On spiking neural P systems

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Abstract This work deals with several aspects concerning the formal verification of SN P systems and the computing power of some variants. A methodology based on the information given by the transition diagram associated with an SN P system is presented. The analysis of the diagram cycles codifies invariants formulae which enable us to establish the soundness and completeness of the system with respect to the problem it tries to resolve. We also study the universality of asynchronous and sequential SN P systems and the capability these models have to generate certain classes of languages. Further, by making a slight modification to the standard SN P systems, we introduce a new variant of SN P systems with a special I/O mode, called SN P modules, and study their computing power. It is demonstrated that, as string language acceptors and transducers, SN P modules can simulate several types of computing devices such as finite automata, a-finite transducers, and systolic trellis automata.

Keywords Spiking neural P system · Formal verification · Universality · Asynchronous · Sequential · Finite automaton · Finite state transducer · Systolic trellis automaton

1 Introduction

Spiking neural P systems (SN P systems, for short) were introduced by Ionescu et al. (2006) with the aim of incorporating in membrane computing ideas of spiking neurons,
which is a promising research line in neural computing (see, e.g., Gerstner and Kistler 2002; Maass 2002).

SN P systems have a biological motivation based on recent discoveries on neural coding. Various studies show evidence of precise temporal correlations between pulses of different neurons and stimulus-dependent synchronisation of the activity in populations of neurons (see, e.g., Eckhorn et al.1988 or Gray and Kistler 2002). The flow of information is carried on the action potentials, which are encoded by objects (spikes) of the same type. These spikes are placed inside the neurons and can be sent from presynaptic to postsynaptic neurons according to specific rules and making use of the time as a support of information.

In the field of membrane computing, SN P systems try to capture the fact that most of the neural impulses are almost identical from an electrical point of view and the intervals of time between signals are crucial.

In this paper we study some interesting topics related to SN P systems, and they are organized as follows. The next section introduces basic concepts about SN P systems. In Section 3, a (restricted) methodology for the formal verification of these systems is presented. Section 4 is devoted to study of some characterizations of generalized models of SN P systems with an extended form of spiking rules. Finally, the computational power of new variants of SN P systems with a special input/output mode is presented in Section 5.

2 Spiking neural P systems

Informally, an SN P system consists of a set of neurons placed in the nodes of a directed graph which send signals (called spikes) along the arcs of the graph (representing the synapses between neurons). The system evolves according to a set of spiking rules and forgetting rules each of which is associated with a neuron that uses the rules for sending or internally consuming spikes. The rules for spiking should take into account all spikes present in a neuron not only part of them, although not all spikes are consumed in this way. The produced spikes are sent (maybe with a delay of some steps) to all neurons where there is a synapse outgoing from the neuron where the rule was applied.

A global clock is assumed and in each time unit each neuron which can use a rule should use one such rule (thus, only one rule is used in each neuron). One of the neurons is considered to be the output neuron, and its spikes are also sent to the environment.

Definition 2.1 A spiking neural P system (abbreviated as SN P system) of degree $m \geq 1$, is a tuple of the form $\Pi = (O, \sigma_1, \ldots, \sigma_m, \text{syn}, \text{out})$, where:

- $O = \{a\}$ is the singleton alphabet (the object $a$ is called spike);
- $\sigma_i = (n_i, R_i), 1 \leq i \leq m$, where $n_i \geq 0$ and $R_i$ is a finite set of rules of the types:
  
  (1) Spiking rules: $E/a^c \rightarrow a ; d , $where $E$ is a regular expression over $a, c \geq 1$, and $d \geq 0$;
  (2) Forgetting rules: $a^s \rightarrow \lambda$, for some $s \geq 1$, with the restriction that for each rule $E/a^c \rightarrow a ; d$ of type (1) from $R_i$, we have $a^d \not\in L(E)$;

- syn $\subseteq \{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\}$ with $(i, i) \not\in \text{syn}$ for $i \in \{1, \ldots, m\}$ (synapses);
- out $\in \{1, 2, \ldots, m\}$.

That is, an SN P system consists of a set of neurons $\{\sigma_1, \ldots, \sigma_m\}$ of the form $\sigma_i = (n_i, R_i)$ where $n_i$ represents the number of spikes initially contained by neuron $\sigma_i$. Besides, syn specifies the set of arcs of a directed graph representing the synapses between neurons, and out indicates the index of the output neuron.
If a spiking rule is of the form \( E / a^e \rightarrow a; d \in R_i \), with \( L(E) = \{ a^e \} \), then such a rule only is applicable when neuron \( \sigma_i \) contains exactly \( k \) spikes. In this case, we denote that rule by \( a^e \rightarrow a; d \).

A spiking rule \( E / a^e \rightarrow a; d \in R_i \), is applicable in a step \( t \) if the neuron \( \sigma_i \) contains exactly \( k \) spikes, \( a^e \in L(E) \) and \( k \geq c \). By applying such a rule, \( c \) spikes are consumed, the neuron \( \sigma_i \) is fired and it produces one spike after \( d \) time units. If \( d = 0 \), then the spikes are emitted immediately, otherwise the spikes are emitted after \( d \) steps. In the case \( d \geq 1 \), in steps \( t, t + 1, t + 2, \ldots, t + d - 1 \) the neuron is closed, and it cannot receive new spikes. In the step \( t + d \), the neuron spikes and becomes again open, hence it can receive spikes. The spike emitted by a neuron \( \sigma_i \) is replicated and it goes to all neurons \( \sigma_j \) such that \( (i, j) \in \text{syn.} \)

A forgetting rule \( a^e \rightarrow \lambda \in R_i \) is applicable in a step \( t \) if the neuron \( \sigma_i \) contains exactly \( s \) spikes in the instant \( t \). By applying such a rule, \( s \) spikes are removed from the neuron \( \sigma_i \).

In each time unit, if some rules from \( R_i \) is applicable, then one rule (and only one) must be applied, chosen non-deterministically among all possible rules applicable to \( \sigma_i \). That is, each neuron processes sequentially its spikes by using, at most, one rule in each time unit.

Let us recall that a spiking rule and a forgetting rule cannot be simultaneously applied to a neuron.

