Determinants of Latin squares of a given pattern

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Abstract
Cycle structure of autotopisms of Latin squares determine all possible patterns of this kind of design. Moreover, given any isotopism, the number of Latin squares containing it in its autotopism group only depends on the cycle structure of this isotopism. This number has been studied in [2] for Latin squares of order up to 7, by following the classification given in [3]. Specifically, regarding each symbol of a Latin square as a variable, any Latin square can be seen as the vector space associated with the solution of an algebraic system of polynomial equations, which can be solved using Gröbner bases, by following the ideas implemented by Bayer [2] to solve the problem of n-colouring a graph. However, computations for orders higher than 7 have shown to be very difficult without using some other combinatorial tools. In this sense, we will see in this paper the possibility of studying the determinants of those Latin squares related to a given cycle structure. Specifically, since the determinant of a Latin square can be seen as a polynomial of degree n in n variables, it will determine a new polynomial equation that can be included into the previous system. Moreover, since determinants of Latin squares of order up to 7 determine their isotopic classes [6], we will study the set of isotopic classes of Latin squares of these orders related to each cycle structure.

Introduction and notation

A Latin square L of order n is an n × n array with elements chosen from a set of n distinct symbols [this paper, it will be the set \{1,2,3,...,n\}] such that each symbol occurs precisely once in each row and each column. The set of Latin squares of order n is denoted by LS(n).

Given L = (Lij) ∈ LS(n), the orthogonal representation of L is the set of n³ triples \((i,j,k)\) s.t. Lij = k. The permutation group on \{n\} by \(L\) is denoted by \(S_n\). 

The permutation group on \{n\} by \(L\) is denoted by \(S_n\). Every permutation \(\sigma \in S_n\) can be uniquely written as a composition of n pairwise-disjoint cycles, \(\sigma = C_1 \circ C_2 \circ ... \circ C_n\), where for all \(i \in \{1,2,3,...,n\}\), \(C_i = (i, \sigma(i), \sigma^2(i), ... , \sigma^{k_i}(i))\), with \(k_1 + k_2 + ... + k_n = n\). The cycle structure of \(\sigma\) is the sequence \(C_1, C_2,..., C_n\), when \(\sigma\) is the number cycle of length \(k\) for all \(i \in \{1,2,3,...,n\}\). Thus, \(L\) is the cardinal of the set of fixed points of \(T\). Fix(\(T\)) = \{i ∈ \{1,2,3,...,n\} | (i, \sigma(i)) = i\}. Given \(\sigma \in S_n\), one defines the conjugate Latin square \(L' = (L')_{ij}\) of \(L\) such that \((i, \sigma(i)) \in L\) then \((i, (L')_{\sigma(i)}\ell) \in L'\), then \((r_{\sigma(i)})T)_{\sigma(r)}(s_{\sigma(i)})T)_{(T)} \in L'\), where \(r_{\sigma(i)}\) gives the \(\sigma\) coordinate of \(T\), for all \(i \in \{1,2,3,...,n\}\). In this way, each Latin square \(L\) has six conjugate Latin squares associated with \(L = L' = L'' = L''' = L'''' = L'''''\). Given \(L = (Lij) \in LS(n)\), it is defined its associated matrix \(X_L\) which is obtained by replacing each element \(L_{ij}\) by the variable \(x_{ij}\). The determinant \(det(L)\) of \(L\) is the homogeneous polynomial of degree n in n variables \(det(X_L)\).

Given an isotopism \(\Theta \in \text{Aut}(L)\), let us consider the set \(det(L) - det(L)\) in \(L \in LS(n)\).

We will say that set \(B\) of polynomials in \(x_{ij}\) is a basis of \(det(L)\) if

i) Two different polynomials of \(B\) are not similar.

ii) Given \(p \in B\), there exists \(L \in LS(n)\) such that \(p \sim det(L)\).

iii) Given \(L \in LS(n)\), there exists \(p \in B\) such that \(det(L) - p\).

Lemma 1. Given \(0 \neq \Theta \in \text{Aut}(L)\), let \(B_1, B_2\) be two different basis of \(det(L)\). Both basis have the same number of elements.

The number of elements of any basis of \(det(L)\) will be called its order and it will be denoted by \(\text{deg}(det(L))\). Since we are interested in those autotopisms \(\Theta \in \text{Aut}(L)\) such that \(\Delta(L) > 0\), it must be \(\text{deg}(det(L)) \geq 1\). Moreover, it is the number of different classes in the quotient set \(det(L)/\sim\).

Proposition 1. Let \((I_{1}, I_{2}, I_{3}) \in C(n)\) and let us consider \(I_{1}, I_{2}, I_{3} \in C(I_{1}, I_{2}, I_{3})\). There exists \(1 \leq \text{deg}(det(L_{I_1})) = \text{deg}(det(L_{I_2})) = \text{deg}(det(L_{I_3})) \leq 2\) as a consequence, a set of polynomials \(B\) in \(x_{ij}\) is a basis of \(det(L_{I_1})\), if and only if it is a basis of \(det(L_{I_2})\).

Theorem 1. The set of determinants B of a given set of patterns \(P = \{X_{I_1}, X_{I_2}, ..., X_{I_n}\}\) is a basis of \(det(L)\), for all \(L \in \text{Aut}(L)\). Moreover, the Latin squares of \(LS(n)\) belong to at least as different isotopic classes of \(\text{Aut}(L)\), as different patterns of \(P\) correspond to a different isotopic class of \(LS(n)\).

Proposition 2. Let \((\ell_{I_1}, \ell_{I_2}, ..., \ell_{I_n}) \leq 2\) be the set of Latin square whose associated matrices constitute the set of patterns \(P\) of the cycle structure \(L_{I_1}\) of an autotopism \(\Theta \in \text{Aut}(L)\). It is verified that \(\text{det}(L_{I_1}, X_{I_1}, X_{I_2}, ..., X_{I_n})\) is a set of patterns of the cycle structure \(L_{I_1}\).

References


