Basic concepts of Lorentz symmetry and Minkowsky isogeometry by using MCIM model

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ABSTRACT

At 1978, R.M. Santilli proposed to generalize the conventional Lie theory by using isotopies. To do it, he considered that the basic unit of any mathematical structure can depend on external factors (such as position, speed, acceleration, time, temperature or density): \( \tilde{f} = f(\tilde{x}, \tilde{\dot{x}}, ..., \mu, \tau, ...) \). In this way, Santilli obtains a more general Lie theory, based on nonassociative, nonlinear and nonlocal-integral systems, which allows him to identify between themselves Euclidean, Riemannian and Minkowskian spaces, when working with extended particles or high energies into unusual physical conditions in exterior or interior dynamic systems. In the 80's and 90's, several mathematicians and physicists have investigated on Lie-Santilli's isotheory. Particularly, Lorentz symmetry and Minkowsky isogeometry have been studied by Santilli and A.K. Aringazin in the 90's. We propose to revise some of these concepts by using the MCIM isotopic model studied at 2001. So, we will need to generalize the construction of isovectorspaces and the isodifferential calculus.
Introduction

Statement of the problem

The mathematics generally used in quantitative sciences in the 20th century were based on ordinary fields with characteristic zero, a trivial unit $I = +1$ and an ordinary associative product $a \times b$ between generic quantities $a, b$ of a structure $(E, \times)$, such as matrices, vector fields, etc. Such a mathematics is known to be linear, local-differential and Hamiltonian, thus solely representing a finite number of isolated point-particles with action-at-a-distance forces derivable from a potential. Such a mathematics was proved to provide an exact and invariant representation of planetary and atomic systems as well as, more generally, of all the so-called exterior dynamical systems in which all constituents can be well approximated as being point-like.

By contrast, the great majority of systems in the physical reality are nonlinear, nonlocal and not entirely representable with a Hamiltonian in the coordinates of the experimenter. This is the case for all systems historically called interior dynamical systems, such as the structure of: planets; strongly interacting particles (such as protons and neutrons); nuclei; molecules; stars; and other systems. The latter systems cannot be consistently reduced to a finite number of isolated point-particles. Therefore, the mathematics so effective for exterior systems is only approximate at best for interior systems.

Santilli’s isotheory

Santilli’s isotopies $[5]$ (1978) constitute a new branch of mathematics characterized by axiom-preserving isotopic lifting of units, products, numbers, fields, topologies, geometries, algebras, groups, etc., with numerous novel applications in physics, chemistry and other quantitative sciences. It was specifically built for the invariant representation of nonlinear, nonlocal and non-Hamiltonian systems. Santilli’s fundamental isotopy was that of the basic unit that was lifted from the trivial value $I = +1$ to a matrix or operator, nowhere singular, Hermitian and positive-definite, but that possesses an otherwise unrestricted, generally nonlinear, nonlocal and non-Hamiltonian functional depend on all needed local variables, such as time $t$, coordinates $x$, velocities $v$, density $\mu$, temperature $\tau$, etc.

$$I \rightarrow \hat{I}(t, e, \mu, \tau, \ldots)$$

Jointly, Santilli lifted the conventional numbers and the associative product $a \times b$ with unit $I$ into the new forms $\hat{a} = a \star \hat{I}$ and $\hat{a} \times \hat{b} = a \times T \times b$, with unit $\hat{I} = I^{-1}$. So, the representation of interior systems via Santilli’s isomathematics requires the knowledge of two quantities, the conventional Hamiltonian $H$ for the representation of conventional linear, local and potential forces, and the isounit $\hat{I}$ for the representation of all nonlinear, nonlocal and non-Hamiltonian effects.
In 2001, Falcón and Nuñez [2] generalized the isotopic model proposed
by Santilli in 1978 although this generalization put stress on the use of several
*-laws and isoalgebras as operations existing in the initial mathematical
structure. Such a model, which from now on will be called MCIM (isoprod
construction model based on the multiplication), was later generalized in [3]
and [4].