The rules are used in a maximally parallel way at the level of the system: at each step, all neurons which can use some rule, must evolve using one rule.

A configuration of an SN P system of degree \( m \geq 1 \) is a tuple \( ((p_1, q_1), \ldots, (p_m, q_m)) \) where \( p_i \) (\( 1 \leq i \leq m \)) describes the number of spikes present in the neuron \( \sigma_i \), and \( q_i \) (\( 1 \leq i \leq m \)) represents the number of steps to count down until it becomes open. The initial configuration of \( \Pi \) is \( ((n_1, 0), \ldots, (n_m, 0)) \), that is, all neurons are open initially. Using the rules of the system in the way described before, a configuration \( C_2 \) can be reached from another configuration \( C_1 \); such a step is called a transition, and we denote it by \( C_1 \Rightarrow C_2 \). Any (finite or infinite) sequence of transitions \( C_0 \Rightarrow C_1 \Rightarrow C_2 \Rightarrow \cdots \) is a computation if \( C_0 \) is the initial configuration of the system, and either \( r = \infty \) or \( r \in \mathbb{N} \) and all neurons are open and no rule can be used (in this case, \( C_r \) is called an halting computation).

Let \( \Pi \) be an SN P system and let \( C = C_0 \Rightarrow C_1 \Rightarrow C_2 \Rightarrow \cdots \) be a computation in \( \Pi \). The spike train of computation \( C \), denoted by \( \text{st}(C) \), is the sequence of steps \( i \) such that \( C_i \) emits a spike out, and we write it in the form \( \text{st}(C) = \langle t_1, t_2, \ldots \rangle \), with \( 1 \leq t_1 < t_2 < \ldots \). That is, we consider the moments when the output neuron spikes not when it fires.

The set of all spike trains (over the set \( \text{COM}(\Pi) \) of all computations) of \( \Pi \) is denoted by \( \text{ST}(\Pi) \), and can be considered as the result of the evolution of the system \( \Pi \). In Sects. 2 and 3 we consider SN P systems as computing devices, which compute sets of natural numbers. One can associate a set of numbers with \( \text{ST}(\Pi) \) in several ways. We denote \( N(\Pi) = \{ n \mid n = t_i - t_{i-1}, \text{ for } 2 \leq i \leq k \} \), and as in Ionescu et al. (2006), we can consider the intervals between consecutive spikes as numbers computed by a computation, with several alternatives:

- **Taking into account only the first \( k \geq 2 \) spikes:**
  
  \[ N_k(\Pi) = \{ n \mid \exists C \in \text{COM}(\Pi)(\text{st}(C) = \langle t_1, t_2, \ldots \rangle, \text{ and st}(C) \text{ has at least } k \text{ spikes, and } n = t_i - t_{i-1}, \text{ for } 2 \leq i \leq k) \} \]

- **Taking into account all spikes of computations with infinite spike trains:**
  
  \[ N_{\infty}(\Pi) = \{ n \mid \exists C \in \text{COM}(\Pi)(\text{st}(C) = \langle t_1, t_2, \ldots \rangle \text{ infinite, and } n = t_i - t_{i-1}, \text{ for } i \geq 2) \} \]
• Taking into account all intervals of all computations:

\[ N_{all}(\Pi) = \bigcup_{k \geq 2} N_k(\Pi) \cup N_\omega(\Pi) \]

**Definition 2.2** An SN P system \( \Pi \) is said to be weakly \( \omega \)-coherent if there is a computation \( C \) in \( \Pi \) such that \( N(C) = N_{all}(\Pi) \).

That is, an SN P system is weakly \( \omega \)-coherent if there exists a computation which provides all numbers which all other computations can provide.

3 Formal verification in SN P systems

This section aims to provide a methodology for the verification of certain SN P systems. The idea is to describe the evolution of the system by a transition diagram in such a way that the analysis of its cycles provide invariant formulae of the (evolutive) process.

**Definition 3.1** Given a P system, \( \Pi \), the computation tree, \( T(\Pi) \), associated with \( \Pi \) is the rooted labelled tree defined as follows: (a) the nodes of the tree are labelled by configurations of \( \Pi \); (b) the label of the root is the initial configuration of \( \Pi \); and (c) the children of a node are labelled by the configurations obtained from the configuration labelling the node through a step of transition.

The maximal branches of the computation tree associated with a P system are called computations of the system.

It is possible to consider an orientation in the computation tree, \( T(\Pi) \), in a natural way through the parent–child relation, that is, \( (u,v) \) is an oriented arc in the tree if and only if \( u \) is the parent of \( v \) (or \( v \) is a child of \( u \)).

Next, we define the transition diagram of an SN P system as a directed graph where the nodes are configurations and an arc is drawn between two nodes/configurations if a transition is possible between them.

**Definition 3.2** Given a SN P system, \( \Pi \), the oriented graph associated with \( \Pi \) is obtained from the (oriented) computation tree of \( \Pi \), by identifying the nodes having the same label.

We intend to show that transition diagrams are an interesting tool to establish the formal verification of SN P systems. Next, we justify this assertion with an example.

3.1 An example

Let us consider the SN P system \( \Pi_1 \) whose graphical representation is depicted in Fig. 1. We aim to prove that the system \( \Pi_1 \) computes the numerical set \( N_{all}(\Pi_1) = \{r + 2i \mid i \geq 1\} \), in the sense described in the previous section.

**Theorem 1**

(a) For each computation, \( C \) of \( \Pi_1 \) we have \( N(C) \subseteq \{r + 2i \mid i \geq 1\} \) (soundness).
(b) For each \( i \geq 1 \) there exists a computation, \( C \) of \( \Pi_1 \) such that \( r + 2i \in N(C) \) (completeness). Moreover, The SN P system \( \Pi_1 \) is weakly \( \omega \)-coherent.
(c) For each \( q \geq 2 \), let \( N_q(\Pi_1) \) be the set
\begin{align*}
\{t_q - t_{q-1}|st(C) = \langle t_1, \ldots, t_{q-1}, t_q, \ldots \rangle, \text{ for some computation } C \text{ of } \Pi_1\}
\end{align*}

Then, we have \( N_q'(\Pi_1) = N_{alt}(\Pi_1) = \{r + 2i|i \geq 1\} \).

