A are those defined in Example 2.3. We already calculated the restricted game v^A in Example 2.7. Now we use the definition of the Shapley authorization value

$$\Phi(v, A) = \phi(v^A) = (3.5, 7, 9.5).$$

We can obtain an expression of the Shapley authorization value that does not involve the restricted game, but the game and the authorization operator separately.

Corollary 2.16 *Let $v \in \mathcal{G}^N$ and $A \in \mathcal{A}^N$. Then, if $\Phi(v, A)$ is considered as a column matrix, it holds that*

$$\Phi(v, A) = Z_A \cdot \Delta_v$$

where Z_A is the matrix in $\mathcal{M}_{n,2^n-1}(\mathbb{R})$ defined by $(Z_A)_{i,E} = \Phi_i(u_E, A)$ for every $i \in N$ and $E \in 2^N \setminus \{\emptyset\}$ and Δ_v is the column matrix given by the Harsanyi dividends of v.

Proof. Making use of the linearity of the Shapley authorization value we can write

$$\begin{aligned}\Phi_i(v, A) &= \Phi_i\left(\sum_{E\in 2^N\setminus\{\emptyset\}} \Delta_v(E)u_E, A\right)\\ &= \sum_{E\in 2^N\setminus\{\emptyset\}} \Delta_v(E)\Phi_i\left(u_E, A\right)\\ &= \sum_{E\in 2^N\setminus\{\emptyset\}} (Z_A)_{i,E}\,\Delta_v(E).\end{aligned}$$

□

Example 2.17 Let us use the expression given in the preceding result to calculate $\Phi(v, A)$ in Example 2.3. Firstly we calculate the characteristic function of u_E^A for each nonempty $E \subseteq \{1, 2, 3\}$.

F	$u_{\{1\}}^A(F)$	$u_{\{2\}}^A(F)$	$u_{\{3\}}^A(F)$	$u_{\{1,2\}}^A(F)$	$u_{\{1,3\}}^A(F)$	$u_{\{2,3\}}^A(F)$	$u_{\{1,2,3\}}^A(F)$
$\{1\}$	0	0	0	0	0	0	0
$\{2\}$	0	0	0	0	0	0	0
$\{3\}$	0	0	1	0	0	0	0
$\{1,2\}$	1	0	0	0	0	0	0
$\{1,3\}$	0	0	1	0	0	0	0
$\{2,3\}$	0	1	1	0	0	1	0
$\{1,2,3\}$	1	1	1	1	1	1	1

We calculate $\Phi(u_E, A)$ for each nonempty $E \subseteq N$

$$\begin{array}{lclcl}
\Phi\left(u_{\{1\}}, A\right) & = & \phi\left(u^A_{\{1\}}\right) & = & \left(\frac{1}{2}, \frac{1}{2}, 0\right) \\
\Phi\left(u_{\{2\}}, A\right) & = & \phi\left(u^A_{\{2\}}\right) & = & \left(0, \frac{1}{2}, \frac{1}{2}\right) \\
\Phi\left(u_{\{3\}}, A\right) & = & \phi\left(u^A_{\{3\}}\right) & = & (0, 0, 1) \\
\Phi\left(u_{\{1,2\}}, A\right) & = & \phi\left(u^A_{\{1,2\}}\right) & = & \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \\
\Phi\left(u_{\{1,3\}}, A\right) & = & \phi\left(u^A_{\{1,3\}}\right) & = & \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \\
\Phi\left(u_{\{2,3\}}, A\right) & = & \phi\left(u^A_{\{2,3\}}\right) & = & \left(0, \frac{1}{2}, \frac{1}{2}\right) \\
\Phi\left(u_{\{1,2,3\}}, A\right) & = & \phi\left(u^A_{\{1,2,3\}}\right) & = & \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)
\end{array}$$

and we write Z_A

$$Z_A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \end{pmatrix}.$$

Now we calculate Δ_v

$$\Delta_v = \begin{pmatrix} \Delta_v(\{1\}) \\ \Delta_v(\{2\}) \\ \Delta_v(\{3\}) \\ \Delta_v(\{1,2\}) \\ \Delta_v(\{1,3\}) \\ \Delta_v(\{2,3\}) \\ \Delta_v(\{1,2,3\}) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 4 \\ 5 \\ 2 \end{pmatrix}.$$

Finally,

$$\Phi(v,A) = Z_A \cdot \Delta_v = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 4 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} \\ 7 \\ \frac{19}{2} \end{pmatrix}.$$

2.3 The Banzhaf authorization value

In the previous section we used the Shapley value to provide an allocation rule for games with authorization structure. In this section we use the Banzhaf value to give another allocation rule for these games.

Definition 2.18 *The Banzhaf authorization value, denoted by* $\mathfrak{B}$*, assigns to each game with authorization structure* (v, A) *the Banzhaf value of* v^A*,*

$$\mathfrak{B}(v,A) = \beta\left(v^A\right) \qquad \textit{for all } v \in \mathcal{G}^N \textit{ and } A \in \mathcal{A}^N.$$

In order to characterize the Banzhaf authorization value we consider the following properties for an allocation rule $\Psi : \mathcal{G}^N \times \mathcal{A}^N \to \mathbb{R}^N$.

- **2-EFFICIENCY**. For every $v \in \mathcal{G}^N$, $A \in \mathcal{A}^N$ and $i, j \in N$ such that every player in $N \setminus \{i, j\}$ is an irrelevant player in (v, A), it holds that

$$\sum_{k \in N} \Psi_k(v,A) = v(A(N)).$$

Suppose $|N| \geqslant 2$. Let i, j be two different players in N. We denote $\widehat{ij}$ a new player, i.e., $\widehat{ij} \notin N$. Let $N^{ij} = (N \setminus \{i,j\}) \cup \{\widehat{ij}\}$.

Let $A \in \mathcal{A}^N$. We define $A^{ij} : 2^{N^{ij}} \longrightarrow 2^{N^{ij}}$ as follows

$$A^{ij}(E) = \begin{cases} A(E) & \text{if } \widehat{ij} \notin E, \\ \left[A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \setminus \{i,j\}\right] \cup \{\widehat{ij}\} & \text{if } \widehat{ij} \in E \text{ and } \{i,j\} \subseteq A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right), \\ A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \setminus \{i,j\} & \text{if } \widehat{ij} \in E \text{ and } \{i,j\} \not\subseteq A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right). \end{cases}$$

It is easy to check that $A^{ij} \in \mathcal{A}^{N^{ij}}$.

Let $v \in \mathcal{G}^N$. We define $v^{ij} \in \mathcal{G}^{N^{ij}}$ as follows

$$v^{ij}(E) = \begin{cases} v(E) & \text{if } \widehat{ij} \notin E, \\ v\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) & \text{if } \widehat{ij} \in E. \end{cases}$$

Definition 2.19 *Let $v \in \mathcal{G}^N$, $A \in \mathcal{A}^N$ and let i, j be two different players in N. The players i and j can be amalgamated in (v, A) if for every $E \subseteq N$ such that $\{i,j\} \subseteq E$ and $\{i,j\} \not\subseteq A(E)$ it holds that*

$$v(A(E)) = v(A(E) \setminus \{i,j\}).$$

- **AMALGAMATION.** For every $v \in \mathcal{G}^N$, $A \in \mathcal{A}^N$ and i, j two different players in N such that i, j can be amalgamated in (v, A), it holds that

$$\Psi_i(v, A) + \Psi_j(v, A) = \Psi_{\widehat{ij}}(v^{ij}, A^{ij}).$$

In the following results we characterize the Banzhaf authorization value.

Theorem 2.20 *The Banzhaf authorization value satisfies the properties of additivity, irrelevant player, veto power over a necessary player, fairness, 2-efficiency and amalgamation.*

Proof. That the Banzhaf authorization value satisfies additivity, irrelevant player, veto power over a necessary player and fairness can be proved in a similar way as we did for the Shapley authorization value in Theorem 2.13. Let us see that the Banzhaf authorization value satisfies the properties of 2-efficiency and amalgamation.

2-EFFICIENCY. Let $v \in \mathcal{G}^N$, $A \in \mathcal{A}^N$ and $i, j \in N$ be such that every player in $N \setminus \{i, j\}$ is an irrelevant player in (v, A). We distinguish two cases.

Case $i = j$. We consider two possibilities:

(a) i is a null player in v. The result derives easily from the fact that all players in $A(N)$ are null players in v.

(b) i is not a null player in v. Taking into consideration that the Banzhaf authorization value satisfies the irrelevant player property, it suffices to prove that $\mathfrak{B}_i(v, A) = v(A(N))$. Since all players in $N \setminus \{i\}$ are irrelevant players in (v, A) and i is not a null player in v, we conclude that i does not depend partially on any player in $N \setminus \{i\}$ in (N, A). From this we conclude that

$$A(E) \cap \{i\} = A(\{i\}) \quad \text{for all } E \subseteq N \text{ with } i \in E. \tag{2.10}$$

Now, taking into consideration that all players in $A(N) \setminus \{i\}$ are null players in v we can write

$$\begin{aligned}
\mathfrak{B}_i(v, A) &= \beta_i\left(v^A\right) = \frac{1}{2^{n-1}} \sum_{\{E \subseteq N:\, i \in E\}} \left[v(A(E)) - v\left(A\left(E \setminus \{i\}\right)\right)\right] \\
&= \frac{1}{2^{n-1}} \sum_{\{E \subseteq N:\, i \in E\}} v(A(E) \cap \{i\}),
\end{aligned}$$

and, using (2.10), we have that

$$\mathfrak{B}_i(v, A) = \frac{1}{2^{n-1}} \sum_{\{E \subseteq N:\, i \in E\}} v(A(\{i\})) = v(A(\{i\})) = v(A(N) \cap \{i\}) = v(A(N)).$$

Case $i \neq j$. Taking into consideration that the Banzhaf authorization value satisfies the irrelevant player property, it suffices to prove that

$$\mathfrak{B}_i(v, A) + \mathfrak{B}_j(v, A) = v(A(N)).$$

It holds that

$$\mathfrak{B}_i(v,A)+\mathfrak{B}_j(v,A)=\beta_i\left(v^A\right)+\beta_j\left(v^A\right)$$

$$\begin{aligned}
&=\frac{1}{2^{n-1}}\sum_{E\subseteq N\setminus\{i\}}\left[v\left(A\left(E\cup\{i\}\right)\right)-v(A(E))\right]\\
&\quad+\frac{1}{2^{n-1}}\sum_{E\subseteq N\setminus\{j\}}\left[v\left(A\left(E\cup\{j\}\right)\right)-v(A(E))\right]\\
&=\frac{1}{2^{n-1}}\sum_{\{E\subseteq N\setminus\{i\}:\,j\in E\}}\left[v\left(A\left(E\cup\{i\}\right)\right)-v(A(E))\right]\\
&\quad+\frac{1}{2^{n-1}}\sum_{\{E\subseteq N\setminus\{i\}:\,j\notin E\}}\left[v\left(A\left(E\cup\{i\}\right)\right)-v(A(E))\right]\\
&\quad+\frac{1}{2^{n-1}}\sum_{\{E\subseteq N\setminus\{j\}:\,i\in E\}}\left[v\left(A\left(E\cup\{j\}\right)\right)-v(A(E))\right]\\
&\quad+\frac{1}{2^{n-1}}\sum_{\{E\subseteq N\setminus\{j\}:\,i\notin E\}}\left[v\left(A\left(E\cup\{j\}\right)\right)-v(A(E))\right].
\end{aligned}\tag{2.11}$$

In order to evaluate this sum, we distinguish several cases:

(a) Neither i nor j is a null player in v. Taking into account that all players in $A(N)\setminus\{i,j\}$ are null players in v, the sum (2.11) is equal to

$$\begin{aligned}
&\frac{1}{2^{n-1}}\sum_{\{E\subseteq N\setminus\{i\}:\,j\in E\}}\left[v\left(A\left(E\cup\{i\}\right)\cap\{i,j\}\right)-v\left(A(E)\cap\{j\}\right)\right]\\
&+\frac{1}{2^{n-1}}\sum_{E\subseteq N\setminus\{i,j\}}v\left(A\left(E\cup\{i\}\right)\cap\{i\}\right)\\
&+\frac{1}{2^{n-1}}\sum_{\{E\subseteq N\setminus\{j\}:\,i\in E\}}\left[v\left(A\left(E\cup\{j\}\right)\cap\{i,j\}\right)-v\left(A(E)\cap\{i\}\right)\right]\\
&+\frac{1}{2^{n-1}}\sum_{E\subseteq N\setminus\{i,j\}}v\left(A\left(E\cup\{j\}\right)\cap\{j\}\right).
\end{aligned}\tag{2.12}$$

Since the players in $N\setminus\{i,j\}$ are irrelevant players in (v,A) and i and j are not null players in v, it follows that neither i nor j depends partially on any player in $N\setminus\{i,j\}$. This implies that

$$A(F) \cap \{i,j\} = A(\{i,j\}) \text{ for all } F \subseteq N \text{ with } \{i,j\} \subseteq F,$$

$$A(G) \cap \{i\} = A(\{i\}) \text{ for all } G \subseteq N \text{ with } i \in G \subseteq N \setminus \{j\},$$

and

$$A(H) \cap \{j\} = A(\{j\}) \text{ for all } H \subseteq N \text{ with } j \in H \subseteq N \setminus \{i\}.$$

So the sum (2.12) is equal to

$$\begin{aligned}
& \frac{1}{2^{n-1}} \sum_{\{E \subseteq N \setminus \{i\} : j \in E\}} \left[v\left(A\left(\{i,j\}\right)\right) - v\left(A\left(\{j\}\right)\right)\right] + \frac{1}{2^{n-1}} \sum_{E \subseteq N \setminus \{i,j\}} v\left(A\left(\{i\}\right)\right) \\
& + \frac{1}{2^{n-1}} \sum_{\{E \subseteq N \setminus \{j\} : i \in E\}} \left[v\left(A\left(\{i,j\}\right)\right) - v\left(A\left(\{i\}\right)\right)\right] + \frac{1}{2^{n-1}} \sum_{E \subseteq N \setminus \{i,j\}} v\left(A\left(\{j\}\right)\right) \\
= \; & \frac{1}{2}\left[v\left(A\left(\{i,j\}\right)\right) - v\left(A\left(\{j\}\right)\right)\right] + \frac{1}{2} v\left(A\left(\{i\}\right)\right) \\
& + \frac{1}{2}\left[v\left(A\left(\{i,j\}\right)\right) - v\left(A\left(\{i\}\right)\right)\right] + \frac{1}{2} v\left(A\left(\{j\}\right)\right) \\
= \; & v\left(A\left(\{i,j\}\right)\right) = v\left(A(N) \cap \{i,j\}\right) = v(A(N)).
\end{aligned}$$

(b) i is not a null player in v and j is a null player in v. Taking into account that all players in $A(N) \setminus \{i\}$ are null players in v, the sum (2.11) is equal to

$$\begin{aligned}
& \frac{1}{2^{n-1}} \sum_{\{E \subseteq N \setminus \{i\} : j \in E\}} v\left(A\left(E \cup \{i\}\right) \cap \{i\}\right) \\
& + \frac{1}{2^{n-1}} \sum_{E \subseteq N \setminus \{i,j\}} v\left(A\left(E \cup \{i\}\right) \cap \{i\}\right) \\
& + \frac{1}{2^{n-1}} \sum_{\{E \subseteq N \setminus \{j\} : i \in E\}} \left[v\left(A\left(E \cup \{j\}\right) \cap \{i\}\right) - v\left(A(E) \cap \{i\}\right)\right].
\end{aligned} \tag{2.13}$$

Since the players in $N \setminus \{i,j\}$ are irrelevant players in (v, A) and i is not a null player in v we conclude that i does not depend partially on any player in $N \setminus \{i,j\}$. This implies that

$$A(F) \cap \{i\} = A(\{i,j\}) \cap \{i\} \text{ for all } F \subseteq N \text{ with } \{i,j\} \subseteq F$$

and

$$A(G) \cap \{i\} = A(\{i\}) \text{ for all } G \subseteq N \text{ with } i \in G \subseteq N \setminus \{j\}.$$

So the sum (2.13) is equal to

$$\begin{aligned} & \frac{1}{2^{n-1}} \sum_{\{E \subseteq N \setminus \{i\} : \, j \in E\}} v\left(A\left(\{i,j\}\right) \cap \{i\}\right) + \frac{1}{2^{n-1}} \sum_{E \subseteq N \setminus \{i,j\}} v\left(A\left(\{i\}\right)\right) \\ & + \frac{1}{2^{n-1}} \sum_{\{E \subseteq N \setminus \{j\} : \, i \in E\}} \left[v\left(A\left(\{i,j\}\right) \cap \{i\}\right) - v\left(A\left(\{i\}\right)\right)\right] \\ = \; & \frac{1}{2} v\left(A\left(\{i,j\}\right) \cap \{i\}\right) + \frac{1}{2} v\left(A\left(\{i\}\right)\right) + \frac{1}{2}\left[v\left(A\left(\{i,j\}\right) \cap \{i\}\right) - v\left(A\left(\{i\}\right)\right)\right] \\ = \; & v\left(A\left(\{i,j\}\right) \cap \{i\}\right) = v\left(A(N) \cap \{i\}\right) = v(A(N)). \end{aligned}$$

(c) i is a null player in v and j is not a null player in v. Analogous to the previous case.

(d) i and j are null players in v. The result derives easily from the fact that all players in $A(N)$ are null players in v.

Amalgamation. Let $v \in \mathcal{G}^N$, $A \in \mathcal{A}^N$ and let i, j be two different players in N such that i and j can be amalgamated in (v, A). We must prove that

$$\mathfrak{B}_i(v,A) + \mathfrak{B}_j(v,A) = \mathfrak{B}_{\widehat{ij}}(v^{ij}, A^{ij}). \tag{2.14}$$

Let's start from the right-hand side.

$$\mathfrak{B}_{\widehat{ij}}(v^{ij}, A^{ij}) = \frac{1}{2^{n-2}} \sum_{E \subseteq N \setminus \{i,j\}} \left[v^{ij}\left(A^{ij}\left(E \cup \{\widehat{ij}\}\right)\right) - v^{ij}\left(A^{ij}\left(E\right)\right)\right]$$

$$= \frac{1}{2^{n-2}} \sum_{\{E \subseteq N \setminus \{i,j\} : \, \{i,j\} \subseteq A(E \cup \{i,j\})\}} \left[v^{ij}\left(A^{ij}\left(E \cup \{\widehat{ij}\}\right)\right) - v^{ij}\left(A^{ij}\left(E\right)\right)\right]$$

$$
\begin{aligned}
&+\frac{1}{2^{n-2}} \sum_{\{E\subseteq N\setminus\{i,j\}:\,\{i,j\}\nsubseteq A(E\cup\{i,j\})\}} \left[v^{ij}\left(A^{ij}\left(E\cup\{\widehat{ij}\}\right)\right)-v^{ij}\left(A^{ij}\left(E\right)\right)\right] \\
&=\frac{1}{2^{n-2}} \sum_{\{E\subseteq N\setminus\{i,j\}:\,\{i,j\}\subseteq A(E\cup\{i,j\})\}} \left[v^{ij}\left(\left(A\left(E\cup\{i,j\}\right)\setminus\{i,j\}\right)\cup\{\widehat{ij}\}\right)-v^{ij}\left(A\left(E\right)\right)\right] \\
&+\frac{1}{2^{n-2}} \sum_{\{E\subseteq N\setminus\{i,j\}:\,\{i,j\}\nsubseteq A(E\cup\{i,j\})\}} \left[v^{ij}\left(A\left(E\cup\{i,j\}\right)\setminus\{i,j\}\right)-v^{ij}\left(A\left(E\right)\right)\right] \\
&=\frac{1}{2^{n-2}} \sum_{\{E\subseteq N\setminus\{i,j\}:\,\{i,j\}\subseteq A(E\cup\{i,j\})\}} \left[v\left(A\left(E\cup\{i,j\}\right)\right)-v\left(A\left(E\right)\right)\right] \\
&+\frac{1}{2^{n-2}} \sum_{\{E\subseteq N\setminus\{i,j\}:\,\{i,j\}\nsubseteq A(E\cup\{i,j\})\}} \left[v\left(A\left(E\cup\{i,j\}\right)\setminus\{i,j\}\right)-v\left(A\left(E\right)\right)\right].
\end{aligned}
$$

Now, as i and j can be amalgamated in (v, A), we know that for every $E \subseteq N \setminus \{i,j\}$ such that $\{i,j\} \nsubseteq A(E \cup \{i,j\})$ it holds

$$v\left(A\left(E\cup\{i,j\}\right)\setminus\{i,j\}\right)=v\left(A\left(E\cup\{i,j\}\right)\right).$$

Substituting into the sum above we obtain

$$
\begin{aligned}
\mathfrak{B}_{\widehat{ij}}(v^{ij},A^{ij}) &= \frac{1}{2^{n-2}} \sum_{\{E\subseteq N\setminus\{i,j\}:\,\{i,j\}\subseteq A(E\cup\{i,j\})\}} \left[v\left(A\left(E\cup\{i,j\}\right)\right)-v\left(A\left(E\right)\right)\right] \\
&\quad+\frac{1}{2^{n-2}} \sum_{\{E\subseteq N\setminus\{i,j\}:\,\{i,j\}\nsubseteq A(E\cup\{i,j\})\}} \left[v\left(A\left(E\cup\{i,j\}\right)\right)-v\left(A\left(E\right)\right)\right] \\
&= \frac{1}{2^{n-2}} \sum_{E\subseteq N\setminus\{i,j\}} \left[v\left(A\left(E\cup\{i,j\}\right)\right)-v\left(A\left(E\right)\right)\right].
\end{aligned}
$$

Now let's focus on the left-hand side of (2.14). We can write

$$
\begin{aligned}
\mathfrak{B}_i(v,A)+\mathfrak{B}_j(v,A) &= \frac{1}{2^{n-1}} \sum_{E\subseteq N\setminus\{i\}} \left[v\left(A\left(E\cup\{i\}\right)\right)-v(A(E))\right] \\
&\quad+\frac{1}{2^{n-1}} \sum_{E\subseteq N\setminus\{j\}} \left[v\left(A\left(E\cup\{j\}\right)\right)-v(E)\right] \\
&= \frac{1}{2^{n-1}} \sum_{E\subseteq N\setminus\{i,j\}} \left[v\left(A\left(E\cup\{i,j\}\right)\right)-v\left(A\left(E\cup\{j\}\right)\right)\right]
\end{aligned}
$$

$$+\frac{1}{2^{n-1}} \sum_{E \subseteq N \setminus \{i,j\}} \left[v\left(A\left(E \cup \{i\}\right)\right) - v\left(A\left(E\right)\right) \right]$$
$$+\frac{1}{2^{n-1}} \sum_{E \subseteq N \setminus \{i,j\}} \left[v\left(A\left(E \cup \{i,j\}\right)\right) - v\left(A\left(E \cup \{i\}\right)\right) \right]$$
$$+\frac{1}{2^{n-1}} \sum_{E \subseteq N \setminus \{i,j\}} \left[v\left(A\left(E \cup \{j\}\right)\right) - v\left(A\left(E\right)\right) \right]$$
$$=\frac{1}{2^{n-2}} \sum_{E \subseteq N \setminus \{i,j\}} \left[v\left(A\left(E \cup \{i,j\}\right)\right) - v\left(A\left(E\right)\right) \right].$$

□

We have already seen that the Banzhaf authorization value satisfies the six properties. Now we see that such properties uniquely determine the Banzhaf authorization value.

Theorem 2.21 *An allocation rule for games with authorization structure is equal to the Banzhaf authorization value if it satisfies the properties of additivity, irrelevant player, veto power over a necessary player, fairness, 2-efficiency and amalgamation.*

Proof. Let Ψ be an allocation rule for games with authorization structure satisfying the properties of additivity, irrelevant player, veto power over a necessary player, fairness, 2-efficiency and amalgamation. We must prove that $\Psi = \mathfrak{B}$.
Firstly, we show that

$$\Psi(cu_E, A) = \mathfrak{B}(cu_E, A) \quad \text{for all } E \in 2^N \setminus \{\emptyset\},\ A \in \mathcal{A}^N \text{ and } c > 0. \tag{2.15}$$

We proceed by induction on the number of players.

1. Base case. If $n = 1$ the equality follows directly from the property of 2-efficiency.

2. Inductive step. Let $E \in 2^N \setminus \{\emptyset\}$ and $c > 0$. We must prove that

$$\Psi(cu_E, A) = \mathfrak{B}(cu_E, A) \quad \text{for all } A \in \mathcal{A}^N. \tag{2.16}$$

We recall that

$$m(A) = \sum_{F \subseteq N} |A(F)| \quad \text{for every } A \in \mathcal{A}^N.$$

We prove (2.16) by induction on $m(A)$.

2.1 Base case. Let $A \in \mathcal{A}^N$ be such that $m(A) = 0$. In this case all players are irrelevant. Since Ψ and $\mathfrak{B}$ satisfy the irrelevant player property, it holds that $\Psi_i(cu_E, A) = \mathfrak{B}_i(cu_E, A) = 0$ for every $i \in N$.

2.2 Inductive step. Let $A \in \mathcal{A}^N$. We consider the three following sets:

$$H_1 = \{i \in N : i \text{ is an irrelevant player in } (cu_E, A)\},$$
$$H_2 = \{i \in N : \text{ there exists } j \in E \text{ such that } i \text{ has veto power over } j \text{ in } (N, A)\},$$
$$H_3 = N \setminus (H_1 \cup H_2).$$

Since $\mathfrak{B}$ and Ψ satisfy the irrelevant player property it holds that

$$\mathfrak{B}_i(cu_E, A) = 0 \quad \text{for all } i \in H_1, \tag{2.17}$$
$$\Psi_i(cu_E, A) = 0 \quad \text{for all } i \in H_1. \tag{2.18}$$

And from the property of veto power over a necessary player we can derive that there exist $b, b' \in \mathbb{R}$ such that

$$\mathfrak{B}_i(cu_E, A) = b \quad \text{for all } i \in H_2, \tag{2.19}$$
$$\Psi_i(cu_E, A) = b' \quad \text{for all } i \in H_2. \tag{2.20}$$

It can be proved, in the same way as in Theorem 2.14, that

$$\mathfrak{B}_i(cu_E, A) - \Psi_i(cu_E, A) = b - b' \quad \text{for all } i \in H_3. \tag{2.21}$$

Now we consider different cases.

(i) Case $|E| \geqslant 2$. Take j and k two different players in E. It is clear that j and k can be amalgamated in (cu_E, A). We can write

$$\begin{aligned} b+b &= \mathfrak{B}_j(cu_E,A)+\mathfrak{B}_k(cu_E,A) \;= \mathfrak{B}_{\widehat{jk}}\left((cu_E)^{jk},A^{jk}\right) \\ &= \mathfrak{B}_{\widehat{jk}}\left(cu_{(E\setminus\{j,k\})\cup\left\{\widehat{jk}\right\}},A^{jk}\right) = \Psi_{\widehat{jk}}\left(cu_{(E\setminus\{j,k\})\cup\left\{\widehat{jk}\right\}},A^{jk}\right) \\ &= \Psi_{\widehat{jk}}\left((cu_E)^{jk},A^{jk}\right) = \Psi_j(cu_E,A)+\Psi_k(cu_E,A) = b'+b', \end{aligned}$$

and we obtain $b = b'$. Using (2.17), (2.18), (2.19), (2.20) and (2.21) we conclude $\Psi(cu_E, A) = \mathfrak{B}(cu_E, A)$.

(ii) Case $|E| = 1$ and $|(H_2 \setminus E) \cup H_3| \geqslant 2$. Take j and k two different players in $(H_2 \setminus E) \cup H_3$. It is easy to check that j and k can be amalgamated in (cu_E, A). It holds that

$$\begin{aligned} \mathfrak{B}_j(cu_E,A)+\mathfrak{B}_k(cu_E,A) &= \mathfrak{B}_{\widehat{jk}}\left((cu_E)^{jk},A^{jk}\right) = \mathfrak{B}_{\widehat{jk}}\left(cu_E,A^{jk}\right) \\ &= \Psi_{\widehat{jk}}\left(cu_E,A^{jk}\right) = \Psi_{\widehat{jk}}\left((cu_E)^{jk},A^{jk}\right) \\ &= \Psi_j(cu_E,A)+\Psi_k(cu_E,A), \end{aligned}$$

and thus

$$\mathfrak{B}_j(cu_E,A)-\Psi_j(cu_E,A) = \Psi_k(cu_E,A)-\mathfrak{B}_k(cu_E,A).$$

Using (2.19), (2.20) and (2.21), the previous equality can be written as $b-b' = b'-b$. So we obtain $b = b'$, that leads to $\Psi(cu_E, A) = \mathfrak{B}(cu_E, A)$.

(iii) Case $|E| = 1$ and $|(H_2 \setminus E) \cup H_3| \leqslant 1$. On the one hand, using the 2-efficiency property, we can write

$$\sum_{i\in N}\mathfrak{B}_i(cu_E,A)-\sum_{i\in N}\Psi_i(cu_E,A) = cu_E(A(N))-cu_E(A(N)) = 0,$$

and, on the other hand, from (2.17), (2.18), (2.19), (2.20) and (2.21) we obtain

$$\sum_{i\in N}\mathfrak{B}_i(cu_E,A)-\sum_{i\in N}\Psi_i(cu_E,A) = (b-b')|H_2\cup H_3|.$$

From both expressions we obtain $b = b'$ and, therefore, $\Psi(cu_E, A) = \mathfrak{B}(cu_E, A)$.

So we have proved (2.15). Now, using additivity and reasoning as we did in the proof of Theorem 2.14 we conclude $\Psi = \mathfrak{B}$. □

Chapter 3

Games with fuzzy authorization structure

In the previous chapter we studied games in which there are dependency relationships among the players. These dependency relationships were considered to be complete, in the sense that, when a coalition is formed, a player in the coalition either can fully cooperate within the coalition or cannot cooperate at all. Nevertheless, in some situations it is possible to consider another option: that a player has a degree of freedom to cooperate within the coalition. In this chapter we aim to present a model for these situations.

3.1 Fuzzy authorization structures

In order to deal with fuzzy relationships among the players we extend the concept of authorization operator.

Definition 3.1 *A fuzzy authorization operator on N is a mapping $a : 2^N \to [0,1]^N$ that satisfies the following conditions:*

1. $a(E) \subseteq \mathbf{1}_E$ *for any* $E \subseteq N$,
2. *If* $E \subset F$ *then* $a(E) \subseteq a(F)$.

The pair (N, a) is called a fuzzy authorization structure. The set of all fuzzy authorization operators on N is denoted by $\mathcal{FA}^N$.

Suppose that a is a fuzzy authorization operator and v is a game on N. Then, given $E \subseteq N$ and $i \in N$, we interpret $a_i(E)$ as the proportion of the whole operating capacity of player i that he is allowed to use within coalition E.

Definition 3.2 *A fuzzy authorization operator a is said to be normal if $a(N) = \mathbf{1}_N$. The set of normal fuzzy authorization operators is denoted by $\widetilde{\mathcal{FA}}^N$.*

Example 3.3 Let us go back to example 2.3. We introduce some changes. Supplier 1 admits that they use a technology patented by 2. However, 1 demonstrates that they are capable of producing component 1 without using that technology. But in that case they would only be able to produce six hundred thousand units before the deadline. Something similar happens with the other dispute. Supplier 2 can produce component 2 without the technology patented by 3, but if they do so then they only will be able to produce nine hundred thousand units within the stipulated time.

For any $E \subseteq \{1,2,3\}$ and $i \in \{1,2,3\}$ the expression $a_i(E)$ denotes the proportion of his maximum production that supplier i can reach within coalition E. We can represent a with the following table.

E	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$a(E)$	$(0.6,0,0)$	$(0,0.9,0)$	$(0,0,1)$	$(1,0.9,0)$	$(0.6,0,1)$	$(0,1,1)$	$(1,1,1)$

Definition 3.4 *A game with fuzzy authorization structure on N is a pair (v,a) where $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$.*

In a similar way as we did in the crisp case, given a game with fuzzy authorization structure we can define a characteristic function that gathers the information from the game and the structure in a reasonable way.

Definition 3.5 *Let $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$. The restricted game of (v,a) is the game $v^a \in \mathcal{G}^N$ defined as follows*

$$v^a(E) = \int a(E)\,dv \quad \text{for all } E \subseteq N,$$

where $\int a(E)\,dv$ *denotes the Choquet integral of* $a(E)$ *with respect to* v.

The number $v^a(E)$ is the worth of E in the game with fuzzy authorization structure (v, a).

Example 3.6 Let us calculate the restricted game of the game with fuzzy authorization structure (v, a) given in Example 3.3.

$$\begin{aligned}
v^a(\{1\}) &= 0.6\,v(\{1\}) = 0.6,\\
v^a(\{2\}) &= 0.9\,v(\{2\}) = 1.8,\\
v^a(\{3\}) &= \;v(\{3\}) = 3,\\
v^a(\{1,2\}) &= 0.9\,v(\{1,2\}) + 0.1\,v(\{1\}) = 5.5,\\
v^a(\{1,3\}) &= 0.6\,v(\{1,3\}) + 0.4\,v(\{3\}) = 6,\\
v^a(\{2,3\}) &= v(\{2,3\}) = 10,\\
v^a(\{1,2,3\}) &= v(\{1,2,3\}) = 20.
\end{aligned}$$

Suppose we want to determine the worth of a coalition E in a game with fuzzy authorization structure (v, a). The calculation of the corresponding Choquet integral implies summation over a set of indexed numbers. A priori, this set depends on the coalition E. Nevertheless the following lemma will allow us to consider the same set for all the coalitions.

Lemma 3.7 *Let* $v \in \mathcal{G}^N$ *and* $f \in [0,1]^N$. *Let* $\{h_l : l = 0, \ldots, m\} \subset [0,1]$ *be such that* $\{h_l : l = 0, \ldots, m\} \supseteq \{f_i : i \in N\}$ *with* $0 = h_0 < \ldots < h_m$. *Then it holds that*

$$\int f\,dv = \sum_{l=1}^{m} (h_l - h_{l-1})\, v\left([f]_{h_l}\right).$$

where $[f]_{h_l} = \{i \in N : f_i \geq h_l\}$ *for* $l = 1, \ldots, m$.

Proof. Let $r \in \{0,\dots,m\}$ and $l_0,\dots,l_r \in \{0,\dots,m\}$ be such that $0 = l_0 < \dots < l_r$ and $\{h_{l_k} : k = 0,\dots,r\} = \{f_i : i \in N\}$. It holds that

$$\begin{aligned}\int f\,dv &= \sum_{k=1}^{r}(h_{l_k} - h_{l_{k-1}})\,v\left([f]_{h_{l_k}}\right)\\ &= \sum_{k=1}^{r}\sum_{j=l_{k-1}+1}^{l_k}(h_j - h_{j-1})\,v\left([f]_{h_{l_k}}\right)\\ &= \sum_{k=1}^{r}\sum_{j=l_{k-1}+1}^{l_k}(h_j - h_{j-1})\,v\left([f]_{h_j}\right)\\ &= \sum_{j=1}^{l_r}(h_j - h_{j-1})\,v\left([f]_{h_j}\right)\\ &= \sum_{j=1}^{m}(h_j - h_{j-1})\,v\left([f]_{h_j}\right).\end{aligned}$$

□

Using the lemma above we can obtain the expression of the restricted game given in the following remark.

Remark 3.8 *Let $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$. Let $\{h_l : l = 0,\dots,m\} \subset [0,1]$ be such that $\{h_l : l = 0,\dots,m\} \supseteq \{a_k(F) : F \subseteq N,\, k \in N\}$ with $0 = h_0 < \dots < h_m$. It holds that*

$$v^a(E) = \sum_{l=1}^{m}(h_l - h_{l-1})\,v\left([a(E)]_{h_l}\right) \qquad \textit{for all } E \subseteq N.$$

Let $a \in \mathcal{FA}^N$ and $t \in [0,1]$. We define $a^t \in \mathcal{A}^N$ as follows

$$a^t(E) = [a(E)]_t = \{k \in E : a_k(E) \geqslant t\} \qquad \text{for all } E \subseteq N.$$

The following expression of the restricted game will turn out to be very useful in order to prove the results in this chapter.

Remark 3.9 *Let $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$. Let $\{h_l : l = 0,\dots,m\} \subset [0,1]$ be such that*

$\{h_l : l = 0, \ldots, m\} \supseteq \{a_k(F) : F \subseteq N, k \in N\}$ *with* $0 = h_0 < \ldots < h_m$*. Then it holds that*

$$v^a = \sum_{l=1}^{m} (h_l - h_{l-1})\, v^{a^{h_l}}.$$

3.2 The Shapley fuzzy authorization value

An allocation rule for games with fuzzy authorization structure assigns to every game with fuzzy authorization structure a payoff vector. In this section we define and characterize an allocation rule for games with fuzzy authorization structure.

Definition 3.10 *The Shapley fuzzy authorization value, denoted by* φ*, assigns to each game with fuzzy authorization structure* (v, a) *the Shapley value of* v^a*,*

$$\varphi(v, a) = \phi\left(v^a\right) \qquad \text{for all } v \in \mathcal{G}^N \text{ and } a \in \mathcal{FA}^N.$$

The Shapley fuzzy authorization value has been studied in Gallardo, Jiménez, Jiménez-Losada and Lebrón [38]. We give another expression of φ in the following lemma.

Lemma 3.11 *Let* $v \in \mathcal{G}^N$ *and* $a \in \mathcal{FA}^N$*. Let* $\{h_l : l = 0, \ldots, m\} \subset [0, 1]$ *be such that* $\{h_l : l = 0, \ldots, m\} \supseteq \{a_k(F) : F \subseteq N, k \in N\}$ *with* $0 = h_0 < \ldots < h_m$*. It holds that*

$$\varphi(v, a) = \sum_{l=1}^{m} (h_l - h_{l-1})\, \Phi(v, a^{h_l}),$$

where Φ *is the Shapley authorization value.*

Proof. Taking into account Remark 3.9 and the linearity of the Shapley value, we have that

$$\begin{aligned}
\varphi(v, a) &= \phi\left(v^a\right) \\
&= \phi\left(\sum_{l=1}^{m} (h_l - h_{l-1})\, v^{a^{h_l}}\right)
\end{aligned}$$

$$= \sum_{l=1}^{m} (h_l - h_{l-1}) \; \phi\left(v^{a^{h_l}}\right)$$
$$= \sum_{l=1}^{m} (h_l - h_{l-1}) \, \Phi(v, a^{h_l}).$$

□

We aim to characterize the Shapley fuzzy authorization value. To that end, we consider the following properties.

- **EFFICIENCY**. For every $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$ with $a(N) \in \{0,1\}^N$ it holds that

$$\sum_{k \in N} \psi_k(v, a) = v(supp(a(N))).$$

- **ADDITIVITY**. For every $v, w \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$ it holds that

$$\psi(v + w, a) = \psi(v, a) + \psi(w, a).$$

Definition 3.12 *Let $a \in \mathcal{FA}^N$ and $i, j \in N$. A player j depends partially on i in (N, a) if there exists $E \subseteq N$ such that $a_j(E) > a_j(E \setminus \{i\})$.*

Definition 3.13 *Let $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$ and $i \in N$. A player i is irrelevant in (v, a) if for every $j \in N$ such that j depends partially on i in (N, a) it holds that j is a null player in v.*

Note that if i is an irrelevant player in (v, a) then, in particular, is a null player in v, since i depends partially on i in (N, a).

- **IRRELEVANT PLAYER PROPERTY**. For every $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$ and $i \in N$ such that i is an irrelevant player in (v, a), it holds that

$$\psi_i(v, a) = 0.$$

Definition 3.14 *Let $a \in \mathcal{FA}^N$ and $i, j \in N$. A player i has veto power over j in (N, a) if $a_j(N \setminus \{i\}) = 0$.*

- **PROPERTY OF VETO POWER OVER A NECESSARY PLAYER.** For every monotonic $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$ and $i, j \in N$ such that j is a necessary player in v and i has veto power over j in (N, a), it holds that

$$\psi_i(v, a) \geqslant \psi_k(v, a) \quad \text{for all } k \in N.$$

Given $a \in \mathcal{FA}^N$, $T \in 2^N \setminus \{\emptyset\}$, $i \in T$ and $s \in [0, 1]$, we define

$$\begin{array}{rcll} a^{T,i,s}: & 2^N & \rightarrow & [0,1]^N \\ & E & \rightarrow & a_k^{T,i,s}(E) = \begin{cases} \max(s, a_i(E)) & \text{if } k = i \text{ and } T \subseteq E, \\ a_k(E) & \text{otherwise.} \end{cases} \end{array}$$

It is clear that $a^{T,i,s} \in \mathcal{FA}^N$.