Particularly, it is defined an isotopy or isotopic lifting as a corre-
respondence (not necessarily a map, in general) between a mathematical struc-
ture \((E, +, \times)\) and another \((\overline{E}, \overline{+}, \overline{\times})\) (isostructure), such that it verifies the
same properties, by using a general set \((V, *, \star)\) and a set of external factors
\(F = \{t, \mu, \tau, \ldots\}\), such that:

\[
\begin{align*}
I & \rightarrow \overline{I}; \quad E \rightarrow \overline{E}: x \rightarrow \overline{x} = x \ast \overline{(F_0)}; \quad T = \overline{T}; \\
V & = E \cup \overline{E} \cup \{T\} \cup E_T; \quad E_T = \{a_T - a \ast T: a \in \overline{E}\}; \\
\Phi_{\ast} : F \times F & \rightarrow F : (F_\alpha, F_\beta) \rightarrow \Phi_{\ast}(F_\alpha, F_\beta);
\end{align*}
\]

\[
\begin{align*}
(a \ast \overline{(F_0)} \overline{T}(b \ast \overline{(F_0)}) =
\overline{\left[(a \ast \overline{\Phi}(F_0)) \ast T(F_0) \right]} \ast \overline{\Phi_{\ast}(F_\alpha, F_\beta)}; \\
(a \ast \overline{\Phi}(F_0)) \overline{T}(b \ast \overline{\Phi}(F_0)) =
\overline{\left[(a \ast \overline{\Phi}(F_0)) \ast T(F_0) \right]} \ast \overline{\Phi_{\ast}(F_{\alpha}, F_{\beta})};
\end{align*}
\]

where, fixed \(F_0 \in F\):

\[
\begin{align*}
\overline{\Phi}(F_0) \ast T(F_0) = I = T(F_0) \ast \overline{\Phi}(F_0) \\
\left[(a \ast \overline{\Phi}(F_0)) \ast T(F_0) \right] \ast \overline{\Phi_{\ast}(F_{\alpha}, F_{\beta})} = a \ast \overline{\Phi}(F_0)
\end{align*}
\]

By the other way, an isotopic lifting of the structure \(E\) will be injec-
tive if \(a = b\), for all \(a, b \in E\) such that \(\overline{a} = \overline{b}\) and it will be compatible with
respect to a law \(\circ\) on \(E\) if for all \(a \ast \overline{\Phi}(F_0), b \ast \overline{\Phi}(F_0) \in \overline{E}\) we have:

\[
\left(a \ast \overline{\Phi}(F_0)\right) \overline{T}(b \ast \overline{\Phi}(F_0)) = (a \circ b) \ast \overline{\Phi_{\ast}(F_{\alpha}, F_{\beta})}
\]

All this construction must be made for every mathematical structure in use.
The first significant application of the isotopies of the unit was that for the liftings of conventional numbers and fields. The second one was the lifting of the conventional vector and metric spaces, first presented in paper [6] of 1983. The Euclidean and Minkowskian isogeometries were studied in details in monograph [9].

However, all these concepts must be revised if we use the MCIM isotopic model:

### Isovector spaces

Let be given a \((\mathbb{R}, + \times)\)-space \((U, +, \circ)\). By using the MCIM isotopic level, we can obtain an isofield \((\mathbb{R}, +, \times)\), with the elements \(+, \times, \circ\) and \(\tilde{F} = \tilde{F}(F)\).

Similarly, we can use other elements \(\diamond, \tilde{S}, \Box, \Diamond, \tilde{P} = \tilde{P}(F') = T^{m-'}\phi_+ \text{ and } \Phi_*\), such that:

\[
\tilde{X} = X \Diamond \tilde{P}(F'_X);
\]

\[
(X \Diamond \tilde{P}(F'_X)) \Box (Y \Diamond \tilde{P}(F'_Y)) = (X \circ Y) \Diamond \tilde{P}(\Phi_+ (F'_X, F'_Y));
\]

\[
(a \circ \tilde{F}(F_0)) \Box (X \Diamond \tilde{P}(F'_X)) = (a \Box X) \Diamond \tilde{P}(\Phi_+ (F_0, F'_X));
\]

\[
F' = [0, 2\pi];
\]

\[
x \Diamond \tilde{P}(x, t) = (x \cos(t), 0, x \sin(t))
\]

\[
(x, y) \Diamond \tilde{P}((x, y), t) = (x \cos(t), y, x \sin(t))
\]
Let consider a $\mathbb{R}$-space $U$ $n$-dimensional, endowed with a metric $g \in (M_n(\mathbb{R}), +, \cdot)$. We fix a isofield $\overline{\mathbb{R}}$ and so, we have $M_n(\overline{\mathbb{R}}, +, \cdot)$ too. If we want to get a $\overline{\mathbb{R}}$-space $\overline{U}$ endowed with a metric $g' \in M_n(\overline{\mathbb{R}})$ depending on external factors $F'$, we will find in the first place an isounit $\overline{P}$ such that $g' = g \oplus \overline{P}(F') - \overline{g}$.

(Santilli proposed $g' = T \cdot g$ in [6]).

Therefore, we have:

$$\overline{X} = X \oplus \overline{P}(F'_X) = X \oplus g^{-1} \oplus \overline{g}; \quad \overline{X} \overline{Y} = (\overline{X} g \overline{Y} \overline{F}) \oplus \overline{P}(\Phi(F_X, F_Y)).$$

Particularly, if $\oplus$ is associative, we will have the isometric $\overline{g}(\overline{X}, \overline{Y}) = g(X, Y)$, which depends on external factors.

$$\overline{x} = \begin{cases} 0, & \text{si } x = 0 \\ \frac{1}{x}, & \text{si } x \neq 0 \end{cases}; \quad \overline{y} = \begin{cases} 0, & \text{si } y = 0 \\ \frac{1}{y^3}, & \text{si } y \neq 0 \end{cases}.$$ 

$$(x, y) \oplus \overline{P}(x, y) = \left(\frac{1}{x}, \frac{1}{y^3}\right).$$

$r \equiv y = 2x; \quad \overline{r} \equiv y = \frac{x^3}{8}$

Straight isoline

1-dimensional isospheres
Minkowsky isogeometry

Let \((M(x, \eta, \mathbb{R}))\) be a Minkowskian \(\mathbb{R}\)-space, with local chart \(x = (x^b) = (\tau, x^a)\) (being \(\tau \in \mathbb{R}^3\) and \(x^4 = ct\)) and \(n\)-dimensional metric \(\eta = \text{diag}(1, 1, 1, -1)\). It will be \(x^2 = x_{12}^2 + x_{23}^2 + x_{34}^2 - x_{44}^2\).

In the absence of gravity, space-time is determined as a smooth flat manifold endowed with the Minkowsky metric \(\eta\). Any modification of \(\eta\) requires non canonical transformations, by using non invariant spacial-temporal units.

To solve it, Santilli proposes in [6] the Minkowskian \(\mathbb{R}\)-isospace \((\bar{M}(\bar{x}, \bar{\eta}, \mathbb{R}))\), by using:

\[
\bar{\eta} = \bar{\eta}(x, v, \mu, \tau, \ldots) = \text{diag}(\bar{\eta}_{12}, \bar{\eta}_{23}, \bar{\eta}_{34}) - T(x, v, \mu, \tau, \ldots) \cdot \eta;
\]

\[
x^2 = \sum_{i=1}^{4} x_i^4 \bar{\eta}_{ii} x_i^4.
\]

It represents all modifications of the Minkowsky metric as encountered, by example, in particle physics, conventional exterior gravitational line elements with \(\bar{\eta} = \eta(\bar{x}, \bar{v}, \mu, \tau, \ldots)\) (such as the full Schwarzschild line element [19]), all its possible generalizations for the interior problem, ...