**Proof** A configuration \( C \) of \( \Pi_1 \) can be described by a tuple \( C = (C(1), \ldots, C(6 + r)) \) where \( C(j) \) is the multiset over \( \{a\} \) contained in neuron \( j \), for \( 1 \leq j \leq 6 \), in neuron \( c_{j-6} \), for \( 7 \leq j \leq 5 + r \), or in neuron \( c'_{r-1} \) for \( j = 6 + r \). We denote by \( C^d \) (with \( d = 0, 1 \)) the configuration obtained from \( C \) applying in neuron 2 the rule \( a \rightarrow a; \delta \).

It is easy to prove that the only configurations sending a spike to the neighbouring neurons are \( C_1 \) and \( C_{r+3} \), and that \( C^{00}_{r+4} = C^{0}_{r+2} \) and \( C^{01}_{r+4} = C^{1}_{r+2} \). The transition diagram associated with \( \Pi_1 \) is depicted in Fig. 2, and we deduce that there is no halting computation.

Let us denote by \( \sigma \) the path \( C_0 \rightarrow C_1 \rightarrow \cdots \rightarrow C_r \rightarrow C_{r+1} \), and let \( \tau \) be the path \( C^1_{r+3} \rightarrow C^1_{r+4} \rightarrow \cdots \rightarrow C^1_{2r+1} \rightarrow C_r \rightarrow C_{r+1} \). Then, \( \text{Length}(\sigma) = r + 1 \) and \( \text{Length}(\tau) = r \).

For each \( j \geq 0 \), let \( \gamma(j) \) the path \( C_{r+1} \rightarrow C^{00}_{r+2} \rightarrow C^{00}_{r+3} \rightarrow C^{01}_{r+2} \rightarrow C^{11}_{r+3} \), meaning that the computation goes through \( C^0_{r+2} \) exactly \( j \) times. Then, \( \text{Length}(\gamma(j)) = 2(j + 1) \).

Taking into account that the computations of \( \Pi_1 \) are the maximal branches of the computation tree, for every computation, \( C \), of \( \Pi_1 \) there exists an infinite sequence of natural numbers \( \{i_k | k \geq 1\} \) such that \( C \) can be described through the following path in the transition diagram associated with \( \Pi_1 \) (we denote it by \( C(\{i_k | k \geq 1\}) \)):

\[ \sigma \gamma(i_1) \tau \gamma(i_2) \gamma(i_3) \tau \gamma(i_4) \tau \gamma(i_5) \cdots \]
Next, we describe the spike train of the computation $C(\{i_k \mid k \geq 1\}) \equiv C_0 \Rightarrow C_1 \Rightarrow C_2 \Rightarrow C_3 \ldots$, through the sequence of steps $i$ such that the configuration $C_i$ sends a spike out. We have:

$$\begin{cases} t_1 = 1 \\ t_{k+1} = t_k + r + 2(i_k + 1) \end{cases}$$

Hence, $N_{all}(C(\{i_k \mid k \geq 1\})) = \{t_{k+1} - t_k \mid k \geq 1\} = \{r + 2(i_k + 1) \mid k \geq 1\}$.

(a) Let $C$ be a computation of $\Pi_1$. Then there exists an infinite sequence of natural numbers $\{i_k \mid k \geq 1\}$ such that $C = C(\{i_k \mid k \geq 1\})$. Then, $N(C(\{i_k \mid k \geq 1\})) = \{r + 2(i_k + 1) \mid k \geq 1\} \subseteq \{r + 2i \mid i \geq 1\}$. Hence, $N_{all}(\Pi_1) \subseteq \{r + 2i \mid i \geq 1\}$.

(b) Let us prove that there exists a computation $C$ of $\Pi_1$ such that $N(C) = N_{all}(\Pi_1) = \{r + 2i \mid i \geq 1\}$. Indeed, let $s$ be the infinite sequence $\{i_k \mid k \geq 1\}$ such that $i_k = k - 1$, for each $k \geq 1$. Then, $N(C(s)) = \{r + 2(i_k + 1) \mid k \geq 1\} = \{r + 2i \mid i \geq 1\}$. Hence, $\{r + 2i \mid i \geq 1\} = N(C(s)) \subseteq N_{all}(\Pi_1) \subseteq \{r + 2i \mid i \geq 1\}$.

(c) If $\{i_k \mid k \geq 1\}$ is an infinite sequence of natural numbers, then

$$N(C(\{i_k \mid k \geq 1\})) = \{r + 2(i_k + 1) \mid k \geq 1\}$$

For each $j \geq 0$ let $C_j = C(\{i_k \mid k \geq 1\})$, where $i_k = j$, for every $k \geq 1$.

Let $q \geq 2$. For each $j \geq 0$, we have $N_{q'}(C_j) = \{t_q - t_{q-1} = \{r + 2(j + 1)\}$. Then, $\{r + 2i \mid i \geq 1\} = \{r + 2(j + 1) \mid j \geq 0\} = \{N_{q'}(C_j) \mid j \geq 0\}$. Hence, $N_{q'}(\Pi_1) \subseteq N_{all}(\Pi_1) = \{r + 2i \mid i \geq 1\}$.

4 Characterizations

In this section, we will discuss some characterizations of SN P systems. We will mainly look at a generalized model of SN P systems with an extended form of spiking rules. This model was introduced and studied in Chen et al. (2008) and Păun and Păun (2007).
An extended rule has the form $E/a^i \rightarrow a^p; d$. This rule operates in the same manner as before except that firing sends $p$ spikes along each outgoing synapse (and these $p$ spikes are received simultaneously by each neighboring neuron). Clearly, when $p = 1$ the extended rules reduce to the standard (or non-extended) rules in the original definition. Note also that forgetting rules are just a special case of firing rules, i.e., when $p = 0$.

We will consider systems with three types of neurons:

1. A neuron is \textit{bounded} if every rule in the neuron is of the form $a^i/a^j \rightarrow a^p; d$, where $1 \leq j \leq i$, $p \geq 0$, and $d \geq 0$. There can be several such rules in the neuron. These rules are called \textit{bounded rules}.