- **FAIRNESS.** For every $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$, $T \in 2^N \setminus \{\emptyset\}$, $i \in T$ and $s \in [0, 1]$ it holds that

$$\psi_j\left(v, a^{T,i,s}\right) - \psi_j(v, a) = \psi_i\left(v, a^{T,i,s}\right) - \psi_i(v, a) \quad \text{for all } j \in T.$$

Notice that if $s \leqslant a_i(T)$ then $a^{T,i,s} = a$. Therefore, the expression above is non trivial only if $s \in (a_i(T), 1]$. This property is an extension of the fairness property of the Shapley authorization value. If a coalition acquires the power to increase the capacity of cooperation of one of the players within the coalition then all the players in the coalition will benefit equally.

- **REDUCTION.** For every $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$ and $t \in (0, 1)$ it holds that

$$\psi(v, a) = t\, \psi(v, a^{[0,t]}) + (1 - t)\, \psi(v, a^{[t,1]}),$$

where, for all $i \in N$ and $E \subseteq N$,

$$\begin{aligned} a_i^{[0,t]}(E) &= \min\left(1, \frac{a_i(E)}{t}\right), \\ a_i^{[t,1]}(E) &= \max\left(0, \frac{a_i(E) - t}{1 - t}\right). \end{aligned}$$

Our aim is to see that the Shapley fuzzy authorization value is uniquely determined by the six properties seen above.

Theorem 3.15 *The Shapley fuzzy authorization value satisfies the properties of additivity, efficiency, irrelevant player, veto power over a necessary player, fairness and reduction.*

Proof. We prove that φ satisfies the six properties.

EFFICIENCY. Let $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$ with $a(N) \in \{0,1\}^N$. It holds that

$$\sum_{k\in N} \varphi_k(v,a) = \sum_{k\in N} \phi_k\left(v^a\right) = v^a(N) = v\left(supp(a(N))\right).$$

ADDITIVITY. Let $v, w \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$. It is easy to check that $(v+w)^a = v^a + w^a$. We can derive that

$$\varphi(v+w,a) = \phi\left((v+w)^a\right) = \phi(v^a + w^a) = \phi(v^a) + \phi(w^a) = \varphi(v,a) + \varphi(w,a).$$

IRRELEVANT PLAYER PROPERTY. Let $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$ and $i \in N$ an irrelevant player in (v,a). We must prove that $\varphi_i(v,a) = 0$. Let $\{t_l : l = 0,\ldots,r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r$. Taking into consideration that Φ satisfies the irrelevant player property, it is clear from Remark 3.9 that it is enough to prove that i is an irrelevant player in (v, a_{t_l}) for every $l = 1,\ldots,r$. So take $l \in \mathbb{N}$ with $l \leqslant r$. Suppose that $j \in N$ depends partially on i in (N, a^{t_l}). This means that there exists $E \subseteq N$ such that $j \in a^{t_l}(E) \setminus a^{t_l}(E \setminus \{i\})$. Therefore, $a_j(E) \geqslant t_l > a_j(E \setminus \{i\})$. It follows that j depends partially on i in (N,a). Since i is an irrelevant player in (v,a), we conclude that j is a null player in v. So we have proved that i is an irrelevant player in (v, a^{t_l}).

PROPERTY OF VETO POWER OVER A NECESSARY PLAYER. Let $v \in \mathcal{G}^N$ be a monotonic game, $a \in \mathcal{FA}^N$, $\{t_l : l = 0,\ldots,r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r$ and $i, j \in N$ be such that j is a necessary player in v and i has veto power over j in (N,a). We must prove that $\varphi_i(v,a) \geqslant \varphi_k(v,a)$ for all $k \in N$. It follows from Remark 3.9 and the fact that Φ satisfies the property of veto power over a necessary player that it is enough to prove that i has veto power over j in (N, a^{t_l}) for every $l = 1,\ldots,r$. And this is a clear consequence of the fact that i has veto power over j in (N,a).

FAIRNESS. Let $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$, $\{t_l : l = 0, \ldots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r$, $T \in 2^N \setminus \{\emptyset\}$, $i, j \in T$ and $s \in [0,1]$. We have to prove that

$$\varphi_j\left(v, a^{T,i,s}\right) - \varphi_j(v,a) = \varphi_i\left(v, a^{T,i,s}\right) - \varphi_i(v,a).$$

We know that if $s \leqslant a_i(T)$ then $a^{T,i,s} = a$ and the equality above would be trivial. So we can suppose that $s > a_i(T)$. Let p, q be such that $t_p = a_i(T)$ and $s \in (t_{q-1}, t_q]$. It must be $p < q$.
On the one hand, from Lemma 3.11 we can obtain that

$$\begin{aligned}\varphi(v, a^{T,i,s}) = &\sum_{l=1}^{p}(t_l - t_{l-1})\Phi\left(v, \left(a^{T,i,s}\right)^{t_l}\right) + \sum_{l=p+1}^{q-1}(t_l - t_{l-1})\Phi\left(v, \left(a^{T,i,s}\right)^{t_l}\right)\\ &+ (s - t_{q-1})\Phi\left(v, \left(a^{T,i,s}\right)^{s}\right) + (t_q - s)\Phi\left(v, \left(a^{T,i,s}\right)^{t_q}\right)\\ &+ \sum_{l=q+1}^{r}(t_l - t_{l-1})\Phi\left(v, \left(a^{T,i,s}\right)^{t_l}\right).\end{aligned}$$

Now we observe that

1. $\left(a^{T,i,s}\right)^{t_l} = a^{t_l}$ for every $l = 1, \ldots, p$,
2. $\left(a^{T,i,s}\right)^{t_l} = \left(a^{t_l}\right)^{T,i}$ for every $l = p+1, \ldots, q-1$,
3. $\left(a^{T,i,s}\right)^{s} = \left(a^{t_q}\right)^{T,i}$,
4. $(t_q - s)\Phi\left(v, \left(a^{T,i,s}\right)^{t_q}\right) = (t_q - s)\Phi\left(v, a^{t_q}\right)$. Notice that if $s < t_q$ then $\left(a^{T,i,s}\right)^{t_q} = a^{t_q}$,
5. $\left(a^{T,i,s}\right)^{t_l} = a^{t_l}$ for every $l = q+1, \ldots, r$.

Substituting into the sum above we obtain that

$$\begin{aligned}\varphi(v, a^{T,i,s}) = &\sum_{l=1}^{p}(t_l - t_{l-1})\Phi\left(v, a^{t_l}\right) + \sum_{l=p+1}^{q-1}(t_l - t_{l-1})\Phi\left(v, \left(a^{t_l}\right)^{T,i}\right)\\ &+(s - t_{q-1})\Phi\left(v, \left(a^{t_q}\right)^{T,i}\right) + (t_q - s)\Phi\left(v, a^{t_q}\right)\\ &+ \sum_{l=q+1}^{r}(t_l - t_{l-1})\Phi\left(v, a^{t_l}\right).\end{aligned}$$

On the other hand it holds that

$$\begin{aligned}\varphi(v,a) &= \sum_{l=1}^{r}(t_l-t_{l-1})\Phi(v,a_{t_l})\\ &= \sum_{l=1}^{p}(t_l-t_{l-1})\Phi\left(v,a^{t_l}\right)+\sum_{l=p+1}^{q-1}(t_l-t_{l-1})\Phi\left(v,a^{t_l}\right)\\ &\quad +(s-t_{q-1})\Phi\left(v,a^{t_q}\right)+(t_q-s)\Phi\left(v,a^{t_q}\right)+\sum_{l=q+1}^{r}(t_l-t_{l-1})\Phi\left(v,a^{t_l}\right).\end{aligned}$$

Subtracting we get

$$\begin{aligned}\varphi(v,a^{T,i,s})-\varphi(v,a) &= \sum_{l=p+1}^{q-1}(t_l-t_{l-1})\left(\Phi\left(v,\left(a^{t_l}\right)^{T,i}\right)-\Phi(v,a^{t_l})\right)\\ &\quad +(s-t_{q-1})\left(\Phi\left(v,\left(a^{t_q}\right)^{T,i}\right)-\Phi(v,a^{t_q})\right).\end{aligned}$$

Now, keeping in mind that Φ satisfies the fairness property, we can write

$$\begin{aligned}\varphi_j(v,a^{T,i,s})-\varphi_j(v,a) &= \sum_{l=p+1}^{q-1}(t_l-t_{l-1})\left(\Phi_j\left(v,\left(a^{t_l}\right)^{T,i}\right)-\Phi_j(v,a^{t_l})\right)\\ &\quad +(s-t_{q-1})\left(\Phi_j\left(v,\left(a^{t_q}\right)^{T,i}\right)-\Phi_j(v,a^{t_q})\right)\\ &= \sum_{l=p+1}^{q-1}(t_l-t_{l-1})\left(\Phi_i\left(v,\left(a^{t_l}\right)^{T,i}\right)-\Phi_i(v,a^{t_l})\right)\\ &\quad +(s-t_{q-1})\left(\Phi_i\left(v,\left(a^{t_q}\right)^{T,i}\right)-\Phi_i(v,a^{t_q})\right)\\ &= \varphi_i(v,a^{T,i,s})-\varphi_i(v,a).\end{aligned}$$

REDUCTION. Let $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$, $t \in (0,1)$ and, for all $i \in N$ and $E \subseteq N$,

$$\begin{aligned}a_i^{[0,t]}(E) &= \min\left(1,\frac{a_i(E)}{t}\right),\\ a_i^{[t,1]}(E) &= \max\left(0,\frac{a_i(E)-t}{1-t}\right).\end{aligned}$$

We must prove that

$$\varphi(v,a) = t\,\varphi(v,a^{[0,t]}) + (1-t)\,\varphi(v,a^{[t,1]}).$$

Taking into consideration the definition of φ and the linearity of the Shapley value, it suffices to prove that

$$v^a = t\,v^{a^{[0,t]}} + (1-t)\,v^{a^{[t,1]}}.$$

To this end, take $\{t_l : l = 0,\ldots,r\} = \{a_k(F) : F \subseteq N, k \in N\} \cup \{t\}$ with $0 = t_0 < \ldots < t_r = 1$. Let $p \in \mathbb{N}$ be such that $t = t_p$. For every $E \subseteq N$ it holds that

$$\begin{aligned}\left(t\,v^{a^{[0,t]}} + (1-t)\,v^{a^{[t,1]}}\right)(E) &= t\sum_{l=1}^{p}\left(\frac{t_l}{t} - \frac{t_{l-1}}{t}\right) v\left(\left[a^{[0,t]}(E)\right]_{\frac{t_l}{t}}\right)\\ &\quad + (1-t)\sum_{l=p+1}^{r}\left(\frac{t_l - t}{1-t} - \frac{t_{l-1}-t}{1-t}\right) v\left(\left[a^{[t,1]}(E)\right]_{\frac{t_l - t}{1-t}}\right)\\ &= t\sum_{l=1}^{p}\left(\frac{t_l}{t} - \frac{t_{l-1}}{t}\right) v\left([a(E)]_{t_l}\right)\\ &\quad + (1-t)\sum_{l=p+1}^{r}\left(\frac{t_l - t}{1-t} - \frac{t_{l-1}-t}{1-t}\right) v\left([a(E)]_{t_l}\right)\\ &= v^a(E).\end{aligned}$$

□

We have proved that the Shapley fuzzy authorization value satisfies the six properties. We see that such properties uniquely determine φ.

Theorem 3.16 *An allocation rule for games with fuzzy authorization structure is equal to the Shapley fuzzy authorization value if it satisfies the properties of additivity, efficiency, irrelevant player, veto power over a necessary player, fairness and reduction.*

Proof. Let ψ be an allocation rule for games with fuzzy authorization structure satisfying the properties of additivity, efficiency, irrelevant player, veto power over a necessary player, fairness and reduction. We must prove that

$$\psi(v,a) = \varphi(v,a) \quad \text{for every } n \in \mathbb{N},\ v \in \mathcal{G}^N \text{ and } a \in \mathcal{FA}^N.$$

We proceed by strong induction on $\lceil(a)$ where

$$\lceil(a) = |\{a_k(F) : F \subseteq N, k \in N\} \setminus \{0,1\}| \quad \text{for all } a \in \mathcal{FA}^N.$$

1. BASE CASE. $\lceil(a) = 0$.

   Notice that we can identify $\mathcal{A}^N$ with the set $\{a \in \mathcal{FA}^N : im(a) \subseteq \{0,1\}^N\}$. From this point of view, we can say that the restriction of ψ to the set of games with fuzzy authorization structure (v,a) with $\lceil(a) = 0$ is an allocation rule for games with authorization structure. It is easy to check that such restriction satisfies the properties of efficiency, additivity, irrelevant player, veto power over a necessary player and fairness. Therefore, using Theorem 2.14 we conclude that

   $$\psi(v,a) = \varphi(v,a) \quad \text{for every } n \in \mathbb{N},\ v \in \mathcal{G}^N \text{ and } a \in \mathcal{FA}^N \text{ with } \lceil(a) = 0.$$

2. INDUCTIVE STEP. Let $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$ with $\lceil(a) > 0$. We want to prove that $\psi(v,a) = \varphi(v,a)$. Take $t \in \{a_k(F) : F \subseteq N, k \in N\} \setminus \{0,1\}$. Since ψ satisfies the reduction property it holds that

   $$\psi(v,a) = t\,\psi(v,a^{[0,t]}) + (1-t)\,\psi(v,a^{[t,1]}).$$

   Since $\lceil(a^{[0,t]}) < \lceil(a)$ and $\lceil(a^{[t,1]}) < \lceil(a)$ it follows by induction hypothesis that $\psi(v,a^{[0,t]}) = \varphi(v,a^{[0,t]})$ and $\psi(v,a^{[t,1]}) = \varphi(v,a^{[t,1]})$. Hence

   $$\psi(v,a) = t\,\varphi(v,a^{[0,t]}) + (1-t)\,\varphi(v,a^{[t,1]}) = \varphi(v,a).$$

□

Example 3.17 Let us calculate $\varphi(v,a)$ where v and a are those defined in Example 3.3. We already calculated the restricted game v^a in Example 3.6. Now we calculate the Shapley fuzzy authorization value

$$\varphi(v,a)=\phi(v^a)=(4.65,7.25,8.1).$$

In a similar way as we did in the crisp case, we can obtain an expression of the Shapley fuzzy authorization value that does not involve the restricted game, but the game and the fuzzy authorization operator separately.

Corollary 3.18 *Let $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}^N$. Then, if $\varphi(v,a)$ is considered as a column matrix, it holds that*

$$\varphi(v,a)=\zeta_a \cdot \Delta_v$$

where ζ_a is the matrix in $\mathcal{M}_{n,2^n-1}(\mathbb{R})$ defined by $(\zeta_a)_{i,E} = \varphi_i(u_E,a)$ for every $i \in N$ and $E \in 2^N \setminus \{\emptyset\}$ and Δ_v is the column matrix given by the Harsanyi dividends of v. Moreover, if $\{t_l : l=0,\dots,r\}=\{a_k(F) : F\subseteq N,\, k\in N\}$ with $0=t_0<\dots<t_r$ then

$$\zeta_a=\sum_{l=1}^{r}(t_l-t_{l-1})\, Z_{a^{t_l}}.$$

Example 3.19 Let us use the expression given in the preceding result to calculate $\varphi(v,a)$ in Example 3.17. Firstly we calculate the characteristic function of u_E^a for each nonempty $E\subseteq\{1,2,3\}$.

F	$u^a_{\{1\}}(F)$	$u^a_{\{2\}}(F)$	$u^a_{\{3\}}(F)$	$u^a_{\{1,2\}}(F)$	$u^a_{\{1,3\}}(F)$	$u^a_{\{2,3\}}(F)$	$u^a_{\{1,2,3\}}(F)$
$\{1\}$	0.6	0	0	0	0	0	0
$\{2\}$	0	0.9	0	0	0	0	0
$\{3\}$	0	0	1	0	0	0	0
$\{1,2\}$	1	0.9	0	0.9	0	0	0
$\{1,3\}$	0.6	0	1	0	0.6	0	0
$\{2,3\}$	0	1	1	0	0	1	0
$\{1,2,3\}$	1	1	1	1	1	1	1

We calculate $\varphi(u_E, a)$ for each nonempty $E \subseteq \{1,2,3\}$

$$\begin{aligned}
\varphi\left(u_{\{1\}}, a\right) &= \phi\left(u^a_{\{1\}}\right) &&= \left(\frac{4}{5}, \frac{1}{5}, 0\right) \\
\varphi\left(u_{\{2\}}, a\right) &= \phi\left(u^a_{\{2\}}\right) &&= \left(0, \frac{19}{20}, \frac{1}{20}\right) \\
\varphi\left(u_{\{3\}}, a\right) &= \phi\left(u^a_{\{3\}}\right) &&= (0,0,1) \\
\varphi\left(u_{\{1,2\}}, a\right) &= \phi\left(u^a_{\{1,2\}}\right) &&= \left(\frac{29}{60}, \frac{29}{60}, \frac{1}{30}\right) \\
\varphi\left(u_{\{1,3\}}, a\right) &= \phi\left(u^a_{\{1,3\}}\right) &&= \left(\frac{13}{30}, \frac{2}{15}, \frac{13}{30}\right) \\
\varphi\left(u_{\{2,3\}}, a\right) &= \phi\left(u^a_{\{2,3\}}\right) &&= \left(0, \frac{1}{2}, \frac{1}{2}\right) \\
\varphi\left(u_{\{1,2,3\}}, a\right) &= \phi\left(u^a_{\{1,2,3\}}\right) &&= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)
\end{aligned}$$

and we write ζ_a

$$\zeta_a = \begin{pmatrix} \frac{4}{5} & 0 & 0 & \frac{29}{60} & \frac{13}{30} & 0 & \frac{1}{3} \\ \frac{1}{5} & \frac{19}{20} & 0 & \frac{29}{60} & \frac{2}{15} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{20} & 1 & \frac{1}{30} & \frac{13}{30} & \frac{1}{2} & \frac{1}{3} \end{pmatrix}.$$

Finally,

$$\varphi(v,a) = \zeta_a \cdot \Delta_v = \begin{pmatrix} \frac{4}{5} & 0 & 0 & \frac{29}{60} & \frac{13}{30} & 0 & \frac{1}{3} \\ \frac{1}{5} & \frac{19}{20} & 0 & \frac{29}{60} & \frac{2}{15} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{20} & 1 & \frac{1}{30} & \frac{13}{30} & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 4 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{93}{20} \\ \frac{29}{4} \\ \frac{81}{10} \end{pmatrix}.$$

Now we give a result regarding the continuity of φ.

Theorem 3.20 *The Shapley fuzzy authorization value is a continuous function.*

Proof. We intend to prove that

$$\begin{array}{rcl} \varphi: \; \mathcal{G}^N \times \mathcal{FA}^N & \rightarrow & \mathbb{R}^N \\ (v,a) & \rightarrow & \phi(v^a) \end{array}$$

is continuous. Taking into consideration the continuity of the Shapley value, it is enough to prove that the application

$$\begin{array}{rcl} \mathfrak{R}: \; \mathcal{G}^N \times \mathcal{FA}^N & \rightarrow & \mathcal{G}^N \\ (v,a) & \rightarrow & v^a \end{array}$$

is continuous. To that end, take $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$ and $\varepsilon > 0$. Let $w \in \mathcal{G}^N$ and $b \in \mathcal{FA}^N$ be such that

$$|v(E) - w(E)| < \varepsilon \quad \text{for all } \; E \subseteq N,$$

$$|a_i(E) - b_i(E)| < \delta = \frac{\varepsilon}{n2^n(2M+\varepsilon)} \quad \text{for all } \; E \subseteq N \text{ and } i \in N,$$

where $M = \max\{|v(E)| : \, E \subseteq N\}$. Let

$$\{t_l : l = 0, \ldots, r\} = \{a_k(E) : \, E \subseteq N, \, k \in N\} \cup \{b_k(E) : \, E \subseteq N, \, k \in N\},$$

with $0 = t_0 < \ldots < t_r = 1$. Take $F \subseteq N$. Let

$$\begin{array}{rcl} L & = & \{l \in \mathbb{N} : \, 1 \leqslant l \leqslant r \;\text{ and }\; [a(F)]_{t_l} = [b(F)]_{t_l}\}\,, \\ L' & = & \{l \in \mathbb{N} : \, 1 \leqslant l \leqslant r \;\text{ and }\; [a(F)]_{t_l} \neq [b(F)]_{t_l}\}\,. \end{array}$$

Notice that if $\; l \in L' \;$ then $\; t_l - t_{l-1} < \delta$. It holds that

$$\begin{aligned} |v^a(F) - w^b(F)| &= \left| \sum_{l=1}^{r} (t_l - t_{l-1}) \; \left(v\left([a(F)]_{t_l}\right) - w\left([b(F)]_{t_l}\right)\right) \right| \\ &\leqslant \sum_{l \in L} (t_l - t_{l-1}) \; \left|v\left([a(F)]_{t_l}\right) - w\left([b(F)]_{t_l}\right)\right| \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{l \in L'} (t_l - t_{l-1}) \left| v\left([a(F)]_{t_l}\right) - w\left([b(F)]_{t_l}\right) \right| \\
&\leqslant \varepsilon + r\delta(2M + \varepsilon) \\
&\leqslant \varepsilon + n2^n \delta(2M + \varepsilon) \\
&\leqslant 2\varepsilon.
\end{aligned}
$$

□

3.3 The Banzhaf fuzzy authorization value

We aim to define an allocation rule for games with fuzzy authorization structure that extends the Banzhaf authorization value defined in the previous chapter.

Definition 3.21 *The Banzhaf fuzzy authorization value, denoted by* $\mathfrak{b}$*, assigns to each game with fuzzy authorization structure* (v, a) *the Banzhaf value of* v^a*,*

$$\mathfrak{b}(v, a) = \beta\left(v^a\right) \qquad \text{for all } v \in \mathcal{G}^N \text{ and } a \in \mathcal{FA}^N.$$

In a similar way as we did for the Shapley fuzzy authorization value, we can give another expression for the Banzhaf fuzzy authorization value.

Lemma 3.22 *Let* $v \in \mathcal{G}^N$*,* $a \in \mathcal{FA}^N$ *and* $\{h_l : l = 0, \ldots, m\} \subset [0, 1]$ *be such that* $\{h_l : l = 0, \ldots, m\} \supseteq \{a_k(F) : F \subseteq N, k \in N\}$ *with* $0 = h_0 < \ldots < h_m$*. It holds that*

$$\mathfrak{b}(v, a) = \sum_{l=1}^{m} (h_l - h_{l-1}) \, \mathfrak{B}(v, a^{h_l}),$$

where $\mathfrak{B}$ *is the Banzhaf authorization value.*

Proof. It is evident taking into account Remark 3.9 and the linearity of the Banzhaf value. □

Our goal is to characterize the Banzhaf fuzzy authorization value. To that end we consider the following two properties for an allocation rule for games with fuzzy authorization structure.

- **2-EFFICIENCY.** For every $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$ with $a(N) \in \{0,1\}^N$ and $i, j \in N$ such that every player in $N \setminus \{i, j\}$ is an irrelevant player in (v, a), it holds that

$$\sum_{k \in N} \psi_k(v, a) = v(supp(a(N))).$$

Suppose $|N| \geqslant 2$. Let $a \in \mathcal{FA}^N$ and let i, j be two different players in N. We define $a^{ij} : 2^{N^{ij}} \longrightarrow [0,1]^{N^{ij}}$ as follows

$$a_k^{ij}(E) = \begin{cases} a_k(E) & \text{if } \widehat{ij} \notin E \text{ and } k \neq \widehat{ij}, \\ 0 & \text{if } \widehat{ij} \notin E \text{ and } k = \widehat{ij}, \\ a_k\left(\left(E \setminus \{\widehat{ij}\}\right) \cup \{i,j\}\right) & \text{if } \widehat{ij} \in E \text{ and } k \neq \widehat{ij}, \\ \min\left(a_i\left(\left(E \setminus \{\widehat{ij}\}\right) \cup \{i,j\}\right), a_j\left(\left(E \setminus \{\widehat{ij}\}\right) \cup \{i,j\}\right)\right) & \text{if } \widehat{ij} \in E \text{ and } k = \widehat{ij}. \end{cases}$$

It is easy to check that $a^{ij} \in \mathcal{FA}^{N^{ij}}$.

Definition 3.23 *Let $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$ and let i, j be two different players in N. The players i and j can be amalgamated in (v, a) if they can be amalgamated in (v, a^t) for every $t \in (0, 1]$.*

- **AMALGAMATION.** For every $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$ and i, j two different players in N such that i, j can be amalgamated in (v, a), it holds that

$$\psi_i(v, a) + \psi_j(v, a) = \psi_{\widehat{ij}}(v^{ij}, a^{ij}).$$

In the following results we give a characterization of the Banzhaf fuzzy authorization value.

Theorem 3.24 *The Banzhaf fuzzy authorization value satisfies the properties of additivity, irrelevant player, veto power over a necessary player, fairness, 2-efficiency, amalgamation and reduction.*

Proof. That the Banzhaf fuzzy authorization value satisfies additivity, irrelevant player, veto power over a necessary player, fairness and reduction can be proved in a similar way as we did for the Shapley fuzzy authorization value in Theorem 3.15. Let us see that the Banzhaf fuzzy authorization value satisfies 2-efficiency and amalgamation.

2-EFFICIENCY. Let $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$ and $i, j \in N$ be such that $a(N) \in \{0,1\}^N$ and every player in $N \setminus \{i,j\}$ is an irrelevant player in (v, a). If $a(N) = 0$ the result is trivial. We assume $a(N) \neq 0$. It is easy to check that every player in $N \setminus \{i,j\}$ is an irrelevant player in (v, a^t) for every $t \in (0,1]$. Take $\{t_l : l = 0, \ldots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r = 1$. Using Lemma 3.22 and the fact that the Banzhaf authorization value satisfies 2-efficiency we can write

$$\begin{aligned}\sum_{k \in N} \mathfrak{b}_k(v,a) &= \sum_{k \in N}\sum_{l=1}^{r}(t_l - t_{l-1})\mathfrak{B}_k(v, a^{t_l}) = \sum_{l=1}^{r}(t_l - t_{l-1})\sum_{k \in N}\mathfrak{B}_k(v, a^{t_l}) \\ &= \sum_{l=1}^{r}(t_l - t_{l-1})v\left(a^{t_l}(N)\right) = \sum_{l=1}^{r}(t_l - t_{l-1})v\left(supp(a(N))\right) = v\left(supp(a(N))\right).\end{aligned}$$

AMALGAMATION. Let $v \in \mathcal{G}^N$, $a \in \mathcal{FA}^N$ and let i, j be two different players in N such that i and j can be amalgamated in (v, a). Take $\{t_l : l = 0, \ldots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r$. It is easy to check that

$$(a^t)^{ij} = (a^{ij})^t \quad \text{for any } t \in (0,1]. \tag{3.1}$$

Using Lemma 3.22, the fact that the Banzhaf authorization value satisfies amalgamation and (3.1) we can write

$$\begin{aligned}\mathfrak{b}_i(v,a) + \mathfrak{b}_j(v,a) &= \sum_{l=1}^{r}(t_l - t_{l-1})\left(\mathfrak{B}_i(v, a^{t_l}) + \mathfrak{B}_j(v, a^{t_l})\right) \\ &= \sum_{l=1}^{r}(t_l - t_{l-1})\,\mathfrak{B}_{\widehat{ij}}(v^{ij}, (a^{t_l})^{ij}) \\ &= \sum_{l=1}^{r}(t_l - t_{l-1})\,\mathfrak{B}_{\widehat{ij}}(v^{ij}, (a^{ij})^{t_l}) \\ &= \mathfrak{b}_{\widehat{ij}}(v^{ij}, a^{ij}).\end{aligned}$$

Thus, we have proved that the Banzhaf fuzzy authorization value satisfies all the properties stated in the theorem. □

Theorem 3.25 *An allocation rule for games with fuzzy authorization structure is equal to the Banzhaf fuzzy authorization value if it satisfies the properties of additivity, irrelevant player, veto power over a necessary player, fairness, 2-efficiency, amalgamation and reduction.*

Proof. Let ψ be an allocation rule for games with fuzzy authorization structure satisfying the properties of additivity, irrelevant player, veto power over a necessary player, fairness, 2-efficiency, amalgamation and reduction. We must prove that

$$\psi(v,a)=\mathfrak{b}(v,a) \quad \text{for every } n\in\mathbb{N},\ v\in\mathcal{G}^N \text{ and } a\in\mathcal{FA}^N.$$

We proceed by strong induction on $\lceil(a)$ where

$$\lceil(a)=|\{a_k(F):\ F\subseteq N, k\in N\}\setminus\{0,1\}| \quad \text{for all } a\in\mathcal{FA}^N.$$

1. BASE CASE. $\lceil(a)=0$.

   Notice that we can identify $\mathcal{A}^N$ with the set $\{a\in\mathcal{FA}^N:\ im(a)\subseteq\{0,1\}^N\}$. From this point of view, we can say that the restriction of ψ to the set of games with fuzzy authorization structure (v,a) with $\lceil(a)=0$ is an allocation rule for games with authorization structure. It is easy to check that such restriction satisfies the properties of additivity, irrelevant player, veto power over a necessary player, fairness, 2-efficiency and amalgamation. Therefore, using Theorem 2.21 we conclude that

   $$\psi(v,a)=\mathfrak{b}(v,a) \quad \text{for every } n\in\mathbb{N},\ v\in\mathcal{G}^N \text{ and } a\in\mathcal{FA}^N \text{ with } \lceil(a)=0.$$

2. INDUCTIVE STEP. Let $v\in\mathcal{G}^N$ and $a\in\mathcal{FA}^N$ with $\lceil(a)>0$. We want to prove that $\psi(v,a)=\mathfrak{b}(v,a)$. Take $t\in\{a_k(F):\ F\subseteq N, k\in N\}\setminus\{0,1\}$. Since ψ satisfies the reduction property it holds that

   $$\psi(v,a)=t\,\psi(v,a^{[0,t]})+(1-t)\,\psi(v,a^{[t,1]}).$$

   Since $\lceil(a^{[0,t]})<\lceil(a)$ and $\lceil(a^{[t,1]})<\lceil(a)$ it follows by induction hypothesis that

$\psi(v, a^{[0,t]}) = \mathfrak{b}(v, a^{[0,t]})$ and $\psi(v, a^{[t,1]}) = \mathfrak{b}(v, a^{[t,1]})$. Hence

$$\psi(v, a) = t\, \mathfrak{b}(v, a^{[0,t]}) + (1 - t)\, \mathfrak{b}(v, a^{[t,1]}) = \mathfrak{b}(v, a).$$

□

Chapter 4

Relational power in authorization structures

If we consider an authorization structure, some agents may be in more advantageous position than others. On the one hand, some of them will be less autonomous than others, in the sense that they will need the permission of other agents in order to actively participate in a coalition. And, on the other hand, some will be more influential than others, in the sense that they will be able to prevent other agents from cooperating within a coalition. Our goal in this chapter will be to study how favorable the situation of each agent in an authorization structure is. To do this, we will define a value that will allow us to measure the influence and the sovereignty of each agent.

4.1 Allocation rules for authorization structures

Firstly we need to recall the concept of set game, which was introduced by Aarts, Funaki and Hoede [2]. A *set game* is a triple $(N, v, \mathcal{U})$ where N is a set of cardinality n with $n \in \mathbb{N}$, $\mathcal{U}$ is a set and $v : 2^N \to 2^{\mathcal{U}}$ is a mapping satisfying $v(\emptyset) = \emptyset$. The mapping v is called the *characteristic function* of the set game. Usually, the set game $(N, v, \mathcal{U})$ is identified with the characteristic function v. The worth $v(E)$ of a coalition E can be interpreted as the set of items that can be obtained by the players in E if they cooperate. A value ψ for set games is a mapping that assigns to each set game v an element $\psi(v) \in (2^{\mathcal{U}})^N$, where, for any $i \in N$, $\psi_i(v)$ is interpreted as the set of items that are given to player i. In 1997, Aarts, Funaki and Hoede [1] defined and characterized a value μ for monotonic set games, called the *marginalistic value*. Given a set game v, the marginalistic value of v is defined as

$$\mu_i(v) = \bigcup_{\{E \subseteq N:\, i \in E\}} [v(E) \setminus v(E \setminus \{i\})] \quad \text{for every } i \in N.$$

Notice that authorization operators are monotonic set games. So, if we are looking for a value for authorization structures, our first idea might be to use the marginalistic value for monotonic set games introduced in [2]. However, this value does not seem to be sufficiently sensitive, as we try to show with the following example.

Example 4.1 Let $N = \{1,2,3,4\}$. Let A, B and C be the authorization operators considered in Example 2.4.

E	$A(E)$	$B(E)$	$C(E)$
$E : 4 \notin E$	E	E	E
$\{4\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1,4\}$	$\{1\}$	$\{1,4\}$	$\{1\}$
$\{2,4\}$	$\{2\}$	$\{2,4\}$	$\{2\}$
$\{3,4\}$	$\{3\}$	$\{3,4\}$	$\{3\}$
$\{1,2,4\}$	$\{1,2\}$	$\{1,2,4\}$	$\{1,2,4\}$
$\{1,3,4\}$	$\{1,3\}$	$\{1,3,4\}$	$\{1,3,4\}$
$\{2,3,4\}$	$\{2,3\}$	$\{2,3,4\}$	$\{2,3,4\}$
$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$

If $W \in \mathcal{A}^N$, the marginalistic value of W is the element in $(2^N)^N$ defined as

$$\mu_i(W) = \bigcup_{\{E \subseteq N:\, i \in E\}} [W(E) \setminus W(E \setminus \{i\})] \quad \text{for every } i \in N.$$

Notice that $\mu(W)$ assigns to each agent the set of agents that depend partially on him in (N, W).

It is clear that if we calculate $\mu(A)$, $\mu(B)$ and $\mu(C)$ we obtain the following

$$\begin{array}{lll} \mu_1(A) = \{1,4\}, & \mu_1(B) = \{1,4\}, & \mu_1(C) = \{1,4\}, \\ \mu_2(A) = \{2,4\}, & \mu_2(B) = \{2,4\}, & \mu_2(C) = \{2,4\}, \\ \mu_3(A) = \{3,4\}, & \mu_3(B) = \{3,4\}, & \mu_3(C) = \{3,4\}, \\ \mu_4(A) = \{4\}, & \mu_4(B) = \{4\}, & \mu_4(C) = \{4\}. \end{array}$$

The reason why μ does not distinguish between A, B and C is that μ registers whether an agent depends partially on another one, but not how strong such dependence is.

Suppose that we have $\sigma : \mathcal{A}^N \to (2^N)^N$, $A \in \mathcal{A}^N$ and $i, j \in N$. If we wanted to use $\sigma(A)$ to evaluate the dependence of agent j on agent i in (N, A), we would check whether j is in $\sigma_i(A)$ or not. But it is clear that, in general, this evaluation would not inform us about how strongly j depends on i. If we want to define a more sensitive value, we need to consider degrees of membership. This is the motive of the following definition.

Definition 4.2 *An allocation rule for authorization structures assigns to each authorization operator on N a mapping in $\left([0,1]^N\right)^N$.*

4.2 The Shapley authorization correspondence

We aim to use the Shapley value to define an allocation rule for authorization structures. Notice that given $A \in \mathcal{A}^N$ and $j \in N$ we can define the simple game

$$\begin{array}{rccl} A_j : & 2^N & \to & \{0,1\} \\ & E & \to & A_j(E) = \left|A(E) \cap \{j\}\right|. \end{array}$$

Now we can define the value for authorization structures that we propose.

Definition 4.3 *The Shapley authorization correspondence assigns to each authorization operator $A \in \mathcal{A}^N$ the mapping $\Xi(A) \in \left([0,1]^N\right)^N$ defined as*

$$\Xi_i(A) = \left(\phi_i(A_j)\right)_{j \in N} \qquad \textit{for every } i \in N,$$

where ϕ is the Shapley value.

We denote $\Xi_{ij}(A) = \phi_i(A_j)$. Notice that $\Xi(A)$ can be identified with a matrix in $\mathcal{M}_n(\mathbb{R})$. Furthermore, $\Xi(A)$ is a submatrix of the matrix Z_A defined in Corollary 2.16, since for every $A \in \mathcal{A}^N$ and $i, j \in N$ it holds that

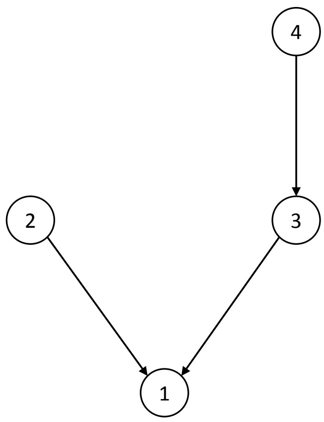
$$\Xi_{ij}(A) = \phi_i(A_j) = \phi_i\left(u^A_{\{j\}}\right) = \Phi_i(u_{\{j\}}, A) = (Z_A)_{i,\{j\}}.$$

Also notice that $\Xi(A)$ can be interpreted as a fuzzy cognitive map on N, Mordeson and Nair [51], in the sense that for any $i, j \in N$ with $i \neq j$ the number $\Xi_{ij}(A)$ measures how much the behavior of agent i affects agent j, whereas $\Xi_{ii}(A)$ measures the autonomy of agent i.

Example 4.4 Let $N = \{1, 2, 3, 4\}$ and A the authorization operator on N defined as

$$A(E) = \begin{cases} E & \text{if } \{3,4\} \subseteq E, \\ E \setminus \{3\} & \text{if } \{3,4\} \not\subseteq E \text{ and } 2 \in E, \\ E \setminus \{1,3\} & \text{if } \{3,4\} \not\subseteq E \text{ and } 2 \notin E. \end{cases}$$

Notice that, for any $E \subseteq \{1, 2, 3, 4\}$, $A(E)$ is equal to the sovereign part of E in the disjunctive hierarchy represented by the digraph below



We proceed to calculate the characteristic functions A_1, A_2, A_3 and A_4.

E	$A_1(E)$	$A_2(E)$	$A_3(E)$	$A_4(E)$
$\{1\}$	0	0	0	0
$\{2\}$	0	1	0	0
$\{3\}$	0	0	0	0
$\{4\}$	0	0	0	1
$\{1,2\}$	1	1	0	0
$\{1,3\}$	0	0	0	0
$\{1,4\}$	0	0	0	1
$\{2,3\}$	0	1	0	0
$\{2,4\}$	0	1	0	1
$\{3,4\}$	0	0	1	1
$\{1,2,3\}$	1	1	0	0
$\{1,2,4\}$	1	1	0	1
$\{1,3,4\}$	1	0	1	1
$\{2,3,4\}$	0	1	1	1
$\{1,2,3,4\}$	1	1	1	1

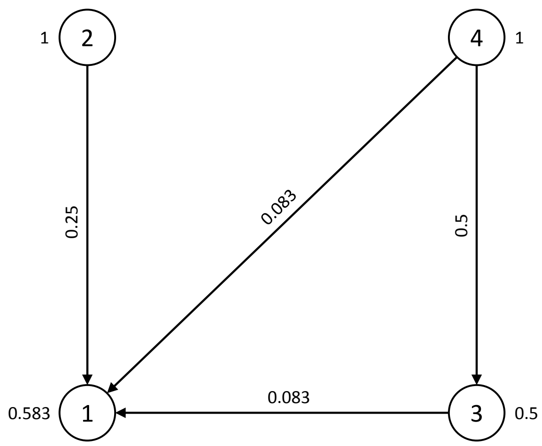
It holds that

$$\begin{aligned}
\phi(A_1) &= \left(\frac{7}{12}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}\right), \\
\phi(A_2) &= (0,1,0,0)\,, \\
\phi(A_3) &= \left(0, 0, \frac{1}{2}, \frac{1}{2}\right), \\
\phi(A_4) &= (0,0,0,1)\,.
\end{aligned}$$

So we have obtained that

$$\Xi(A) = \begin{pmatrix} \frac{7}{12} & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{12} & 0 & \frac{1}{2} & 0 \\ \frac{1}{12} & 0 & \frac{1}{2} & 1 \end{pmatrix}.$$

Rounding to the nearest thousandth, we can identify $\Xi(A)$ with the fuzzy cognitive map represented by the following fuzzy digraph



Our first goal will be to characterize the Shapley authorization correspondence. To do that, we consider the properties stated below. In the statement of these properties Ψ is an allocation rule for authorization structures.

- **EFFICIENCY**. For every $A \in \mathcal{A}^N$ it holds that

$$\sum_{k \in N} \Psi_k(A) = \mathbf{1}_{A(N)}.$$

Definition 4.5 *Let $A \in \mathcal{A}^N$. An agent $i \in N$ is said to be null in (N, A) if $A(E) = A(E \setminus \{i\})$ for all $E \subseteq N$.*

- NULL AGENT PROPERTY. For every $A \in \mathcal{A}^N$ and i null agent in (N, A), it holds that

$$\Psi_i(A) = 0.$$

The following property is inspired by the transfer property introduced by Dubey [35] to characterize the Shapley-Shubik index. It states that equal changes in the dependency relationships among the agents produce equal changes in the allocations.

