0/1-Polytopes related to Latin squares autotopisms.*

R. M. Falcón
Department of Applied Mathematics I.
Technical Architecture School. University of Seville.
Avda. Reina Mercedes, 4A - 41012, Seville (Spain).
rafalgan@us.es

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Abstract

The set $LS(n)$ of Latin squares of order $n$ can be represented in $\mathbb{R}^{n^2}$ as a $(n - 1)^3$-dimensional 0/1-polytope. Given an autotopism $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{A}_n$, we study in this paper the 0/1-polytope related to the subset of $LS(n)$ having $\Theta$ in their autotopism group. Specifically, we prove that this polyhedral structure is generated by a polytope in $\mathbb{R}^{{\delta}(\alpha, \beta, \gamma) \cdot n^2 + \delta \cdot n^2}$, where $n_\alpha$ and $n_\beta$ are the number of cycles of $\alpha$ and $\beta$, respectively, and $\delta$ is the number of fixed points of $\delta$, for all $\delta \in \{\alpha, \beta, \gamma\}$. Moreover, we study the dimension of these two polytopes for Latin squares of order up to 9.

Key words: 0/1-polytope, Latin Square, Autotopism group.

1 Introduction

A 0/1-polytope [9] in $\mathbb{R}^d$ is the convex hull $P$ of a finite set of points with 0/1-coordinates. Equivalently, it is a polytope with all its vertices in the vertex set of the unit cube $C_d = [0,1]^d$. Thus, if we consider these vertices as the column vectors of a matrix $V \in \{0,1\}^{d \times n}$, it is verified that $P = \text{conv}(V) = \{V \cdot (x_1, x_2, ..., x_n)^t \mid x_i \geq 0, i \in [n] \text{ and } \sum_{i=1}^{n} x_i = 1\}$, where $[n]$ will denote from now on the set $\{1, 2, ..., n\}$. The dimension of $P$ is the maximum number of affinely independent points in $P$ minus 1. Permuting coordinates and switching (replacing $x_i$ by $1 - x_i$) coordinates transform 0/1-polytopes into 0/1-polytopes. Two 0/1-polytopes are said to be 0/1-equivalent if there exists a sequence of the two previous operations transforming one of them into the other one. In combinatorial optimization there are several examples of 0/1-polytopes like the salesman polytope [8], the cut polytope [2] or the Latin square polytope [3]. In this paper, we are interested in the last one, which appears in the 3-dimensional planar assignment problem (3PAP$_n$):

$$\min \sum_{i \in I, j \in J, k \in K} w_{ijk} \cdot x_{ijk}, \text{ s.t. } \begin{cases} \sum_{i \in I} x_{ijk} = 1, \forall j \in J, k \in K. \\
\sum_{j \in J} x_{ijk} = 1, \forall i \in I, k \in K. \\
\sum_{k \in K} x_{ijk} = 1, \forall i \in I, j \in J. \\
x_{ijk} \in \{0,1\}, \forall i \in I, j \in J, k \in K. \end{cases} \quad (1)$$

where \( w_{ijk} \) are real weights and \( I, J, K \) are three disjoint \( n \)-sets.

Euler et al. [3] observed that there exists a 1-1 correspondence between the set \( LS(n) \) of Latin squares of order \( n \) and the set \( FS(n) \) of feasible solutions of the 3PAP\(_n\). Specifically, a Latin square \( L \) of order \( n \) is an \( n \times n \) array with elements chosen from a set of \( n \) distinct symbols such that each symbol occurs precisely once in each row and each column. From now on, we will assume \( |n| \) as this set of symbols. Given \( L = (i_{ij}) \in LS(n) \), the orthogonal array representation of \( L \) is the set of \( n^2 \) triples \( \{(i, j, l_{ij}) \mid i, j \in [n]\} \). So, by taking \( I = J = K = [n] \) and by considering the lexicographical order in \( I \times J \times K \), it can be defined the 1-1 correspondence \( \Phi : LS(n) \to FS(n) \subseteq \mathbb{R}^{n^3} \), such that, given \( L = (i_{ij}) \in LS(n) \), it is \( \Phi(L) = (x_{111}, x_{112}, \ldots, x_{1mn}, x_{211}, \ldots, x_{nmn}) \), where \( x_{ijk} = \begin{cases} 1, & \text{if } l_{ij} = k, \\ 0, & \text{otherwise.} \end{cases} \) Moreover, if \( A \) is the constraint matrix of the system of equations (1), it is defined the Latin square polytope, \( P_{LS(n)} = \text{conv}\{FS(n)\} = \text{conv}\{x \in \{0, 1\}^{n^3} \mid A \cdot x = e\} \), where \( e = (1, \ldots, 1)^T \) with \( 3 \cdot n^2 \) entries. Thus, every point of \( P_{LS} \cap C_{n^3} \) is a Latin square of order \( n \) and vice versa. By obtaining the minimal equation system for \( P_{LS} \), Euler et al. proved that this polytope is \((n - 1)^3\)-dimensional and they gave some general results about its facial structure.