2. A neuron is \textit{unbounded} if every rule in the neuron is of the form $a^i(a^k)^* / a^j \rightarrow a^p; d$, where $i \geq 0$, $k \geq 1$, $j \geq 1$, $p \geq 0$, $d \geq 0$. Again, there can be several such rules in the neuron. These rules are called \textit{unbounded rules}.

3. A neuron is \textit{general} if it can have \textit{general rules}, i.e., bounded as well as unbounded rules.

One can allow rules like $z_1 + \cdots + z_n \rightarrow a^p; d$ in the neuron, where all $z_i$’s have bounded (resp., unbounded) regular expressions as defined above. But such a rule is equivalent to putting $n$ rules $z_i \rightarrow a^p; d$ ($1 \leq i \leq n$) in the neuron. It is known that any regular set over a 1-letter symbol $a$ can be expressed as a finite union of regular sets of the form $\{a^i(a^j)^k | k \geq 0\}$ for some $i, j \geq 0$. Note such a set is finite if $j = 0$. We can define three types of SN P systems:

1. \textit{Bounded SN P system}—a system in which every neuron is bounded.

2. \textit{Unbounded SN P system}—a system in which every neuron is either bounded or unbounded.

3. \textit{General SN P system}—a system with general neurons (i.e., each neuron can contain both bounded and unbounded rules).

Let $k \geq 1$. A \textit{k-output SN P system} has $k$ output neurons, $O_1, \ldots, O_k$. We say that the system generates a \textit{k-tuple} $(n_1, \ldots, n_k) \in \mathbb{N}^k$ if, starting from the initial configuration, there is a sequence of steps such that each output neuron $O_i$ generates (sends out to the environment) exactly $n_i$ spikes and then the system eventually halts. We will consider systems with delays and systems without delays (i.e., $d = 0$ in all rules).

4.1 Asynchronous general SN P systems

The standard model of SN P systems is synchronized, meaning that all neurons fire at each step of the computation whenever they are fireable. This synchronization is quite powerful: It is known that a set $Q \subseteq \mathbb{N}^3$ is recursively enumerable if and only if it can be generated by a 1-output general SN P system (with or without delays) (Ionescu et al. 2006; Ibarra et al. 2007). This result holds for systems with standard rules or extended rules, and it generalizes to systems with multiple outputs. Thus, such systems are universal.

In Cavaliere et al. (2007) the computational power of SN P systems that operate in an asynchronous mode was introduced and studied. In an \textit{asynchronous SN P system}, we do not require the neurons to fire at each step. During each step, any number of fireable neurons are fired (including the possibility of firing no neurons). When a neuron is fireable it may (or may not) choose to fire during the current step. If the neuron chooses not to fire, it may fire in any later step as long as the rule is still applicable. (The neuron may still receive spikes while it is waiting, which may cause the neuron to no longer be fireable.) Hence there is no restriction on the time interval for firing a neuron. Once a neuron chooses
to fire, the appropriate number of spikes are sent out after a delay of exactly $d$ time steps and are received by the neighboring neurons during the step when they are sent.

We recall the definition of a counter machine. A nondeterministic multicontroller machine (CM) $M$ is a nondeterministic finite automaton with a finite number of counters (it has no input tape). Each counter can only hold a nonnegative integer. The machine starts in a fixed initial state with all counters set to zero. During the computation, each counter can be incremented by 1, decremented by 1, or tested for zero. A distinguished set of $k$ counters (for some $k \geq 1$) is designated as the output counters. The output counters are non-decreasing (i.e., cannot be decremented). A $k$-tuple $(n_1, \ldots, n_k) \in \mathbb{N}^k$ is generated if $M$ eventually halts in an accepting state, where all non-output counters are set to zero and the contents of the output counters are $n_1, \ldots, n_k$, respectively. We will refer to a CM with $k$ output counters (the other counters are auxiliary counters) as a $k$-output CM.

It is well-known that a set $Q \subseteq \mathbb{N}^k$ is generated by a $k$-output CM if and only if $Q$ is recursively enumerable. Hence, $k$-output CMs are universal.

The following result was shown in Cavaliere et al. (2007). It says that SN P systems which operate in an asynchronous mode of computation are still universal provided that the neurons are allowed to use extended rules.

**Theorem 2** A set $Q \subseteq \mathbb{N}^k$ is recursively enumerable if and only if it can be generated by an asynchronous $k$-output general SN P system with extended rules. The result holds for systems with or without delays.

It remains an open question whether the above result holds for the case when the system uses only standard (i.e., non-extended) rules.

### 4.2 Asynchronous unbounded SN P systems

Next, we will study unbounded SN P systems, again assuming the use of extended rules. Recall that these systems can only use bounded and unbounded neurons (i.e., no general neurons are allowed). In contrast to Theorem 2, these systems can be characterized by partially blind multicontroller machines (PBCMs).

A partially blind $k$-output multicontroller machine ($k$-output PBCM) (Greibach 1978) is a $k$-output CM, where the counters cannot be tested for zero. The output counters are non-decreasing. The other counters can be incremented by 1 or decremented by 1, but if there is an attempt to decrement a zero counter then the computation aborts (i.e., the computation becomes invalid). Again, by definition, a successful generation of a $k$-tuple requires that the machine enters an accepting state with all non-output counters set to zero.

It is well known that $k$-output PBCMs can be simulated by $k$-dimensional vector addition systems, and vice-versa (Greibach 1978). Hence, such counter machines are not universal. In particular, a $k$-output PBCM can generate the reachability set of a vector addition system.

#### 4.2.1 Systems without delays

In Cavaliere et al. (2007), asynchronous unbounded SN P systems without delays were investigated. The systems considered in Cavaliere et al. (2007) are restricted to halt in a pre-defined configuration. Specifically, a computation is valid if, at the time of halting, the numbers of spikes that remain in the neurons are equal to pre-defined values; if the system halts but the neurons do not have the pre-defined values, the computation is considered...
invalid and the output is ignored. These systems were shown to be equivalent to PBCMs in Cavaliere et al. (2007). However, it was left as an open question whether the ‘pre-defined halting’ requirement was necessary to prove this result. It was recently shown in Ibarra and Woodworth (2007b) (see also Woodworth 2007) that this condition is, in fact, not necessary. Note that for these systems, firing zero or more neurons at each step is equivalent to firing one or more neurons at each step (otherwise, since there are no delays, the configuration stays the same when no neuron is fired).

**Theorem 3** A set \( Q \subseteq \mathbb{N}^k \) is generated by a \( k \)-output PBCM if and only if it can be generated by an asynchronous \( k \)-output unbounded SN P system without delays. Hence, such SN P systems are not universal.

Note that by Theorem 2, if we allow both bounded rules and unbounded rules to be present in the neurons, SN P systems become universal. Again, it is an open problem whether the above theorem holds for the case when the system uses only standard rules.

It is known that PBCMs with only one output counter can only generate semilinear sets of numbers. Hence:

**Corollary 4.1** Asynchronous 1-output unbounded SN P systems without delays can only generate semilinear sets of numbers.

The results in the following corollary can be obtained using Theorem 3 and the fact that they hold for \( k \)-output PBCMs.

**Corollary 4.2**

1. The family of \( k \)-tuples generated by asynchronous \( k \)-output unbounded SN P systems without delays is closed under union and intersection, but not under complementation.
2. The membership, emptiness, infiniteness, disjointness, and reachability problems are decidable for asynchronous \( k \)-output unbounded SN P systems without delays; but containment and equivalence are undecidable.

### 4.2.2 Systems with delays

Theorem 3 assumed an asynchronous SN P system without delays. However, it is possible that allowing delays would give additional power. For asynchronous unbounded SN P systems with delays, we can no longer assume that firing zero or more neurons at each step is equivalent to firing one or more neurons at each step.

Note that not every step in a computation has at least one neuron with a fireable rule. In a given configuration, if no neuron is fireable but at least one neuron is closed, we say that the system is in a *dormant* step. If there is at least one fireable neuron in a given configuration, we say the system is in a *non-dormant* step. (Of course, if a given configuration has no fireable neuron, and all neurons are open, we are in a halting configuration.) Thus, an SN P system with delays might be dormant at some point in the computation until a rule becomes fireable. However, the clock will keep on ticking. Interestingly, the addition of delays does not increase the power of the system (Ibarra and Woodworth 2007b; see also Woodworth 2007):

**Theorem 4** A set \( Q \subseteq \mathbb{N}^k \) is generated by \( k \)-output PBCM if and only if it can be generated by an asynchronous \( k \)-output unbounded SN P system with delays.
This result contrasts the result in Ibarra et al. (2007) which shows that synchronous unbounded SN P systems with delays and standard rules (but also standard output) are universal.

4.3 Asynchronous bounded SN P systems

In this section we consider asynchronous SN P systems, where the neurons can only use bounded rules. We show that these bounded SN P systems with extended rules generate precisely the semilinear sets.

A k-output monotonic CM is a nondeterministic machine with k counters, all of which are output counters. The counters are initially set to zero and can only be incremented by 1 or 0 (they cannot be decremented). When the machine halts in an accepting state, the k-tuple of values in the k-counter is said to be generated by the machine. Clearly, a k-output monotonic CM is a special case of a PBCM, where all the counters are output counters and all the instructions are addition instructions.

It is known that a set $Q \subseteq \mathbb{N}^k$ is semilinear if and only if it can be generated by a k-output monotonic CM (Harju et al. 2002). We can show the following (see, e.g., Ibarra and Woodworth 2006; Woodworth 2007):

**Theorem 5** $Q \subseteq \mathbb{N}^k$ can be generated by a k-output monotonic CM if and only if it can be generated by a k-output asynchronous bounded SN P system with extended rules. The result holds for systems with or without delays.

At present, we do not know whether Theorem 5 holds when the system is restricted to use only standard (non-extended) rules. However, we can show the result holds for synchronous bounded SN P systems using only standard rules (Woodworth 2007).

4.4 Sequential SN P systems

Sequential SN P systems are another closely related model introduced in Ibarra et al. (2006). These are systems that operate in a sequential mode. This means that at every step of the computation, if there is at least one neuron with at least one rule that is fireable, we only allow one such neuron and one such rule (both nondeterministically chosen) to be fired. If there is no fireable rule, then the system is dormant until a rule becomes fireable. However, the clock will keep on ticking. The system is called strongly sequential if at every step, there is at least one neuron with a fireable rule.

Unlike for asynchronous systems (considered in the previous section), where the results relied on the fact that the systems use extended rules, the following results in Ibarra et al. (2006) and Woodworth (2007) hold for systems that use standard rules as well as for systems that use extended rules.

**Theorem 6** The following results hold for systems with delays.

1. Sequential k-output unbounded SN P systems with standard rules are universal.
2. Strongly sequential k-output general SN P systems with standard rules are universal.
3. Strongly sequential k-output unbounded SN P systems with standard rules and k-output PBCMs are equivalent.

The above results also hold for systems with extended rules.
Item 3 in the above theorem improves the result in Ibarra et al. (2006) which required a special halting configuration similar to the halting configuration in Cavaliere et al. (2007). In fact, this halting requirement is not necessary (Ibarra and Woodworth 2007b).

4.5 SN P systems as language generators

SN P systems can be used as language generators, as described in two recent papers (Chen et al. 2006a, b). Consider an SN P system $\Pi$ with output neuron, $\text{out}$, which is bounded. Interpret the output as follows. At times when $\text{out}$ spikes, $a$ is interpreted to be 1, and at times when it does not spike, interpret the output to be 0. We say that a binary string $x = a_1 \ldots a_n$, where $n \geq 1$, is generated by $\Pi$ if starting in its initial configuration, it outputs $x$ and halts. We assume that the SN P systems use standard rules.

4.5.1 Regular languages

It was recently shown in Chen et al. (2006a) that for any finite binary language $F$, the language $F^1$ (i.e., with a supplementary suffix of 1) can be generated by a bounded SN P system. This is not true in general if $F$ is an infinite regular language as was shown in Ibarra and Woodworth (2007a):

Observation 4.1 Let $F = 0^*$. Then $F^1$ cannot be generated by a bounded SN P system.

However, it was also shown in Ibarra and Woodworth (2007a) that by just modifying the previous language $F$ to begin with at least one zero, one can generate $F^1$:

Observation 4.2 Let $F = 0^+$. Then $F^1$ can be generated by a bounded SN P system. Thus, it is possible to generate some languages where $F$ is an infinite language.

The following result, also shown in Ibarra and Woodworth (2007a), contrasts Observation 4.1:

Observation 4.3 Let $F = 0^*$. Then $1F$ can be generated by a bounded SN P system.

Observation 4.3 actually generalizes to the following rather surprising result shown in Ibarra and Woodworth (2007a):

**Theorem 7** Let $L \subseteq (0 + 1)^*$. Then the language $1L$ (i.e., with a supplementary prefix 1) can be generated by a bounded SN P system if and only if $L$ is regular. (The result holds also for $0L$, i.e., the supplementary prefix is 0 instead of 1.)

4.5.2 Another way of generating languages

We can define another way of “generating” a string. We say that a binary string $x = a_1 \ldots a_n$, where $n \geq 0$, is generated by $\Pi$ if it outputs $1x10^d$, for some $d$ which may depend on $x$, and halts. Thus, in the generation, $\Pi$ outputs a 1 before generating $x$, followed by $10^d$ for some $d$. (Note that the prefix 1 and the suffix $10^d$ are not considered as part of the string.) The set $L(\Pi)$ of binary strings generated in the manner described is the language generated by $\Pi$. Note that $d$ provides the “space” needed for the computation, just like worktape space in Turing machine computations. One can characterize various classes of languages by putting restrictions on the value of $d$ as a function of the length of $x$. See Ibarra and Woodworth (2007a).
5 A new variant of SN P systems with I/O mode

In the study of SN P systems there are several manners of defining string languages generated or accepted by those systems. One can consider as language the set of binary strings (studied in the previous section) or traces associated with halting computations (Chen et al. 2006b). Another way is to associate a string over an arbitrary alphabet in such a way that a symbol $a_i$ is associated with a step when a certain designated neuron (output neuron) emits $i$ spikes, leading to a string language over the alphabet generated by an extended SN P system (Chen et al. 2008).

Based on the idea suggested from the latter manner, in this section we introduce a new variant of SN P systems with a special I/O mode, called SN P modules, and study their computing power, to show that several types of computing devices based on finite state control can be simulated by SN P modules.

5.1 Spiking neural P module

A spiking neural P module (in short, an SN P module), of degree $m \geq 1$, is a construct of the form:

$$\Pi = (\{a\}, \sigma_1, \ldots, \sigma_m, \text{syn}, N_i, N_o),$$

where

1. $\{a\}$ is the singleton alphabet ($a$ is called spike);
2. $\sigma_1, \ldots, \sigma_m$ are neurons, of the form $\sigma_i = (n_i, R_i)$, $1 \leq i \leq m$, where
   - (a) $n_i \geq 0$ is the initial number of spikes contained by the neuron;
   - (b) $R_i$ is a finite set of rules of the following form: $E/a^c \rightarrow a^p$ with a regular expression $E$ over $\{a\}$ and $c \geq p \geq 1$;
3. $\text{syn} \subseteq \{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\}$ with $(i, i) \notin \text{syn}$ for $1 \leq i \leq m$ (synapses);
4. $N_i$ and $N_o(\subseteq \{1, 2, \ldots, m\})$ indicate the sets of input neurons and output neurons, respectively.

An SN P module $\Pi$ behaves in the usual way: An application of a rule of the form $E/a^c \rightarrow a^p$ implies that $c$ spikes are consumed and $p$ spikes are produced, provided the number of spikes in the neuron is covered by $E$, and so forth. (Note that each rule in $\Pi$ has no delay and both firing and spiking occur without any time delay.)

In contrast to the usual SN P systems, however, an SN P module $\Pi$ has the following distinguished feature:

In each step of computation, each input neuron $\sigma_{c_{i}}$ ($c_{i}$ in $N_{i}$) takes as input a number of a’s at a time (from the environment, denoted by Env), while each output neuron $\sigma_{c_{o}}$ ($c_{o}$ in $N_{o}$) produces as output $a^p$ to Env, if a rule of the form $E/a^c \rightarrow a^p$ is successfully applied in $\sigma_{c_{o}}$. (Note that $N_{i}$ and $N_{o}$ may share some neurons.)

Thus, an SN P module is a special form of an extended SN P system having neurons with input–output mode.

In the sequel, we only deal with the case when a regular language $L(E)$ is a singleton $\{a^m\}$; therefore, each rule is given in the form $a^m/a^c \rightarrow a^p$. In particular, for a rule of the form $a^c/a^c \rightarrow a^p$, we simply denote it by $a^c \rightarrow a^p$. 

5.2 Computing with SN P modules

As is shown below, SN P modules can simulate in a direct manner several types of computing devices based on finite state transitions.

5.2.1 Simulating finite automata

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite state automaton (DFA), where $\Sigma = \{b_1, \ldots, b_m\}$, $Q = \{q_1, \ldots, q_n\}$ and $q_0$ is the initial state. We demonstrate that DFA $M$ can be simulated by an SN P module.

Consider an SN P module:

$$\Pi_a = \{\{a\}, \sigma_1, \sigma_2, \sigma_3, \text{syn}, \{3\}, \{3\}\},$$

where

$$\sigma_1 = (n, \{a^n \rightarrow a^n\}),$$

$$\sigma_2 = (n, \{a^n \rightarrow a^n\}),$$

$$\sigma_3 = (n, \{a^{2n+i+k} / a^{2n+i+k-j} \rightarrow a^i|\delta(q_i, b_k) = q_j\}),$$

$$\text{syn} = \{(1, 2), (2, 1), (1, 3)\}.$$  

The module is given in a pictorial way in (a) of Fig. 3. Note that $n$ and $m$ are given fixed numbers, and that for each $1 \leq i \leq n$, $q_i$ in $Q$ is represented by $a^i$, while for each $1 \leq k \leq m$, $b_k$ in $\Sigma$ is represented by $a^{n+k}$. The number of spikes $a^i$ in neuron 3 is referred to (or identified) as a state of $\Pi_a$.

The manner of constructing $\Pi_a$ is a modification of the one presented for the generating SN P system in Chen et al. (2008). This system works as follows. Neurons 1 and 2 can fire and spike in the first step, and exchange each other. At the same time, neuron 1 emits $a^i$ to neuron 3. This action is repeatedly performed in arbitrary times.

Suppose that, in the first step, neuron 3 contains $a^i$ and is ready to receive input $a^{n+k}$ (representing $b_k$ in $\Sigma$) from Env. In the next step, it can fire by receiving $a^n$ (from neuron 1).

![Fig. 3](image-url) (a) SN P module $\Pi_a$ simulating a finite automaton $M$: $\Pi_a$ is currently in the state $q_i$ (represented by $a^i$). If input $b_k$ (represented by $a^{n+k}$) is read, then $\Pi_a$ changes its state to $q_j$ (represented by $a^j$), and at the same time produces it as output symbol. Thus, one transition $\delta(q_i, b_k) = q_j$ is simulated and the latest state $(q_j)$ is available (on demand). (b) SN P module $\Pi_a$ simulating a finite transducer $S$: $\Pi_a$ is currently in the state $q_i$ (represented by $a^i$). If input $b_k$ (represented by $a^{n+k}$) is read, then $\Pi_a$ changes its state to $q_j$ (represented by $a^j$) and produces as output $c_s$ (represented by $a^{n+k}$). In this moment, there remains $a^s$ in neuron 3 keeping the state $q_j$. Thus, one transition process $\delta(q_i, b_k) = (q_j, c_s)$ is simulated.
together with input $a^{n+k}$, and emits $a_j$ (representing $q_j$) to Env by consuming $(2n + i + k - j)$ spikes, leaving $a'$ in neuron 3. Hence, one state transition $\delta(q,b_k) = q_i$ is simulated.

Thus, (with 1 step delay) for a given input $w = b_1, \ldots, b_l$ in $\Sigma^*$, $\Pi_d$ produces a sequence of states: $z = q_{i_1}, \ldots, q_{i_r}$ (represented by $a^{i_1}, \ldots, a^{i_r}$) such that $\delta(q_{i_1}, b_{i_1}) = q_{i_{l+1}}$ for each $\ell = 0, \ldots, r$ where $q_{i_0} = q_0$. We denote this configuration by $z = \Pi_d(w)$. Then, it holds that $w$ is accepted by $M$ (i.e., $\delta(q_n, w) \in F$) iff $z = \Pi_d(w)$ ends up with a state in $F$ (i.e., $q_i$ is in $F$).

We now define the language accepted by $\Pi_d$ as:

$$L(\Pi_d) = \{ w \in \Sigma^* | \Pi_d(w) \text{ is in } Q'F \}.$$  

Then, the following is clearly proved.

**Theorem 8** Any regular language $L$ can be expressed as $L = L(\Pi_d)$ for some SN P module $\Pi_d$.

### 5.2.2 Simulating finite state transducers

By slightly modified construction presented above, one can construct an SN P module which simulates a finite state transducer (or sequential machine) as well.

Let $S = (Q, \Sigma, \Delta, \delta, q_n, F)$ be a deterministic finite state transducer (with accepting states), where $\Sigma = \{b_1, \ldots, b_m\}$, $\Delta = \{c_1, \ldots, c_i\}$, $Q = \{q_1, \ldots, q_n\}$ and $q_n$ is the initial state.

We now construct an SN P module:

$$\Pi_s = (\{\sigma\}, \sigma_1, \sigma_2, \sigma_3, \text{syn}, \{3\}, \{3\}),$$

where

\[
\begin{align*}
\sigma_1 & = (n, \{a^n \rightarrow a^n\}), \\
\sigma_2 & = (n, \{a^n \rightarrow a^n\}), \\
\sigma_3 & = (n, \{a^{2n+i+t+k} / a^{2n+i+k-j} \rightarrow a^{n+s}|\delta(q_i, b_k) = (q_j, c_s)\}), \\
\text{syn} & = \{(1,2), (2,1), (1,3)\}.
\end{align*}
\]

It should be noted that $n$, $m$ and $t$ are given fixed numbers. Further, for each $1 \leq i \leq m$ ($1 \leq s \leq t$), $q_i$ in $Q(c_s)$ in $\Delta$ is represented by $a'$ ($a^{n+s}$, respectively), while for each $1 \leq k \leq m$, $b_k$ in $\Sigma$ is represented by $a^{n+i+k}$.

The module is given in Fig. 3b. The manner of constructing $\Pi_s$ is a modification of the one for $\Pi_d$ presented above, and the computation process of an input $w$ (in $\Sigma^*$) by $\Pi_s$ is in parallel to the one by $\Pi_d$, with the difference that the former produces an output symbol in $\Delta$, while the latter provides the latest state in $Q$.

From the manner of constructing $\Pi_s$ and the previous argument to claim that $L = L(\Pi_d)$, we have the following.

**Theorem 9** Any finite state transducer $S$ can be simulated by some SN P module $\Pi_s$.

### 5.2.3 Simulating systolic trellis automata

Systolic automata are parallel computing models in the form of regular network structures of simple processors with 1-way flow of data. There are two types of underlying structures for regular networks: systolic trees and systolic trellis, while both employ in common a simple finite state control devices as functional elements (of processors).
Here we shall show that systolic trellis automata can be simulated by regular networks of SN P modules. (Note that the developed construction here also apply to simulate systolic tree automata in a straightforward manner.)

Formally, a homogeneous systolic trellis automaton (trellis automaton, in short) is a construct

\[ K = (\Sigma, \Gamma, \Gamma_0, f) \]

where \( \Sigma, \Gamma \) and \( \Gamma_0 \) (with \( \Sigma \subseteq \Gamma, \Gamma_0 \subseteq \Gamma \)) are finite alphabets of terminal, operating and accepting symbols, respectively, and \( f : \Gamma \times \Gamma \rightarrow \Gamma \) is the transition function. The domain of \( f \) is extended to \( \Gamma^* \) as follows: If \( |w| = 1 \), then \( f(w) = w = f^0(w) \), and if \( w = x_1 \ldots x_n \in \Gamma^n (n \geq 2) \), then

\[ f(w) = f(x_1, x_2)f(x_2, x_3) \ldots f(x_{n-1}, x_n). \]

The language accepted by \( K \) is defined by: \( L(K) = \{ w \in \Sigma^* | f^{\text{inw}\rightarrow\rightarrow}(w) \in \Gamma_0 \} \) (Culik et al. 1996).

Example Consider a trellis automaton \( K = (\{a, b\}, \{a, b, A, B, X, Y\}, \{X\}, f) \) where

\[
\begin{align*}
    f(a,a) &= A, & f(a,b) &= X, & f(b,a) &= Y, & f(A,A) &= A, \\
    f(A,B) &= X, & f(A,X) &= A, & f(B,B) &= B, & f(X,B) &= B
\end{align*}
\]

and for any other pair \((x, y), f(x, y) = Y\). Note that \( X \) is the only accepting symbol, while \( Y \) is a trap symbol to behave in such a way that once \( Y \) is introduced, it leads to a rejecting computation of an input. Figure 4 illustrates examples of both accepting and rejecting computations in \( K \). It is easy to see that \( L(K) = \{a^n b^n | n \geq 1\} \).

We demonstrate that \( K \) can be simulated by a network of SN P modules with two input neurons and one output neuron.

Consider an SN P module:

\[ \Pi_f = (\{a\}, \sigma_{in1}, \sigma_{in2}, \sigma_{out}, \sigma_1, \ldots, \sigma_{p+1}, \text{syn}, \{in1, in2\}, \{out\}) \]

where \( p \) is the cardinality of \( \Gamma \), and the details of the module is given in Fig. 5a.

Note that \( p \) is a given fixed number, and that for each \( 1 \leq i \leq p, qi \) in \( \Gamma \) is represented by \( a^i \).

The module \( \Pi_f \) works as follows. Suppose that, in \( t \)-th step, neuron in1 (in2) takes as input \( a^i (a^j) \) from input 1 (input 2, respectively). Then, \( a^i \) of in1 is passed through neuron \( (p+1) \) in \( (t+1) \)-th step and will reach neuron “out” in \( (t+2) \)th step, while \( a^j \) of in2 is

![Fig. 4](image-url) (a) An accepting computation of aaabbb; (b) A rejecting computation of aaabb
distributed to all neurons 1 through \( p \) in \( (t + 1) \)th step, and the sum of \( p \) copies of it (i.e., \( a'^p \)) will reach neuron “out” in \( (t + 2) \)th step.

On the other hand, in \( t \)th step neuron “out” contains no spike and is ready to receive both \( a' \) from neuron \( (p + 1) \) and the total amount of \( a'^p \) from neurons 1 through \( p \), where neuron \( (p + 1) \) will emit \( a' \) and each neuron \( i \) (\( 1 \leq i \leq p \)) will emit \( a'_i \) in \( (t + 1) \)th step. Therefore, in \( (t + 2) \)th step, neuron “out” can fire and emit \( a_k \) to the environment, by fully exhausting \( (i + jp) \) spikes and leaving no spike there. Hence, one state transition \( f(q_i, q_j) = q_k \) is simulated by \( \Pi_f \) in three steps.

In order to simulate the global process of computation in \( K \), we have only to allocate \( \Pi_f \) to all nodes of the underlying network structure of \( K \) (Fig. 5b) so that the function \( f \) is replaced by \( \Pi_f \). Let \( NW(\Pi_f) \) be such a trellis network consisting of \( \Pi_f \)s.

Let \( w = q_{i_1}, \ldots, q_{i_r} \) in \( \sum^* \) be an input which is simultaneously fed, in the form \( a_1 \ldots a^n \), from the frontier of the network (shown in Fig. 5b). Then, an input \( w \) is accepted by \( K \) iff \( j^{h\Pi_f}(w) \) is in \( \Gamma_0 \) iff after \( 3(|w| - 1) \) steps the module \( \Pi_f \) placed at the root of \( NW(\Pi_f) \) produces as output \( a^h \) representing some \( q_h \) in \( \Gamma_0 \).

**Theorem 10** Any trellis automaton \( K \) with the transition function \( f \) can be simulated by a trellis network \( NW(\Pi_f) \) for some SN P module \( \Pi_f \).

### 6 Conclusions

In this work, we have studied some topics related to SN P systems. Firstly, we presented the transition diagram, associated with a SN P system, as a tool to formally verify that such systems solve a given problem. The methodology is based on the information, encoded by the diagram, about some invariant formulae of the evolution.

Secondly, different characterizations of the power of SN P systems with an extended form of spiking rules were presented. Specifically, we focused on asynchronous (with or without delays), sequential SN P systems and we also used the models as language generators.

Finally, we have started a new research direction to explore the potential computing power of SN P systems with IO mode. It is known that the class of homogeneous trellis

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**Fig. 5** (a) SN P module \( \Pi_f \) simulating a transition function \( f \) of trellis automaton \( K \): \( \Pi_f \) takes \( q_i \) (represented by \( a' \)) for input 1 and \( q_j \) (represented by \( a'^p \)) for input 2, and produces as output \( q_k \) (represented by \( a'^k \)). Thus, one transition process \( f(q_i, q_j) = q_k \) is simulated. (b) The underlying network structure of \( K \)

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languages contains the class of linear languages and is closed under Boolean operations, and is also recognized in time \(O(n^2)\) by Turing machines (Culik et al. 1986; Ibarra and Kim 1984). Furthermore, as mentioned earlier, the class of systolic tree languages can be simulated in terms of networks of SN P modules in a natural manner.

In these respects, there remains much to be investigated on SN P modules \(\Pi\) and their networks \(NW(\Pi)\) with systolic structures: how far their computing power goes beyond regularity, and their closure and decidability properties, and so on.

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