- TRANSFER PROPERTY. For every $A, \hat{A}, B, \hat{B} \in \mathcal{A}^N$ such that, for every $E \subseteq N$, $A(E) \setminus \hat{A}(E) = B(E) \setminus \hat{B}(E)$ and $\hat{A}(E) \setminus A(E) = \hat{B}(E) \setminus B(E)$, it holds that

$$\Psi(A) - \Psi(\hat{A}) = \Psi(B) - \Psi(\hat{B}).$$

- EQUAL TREATMENT PROPERTY. For every $A \in \mathcal{A}^N$ and $i, j \in N$ such that $A(E \cup \{i\}) = A(E \cup \{j\})$ for all $E \subseteq N \setminus \{i, j\}$, it holds that

$$\Psi_i(A) = \Psi_j(A).$$

In the following theorem we see that these four properties stated above uniquely determine the Shapley authorization correspondence.

Theorem 4.6 *An allocation rule for authorization structures is equal to the Shapley authorization correspondence if and only if it satisfies the properties of efficiency, null agent, transfer and equal treatment.*

Proof. Firstly, it is proved that the Shapley authorization correspondence satisfies the four properties mentioned.

EFFICIENCY. Let $A \in \mathcal{A}^N$. For every $j \in N$ it holds that

$$\left(\sum_{k\in N} \Xi_k(A)\right)_j = \sum_{k\in N} \Xi_{kj}(A) = \sum_{k\in N} \Phi_k(u_{\{j\}}, A) = u_{\{j\}}(A(N)) = \left(\mathbf{1}_{A(N)}\right)_j .$$

NULL AGENT PROPERTY. Let $A \in \mathcal{A}^N$ and $i \in N$ be such that i is a null agent in (N, A). It is clear that, for every $j \in N$, i is an irrelevant player in $(u_{\{j\}}, A)$. From the irrelevant player property of the Shapley authorization value, it follows that $\Phi_i(u_{\{j\}}, A) = 0$ for all $j \in N$, and hence $\Xi_i(A) = 0$.

TRANSFER PROPERTY. Let $A, \hat{A}, B, \hat{B} \in \mathcal{A}^N$ be such that $A(E) \setminus \hat{A}(E) = B(E) \setminus \hat{B}(E)$ and $\hat{A}(E) \setminus A(E) = \hat{B}(E) \setminus B(E)$ for every $E \subseteq N$. It is clear that $A_j - \hat{A}_j = B_j - \hat{B}_j$ for every $j \in N$. So, for every $i, j \in N$, it holds that

$$\begin{aligned}
\Xi_{ij}(A) - \Xi_{ij}(\hat{A}) &= \phi_i(A_j) - \phi_i(\hat{A}_j) = \phi_i(A_j - \hat{A}_j) \\
&= \phi_i(B_j - \hat{B}_j) = \phi_i(B_j) - \phi_i(\hat{B}_j) \\
&= \Xi_{ij}(B) - \Xi_{ij}(\hat{B}).
\end{aligned}$$

EQUAL TREATMENT PROPERTY. Let $A \in \mathcal{A}^N$ and $i, j \in N$ be such that $A(E \cup \{i\}) = A(E \cup \{j\})$ for all $E \subseteq N \setminus \{i, j\}$. It is clear that for every $k \in N$, it holds that $A_k(E \cup \{i\}) = A_k(E \cup \{j\})$ for all $E \subseteq N \setminus \{i, j\}$. Making use of the equal treatment property of the Shapley value it follows that $\phi_i(A_k) = \phi_j(A_k)$ for all $k \in N$, and hence $\Xi_i(A) = \Xi_j(A)$.

Now we show that the properties in the theorem uniquely determine the Shapley authorization correspondence. Let Ψ be an allocation rule for authorization structures satisfying the properties of efficiency, null agent, transfer and equal treatment. We must prove that $\Psi = \Xi$.

Let $n \in \mathbb{N}$ and let N be a set of cardinality n. For every $T \in 2^N \setminus \{\emptyset\}$ and $i \in T$ we consider

$$\begin{array}{rccl}
C_{T,i}: & 2^N & \to & 2^N \\
& E & \to & C_{T,i}(E) = \begin{cases} \{i\} & \text{if } T \subseteq E, \\ \emptyset & \text{if } T \not\subseteq E. \end{cases}
\end{array}$$

It is clear that $C_{T,i} \in \mathcal{A}^N$.

Let $A \in \mathcal{A}^N$. If $A(N) = \emptyset$, we can easily deduce, from the null agent property, that $\Psi(A) = \Xi(A) = 0$. If $A(N) \neq \emptyset$, we can write

$$A = \bigcup_{\{(T,i)\in 2^N \times N : \, i \in A(T)\}} C_{T,i}.$$

So it is enough to show that for every $m \in \mathbb{N}$, $T_1, \ldots, T_m \in 2^N \setminus \{\emptyset\}$ and $i_1, \ldots, i_m \in N$ with $i_k \in T_k$ for all $k = 1, \ldots, m,$ it holds that

$$\Psi\left(\bigcup_{k=1}^{m} C_{T_k,i_k}\right) = \Xi\left(\bigcup_{k=1}^{m} C_{T_k,i_k}\right).$$

Let us prove this equality by strong induction on m.

1. Base case. Let $T \in 2^N \setminus \{\emptyset\}$ and $i \in T$. From the null agent property, we can easily derive that

$$\Psi_j(C_{T,i}) = 0 \quad \text{for all } j \in N \setminus T. \tag{4.1}$$

From the equal treatment property we can deduce that

$$\Psi_j(C_{T,i}) = \Psi_l(C_{T,i}) \quad \text{for all } j, l \in T. \tag{4.2}$$

From (4.1) and (4.2) we conclude, using the efficiency property, that

$$\Psi_j(C_{T,i}) = \begin{cases} \dfrac{1}{|T|}\,\mathbf{1}_{\{i\}} & \text{if } j \in T, \\ \\ 0 & \text{if } j \in N \setminus T. \end{cases}$$

Since Ξ also satisfies the properties used, it is clear that $\Psi(C_{T,i}) = \Xi(C_{T,i})$.

2. Inductive step. Take $T_1, \ldots, T_{m+1} \in 2^N \setminus \{\emptyset\}$ and $i_1, \ldots, i_{m+1} \in N$ with $i_k \in T_k$ for all

$k = 1, \ldots, m+1$. Let

$$A = \bigcup_{k=1}^{m+1} C_{T_k,i_k}, \qquad \hat{A} = \bigcup_{k=1}^{m} C_{T_k,i_k},$$

$$B = C_{T_{m+1},i_{m+1}}, \qquad \hat{B} = \bigcup_{k=1}^{m} (C_{T_k,i_k} \cap C_{T_{m+1},i_{m+1}}).$$

Since Ψ and Ξ satisfy the transfer property, we have

$$\Psi(A) - \Psi(\hat{A}) = \Psi(B) - \Psi(\hat{B}), \tag{4.3}$$

$$\Xi(A) - \Xi(\hat{A}) = \Xi(B) - \Xi(\hat{B}). \tag{4.4}$$

We already know that

$$\Psi(B) = \Xi(B). \tag{4.5}$$

By induction hypothesis, it holds

$$\Psi(\hat{A}) = \Xi(\hat{A}). \tag{4.6}$$

Now, observe that if $i_k \neq i_{m+1}$ then we can eliminate $C_{T_k,i_k} \cap C_{T_{m+1},i_{m+1}}$ in the expression of $\hat{B}$, and if $i_k = i_{m+1}$ then $C_{T_k,i_k} \cap C_{T_{m+1},i_{m+1}} = C_{T_k \cup T_{m+1},i_k}$. Then, by induction hypothesis, we have

$$\Psi(\hat{B}) = \Xi(\hat{B}). \tag{4.7}$$

From (4.3), (4.4), (4.5), (4.6) and (4.7) we deduce

$$\Psi\left(\bigcup_{k=1}^{m+1} C_{T_k,i_k}\right) = \Xi\left(\bigcup_{k=1}^{m+1} C_{T_k,i_k}\right).$$

□

4.3 Indices for authorization structures

Let $A \in \mathcal{A}^N$ and $i, j \in N$. It holds that

$$\Xi_{ij}(A) = \phi_i(A_j) = \sum_{\{E \subseteq N:\, i \in E\}} p_E \left[A_j(E) - A_j(E \setminus \{i\}) \right],$$

where the numbers p_E are the coefficients of the Shapley value.

It is clear that $\Xi_{ij}(A) > 0$ if and only if j depends partially on i in (N, A). In fact, $\Xi_{ij}(A)$ can measure how strongly agent j depends on agent i in the structure (N, A). This motivates the following definitions.

Let $A \in \mathcal{A}^N$ and $i, j \in N$ with $i \neq j$. The number $\Xi_{ij}(A)$ is called *the influence index of i over j in (N, A)*.

Definition 4.7 *Let $A \in \mathcal{A}^N$ and $i \in N$. The sum of the influence indices of i over each one of the rest of agents is called the influence index of i in (N, A) and is denoted by $inf_i(A)$, that is,*

$$inf_i(A) = \sum_{j \in N \setminus \{i\}} \Xi_{ij}(A).$$

Definition 4.8 *Let $A \in \mathcal{A}^N$ and $i \in N$. The number $\Xi_{ii}(A)$ is called the sovereignty index of i in (N, A) and is denoted by $sov_i(A)$.*

Definition 4.9 *Let $A \in \mathcal{A}^N$ and $i \in N$. The addition of the influence and sovereignty indices of i is called the power index of i in (N, A) and is denoted by $pow_i(A)$, that is,*

$$pow_i(A) = sov_i(A) + inf_i(A).$$

Example 4.10 Let us calculate the sovereignty, influence and power indices in Example 4.4. It is easy to check that

$$sov_1(A) = \frac{7}{12}, \qquad sov_2(A) = 1, \qquad sov_3(A) = \frac{1}{2}, \qquad sov_4(A) = 1,$$

$$inf_1(A) = 0, \qquad inf_2(A) = \frac{1}{4}, \qquad inf_3(A) = \frac{1}{12}, \qquad inf_4(A) = \frac{7}{12},$$

$$pow_1(A) = \frac{7}{12}, \qquad pow_2(A) = \frac{5}{4}, \qquad pow_3(A) = \frac{7}{12}, \qquad pow_4(A) = \frac{19}{12}.$$

We want to show the most important properties of the sovereignty, influence and power indices. To that end, it is convenient to introduce some notation. Let $A \in \mathcal{A}^N$ and $i \in N$. We denote

$$\mathtt{V}_i(A) = \{j \in N : j \text{ has veto power over } i \text{ in } (N, A)\},$$

$$\mathtt{P}_i(A) = \{j \in N : i \text{ depends partially on } j \text{ in } (N, A)\}.$$

Firstly, we see a characterization of the sovereignty index. To that end, we consider the properties stated below. In the statement of these properties Ψ is a mapping from $\mathcal{A}^N$ into $\mathbb{R}^N$.

Definition 4.11 *Let $A \in \mathcal{A}^N$ and $i \in N$. An agent i is said to be inactive in (N, A) if $i \notin A(N)$.*

- **INACTIVE AGENT PROPERTY**. For every $A \in \mathcal{A}^N$ and $i \in N$ such that i is inactive in (N, A), it holds that

$$\Psi_i(A) = 0.$$

If we wanted to evaluate the freedom of an agent i in an authorization structure (N, A), perhaps the first two measures we would think of would be the numbers $\frac{1}{|\mathtt{V}_i(A)|}$ and $\frac{1}{|\mathtt{P}_i(A)|}$. Notice that both are extreme in the sense that with the first number we would ignore all the dependency relationships that are not veto relationships, whereas with the second number we would equally value all the dependency relationships. The following property distinguishes the indices that are in between those two.

- **MAXIMUM AND MINIMUM SOVEREIGNTY**. For every $A \in \mathcal{A}^N$ and $i \in N$ such that i is not inactive in (N, A), it holds that

$$\frac{1}{|\mathtt{P}_i(A)|} \leqslant \Psi_i(A) \leqslant \frac{1}{|\mathtt{V}_i(A)|}.$$

- **TRANSFER PROPERTY**. For every $A, \hat{A}, B, \hat{B} \in \mathcal{A}^N$ such that, for every $E \subseteq N$, $A(E) \setminus \hat{A}(E) = B(E) \setminus \hat{B}(E)$ and $\hat{A}(E) \setminus A(E) = \hat{B}(E) \setminus B(E)$, it holds that

$$\Psi(A) - \Psi(\hat{A}) = \Psi(B) - \Psi(\hat{B}).$$

In the following theorem we see that these properties uniquely determine the sovereignty index. We need the following result.

Lemma 4.12 *Let $T \in 2^N \setminus \{\emptyset\}$. Then*

$$\sum_{\{E\subseteq N:\, T\subseteq E\}} p_E = \frac{1}{|T|}$$

where the numbers p_E are the coefficients of the Shapley value.

Theorem 4.13 *A mapping $\Psi : \mathcal{A}^N \to \mathbb{R}^N$ is equal to the sovereignty index if and only if it satisfies the properties of inactive agent, maximum and minimum sovereignty and transfer.*

Proof. Firstly, we prove that the sovereignty index satisfies the properties mentioned in the theorem.

INACTIVE AGENT PROPERTY. Let $A \in \mathcal{A}^N$ and $i \in N$ be such that i is inactive in (N, A). It follows that $A_i(E) = 0$ for all $E \subseteq N$. It holds that $sov_i(A) = \Xi_{ii}(A) = \phi_i(A_i) = 0$.

MAXIMUM AND MINIMUM SOVEREIGNTY. Let $A \in \mathcal{A}^N$ and $i \in N$ be such that i is not inactive in (N, A). It holds that

$$\begin{aligned} sov_i(A) &= \Xi_{ii}(A) = \phi_i(A_i) = \sum_{\{E\subseteq N:\, i\in E\}} p_E\, [A_i(E) - A_i(E \setminus \{i\})] \\ &= \sum_{\{E\subseteq N:\, i\in E\}} p_E\, A_i(E) = \sum_{\{E\subseteq N:\, i\in A(E)\}} p_E. \end{aligned} \quad (4.8)$$

Firstly, we prove that $\dfrac{1}{|\mathtt{P}_i(A)|} \leqslant sov_i(A)$.

Notice that for every $E \subseteq N$ with $\mathtt{P}_i(A) \subseteq E$ it holds that $i \in A(E)$ (otherwise, we would have $i \in A(N) \setminus A(E)$ from what we could easily deduce that there would exist $j \in N \setminus E$ such that i depends partially on j, which would be absurd). Therefore, the sum in (4.8) is greater or equal to

$$\sum_{\{E \subseteq N \colon \mathtt{P}_i(A) \subseteq E\}} p_E$$

which, from Lemma 4.12, is equal to $\dfrac{1}{|\mathtt{P}_i(A)|}$.

Now we prove that $sov_i(A) \leqslant \dfrac{1}{|\mathtt{V}_i(A)|}$.

It is clear that for every $E \subseteq N$ with $i \in A(E)$ it holds that $\mathtt{V}_i(A) \subseteq E$. Therefore, the sum in (4.8) is less or equal to

$$\sum_{\{E \subseteq N \colon \mathtt{V}_i(A) \subseteq E\}} p_E$$

which, from Lemma 4.12, is equal to $\dfrac{1}{|\mathtt{V}_i(A)|}$.

Transfer property. It is a direct consequence of the fact that the Shapley authorization correspondence satisfies an analogous transfer property.

Now we show that these properties uniquely determine the sovereignty index. The reasoning is similar to that followed in the case of the Shapley authorization correspondence. Suppose that $\Psi : \mathcal{A}^N \to \mathbb{R}^N$ satisfies the properties of inactive agent, maximum and minimum sovereignty and transfer. We must prove that Ψ is equal to the sovereignty index.

Let $A \in \mathcal{A}^N$. If $A(N) = \emptyset$, we know, from the inactive agent property, that $\Psi(A) = sov(A) = 0$. If $A(N) \neq \emptyset$, we can write

$$A = \bigcup_{\{(T,i) \in 2^N \times N \colon i \in A(T)\}} C_{T,i}.$$

So, if we want to prove that $\Psi = sov$, it is enough to show that for every $m \in \mathbb{N}$, $T_1, \ldots, T_m \in 2^N \setminus \{\emptyset\}$ and $i_1, \ldots, i_m \in N$ with $i_k \in T_k$ for all $k = 1, \ldots, m,$ it holds that

$$\Psi\left(\bigcup_{k=1}^{m} C_{T_k,i_k}\right) = sov\left(\bigcup_{k=1}^{m} C_{T_k,i_k}\right).$$

Let us prove this equality by strong induction on m.

1. BASE CASE. Let $T \in 2^N \setminus \{\emptyset\}$ and $i \in T$. From the inactive agent property we obtain that

$$\Psi_j(C_{T,i}) = 0 \quad \text{for all } j \in N \setminus \{i\}. \tag{4.9}$$

Notice that $\mathtt{P}_i(C_{T,i}) = \mathtt{V}_i(C_{T,i}) = T$. From the property of maximum and minimum sovereignty it follows that

$$\Psi_i(C_{T,i}) = \frac{1}{|T|}. \tag{4.10}$$

From (4.9) and (4.10) we have that $\Psi(C_{T,i}) = \frac{1}{|T|}\mathbf{1}_{\{i\}}$. Since the sovereignty index also satisfies the properties used, we obtain that

$$\Psi(C_{T,i}) = sov(C_{T,i}).$$

2. INDUCTIVE STEP. The reasoning is equal to that followed in the case of the Shapley authorization correspondence.

□

There is something to comment about the property of maximum and minimum sovereignty. Actually, the two inequalities given are strict except in the case that $\mathtt{P}_i(A) = \mathtt{V}_i(A)$. Let us see this. Let $A \in \mathcal{A}^N$ and $i \in N$ such that i is not inactive in (N, A). In the proof of the property referred to we used that

$$sov_i(A) = \sum_{\{E \subseteq N :\, i \in A(E)\}} p_E, \tag{4.11}$$

$$\frac{1}{|\mathtt{P}_i(A)|} = \sum_{\{E \subseteq N :\, \mathtt{P}_i(A) \subseteq E\}} p_E, \tag{4.12}$$

$$\frac{1}{|\mathtt{V}_i(A)|} = \sum_{\{E \subseteq N :\, \mathtt{V}_i(A) \subseteq E\}} p_E, \tag{4.13}$$

and

$$\{E \subseteq N : \mathtt{P}_i(A) \subseteq E\} \subseteq \{E \subseteq N : i \in A(E)\} \subseteq \{E \subseteq N : \mathtt{V}_i(A) \subseteq E\}. \tag{4.14}$$

Suppose that $sov_i(A) = \dfrac{1}{|\mathtt{P}_i(A)|}$. In this case, from (4.11), (4.12) and (4.14), it follows that $i \in A(E)$ if and only if $\mathtt{P}_i(A) \subseteq E$. It easily derives that $\mathtt{P}_i(A) = \mathtt{V}_i(A)$.
Suppose that $sov_i(A) = \dfrac{1}{|\mathtt{V}_i(A)|}$. In this case, from (4.11), (4.13) and (4.14) it follows that $i \in A(E)$ if and only if $\mathtt{V}_i(A) \subseteq E$. Consequently, it holds that $\mathtt{P}_i(A) = \mathtt{V}_i(A)$.

It is clear that the sovereignty index of an agent is equal to 0 if and only if the agent is inactive in the structure. Let us see when the sovereignty index of an agent is equal to 1.

Definition 4.14 *Let $A \in \mathcal{A}^N$ and $i \in N$. An agent i is said to be sovereign in (N, A) if $A(\{i\}) = \{i\}$.*

Lemma 4.15 *Let $A \in \mathcal{A}^N$ and $i \in N$. Then, i is sovereign in (N, A) if and only if $sov_i(A) = 1$.*

Proof. Let $A \in \mathcal{A}^N$ and $i \in N$. We saw in (4.8) that

$$sov_i(A) = \sum_{\{E \subseteq N:\, i \in A(E)\}} p_E. \tag{4.15}$$

Moreover, we know from Lemma 4.12 that

$$\sum_{\{E \subseteq N:\, i \in E\}} p_E = 1. \tag{4.16}$$

From (4.15) and (4.16) it easily follows that $sov_i(A)$ is equal to 1 if and only if $i \in A(E)$ for every $E \subseteq N$ with $i \in E$, or, equivalently, $A(\{i\}) = \{i\}$. □

Our next goal will be to give a characterization of the influence index. We consider the following property, where Ψ is a mapping from $\mathcal{A}^N$ into $\mathbb{R}^N$.

- MAXIMUM AND MINIMUM INFLUENCE. For every $A \in \mathcal{A}^N$ and $i \in N$, it holds that

$$\sum_{\{j \in A(N) \setminus \{i\}:\, i \in \mathtt{V}_j(A)\}} \frac{1}{|\mathtt{P}_j(A)|} \leqslant \Psi_i(A) \leqslant \sum_{\{j \in A(N) \setminus \{i\}:\, i \in \mathtt{P}_j(A)\}} \frac{1}{|\mathtt{V}_j(A)|}.$$

In the following theorem we show that this property together with the transfer property uniquely determine the influence index. We need a previous lemma.

Lemma 4.16 *Let $A \in \mathcal{A}^N$ and $i, j \in N$. It holds that*

$$\Xi_{ij}(A) \leqslant sov_j(A)$$

and the equality holds if and only if i has veto power over j in (N, A).

Proof. Let $A \in \mathcal{A}^N$ and $i, j \in N$. Remember that $sov_j(A) = \Xi_{jj}(A) = \phi_j(A_j)$ and $\Xi_{ij}(A) = \phi_i(A_j)$, where $A_j(E) = |A(E) \cap \{j\}|$ for every $E \subseteq N$. It is clear that j is a necessary player in the monotonic game A_j. From the necessary player property of the Shapley value it follows that $\phi_i(A_j) \leqslant \phi_j(A_j)$. So we have proved the inequality given in the lemma.

Now suppose that i has veto power over j in (N, A). It this case i is also a necessary player in A_j. So we conclude that $\phi_i(A_j) = \phi_j(A_j)$. Conversely, suppose that $\phi_i(A_j) = \phi_j(A_j)$. It holds that

$$\phi_j(A_j) = \sum_{\{E \subseteq N : \, j \in A(E)\}} p_E \tag{4.17}$$

and

$$\phi_i(A_j) = \sum_{\{E \subseteq N : \, i \in E\}} p_E \left[A_j(E) - A_j(E \setminus \{i\}) \right] = \sum_{\{E \subseteq N : \, j \in A(E) \setminus A(E \setminus \{i\})\}} p_E. \tag{4.18}$$

We conclude that

$$\{E \subseteq N : \, j \in A(E)\} = \{E \subseteq N : \, j \in A(E) \setminus A(E \setminus \{i\})\}$$

whence it derives that i has veto power over j in (N, A). □

Theorem 4.17 *A mapping $\Psi : \mathcal{A}^N \to \mathbb{R}^N$ is equal to the influence index if and only if it satisfies the properties of maximum and minimum influence and transfer.*

Proof. Firstly, we prove that the influence index satisfies the properties mentioned in the theorem.

Maximum and minimum influence. Let $A \in \mathcal{A}^N$ and $i \in N$. We have that

$$inf_i(A) = \sum_{j \in N \setminus \{i\}} \Xi_{ij}(A)$$

which, taking into consideration that $\Xi_{ij}(A) > 0$ if and only if j depends partially on i in (N, A), is equal to

$$\sum_{\{j \in A(N) \setminus \{i\} : \, i \in \mathtt{P}_j(A)\}} \Xi_{ij}(A)$$

which, from Lemma 4.16, is less or equal to

$$\sum_{\{j \in A(N) \setminus \{i\} : \, i \in \mathtt{P}_j(A)\}} sov_j(A)$$

which, from the property of maximum and minimum sovereignty, is less or equal to

$$\sum_{\{j \in A(N) \setminus \{i\} : \, i \in \mathtt{P}_j(A)\}} \frac{1}{|\mathtt{V}_j(A)|}.$$

So we have proved the right-hand inequality. Let us prove the other one.

$$inf_i(A) = \sum_{\{j \in A(N) \setminus \{i\} : \, i \in \mathtt{P}_j(A)\}} \Xi_{ij}(A) \geqslant \sum_{\{j \in A(N) \setminus \{i\} : \, i \in \mathtt{V}_j(A)\}} \Xi_{ij}(A)$$

which, using the second statement of Lemma 4.16, is equal to

$$\sum_{\{j \in A(N) \setminus \{i\} : \, i \in \mathtt{V}_j(A)\}} sov_j(A)$$

which, from the property of maximum and minimum sovereignty, is greater or equal to

$$\sum_{\{j \in A(N) \setminus \{i\} : \, i \in \mathtt{V}_j(A)\}} \frac{1}{|\mathtt{P}_j(A)|}.$$

It is easy to see that, actually, the two inequalities given in the property are strict except in case that

$$\{j \in A(N) \setminus \{i\} : i \in \mathtt{V}_j(A)\} = \{j \in A(N) \setminus \{i\} : i \in \mathtt{P}_j(A)\}$$

and, moreover, for every j in that set, $\mathtt{P}_j(A) = \mathtt{V}_j(A)$.

TRANSFER PROPERTY. It is a direct consequence of the fact that the Shapley authorization correspondence satisfies an analogous transfer property.

Now we show that these properties uniquely determine the influence index. The reasoning is similar to that followed in the case of the Shapley authorization correspondence. Suppose that $\Psi : \mathcal{A}^N \to \mathbb{R}^N$ satisfies the properties of maximum and minimum influence and transfer. We must prove that Ψ is equal to the influence index.

Let $A \in \mathcal{A}^N$. If $A(N) = \emptyset$, we know, from the property of maximum and minimum influence, that $\Psi(A) = inf(A) = 0$. If $A(N) \neq \emptyset$, we can write

$$A = \bigcup_{\{(T,i)\in 2^N \times N:\, i\in A(T)\}} C_{T,i}.$$

So, if we want to prove that $\Psi = inf$, it is enough to show that for every $m \in \mathbb{N}$, $T_1, \ldots, T_m \in 2^N \setminus \{\emptyset\}$ and $i_1, \ldots, i_m \in N$ with $i_k \in T_k$ for all $k = 1, \ldots, m$ it holds that

$$\Psi\left(\bigcup_{k=1}^{m} C_{T_k,i_k}\right) = inf\left(\bigcup_{k=1}^{m} C_{T_k,i_k}\right).$$

Let us prove this equality by strong induction on m.

1. BASE CASE. Let $T \in 2^N \setminus \{\emptyset\}$ and $i \in T$. We can derive, from the property of maximum and minimum influence, that

$$\Psi_j(C_{T,i}) = \begin{cases} \dfrac{1}{|T|} & \text{if } j \in T \setminus \{i\}, \\ \\ 0 & \text{if } j \in (N \setminus T) \cup \{i\}. \end{cases}$$

   Since the influence index also satisfies the property used, we obtain that

$$\Psi(C_{T,i}) = inf(C_{T,i}).$$

2. INDUCTIVE STEP. The reasoning is equal to that followed in the case of the Shapley authorization correspondence.

□

Now we intend to give a characterization of the power index. For that purpose, we consider the following properties, where Ψ is a mapping from $\mathcal{A}^N$ into $\mathbb{R}^N$.

- **EFFICIENCY**. For every $A \in \mathcal{A}^N$ it holds that

$$\sum_{k \in N} \Psi_k(A) = |A(N)|.$$

- **NULL AGENT PROPERTY**. For every $A \in \mathcal{A}^N$ and $i \in N$ such that i is a null agent in (N, A), it holds that

$$\Psi_i(A) = 0.$$

- **EQUAL TREATMENT PROPERTY**. For every $A \in \mathcal{A}^N$ and $i, j \in N$ such that $A(E \cup \{i\}) = A(E \cup \{j\})$ for all $E \subseteq N \setminus \{i, j\}$, it holds that

$$\Psi_i(A) = \Psi_j(A).$$

These properties together with the transfer property uniquely determine the power index.

Theorem 4.18 *A mapping $\Psi : \mathcal{A}^N \to \mathbb{R}^N$ is equal to the power index if and only if it satisfies the properties of efficiency, null agent, transfer and equal treatment.*

Proof. That the power index satisfies the properties mentioned in the theorem is a direct consequence of the fact that the Shapley authorization correspondence satisfies analogous properties. The proof that the properties in the theorem uniquely determine the power index is similar to the proof of the uniqueness of the Shapley authorization correspondence, so we omit it. □

4.4 The Shapley fuzzy authorization correspondence

In this section we study relational power in fuzzy authorization structures. Our aim is to extend the concepts introduced in the previous section.

Definition 4.19 *An allocation rule for fuzzy authorization structures assigns to each fuzzy authorization operator on N a mapping in $\left([0,1]^N\right)^N$.*

In a similar way as we defined the Shapley authorization correspondence, we make use of the Shapley value to introduce an allocation rule for fuzzy authorization structures.

Definition 4.20 *The Shapley fuzzy authorization correspondence assigns to each fuzzy authorization operator $a \in \mathcal{FA}^N$ the mapping $\xi(a) \in \left([0,1]^N\right)^N$ defined as*

$$\xi_i(a) = \left(\phi_i(a_j)\right)_{j \in N} \qquad \text{for every } i \in N,$$

where ϕ is the Shapley value.

We denote $\xi_{ij}(a) = \phi_i(a_j)$. Observe that $\xi(a)$ can be identified with a matrix in $\mathcal{M}_n(\mathbb{R})$. In fact, $\xi(a)$ is a submatrix of the matrix ζ_a defined in Corollary 3.18, since for every $a \in \mathcal{FA}^N$ and $i, j \in N$ it holds that

$$\xi_{ij}(a) = \phi_i(a_j) = \phi_i\left(u^a_{\{j\}}\right) = \varphi_i(u_{\{j\}}, a) = (\zeta_a)_{i,\{j\}}\,.$$

In a similar way as we did in the crisp case, $\xi(a)$ can be interpreted as a fuzzy cognitive map on N.

Our first goal will be to characterize the Shapley fuzzy authorization correspondence. To do that, we consider the properties stated below. In the statement of these properties ψ is an allocation rule for fuzzy authorization structures.

- **EFFICIENCY**. For every $a \in \mathcal{FA}^N$ it holds that

$$\sum_{k \in N} \psi_k(a) = a(N).$$

Definition 4.21 *Let $a \in \mathcal{FA}^N$. An agent $i \in N$ is said to be null in (N, a) if $a(E) = a(E \setminus \{i\})$ for all $E \subseteq N$.*

- **NULL AGENT PROPERTY**. For every $a \in \mathcal{FA}^N$ and i null agent in (N, a), it holds that

$$\psi_i(a) = 0.$$

- **TRANSFER PROPERTY**. For every $a, \hat{a}, b, \hat{b} \in \mathcal{FA}^N$ such that $a - \hat{a} = b - \hat{b}$, it holds that

$$\psi(a) - \psi(\hat{a}) = \psi(b) - \psi(\hat{b}).$$

- **EQUAL TREATMENT PROPERTY**. For every $a \in \mathcal{FA}^N$ and $i, j \in N$ such that $a(E \cup \{i\}) = a(E \cup \{j\})$ for all $E \subseteq N \setminus \{i, j\}$, it holds that

$$\psi_i(a) = \psi_j(a).$$

- **HOMOGENEITY**. For every $a \in \mathcal{FA}^N$ and $t \in (0, 1)$, it holds that

$$\psi(ta) = t\psi(a).$$

In the following theorem we see that these five properties stated above uniquely determine the Shapley fuzzy authorization correspondence. Before, we make a remark.

Remark 4.22 *Let $a \in \mathcal{FA}^N$ and $i, j \in N$. Let $\{h_l : l = 0, \ldots, m\} \subset [0, 1]$ be such that*

$\{h_l : l = 0, \ldots, m\} \supseteq \{a_k(F) : F \subseteq N, k \in N\}$ *with* $0 = h_0 < \ldots < h_m$. *Then,*

$$\begin{aligned}
\xi_{ij}(a) &= \phi_i(a_j) = \phi_i\left(u^a_{\{j\}}\right) = \varphi_i(u_{\{j\}}, a) \\
&= \sum_{l=1}^{m}(h_l - h_{l-1})\,\Phi_i\left(u_{\{j\}}, a^{h_l}\right) = \sum_{l=1}^{m}(h_l - h_{l-1})\,\Xi_{ij}(a^{h_l}).
\end{aligned}$$

So it holds that

$$\xi(a) = \sum_{l=1}^{m}(h_l - h_{l-1})\,\Xi(a^{h_l}).$$

Theorem 4.23 *An allocation rule for fuzzy authorization structures is equal to the Shapley fuzzy authorization correspondence if and only if it satisfies the properties of efficiency, null agent, transfer, equal treatment and homogeneity.*

Proof. Firstly, we see that the Shapley fuzzy authorization correspondence satisfies the five properties mentioned.

EFFICIENCY. Let $a \in \mathcal{FA}^N$. For every $j \in N$ it holds that

$$\left(\sum_{k \in N} \xi_k(a)\right)_j = \sum_{k \in N} \phi_k(a_j) = a_j(N).$$

NULL AGENT PROPERTY. Let $a \in \mathcal{FA}^N$ and $i \in N$ a null agent in (N, a). It is clear that, for every $j \in N$, i is a null player in a_j. From the null player property of the Shapley value we conclude that $\phi_i(a_j) = 0$ for every $j \in N$. Therefore, $\xi_i(a) = 0$.

TRANSFER PROPERTY. Let $a, \hat{a}, b, \hat{b} \in \mathcal{FA}^N$ be such that $a - \hat{a} = b - \hat{b}$. Let $i, j \in N$. It holds that

$$\begin{aligned}
\xi_{ij}(a) - \xi_{ij}(\hat{a}) &= \phi_i(a_j) - \phi_i(\hat{a}_j) = \phi_i\left(a_j - \hat{a}_j\right) \\
&= \phi_i(b_j - \hat{b}_j) = \phi_i(b_j) - \phi_i(\hat{b}_j) \\
&= \xi_{ij}(b) - \xi_{ij}(\hat{b}).
\end{aligned}$$

EQUAL TREATMENT PROPERTY. Let $a \in \mathcal{FA}^N$ and $i, j \in N$ be such that $a(E \cup \{i\}) = a(E \cup \{j\})$ for all $E \subseteq N \backslash \{i, j\}$. For every $k \in N$, it holds that $a_k(E \cup \{i\}) = a_k(E \cup \{j\})$ for all $E \subseteq N \backslash \{i, j\}$.

Using the equal treatment property of the Shapley value we derive that $\phi_i(a_k) = \phi_j(a_k)$ for every $k \in N$, and hence $\xi_i(a) = \xi_j(a)$.

HOMOGENEITY. It follows easily from the definition of the Shapley fuzzy authorization correspondence.

It remains to prove that the properties in the theorem uniquely determine the Shapley fuzzy authorization correspondence. Let ψ be an allocation rule for fuzzy authorization structures satisfying the properties of efficiency, null agent, transfer, equal treatment and homogeneity. We must prove that

$$\psi(a) = \xi(a) \quad \text{for every } n \in \mathbb{N} \text{ and } a \in \mathcal{FA}^N.$$

We proceed by strong induction on $\lceil(a)$ where

$$\lceil(a) = |\{a_k(F) : F \subseteq N, k \in N\} \setminus \{0,1\}| \quad \text{for all } a \in \mathcal{FA}^N.$$

1. BASE CASE. $\lceil(a) = 0$.

   Notice that we can identify $\mathcal{A}^N$ with the set $\{a \in \mathcal{FA}^N : im(a) \subseteq \{0,1\}^N\}$. From this point of view, we can say that the restriction of ψ to the set of fuzzy authorization operators a with $\lceil(a) = 0$ is an allocation rule for authorization structures. It is easy to check that such restriction satisfies the properties of efficiency, null agent, transfer and equal treatment. Therefore, using Theorem 4.6, it must hold that $\psi(a) = \xi(a)$ for every $n \in \mathbb{N}$ and $a \in \mathcal{FA}^N$ with $\lceil(a) = 0$.

2. INDUCTIVE STEP. Let $n \in \mathbb{N}$ and $a \in \mathcal{FA}^N$ with $\lceil(a) > 0$. We want to prove that $\psi(a) = \xi(a)$. Take $t \in \{a_k(F) : F \subseteq N, k \in N\} \setminus \{0,1\}$. It holds that

   $$a = ta^{[0,t]} + (1-t)a^{[t,1]},$$

   whence, using transfer, homogeneity and null agent, it follows that

   $$\psi(a) = t\,\psi(a^{[0,t]}) + (1-t)\,\psi(a^{[t,1]}), \tag{4.19}$$

   $$\xi(a) = t\,\xi(a^{[0,t]}) + (1-t)\,\xi(a^{[t,1]}). \tag{4.20}$$

Since $\lceil(a^{[0,t]}) < \lceil(a)$ and $\lceil(a^{[t,1]}) < \lceil(a)$ it follows by induction hypothesis that

$$\psi(a^{[0,t]}) = \xi(a^{[0,t]}), \tag{4.21}$$

$$\psi(a^{[t,1]}) = \xi(a^{[t,1]}). \tag{4.22}$$

From (4.19), (4.20), (4.21) and (4.22) it follows that $\psi(a) = \xi(a)$.

□

4.5 Indices for fuzzy authorization structures

In a similar way as we defined the sovereignty, influence and power indices for authorization structures, we can define analogous indices for fuzzy authorization structures.

Let $a \in \mathcal{FA}^N$ and $i, j \in N$ with $i \neq j$. The number $\xi_{ij}(a)$ is called *the influence index of* i *over* j *in* (N, a).

Definition 4.24 *Let $a \in \mathcal{FA}^N$ and $i \in N$. The sum of the influence indices of i over each one of the rest of agents is called the influence index of i in (N, a) and is denoted by $inf_i(a)$, that is,*

$$inf_i(a) = \sum_{j \in N \setminus \{i\}} \xi_{ij}(a).$$

Definition 4.25 *Let $a \in \mathcal{FA}^N$ and $i \in N$. The number $\xi_{ii}(a)$ is called the sovereignty index of i in (N, a) and is denoted by $sov_i(a)$.*

Definition 4.26 *Let $a \in \mathcal{FA}^N$ and $i \in N$. The addition of the influence and sovereignty indices of i is called the power index of i in (N, a) and is denoted by $pow_i(a)$, that is,*

$$pow_i(a) = sov_i(a) + inf_i(a).$$

Remark 4.27 *Let $a \in \mathcal{FA}^N$ and $\{h_l : l = 0, \ldots, m\} \subset [0, 1]$ be such that $\{h_l : l = 0, \ldots, m\} \supseteq \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = h_0 < \ldots < h_m$. From Remark*

4.22, it holds that

$$sov(a) = \sum_{l=1}^{m}(h_l - h_{l-1})\, sov(a^{h_l}), \tag{4.23}$$

$$inf(a) = \sum_{l=1}^{m}(h_l - h_{l-1})\, inf(a^{h_l}), \tag{4.24}$$

$$pow(a) = \sum_{l=1}^{m}(h_l - h_{l-1})\, pow(a^{h_l}). \tag{4.25}$$

We aim to see the most important properties of the sovereignty, influence and power indices. Let $a \in \mathcal{FA}^N$ and $i \in N$. We denote

$$\mathtt{V}_i(a) = \{j \in N : j \text{ has veto power over } i \text{ in } (N,a)\},$$

$$\mathtt{P}_i(a) = \{j \in N : i \text{ depends partially on } j \text{ in } (N,a)\}.$$

Firstly, we see a characterization of the sovereignty index. To that end, we consider the properties stated below. In the statement of these properties ψ is a mapping from $\mathcal{FA}^N$ to $\mathbb{R}^N$.

Definition 4.28 *Let $a \in \mathcal{FA}^N$ and $i \in N$. An agent i is said to be inactive in (N,a) if $a_i(N) = 0$.*

- **INACTIVE AGENT PROPERTY**. For every $a \in \mathcal{FA}^N$ and $i \in N$ such that i is inactive in (N,a), it holds that
$$\psi_i(a) = 0.$$
- **MAXIMUM AND MINIMUM SOVEREIGNTY**. For every $a \in \mathcal{FA}^N$ and $i \in N$ such that i is not inactive in (N,a), it holds that
$$\frac{a_i(N)}{|\mathtt{P}_i(a)|} \leqslant \psi_i(a) \leqslant \frac{a_i(N)}{|\mathtt{V}_i(a)|}.$$
- **TRANSFER PROPERTY**. For every $a, \hat{a}, b, \hat{b} \in \mathcal{FA}^N$ such that $a - \hat{a} = b - \hat{b}$, it holds that
$$\psi(a) - \psi(\hat{a}) = \psi(b) - \psi(\hat{b}).$$

- **HOMOGENEITY**. For every $a \in \mathcal{FA}^N$ and $t \in (0,1)$ it holds that

$$\psi(ta) = t\psi(a).$$

In the following theorem we see that these properties uniquely determine the sovereignty index.