For a better mathematical consistency, the MCM isotopic model proposes the isometric

\[
\bar{\eta} = \bar{\eta}(x, v, \mu, \tau, \ldots) = \bar{\eta} \cdot \bar{T}(x, v, \mu, \tau, \ldots); \quad \bar{\eta} = \bar{\eta}(x, v, \mu, \tau, \ldots);
\]

\[
x^2 = \sum_{i=1}^{4} x_i^4 \bar{\eta}_{ii} = (x \cdot T)^2 \bar{\eta}.
\]

It allows to use the law \(\cdot\) instead of \(-\) and to obtain more general structures. However, it is necessary to change the definition of Santilli’s mordifilential calculus [10] \((\bar{\eta} = \sum_{i=1}^{4} \bar{T}_{ii} dx^i)\) in the following form:

\[
\bar{\eta} = \sum_{i=1}^{4} dx^i \bar{T}_{ii}.
\]

Example

\[
\bar{\eta} = \bar{\eta} - \begin{pmatrix}
1 & -3 & 2 & 0 \\
-3 & 3 & -1 & 0 \\
2 & -1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
(x^1, x^2, x^3, x^4) = (x^1 - 3x^2 + 2x^3, -3x^1 + 3x^4 - x^3, 2x^1 - x^4)
\]

\[
\bar{\eta} = \bar{\eta} - \begin{pmatrix}
1 & -3 & 2 & 0 \\
-3 & 3 & -1 & 0 \\
2 & -1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\frac{\partial}{\partial x} = 2x^1 - dx^1; \quad \frac{\partial}{\partial x} = dx^4.
\]
Transformation group of the space-time which leaves
the space-time interval, \( ds^2 = dx^\alpha \eta_{\alpha\beta} dx^\beta \), invariant is called
Lorentz group. Besides, the Lorentz symmetry is one of the
fundamental symmetries of physical theories. It is correct
into ordinary conditions, but not with extended particles,
high energies or unusual physical conditions [1], such as
non-linear dependence.

To solve it, we can use the Minkowsky isogeometry.
Non-Minkowskian part of the isometric is assumed to de-
scribe geometrically local physical properties of the space-
time, such as non homogeneity, deformations, resistance,
anisotropy and velocity dependence.

In this way, Lorentz isosymmetry was introduced by
Santilli in [6] and studied in detail in monograph [7], at the
classical level, and in monograph [9], at the operator level.
It was conceived to study the global symmetries of gravity.

Particularly, he defines the isolatedentz symmetries
as the isotransformations group:

\[
x' = \widehat{\Lambda} x = \widehat{\Lambda} \ast T \ast x;
\]

\[
\widehat{\Lambda}^\ast \tilde{\eta} \widehat{\Lambda} = \tilde{\eta}^{-1} = \tilde{\Lambda} x = \widehat{\Lambda} \ast T \ast x.
\]

These transformations are formally linear and local
on the Minkowskian isospace \( \widehat{M} \), but generally non-linear
and non-local on the conventional space \( M \). They provide
methods for the explicit construction of (generally nonlinear
but local) symmetries of conventional gravitational metrics,
such as Schwarzschild’s metric. Besides, Einstein’s grav-
itation or any other gravitational theory (not necessarily
Riemannian) with metric \( \tilde{\eta} = T \eta \) (\( det(T) > 0 \)), admits the
conventional Lorentz symmetry as a global isotopic sym-
metry.

If we use the MCIM isotopic model, isotransfor-
mations group become the following:

\[
x' = \widehat{\Lambda} x = \Lambda \ast \overline{(x \ast T)}.
\]

It allows to obtain more general structures, because
the law \( \ast \) can be non associative or non commutative.
Minkowsky isogeometry by using MCIM

Local, linear, associative, commutative...

Non local, non linear, non associative, non commutative...

Non local, non linear, associative, commutative...