In this paper, we are interested in obtaining a similar construction than the above one, in the case of adding some extra conditions to the 3PAP\(_n\). Specifically, we want to study those 0/1-polytopes related to Latin squares having some symmetrical restrictions. To expose the problem, some previous considerations are needed: The permutation group on \([n]\) is denoted by \( S_n \). Every permutation \( \delta \in S_n \) can be uniquely written as a composition of pairwise disjoint cycles, \( \delta = C_{l_1}^\delta \circ C_{l_2}^\delta \circ \cdots \circ C_{l_{\ell_n}}^\delta \), where for all \( i \in [n] \), one has \( C_i^\delta = (\ell_1^\delta, \ell_2^\delta, \ldots, \ell_{l_i^\delta}^\delta) \), with \( l_{i,1}^\delta = \min_j \{c_{i,j}^\delta\} \). The cycle structure of \( \delta \) is the sequence \( l_\delta = (l_1^\delta, l_2^\delta, \ldots, l_{\ell_n}^\delta) \), where \( l_i^\delta \) is the number of cycles of length \( i \) in \( \delta \), for all \( i \in [n] \). Thus, \( l_\delta \) is the cardinal of the set of fixed points of \( \delta \). \( \text{Fix}(\delta) = \{i \in [n] \mid \delta(i) = i\} \). An isotopism of a Latin square \( L = (i_{ij}) \in LS(n) \) is a triple \( \Theta = (\alpha, \beta, \gamma) \in I_n = S_n \times S_n \times S_n \). In this way, \( \alpha, \beta \) and \( \gamma \) are permutations of rows, columns and symbols of \( L \), respectively. The resulting square \( L^\Theta = \{(\alpha(i), \beta(j), \gamma(i_{ij})) \mid i, j \in [n]\} \) is also a Latin square. The cycle structure of \( \Theta \) is the triple \( (l_\alpha, l_\beta, l_\gamma) \).

An isotopism which maps \( L \) to itself is an autotopism. The stabilizer subgroup of \( L \) in \( I_n \) is its autotopism group, \( \mathfrak{A}(L) = \{\Theta \in I_n \mid L^\Theta = L\} \). The set of all autotopisms of Latin squares of order \( n \) is denoted by \( \mathfrak{A}_n \). Given \( \Theta \in \mathfrak{A}_n \), the set of all Latin squares \( L \) such that \( \Theta \in \mathfrak{A}(L) \) is denoted by \( LS(\Theta) \) and the cardinality of \( LS(\Theta) \) is denoted by \( \Delta(\Theta) \). Specifically, if \( \Theta_1 \) and \( \Theta_2 \) are two autotopisms with the same cycle structure, then \( \Delta(\Theta_1) = \Delta(\Theta_2) \). The possible cycle structures of the set of non-trivial autotopisms of Latin squares of order up to 11 were obtained in [4].

Gröbner bases were used in [5] to describe an algorithm that allows one to obtain the number \( \Delta(\Theta) \) in a computational way. This algorithm was implemented in SINGULAR [7] to get the number of Latin squares of order up to 7 related to any autotopism of a given cycle structure. Specifically, the authors followed the ideas implemented by Bayer [1] to solve the problem of an \( n \)-colouring a graph, since every Latin square of order \( n \) is equivalent to an \( n \)-coloured bipartite graph \( K_{n,n} \). More recently, Falcón and Martín-Morales [6] have studied the case \( n > 7 \) by implementing in a new algorithm the 1-1 correspondence between the 3PAP\(_n\) and the set \( LS(n) \). As an immediate consequence, the set of vertices of \( C_{n^3} \) related to \( LS(\Theta) \) can be obtained.

In Section 2, given \( \Theta \in \mathfrak{A}_n \), we study the set of constraints which can be added to
the 3PAP\textsubscript{n} to get a set of feasible solutions equivalent to the set LS(\Theta). In Section 3, we define the 0/1-polytope in \mathbb{R}^n related to LS(\Theta). Moreover, we prove the existence of a 0/1-subpolytope of the previous one which can generate it. We see that these two polytopes do not depend on the autotopism \Theta but on the cycle structure of the autotopism. Finally, we study the dimensions of these polytopes and we give a classification for polytopes related to autotopisms of Latin squares of order up to 9.

2 Constraints related to a Latin square autotopism

Given a autotopism \(\Theta = (\alpha, \beta, \gamma) \in \mathfrak{A}_n\), let \((1)_{\Theta}\) be the set of constraints obtained by adding to \((1)\) the \(n^3\) constraints:

\[
x_{ijk} = x_{\alpha(i)\beta(j)\gamma(k)}, \forall i \in I, j \in J, k \in K. \quad (1)_\Theta
\]

The following results hold:

**Theorem 2.1** There exists a 1-1 correspondence between LS(\Theta) and the set FS(\Theta) of feasible solutions related to a combinatorial optimization problem having \((1)_{\Theta}\) as the set of constraints.

**Proof.** It is enough to consider the restriction to LS(\Theta) of the correspondence \(\Phi\) between LS(n) and FS(n), because then, given \(L = (l_{ij}) \in LS(n)\), it is verified that \(L \in LS(\Theta)\) if and only if, for all \(i, j, k \in [n]\): \(l_{i,j} = k \iff l_{\alpha(i),\beta(j)} = \gamma(k)\). But this last condition is equivalent to say that \(x_{ijk} = 1\) if and only if \(x_{\alpha(i)\beta(j)\gamma(k)} = 1\). That is to say, \(x_{ijk} = x_{\alpha(i)\beta(j)\gamma(k)}\). \(\Box\)

**Corollary 2.2** Every feasible solution of FS(\Theta) verifies that \(x_{ijk} = 0\), for all \(i, j, k \in [n]\) such that one of the following assertions is verified:

\[a)\] \(i \in \text{Fix}(\alpha), j \in \text{Fix}(\beta)\) and \(k \notin \text{Fix}(\gamma)\).

\[b)\] \(i \in \text{Fix}(\alpha), k \in \text{Fix}(\gamma)\) and \(j \notin \text{Fix}(\beta)\).

\[c)\] \(j \in \text{Fix}(\beta), k \in \text{Fix}(\gamma)\) and \(i \notin \text{Fix}(\alpha)\).

**Proof.** From the conjugacy of rows, columns and symbols in Latin squares, it is enough to consider assertion \((a)\). So, let us consider a feasible solution of FS(\Theta) such that \(x_{ijk} = 1\), for some \(i, j, k \in [n]\) verifying assertion \((a)\). From Theorem 2.1, there exists an unique \(L = (l_{ij}) \in LS(\Theta)\) being equivalent with such a feasible solution. Specifically, it must be \(l_{i,j} = k\) and therefore, \(k = l_{i,j} = l_{\alpha(i),\beta(j)} = \gamma(l_{i,j}) = \gamma(k)\), which is a contradiction, because \(k \notin \text{Fix}(\gamma)\). \(\Box\)

Let \(S_{\text{Fix}(\Theta)}\) be the set of triples \((i, j, k) \in [n]^3\) such that one of the assertions of Corollary 2.2 is verified. Since the \(1\alpha \cdot 1\beta \cdot (n - 1\gamma) + 1\alpha \cdot 1\gamma \cdot (n - 1\beta) + 1\beta \cdot 1\gamma \cdot (n - 1\alpha)\) variables \(x_{ijk}\) related to \(S_{\text{Fix}(\Theta)}\) are all nulls, we can reduce the number of variables of the system \((1)_{\Theta}\) in order to obtain a 1–1 correspondence between FS(\Theta) and LS(\Theta). Given \(s, t \in [n]\), the following sets will be useful:

\[
S^{(1,s,t)}_{\text{Fix}(\Theta)} = \{i \in [n] \mid (i, s, t) \in S_{\text{Fix}(\Theta)}\}, \quad S^{(2,s,t)}_{\text{Fix}(\Theta)} = \{j \in [n] \mid (s, j, t) \in S_{\text{Fix}(\Theta)}\}.
\]
\[ S_{F_{\text{fix}}(\Theta)}^{(3, s, t)} = \{ k \in [n] \mid (s, t, k) \in S_{F_{\text{fix}}(\Theta)} \}. \]

Moreover, the symmetrical structure given by the autotopism \( \Theta \) can also be used to reduce the number of variables of \((1)_{\Theta}\). To see it, let us consider:

\[ S_{\Theta} = \left\{ (i, j) \mid i \in S_{\alpha}, j \in \begin{cases} [n], & \text{if } i \notin \text{Fix}(\alpha), \\
_{\beta}, & \text{if } i \in \text{Fix}(\alpha). \end{cases} \right\} \]

as a set of \((n_{\alpha} - l_{\alpha}^{1}) \cdot n + l_{\alpha}^{1} \cdot n_{\beta}\) multi-indices, where \( S_{\alpha} = \{ c_{i, 1}^{\alpha} \mid i \in [n_{\alpha}] \} \) and \( S_{\beta} = \{ c_{j, 1}^{\beta} \mid j \in [n_{\beta}] \} \). The following result is verified:

**Proposition 2.3** Let \( L = (l_{i, j}) \in LS(\Theta) \) be such that all the triples of the Latin subrectangle \( R_{L} = \{(i, j, l_{i, j}) \mid (i, j) \in S_{\Theta}\} \) of \( L \) are known. Then, all the triples of \( L \) are known. Indeed, given \( i, j \in [n] \), there exists an unique element \((i_{\Theta}, j_{\Theta}) \in S_{\Theta}\) such that \( l_{i, j} \) can be obtained starting from \( l_{i_{\Theta}, j_{\Theta}} \).

**Proof.** