New Advances in the Study of Gröbner Bases and the Number of Latin Squares Related to Autotopisms

Jorge Martín-Morales
Department of Mathematics - U.M.A.
University of Zaragoza (Spain)
jorge@unizar.es

Raúl M. Falcón
Department of Applied Mathematics
University of Sevilla (Spain)
rafalgan@us.es

Abstract

Gröbner bases have been used in [4] to describe an algorithm that allows one to compute the number of Latin squares of order up to 7 having a given isotopism in their autotopism group. In order to improve the time of computation of this algorithm, we could use another combination between Gröbner bases and some combinatorial tools. Specifically, we add to the ideal of polynomials defining a Latin square L some polynomials related to the permutations of rows, columns and symbols corresponding to the given autotopism of L. Using this method we could compute the number of some Latin squares of order 8 having an isotopism in their autotopism group.

Introduction and notation

A Latin square L of order n is an n x n array with elements chosen from a set of n distinct symbols {1, ..., n}, such that each symbol occurs exactly once in each row and in each column. The set of Latin squares of order n is denoted by LS(n). A partial Latin square P of order n is an n x n array with elements chosen from a set of n symbols such that each symbol occurs at most once in each row and in each column. The set of partial Latin squares of order n is denoted by PLS(n).

For any given n ∈ N, we denote by [n] the set {1, 2, ..., n} and we assume that the set of symbols of any Latin square of order n is [n]. The symmetric group on n symbols is denoted by Sn. Given a permutation ρ ∈ Sn, it is defined the set of the fixed points Fix(ρ) = {i ∈ [n] | ρ(i) = i}. The cycle structure of a permutation ρ is the sequence l1, l2, ..., ln where ln is the number of cycles of length n, for all i ∈ [1, n]. On the other hand given L = (l1, l2, ..., ln) ∈ Sn, the orthogonal array representation of L is the set of n+1 triples (i, j, l) | i ≠ j ∈ [n]. The previous set is identified with itself, and then, it is written (i, j, l) ∈ L, for all i, j ∈ [n].

An autotopism of a Latin square L ∈ LS(n) is a triple Θ = (0, 0, 0, 0, 0, 0, 0) ∈ Tn = Sn × Sn × Sn. In this way, α, β and γ are permutations of rows, columns and symbols of L, respectively. The resulting square LΘ is also a Latin square and it is said to be isotopic to L. If L = (l1, l2, ..., ln), then L(α, β, γ) = (l1, l2, ..., ln) | i, j ∈ [n]. The cycle structure of an autotopism is (α, β, γ, γ, γ) ∈ Tn, where l1 is the cycle structure of L, for all i ∈ [α, β, γ, γ]. An isotopism which maps L to itself is an autotopism. The possible autotopisms can be obtained by the permutations of the partial Latin squares of order up to 3 were computed in [3].

The stabilizer subgroup of L in Tn is its autotopism group. Aut(L) = {Θ ∈ Tn | L(Θ) = L}. Given L in Tn, the set of all Latin squares L such that L(Θ) = L for all Θ ∈ Tn is denoted by L(Θ). If φ and θ are two autotopisms with the same cycle structure, then L(φ, θ) = L(θ, φ). For L ⊂ LS(n), the number |L(Θ)| is called the number of automorphisms of L.

Gröbner bases and Latin square autotopisms

Given a generic Latin square L = (lij) ∈ LS(n), we can consider the set of n+1 variables {xi, j | i ≠ j ∈ [n]}, where xi, j corresponds to the triple (i, j, l), L, for all i, j ∈ [n]. Now, we define: F(ρ) = \[ \prod_{i \in \text{Fix}(\rho)} (x_{i, i} - m) \]
G(ρ) = \[ \prod_{i \not\in \text{Fix}(\rho)} (x_{i, i} - m) \]

Now, given an autotopism Θ = (α, β, γ, γ, γ) ∈ Tn, let H(Θ) be a polynomial such that H(Θ) = γ(x) for all x ∈ [n]. Following the ideas implemented by Bax [2] (see also [1]) to solve the problem of n-coloring a graph, we have: Tn = \{ (i, j, l) ∈ Tn | l ∈ Fix(γ) \}

Theorem 2. Let S = \{ (i, j, l) | l ∈ Fix(γ) \} be the ideal generated by H(Θ), G(Θ), F(Θ) and \[ \prod_{i \in \text{Fix}(\gamma)} (x_{i, i} - m) \]
Then, F(L) = 1.

Let SΘ the set of all multi-indices corresponding to the set of all Latin squares L ∈ LS(n) having an isotopism in their autotopism group. The following set of multi-indices: \[ \{ (α, β, γ, γ, γ, α, β, γ, γ, γ) \} \]

Gröbner bases on block design

We have just seen that Algorithm 1 allows one to obtain all the elements of the Latin rectangle Rθ of Proposition 1. Next, let us observe that Rθ is indeed the union of k = 3 Latin rectangles:

Gröbner bases and Latin square autotopisms

Every permutation θ ∈ Sn can be uniquely written as a composition of pairwise disjoint cycles, \( \theta = C_{1} \cdots C_{k} \), where

\( n \in |C_{i}| = 1 \quad \text{ if } i \neq j \)
\( n \in |C_{i}| = 1 \quad \text{ if } i = j \)

\( i, j \leq n \quad \text{if } i \neq j \)
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Proposition 1. Let \( (α, β, γ, γ, γ) \in Tn \), be such that the products of the following Latin square of order \( n \leq 7 \) are known:

\[ \begin{bmatrix}
 x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} \\
 x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} \\
 x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} & x_{37} \\
 x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} & x_{47} \\
 x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} & x_{57} \\
 x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66} & x_{67} \\
 x_{71} & x_{72} & x_{73} & x_{74} & x_{75} & x_{76} & x_{77} 
\end{bmatrix} \]

Theorem 3. The set of zeros of the following ideal of \( \mathbb{Q}[x_{ij}] \) corresponds to the set L(Θ):

\[ \mathcal{I}(\Theta) = \langle x_{ij} - x_{kl} | \langle i, j, l, k \rangle \in Tn, \langle i, j, l \rangle \notin \text{Fix}(\gamma) \rangle \]

References