Theorem 4.29 *A mapping $\psi : \mathcal{FA}^N \to \mathbb{R}^N$ is equal to the sovereignty index if and only if it satisfies the properties of inactive agent, maximum and minimum sovereignty, transfer and homogeneity.*

Proof. Firstly, we see that the sovereignty index satisfies the properties mentioned in the theorem.

INACTIVE AGENT PROPERTY. Let $a \in \mathcal{FA}^N$ and $i \in N$ such that i is inactive in (N,a). It follows that $a_i(E) = 0$ for every $E \subseteq N$. It holds that $sov_i(a) = \xi_{ii}(a) = \phi_i(a_i) = 0$.

MAXIMUM AND MINIMUM SOVEREIGNTY. Let $a \in \mathcal{FA}^N$ and $i \in N$ such that i is not inactive in (N,a). Let $\{t_l : l = 0,\dots,r\} = \{a_j(F) : F \subseteq N,\, j \in N\}$ with $0 = t_0 < \ldots < t_r$. Using (4.23) we can write

$$sov_i(a) = \sum_{l=1}^{r}(t_l - t_{l-1})\, sov_i(a^{t_l}).$$

Let $m \in \{1,\dots,r\}$ such that $a_i(N) = t_m$. Notice that, given $l \in \{1,\dots,r\}$, if $l > m$ then i is inactive in (N,a^{t_l}), whereas if $l \leqslant m$ then i is not inactive in (N,a^{t_l}). Taking into consideration the inactive agent property of the (crisp) sovereignty index we can write

$$sov_i(a) = \sum_{l=1}^{m}(t_l - t_{l-1})\, sov_i(a^{t_l}). \tag{4.26}$$

From the property of maximum and minimum sovereignty of the (crisp) sovereignty index it follows that for every $l \in \{1,\dots,m\}$ it holds that

$$\frac{1}{|\mathtt{P}_i(a^{t_l})|} \leqslant sov_i(a^{t_l}) \leqslant \frac{1}{|\mathtt{V}_i(a^{t_l})|}. \tag{4.27}$$

But it is clear that

$$\mathtt{V}_i(a) \subseteq \mathtt{V}_i(a^{t_l}) \quad \text{for every } l \in \{1,\dots,r\} \tag{4.28}$$

and

$$\mathtt{P}_i(a) \supseteq \mathtt{P}_i(a^{t_l}) \quad \text{for every } l \in \{1, \ldots, r\}. \tag{4.29}$$

From (4.27), (4.28) and (4.29) it follows that

$$\frac{1}{|\mathtt{P}_i(a)|} \leqslant sov_i(a^{t_l}) \leqslant \frac{1}{|\mathtt{V}_i(a)|} \quad \text{for every } l \in \{1, \ldots, m\}. \tag{4.30}$$

From (4.26) and (4.30) we obtain

$$\frac{a_i(N)}{|\mathtt{P}_i(a)|} \leqslant sov_i(a) \leqslant \frac{a_i(N)}{|\mathtt{V}_i(a)|}.$$

TRANSFER PROPERTY AND HOMOGENEITY. It is a direct consequence of the fact that the Shapley fuzzy authorization correspondence satisfies analogous properties.

The proof of the uniqueness of the sovereignty index is similar to the proof of the uniqueness of the Shapley fuzzy authorization correspondence. We would proceed by induction on $\lceil(a)$. In the base case, we would use Theorem 4.13 to derive that a mapping satisfying the properties in the theorem must coincide with the (crisp) sovereignty index when it is restricted to authorization structures. In the induction step, we would use transfer, homogeneity and inactive agent to prove the uniqueness.

□

Like in the crisp case, we can wonder when the sovereignty index of an agent is equal to 1. And the answer is similar to that given in the crisp case, as we will see.

Definition 4.30 *Let $a \in \mathcal{FA}^N$ and $i \in N$. An agent i is said to be sovereign in (N, a) if $a_i(\{i\}) = 1$.*

Lemma 4.31 *Let $a \in \mathcal{FA}^N$ and $i \in N$. Then, i is sovereign in (N, a) if and only if $sov_i(a) = 1$.*

Proof. Let $a \in \mathcal{FA}^N$ and $i \in N$. It holds that

$$sov_i(a) = \xi_{ii}(a) = \sum_{\{E \subseteq N :\, i \in E\}} p_E \left[a_i(E) - a_i(E \setminus \{i\})\right] = \sum_{\{E \subseteq N :\, i \in E\}} p_E \, a_i(E). \tag{4.31}$$

We know from Lemma 4.12 that

$$\sum_{\{E\subseteq N:\, i\in E\}} p_E = 1. \tag{4.32}$$

From (4.31) and (4.32) it easily follows that $sov_i(a)$ is equal to 1 if and only if $a_i(E) = 1$ for every $E \subseteq N$ with $i \in E$, or, equivalently, $a_i(\{i\}) = 1$. □

Now we aim to characterize the influence index. We consider the following property, where ψ is a mapping from $\mathcal{FA}^N$ into $\mathbb{R}^N$.

- **MAXIMUM AND MINIMUM INFLUENCE**. For every $a \in \mathcal{FA}^N$ and $i \in N$, it holds that

$$\sum_{\{j\in supp(a(N))\setminus\{i\}:\, i\in \mathtt{V}_j(a)\}} \frac{a_j(N)}{|\mathtt{P}_j(a)|} \leqslant \psi_i(a) \leqslant \sum_{\{j\in supp(a(N))\setminus\{i\}:\, i\in \mathtt{P}_j(a)\}} \frac{a_j(N)}{|\mathtt{V}_j(a)|}.$$

In the following theorem we see that this property together with transfer and homogeneity uniquely determine the influence index. We need a previous lemma.

Lemma 4.32 *Let $a \in \mathcal{FA}^N$ and $i, j \in N$. It holds that*

$$\xi_{ij}(a) \leqslant sov_j(a)$$

and the equality holds if and only if i has veto power over j in (N, a).

Proof. Let $a \in \mathcal{FA}^N$ and $i, j \in N$. Let $\{t_l : l = 0, \ldots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r$. Using Remarks 4.22 and 4.27 we can write

$$\begin{aligned} \xi_{ij}(a) &= \sum_{l=1}^{r} (t_l - t_{l-1})\, \Xi_{ij}(a^{t_l}), \\ sov_j(a) &= \sum_{l=1}^{r} (t_l - t_{l-1})\, sov_j(a^{t_l}). \end{aligned}$$

From these two equalities and Lemma 4.16 it follows that $\xi_{ij}(a) \leqslant sov_j(a)$. Moreover, the equality holds if and only if $\Xi_{ij}(a^{t_l}) = sov_j(a^{t_l})$ for every $l \in \{1, \ldots, r\}$. But, from the second statement of

Lemma 4.16, this is equivalent to $i \in \mathtt{V}_j(a^{t_l})$ for every $l \in \{1, \ldots, r\}$, which holds if and only if i has veto power over j in (N, a). □

Theorem 4.33 *A mapping $\psi : \mathcal{FA}^N \to \mathbb{R}^N$ is equal to the influence index if and only if it satisfies the properties of maximum and minimum influence, transfer and homogeneity.*

Proof. Firstly, we prove that the influence index satisfies the properties mentioned in the theorem.

Maximum and minimum influence. Let $a \in \mathcal{FA}^N$ and $i \in N$. Recall that

$$\xi_{ij}(a) = \phi_i(a_j) = \sum_{\{E \subseteq N :\, i \in E\}} p_E \left[a_j(E) - a_j(E \setminus \{i\})\right].$$

Notice that $\xi_{ij}(a)$ is strictly positive if and only if j depends partially on i in (N, a). So we can write

$$inf_i(a) = \sum_{\{j \in supp(a(N)) \setminus \{i\} :\, i \in \mathtt{P}_j(a)\}} \xi_{ij}(a)$$

which, from Lemma 4.32, is less or equal to

$$\sum_{\{j \in supp(a(N)) \setminus \{i\} :\, i \in \mathtt{P}_j(a)\}} sov_j(a)$$

which, from the property of maximum and minimum sovereignty, is less or equal to

$$\sum_{\{j \in supp(a(N)) \setminus \{i\} :\, i \in \mathtt{P}_j(a)\}} \frac{a_j(N)}{|\mathtt{V}_j(a)|}.$$

So we have proved the right-hand inequality. Let us prove the other one:

$$inf_i(a) = \sum_{\{j \in supp(a(N)) \setminus \{i\} :\, i \in \mathtt{P}_j(a)\}} \xi_{ij}(a) \geqslant \sum_{\{j \in supp(a(N)) \setminus \{i\} :\, i \in \mathtt{V}_j(a)\}} \xi_{ij}(a)$$

which, using the second statement of Lemma 4.32, is equal to

$$\sum_{\{j \in supp(a(N)) \setminus \{i\} :\, i \in \mathtt{V}_j(a)\}} sov_j(a)$$

which, from the property of maximum and minimum sovereignty, is greater or equal to

$$\sum_{\{j \in supp(a(N)) \setminus \{i\} : \, i \in \mathtt{V}_j(a)\}} \frac{a_j(N)}{|\mathtt{P}_j(a)|}.$$

TRANSFER PROPERTY AND HOMOGENEITY. It is a direct consequence of the fact that the Shapley fuzzy authorization correspondence satisfies analogous properties.

The proof of the uniqueness of the influence index is similar to the proof of the uniqueness of the Shapley fuzzy authorization correspondence. We would proceed by induction on $\lceil(a)$. In the base case, we would use Theorem 4.17 to conclude that a mapping satisfying the properties in the theorem must coincide with the (crisp) influence index when it is restricted to authorization structures. In the induction step, we would proceed in a similar way as in the proof referred to. □

Finally, we characterize the power index. For that purpose, we consider the following properties, where ψ is a mapping from $\mathcal{FA}^N$ into $\mathbb{R}^N$.

- **EFFICIENCY**. For every $a \in \mathcal{FA}^N$ it holds

$$\sum_{k \in N} \psi_k(a) = \sum_{k \in N} a_k(N).$$

- **NULL AGENT PROPERTY**. For every $a \in \mathcal{FA}^N$ and i null agent in (N, a) it holds that

$$\psi_i(a) = 0.$$

- **EQUAL TREATMENT PROPERTY**. For every $a \in \mathcal{FA}^N$ and $i, j \in N$ such that $a(E \cup \{i\}) = a(E \cup \{j\})$ for all $E \subseteq N \setminus \{i, j\}$ it holds that

$$\psi_i(a) = \psi_j(a).$$

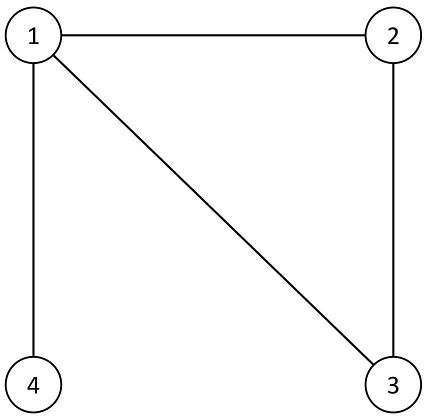
Theorem 4.34 *A mapping $\psi : \mathcal{FA}^N \to \mathbb{R}^N$ is equal to the power index if and only if it satisfies the properties of efficiency, null agent, transfer, equal treatment and homogeneity.*

Proof. That the power index satisfies the properties mentioned in the theorem is a direct consequence of the fact that the Shapley fuzzy authorization correspondence satisfies analogous properties. The proof that the properties in the theorem uniquely determine the power index is almost identical to the proof of the uniqueness of the Shapley fuzzy authorization correspondence, so we omit it. □

Example 4.35 Let $N = \{1,2,3,4\}$. Consider the graph $G = (N, E)$ where

$$E = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\}\}.$$

This graph is illustrated below



Given $E \subseteq N$ and $i \in E$ we denote $C(E,i)$ the connected component of the induced subgraph $G[E]$ that contains i. Let $a : 2^N \to [0,1]^N$ defined as

$$a_i(E) = \begin{cases} \dfrac{|C(E,i)|}{|N|} & \text{if } i \in E, \\ 0 & \text{if } i \notin E. \end{cases}$$

It is easy to check that a is a fuzzy authorization operator on N. The following table gives the vector

$a(E)$ for each nonempty $E \subseteq N$.

E	$a_1(E)$	$a_2(E)$	$a_3(E)$	$a_4(E)$
$\{1\}$	$\frac{1}{4}$	0	0	0
$\{2\}$	0	$\frac{1}{4}$	0	0
$\{3\}$	0	0	$\frac{1}{4}$	0
$\{4\}$	0	0	0	$\frac{1}{4}$
$\{1,2\}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$\{1,3\}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$\{1,4\}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$
$\{2,3\}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0
$\{2,4\}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$
$\{3,4\}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$
$\{1,2,3\}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	0
$\{1,2,4\}$	$\frac{3}{4}$	$\frac{3}{4}$	0	$\frac{3}{4}$
$\{1,3,4\}$	$\frac{3}{4}$	0	$\frac{3}{4}$	$\frac{3}{4}$
$\{2,3,4\}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
$\{1,2,3,4\}$	1	1	1	1

If we calculate the matrix $(\xi_{ij}(a))_{1\leqslant i,j\leqslant 4}$ we obtain

$$\xi(a) = \frac{1}{24}\begin{pmatrix} 15 & 5 & 5 & 7 \\ 3 & 14 & 3 & 2 \\ 3 & 3 & 14 & 2 \\ 3 & 2 & 2 & 13 \end{pmatrix}$$

Let us calculate the sovereignty, influence and power indices.

$$sov_1(a) = \frac{5}{8}, \qquad sov_2(a) = \frac{7}{12}, \qquad sov_3(a) = \frac{7}{12}, \qquad sov_4(a) = \frac{13}{24},$$

$$inf_1(a) = \frac{17}{24}, \qquad inf_2(a) = \frac{1}{3}, \qquad inf_3(a) = \frac{1}{3}, \qquad inf_4(a) = \frac{7}{24},$$

$$pow_1(a) = \frac{4}{3}, \qquad pow_2(a) = \frac{11}{12}, \qquad pow_3(a) = \frac{11}{12}, \qquad pow_4(a) = \frac{5}{6}.$$

Notice that in this example we could use the power index as a centrality index. If we prefer to obtain a centrality index that assigns zero to isolated vertices, then we can consider $b \in \mathcal{FA}^N$ given by

$$b_i(E) = \begin{cases} \dfrac{|C(E,i)|-1}{|N|-1} & \text{if } i \in E, \\ \\ 0 & \text{if } i \notin E. \end{cases}$$

If we calculate $(\xi_{ij}(b))_{1 \leqslant i,j \leqslant 4}$ we obtain

$$\xi(b) = \frac{1}{18}\begin{pmatrix} 9 & 5 & 5 & 7 \\ 3 & 8 & 3 & 2 \\ 3 & 3 & 8 & 2 \\ 3 & 2 & 2 & 7 \end{pmatrix}$$

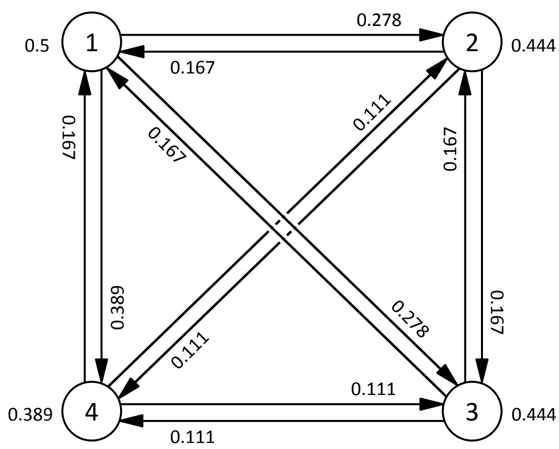
Let us calculate the sovereignty, influence and power indices.

$$sov_1(b) = \frac{1}{2}, \qquad sov_2(b) = \frac{4}{9}, \qquad sov_3(b) = \frac{4}{9}, \qquad sov_4(b) = \frac{7}{18},$$

$$inf_1(b) = \frac{17}{18}, \qquad inf_2(b) = \frac{4}{9}, \qquad inf_3(b) = \frac{4}{9}, \qquad inf_4(b) = \frac{4}{9},$$

$$pow_1(b) = \frac{13}{9}, \qquad pow_2(b) = \frac{8}{9}, \qquad pow_3(b) = \frac{8}{9}, \qquad pow_4(b) = \frac{7}{9}.$$

Rounding to the nearest thousandth, we can identify $\xi(b)$ with the fuzzy cognitive map represented by the following fuzzy digraph



4.6 The Banzhaf authorization correspondence

In a similar way as we used the Shapley value to define the Shapley authorization correspondence, in this section we make use of the Banzhaf value to define another allocation rule for authorization structures.

Definition 4.36 *The Banzhaf authorization correspondence assigns to each authorization operator* $A \in \mathcal{A}^N$ *the mapping* $\Upsilon(A) \in \left([0,1]^N\right)^N$ *defined as*

$$\Upsilon_i(A) = (\beta_i(A_j))_{j \in N} \qquad \textit{for every } i \in N,$$

where β *is the Banzhaf value.*

We denote $\Upsilon_{ij}(A) = \beta_i(A_j)$. Notice that $\Upsilon(A)$ can be identified with a matrix in $\mathcal{M}_n(\mathbb{R})$. Moreover, similarly as we did with $\Xi(A)$, $\Upsilon(A)$ can be interpreted as a fuzzy cognitive map on N. Also notice that for every $A \in \mathcal{A}^N$ and $i, j \in N$ it holds that

$$\Upsilon_{ij}(A) = \beta_i(A_j) = \beta_i\left(u^A_{\{j\}}\right) = \mathfrak{B}_i(u_{\{j\}}, A).$$

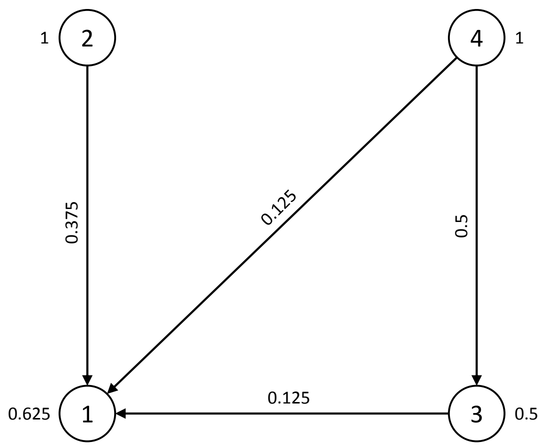
Example 4.37 Let (N, A) be the authorization structure given in Example 4.4. It holds that

$$\begin{aligned}\beta(A_1) &= \left(\frac{5}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right), \\ \beta(A_2) &= (0, 1, 0, 0), \\ \beta(A_3) &= \left(0, 0, \frac{1}{2}, \frac{1}{2}\right), \\ \beta(A_4) &= (0, 0, 0, 1).\end{aligned}$$

So we have obtained that

$$\Upsilon(A) = \begin{pmatrix} \frac{5}{8} & 0 & 0 & 0 \\ \frac{3}{8} & 1 & 0 & 0 \\ \frac{1}{8} & 0 & \frac{1}{2} & 0 \\ \frac{1}{8} & 0 & \frac{1}{2} & 1 \end{pmatrix}.$$

We can identify $\Upsilon(A)$ with the fuzzy cognitive map represented by the following fuzzy digraph



We aim to characterize the Banzhaf authorization correspondence. To do that, we consider the properties stated below. In the statement of these properties Ψ is an allocation rule for authorization structures.

- **2-EFFICIENCY.** For every $A \in \mathcal{A}^N$ and $i, j \in N$ such that every agent in $N \setminus \{i, j\}$ is a null agent in (N, A) it holds that
$$\sum_{k \in N} \Psi_k(A) = \mathbf{1}_{A(N)}.$$

Definition 4.38 *Let $A \in \mathcal{A}^N$ and let i, j be two different agents in N. The agents i and j can be amalgamated in (N, A) if for every $E \subseteq N$ such that $\{i, j\} \subseteq E$ and $\{i, j\} \cap A(E) \neq \emptyset$ it holds that $\{i, j\} \subseteq A(E)$.*

- **AMALGAMATION.** For every $A \in \mathcal{A}^N$ and i, j two different agents in N such that i, j can be amalgamated in (N, A), it holds that
$$\Psi_{ik}(A) + \Psi_{jk}(A) = \Psi_{\widehat{ij}\,k}(A^{ij}) \quad \text{for every } k \in N \setminus \{i, j\}$$
and
$$\Psi_{ik}(A) + \Psi_{jk}(A) = \Psi_{\widehat{ij}\,\widehat{ij}}(A^{ij}) \quad \text{for every } k \in \{i, j\}.$$

In the following theorem we see that these properties together with null agent, transfer and equal treatment uniquely determine the Banzhaf authorization correspondence.

Theorem 4.39 *An allocation rule for authorization structures is equal to the Banzhaf authorization correspondence if and only if it satisfies the properties of null agent, transfer, equal treatment, 2-efficiency and amalgamation.*

Proof. That the Banzhaf authorization correspondence satisfies the properties of null agent, transfer and equal treatment can be proved in a similar way as we did for the Shapley authorization correspondence in Theorem 4.6. Let us see that the Banzhaf authorization correspondence satisfies 2-efficiency and amalgamation.

2-EFFICIENCY. Let $A \in \mathcal{A}^N$ and $i, j \in N$ be such that every agent in $N \setminus \{i, j\}$ is a null agent in (N, A). Let $l \in N$. It is clear that every player in $N \setminus \{i, j\}$ is an irrelevant player in $\left(u_{\{l\}}, A\right)$. From the 2-efficiency property of the Banzhaf authorization value it follows that

$$\sum_{k \in N} \mathfrak{B}_k\left(u_{\{l\}}, A\right) = u_{\{l\}}(A(N)) = \left(\mathbf{1}_{A(N)}\right)_l . \tag{4.33}$$

Also notice that

$$\left(\sum_{k \in N} \Upsilon_k(A)\right)_l = \sum_{k \in N} \Upsilon_{kl}(A) = \sum_{k \in N} \beta_k(A_l) = \sum_{k \in N} \beta_k\left(u^A_{\{l\}}\right) = \sum_{k \in N} \mathfrak{B}_k\left(u_{\{l\}}, A\right). \tag{4.34}$$

From (4.33) and (4.34) we obtain that

$$\left(\sum_{k \in N} \Upsilon_k(A)\right)_l = \left(\mathbf{1}_{A(N)}\right)_l .$$

AMALGAMATION. Let $A \in \mathcal{A}^N$ and let i, j be two different agents in N such that i, j can be amalgamated in (N, A). If $k \in N \setminus \{i, j\}$ we can write

$$\Upsilon_{ik}(A) + \Upsilon_{jk}(A) = \mathfrak{B}_i\left(u_{\{k\}}, A\right) + \mathfrak{B}_j\left(u_{\{k\}}, A\right). \tag{4.35}$$

From the fact that i and j can be amalgamated in (N, A), it follows that they can be amalgamated in $(u_{\{k\}}, A)$. Moreover, it is clear that $\left(u_{\{k\}}\right)^{ij}$ is equal to the unanimity game (on N^{ij}) $u_{\{k\}}$. Using the amalgamation property of the Banzhaf authorization value we can write

$$\mathfrak{B}_i\left(u_{\{k\}}, A\right) + \mathfrak{B}_j\left(u_{\{k\}}, A\right) = \mathfrak{B}_{\widehat{ij}}\left(\left(u_{\{k\}}\right)^{ij}, A^{ij}\right) = \mathfrak{B}_{\widehat{ij}}\left(u_{\{k\}}, A^{ij}\right) = \Upsilon_{\widehat{ij}\,k}(A^{ij}). \tag{4.36}$$

From (4.35) and (4.36) we obtain the first statement of the amalgamation property. As for the second statement, it holds that

$$\Upsilon_{ii}(A) + \Upsilon_{ji}(A) = \mathfrak{B}_i\left(u_{\{i\}}, A\right) + \mathfrak{B}_j\left(u_{\{i\}}, A\right). \tag{4.37}$$

It is clear that i and j can be amalgamated in $(u_{\{i\}}, A)$. Moreover, it holds that $\left(u_{\{i\}}\right)^{ij} = u_{\left\{\widehat{ij}\right\}}$.

Using the amalgamation property of the Banzhaf authorization value we can write

$$\mathfrak{B}_i\left(u_{\{i\}}, A\right) + \mathfrak{B}_j\left(u_{\{i\}}, A\right) = \mathfrak{B}_{\widehat{ij}}\left(\left(u_{\{i\}}\right)^{ij}, A^{ij}\right) = \mathfrak{B}_{\widehat{ij}}\left(u_{\left\{\widehat{ij}\right\}}, A^{ij}\right) = \Upsilon_{\widehat{ij}\,\widehat{ij}}(A^{ij}). \quad (4.38)$$

From (4.37) and (4.38) we obtain

$$\Upsilon_{ii}(A) + \Upsilon_{ji}(A) = \Upsilon_{\widehat{ij}\,\widehat{ij}}(A^{ij}).$$

Now we show that the properties in the theorem uniquely determine the Banzhaf authorization correspondence. Let Ψ be an allocation rule for authorization structures satisfying the properties of null agent, transfer, equal treatment, 2-efficiency and amalgamation. We must prove that $\Psi = \Upsilon$. Let $n \in \mathbb{N}$ and let N be a set of cardinality n. Let $A \in \mathcal{A}^N$. If $A(N) = \emptyset$ we know, from the null agent property, that $\Psi(A) = \Upsilon(A) = 0$. If $A(N) \neq \emptyset$, we can write

$$A = \bigcup_{\{(T,i)\in 2^N \times N\colon\, i\in A(T)\}} C_{T,i}.$$

In order to prove that $\Psi = \Upsilon$, it is enough to show that for every $n, m \in \mathbb{N}$, $T_1, \ldots, T_m \in 2^N \setminus \{\emptyset\}$ and $i_1, \ldots, i_m \in N$ with $i_k \in T_k$ for all $k = 1, \ldots, m,$ it holds that

$$\Psi\left(\bigcup_{k=1}^{m} C_{T_k,i_k}\right) = \Upsilon\left(\bigcup_{k=1}^{m} C_{T_k,i_k}\right).$$

Let us prove this equality by strong induction on m.

1. Base case. We must prove that for every $n \in \mathbb{N}$, $T \in 2^N \setminus \{\emptyset\}$ and $i \in T$ it holds that

   $$\Psi\left(C_{T,i}\right) = \Upsilon\left(C_{T,i}\right).$$

   We prove this equality by induction on $|T|$.

   1.1 Base case. $|T| \leqslant 2$. Let $n \in \mathbb{N}$, $T \in 2^N \setminus \{\emptyset\}$ with $|T| \leqslant 2$ and $i \in T$. It is clear that every agent in $N \setminus T$ is a null agent in $(N, C_{T,i})$. From the 2-efficiency property it follows that

$$\sum_{k \in N} \Psi_k \left(C_{T,i}\right) = \mathbf{1}_{\{i\}}. \tag{4.39}$$

And from the null agent property we obtain that

$$\Psi_l \left(C_{T,i}\right) = 0 \quad \text{for every } l \in N \setminus T. \tag{4.40}$$

Moreover, using the equal treatment property we can derive that

$$\Psi_j \left(C_{T,i}\right) = \Psi_i \left(C_{T,i}\right) \quad \text{for every } j \in T. \tag{4.41}$$

From (4.39), (4.40) and (4.41) we conclude that

$$\Psi_k(C_{T,i}) = \begin{cases} \frac{1}{|T|} \mathbf{1}_{\{i\}} & \text{if } k \in T, \\ 0 & \text{if } k \in N \setminus T. \end{cases}$$

Since Υ also satisfies the properties used, it is clear that $\Psi(C_{T,i}) = \Upsilon(C_{T,i})$.

1.2 Inductive step. Let $n \in \mathbb{N}$, $T \in 2^N \setminus \{\emptyset\}$ with $|T| \geqslant 3$ and $i \in T$. From the null agent property it follows that

$$\Psi_l \left(C_{T,i}\right) = \Upsilon_l \left(C_{T,i}\right) = 0 \quad \text{for every } l \in N \setminus T. \tag{4.42}$$

Since Ψ and Υ satisfy the equal treatment property it holds that

$$\Psi_l \left(C_{T,i}\right) = \Psi_i \left(C_{T,i}\right) \quad \text{for every } l \in T, \tag{4.43}$$

$$\Upsilon_l \left(C_{T,i}\right) = \Upsilon_i \left(C_{T,i}\right) \quad \text{for every } l \in T. \tag{4.44}$$

Take j and k two different agents in $T \setminus \{i\}$. It is clear that j and k can be amalgamated in $(N, C_{T,i})$. From the amalgamation property we obtain that

$$\Psi_{jl} \left(C_{T,i}\right) + \Psi_{kl} \left(C_{T,i}\right) = \Psi_{\widehat{jk}\,l} \left(\left(C_{T,i}\right)^{jk}\right) \quad \text{for every } l \in N \setminus \{j,k\} \tag{4.45}$$

$$\Psi_{jl} \left(C_{T,i}\right) + \Psi_{kl} \left(C_{T,i}\right) = \Psi_{\widehat{jk}\,\widehat{jk}} \left(\left(C_{T,i}\right)^{jk}\right) \quad \text{for every } l \in \{j,k\}, \tag{4.46}$$

$$\Upsilon_{jl}\left(C_{T,i}\right)+\Upsilon_{kl}\left(C_{T,i}\right)=\Upsilon_{\widehat{jk}\,l}\left(\left(C_{T,i}\right)^{jk}\right) \quad \text{for every } l\in N\setminus\{j,k\}, \tag{4.47}$$

$$\Upsilon_{jl}\left(C_{T,i}\right)+\Upsilon_{kl}\left(C_{T,i}\right)=\Upsilon_{\widehat{jk}\,\widehat{jk}}\left(\left(C_{T,i}\right)^{jk}\right) \quad \text{for every } l\in\{j,k\}. \tag{4.48}$$

It is easy to check that $(C_{T,i})^{jk}$ is equal to the authorization operator (on N^{jk}) $C_{(T\setminus\{j,k\})\cup\{\widehat{jk}\},i}$. So we have that

$$\Psi\left(\left(C_{T,i}\right)^{jk}\right) = \Psi\left(C_{(T\setminus\{j,k\})\cup\{\widehat{jk}\},i}\right), \tag{4.49}$$

$$\Upsilon\left(\left(C_{T,i}\right)^{jk}\right) = \Upsilon\left(C_{(T\setminus\{j,k\})\cup\{\widehat{jk}\},i}\right). \tag{4.50}$$

Since $|(T\setminus\{j,k\})\cup\{\widehat{jk}\}|=|T|-1$ we know from the induction hypothesis that

$$\Psi\left(C_{(T\setminus\{j,k\})\cup\{\widehat{jk}\},i}\right)=\Upsilon\left(C_{(T\setminus\{j,k\})\cup\{\widehat{jk}\},i}\right). \tag{4.51}$$

From (4.49), (4.50) and (4.51) we obtain that

$$\Psi\left(\left(C_{T,i}\right)^{jk}\right)=\Upsilon\left(\left(C_{T,i}\right)^{jk}\right). \tag{4.52}$$

From (4.45), (4.46), (4.47), (4.48) and (4.52) we conclude that

$$\Psi_j\left(C_{T,i}\right)+\Psi_k\left(C_{T,i}\right)=\Upsilon_j\left(C_{T,i}\right)+\Upsilon_k\left(C_{T,i}\right). \tag{4.53}$$

From (4.43), (4.44) and (4.53) it follows that

$$\Psi_l\left(C_{T,i}\right)=\Upsilon_l\left(C_{T,i}\right) \quad \text{for every } l\in T. \tag{4.54}$$

Finally, from (4.42) and (4.54) we conclude that $\Psi\left(C_{T,i}\right)=\Upsilon\left(C_{T,i}\right)$.

2. Inductive step. The reasoning is equal to that followed in the case of the Shapley authorization correspondence.

□

4.7 The Banzhaf fuzzy authorization correspondence

In a similar way as we used the Shapley value to define the Shapley fuzzy authorization correspondence, in this section we make use of the Banzhaf value to define another allocation rule for fuzzy authorization structures.

Definition 4.40 *The Banzhaf fuzzy authorization correspondence assigns to each fuzzy authorization operator $a \in \mathcal{FA}^N$ the mapping $\tau(a) \in \left([0,1]^N\right)^N$ defined as*

$$\tau_i(a) = \left(\beta_i(a_j)\right)_{j\in N} \qquad \textit{for every } i \in N,$$

where β is the Banzhaf value.

We denote $\tau_{ij}(a) = \beta_i(a_j)$. Observe that $\tau(a)$ can be identified with a matrix in $\mathcal{M}_n(\mathbb{R})$. Moreover, $\tau(a)$ can be interpreted as a fuzzy cognitive map on N. Also notice that for every $a \in \mathcal{FA}^N$ and $i, j \in N$ it holds that

$$\tau_{ij}(a) = \beta_i(a_j) = \beta_i\left(u^a_{\{j\}}\right) = \mathfrak{B}_i(u_{\{j\}}, a).$$

We aim to characterize the Banzhaf fuzzy authorization correspondence. To do that, we consider the properties stated below. In the statement of these properties ψ is an allocation rule for fuzzy authorization structures.

- **2-EFFICIENCY**. For every $a \in \mathcal{FA}^N$ and $i, j \in N$ such that every agent in $N \setminus \{i,j\}$ is a null agent in (N, a) it holds that
$$\sum_{k\in N} \psi_k(a) = a(N).$$

Definition 4.41 *Let $a \in \mathcal{FA}^N$ and let i, j be two different agents in N. The agents i and j can be amalgamated in (a, N) if for every $E \subseteq N$ such that $\{i,j\} \subseteq E$ it holds that $a_i(E) = a_j(E)$.*

- **AMALGAMATION**. For every $a \in \mathcal{FA}^N$ and i, j two different agents in N such that i, j can be amalgamated in (N, a), it holds that

$$\psi_{ik}(a) + \psi_{jk}(a) = \psi_{\widehat{ij}\,k}(a^{ij}) \quad \text{for every } k \in N \setminus \{i, j\}$$

and

$$\psi_{ik}(a) + \psi_{jk}(a) = \psi_{\widehat{ij}\,\widehat{ij}}(a^{ij}) \quad \text{for every } k \in \{i, j\}.$$

In the following theorem we see that these properties together with null agent, transfer, equal treatment and homogeneity uniquely determine the Banzhaf fuzzy authorization correspondence. Before, we make a remark.

Remark 4.42 *Let $a \in \mathcal{FA}^N$ and $i, j \in N$. Let $\{h_l : l = 0, \ldots, m\} \subset [0, 1]$ be such that $\{h_l : l = 0, \ldots, m\} \supseteq \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = h_0 < \ldots < h_m$. It holds that*

$$\begin{aligned} \tau_{ij}(a) &= \beta_i(a_j) = \beta_i\left(u^a_{\{j\}}\right) = \mathfrak{B}_i(u_{\{j\}}, a) \\ &= \sum_{l=1}^{m}(h_l - h_{l-1})\,\mathfrak{B}_i\left(u_{\{j\}}, a^{h_l}\right) = \sum_{l=1}^{m}(h_l - h_{l-1})\,\Upsilon_{ij}(a^{h_l}). \end{aligned}$$

So we have seen that

$$\tau(a) = \sum_{l=1}^{m}(h_l - h_{l-1})\,\Upsilon(a^{h_l}).$$

Theorem 4.43 *An allocation rule for fuzzy authorization structures is equal to the Banzhaf fuzzy authorization correspondence if and only if it satisfies the properties of null agent, transfer, equal treatment, 2-efficiency, amalgamation and homogeneity.*

Proof. That the Banzhaf fuzzy authorization correspondence satisfies the properties of null agent, transfer, equal treatment and homogeneity can be proved in a similar way as we did for the Shapley fuzzy authorization correspondence in Theorem 4.23. Let us see that the Banzhaf fuzzy authorization correspondence satisfies 2-efficiency and amalgamation.

2-EFFICIENCY. Let $a \in \mathcal{FA}^N$ and $i, j \in N$ be such that every agent in $N \setminus \{i, j\}$ is a null agent in (N, a). Let $\{t_l : l = 0, \ldots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r$. For every

$l \in \{1, \ldots, r\}$ the agents in $N \setminus \{i, j\}$ are null agents in (N, a^{t_l}). From the 2-efficiency property of the Banzhaf authorization correspondence it follows that

$$\sum_{k \in N} \Upsilon_k(a^{t_l}) = \mathbf{1}_{a^{t_l}(N)} \qquad \text{for every } l \in \{1, \ldots, r\}. \tag{4.55}$$

Using Remark 4.42 and (4.55) we have that

$$\begin{aligned} \sum_{k \in N} \tau_k(a) &= \sum_{k \in N} \left(\sum_{l=1}^{r} (t_l - t_{l-1}) \, \Upsilon_k(a^{t_l}) \right) \\ &= \sum_{l=1}^{r} (t_l - t_{l-1}) \left(\sum_{k \in N} \Upsilon_k(a^{t_l}) \right) \\ &= \sum_{l=1}^{r} (t_l - t_{l-1}) \, \mathbf{1}_{a^{t_l}(N)} = a(N). \end{aligned}$$

Amalgamation. Let $a \in \mathcal{FA}^N$ and let i, j be two different agents in N such that i, j can be amalgamated in (N, a). Let $\{t_l : l = 0, \ldots, r\} = \{a_k(F) : F \subseteq N, \, k \in N\}$ with $0 = t_0 < \ldots < t_r$. It is clear that i and j can be amalgamated in (N, a^{t_l}) for every $l \in \{1, \ldots, r\}$. From the amalgamation property of the Banzhaf authorization correspondence it follows that

$$\Upsilon_{ik}(a^{t_l}) + \Upsilon_{jk}(a^{t_l}) = \Upsilon_{\widehat{ij}\,k}\left((a^{t_l})^{ij}\right) \qquad \text{for every } k \in N \setminus \{i, j\} \text{ and } l \in \{1, \ldots, r\}, \tag{4.56}$$

$$\Upsilon_{ik}(a^{t_l}) + \Upsilon_{jk}(a^{t_l}) = \Upsilon_{\widehat{ij}\,\widehat{ij}}\left((a^{t_l})^{ij}\right) \qquad \text{for every } k \in \{i, j\} \text{ and } l \in \{1, \ldots, r\}. \tag{4.57}$$

From (3.1) we know that

$$(a^{t_l})^{ij} = (a^{ij})^{t_l} \qquad \text{for every } l \in \{1, \ldots, r\}. \tag{4.58}$$

Let $k \in N \setminus \{i, j\}$. Using Remark 4.42, (4.56) and (4.58) we can write

$$\begin{aligned} \tau_{ik}(a) + \tau_{jk}(a) &= \sum_{l=1}^{r} (t_l - t_{l-1}) \, \Upsilon_{ik}(a^{t_l}) + \sum_{l=1}^{r} (t_l - t_{l-1}) \, \Upsilon_{ik}(a^{t_l}) \\ &= \sum_{l=1}^{r} (t_l - t_{l-1}) \, \Upsilon_{\widehat{ij}\,k}\left((a^{t_l})^{ij}\right) = \sum_{l=1}^{r} (t_l - t_{l-1}) \, \Upsilon_{\widehat{ij}\,k}\left((a^{ij})^{t_l}\right) = \tau_{\widehat{ij}\,k}\left(a^{ij}\right). \end{aligned}$$

Similarly, using Remark 4.42, (4.57) and (4.58) we can obtain that

$$\tau_{ik}(a) + \tau_{jk}(a) = \tau_{\widehat{ij}\,\widehat{ij}}\left(a^{ij}\right) \qquad \text{for every } k \in \{i,j\}.$$

It remains to prove that the properties in the theorem uniquely determine the Banzhaf fuzzy authorization correspondence. Let ψ be an allocation rule for fuzzy authorization structures satisfying the properties of null agent, transfer, equal treatment, 2-efficiency, amalgamation and homogeneity. We must prove that

$$\psi(a) = \tau(a) \qquad \text{for every } n \in \mathbb{N} \text{ and } a \in \mathcal{FA}^N.$$

We proceed by strong induction on $\lceil(a)$ where

$$\lceil(a) = |\{a_k(F) :\ F \subseteq N, k \in N\} \setminus \{0,1\}| \qquad \text{for all } a \in \mathcal{FA}^N.$$

1. BASE CASE. $\lceil(a) = 0$.

   Notice that we can identify $\mathcal{A}^N$ with the set $\{a \in \mathcal{FA}^N \,:\, im(a) \subseteq \{0,1\}^N\}$. From this point of view, we can say that the restriction of ψ to the set of fuzzy authorization operators a with $\lceil(a) = 0$ is an allocation rule for authorization structures. It is easy to check that such restriction satisfies the properties of null agent, transfer, equal treatment, 2-efficiency and amalgamation. Therefore, using Theorem 4.39, it must hold that $\psi(a) = \tau(a)$ for every $n \in \mathbb{N}$ and $a \in \mathcal{FA}^N$ with $\lceil(a) = 0$.

2. INDUCTIVE STEP. The reasoning is equal to that followed in the case of the Shapley fuzzy authorization correspondence.

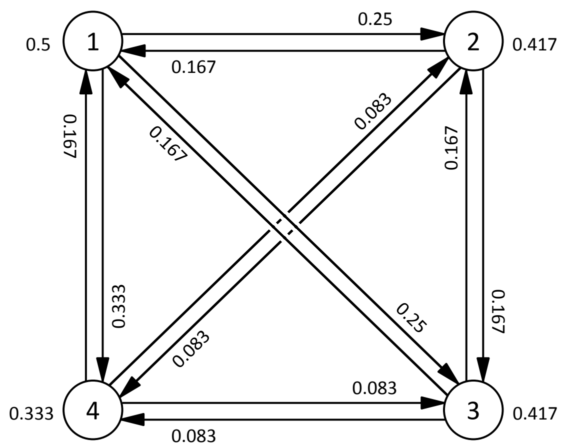
□

Example 4.44 Consider the graph G and the fuzzy authorization operators a and b defined in Example 4.35. It holds that

$$\tau(a) = \frac{1}{16}\begin{pmatrix} 10 & 3 & 3 & 4 \\ 2 & 9 & 2 & 1 \\ 2 & 2 & 9 & 1 \\ 2 & 1 & 1 & 8 \end{pmatrix}$$

$$\tau(b) = \frac{1}{12}\begin{pmatrix} 6 & 3 & 3 & 4 \\ 2 & 5 & 2 & 1 \\ 2 & 2 & 5 & 1 \\ 2 & 1 & 1 & 4 \end{pmatrix}$$

Rounding to the nearest thousandth, we can identify $\tau(b)$ with the fuzzy cognitive map represented by the following fuzzy digraph



Chapter 5

Games with interior operator structure

In the second chapter we introduced authorization operators and provided a sharing value for games with authorization structure. In this chapter we will focus on a particular class of authorization operators: interior operators. Studying interior operators separately is due to a couple of reasons. Firstly, they have, as we will see, especially interesting properties. And secondly, we aim to give a characterization of the Shapley authorization value restricted to interior operator structures.

5.1 Interior operators

The concept of interior operator is very close to that of restriction introduced by Derks and Peters [33], as we can see in the following definition.

Definition 5.1 *An authorization operator $A \in \mathcal{A}^N$ is said to be an interior operator if it satisfies the following conditions:*

1. $A(N) = N$,
2. $A(A(E)) = A(E)$ *for every* $E \subseteq N$.

Observe that interior operators are particular cases of restrictions. Actually, the model proposed by Derks and Peters would not lose generality if they considered just interior operators. Indeed, in their model, given a restriction ρ and a game v on N, the players in $N \setminus \rho(N)$ are irrelevant and receive zero payoff, whereas the players in $\rho(N)$ get the same payoff that would receive if the players in $N \setminus \rho(N)$ were eliminated from the game and we considered the restriction of v to $2^{\rho(N)}$ and the interior operator resulting from restricting ρ to $2^{\rho(N)}$.

Given an interior operator $A \in \mathcal{A}^N$, we can consider the family $\mathcal{O}_A = \{E \subseteq N : A(E) = E\}$. In this way, we can identify an interior operator with a family of coalitions that is union-closed and contains the empty set and the grand coalition. Via this identification, we can say that some of the families of feasible coalitions considered in literature to model games with restricted cooperation are interior operators. For instance, that is the case of antimatroids [3].

If we see the definition of interior operator, the first condition is very clear. But what is the meaning of the second condition? In the following we see that this condition establishes that the veto relations induced by A are transitive. We need a previous definition.

Definition 5.2 *Let $A \in \mathcal{A}^N$, $E \subseteq N$ and $i \in N$. A coalition E has veto power over i in (N, A) if $i \notin A(N \setminus E)$.*

Suppose that we have a game $v \in \mathcal{G}^N$ with authorization structure (N, A). Take $E \subseteq N$ and $i \in N$. In this context, saying that E has veto power over i in (N, A) means that player i will not be allowed to play the game v in any coalition disjoint from E. Remember that in Definition 2.12 we stated that, given an authorization operator $A \in \mathcal{A}^N$ and two players $i, j \in N$, i has veto power over j in (N, A) if $j \notin A(N \setminus \{i\})$. Notice that this is equivalent to saying that $\{i\}$ has veto power over j in (N, A).

Let $A \in \mathcal{A}^N$. We define the following relation on 2^N

$$E \overset{A}{\triangleright} F \textit{ iff } E \textit{ has veto power over every player in } F \textit{ in } (N, A)$$

for all $E, F \subseteq N$.

Proposition 5.3 *Let $A \in \tilde{\mathcal{A}}^N$. The following statements are equivalent:*

1. *The authorization operator A is an interior operator.*

2. *The relation $\overset{A}{\triangleright}$ is transitive.*

Proof. Let $A \in \tilde{\mathcal{A}}^N$.

$1 \implies 2$. Suppose that A is an interior operator on N. Let $E, F, H \subseteq N$ such that $E \overset{A}{\triangleright} F$ and $F \overset{A}{\triangleright} H$. By definition, $E \overset{A}{\triangleright} F$ means that $A(N \setminus E) \subseteq N \setminus F$. From the monotonicity of A it follows that $A(A(N \setminus E)) \subseteq A(N \setminus F)$. Now, taking into consideration that A is an interior operator and $F \overset{A}{\triangleright} H$ we can derive that

$$A(N \setminus E) = A(A(N \setminus E)) \subseteq A(N \setminus F) \subseteq N \setminus H.$$

Thus, $A(N \setminus E) \subseteq N \setminus H$, which is equivalent to $E \overset{A}{\triangleright} H$.

$2 \implies 1$. Suppose that A is such that the relation $\overset{A}{\triangleright}$ is transitive. Let $E \subseteq N$. By definition, it is clear that $N \setminus E \overset{A}{\triangleright} N \setminus A(E)$. Similarly, it holds that $N \setminus A(E) \overset{A}{\triangleright} N \setminus A(A(E))$. From the transitivity of $\overset{A}{\triangleright}$ we can derive $N \setminus E \overset{A}{\triangleright} N \setminus A(A(E))$, which means that $A(E) \subseteq A(A(E))$. □

Now we intend to show that interior operators can be identified with certain functions called interior functions.

Definition 5.4 *A mapping $\Lambda : 2^N \to \{0, 1\}$ is an interior function on N if it satisfies the following conditions:*

1. $\Lambda(N) = \Lambda(\emptyset) = 1$,

2. $\Lambda(E \cup F) \geqslant \min\{\Lambda(E), \Lambda(F)\}$ *for all* $E, F \subseteq N$.

The set of all interior functions on N is denoted by $\mathcal{I}^N$.

Remark 5.5 *If $\Lambda \in \mathcal{I}^N$ and $E_1, \ldots, E_r \subseteq N$ then*

$$\Lambda(E_1 \cup \ldots \cup E_r) \geqslant \min(\Lambda(E_1), \ldots, \Lambda(E_r)).$$

Let $\Lambda \in \mathcal{I}^N$. Consider $int^\Lambda : 2^N \to 2^N$ defined as

$$int^\Lambda(E) = \bigcup_{\{F \subseteq E :\, \Lambda(F)=1\}} F \quad \text{for all } E \subseteq N.$$

The following two results show that there is an identification between interior functions and interior operators.

Proposition 5.6 *Let $\Lambda \in \mathcal{I}^N$. It holds that int^Λ is an interior operator on N.*

Proof. It is clear that $int^\Lambda \in \tilde{\mathcal{A}}^N$. Now take $E \subseteq N$. We must prove that $int^\Lambda(int^\Lambda(E)) = int^\Lambda(E)$. It suffices to prove that $\Lambda(int^\Lambda(E)) = 1$. Notice that

$$\Lambda(int^\Lambda(E)) = \Lambda\left(\bigcup_{\{F \subseteq E \colon \Lambda(F)=1\}} F\right) \geqslant \min\{\Lambda(F) : F \subseteq E \text{ and } \Lambda(F) = 1\} = 1.$$

□

Proposition 5.7 *Let $A : 2^N \to 2^N$ be an interior operator on N. Then, there exists a unique $\Lambda \in \mathcal{I}^N$ such that $A = int^\Lambda$.*

Proof. Let A be an interior operator on N. Take $\Lambda : 2^N \to \{0, 1\}$ defined by

$$\Lambda(E) = \begin{cases} 1 & \text{if } A(E) = E, \\ 0 & \text{otherwise.} \end{cases}$$

We want to check that $\Lambda \in \mathcal{I}^N$. It is clear that $\Lambda(N) = \Lambda(\emptyset) = 1$. Let $E, F \subseteq N$. We must see that $\Lambda(E \cup F) \geqslant \min(\Lambda(E), \Lambda(F))$. It can be assumed that $\Lambda(E) = \Lambda(F) = 1$. It holds that

$$A(E \cup F) \supseteq A(E) \cup A(F) = E \cup F.$$

We conclude that $A(E \cup F) = E \cup F$ and, therefore, $\Lambda(E \cup F) = 1$.

Now we show that $int^\Lambda = A$. Let $E \subseteq N$. On the one hand, it holds that

$$int^\Lambda(E) = \bigcup_{\{F \subseteq E \colon \Lambda(F)=1\}} F = \bigcup_{\{F \subseteq E \colon A(F)=F\}} F = \bigcup_{\{F \subseteq E \colon A(F)=F\}} A(F) \subseteq A(E),$$

and, on the other hand, since $A(A(E)) = A(E)$ it holds that $\Lambda(A(E)) = 1$ and hence

$$A(E) \subseteq \bigcup_{\{F \subseteq E\colon \Lambda(F)=1\}} F = int^{\Lambda}(E).$$

So we have obtained that $int^{\Lambda}(E) = A(E)$.

Finally, let $\Lambda \in \mathcal{I}^N$ be such that $A = int^{\Lambda}$. Let $E \subseteq N$. Notice that if $\Lambda(E) = 1$ then $int^{\Lambda}(E) = E$ and, therefore, $A(E) = E$. Conversely, if $A(E) = E$ it holds that

$$\begin{aligned} \Lambda(E) &= \Lambda(A(E)) = \Lambda(int^{\Lambda}(E)) = \Lambda\left(\bigcup_{\{F \subseteq E\colon \Lambda(F)=1\}} F\right) \\ &\geqslant \min\left\{\Lambda(F) : F \subseteq E \text{ and } \Lambda(F) = 1\right\} = 1, \end{aligned}$$

whence $\Lambda(E) = 1$. We have proved that $\Lambda(E) = 1$ if and only if $A(E) = E$, hence we have uniqueness. □

Definition 5.8 *A game with interior operator structure on N is a pair (v, Λ) where $v \in \mathcal{G}^N$ and $\Lambda \in \mathcal{I}^N$.*

Given a game with interior operator structure, we can define a characteristic function that gathers the information from the game and the interior operator structure in a reasonable way.

Definition 5.9 *Let $v \in \mathcal{G}^N$ and $\Lambda \in \mathcal{I}^N$. The restricted game of (v, Λ), denoted by v^{Λ}, is defined as the restricted game of $\left(v, int^{\Lambda}\right)$, that is,*

$$v^{\Lambda}(E) = v^{int^{\Lambda}}(E) = v(int^{\Lambda}(E)) \quad \textit{for all } E \subseteq N.$$

The number $v^{\Lambda}(E)$ is the worth of E in the game with interior operator structure (v, Λ).

5.2 The Shapley interior value

An allocation rule for games with interior operator structure is a mapping that assigns to every game with interior operator structure a payoff vector. In this section we define and characterize an allocation rule for games with interior operator structure.

Definition 5.10 *The Shapley interior value, denoted by Φ^{int}, assigns to every game with interior operator structure (v, Λ) the Shapley value of the restricted game v^{Λ},*

$$\Phi^{int}(v,\Lambda) = \phi\left(v^{\Lambda}\right) \qquad \textit{for all } v \in \mathcal{G}^N \textit{ and } \Lambda \in \mathcal{I}^N.$$

Notice that

$$\Phi^{int}(v,\Lambda) = \Phi\left(v, int^{\Lambda}\right) \qquad \text{for all } v \in \mathcal{G}^N \text{ and } \Lambda \in \mathcal{I}^N,$$

where Φ is the Shapley authorization value.

Although the Shapley interior value can be seen as the restriction of the Shapley authorization value to the set of games with interior operator structure, the characterization given in the second chapter is not valid here. For instance, notice that the fairness property stated in that chapter cannot be used when we restrict to games with interior operator structure, since, given an interior operator $A \in \mathcal{A}^N$, $T \subseteq N$ and $i \in T$, the authorization operator $A^{T,i}$ is not necessarily an interior operator. We give a characterization of the Shapley interior value based on the characterization of the Shapley value for games with restrictions provided by Derks and Peters [33]. Let us consider the following properties.

- NECESSARY PLAYER PROPERTY. For every monotonic $v \in \mathcal{G}^N$, $\Lambda \in \mathcal{I}^N$ and $i \in N$ a necessary player in v, it holds that

$$\Psi_i(v,\Lambda) \geqslant \Psi_k(v,\Lambda) \qquad \text{for all } k \in N.$$

Definition 5.11 *Let $\Lambda \in \mathcal{I}^N$, $E \subseteq N$ and $i, j \in E$. A player j depends on i within E according to Λ if $j \in int^{\Lambda}(E) \setminus int^{\Lambda}(E \setminus \{i\})$.*

Definition 5.12 *Let $v \in \mathcal{G}^N$, $E \subseteq N$ and $j \in E$. A player j is a null player for v within E if $v(F) = v(F \setminus \{j\})$ for every $F \subseteq E$.*

Definition 5.13 *Let $v \in \mathcal{G}^N$, $\Lambda \in \mathcal{I}^N$ and $i \in N$. A player i is an unnecessary player in (v, Λ) if for every $E \subseteq N$ and $j \in E$ such that j depends on i within E according to Λ it holds that j is a null player for v within E.*

- UNNECESSARY PLAYER PROPERTY. For every $v \in \mathcal{G}^N$, $\Lambda \in \mathcal{I}^N$ and $i \in N$ such that i is an unnecessary player in (v, Λ), it holds that

$$\Psi_i(v, \Lambda) = 0.$$

Definition 5.14 *Let $\Lambda \in \mathcal{I}^N$ and $E \subset N$. A coalition E is an inessential coalition for Λ if $\Lambda(E) = 0$.*

- INESSENTIAL COALITION PROPERTY. For every $v, w \in \mathcal{G}^N$, $\Lambda \in \mathcal{I}^N$ and $E \subset N$ an inessential coalition for Λ such that $v(F) = w(F)$ for all $F \in 2^N \setminus \{E\}$, it holds that

$$\Psi(v, \Lambda) = \Psi(w, \Lambda).$$

In the following theorem we provide a characterization of the Shapley interior value.

Theorem 5.15 *An allocation rule for games with interior operator structure is equal to the Shapley interior value if and only if it satisfies the properties of efficiency, additivity, necessary player, unnecessary player and inessential coalition.*

Proof. That the Shapley interior value satisfies the properties of efficiency, additivity and necessary player follows from the fact that the Shapley interior value is the restriction of the Shapley authorization value to the set of games with interior operator structure and the fact that the Shapley authorization value satisfies such properties. Let us prove that Φ^{int} satisfies the properties of unnecessary player and inessential coalition.

UNNECESSARY PLAYER PROPERTY. Let $v \in \mathcal{G}^N$, $\Lambda \in \mathcal{I}^N$ and $i \in N$ an unnecessary player in (v, Λ). We must prove that $\Phi_i^{int}(v, \Lambda) = 0$. Taking into consideration the definition of Φ^{int}, it is enough to show that i is a null player in v^Λ. For that, take $E \subseteq N$. We must check that $v^\Lambda(E) = v^\Lambda(E \setminus \{i\})$.

Observe that

$$int^\Lambda(E) \setminus int^\Lambda(E \setminus \{i\}) = \{j \in N : j \text{ depends on } i \text{ within } E \text{ according to } \Lambda\}.$$

Since i is an unnecessary player in (v, Λ), every player that depends on i within E according to Λ is a null player for v within E. So it holds that

$$int^\Lambda(E) \setminus int^\Lambda(E \setminus \{i\}) \subseteq \{j \in N : j \text{ is a null player for } v \text{ within } E\}$$

and, therefore

$$v(int^\Lambda(E)) = v\left(int^\Lambda(E \setminus \{i\})\right)$$

or, equivalently

$$v^\Lambda(E) = v^\Lambda(E \setminus \{i\}).$$

INESSENTIAL COALITION PROPERTY. Let $v, w \in \mathcal{G}^N$, $\Lambda \in \mathcal{I}^N$ and $E \subset N$ an inessential coalition for Λ be such that $v(F) = w(F)$ for all $F \in 2^N \setminus \{E\}$. We must prove that $\Phi^{int}(v, \Lambda) = \Phi^{int}(w, \Lambda)$. It holds that

$$\Phi_i^{int}(v, \Lambda) = \phi_i(v^\Lambda) = \sum_{\{H \subseteq N:\, i \in H\}} p_H \left[v(int^\Lambda(H)) - v(int^\Lambda(H \setminus \{i\}))\right],$$

$$\Phi_i^{int}(w, \Lambda) = \phi_i(w^\Lambda) = \sum_{\{H \subseteq N:\, i \in H\}} p_H \left[w(int^\Lambda(H)) - w(int^\Lambda(H \setminus \{i\}))\right],$$

where the numbers p_H are the coefficients of the Shapley value. Therefore, in order to see that $\Phi^{int}(v, \Lambda) = \Phi^{int}(w, \Lambda)$ it suffices to prove that $v(int^\Lambda(H)) = w(int^\Lambda(H))$ for all $H \subseteq N$. And this is elementary because $\Lambda(int^\Lambda(H)) = 1$.

We have seen that the Shapley interior value satisfies the five properties in the theorem. Now we show that it is uniquely determined by those properties.

Let Ψ be an allocation rule for games with interior operator structure satisfying the properties of efficiency, additivity, necessary player, unnecessary player and inessential coalition. We must see that $\Psi = \Phi^{int}$.

Let $n \in \mathbb{N}$ and let N be a set of cardinality n. Let $\Lambda \in \mathcal{I}^N$. We show that $\Psi(v, \Lambda) = \Phi^{int}(v, \Lambda)$ for every $v \in \mathcal{G}^N$.

Firstly, we prove that $\Psi(cu_E, \Lambda) = \Phi^{int}(cu_E, \Lambda)$ for all $c > 0$ and $E \in 2^N \setminus \{\emptyset\}$ with $\Lambda(E) = 1$.

Let $c > 0$ and $E \in 2^N \setminus \{\emptyset\}$ with $\Lambda(E) = 1$. On the one hand, since Ψ satisfies the necessary player property, it is clear that there exists $b \in \mathbb{R}$ such that $\Psi_i(cu_E, \Lambda) = b$ for every $i \in E$. On the other hand, it is easy to check that the players in $N \setminus E$ are unnecessary players in (cu_E, Λ). So, by the unnecessary player property, it must be $\Psi_i(cu_E, \Lambda) = 0$ for every $i \in N \setminus E$. Now, if we use the efficiency of Ψ, we can conclude that

$$\Psi_i(cu_E, \Lambda) = \begin{cases} \dfrac{c}{|E|} & \text{if } i \in E, \\ \\ 0 & \text{if } i \notin E. \end{cases}$$

Since the Shapley interior value also satisfies the properties used, it holds that

$$\Psi(cu_E, \Lambda) = \Phi^{int}(cu_E, \Lambda) \text{ for all } c > 0 \text{ and } E \in 2^N \setminus \{\emptyset\} \text{ with } \Lambda(E) = 1.$$

Now take $c < 0$ and $E \in 2^N \setminus \{\emptyset\}$ with $\Lambda(E) = 1$. We intend to show that $\Psi(cu_E, \Lambda) = \Phi^{int}(cu_E, \Lambda)$. Firstly, it is clear, from the unnecessary player property, that $\Psi(0, \Lambda) = 0$. Now, by additivity, we can write

$$0 = \Psi(0, \Lambda) = \Psi(cu_E, \Lambda) + \Psi(-cu_E, \Lambda)$$

and, therefore

$$\Psi(cu_E, \Lambda) = -\Psi(-cu_E, \Lambda).$$

Similarly,

$$\Phi^{int}(cu_E, \Lambda) = -\Phi^{int}(-cu_E, \Lambda)$$

and, since $\Psi(-cu_E, \Lambda) = \Phi^{int}(-cu_E, \Lambda)$, we conclude that $\Psi(cu_E, \Lambda) = \Phi^{int}(cu_E, \Lambda)$.

So we have seen that $\Psi(cu_E, \Lambda) = \Phi^{int}(cu_E, \Lambda)$ for all $c \in \mathbb{R}$ and $E \in 2^N \setminus \{\emptyset\}$ with $\Lambda(E) = 1$.

Now, let $v \in \mathcal{G}^N$. We define $\Delta : 2^N \to \mathbb{R}$ in the following way,

$$\Delta(E) = \begin{cases} 0 & \text{if } E = \emptyset \text{ or } \Lambda(E) = 0, \\ v(E) - \sum_{F \subsetneq E} \Delta(F) & \text{otherwise.} \end{cases}$$

Take

$$w = \sum_{\{E \in 2^N \setminus \{\emptyset\} : \Lambda(E)=1\}} \Delta(E)\, u_E.$$

It is clear that $w(H) = v(H)$ for every $H \in 2^N$ with $\Lambda(H) = 1$. Successively applying the inessential coalition property it follows that $\Psi(v, \Lambda) = \Psi(w, \Lambda)$ and $\Phi^{int}(v, \Lambda) = \Phi^{int}(w, \Lambda)$. Finally, we can write

$$\begin{aligned} \Psi(v, \Lambda) &= \Psi(w, \Lambda) \\ &= \Psi\left(\sum_{\{E \in 2^N \setminus \{\emptyset\} : \Lambda(E)=1\}} \Delta(E)\, u_E, \Lambda \right) \\ &= \sum_{\{E \in 2^N \setminus \{\emptyset\} : \Lambda(E)=1\}} \Psi(\Delta(E)\, u_E, \Lambda) \\ &= \sum_{\{E \in 2^N \setminus \{\emptyset\} : \Lambda(E)=1\}} \Phi^{int}(\Delta(E)\, u_E, \Lambda) \\ &= \Phi^{int}\left(\sum_{\{E \in 2^N \setminus \{\emptyset\} : \Lambda(E)=1\}} \Delta(E)\, u_E, \Lambda \right) \\ &= \Phi^{int}(w, \Lambda) \\ &= \Phi^{int}(v, \Lambda). \end{aligned}$$

□

Now, in this section we see that the Shapley interior value satisfies structural monotonicity, that is, if we have a monotonic game with interior operator structure and a player that has veto power over another player, then the first player will be given (according to the Shapley interior value) a payoff greater or equal to the payoff received by the second player. Firstly we show that this property is not true, in general, for the Shapley authorization value and games with authorization structure.

Example 5.16 Let $N = \{1,2,3\}$. Let $A \in \mathcal{A}^N$ be the authorization operator defined in Example 2.3.

E	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$A(E)$	$\emptyset$	$\emptyset$	$\{3\}$	$\{1\}$	$\{3\}$	$\{2,3\}$	$\{1,2,3\}$

Consider the unanimity game $u_{\{1\}} \in \mathcal{G}^N$. It holds that

$$\Phi\left(u_{\{1\}}, A\right) = (0.5, 0.5, 0).$$

So we have that player 3 has veto power over player 2 in (N, A) and

$$\Phi_3\left(u_{\{1\}}, A\right) < \Phi_2\left(u_{\{1\}}, A\right).$$

Proposition 5.17 *Let $v \in \mathcal{G}^N$ be a monotonic game, $\Lambda \in \mathcal{I}^N$ and $i, j \in N$ be such that i has veto power over j in (N, int^Λ). Then $\Phi_i^{int}(v,\Lambda) \geqslant \Phi_j^{int}(v,\Lambda)$.*

Proof. Take a monotonic $v \in \mathcal{G}^N$, $\Lambda \in \mathcal{I}^N$ and $i, j \in N$ such that i has veto power over j in (N, int^Λ). It holds that

$$\Phi_j^{int}(v,\Lambda) = \phi_j(v^\Lambda) = \sum_{\{E \subseteq N:\, j \in E\}} p_E\left[v(int^\Lambda(E)) - v(int^\Lambda(E \setminus \{j\}))\right].$$

Let $E \subseteq N$. Since i has veto power over j in (N, int^Λ), it holds that $j \notin int^\Lambda(E \setminus \{i\})$. Hence, $int^\Lambda(E \setminus \{i\}) \subseteq E \setminus \{j\}$, what leads to $int^\Lambda(E \setminus \{i\}) \subseteq int^\Lambda(E \setminus \{j\})$. Using the monotonicity of v, it follows that $v(int^\Lambda(E \setminus \{i\})) \leqslant v(int^\Lambda(E \setminus \{j\}))$. So we can write

$$\begin{aligned}\Phi_j^{int}(v,\Lambda) &\leqslant \sum_{\{E \subseteq N:\, j \in E\}} p_E\left[v(int^\Lambda(E)) - v(int^\Lambda(E \setminus \{i\}))\right] \\ &= \sum_{\{E \subseteq N:\, i,j \in E\}} p_E\left[v(int^\Lambda(E)) - v(int^\Lambda(E \setminus \{i\}))\right]\end{aligned}$$

$$\leqslant \sum_{\{E \subseteq N \colon i \in E\}} p_E \left[v(int^{\Lambda}(E)) - v(int^{\Lambda}(E \setminus \{i\}))\right]$$
$$= \Phi_i^{int}(v, \Lambda).$$

□

Corollary 5.18 *Let A be an interior operator on N and let $i, j \in N$ be such that i has veto power over j in (N, A). Then $\Xi_i(A) \geqslant \Xi_j(A)$.*

Proof. Let $\Lambda \in \mathcal{I}^N$ be such that $A = int^{\Lambda}$. Let $k \in N$. It holds that

$$\Xi_{ik}(A) = \Phi_i\left(u_{\{k\}}, A\right) = \Phi_i^{int}\left(u_{\{k\}}, \Lambda\right)$$

which, from Proposition 5.17 is greater or equal than

$$\Phi_j^{int}\left(u_{\{k\}}, \Lambda\right) = \Phi_j\left(u_{\{k\}}, A\right) = \Xi_{jk}(A).$$

□

Finally, we see an example of games with interior operator structure: the *information market games* introduced by Muto, Potters and Tijs [52].

Example 5.19 Let $N = \{1, \ldots, n\}$ with $n \geqslant 3$. Suppose that the elements of N are companies that want to sell their products in a certain market. Let $(\omega_1, \ldots, \omega_n) \in \mathbb{R}^N_{++}$ be the vector whose components represent the profit that each company will get from selling their products in the market. Suppose that there exists a subset I of companies that have certain necessary information about the market. We assume $I \subsetneq N$ and $|I| \geqslant 2$. If a coalition does not contain any element in I then it will not be able to sell in the market. So the profit that can be made by each coalition in 2^N is given by

$$v(E) = \begin{cases} \sum_{i \in E} \omega_i & \text{if } E \cap I \neq \emptyset, \\ \\ 0 & \text{if } E \cap I = \emptyset. \end{cases}$$

Notice that $v = \omega^A$, where ω is the additive game given by the vector $(\omega_1, \ldots, \omega_n)$ and A is the interior operator on N defined as

$$A(E) = \begin{cases} E & \text{if } E \cap I \neq \emptyset, \\ \emptyset & \text{if } E \cap I = \emptyset. \end{cases}$$

It is clear that $A = int^{\Lambda}$ where Λ is the interior function on N defined by

$$\Lambda(E) = \begin{cases} 1 & \text{if } E \cap I \neq \emptyset \text{ or } E = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We calculate $\Phi^{int}(\omega, \Lambda)$, that is equal to $\Phi(\omega, A)$. In order to calculate $\Phi(\omega, A)$ we use the expression given in Corollary 2.16. Notice that in our case, since ω is additive, we just need to calculate $\Phi_i\left(u_{\{j\}}, A\right)$ for every $i, j \in N$. Moreover, observe that

$$\Phi_i\left(u_{\{j\}}, A\right) = \phi_i\left(\left(u_{\{j\}}\right)^A\right) = \phi_i(A_j) = \Xi_{ij}(A),$$

where $\Xi_{ij}(A)$ is the influence of i over j in (N, A). So, it holds that

$$\Phi(\omega, A) = \begin{pmatrix} \Xi_{11}(A) & \ldots & \Xi_{1n}(A) \\ \vdots & \ddots & \vdots \\ \Xi_{n1}(A) & \ldots & \Xi_{nn}(A) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}. \tag{5.1}$$

We proceed to calculate $\Xi_{ij}(A)$ for every $i, j \in N$. We distinguish two cases.

Case $j \in I$. In this case it is clear that $A_j = u_{\{j\}}$. It follows that $\Xi_{jj}(A) = 1$ and $\Xi_{ij}(A) = 0$ for every $i \in N \setminus \{j\}$.

Case $j \notin I$. For every $E \in 2^N \setminus \{\emptyset\}$ we define

$$\Delta(E) = \begin{cases} (-1)^{|H|+1} & \text{if there exists } H \subseteq I, H \neq \emptyset \text{ such that } E = H \cup \{j\}, \\ 0 & \text{otherwise .} \end{cases}$$

We want to prove that

$$\Delta_{A_j}(E) = \Delta(E) \quad \text{for every } E \in 2^N \setminus \{\emptyset\}. \tag{5.2}$$

To do this, it suffices to show that

$$A_j(F) = \sum_{\{E\in 2^N\setminus\{\emptyset\}:\, E\subseteq F\}} \Delta(E) \quad \text{for every } F \in 2^N \setminus \{\emptyset\}. \tag{5.3}$$

Let $F \in 2^N \setminus \{\emptyset\}$. If $j \notin F$ or $F \cap I = \emptyset$, it is clear that (5.3) holds. Suppose now that $j \in F$ and $F \cap I \neq \emptyset$. It holds that

$$\sum_{\{E\in 2^N\setminus\{\emptyset\}:\, E\subseteq F\}} \Delta(E) = \sum_{\{H\in 2^N\setminus\{\emptyset\}:\, H\subseteq F\cap I\}} (-1)^{|H|+1} = \sum_{k=1}^{|F\cap I|} \binom{|F\cap I|}{k} (-1)^{k+1} = 1 = A_j(F).$$

So we have proved (5.3) and, consequently, (5.2). We proceed to calculate $\Xi_{jj}(A)$

$$\begin{aligned}
\Xi_{jj}(A) = \phi_j(A_j) &= \sum_{\{E\subseteq N:\, j\in E\}} \frac{\Delta_{A_j}(E)}{|E|} \\
&= \sum_{\{H\in 2^N\setminus\{\emptyset\}:\, H\subseteq I\}} \frac{(-1)^{|H|+1}}{|H|+1} = \frac{1}{|I|+1}\sum_{k=1}^{|I|} \binom{|I|+1}{k+1} (-1)^{k+1} \\
&= \sum_{k=1}^{|I|} \binom{|I|}{k} \frac{(-1)^{k+1}}{k+1} = \frac{1}{|I|+1}\sum_{k=1}^{|I|} \binom{|I|+1}{k+1} (-1)^{k+1} \\
&= \frac{1}{|I|+1}\sum_{k=2}^{|I|+1} \binom{|I|+1}{k} (-1)^{k} = \frac{|I|}{|I|+1}.
\end{aligned} \tag{5.4}$$

It remains to calculate $\Xi_{ij}(A)$ for every $i \in N\setminus\{j\}$. On the one hand, notice that if $i \in N\setminus(I\cup\{j\})$ then i is a null player in A_j. Therefore

$$\Xi_{ij}(A) = 0 \quad \text{for every } i \in N \setminus (I \cup \{j\}). \tag{5.5}$$

On the other hand, from the equal treatment property of the Shapley value it follows that

$$\Xi_{ij}(A) = \Xi_{lj}(A) \quad \text{for every } i, l \in I. \tag{5.6}$$

From (5.4), (5.5) and (5.6) we conclude, using the property of efficiency of the Shapley authorization correspondence, that

$$\Xi_{ij}(A) = \frac{1}{(|I|+1)\,|I|} \quad \text{for every } i \in I. \tag{5.7}$$

Once we have calculated $\Xi(A)$, if we go back to (5.1) we obtain

$$\Phi_i^{int}(\omega, \Lambda) = \Phi_i(\omega, A) = \begin{cases} \omega_i + \dfrac{\omega(N \setminus I)}{(|I|+1)\,|I|} & \text{if } i \in I, \\ \\ \dfrac{|I|}{|I|+1}\,\omega_i & \text{if } i \in N \setminus I. \end{cases}$$

5.3 Fuzzy interior operators

We know that any interior operator has the form int^{Λ} where Λ is an interior function. We make use of this idea to introduce the concept of fuzzy interior operator. Firstly, we will define fuzzy interior functions by generalizing Definition 5.4 in a natural way, and, then, we will associate what we will call a fuzzy interior operator to each fuzzy interior function, in a similar way as we have associated an interior operator to each interior function.

Definition 5.20 *A fuzzy interior function on N is a mapping $\lambda : 2^N \to [0,1]$ that satisfies the following conditions:*

1. $\lambda(N) = \lambda(\emptyset) = 1$,
2. $\lambda(E \cup F) \geqslant \min\{\lambda(E), \lambda(F)\}$ *for all* $E, F \subseteq N$.

The set of all fuzzy interior functions on N is denoted by $\mathcal{FI}^N$.

Let $\lambda \in \mathcal{FI}^N$. Consider $int^{\lambda} : 2^N \to [0,1]^N$ defined as

$$int^{\lambda}(E) = \bigcup_{F \subseteq E} \lambda(F)\,\mathbf{1}_F \quad \text{for every } E \subseteq N.$$

It is easy to check that int^{λ} is a fuzzy authorization operator.

Definition 5.21 *Let $a \in \mathcal{FA}^N$. The fuzzy authorization structure a is a fuzzy interior operator on N if there exists $\lambda \in \mathcal{FI}^N$ such that $a = int^\lambda$.*

Proposition 5.22 *Let $a : 2^N \to [0,1]^N$ be a fuzzy interior operator on N. Then there exists a unique $\lambda \in \mathcal{FI}^N$ such that $a = int^\lambda$.*

Proof. It is enough to prove that if $\lambda : 2^N \to [0,1]$ is a fuzzy interior function on N and $a = int^\lambda$ then

$$\lambda(E) = \min\{a_i(E) : i \in E\} \qquad \text{for all } E \in 2^N \setminus \{\emptyset\}.$$

So, let $\lambda : 2^N \to [0,1]$ be a fuzzy interior function on N and $E \in 2^N \setminus \{\emptyset\}$. We must see that

$$\lambda(E) = \min\left\{int_i^\lambda(E) : i \in E\right\}.$$

By definition, $int_i^\lambda(E) = \max\{\lambda(F) : i \in F \subseteq E\}$. So we have to check that

$$\lambda(E) = \min\{\max\{\lambda(F) : i \in F \subseteq E\} : i \in E\}.$$

It is clear that $\lambda(E) \leqslant \min\{\max\{\lambda(F) : i \in F \subseteq E\} : i \in E\}$. It only remains to prove that $\lambda(E) \geqslant \min\{\max\{\lambda(F) : i \in F \subseteq E\} : i \in E\}$. Take, for every $i \in E$, a set F_i such that $i \in F_i \subseteq E$ and $\lambda(F_i) = \max\{\lambda(F) : i \in F \subseteq E\}$. It holds that

$$\lambda(E) = \lambda\left(\bigcup_{i \in E} F_i\right) \geqslant \min\{\lambda(F_i) : i \in E\} = \min\{\max\{\lambda(F) : i \in F \subseteq E\} : i \in E\}.$$

□

In a similar way as we did with interior operators, we study the transitivity of the dependency relations induced by a fuzzy interior operator. Firstly we see that a fuzzy interior function determines some certain interior functions. Let $\lambda : 2^N \to [0,1]$. For any $t \in (0,1]$ consider

$$\begin{array}{rccl} \lambda_t : & 2^N & \to & \{0,1\} \\ & E & \to & \lambda_t(E) = \begin{cases} 1 & \text{if } \lambda(E) \geqslant t, \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Proposition 5.23 *Let $\lambda : 2^N \to [0,1]$. Then the following statements are equivalent:*

1. *The function λ is a fuzzy interior function.*

2. *The function λ_t is an interior function for every $t \in (0,1]$.*

Proof. Let $\lambda : 2^N \to [0,1]$.
$1 \implies 2$. Suppose that λ is a fuzzy interior function on N. Let $t \in (0,1]$. It is clear that $\lambda_t(N) = \lambda_t(\emptyset) = 1$. Let $E, F \subseteq N$ be such that $\lambda_t(E) = \lambda_t(F) = 1$. It holds that $\lambda(E \cup F) \geqslant \min\{\lambda(E), \lambda(F)\} \geqslant t$, and hence $\lambda_t(E \cup F) = 1$.

$2 \implies 1$. Suppose that λ is such that λ_t is an interior function for every $t \in (0,1]$. It is clear that $\lambda(N) = \lambda(\emptyset) = 1$. Let $E, F \subseteq N$ and $s = \min\{\lambda(E), \lambda(F)\}$. Suppose that $s > 0$. It holds that $\lambda_s(E) = \lambda_s(F) = 1$. Since λ_s is an interior function, it holds that $\lambda_s(E \cup F) = 1$, and hence $\lambda(E \cup F) \geqslant s$. □

Proposition 5.24 *Let $\lambda \in \mathcal{FI}^N$. It holds that*

$$\left(int^\lambda\right)^t = int^{\lambda_t} \quad \text{for every } t \in (0,1].$$

Proof. Let $\lambda \in \mathcal{FI}^N$, $t \in (0,1]$ and $E \subseteq N$. We prove that

$$[int^\lambda(E)]_t = int^{\lambda_t}(E).$$

Let $i \in [int^\lambda(E)]_t$. This means that $int_i^\lambda(E) \geqslant t$. It follows that there exists $F \subseteq E$ such that $i \in F$ and $\lambda(F) \geqslant t$. From $F \subseteq E$ and $\lambda_t(F) = 1$ we can derive that $F \subseteq int^{\lambda_t}(E)$. Therefore, $i \in int^{\lambda_t}(E)$.

Now take $i \in int^{\lambda_t}(E)$. It holds that $\lambda_t(int^{\lambda_t}(E)) = 1$. Equivalently, $\lambda(int^{\lambda_t}(E)) \geqslant t$. From $i \in int^{\lambda_t}(E)$, $int^{\lambda_t}(E) \subseteq E$ and $\lambda(int^{\lambda_t}(E)) \geqslant t$, it follows that $int_i^\lambda(E) \geqslant t$. □

Proposition 5.25 *Let $a \in \mathcal{FA}^N$. The following statements are equivalent:*

1. *The fuzzy authorization operator a is a fuzzy interior operator.*

2. *The authorization operator a^t is an interior operator for every $t \in (0,1]$.*

Proof. Let $a \in \mathcal{FA}^N$.

$1 \Longrightarrow 2$. Suppose that a is a fuzzy interior operator on N. There exists $\lambda \in \mathcal{FI}^N$ such that $a = int^{\lambda}$. Take $t \in (0,1]$. From Proposition 5.24 we obtain that $a^t = int^{\lambda_t}$, thus a^t is an interior operator.

$2 \Longrightarrow 1$. Suppose that a is such that a^t is an interior operator for every $t \in (0,1]$. Consider $\lambda : 2^N \to [0,1]$ defined as

$$\lambda(E) = \max\left\{t \in [0,1] : a^t(E) = E\right\} \quad \text{for every } E \subseteq N.$$

It is clear that $\lambda \in \mathcal{FI}^N$. In order to conclude that a is a fuzzy interior operator, it suffices to prove that $a = int^{\lambda}$. Take $E \in 2^N \setminus \{\emptyset\}$ and $i \in E$. It holds that

$$\begin{aligned} int_i^{\lambda}(E) &= \max\{\lambda(F) : i \in F \subseteq E\} \\ &= \max\left\{\max\left\{t \in [0,1] : a^t(F) = F\right\} : i \in F \subseteq E\right\} \\ &= \max\left\{t \in [0,1] : \text{there exists } F \in 2^N \text{ such that } i \in F \subseteq E \text{ and } a^t(F) = F\right\} \\ &= \max\left\{t \in [0,1] : i \in a^t(E)\right\} = a_i(E). \end{aligned}$$

□

Let $a \in \mathcal{FA}^N$. We define the following fuzzy relation on 2^N

$$R_a(E,F) = 1 - \max\{a_i(N \setminus E) : i \in F\} \quad \text{for all } E, F \subseteq N.$$

Proposition 5.26 *Let $a \in \mathcal{FA}^N$. The following statements are equivalent:*

1. *The fuzzy relation R_a is transitive.*
2. *The relation $\overset{a^t}{\triangleright}$ is transitive for every $t \in (0,1]$.*

Proof. Let $a \in \mathcal{FA}^N$.

$1 \Longrightarrow 2$. Suppose that a is such that R_a is transitive. Let $t \in (0,1]$. If $E, F, G \subseteq N$ are such that $E \overset{a^t}{\triangleright} F$ and $F \overset{a^t}{\triangleright} G$ it holds that $F \cap a^t(N \setminus E) = \emptyset$ and $G \cap a^t(N \setminus F) = \emptyset$. We can write

$$a_i(N \setminus E) < t \quad \text{for every } i \in F,$$

$$a_j(N \setminus F) < t \quad \text{for every } j \in G,$$

whence $R_a(E,F) > 1-t$ and $R_a(F,G) > 1-t$. Since R_a is transitive we conclude that $R_a(E,G) > 1-t$, from where we get $E \overset{a^t}{\triangleright} G$.

$2 \implies 1$. Suppose a is such that $\overset{a^t}{\triangleright}$ is transitive for every $t \in (0,1]$. Take $E,F,G \subseteq N$. We must see that $R_a(E,G) \geqslant \min\{R_a(E,F), R_a(F,G)\}$. It suffices to prove that for every $t \in (0, \min\{R_a(E,F), R_a(F,G)\})$ it holds that $R_a(E,G) > t$. If $t \in (0, \min\{R_a(E,F), R_a(F,G)\})$ it is clear that

$$a_i(N \setminus E) < 1-t \quad \text{for every } i \in F,$$

$$a_j(N \setminus F) < 1-t \quad \text{for every } j \in G.$$

Therefore, $E \overset{a^{1-t}}{\triangleright} F$ and $F \overset{a^{1-t}}{\triangleright} G$. Since $\overset{a^{1-t}}{\triangleright}$ is transitive $E \overset{a^{1-t}}{\triangleright} G$, whence $R_a(E,G) > t$. □

Proposition 5.27 *Let $a \in \widetilde{\mathcal{FA}}^N$. The following statements are equivalent:*

1. *The fuzzy authorization operator a is a fuzzy interior operator.*

2. *The relation R_a is transitive.*

Proof. The result follows from Propositions 5.3, 5.25 and 5.26. □

Definition 5.28 *A game with fuzzy interior operator structure on N is a pair (v,λ) where $v \in \mathcal{G}^N$ and $\lambda \in \mathcal{FI}^N$.*

In a similar way as we did in the crisp case, given a game with fuzzy interior operator structure, we can define a characteristic function that gathers the information from the game and the fuzzy interior operator structure.

Definition 5.29 *Let $v \in \mathcal{G}^N$ and $\lambda \in \mathcal{FI}^N$. The restricted game of (v,λ), denoted by v^λ, is defined as the restricted game of (v, int^λ), that is,*

$$v^\lambda(E) = v^{int^\lambda}(E) = \int int^\lambda(E)\, dv \quad \textit{for all } E \subseteq N.$$

The number $v^{\lambda}(E)$ is the worth of E in the game with fuzzy interior operator structure.

Remark 5.30 *Let $v \in \mathcal{G}^N$ and $\lambda \in \mathcal{FI}^N$. Let $\{t_l : l = 0, \ldots, r\} = im(\lambda) \cup \{0\}$ with $0 = t_0 < \ldots < t_r = 1$. Then it holds that*

$$v^{\lambda}(E) = \sum_{l=1}^{r} (t_l - t_{l-1})\, v\left([int^{\lambda}(E)]_{t_l}\right) \qquad \text{for all } E \subseteq N,$$

where $[int^{\lambda}(E)]_{t_l} = \left\{i \in N : int_i^{\lambda}(E) \geq t_l\right\}$ for $l = 1, \ldots, r$.

5.4 The Shapley fuzzy interior value

An allocation rule for games with fuzzy interior operator structure is a mapping that assigns to every game with fuzzy interior operator structure a payoff vector. We introduce a particular allocation rule.

Definition 5.31 *The Shapley fuzzy interior value, denoted by φ^{int}, assigns to every game with fuzzy interior operator structure (v, λ) the Shapley value of the restricted game v^{λ},*

$$\varphi^{int}(v, \lambda) = \phi\left(v^{\lambda}\right) \qquad \text{for all } v \in \mathcal{G}^N \text{ and } \lambda \in \mathcal{FI}^N.$$

Notice that

$$\varphi^{int}(v, \lambda) = \varphi\left(v, int^{\lambda}\right) \qquad \text{for all } v \in \mathcal{G}^N \text{ and } \lambda \in \mathcal{FI}^N,$$

where φ is the Shapley fuzzy authorization value.

Lemma 5.32 *Let $\lambda \in \mathcal{FI}^N$ and $\{t_l : l = 0, \ldots, r\} = im(\lambda) \cup \{0\}$ with $0 = t_0 < \ldots < t_r = 1$ and let $v \in \mathcal{G}^N$. It holds that*

$$\varphi^{int}(v, \lambda) = \sum_{l=1}^{r} (t_l - t_{l-1})\, \Phi^{int}(v, \lambda_{t_l}),$$

where Φ^{int} is the Shapley interior value.

Proof. Let $\lambda \in \mathcal{FI}^N$ and $\{t_l : l = 0, \dots, r\} = im(\lambda) \cup \{0\}$ with $0 = t_0 < \dots < t_r = 1$. Let $v \in \mathcal{G}^N$. Using Lemma 3.11 we can write

$$\varphi^{int}(v, \lambda) = \varphi\left(v, int^{\lambda}\right) = \sum_{l=1}^{r}(t_l - t_{l-1})\, \Phi(v, (int^{\lambda})^{t_l}).$$

Using Proposition 5.24 we conclude that the sum before is equal to

$$\sum_{l=1}^{r}(t_l - t_{l-1})\, \Phi(v, int^{\lambda_{t_l}}) = \sum_{l=1}^{r}(t_l - t_{l-1})\, \Phi^{int}(v, \lambda_{t_l}).$$

□

We aim to characterize the Shapley fuzzy interior value. To that end, we consider the following properties.

- NECESSARY PLAYER PROPERTY. For every monotonic $v \in \mathcal{G}^N$, $\lambda \in \mathcal{FI}^N$ and $i \in N$ a necessary player in v, it holds that

$$\psi_i(v, \lambda) \geqslant \psi_k(v, \lambda) \quad \text{for all } k \in N.$$

Definition 5.33 *Let $\lambda \in \mathcal{FI}^N$, $E \subseteq N$ and $i, j \in E$. A player j depends on i within E according to λ if $int_j^{\lambda}(E) > int_j^{\lambda}(E \setminus \{i\})$*

Definition 5.34 *Let $v \in \mathcal{G}^N$, $E \subseteq N$ and $j \in E$. A player j is a null player for v within E if $v(F) = v(F \setminus \{j\})$ for every $F \subseteq E$.*

Definition 5.35 *Let $v \in \mathcal{G}^N$, $\lambda \in \mathcal{FI}^N$ and $i \in N$. A player i is an unnecessary player in (v, λ) if for every $E \subseteq N$ and $j \in E$ such that j depends on i within E according to λ it holds that j is a null player for v within E.*

- UNNECESSARY PLAYER PROPERTY. For every $v \in \mathcal{G}^N$, $\lambda \in \mathcal{FI}^N$ and $i \in N$ an unnecessary player in (v, λ), it holds that

$$\psi_i(v, \lambda) = 0.$$

Definition 5.36 *Let $\lambda \in \mathcal{FI}^N$ and $E \subset N$. A coalition E is an inessential coalition for λ if $\lambda(E) = 0$.*

- INESSENTIAL COALITION PROPERTY. For every $v, w \in \mathcal{G}^N$, $\lambda \in \mathcal{FI}^N$ and $E \subset N$ an inessential coalition for λ such that $v(F) = w(F)$ for all $F \in 2^N \setminus \{E\}$, it holds that

$$\psi(v, \lambda) = \psi(w, \lambda).$$

- REDUCTION. For every $v \in \mathcal{G}^N$, $\lambda \in \mathcal{FI}^N$ and $t \in (0, 1)$, it holds that

$$\psi(v, \lambda) = t\psi(v, \lambda^{[0,t]}) + (1 - t)\psi(v, \lambda^{[t,1]}),$$

where

$$\lambda^{[0,t]} = \min\left(1, \frac{\lambda}{t}\right),$$
$$\lambda^{[t,1]} = \max\left(0, \frac{\lambda - t}{1 - t}\right).$$

Theorem 5.37 *An allocation rule for games with fuzzy interior operator structure is equal to the Shapley fuzzy interior value if and only if it satisfies the properties of efficiency, additivity, necessary player, unnecessary player, inessential coalition and reduction.*

Proof. That the Shapley fuzzy interior value satisfies the properties of efficiency, additivity and necessary player follows from the fact that the Shapley fuzzy interior value is the restriction of the Shapley fuzzy authorization value to the set of games with fuzzy interior operator structure and the fact that the Shapley fuzzy authorization value satisfies such properties. Let us prove that φ^{int} satisfies the properties of unnecessary player, inessential coalition and reduction.

UNNECESSARY PLAYER PROPERTY. Let $v \in \mathcal{G}^N$, $\lambda \in \mathcal{FI}^N$, $\{t_l : l = 0, \ldots, r\} = im(\lambda) \cup \{0\}$ with $0 = t_0 < \ldots < t_r = 1$ and $i \in N$ an unnecessary player in (v, λ). We must prove that $\varphi_i^{int}(v, \lambda) = 0$. Taking into consideration Lemma 5.32 it is enough to see that $\Phi_i^{int}(v, \lambda_{t_l}) = 0$ for all $l = 1, \ldots, r$. Since Φ^{int} satisfies the unnecessary player property, it suffices to prove that i is an unnecessary player in (v, λ_{t_l}) for all $l = 1, \ldots, r$. Take $l \in \{1, \ldots, r\}$. Let $E \subseteq N$ and $j \in E$ be such that j depends on i within E according to λ_{t_l}. This means that

$j \in int^{\lambda_{t_l}}(E) \setminus int^{\lambda_{t_l}}(E \setminus \{i\})$. By Proposition 5.24, this is equivalent to write that $j \in [int^{\lambda}(E)]_{t_l} \setminus [int^{\lambda}(E \setminus \{i\})]_{t_l}$ and hence, $int^{\lambda}_j(E) \geqslant t_l > int^{\lambda}_j(E \setminus \{i\})$. From this and the fact that i is an unnecessary player in (v, λ) we can derive that j is a null player for v within E. So we have proved that i is an unnecessary player in (v, λ_{t_l}).

Inessential coalition property. Let $v, w \in \mathcal{G}^N$, $\lambda \in \mathcal{FI}^N$, $\{t_l : l = 0, \ldots, r\} = im(\lambda) \cup \{0\}$ with $0 = t_0 < \ldots < t_r = 1$ and $E \subset N$ an inessential coalition for λ such that $v(F) = w(F)$ for all $F \in 2^N \setminus \{E\}$. We have to prove that $\varphi^{int}(v, \lambda) = \varphi^{int}(w, \lambda)$. By Lemma 5.32, it suffices to prove that $\Phi^{int}(v, \lambda_{t_l}) = \Phi^{int}(w, \lambda_{t_l})$ for all $l = 1, \ldots, r$. And this is elementary from the facts that Φ^{int} satisfies the inessential coalition property and E is an inessential coalition for λ_{t_l} for all $l = 1, \ldots, r$.

Reduction. Let $v \in \mathcal{G}^N$, $\lambda \in \mathcal{FI}^N$ and $t \in (0, 1)$. Using the property of reduction of the Shapley fuzzy authorization value we can write

$$\varphi^{int}(v, \lambda) = \varphi\left(v, int^{\lambda}\right) = t\varphi\left(v, \left(int^{\lambda}\right)^{[0,t]}\right) + (1-t)\varphi\left(v, \left(int^{\lambda}\right)^{[t,1]}\right).$$

It is easy to check that $\left(int^{\lambda}\right)^{[0,t]} = int^{\left(\lambda^{[0,t]}\right)}$ and $\left(int^{\lambda}\right)^{[t,1]} = int^{\left(\lambda^{[t,1]}\right)}$. Therefore the sum above is equal to

$$t\varphi\left(v, int^{\left(\lambda^{[0,t]}\right)}\right) + (1-t)\varphi\left(v, int^{\left(\lambda^{[t,1]}\right)}\right) = t\varphi^{int}\left(v, \lambda^{[0,t]}\right) + (1-t)\varphi^{int}\left(v, \lambda^{[t,1]}\right).$$

We have seen that the Shapley fuzzy interior value satisfies the six properties. It remains to prove that such properties uniquely determine φ^{int}.

Let ψ be an allocation rule for games with fuzzy interior operator structure satisfying the properties of efficiency, additivity, necessary player, unnecessary player, inessential coalition and reduction. We must prove that

$$\psi(v, \lambda) = \varphi^{int}(v, \lambda) \quad \text{for every } v \in \mathcal{G}^N \text{ and } \lambda \in \mathcal{FI}^N.$$

We proceed by strong induction on $\lceil(\lambda)$ where

$$\lceil(\lambda) = |im(\lambda) \setminus \{0, 1\}| \quad \text{for all } \lambda \in \mathcal{FI}^N.$$

1. Base case. $\lceil(\lambda) = 0$.

   Notice that we can identify $\mathcal{I}^N$ with the set $\{\lambda \in \mathcal{FI}^N : im(\lambda) \subseteq \{0, 1\}\}$. So we can say that the restriction of ψ to the set of games with fuzzy interior operator structure (v, λ) with $\lceil(\lambda) = 0$ is an allocation rule for games with interior operator structure. It is easy to check that such restriction satisfies the properties of efficiency, additivity, necessary player, unnecessary player and inessential coalition. Therefore, using Theorem 5.15 we conclude that

$$\psi(v, \lambda) = \varphi^{int}(v, \lambda) \text{ for every } v \in \mathcal{G}^N \text{ and } \lambda \in \mathcal{FI}^N \text{ with } \lceil(\lambda) = 0.$$

2. Inductive step. Let $v \in \mathcal{G}^N$ and $\lambda \in \mathcal{FI}^N$ with $\lceil(\lambda) > 0$. We must prove that $\psi(v, \lambda) = \varphi^{int}(v, \lambda)$. Take $t \in im(\lambda) \setminus \{0, 1\}$. Since ψ satisfies the reduction property it holds that

$$\psi(v, \lambda) = t\, \psi(v, \lambda^{[0,t]}) + (1 - t)\, \psi(v, \lambda^{[t,1]}).$$

Since $\lceil(\lambda^{[0,t]}) < \lceil(\lambda)$ and $\lceil(\lambda^{[t,1]}) < \lceil(\lambda)$ it follows by induction hypothesis that $\psi(v, \lambda^{[0,t]}) = \varphi^{int}(v, \lambda^{[0,t]})$ and $\psi(v, \lambda^{[t,1]}) = \varphi^{int}(v, \lambda^{[t,1]})$. Hence

$$\psi(v, \lambda) = t\, \varphi^{int}(v, \lambda^{[0,t]}) + (1 - t)\, \varphi^{int}(v, \lambda^{[t,1]}) = \varphi^{int}(v, \lambda).$$

□

Chapter 6

Games with conjunctive authorization structure

Given an authorization structure and a coalition, the set of agents who are dominated by the coalition can be larger than the union of the sets of agents who are dominated by each one of the members of the coalition. There is a kind of authorization structures in which both sets are always equal. They are called conjunctive authorization structures. In these structures all the dependency relationships are bilateral. For this reason they are the most simple authorization structures. In this chapter we define and characterize allocation rules for games with conjunctive authorization structure.

6.1 Conjunctive authorization structures

We aim to consider authorization structures in which the set of agents who are dominated by the union of two coalitions coincides with the set of agents that are dominated by at least one of the coalitions. This can be expressed with the following definition.

Definition 6.1 *An authorization operator $A \in \mathcal{A}^N$ is said to be conjunctive if*

$$A(E \cap F) = A(E) \cap A(F) \quad \text{for every } E, F \subseteq N.$$

If A is a conjunctive authorization operator on N, the pair (N, A) is said to be a conjunctive authorization structure. The set of all conjunctive authorization operators is denoted by $\mathcal{A}_c^N$.

The conjunctive permission structures introduced by Gilles, Owen and van den Brink [40] are an example of conjunctive authorization structure. Given a permission structure, the operator that assigns to each coalition its conjunctive sovereign part is a conjunctive authorization operator. The conjunctive permission structures were generalized in Algaba, Bilbao, van den Brink and Jiménez-Losada [5] with the so called *poset antimatroids*, which are also conjunctive authorization structures.

Firstly we see a characterization of conjunctive authorization operators in terms of the veto relationships induced by these operators.

Proposition 6.2 *Let $A \in \mathcal{A}^N$. The following statements are equivalent:*

1. *The authorization operator A is conjunctive.*
2. *For every $i, j \in N$ such that j depends partially on i in (N, A), it holds that i has veto power over j in (N, A).*
3. *For every $E \in 2^N \setminus \{\emptyset\}$ and $j \in N$ such that E has veto power over j in (N, A), there exists $i \in E$ such that i has veto power over j in (N, A).*

Proof. Let $A \in \mathcal{A}^N$.
$1 \Longrightarrow 2$. Suppose that A is conjunctive and let $i, j \in N$ such that j depends partially on i in (N, A). There must exist $E \subseteq N$ such that $j \in A(E) \setminus A(E \setminus \{i\})$. We have

$$\begin{aligned} j \in A(E) \setminus A(E \setminus \{i\}) &= A(E) \setminus A(E \cap (N \setminus \{i\})) \\ &= A(E) \setminus (A(E) \cap A(N \setminus \{i\})) \\ &= A(E) \setminus A(N \setminus \{i\}). \end{aligned}$$

Thereby $j \notin A(N \setminus \{i\})$, that is, i has veto power over j in (N, A).

$2 \Longrightarrow 3$. Let $E \in 2^N \setminus \{\emptyset\}$ and $j \in N$ such that E has veto power over j in (N, A). If $j \notin A(N)$ any player in E has veto power over j. Suppose $j \in A(N)$. Let $E = \{i_1, \ldots, i_l\}$. Take

$$m = \min \{r \in \{1, \ldots, l\} : j \notin A(N \setminus \{i_1, \ldots, i_r\})\}.$$

It is clear that j depends partially on i_m in (N, A). Using the hypothesis, we conclude that i_m has veto power over j in (N, A).

$3 \Longrightarrow 1$. Let $E, F \subseteq N$. We must prove that $A(E \cap F) = A(E) \cap A(F)$. From monotonicity of authorization operators, it is clear that $A(E \cap F) \subseteq A(E) \cap A(F)$. Let us prove the other inclusion. If $A(E) \cap A(F) = \emptyset$ or $E = F = N$ we have finished. Otherwise, take $j \in A(E) \cap A(F)$. Let us see that $j \in A(E \cap F)$. Suppose $j \notin A(E \cap F)$. This means that $N \setminus (E \cap F)$ has veto power over j. Using the hypothesis, we conclude that there exists $i \in N \setminus (E \cap F)$ such that i has veto power over j in (N, A). It holds that or $i \notin E$ or $i \notin F$. Suppose $i \notin E$ (the case $i \notin F$ is similar). Since $j \notin A(N \setminus \{i\})$ and $E \subseteq N \setminus \{i\}$, we can derive $j \notin A(E)$, but this is a contradiction. □

The calculation of the sovereignty, influence and power indices is particularly easy in the case of conjunctive authorization structures, as we see in the following proposition.

Proposition 6.3 *Let $A \in \mathcal{A}_c^N$. Then, it holds that*

(a) $sov_i(A) = \dfrac{1}{|\mathtt{V}_i(A)|}$ *for every $i \in N$ such that i is not inactive in (N, A).*

(b) $inf_i(A) = \displaystyle\sum_{\{j \in A(N)\setminus\{i\} :\, i \in \mathtt{V}_j(A)\}} \frac{1}{|\mathtt{V}_j(A)|}$ *for every $i \in N$.*

(c) $pow_i(A) = \displaystyle\sum_{\{j \in A(N) :\, i \in \mathtt{V}_j(A)\}} \frac{1}{|\mathtt{V}_j(A)|}$ *for every $i \in N$.*

Proof. It is easy to check that *(a)* and *(b)* are a direct consequence of Proposition 6.2 and the properties of maximum and minimum sovereignty and maximum and minimum influence seen in Chapter 4. Finally, *(c)* derives from *(a)* and *(b)*. □

An important advantage of conjunctive operators is the fact that they preserve convexity. This means that if we have a nonnegative convex game on a conjunctive structure, then the restricted game is convex as well. In fact, this property characterizes conjunctive operators.

Proposition 6.4 *Let $A \in \mathcal{A}^N$. The following statements are equivalent:*

1. *The authorization operator A is conjunctive.*
2. *For every nonnegative and convex $v \in \mathcal{G}^N$ it holds that v^A is convex.*

3. *For every nonnegative and additive $\omega \in \mathcal{G}^N$ it holds that ω^A is convex.*

Proof. Let $A \in \mathcal{A}^N$.

$1 \Longrightarrow 2$. Suppose that A is conjunctive and let $v \in \mathcal{G}^N$ be nonnegative and convex. Let $E, F \subseteq N$. It holds that

$$\begin{aligned} v^A(E \cup F) + v^A(E \cap F) &= v(A(E \cup F)) + v(A(E \cap F)) \\ &= v(A(E \cup F)) + v(A(E) \cap A(F)) \\ &\geqslant v(A(E) \cup A(F)) + v(A(E) \cap A(F)) \\ &\geqslant v(A(E)) + v(A(F)) \\ &= v^A(E) + v^A(F). \end{aligned}$$

$2 \Longrightarrow 3$. This implication is trivial.

$3 \Longrightarrow 1$. Let $E, F \subseteq N$. We must prove that $A(E \cap F) = A(E) \cap A(F)$. It is clear that the inclusion $A(E \cap F) \subseteq A(E) \cap A(F)$ is true for any authorization operator. We need to show that $A(E \cap F) \supseteq A(E) \cap A(F)$. If $A(E) \cap A(F) = \emptyset$ we would have finished. Otherwise, take $i \in A(E) \cap A(F)$. We must prove that $i \in A(E \cap F)$. Consider the unanimity game $u_{\{i\}}$ which is an additive game. By hypothesis we know that $u^A_{\{i\}}$ is a convex game. Therefore it holds that

$$u^A_{\{i\}}(E) + u^A_{\{i\}}(F) \leqslant u^A_{\{i\}}(E \cap F) + u^A_{\{i\}}(E \cup F)$$

whence it follows that $i \in A(E \cap F)$. □

Definition 6.5 *A game with conjunctive authorization structure on N is a pair (v, A) where $v \in \mathcal{G}^N$ and $A \in \mathcal{A}^N_c$.*

6.2 The Shapley conjunctive authorization value

An allocation rule for games with conjunctive authorization structure assigns to every game with conjunctive authorization structure a payoff vector. In this section we define and characterize an allocation rule for games with conjunctive authorization structure.

Definition 6.6 *The Shapley conjunctive authorization value, denoted by Φ^c, assigns to each game with conjunctive authorization structure (v, A) the Shapley value of v^A,*

$$\Phi^c(v, A) = \phi\left(v^A\right) \quad \text{for all } v \in \mathcal{G}^N \text{ and } A \in \mathcal{A}_c^N.$$

Notice that

$$\Phi^c(v, A) = \Phi(v, A) \quad \text{for all } v \in \mathcal{G}^N \text{ and } A \in \mathcal{A}_c^N,$$

where Φ is the Shapley authorization value.

In the following theorem we show a characterization of the Shapley conjunctive authorization value.

Theorem 6.7 *An allocation rule for games with conjunctive authorization structure is equal to the Shapley conjunctive authorization value if and only if it satisfies the properties of efficiency, additivity, irrelevant player and veto power over a necessary player.*

Proof. That the Shapley conjunctive authorization value satisfies the properties of efficiency, additivity, irrelevant player and veto power over a necessary player follows from the fact that the Shapley conjunctive authorization value is the restriction of the Shapley authorization value to the set of games with conjunctive authorization structure and the fact that the Shapley authorization value satisfies such properties.

We have already seen that the Shapley conjunctive authorization value satisfies the four properties. Now we see that such properties uniquely determine the Shapley conjunctive authorization value. Let Ψ be an allocation rule for games with conjunctive authorization structure satisfying the properties of efficiency, additivity, irrelevant player and veto power over a necessary player. We must prove that $\Psi = \Phi^c$.

Let $n \in \mathbb{N}$ and let N be a set of cardinality n. Our first goal will be to show that $\Psi(\alpha u_E, A) = \Phi^c(\alpha u_E, A)$ for all $\alpha > 0$, $E \in 2^N \setminus \{\emptyset\}$ and $A \in \mathcal{A}_c^N$. So, take $\alpha > 0$ and $E \in 2^N \setminus \{\emptyset\}$ and $A \in \mathcal{A}_c^N$. Consider the set

$$F = \{i \in N : \text{ there exists } j \in E \text{ such that } i \text{ has veto power over } j \text{ in } (N, A)\}.$$

From the fact that Ψ satisfies the property of veto power over a necessary player, we can derive that there exists $b \in \mathbb{R}$ such that

$$\Psi_i(\alpha u_E, A) = b \quad \text{for all } i \in F. \tag{6.1}$$

Take $k \in N \setminus F$. Notice that k is an irrelevant player in $(\alpha u_E, A)$. Indeed, if a player j depends partially on k in (N, A), then, from Proposition 6.2, k has veto power over j in (N, A). And, since $k \notin F$, we can derive that $j \notin E$, and, therefore, j is a null player in αu_E. Now, using that Ψ satisfies the irrelevant player property, we conclude that

$$\Psi_i(\alpha u_E, A) = 0 \quad \text{for all } i \in N \setminus F. \tag{6.2}$$

From (6.1), (6.2) and the fact that Ψ satisfies efficiency we obtain that

$$\Psi_i(\alpha u_E, A) = \begin{cases} \dfrac{\alpha u_E(A(N))}{|F|} & \text{if } i \in F, \\ \\ 0 & \text{if } i \notin F. \end{cases}$$

Since Φ^c also satisfies the properties used, it holds that $\Psi(\alpha u_E, A) = \Phi^c(\alpha u_E, A)$.
So we have proved that $\Psi(\alpha u_E, A) = \Phi^c(\alpha u_E, A)$ for all $\alpha > 0$, $E \in 2^N \setminus \{\emptyset\}$ and $A \in \mathcal{A}_c^N$.
Finally, using the same linearity argument that we have used in several proofs before, we obtain $\Psi = \Phi^c$. □

We can give another expression of the Shapley conjunctive authorization value. Before, we see some results.

Let $A \in \mathcal{A}^N$ and $E \in 2^N \setminus \{\emptyset\}$. We define

$$\mathtt{V}_E(A) = \{i \in N : \, E \nsubseteq A\,(N \setminus \{i\})\}\,.$$

It is clear that $\mathtt{V}_E(A) = \bigcup_{i \in E} \mathtt{V}_i(A)$. Moreover, if $E \in 2^{A(N)} \setminus \{\emptyset\}$ then

$$\mathtt{V}_E(A) = \bigcap_{\{F \subseteq N : \, E \subseteq A(F)\}} F. \tag{6.3}$$

Lemma 6.8 *Let $A \in \mathcal{A}^N$. The following statements are equivalent:*

1. *The authorization operator A is conjunctive.*
2. *For every $E \in 2^{A(N)} \setminus \{\emptyset\}$ it holds that $E \subseteq A(\mathtt{V}_E(A))$.*

Proof. Let $A \in \mathcal{A}^N$.
$1 \Longrightarrow 2$. Suppose that A is conjunctive and let $E \in 2^{A(N)} \setminus \{\emptyset\}$. If $\mathtt{V}_E(A) = N$ we would have finished, since $E \subseteq A(N)$. Suppose $\mathtt{V}_E(A) \neq N$. Notice that for every $i \in N \setminus \mathtt{V}_E(A)$ it holds that $E \subseteq A(N \setminus \{i\})$. So we have

$$E \subseteq \bigcap_{i \in N \setminus \mathtt{V}_E(A)} A(N \setminus \{i\}). \tag{6.4}$$

But, since A is conjunctive it holds that

$$\bigcap_{i \in N \setminus \mathtt{V}_E(A)} A(N \setminus \{i\}) = A\left(\bigcap_{i \in N \setminus \mathtt{V}_E(A)} (N \setminus \{i\}) \right) = A(\mathtt{V}_E(A)). \tag{6.5}$$

$2 \Longrightarrow 1$. Let $F, H \subseteq N$. We must prove that $A(F) \cap A(H) \subseteq A(F \cap H)$. If $A(F) \cap A(H) = \emptyset$ we would have finished. Suppose that $A(F) \cap A(H) \neq \emptyset$. From $A(F) \cap A(H) \subseteq A(F)$ and $A(F) \cap A(H) \subseteq A(H)$ it follows, respectively, that $\mathtt{V}_{A(F)\cap A(H)}(A) \subseteq F$ and $\mathtt{V}_{A(F)\cap A(H)}(A) \subseteq H$. Therefore, $\mathtt{V}_{A(F)\cap A(H)}(A) \subseteq F \cap H$, whence we derive that $A\left(\mathtt{V}_{A(F)\cap A(H)}(A)\right) \subseteq A(F \cap H)$. Using the hypothesis we obtain that $A(F) \cap A(H) \subseteq A(F \cap H)$. □

Lemma 6.9 *Let $v \in \mathcal{G}^N$ and $A \in \mathcal{A}_c^N$. Then it holds*

$$\Delta_{v^A}(F) = \sum_{\left\{E \in 2^{A(N)} \setminus \{\emptyset\} : \, \mathtt{V}_E(A) = F\right\}} \Delta_v(E) \quad \textit{for all nonempty } F \subseteq N,$$

where Δ_v are the dividends of Harsanyi of the game v.

Proof. Let $T \in 2^N \setminus \{\emptyset\}$. It suffices to prove that

$$v^A(T) = \sum_{F \subseteq T} \sum_{\left\{E \in 2^{A(N)} \setminus \{\emptyset\} : \, \mathtt{V}_E(A) = F\right\}} \Delta_v(E).$$

Let us calculate,

$$\sum_{F\subseteq T}\ \sum_{\left\{E\in 2^{A(N)}\setminus\{\emptyset\}:\, \mathtt{V}_E(A)=F\right\}} \Delta_v(E) = \sum_{\left\{E\in 2^{A(N)}\setminus\{\emptyset\}:\, \mathtt{V}_E(A)\subseteq T\right\}} \Delta_v(E)$$

which, from (6.3) and Lemma 6.8, is equal to

$$\sum_{\{E\in 2^N\setminus\{\emptyset\}:\, E\subseteq A(T)\}} \Delta_v(E) = v(A(T)).$$

□

Proposition 6.10 *Let* $A \in \mathcal{A}_c^N$ *and* $v \in \mathcal{G}^N$*. Then*

$$\Phi_i^c(v, A) = \sum_{\left\{E\in 2^{A(N)}\setminus\{\emptyset\}:\, i\in\mathtt{V}_E(A)\right\}} \frac{\Delta_v(E)}{|\mathtt{V}_E(A)|}.$$

Proof. It holds that

$$\Phi_i^c(v, A) = \phi_i(v^A) = \sum_{\{F\subseteq N:\, i\in F\}} \frac{\Delta_{v^A}(F)}{|F|}$$

which, from Lemma 6.9, is equal to

$$\sum_{\{F\subseteq N:\, i\in F\}} \left(\frac{1}{|F|} \sum_{\left\{E\in 2^{A(N)}\setminus\{\emptyset\}:\, \mathtt{V}_E(A)=F\right\}} \Delta_v(E) \right)$$

$$= \sum_{\{F\subseteq N:\, i\in F\}}\ \sum_{\left\{E\in 2^{A(N)}\setminus\{\emptyset\}:\, \mathtt{V}_E(A)=F\right\}} \frac{\Delta_v(E)}{|\mathtt{V}_E(A)|}$$

$$= \sum_{\left\{E\in 2^{A(N)}\setminus\{\emptyset\}:\, i\in\mathtt{V}_E(A)\right\}} \frac{\Delta_v(E)}{|\mathtt{V}_E(A)|}.$$

□

6.3 The Banzhaf conjunctive authorization value

In Section 6.2 we used the Shapley value to provide an allocation rule for games with conjunctive authorization structure. In this section we use the Banzhaf value to give another allocation rule for these games.

Definition 6.11 *The Banzhaf conjunctive authorization value, denoted by $\mathfrak{B}^c$, assigns to each game with conjunctive authorization structure (v, A) the Banzhaf value of v^A,*

$$\mathfrak{B}^c(v, A) = \beta\left(v^A\right) \quad \textit{for all } v \in \mathcal{G}^N \textit{ and } A \in \mathcal{A}_c^N.$$

Notice that

$$\mathfrak{B}^c(v, A) = \mathfrak{B}\left(v, A\right) \quad \text{for all } v \in \mathcal{G}^N \text{ and } A \in \mathcal{A}_c^N,$$

where $\mathfrak{B}$ is the Banzhaf authorization value.

In the following theorem we characterize the Banzhaf conjunctive authorization value.

Theorem 6.12 *An allocation rule for games with conjunctive authorization structure is equal to the Banzhaf conjunctive authorization value if and only if satisfies the properties of additivity, irrelevant player, veto power over a necessary player, 2-efficiency and amalgamation.*

Proof. That the Banzhaf conjunctive authorization value satisfies the properties of additivity, irrelevant player, veto power over a necessary player and 2-efficiency follows from the fact that the Banzhaf conjunctive authorization value is the restriction of the Banzhaf authorization value to the set of games with conjunctive authorization structure and the fact that the Banzhaf authorization value satisfies such properties.

In the case of the property of amalgamation we can use the same reasoning, but before we must check that if A is a conjunctive authorization operator then A^{ij} is also conjunctive. So let $A \in \mathcal{A}_c^N$ and $i, j \in N$. Take $E, F \subseteq N^{ij}$. We want to prove that $A^{ij}(E \cap F) = A^{ij}(E) \cap A^{ij}(F)$. We distinguish several cases:

(a) $\widehat{ij} \notin E, \widehat{ij} \notin F$. It holds that

$$A^{ij}(E) \cap A^{ij}(F) = A(E) \cap A(F) = A(E \cap F) = A^{ij}(E \cap F).$$

(b) $\widehat{ij} \in E$, $\{i,j\} \not\subseteq A((E \setminus \{\widehat{ij}\}) \cup \{i,j\})$ and $\widehat{ij} \notin F$. It holds that

$$\begin{aligned} A^{ij}(E) \cap A^{ij}(F) &= \left[A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \setminus \{i,j\}\right] \cap A(F) \\ &= A\left((E \setminus \{\widehat{ij}\} \cup \{i,j\}\right) \cap A(F) \\ &= A\left(\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \cap F\right) \\ &= A(E \cap F) \\ &= A^{ij}(E \cap F). \end{aligned}$$

(c) $\widehat{ij} \in E$, $\{i,j\} \subseteq A((E \setminus \{\widehat{ij}\}) \cup \{i,j\})$ and $\widehat{ij} \notin F$. It holds that

$$\begin{aligned} A^{ij}(E) \cap A^{ij}(F) &= \left[\left(A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \setminus \{i,j\}\right) \cup \{\widehat{ij}\}\right] \cap A(F) \\ &= A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \cap A(F) \\ &= A\left(\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \cap F\right) \\ &= A(E \cap F) \\ &= A^{ij}(E \cap F). \end{aligned}$$

(d) $\widehat{ij} \notin E, \widehat{ij} \in F$ and $\{i,j\} \not\subseteq A((F \setminus \{\widehat{ij}\}) \cup \{i,j\})$. Analogous to case (b).

(e) $\widehat{ij} \in E$, $\{i,j\} \not\subseteq A((E \setminus \{\widehat{ij}\}) \cup \{i,j\})$, $\widehat{ij} \in F$ and $\{i,j\} \not\subseteq A((F \setminus \{\widehat{ij}\}) \cup \{i,j\})$. It holds that $A^{ij}(E) \cap A^{ij}(F)$ is equal to

$$\begin{aligned} & \left[A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \setminus \{i,j\}\right] \cap \left[A\left((F \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \setminus \{i,j\}\right] \\ = \; & \left[A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \cap A\left((F \setminus \{\widehat{ij}\}) \cup \{i,j\}\right)\right] \setminus \{i,j\} \\ = \; & A\left(\left((E \cap F) \setminus \{\widehat{ij}\}\right) \cup \{i,j\}\right) \setminus \{i,j\} \\ = \; & A^{ij}(E \cap F). \end{aligned}$$

(f) $\widehat{ij} \in E$, $\{i,j\} \subseteq A((E \setminus \{\widehat{ij}\}) \cup \{i,j\})$, $\widehat{ij} \in F$ and $\{i,j\} \nsubseteq A((F \setminus \{\widehat{ij}\}) \cup \{i,j\})$. It holds that $A^{ij}(E) \cap A^{ij}(F)$ is equal to

$$\begin{aligned}
&\left[\left(A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \setminus \{i,j\}\right) \cup \{\widehat{ij}\}\right] \cap \left[A\left((F \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \setminus \{i,j\}\right] \\
= &\left[A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \cap A\left((F \setminus \{\widehat{ij}\}) \cup \{i,j\}\right)\right] \setminus \{i,j\} \\
= &A\left(\left((E \cap F) \setminus \{\widehat{ij}\}\right) \cup \{i,j\}\right) \setminus \{i,j\} = A^{ij}(E \cap F).
\end{aligned}$$

(g) $\widehat{ij} \notin E$, $\widehat{ij} \in F$ and $\{i,j\} \subseteq A((F \setminus \{\widehat{ij}\}) \cup \{i,j\})$. Analogous to case (c).

(h) $\widehat{ij} \in E$, $\{i,j\} \nsubseteq A((E \setminus \{\widehat{ij}\}) \cup \{i,j\})$, $\widehat{ij} \in F$ and $\{i,j\} \subseteq A((F \setminus \{\widehat{ij}\}) \cup \{i,j\})$. Analogous to case (f).

(i) $\widehat{ij} \in E$, $\{i,j\} \subseteq A((E \setminus \{\widehat{ij}\}) \cup \{i,j\})$, $\widehat{ij} \in F$ and $\{i,j\} \subseteq A((F \setminus \{\widehat{ij}\}) \cup \{i,j\})$. Notice that in this case it holds that

$$\{i,j\} \subseteq A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \ \cap \ A\left((F \setminus \{\widehat{ij}\}) \cup \{i,j\}\right)$$

and

$$A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \ \cap \ A\left((F \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) = A\left(\left((E \cap F) \setminus \{\widehat{ij}\}\right) \cup \{i,j\}\right).$$

We have that $A^{ij}(E) \cap A^{ij}(F)$ is equal to

$$\begin{aligned}
&\left[\left(A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \setminus \{i,j\}\right) \cup \{\widehat{ij}\}\right] \cap \left[\left(A\left((F \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \setminus \{i,j\}\right) \cup \{\widehat{ij}\}\right] \\
= &\left[\left(A\left((E \setminus \{\widehat{ij}\}) \cup \{i,j\}\right) \cap A\left((F \setminus \{\widehat{ij}\}) \cup \{i,j\}\right)\right) \setminus \{i,j\}\right] \cup \{\widehat{ij}\} \\
= &\left[A\left(\left((E \cap F) \setminus \{\widehat{ij}\}\right) \cup \{i,j\}\right) \setminus \{i,j\}\right] \cup \{\widehat{ij}\} \\
= &A^{ij}(E \cap F).
\end{aligned}$$

We have seen that the Banzhaf conjunctive authorization value satisfies the five properties mentioned in the theorem. Now we see that such properties uniquely determine the Banzhaf conjunctive authorization value. Let Ψ be an allocation rule for games with conjunctive authorization structure satisfying the properties of additivity, irrelevant player, veto power over a necessary player, 2-efficiency and amalgamation. We must prove that $\Psi = \mathfrak{B}^c$.

Firstly we show that

$$\Psi(\alpha u_E, A) = \mathfrak{B}^c(\alpha u_E, A) \quad \text{for all } E \in 2^N \setminus \{\emptyset\}, A \in \mathcal{A}_c^N \text{ and } \alpha > 0. \tag{6.6}$$

We proceed by induction on the number of players.

1. Base case. If $n = 1$ the equality follows directly from the property of 2-efficiency.

2. Inductive step. Let $E \in 2^N \setminus \{\emptyset\}$, $A \in \mathcal{A}_c^N$ and $\alpha > 0$. Consider the set

$$F = \{i \in N : \text{ there exists } j \in E \text{ such that } i \text{ has veto power over } j \text{ in } (N, A)\}.$$

From the fact that $\mathfrak{B}^c$ and Ψ satisfy the property of veto power over a necessary player, we can derive that there exist $b, b' \in \mathbb{R}$ such that

$$\mathfrak{B}_i^c(\alpha u_E, A) = b \quad \text{for all } i \in F, \tag{6.7}$$

$$\Psi_i(\alpha u_E, A) = b' \quad \text{for all } i \in F. \tag{6.8}$$

Suppose that $i \in N \setminus F$. Notice that i is an irrelevant player in $(\alpha u_E, A)$. Indeed, if a player j depends partially on i in (N, A), then, from Proposition 6.2, i has veto power over j in (N, A). And, since $i \notin F$, we can derive that $j \notin E$, and, therefore, j is a null player in αu_E. Now, using that $\mathfrak{B}^c$ and Ψ satisfy the irrelevant player property, we conclude that

$$\mathfrak{B}_i^c(\alpha u_E, A) = 0 \quad \text{for all } i \in N \setminus F, \tag{6.9}$$

$$\Psi_i(\alpha u_E, A) = 0 \quad \text{for all } i \in N \setminus F. \tag{6.10}$$

Now we consider different cases.

(i) Case $|E| \geqslant 2$. Take j and k two different players in E. It is clear that j and k can be amalgamated in $(\alpha u_E, A)$. We can write

$$\begin{aligned} b+b &= \mathfrak{B}^c_j(\alpha u_E, A) + \mathfrak{B}^c_k(\alpha u_E, A) = \mathfrak{B}^c_{\widehat{jk}}\left((\alpha u_E)^{jk}, A^{jk}\right) \\ &= \mathfrak{B}^c_{\widehat{jk}}\left(\alpha u_{(E\setminus\{j,k\})\cup\left\{\widehat{jk}\right\}}, A^{jk}\right) = \Psi_{\widehat{jk}}\left(\alpha u_{(E\setminus\{j,k\})\cup\left\{\widehat{jk}\right\}}, A^{jk}\right) \\ &= \Psi_{\widehat{jk}}\left((\alpha u_E)^{jk}, A^{jk}\right) = \Psi_j(\alpha u_E, A) + \Psi_k(\alpha u_E, A) = b' + b', \end{aligned}$$

and we obtain $b = b'$. Using (6.7), (6.8), (6.9) and (6.10) we can conclude that $\Psi(\alpha u_E, A) = \mathfrak{B}^c(\alpha u_E, A)$.

(ii) Case $|E| = 1$ and $|F \setminus E| \geqslant 2$. Take j and k two different players in $F \setminus E$. It is easy to check that j and k can be amalgamated in $(\alpha u_E, A)$. It holds that

$$\begin{aligned} b+b &= \mathfrak{B}^c_j(\alpha u_E, A) + \mathfrak{B}^c_k(\alpha u_E, A) = \mathfrak{B}^c_{\widehat{jk}}\left((\alpha u_E)^{jk}, A^{jk}\right) \\ &= \mathfrak{B}^c_{\widehat{jk}}\left(\alpha u_E, A^{jk}\right) = \Psi_{\widehat{jk}}\left(\alpha u_E, A^{jk}\right) \\ &= \Psi_{\widehat{jk}}\left((\alpha u_E)^{jk}, A^{jk}\right) = \Psi_j(\alpha u_E, A) + \Psi_k(\alpha u_E, A) = b' + b', \end{aligned}$$

and we obtain $b = b'$. Using (6.7), (6.8), (6.9) and (6.10) we conclude $\Psi(\alpha u_E, A) = \mathfrak{B}^c(\alpha u_E, A)$.

(iii) Case $|E| = 1$ and $|F \setminus E| \leqslant 1$. On the one hand, using the 2-efficiency property, we can write

$$\sum_{i\in N} \mathfrak{B}^c_i(\alpha u_E, A) - \sum_{i\in N} \Psi_i(\alpha u_E, A) = \alpha u_E(A(N)) - \alpha u_E(A(N)) = 0,$$

and, on the other hand, from (6.7), (6.8), (6.9) and (6.10) we obtain

$$\sum_{i\in N} \mathfrak{B}^c_i(\alpha u_E, A) - \sum_{i\in N} \Psi_i(\alpha u_E, A) = (b - b')|F|.$$

From both expressions we obtain $b = b'$ and, therefore, $\Psi(\alpha u_E, A) = \mathfrak{B}^c(\alpha u_E, A)$.

So we have proved (6.6). Now, using additivity and reasoning as we did in the proof of Theorem 2.14 we conclude $\Psi = \mathfrak{B}^c$. □

6.4 Conjunctive fuzzy authorization structures

In a similar way as we introduced conjunctive authorization structures, we aim to introduce conjunctive fuzzy authorization structures.

Definition 6.13 *A fuzzy authorization operator $a \in \mathcal{FA}^N$ is said to be conjunctive if*

$$a(E \cap F) = a(E) \cap a(F) \quad \text{for every } E, F \subseteq N.$$

If a is a conjunctive fuzzy authorization operator on N, the pair (N, a) is said to be a conjunctive fuzzy authorization structure. The set of all conjunctive fuzzy authorization operators is denoted by $\mathcal{FA}_c^N$.

In the following result the conjunctive fuzzy authorization operators are characterized in terms of conjunctive authorization operators.

Proposition 6.14 *Let $a \in \mathcal{FA}^N$. The following statements are equivalent:*

1. *The fuzzy authorization operator a is conjunctive.*

2. *The authorization operator a^t is conjunctive for every $t \in (0, 1]$.*

Proof. Let $a \in \mathcal{FA}^N$.
$1 \Longrightarrow 2$. Suppose that a is conjunctive and let $t \in (0, 1]$. Let $E, F \subseteq N$. It holds that

$$a^t(E \cap F) = [a(E \cap F)]_t = [a(E) \cap a(F)]_t = [a(E)]_t \cap [a(F)]_t = a^t(E) \cap a^t(F).$$

$2 \Longrightarrow 1$. Suppose that a^t is conjunctive for every $t \in (0, 1]$ and let $E, F \subseteq N$. It suffices to show that $[a(E \cap F)]_t = [a(E) \cap a(F)]_t$ for every $t \in (0, 1]$. Let us see this.

$$[a(E \cap F)]_t = a^t(E \cap F) = a^t(E) \cap a^t(F) = [a(E)]_t \cap [a(F)]_t = [a(E) \cap a(F)]_t.$$

□

Conjunctive fuzzy authorization structures preserve convexity, as we see in the following proposition.

Definition 6.15 *A game with conjunctive fuzzy authorization structure on N is a pair (v, a) where $v \in \mathcal{G}^N$ and $a \in \mathcal{FA}_c^N$.*

6.5 The Shapley conjunctive fuzzy authorization value

An allocation rule for games with conjunctive fuzzy authorization structure assigns to every game with conjunctive fuzzy authorization structure a payoff vector.

Definition 6.16 *The Shapley conjunctive fuzzy authorization value, denoted by φ^c, assigns to each game with conjunctive fuzzy authorization structure (v, a) the Shapley value of the restricted game v^a,*

$$\varphi^c(v, a) = \phi(v^a) \quad \text{for all } v \in \mathcal{G}^N \text{ and } a \in \mathcal{FA}_c^N.$$

Notice that

$$\varphi^c(v, a) = \varphi(v, a) \quad \text{for all } v \in \mathcal{G}^N \text{ and } a \in \mathcal{FA}_c^N,$$

where φ is the Shapley fuzzy authorization value.

The Shapley conjunctive fuzzy authorization value has been studied in Gallardo, Jiménez, Jiménez-Losada and Lebrón [39]. We give a characterization of φ^c in the following theorem.

Theorem 6.17 *An allocation rule for games with conjunctive fuzzy authorization structure is equal to the Shapley conjunctive fuzzy authorization value if and only if it satisfies the properties of efficiency, additivity, irrelevant player, veto power over a necessary player and reduction.*

Proof. That the Shapley conjunctive fuzzy authorization value satisfies the properties of efficiency, additivity, irrelevant player and veto power over a necessary player follows from the fact that the Shapley conjunctive fuzzy authorization value is the restriction of the Shapley fuzzy authorization value to the set of games with conjunctive fuzzy authorization structure and the fact that the Shapley fuzzy authorization value satisfies such properties.

In the case of the property of reduction we can use the same reasoning, but before we must prove that if $a \in \mathcal{FA}_c^N$ and $t \in (0,1)$ then $a^{[0,t]}, a^{[t,1]} \in \mathcal{FA}_c^N$. So let $a \in \mathcal{FA}_c^N$ and $t \in (0,1)$. It is easy to check that

$$\begin{aligned} \left(a^{[0,t]}\right)^s &= a^{ts} \quad \text{for every } s \in (0,1], \\ \left(a^{[t,1]}\right)^s &= a^{t+s(1-t)} \quad \text{for every } s \in (0,1]. \end{aligned}$$

From these two equalities and Proposition 6.14 we can easily derive that $a^{[0,t]}$ and $a^{[t,1]}$ are conjunctive fuzzy authorization operators.

Uniqueness can be proved the same way as we did it for the Shapley fuzzy authorization value in Theorem 3.16. □

6.6 The Banzhaf conjunctive fuzzy authorization value

We aim to define an allocation rule for games with conjunctive fuzzy authorization structure that extends the Banzhaf conjunctive authorization value defined in Section 6.3.

Definition 6.18 *The Banzhaf conjunctive fuzzy authorization value, denoted by β^c, assigns to each game with conjunctive fuzzy authorization structure (v,a) the Banzhaf value of the restricted game v^a,*

$$\mathfrak{b}^c(v,a) = \beta\left(v^a\right) \quad \text{for all } v \in \mathcal{G}^N \text{ and } a \in \mathcal{FA}_c^N.$$

Notice that

$$\mathfrak{b}^c(v,a) = \mathfrak{b}\left(v,a\right) \quad \text{for all } v \in \mathcal{G}^N \text{ and } a \in \mathcal{FA}_c^N,$$

where $\mathfrak{b}$ is the Banzhaf fuzzy authorization value.

In the following theorem we give a characterization of the Banzhaf conjunctive fuzzy authorization value.

Theorem 6.19 *An allocation rule for games with conjunctive fuzzy authorization structure is equal to the Banzhaf conjunctive fuzzy authorization value if and only if it satisfies the properties of additivity, irrelevant player, veto power over a necessary player, 2-efficiency, amalgamation and reduction.*

Proof. That the Banzhaf conjunctive fuzzy authorization value satisfies the properties mentioned in the theorem follows from the fact that the Banzhaf conjunctive fuzzy authorization value is the restriction of the Banzhaf fuzzy authorization value to the set of games with conjunctive fuzzy authorization structure and the fact that the Banzhaf fuzzy authorization value satisfies such properties. In the case of the property of amalgamation, take into consideration that if a is a conjunctive fuzzy authorization operator then a^{ij} is also conjunctive (this is a consequence of (3.1), Proposition 6.14 and the analogous result for conjunctive authorization operators seen in the proof of Theorem 6.12).

Uniqueness can be proved the same way as we did it for the Banzhaf fuzzy authorization value in Theorem 3.25. □

Chapter 7

NTU games with authorization structure

In a similar way as, in previous chapters, we have made use of the Shapley value and the Banzhaf value for TU games to provide allocation rules for TU games with authorization structure, in this chapter we use two well known solutions for NTU games, the Shapley correspondence and the Harsanyi configuration correspondence, to introduce solutions for NTU games with authorization structure.

7.1 NTU games and authorization structures

Let $n \in \mathbb{N}$ and let N be a set of cardinality n. Recall that an NTU game on N is a correspondence V that assigns to each nonempty $E \subseteq N$ a nonempty subset $V(E) \subseteq \mathbb{R}^E$.

Definition 7.1 *An NTU game with authorization structure is a pair (V, A) where V is an NTU game on N and $A \in \tilde{\mathcal{A}}^N$.*

In a similar way as we did it with games with authorization structure, given an NTU game with authorization structure we define an NTU game that gathers the information from both the game and the structure.

Definition 7.2 *Let V be an NTU game on N and $A \in \tilde{\mathcal{A}}^N$. The restricted game of (V, A) is the*

NTU game V^A given by

$$V^A(E) = \begin{cases} V(E) & \text{if } A(E) = E, \\ (-\infty, 0]^E & \text{if } A(E) = \emptyset, \\ V(A(E)) \times (-\infty, 0]^{E \setminus A(E)} & \text{otherwise.} \end{cases}$$

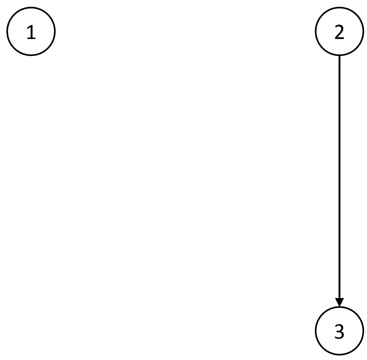
Example 7.3 Let $N = \{1, 2, 3\}$. Let V be an NTU game defined by

$$\begin{aligned} V(\{i\}) &= \left\{z_i \in \mathbb{R}^{\{i\}} : z_i \leqslant 1\right\} \text{ for every } i \in N, \\ V(\{i,j\}) &= \left\{z \in \mathbb{R}^{\{i,j\}} : z \leqslant (2,2)\right\} \text{ for every } i, j \in N \text{ with } i \neq j, \\ V(N) &= \left\{(z_1, z_2, z_3) \in \mathbb{R}^N : z_1 + z_2 + z_3 \leqslant 6\right\}. \end{aligned}$$

Let $A \in \tilde{\mathcal{A}}^N$ be given by

$$A(E) = \begin{cases} E \setminus \{3\} & \text{if } 2 \notin E, \\ E & \text{otherwise.} \end{cases}$$

Notice that (N, A) is a conjunctive authorization structure with one non-trivial veto relationship: 2 has veto power over 3. We can represent it with the following digraph



Consider the NTU game with authorization structure (V, A). Let us calculate V^A.

$$V^A(\{1\}) = \left\{z_1 \in \mathbb{R}^{\{1\}} : z_1 \leqslant 1\right\},$$
$$V^A(\{2\}) = \left\{z_2 \in \mathbb{R}^{\{2\}} : z_2 \leqslant 1\right\},$$

$$
\begin{aligned}
V^A(\{3\}) &= \left\{ z_3 \in \mathbb{R}^{\{3\}} : z_3 \leqslant 0 \right\}, \\
V^A(\{1,2\}) &= \left\{ (z_1, z_2) \in \mathbb{R}^{\{1,2\}} : (z_1, z_2) \leqslant (2,2) \right\}, \\
V^A(\{1,3\}) &= \left\{ (z_1, z_3) \in \mathbb{R}^{\{1,3\}} : z_1 \leqslant 1,\, z_3 \leqslant 0 \right\}, \\
V^A(\{2,3\}) &= \left\{ (z_2, z_3) \in \mathbb{R}^{\{2,3\}} : (z_2, z_3) \leqslant (2,2) \right\}, \\
V^A(\{1,2,3\}) &= \left\{ (z_1, z_2, z_3) \in \mathbb{R}^{\{1,2,3\}} : z_1 + z_2 + z_3 \leqslant 6 \right\}.
\end{aligned}
$$

7.2 The Shapley authorization NTU correspondence

In this section we only consider the NTU games V satisfying the following conditions:

(i) $V(E)$ is convex and comprehensive for all nonempty $E \subseteq N$.

(ii) $V(N)$ is a proper, closed and smooth subset of $\mathbb{R}^N$.

(iii) For every $x \in \partial(V(N))$ it holds $\{y \in \mathbb{R}^N : y \geqslant x\} \cap V(N) = \{x\}$.

(iv) There exists $x \in \mathbb{R}^N$ such that $V(E) \times \{0\}^{N \setminus E} \subseteq x + V(N)$ for every nonempty $E \subseteq N$.

(v) For every nonempty $E \subseteq N$ and $\lambda \in \mathbb{R}^E_{++}$, the set $\{\lambda \cdot x : x \in V(E)\}$ is closed.

We denote Γ^N the set of NTU games satisfying (i), (ii), (iii), (iv) and (v).

It is easy to prove the following proposition.

Proposition 7.4 *Let $V \in \Gamma^N$ and $A \in \tilde{\mathcal{A}}^N$. It holds that $V^A \in \Gamma^N$.*

We aim to give a correspondence that assigns to each NTU game with authorization structure (V, A) with $V \in \Gamma^N$ and $A \in \tilde{\mathcal{A}}^N$ a set of payoff vectors.

Definition 7.5 *The Shapley authorization NTU correspondence is given by*

$$\Theta(V,A) = SH\left(V^{A}\right) \qquad \text{for every } V \in \Gamma^N \text{ and } A \in \tilde{\mathcal{A}}^N,$$

where SH denotes the Shapley NTU correspondence.

Our goal is to provide a characterization of the Shapley authorization NTU correspondence. To that end, we consider the properties that we state below. In the statement of these properties, Ψ is a correspondence that assigns to each $V \in \Gamma^N$ and $A \in \tilde{\mathcal{A}}^N$ a subset $\Psi(V,A) \subseteq \mathbb{R}^N$.

- NON-EMPTINESS. For every $V \in \Gamma^N$ such that $\partial(V(N))$ is a hyperplane and $A \in \tilde{\mathcal{A}}^N$, it holds that
$$\Psi(V,A) \neq \emptyset.$$

- EFFICIENCY. For every $V \in \Gamma^N$ and $A \in \tilde{\mathcal{A}}^N$, it holds that
$$\Psi(V,A) \subseteq \partial(V(N)).$$

- CONDITIONAL ADDITIVITY. For every $A \in \tilde{\mathcal{A}}^N$ and $V, W \in \Gamma^N$ such that $V + W \in \Gamma^N$, it holds that
$$(\Psi(V,A) + \Psi(W,A)) \cap \partial\left((V+W)(N)\right) \subseteq \Psi(V+W,A).$$

- SCALE COVARIANCE. For every $V \in \Gamma^N$, $A \in \tilde{\mathcal{A}}^N$ and $\alpha \in \mathbb{R}^N_{++}$, it holds that
$$\Psi(\alpha * V, A) = \alpha * \Psi(V,A).$$

- INDEPENDENCE OF IRRELEVANT ALTERNATIVES. For every $A \in \tilde{\mathcal{A}}^N$ and $V, W \in \Gamma^N$ such that $V(N) \subseteq W(N)$ and $V(E) = W(E)$ for every $E \neq N$, it holds that
$$\Psi(W,A) \cap V(N) \subseteq \Psi(V,A).$$

- **CONSISTENCY WITH THE SHAPLEY AUTHORIZATION VALUE.** For all $v \in \mathcal{G}^N$ and $A \in \tilde{\mathcal{A}}^N$, it holds that
$$\Psi(V_v, A) = \{\Phi(v, A)\},$$
where Φ is the Shapley authorization value.

In the following theorem we show that these properties uniquely determine the Shapley authorization NTU correspondence.

Theorem 7.6 *A mapping $\Psi : \Gamma^N \times \tilde{\mathcal{A}}^N \to 2^{\mathbb{R}^N}$ is equal to the Shapley authorization NTU correspondence if and only if it satisfies the properties of non-emptiness, efficiency, conditional additivity, scale covariance, independence of irrelevant alternatives and consistency with the Shapley authorization value.*

Proof. Firstly, we prove that the Shapley authorization NTU correspondence satisfies such properties.

NON-EMPTINESS. Let $V \in \Gamma^N$ be such that $\partial(V(N))$ is a hyperplane. Using property (iii) of the games in Γ^N we can derive that there exist $\lambda \in \mathbb{R}^N_{++}$ and $\alpha \in \mathbb{R}$ such that

$$V(N) = \left\{y \in \mathbb{R}^N : \lambda \cdot y \leqslant \alpha\right\}.$$

From property (iv) of the games in Γ^N it follows that

$$\sup\left\{\lambda^E \cdot z : z \in V^A(E)\right\} < +\infty \quad \text{for all nonempty } E \subseteq N.$$

Let $w \in \mathcal{G}^N$ given by

$$w(E) = \sup\left\{\lambda^E \cdot z : z \in V^A(E)\right\} \quad \text{for all nonempty } E \subseteq N \text{ and } w(\emptyset) = 0,$$

and take $x \in \mathbb{R}^N$ defined by $x_i = \dfrac{\phi_i(w)}{\lambda_i}$. It is clear that $x \in \Theta(V, A)$. So we have proved that Θ satisfies non-emptiness.

EFFICIENCY. Let $V \in \Gamma^N$ and $A \in \tilde{\mathcal{A}}^N$. Using the efficiency property of the Shapley NTU correspondence we can write

$$\Theta(V, A) = SH\left(V^A\right) \subseteq \partial\left(V^A(N)\right) = \partial(V(N)).$$

CONDITIONAL ADDITIVITY. Let $A \in \tilde{\mathcal{A}}^N$. Let $V, W \in \Gamma^N$ be such that $V + W \in \Gamma^N$. It holds

$$(\Theta(V, A) + \Theta(W, A)) \cap \partial((V+W)(N)) = \left(SH\left(V^A\right) + SH\left(W^A\right)\right) \cap \partial\left((V+W)^A(N)\right)$$

which, using $(V+W)^A = V^A + W^A$, is equal to

$$\left(SH\left(V^A\right) + SH\left(W^A\right)\right) \cap \partial\left((V^A + W^A)(N)\right) \subseteq SH\left(V^A + W^A\right)$$

where we have used that the Shapley NTU correspondence satisfies conditional additivity. Finally it suffices to notice that

$$SH\left(V^A + W^A\right) = SH\left((V+W)^A\right) = \Theta(V + W, A).$$

SCALE COVARIANCE. Let $V \in \Gamma^N$, $A \in \tilde{\mathcal{A}}^N$ and $\alpha \in \mathbb{R}^N_{++}$. It holds

$$\Theta(\alpha * V, A) = SH\left((\alpha * V)^A\right) = SH\left(\alpha * V^A\right)$$

which, using that the Shapley NTU correspondence satisfies scale covariance, is equal to

$$\alpha * SH\left(V^A\right) = \alpha * \Theta(V, A).$$

INDEPENDENCE OF IRRELEVANT ALTERNATIVES. Let $A \in \tilde{\mathcal{A}}^N$. Let $V, W \in \Gamma^N$ be such that $V(N) \subseteq W(N)$ and $V(E) = W(E)$ for every $E \neq N$. It holds

$$\Theta(W, A) \cap V(N) = SH\left(W^A\right) \cap V^A(N) \subseteq SH\left(V^A\right) = \Theta(V, A)$$

where the inclusion follows from the fact that the Shapley NTU correspondence satisfies the property of independence of irrelevant alternatives.

Consistency with the Shapley authorization value. Let $v \in \mathcal{G}^N$ and $A \in \tilde{\mathcal{A}}^N$. Since $\Theta\left(V_v, A\right) = SH\left((V_v)^A\right)$, we calculate $SH\left((V_v)^A\right)$. A vector $x \in \mathbb{R}^N$ belongs to $SH\left((V_v)^A\right)$ if and only if there exists $\lambda \in \mathbb{R}^N_{++}$ such that

(1) $x \in (V_v)^A(N)$,

(2) $\sup\left\{\lambda^E \cdot z : z \in (V_v)^A(E)\right\} < +\infty$ for all nonempty $E \subseteq N$,

(3) $\lambda * x = \phi(w_\lambda)$ where w_λ is the TU game defined by

$$w_\lambda(E) = \sup\left\{\lambda^E \cdot z : z \in (V_v)^A(E)\right\} \quad \text{for all nonempty } E \subseteq N.$$

Taking into consideration that

$$(V_v)^A(N) = V_v(N) = \left\{z \in \mathbb{R}^N : \sum_{k\in N} z_k \leqslant v(N)\right\},$$

it is clear that

$$\sup\left\{\lambda \cdot z : z \in (V_v)^A(N)\right\} = +\infty \quad \text{for all } \lambda \in \mathbb{R}^N_{++} \setminus \{1_N\}.$$

Hence, the only element in $SH\left((V_v)^A\right)$ is the one that is obtained with $\lambda = 1_N$. So it holds $SH\left((V_v)^A\right) = \{\phi(w_{1_N})\}$. It is clear that $w_{1_N}(E) = v(A(E))$ for all nonempty $E \subseteq N$. Therefore, we have

$$SH\left((V_v)^A\right) = \{\phi(w_{1_N})\} = \{\phi(v^A)\} = \{\Phi(v, A)\}.$$

We have proved that Θ satisfies the properties in the theorem. Now we want to show that these properties uniquely determine the Shapley authorization NTU correspondence.

Let $\Psi : \Gamma^N \times \tilde{\mathcal{A}}^N \rightarrow 2^{\mathbb{R}^N}$ be a mapping satisfying the properties of non-emptiness, efficiency, conditional additivity, scale covariance, independence of irrelevant alternatives and consistency with the Shapley authorization value. Now, take $V \in \Gamma^N$ and $A \in \tilde{\mathcal{A}}^N$. We intend to show that $\Psi(V, A) = \Theta(V, A)$. We prove both inclusions.

Firstly, we prove that $\Psi(V, A) \subseteq \Theta(V, A)$. Let $x \in \Psi(V, A)$. Since Ψ satisfies efficiency, it holds that $x \in \partial(V(N))$. From $x \in \partial(V(N))$ and properties (i), (ii) and (iii) of the games in Γ^N it is easy to derive that there exists $\lambda \in \mathbb{R}^N_{++}$ such that

$$\lambda \cdot y \leqslant \lambda \cdot x \text{ for every } y \in V(N). \tag{7.1}$$

Using the property of scale covariance, it follows that

$$\lambda * x \in \Psi(\lambda * V, A). \tag{7.2}$$

Let V_0 be the NTU game corresponding to the TU game that is identically zero. From the fact that Ψ satisfies consistency with the Shapley authorization value it follows that

$$\Psi(V_0, A) = \{0\}. \tag{7.3}$$

From (7.2) and (7.3) we obtain

$$\lambda * x \in \Psi(\lambda * V, A) + \Psi(V_0, A). \tag{7.4}$$

Now, take $y \in V(N)$ and $z \in V_0(N)$. Taking into consideration the definition of V_0 and (7.1) we have

$$\sum_{k \in N} (\lambda * y + z)_k = \lambda \cdot y + \sum_{k \in N} z_k \leqslant \lambda \cdot y \leqslant \lambda \cdot x = \sum_{k \in N} (\lambda * x)_k$$

from where we derive that $\lambda * x \in \partial\left((\lambda * V + V_0)(N)\right)$. Therefore, using (7.4), we have obtained

$$\lambda * x \in \left(\Psi(\lambda * V, A) + \Psi(V_0, A)\right) \cap \partial\left((\lambda * V + V_0)(N)\right),$$

and hence, since Ψ satisfies conditional additivity, we conclude that

$$\lambda * x \in \Psi\left(\lambda * V + V_0, A\right).$$

Using properties (iv) and (v) of the games in Γ^N, it is easy to prove that $\lambda * V + V_0 = V_v$ where v is the TU game given by

$$v(E) = \sup\left\{\lambda^E \cdot t : \, t \in V(E)\right\} \text{ for all nonempty } E \subseteq N.$$

Since Ψ satisfies consistency with the Shapley authorization value we can write

$$\lambda * x \in \Psi\left(\lambda * V + V_0, A\right) = \Psi\left(V_v, A\right) = \{\Phi(v, A)\}$$

hence, $\lambda * x = \Phi(v, A) = \phi(v^A)$. So we have

(1) $x \in V(N) = V^A(N)$,

(2) $\sup\left\{\lambda^E \cdot t : \, t \in V^A(E)\right\} < +\infty$ for all nonempty $E \subseteq N$,

(3) $\lambda * x = \phi(v^A)$ where, for every nonempty $E \subseteq N$,

$$v^A(E) = \begin{cases} \sup\left\{\lambda^{A(E)} \cdot t : \, t \in V(A(E))\right\} & \text{if } A(E) \neq \emptyset, \\ 0 & \text{if } A(E) = \emptyset, \end{cases}$$

or equivalently

$$v^A(E) = \sup\left\{\lambda^E \cdot t : \, t \in V^A(E)\right\}.$$

Hence, (1), (2) and (3) mean that $x \in SH(V^A) = \Theta(V, A)$.

Now, we prove that $\Theta(V, A) \subseteq \Psi(V, A)$. Let $x \in \Theta(V, A)$. By definition there exists $\lambda \in \mathbb{R}^N_{++}$ such that

(1) $x \in V^A(N)$,

(2) $\sup\left\{\lambda^E \cdot z : \, z \in V^A(E)\right\} < +\infty$ for all nonempty $E \subseteq N$,

(3) $\lambda * x = \phi(w_\lambda)$ where w_λ is the TU game defined by

$$w_\lambda(E) = \sup\left\{\lambda^E \cdot z : \, z \in V^A(E)\right\} \text{ for all nonempty } E \subseteq N.$$

Consider the NTU game $W \in \Gamma^N$ defined as

$$W(E) = \begin{cases} \left\{y \in \mathbb{R}^N : \, \lambda \cdot y \leqslant \lambda \cdot x\right\} & \text{if } E = N, \\ V(E) & \text{if } E \neq N. \end{cases}$$

It is easy to check that $\Theta(W, A) = \{x\}$. We know that $\Psi(W, A) \subseteq \Theta(W, A)$. Besides, from non-emptiness, it must be that $\Psi(W, A) \neq \emptyset$. Therefore, $\Psi(W, A) = \{x\}$. Finally, using the property of independence of irrelevant alternatives, we conclude that $x \in \Psi(V, A)$. □

In practice, if we have $V \in \Gamma^N$ and $A \in \tilde{\mathcal{A}}^N$, we do not need to obtain V^A to calculate $\Theta(V, A)$. Let us see this. Let $\lambda \in \mathbb{R}^N_{++}$ be such that

$$\sup\{\lambda \cdot z : z \in V(N)\} < +\infty.$$

Consider $w_\lambda, v_\lambda \in \mathcal{G}^N$ defined by

$$\begin{aligned} w_\lambda(E) &= \sup\left\{\lambda^E \cdot z : z \in V^A(E)\right\}, \\ v_\lambda(E) &= \sup\left\{\lambda^E \cdot z : z \in V(E)\right\}, \end{aligned}$$

for every nonempty $E \subseteq N$. It is clear that $w_\lambda = v^A_\lambda$. We can use this to give a definition of $\Theta(V, A)$ that does not involve the restricted game V^A, as we see in the following remark.

Remark 7.7 *Let $V \in \Gamma^N$ and $A \in \tilde{\mathcal{A}}^N$. A vector $x \in \mathbb{R}^N$ belongs to $\Theta(V, A)$ if and only if there exists $\lambda \in \mathbb{R}^N_{++}$ such that*

(1) $x \in V(N)$,

(2) $\sup\{\lambda \cdot z : z \in V(N)\} < +\infty$,

(3) $\lambda * x = \Phi(v_\lambda, A)$ *where v_λ is the TU game defined by*

$$v_\lambda(E) = \sup\left\{\lambda^E \cdot z : z \in V(E)\right\} \text{ for all nonempty } E \subseteq N.$$

Example 7.8 Let (V, A) be the NTU game with authorization structure considered in Example 7.3. Let us calculate $\Theta(V, A)$ by means of Remark 7.7, without using the expression of the restricted game V^A. Since $\partial(V(N))$ is a hyperplane, it is plain to see that $\Theta(V, A)$ is a singleton $\{x\}$. It is clear that x is associated to the comparison vector $\lambda = (1, 1, 1)$. We calculate x by using Remark 7.7. Firstly, we proceed to calculate the characteristic function v_λ

$$\begin{gathered} v_\lambda(\{1\}) = v_\lambda(\{2\}) = v_\lambda(\{3\}) = 1, \\ v_\lambda(\{1,2\}) = v_\lambda(\{1,3\}) = v_\lambda(\{2,3\}) = 4, \\ v_\lambda(\{1,2,3\}) = 6. \end{gathered}$$

Now we calculate v_λ^A

$$v_\lambda^A(\{1\}) = 1, \quad v_\lambda^A(\{2\}) = 1, \quad v_\lambda^A(\{3\}) = 0,$$
$$v_\lambda^A(\{1,2\}) = 4, \quad v_\lambda^A(\{1,3\}) = 1, \quad v_\lambda^A(\{2,3\}) = 4,$$
$$v_\lambda^A(\{1,2,3\}) = 6.$$

It holds that

$$\Phi(v_\lambda, A) = \phi\left(v_\lambda^A\right) = \frac{1}{6}(10, 19, 7).$$

and from Remark 7.7 we conclude that

$$x = \frac{1}{6}(10, 19, 7).$$

So we have obtained that

$$\Theta(V, A) = \left\{\frac{1}{6}(10, 19, 7)\right\}.$$

7.3 NTU games and fuzzy authorization structures

In a similar way as we did it with TU games, we introduce NTU games with fuzzy authorization structure.

Definition 7.9 *An NTU game with fuzzy authorization structure is a pair (V, a) where V is an NTU game on N and $a \in \tilde{\mathcal{FA}}^N$.*

Definition 7.10 *Let $V \in \Gamma^N$, $a \in \widetilde{\mathcal{FA}}^N$ and $\{t_l : l = 0, \ldots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r = 1$. The restricted game of (V, a) is the NTU game V^a given by*

$$V^a(E) = \sum_{l=1}^{r} (t_l - t_{l-1}) V^{a^{t_l}}(E),$$

where for every $t \in (0, 1]$, $a^t \in \tilde{\mathcal{A}}^N$ is defined as $a^t(E) = \{k \in E : a_k(E) \geqslant t\}$ for all $E \subseteq N$.

Example 7.11 Let V be the NTU game given in Example 7.3. Let a be the fuzzy authorization operator on $\{1,2,3\}$ defined in the following table.

E	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$a(E)$	$(1,0,0)$	$(0,1,0)$	$(0,0,0.4)$	$(1,1,0)$	$(1,0,0.6)$	$(0,1,0.8)$	$(1,1,1)$

Consider the NTU game with fuzzy authorization structure (V,a). Let us calculate the restricted game V^a,

$$\begin{aligned}
V^a(\{1\}) &= \left\{z_1 \in \mathbb{R}^{\{1\}} : z_1 \leqslant 1\right\}, \\
V^a(\{2\}) &= \left\{z_2 \in \mathbb{R}^{\{2\}} : z_2 \leqslant 1\right\}, \\
V^a(\{3\}) &= \left\{z_3 \in \mathbb{R}^{\{3\}} : z_3 \leqslant 0.4\right\}, \\
V^a(\{1,2\}) &= \left\{(z_1,z_2) \in \mathbb{R}^{\{1,2\}} : (z_1,z_2) \leqslant (2,2)\right\}, \\
V^a(\{1,3\}) &= \left\{(z_1,z_3) \in \mathbb{R}^{\{1,3\}} : z_1 \leqslant 1.6,\ z_3 \leqslant 1.2\right\}, \\
V^a(\{2,3\}) &= \left\{(z_2,z_3) \in \mathbb{R}^{\{2,3\}} : z_2 \leqslant 1.8,\ z_3 \leqslant 1.6\right\}, \\
V^a(\{1,2,3\}) &= \left\{(z_1,z_2,z_3) \in \mathbb{R}^{\{1,2,3\}} : z_1+z_2+z_3 \leqslant 6\right\}.
\end{aligned}$$

7.4 The Shapley fuzzy authorization NTU correspondence

We aim to give a correspondence that assigns to each NTU game with fuzzy authorization structure (V,a) with $V \in \Gamma^N$ and $a \in \tilde{\mathcal{FA}}^N$ a set of payoff vectors. We need a previous result.

Proposition 7.12 *Let $V \in \Gamma^N$ and $a \in \widetilde{\mathcal{FA}}^N$. Then, it holds that*

(a) $V^a(N) = V(N)$,

(b) $V^a \in \Gamma^N$.

Proof. Let $V \in \Gamma^N$, $a \in \widetilde{\mathcal{FA}}^N$ and $\{t_l : l = 0, \ldots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r = 1$.

(a) It holds

$$V^a(N) = \sum_{l=1}^{r} (t_l - t_{l-1}) V^{a^{t_l}}(N) = \sum_{l=1}^{r} (t_l - t_{l-1}) V(N)$$

which, taking into consideration that $V(N)$ is convex, is equal to $V(N)$.

(b) We must prove that V^a satisfies the five properties that characterize the NTU games in Γ^N. From $V \in \Gamma^N$ and $V^a(N) = V(N)$ it follows that V^a satisfies (ii) and (iii).

Let E be a nonempty subset of N. From the fact that $V^{a^{t_l}}(E)$ is convex and comprehensive for all $l = 1, \ldots, r$ it can be derived that $V^a(E)$ is convex and comprehensive. Hence V^a satisfies (i).

Since $V \in \Gamma^N$ there exists $x \in \mathbb{R}^N$ such that $V(F) \times \{0\}^{N \setminus F} \subseteq x + V(N)$ for every nonempty $F \subseteq N$. Since $V(N)$ is comprehensive, it is clear that we can assume that $0 \in x + V(N)$. In these conditions, it is easy to check, making use of the comprehensiveness of $V(N)$, that $V^{a^{t_l}}(E) \times \{0\}^{N \setminus E} \subseteq x + V(N)$ for all $l = 1, \ldots, r$. It holds that

$$\begin{aligned} V^a(E) \times \{0\}^{N \setminus E} &= \left(\sum_{l=1}^{r} (t_l - t_{l-1}) V^{a^{t_l}}(E) \right) \times \{0\}^{N \setminus E} \\ &= \sum_{l=1}^{r} (t_l - t_{l-1}) \left(V^{a^{t_l}}(E) \times \{0\}^{N \setminus E} \right) \\ &\subseteq \sum_{l=1}^{r} (t_l - t_{l-1}) (x + V(N)) \end{aligned}$$

which, using the convexity of $V(N)$, is equal to $x + V(N)$. Therefore, V^a satisfies (iv).

Let $\lambda \in \mathbb{R}^E_{++}$. It holds that

$$\lambda \cdot V^a(E) = \lambda \cdot \left(\sum_{l=1}^{r} (t_l - t_{l-1}) V^{a^{t_l}}(E) \right) = \sum_{l=1}^{r} (t_l - t_{l-1}) \left(\lambda \cdot V^{a^{t_l}}(E) \right)$$

which is closed, since it is a sum of closed intervals in the real line. So V^a satisfies (v).

□

Definition 7.13 *The Shapley fuzzy authorization NTU correspondence is given by*

$$\theta(V,a) = SH\left(V^a\right) \quad \text{for every } V \in \Gamma^N \text{ and } a \in \widetilde{\mathcal{FA}}^N,$$

where SH denotes the Shapley NTU correspondence.

We aim to give a characterization of the Shapley fuzzy authorization NTU correspondence. To that end, we consider the properties that we state below. In the statement of these properties, ψ is a correspondence that assigns to each $V \in \Gamma^N$ and $a \in \widetilde{\mathcal{FA}}^N$ a subset $\psi(V,a) \subseteq \mathbb{R}^N$.

- **NON-EMPTINESS**. For every $V \in \Gamma^N$ such that $\partial(V(N))$ is a hyperplane and $a \in \widetilde{\mathcal{FA}}^N$, it holds that
$$\psi(V,a) \neq \emptyset.$$

- **EFFICIENCY**. For all $V \in \Gamma^N$ and $a \in \widetilde{\mathcal{FA}}^N$, it holds that
$$\psi(V,a) \subseteq \partial(V(N)).$$

- **CONDITIONAL ADDITIVITY**. For every $a \in \widetilde{\mathcal{FA}}^N$ and $V, W \in \Gamma^N$ such that $V + W \in \Gamma^N$, it holds that
$$(\psi(V,a) + \psi(W,a)) \cap \partial\left((V+W)(N)\right) \subseteq \psi(V+W,a).$$

- **SCALE COVARIANCE**. For all $V \in \Gamma^N$, $a \in \widetilde{\mathcal{FA}}^N$ and $\alpha \in \mathbb{R}^N_{++}$, it holds that
$$\psi(\alpha * V, a) = \alpha * \psi(V,a).$$

- **INDEPENDENCE OF IRRELEVANT ALTERNATIVES**. For all $a \in \widetilde{\mathcal{FA}}^N$ and $V, W \in \Gamma^N$ such that $V(N) \subseteq W(N)$ and $V(E) = W(E)$ for every $E \neq N$, it holds that
$$\psi(W,a) \cap V(N) \subseteq \psi(V,a).$$

- CONSISTENCY WITH THE SHAPLEY FUZZY AUTHORIZATION VALUE. For every $v \in \mathcal{G}^N$ and $a \in \widetilde{\mathcal{FA}}^N$, it holds that
$$\psi(V_v, a) = \{\varphi(v, a)\},$$
where φ is the Shapley fuzzy authorization value.

Next we show that these properties characterize the Shapley fuzzy authorization NTU correspondence.

Theorem 7.14 *A mapping $\psi : \Gamma^N \times \widetilde{\mathcal{FA}}^N \to 2^{\mathbb{R}^N}$ is equal to the Shapley fuzzy authorization NTU correspondence if and only if it satisfies the properties of non-emptiness, efficiency, conditional additivity, scale covariance, independence of irrelevant alternatives and consistency with the Shapley fuzzy authorization value.*

Proof. That the Shapley fuzzy authorization NTU correspondence satisfies non-emptiness, efficiency, conditional additivity, scale covariance and independence of irrelevant alternatives can be proved in a similar way as was proved for the Shapley authorization NTU correspondence.

Let us see that θ satisfies consistency with the Shapley fuzzy authorization value. Let $v \in \mathcal{G}^N$, $a \in \widetilde{\mathcal{FA}}^N$ and $\{t_l : l = 0, \ldots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r = 1$. Since $\theta(V_v, a) = SH((V_v)^a)$, we calculate $SH((V_v)^a)$. A vector $x \in \mathbb{R}^N$ belongs to $SH((V_v)^a)$ if and only if there exists $\lambda \in \mathbb{R}^N_{++}$ such that

(1) $x \in (V_v)^a(N)$,

(2) $\sup\left\{\lambda^E \cdot z : z \in (V_v)^a(E)\right\} < +\infty$ for all nonempty $E \subseteq N$,

(3) $\lambda * x = \phi(w_\lambda)$ where w_λ is the TU game defined by

$$w_\lambda(E) = \sup\left\{\lambda^E \cdot z : z \in (V_v)^a(E)\right\} \text{ for all nonempty } E \subseteq N.$$

Taking into consideration that

$$(V_v)^a(N) = V_v(N) = \left\{z \in \mathbb{R}^N : \sum_{k \in N} z_k \leqslant v(N)\right\}.$$

It is clear that

$$\sup\{\lambda \cdot z : z \in (V_v)^a(N)\} = +\infty \quad \text{for all } \lambda \in \mathbb{R}^N_{++} \setminus \{1_N\}.$$

Hence, the only element in $SH((V_v)^a)$ is the one that is obtained with $\lambda = 1_N$. So we have that $\theta(V_v, a) = \{\phi(w_{1_N})\}$. If we get to prove that $w_{1_N} = v^a$ we will have finished, since $\phi(v^a) = \varphi(v, a)$. To that end, take E a nonempty subset of N. If $a(E) = 0$ it is clear that $w_{1_N}(E) = v^a(E) = 0$. If $a(E) \neq 0$, denote $m = \max\left\{l : a^{t_l}(E) \neq \emptyset\right\}$. It holds that

$$\begin{aligned}
w_{1_N}(E) &= \sup\{z(E) : z \in (V_v)^a(E)\} \\
&= \sup\left\{\sum_{l=1}^{r}(t_l - t_{l-1})\, z_l(E) : z_l \in (V_v)^{a^{t_l}}(E) \text{ for all } l = 1, \ldots, m\right\} \\
&= \sum_{l=1}^{r}(t_l - t_{l-1})\, \sup\left\{\, z(E) : z \in (V_v)^{a^{t_l}}(E)\right\} \\
&= \sum_{l=1}^{m}(t_l - t_{l-1})\, \sup\left\{\, y(a^{t_l}(E)) : y \in V_v(a^{t_l}(E))\right\} \\
&= \sum_{l=1}^{m}(t_l - t_{l-1})\, v\left(a^{t_l}(E)\right) \\
&= v^a(E).
\end{aligned}$$

We have proved that θ satisfies the properties in the theorem. Now we aim to see that these properties uniquely determine the Shapley fuzzy authorization NTU correspondence.

Let $\psi : \Gamma^N \times \widetilde{\mathcal{FA}}^N \to 2^{\mathbb{R}^N}$ be a mapping satisfying the properties of non-emptiness, efficiency, conditional additivity, scale covariance, independence of irrelevant alternatives and consistency with the Shapley fuzzy authorization value. Now, take $V \in \Gamma^N$ and $a \in \widetilde{\mathcal{FA}}^N$. We show that $\psi(V, a) = \theta(V, a)$. We prove both inclusions.

Firstly, we prove that $\psi(V, a) \subseteq \theta(V, a)$. Let $x \in \psi(V, a)$. Proceeding in the same way as we did in the case of the Shapley authorization NTU correspondence, we can obtain $\lambda \in \mathbb{R}^N_{++}$ such that

$$\lambda * x \in \psi(\lambda * V + V_0, a)$$

where V_0 is the NTU game corresponding to the TU game that is identically zero.

Remember that $\lambda * V + V_0 = V_v$ where v is the TU game given by

$$v(E) = \sup\left\{\lambda^E \cdot z : \, z \in V(E)\right\} \quad \text{for all nonempty } E \subseteq N.$$

Since ψ satisfies consistency with the Shapley fuzzy authorization value we can write

$$\lambda * x \in \psi\left(\lambda * V + V_0, a\right) = \psi\left(V_v, a\right) = \{\varphi(v,a)\}$$

hence, $\lambda * x = \varphi(v,a) = \phi(v^a)$. So we have

(1) $x \in V(N) = V^a(N)$,

(2) $\sup\left\{\lambda^E \cdot z : \, z \in V^a(E)\right\} < +\infty$ for all nonempty $E \subseteq N$,

(3) $\lambda * x = \phi(v^a)$.

If we prove that

$$v^a(E) = \sup\left\{\lambda^E \cdot z : \, z \in V^a(E)\right\} \qquad \text{for all nonempty } E \subseteq N, \tag{7.5}$$

we will have finished since in that case (1), (2) and (3) will mean that, by definition, $x \in SH(V^a) = \theta(V,a)$. In order to prove (7.5) take E a nonempty subset of N. If $a(E) = 0$ the equality is clear. So we assume $a(E) \neq 0$. Let $\{t_l : \, l = 0, \ldots, r\} = \{a_k(F) : \, F \subseteq N, \, k \in N\}$ with $0 = t_0 < \ldots < t_r = 1$. Name $m = \max\left\{l : \, a^{t_l}(E) \neq \emptyset\right\}$. It holds that

$$\begin{aligned} v^a(E) &= \sum_{l=1}^{m}(t_l - t_{l-1})\, v\left(a^{t_l}(E)\right) \\ &= \sum_{l=1}^{m}(t_l - t_{l-1})\, \sup\left\{\lambda^{a^{t_l}(E)} \cdot y : \, y \in V\left(a^{t_l}(E)\right)\right\} \\ &= \sum_{l=1}^{r}(t_l - t_{l-1})\, \sup\left\{\lambda^{E} \cdot z : \, z \in V^{a^{t_l}}(E)\right\} \\ &= \sup\left\{\lambda^E \cdot \left(\sum_{l=1}^{r}(t_l - t_{l-1})\, z_l\right) : \, z_l \in V^{a^{t_l}}(E) \;\text{ for all } l = 1, \ldots, m\right\} \\ &= \sup\left\{\lambda^E \cdot z : \, z \in V^a(E)\right\}. \end{aligned}$$

It remains to prove that $\theta(V,a) \subseteq \psi(V,a)$. The same reasoning that we followed in the case of the Shapley authorization NTU correspondence can be used. $\square$

In a similar way as in the crisp case, in practice, if we have $V \in \Gamma^N$ and $a \in \widetilde{\mathcal{FA}}^N$, we do not need to obtain V^a to calculate $\theta(V,a)$. Let $\lambda \in \mathbb{R}^N_{++}$ be such that $\sup\{\lambda \cdot z : z \in V(N)\} < +\infty$. Consider $w_\lambda, v_\lambda \in \mathcal{G}^N$ defined by

$$\begin{aligned} w_\lambda(E) &= \sup\left\{\lambda^E \cdot z : z \in V^a(E)\right\}, \\ v_\lambda(E) &= \sup\left\{\lambda^E \cdot z : z \in V(E)\right\}, \end{aligned}$$

for every nonempty $E \subseteq N$. Let us see that

$$w_\lambda = v_\lambda^a. \tag{7.6}$$

Let $\{t_l : l = 0, \ldots, r\} = \{a_k(F) : F \subseteq N, k \in N\}$ with $0 = t_0 < \ldots < t_r = 1$ and let $E \in 2^N \backslash \{\emptyset\}$. We must prove that $w_\lambda(E) = v_\lambda^a(E)$. It is easy to check that

$$\sup\left\{\lambda^E \cdot z : z \in V^{a^{t_l}}(E)\right\} = v_\lambda^{a^{t_l}}(E) \quad \text{for every } l = 1, \ldots, r. \tag{7.7}$$

It holds that

$$\begin{aligned} w_\lambda(E) &= \sup\left\{\lambda^E \cdot z : z \in V^a(E)\right\} \\ &= \sup\left\{\sum_{l=1}^{r}(t_l - t_{l-1})\left(\lambda^E \cdot z_l\right) : z_l \in V^{a^{t_l}}(E) \quad \text{for all } l = 1, \ldots, r\right\} \\ &= \sum_{l=1}^{r}(t_l - t_{l-1}) \sup\left\{\lambda^E \cdot z : z \in V^{a^{t_l}}(E)\right\} \end{aligned}$$

which, from (7.7), is equal to

$$\sum_{l=1}^{r}(t_l - t_{l-1})\, v_\lambda^{a^{t_l}}(E) = v_\lambda^a(E).$$

We can use (7.6) to give a definition of $\theta(V,a)$ that does not involve the restricted game V^a, as we see in the following remark.

Remark 7.15 *Let $V \in \Gamma^N$ and $a \in \widetilde{\mathcal{FA}}^N$. A vector $x \in \mathbb{R}^N$ belongs to $\theta(V,a)$ if and only if there exists $\lambda \in \mathbb{R}^N_{++}$ such that*

(1) $x \in V(N)$,

(2) $\sup\{\lambda \cdot z : z \in V(N)\} < +\infty$,

(3) $\lambda * x = \varphi(v_\lambda, a)$ *where v_λ is the TU game defined by*

$$v_\lambda(E) = \sup\left\{\lambda^E \cdot z : z \in V(E)\right\} \quad \text{for all nonempty } E \subseteq N.$$

Example 7.16 Let (V,a) be the NTU game with fuzzy authorization structure considered in Example 7.11. Let us calculate $\theta(V,a)$ without using the expression of the restricted game. We use Remark 7.15. Since $\partial(V(N))$ is a hyperplane, it is plain to see that $\theta(V,a)$ is a singleton $\{x\}$. It is clear that x is associated to the comparison vector $\lambda = (1,1,1)$. We calculated v_λ in Example 7.3. We proceed to calculate v^a_λ,

$$\begin{aligned}
v^a_\lambda(\{1\}) &= v_\lambda(\{1\}) = 1\\
v^a_\lambda(\{2\}) &= v_\lambda(\{2\}) = 1\\
v^a_\lambda(\{3\}) &= 0.4\, v_\lambda(\{3\}) = 0.4\\
v^a_\lambda(\{1,2\}) &= v_\lambda(\{1,2\}) = 4\\
v^a_\lambda(\{1,3\}) &= 0.6\, v_\lambda(\{1,3\}) + 0.4\, v_\lambda(\{1\}) = 2.8\\
v^a_\lambda(\{2,3\}) &= 0.8\, v_\lambda(\{2,3\}) + 0.2\, v_\lambda(\{2\}) = 3.4\\
v^a_\lambda(\{1,2,3\}) &= v_\lambda(\{1,2,3\}) = 6
\end{aligned}$$

It holds that

$$\varphi(v_\lambda, a) = \phi(v^a_\lambda) = (2.1, 2.4, 1.5).$$

From Remark 7.15 we conclude that

$$x = (2.1, 2.4, 1.5).$$

So we have obtained that

$$\theta(V,a) = \{(2.1, 2.4, 1.5)\}.$$

7.5 The Harsanyi configuration correspondence for NTU games with interior operator structure

We aim to define and characterize a Harsanyi solution for NTU games with interior operator structure. In this section we only consider the NTU games V satisfying the following conditions:

(i) $V(E)$ is closed, convex and comprehensive for all nonempty $E \subseteq N$.

(ii) $V(N)$ is smooth.

(iii) For every $x \in \partial(V(N))$ it holds $\{y \in \mathbb{R}^N : y \geqslant x\} \cap V(N) = \{x\}$.

We denote Ω^N the set of NTU games satisfying (i), (ii) and (iii).

It is easy to prove the following proposition.

Proposition 7.17 *Let $V \in \Omega^N$ and $A \in \tilde{\mathcal{A}}^N$. It holds that $V^A \in \Omega^N$.*

A configuration correspondence Ψ for NTU games with interior operator structure assigns to each $V \in \Omega^N$ and A an interior operator on N a set $\Psi(V, A) \subset \prod_{E \in 2^N \setminus \{\emptyset\}} \mathbb{R}^E$. We aim to provide a configuration correspondence for NTU games with interior operator structure with good properties.

Definition 7.18 *The Harsanyi configuration correspondence for NTU games with interior operator structure, denoted by $\mathcal{H}$, is defined by*

$$\mathcal{H}(V, A) = H\left(V^A\right) \qquad \text{for every } V \in \Omega^N \text{ and } A \text{ interior operator on } N,$$

where H denotes the Harsanyi configuration correspondence for NTU games.

Bearing in mind the definition of the Harsanyi configuration correspondence for NTU games, we can give another version of the definition of the Harsanyi configuration correspondence for NTU games with interior operator structure.

Remark 7.19 *Let $V \in \Omega^N$ and A an interior operator on N. A payoff configuration $(x_E)_{E \in 2^N \setminus \{\emptyset\}}$ belongs to $\mathcal{H}(V, A)$ if there exists $\lambda \in \mathbb{R}^N_{++}$ such that*

(1) $x_E \in \partial\left(V^A(E)\right)$ for all $E \in 2^N \setminus \{\emptyset\}$,

(2) $\lambda \cdot x_N = \max\{\lambda \cdot y : \, y \in V(N)\}$,

(3) if w is the TU game given by $w(F) = \lambda^F \cdot x_F$ for every $F \in 2^N \setminus \{\emptyset\}$, then $\lambda^E * x_E = \phi(w_{|E})$ for all $E \in 2^N \setminus \{\emptyset\}$.

We introduce a notation that will be useful in the rest of the chapter. Given $A \in \mathcal{A}^N$ we denote

$$aut(A) = \left\{E \in 2^N \setminus \{\emptyset\} : \, A(E) = E\right\}.$$

We aim to give a characterization of the Harsanyi configuration correspondence for NTU games with interior operator structure. To that end, we consider the properties stated below. In the statement of these properties, Ψ is a configuration correspondence for NTU games with interior operator structure.

- **EFFICIENCY**. For every $V \in \Omega^N$, A an interior operator on N and $(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V, A)$, it holds that
$$x_E \in \partial\left(V(E)\right) \quad \text{for all } E \in aut(A).$$

- **CONDITIONAL ADDITIVITY**. For every $V, W \in \Omega^N$ such that $V + W \in \Omega^N$, A an interior operator on N and $(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V, A) + \Psi(W, A)$ such that $x_E \in \partial((V + W)(E))$ for all $E \in aut(A)$ it holds that
$$(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V + W, A).$$

- **SCALE COVARIANCE**. For every $V \in \Omega^N$, A an interior operator on N and $\alpha \in \mathbb{R}^N_{++}$, it holds that
$$\Psi(\alpha * V, A) = \left\{(\alpha^E * x_E)_{E \in 2^N \setminus \{\emptyset\}} : (x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V, A)\right\}.$$

- **INDEPENDENCE OF IRRELEVANT ALTERNATIVES**. Given $V, W \in \Omega^N$, A an interior operator on N such that $V(E) \subseteq W(E)$ for all $E \in aut(A)$ and $(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(W, A)$ such that

$x_E \in V(E)$ for all $E \in aut(A)$ it holds that

$$(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V, A).$$

- **CONSISTENCY WITH THE SHAPLEY AUTHORIZATION VALUE.** For $v \in \mathcal{G}^N$ and A an interior operator on N, it holds that
$$\Psi(V_v, A) = \left\{ \left(\Phi(v_{|E}, A_{|E})\right)_{E \in 2^N \setminus \{\emptyset\}} \right\}.$$

- **ZERO INESSENTIAL GAMES.** For $V \in \Omega^N$, A an interior operator on N and $0 \in \partial(V(E))$ for all $E \in aut(A)$, it holds that
$$(0)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V, A).$$

In the next theorem we prove that these properties uniquely determine the Harsanyi configuration correspondence for NTU games with interior operator structure. Before, we see a proposition that will be useful in the proof of the theorem.

Proposition 7.20 *Let $V \in \Omega^N$, A an interior operator on N and $(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \mathcal{H}(V, A)$. Then, it holds that*

(a) $x_E = 0$ for every $E \in 2^N \setminus \{\emptyset\}$ with $A(E) = \emptyset$.

(b) For every $E \in 2^N$ with $A(E) \neq \emptyset$,

$$(x_E)_i = \begin{cases} \left(x_{A(E)}\right)_i & \text{if } i \in A(E), \\ 0 & \text{if } i \in E \setminus A(E). \end{cases}$$

Proof. Let $V \in \Omega^N$, A an interior operator on N and $(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \mathcal{H}(V, A)$.

(a) We proceed by strong induction on $|E|$.

1. Suppose $|E| = 1$. It holds that $E = \{i\}$ and $V^A(E) = (-\infty, 0]^{\{i\}}$. Since $x_E \in \partial\left(V^A(E)\right)$ it must be $x_E = 0$.
2. Suppose $|E| > 1$. By induction hypothesis it holds that

$$x_F = 0 \quad \text{for all } F \subsetneq E \text{ with } F \neq \emptyset. \tag{7.8}$$

Let $\lambda \in \mathbb{R}^N_{++}$ the vector associated to the payoff configuration $(x_E)_{E\in 2^N\setminus\{\emptyset\}}$. From condition *(3)* of the definition of $\mathcal{H}$ in Remark 7.19 we have that

$$\lambda^E * x_E = \phi(w_{|E}), \tag{7.9}$$

where $w(F) = \lambda^F \cdot x_F$ for all $F \in 2^N \setminus \{\emptyset\}$. From (7.8) we know that $w(F) = 0$ for all $F \subsetneqq E$. Therefore

$$\phi(w_{|E}) = \left\{\frac{w(E)}{|E|}\right\}^E, \tag{7.10}$$

and from (7.9) and (7.10) we obtain that

$$(x_E)_i = \frac{w(E)}{|E|\lambda_i} \quad \text{for every } i \in E.$$

But from condition *(1)* in Remark 7.19 we know that $x_E \in \partial\left(V^A(E)\right) = \partial\left((-\infty,0]^E\right)$. Consequently it holds that $w(E) = 0$ and hence $x_E = 0$.

(b) We proceed by strong induction on $|E|$.

1. If $|E| = 1$ there is nothing to prove.

2. Suppose $|E| > 1$. We can assume that $A(E) \neq E$ because otherwise there is nothing to prove. Let $\lambda \in \mathbb{R}^N_{++}$ the vector associated to the payoff configuration $(x_E)_{E\in 2^N\setminus\{\emptyset\}}$. Let w be the TU game given by $w(F) = \lambda^F \cdot x_F$ for every $F \in 2^N \setminus \{\emptyset\}$. From (a) and the induction hypothesis we can easily derive that

$$w(F) = w(A(F)) \quad \text{for every } F \subsetneqq E. \tag{7.11}$$

Notice that if $F \subseteq E$ then $A(F) = A\left(A(F)\right) \subseteq A\left(F \cap A(E)\right)$. So it is clear that

$$A(F) = A\left(F \cap A(E)\right) \quad \text{for every } F \subseteq E. \tag{7.12}$$

From (7.11) and (7.12) we can easily derive that

$$w(F) = w\left(F \cap A(E)\right) \quad \text{for every } F \subsetneqq E. \tag{7.13}$$

For each nonempty $F \subseteq E$ we define

$$\Delta(F) = \begin{cases} w(E) - w(A(E)) & \text{if } F = E, \\ 0 & \text{if } F \neq E \text{ and } F \nsubseteq A(E), \\ \Delta_w(F) & \text{if } F \subseteq A(E). \end{cases}$$

Using (7.13) it is clear that

$$w(H) = \sum_{\{F \in 2^N \setminus \{\emptyset\}:\, F \subseteq H\}} \Delta(F) \quad \text{for every nonempty } H \subseteq E,$$

hence

$$\Delta(F) = \Delta_w(F) \quad \text{for every nonempty } F \subseteq E.$$

If $i \in E$ it holds that

$$\begin{aligned} \phi_i(w_{|E}) &= \sum_{\{F \subseteq E:\, i \in F\}} \frac{\Delta_w(F)}{|F|} = \sum_{\{F \subseteq E:\, i \in F\}} \frac{\Delta(F)}{|F|} \\ &= \sum_{\{F \subseteq A(E):\, i \in F\}} \frac{\Delta_w(F)}{|F|} + \frac{w(E) - w(A(E))}{|E|}, \end{aligned}$$

whence we obtain that

$$\phi_i(w_{|E}) = \begin{cases} \phi_i(w_{|A(E)}) + \dfrac{w(E) - w(A(E))}{|E|} & \text{if } i \in A(E), \\ \\ \dfrac{w(E) - w(A(E))}{|E|} & \text{if } i \in E \setminus A(E). \end{cases}$$

Taking into consideration that $\lambda^E * x_E = \phi(w_{|E})$ and $\lambda^{A(E)} * x_{A(E)} = \phi(w_{|A(E)})$ it follows that

$$(x_E)_i = \begin{cases} \left(x_{A(E)}\right)_i + \dfrac{w(E) - w(A(E))}{|E|\lambda_i} & \text{if } i \in A(E), \\ \\ \dfrac{w(E) - w(A(E))}{|E|\lambda_i} & \text{if } i \in E \setminus A(E). \end{cases}$$

But, taking into account that

$$x_{A(E)} \in \partial\left(V(A(E))\right)$$

and

$$x_E \in \partial\left(V^A(E)\right) = \partial\left(V(A(E)) \times (-\infty, 0]^{E\setminus A(E)}\right),$$

it is easy to see that $w(E) - w(A(E)) = 0$, which completes the proof.

□

Theorem 7.21 *A configuration correspondence for NTU games with interior operator structure is equal to the Harsanyi configuration correspondence for NTU games with interior operator structure if and only if it satisfies the properties of efficiency, conditional additivity, scale covariance, independence of irrelevant alternatives, consistency with the Shapley authorization value and zero inessential games.*

Proof. Firstly, we prove that the Harsanyi configuration correspondence for NTU games with interior operator structure satisfies the properties in the theorem.

EFFICIENCY. Let $V \in \Omega^N$, A an interior operator on N, $(x_E)_{E\in 2^N\setminus\{\emptyset\}} \in \mathcal{H}(V, A)$ and $E \in aut(A)$. Since $\mathcal{H}(V, A) = H(V^A)$ and H satisfies efficiency it holds that $x_E \in \partial\left(V^A(E)\right) = \partial(V(E))$.

CONDITIONAL ADDITIVITY. Let $V, W \in \Omega^N$ such that $V + W \in \Omega^N$ and A an interior operator on N. Let $(x_E)_{E\in 2^N\setminus\{\emptyset\}} \in \mathcal{H}(V, A) + \mathcal{H}(W, A)$ such that $x_E \in \partial((V + W)(E))$ for all $E \in aut(A)$. Let $(y_E)_{E\in 2^N\setminus\{\emptyset\}} \in \mathcal{H}(V, A)$ and $(z_E)_{E\in 2^N\setminus\{\emptyset\}} \in \mathcal{H}(W, A)$ such that $x_E = y_E + z_E$ for every $E \in 2^N \setminus \{\emptyset\}$. Take $F \in 2^N \setminus \{\emptyset\}$. We want to see that $x_F \in \partial\left((V^A + W^A)(F)\right)$. We distinguish two cases:

(a) If $A(F) = \emptyset$ we know, from Proposition 7.20, that $y_F = z_F = 0$ and hence $x_F = 0$. Notice that $(V^A + W^A)(F) = (-\infty, 0]^F$. So it holds that $x_F \in \partial\left((V^A + W^A)(F)\right)$.

(b) If $A(F) \neq \emptyset$ we know, from Proposition 7.20, that

$$(y_F)_i = \begin{cases} \left(y_{A(F)}\right)_i & \text{if } i \in A(F), \\ 0 & \text{if } i \in F \setminus A(F), \end{cases}$$

and

$$(z_F)_i = \begin{cases} \left(z_{A(F)}\right)_i & \text{if } i \in A(F), \\ 0 & \text{if } i \in F \setminus A(F), \end{cases}$$

whence it follows that

$$(x_F)_i = \begin{cases} \left(x_{A(F)}\right)_i & \text{if } i \in A(F), \\ 0 & \text{if } i \in F \setminus A(F). \end{cases} \tag{7.14}$$

By hypothesis, we know that

$$x_{A(F)} \in \partial \left((V + W)(A(F))\right). \tag{7.15}$$

Moreover, it holds that

$$(V^A + W^A)(F) = (V + W)^A(F) = (V + W)(A(F)) \times (-\infty, 0]^{F \setminus A(F)}. \tag{7.16}$$

From (7.14), (7.15) and (7.16) it follows that $x_F \in \partial\left((V^A + W^A)(F)\right)$.

We have that $(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in H(V^A) + H(W^A)$ and $x_E \in \partial\left((V^A + W^A)(E)\right)$ for every $E \in 2^N \setminus \{\emptyset\}$. Since H satisfies conditional additivity it holds that

$$(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in H\left(V^A + W^A\right) = H\left((V + W)^A\right) = \mathcal{H}(V + W, A).$$

Scale covariance. Let $V \in \Omega^N$, A an interior operator on N and $\alpha \in \mathbb{R}^N_{++}$. It holds that

$$\mathcal{H}(\alpha * V, A) = H\left((\alpha * V)^A\right) = H\left(\alpha * V^A\right)$$

which, using that the Harsanyi configuration correspondence for NTU games satisfies scale covariance, is equal to

$$\left\{(\alpha^E * x_E)_{E \in 2^N \setminus \{\emptyset\}} : (x_E)_{E \in 2^N \setminus \{\emptyset\}} \in H(V^A)\right\},$$

or equivalently,

$$\left\{(\alpha^E * x_E)_{E \in 2^N \setminus \{\emptyset\}} : (x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \mathcal{H}(V, A)\right\}.$$

Independence of irrelevant alternatives. Let $V, W \in \Omega^N$, A an interior operator on N such that $V(E) \subseteq W(E)$ for all $E \in aut(A)$ and $(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \mathcal{H}(W, A)$ such that $x_E \in V(E)$ for all $E \in aut(A)$. From $V(E) \subseteq W(E)$ for all $E \in aut(A)$, it easily derives that

$$V^A(E) \subseteq W^A(E) \quad \text{for every } E \in 2^N \setminus \{\emptyset\}. \tag{7.17}$$

Let $F \in 2^N \setminus \{\emptyset\}$. We want to see that $x_F \in V^A(F)$. We distinguish two cases:

(a) If $A(F) = \emptyset$ we know, from Proposition 7.20, that $x_F = 0$. In this case, $V^A(F) = (-\infty, 0]^F$, so $x_F \in V^A(F)$.

(b) If $A(F) \neq \emptyset$ we know, from Proposition 7.20, that

$$(x_F)_i = \begin{cases} \left(x_{A(F)}\right)_i & \text{if } i \in A(F), \\ 0 & \text{if } i \in F \setminus A(F). \end{cases} \tag{7.18}$$

By hypothesis, we know that

$$x_{A(F)} \in V(A(F)). \tag{7.19}$$

From (7.18), (7.19) and $V^A(F) = V(A(F)) \times (-\infty, 0]^{F \setminus A(F)}$ it follows that $x_F \in V^A(F)$.

We have that $x_E \in V^A(E)$ for every $E \in 2^N \setminus \{\emptyset\}$. From this fact, (7.17) and the fact that H satisfies independence of irrelevant alternatives it follows that $(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \mathcal{H}(V, A)$.

Consistency with the Shapley authorization value. Let $v \in \mathcal{G}^N$ and A an interior operator on N. If we see condition (2) of the definition of $\mathcal{H}$ in Remark 7.19 and take into account that

$$V_v(N) = \left\{ y \in \mathbb{R}^N : \sum_{k \in N} y_k \leqslant v(N) \right\},$$

it is clear that any payoff configuration in $\mathcal{H}(V_v, A)$ must be associated to $\lambda = 1_N$. If we write the three conditions for $\lambda = 1_N$ we have that a payoff configuration $(x_E)_{E \in 2^N \setminus \{\emptyset\}}$ belongs to $\mathcal{H}(V_v, A)$ if

(1) $x_E \in \partial \left(V_v^A(E)\right)$ for all $E \in 2^N \setminus \{\emptyset\}$,

(2) $\sum_{k \in N} (x_N)_k = v(N)$,

(3) if w is the TU game given by $w(F) = \sum_{k\in F}(x_F)_k$ for every $F \in 2^N \setminus \{\emptyset\}$, then $x_E = \phi(w_{|E})$ for all $F \in 2^N \setminus \{\emptyset\}$.

Notice that if $(x_E)_{E\in 2^N\setminus\{\emptyset\}} \in \mathcal{H}(V_v, A)$ and $H \in aut(A)$ then, from condition (1), it holds that

$$x_H \in \partial\left(V_v(H)\right) = \left\{y \in \mathbb{R}^H : \sum_{k\in H} y_k = v(H)\right\}.$$

So it follows that $w(H) = v(H)$. Moreover, from Proposition 7.20 we know that $w(E) = w(A(E))$ for all $E \subseteq N$. Therefore

$$w(E) = w(A(E)) = v(A(E)) = v^A(E) \quad \text{for every } E \subseteq N,$$

whence, taking into account condition (1), we obtain

$$x_E = \phi(w_{|E}) = \phi\left({v^A}_{|E}\right) = \Phi\left(v_{|E}, A_{|E}\right) \quad \text{for every } E \in 2^N \setminus \{\emptyset\}.$$

We have seen that $\left(\Phi(v_{|E}, A_{|E})\right)_{E\in 2^N\setminus\{\emptyset\}}$ is the only possible payoff configuration in $\mathcal{H}\left(V_v, A\right)$. Conversely, it is plain to see that $\left(\Phi(v_{|E}, A_{|E})\right)_{E\in 2^N\setminus\{\emptyset\}}$ satisfies (1), (2) and (3).

Zero inessential games. Take $V \in \Omega^N$ and A an interior operator on N such that $0 \in \partial(V(F))$ for all $F \in aut(A)$. It is clear that $0 \in \partial\left(V^A(E)\right)$ for every $E \in 2^N \setminus \{\emptyset\}$. Since the Harsanyi configuration correspondence for NTU games satisfies the property of zero inessential games it holds that

$$(0)_{E\in 2^N\setminus\{\emptyset\}} \in H(V^A) = \mathcal{H}(V, A).$$

We have proved that $\mathcal{H}$ satisfies the properties in the theorem. Now we see that these properties uniquely determine the Harsanyi configuration correspondence for NTU games with interior operator structure.

Let Υ and Ψ be configuration correspondences for NTU games with interior operator structure satisfying the properties of efficiency, conditional additivity, scale covariance, independence of irrelevant alternatives, consistency with the Shapley authorization value and zero inessential games. We must prove that $\Upsilon = \Psi$. Take $V \in \Omega^N$ and A an interior operator on N. We want to verify that

$\Upsilon(V, A) = \Psi(V, A)$. For symmetry, it suffices to prove one inclusion.

Let $(x_E)_{E\in 2^N\setminus\{\emptyset\}} \in \Upsilon(V, A)$. Consider $W_1 \in \Omega^N$ given by

$$W_1(E) = V(E) - \{x_E\} \quad \text{for every } E \in 2^N \setminus \{\emptyset\}.$$

Since Υ satisfies efficiency it holds that $x_F \in \partial(V(F))$ for every $F \in aut(A)$, whence we obtain that $0 \in \partial(W_1(F))$ for every $F \in aut(A)$. From the property of zero inessential games we derive that

$$(0)_{E\in 2^N\setminus\{\emptyset\}} \in \Psi(W_1, A). \tag{7.20}$$

From properties (i), (ii) and (iii) of the games in Ω^N we can obtain that there exists $\lambda \in \mathbb{R}^N_{++}$ such that

$$\lambda \cdot y \leqslant \lambda \cdot x_N \quad \text{for all } y \in V(N).$$

Take $W_2 \in \Omega^N$ defined by

$$W_2(E) = \begin{cases} V(N) & \text{if } E = N, \\ \left\{y \in \mathbb{R}^E : y \leqslant x_E\right\} & \text{if } E \subsetneq N, E \neq \emptyset. \end{cases}$$

Since Υ satisfies independence of irrelevant alternatives it holds that $(x_E)_{E\in 2^N\setminus\{\emptyset\}} \in \Upsilon(W_2, A)$. Using scale covariance it follows that

$$(\lambda^E * x_E)_{E\in 2^N\setminus\{\emptyset\}} \in \Upsilon(\lambda * W_2, A). \tag{7.21}$$

Let V_0 be the NTU game corresponding to the TU game that is identically zero. From the fact that Υ satisfies consistency with the Shapley authorization value it follows that

$$\Upsilon(V_0, A) = \left\{(0)_{E\in 2^N\setminus\{\emptyset\}}\right\}. \tag{7.22}$$

From (7.21) and (7.22) we obtain that $(\lambda^E * x_E)_{E\in 2^N\setminus\{\emptyset\}} \in \Upsilon(\lambda * W_2, A) + \Upsilon(V_0, A)$. Notice that $\lambda * W_2 + V_0 = V_v$ where v is the TU game given by $v(E) = \lambda^E \cdot x_E$ for all nonempty $E \subseteq N$. From conditional additivity it follows that $(\lambda^E * x_E)_{E\in 2^N\setminus\{\emptyset\}} \in \Upsilon(V_v, A)$. Since Υ and Ψ satisfy the property of consistency with the Shapley authorization value it holds that

$\Upsilon(\lambda * W_2 + V_0, A) = \Psi(\lambda * W_2 + V_0, A)$. Therefore,

$$(\lambda^E * x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(\lambda * W_2 + V_0, A). \tag{7.23}$$

Consider $W_3 \in \Omega^N$ given by

$$W_3(E) = \begin{cases} \{y \in \mathbb{R}^N : \lambda \cdot y \leqslant \lambda \cdot x_N\} & \text{if } E = N, \\ \{y \in \mathbb{R}^E : y \leqslant x_E\} & \text{if } E \subsetneqq N, E \neq \emptyset. \end{cases}$$

It is clear that

$$(\lambda * W_3)(E) \subseteq (\lambda * W_2 + V_0)(E) \quad \text{for all } E \in 2^N \setminus \{\emptyset\}. \tag{7.24}$$

From (7.23), (7.24), the fact that $\lambda^E * x_E \in (\lambda * W_3)(E)$ for all $E \in 2^N \setminus \{\emptyset\}$ and the property of independence of irrelevant alternatives it follows that

$$(\lambda^E * x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(\lambda * W_3, A),$$

which, from scale covariance, implies that

$$(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(W_3, A). \tag{7.25}$$

From (7.20) , (7.25) and the property of conditional additivity we obtain

$$(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(W_1 + W_3, A).$$

But notice that

$$(W_1 + W_3)(E) = \begin{cases} \{y \in \mathbb{R}^N : \lambda \cdot y \leqslant \lambda \cdot x_N\} & \text{if } E = N, \\ V(E) & \text{if } E \subsetneqq N, E \neq \emptyset, \end{cases}$$

hence $V(E) \subseteq (W_1 + W_3)(E)$ for every $E \in 2^N \setminus \{\emptyset\}$. Finally, using the property of independence of irrelevant alternatives we obtain that $(x_E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V, A)$. □

In practice, given $V \in \Omega^N$ and A an interior operator on N, we do not need to obtain V^A to calculate $\mathcal{H}(V, A)$. From Remark 7.19 and Proposition 7.20 we can derive an alternative definition of $\mathcal{H}(V,A)$ that does not involve the restricted game V^A. We give that definition in the following remark.

Remark 7.22 *Let $V \in \Omega^N$ and A an interior operator on N. A payoff configuration $(x_E)_{E\in 2^N\setminus\{\emptyset\}}$ belongs to $\mathcal{H}(V,A)$ if there exists $\lambda \in \mathbb{R}^N_{++}$ such that*

(1) $x_E \in \partial\left(V(E)\right)$ *for all* $E \in aut(A)$,

(2) $\lambda \cdot x_N = \max\left\{\lambda \cdot y : y \in V(N)\right\}$,

(3) if w is the TU game given by $w(F) = \lambda^F \cdot x_T$ for every $F \in 2^N \setminus \{\emptyset\}$, then $\lambda^E * x_E = \phi(w_{|E})$ for all $E \in aut(A)$,

(4) $x_E = 0$ *for all* $E \in 2^N \setminus \{\emptyset\}$ *with* $A(E) = \emptyset$,

(5) For every $E \in 2^N$ with $A(E) \neq \emptyset$,

$$(x_E)_i = \begin{cases} \left(x_{A(E)}\right)_i & \text{if } i \in A(E), \\ 0 & \text{if } i \in E \setminus A(E). \end{cases}$$

Example 7.23 Let us calculate $\mathcal{H}(V,A)$ where V and A are those given in Example 7.3. Since $\partial(V(N))$ is a hyperplane, it is easy to check that $\mathcal{H}(V,A)$ contains exactly one payoff configuration $(x_E)_{E\in 2^N\setminus\{\emptyset\}}$. It is clear that this payoff configuration is associated to the comparison vector $\lambda = (1,1,1)$. Let w be the TU game given by $w(F) = \lambda^F \cdot x_F$ for every nonempty $F \subseteq N$. Let us calculate $(x_E)_{E\in 2^N\setminus\{\emptyset\}}$ by using the preceding remark.
From Remark 7.22 (1), it follows that

$$x_{\{1\}} = 1,$$
$$x_{\{2\}} = 1.$$

From Remark 7.22 (4), it follows that

$$x_{\{3\}} = 0.$$

We have that $w(\{1\}) = \lambda^{\{1\}} \cdot x_{\{1\}} = 1$ and $w(\{2\}) = \lambda^{\{2\}} \cdot x_{\{2\}} = 1$. So it holds that

$$\phi\left(w_{|\{1,2\}}\right) = \left(\frac{w(\{1,2\})}{2}, \frac{w(\{1,2\})}{2}\right).$$

Using Remark 7.22 (3), we conclude that

$$x_{\{1,2\}} = \left(\frac{w(\{1,2\})}{2}, \frac{w(\{1,2\})}{2}\right).$$

From this fact and Remark 7.22 (1), we have that

$$x_{\{1,2\}} \in \partial\left(V\left(\{1,2\}\right)\right) \cap \left\{(\alpha,\alpha) \in \mathbb{R}^{\{1,2\}} : \alpha \in \mathbb{R}\right\} = \{(2,2)\}.$$

So we have obtained that

$$x_{\{1,2\}} = (2,2).$$

From Remark 7.22 (5), it follows that

$$x_{\{1,3\}} = (1,0).$$

We have that $w(\{2\}) = 1$ and $w(\{3\}) = \lambda^{\{3\}} \cdot x_{\{3\}} = 0$. So it holds that

$$\phi\left(w_{|\{2,3\}}\right) = \left(1 + \frac{w(\{2,3\}) - 1}{2}, \frac{w(\{2,3\}) - 1}{2}\right).$$

Using Remark 7.22 (3), we conclude that

$$x_{\{2,3\}} = \left(1 + \frac{w(\{2,3\}) - 1}{2}, \frac{w(\{2,3\}) - 1}{2}\right).$$

From this fact and Remark 7.22 (1), we have that

$$x_{\{2,3\}} \in \partial\left(V\left(\{2,3\}\right)\right) \cap \left\{(1+\alpha,\alpha) \in \mathbb{R}^{\{2,3\}} : \alpha \in \mathbb{R}\right\} = \{(2,1)\}.$$

So we have obtained that

$$x_{\{2,3\}} = (2,1).$$

We have that

$$
\begin{aligned}
&w(\{1\}) = 1, \\
&w(\{2\}) = 1, \\
&w(\{3\}) = 0, \\
&w(\{1,2\}) = \lambda^{\{1,2\}} \cdot x_{\{1,2\}} = 4, \\
&w(\{1,3\}) = \lambda^{\{1,3\}} \cdot x_{\{1,3\}} = 1, \\
&w(\{2,3\}) = \lambda^{\{2,3\}} \cdot x_{\{2,3\}} = 3.
\end{aligned}
$$

Moreover, since $x_N \in \partial(V(N))$, it follows that $\lambda \cdot x_N = 6$, that is, $w(N) = 6$. It holds that $\phi(w) = (2,3,1)$, and using Remark 7.22 (3), we conclude that

$$x_N = (2,3,1).$$

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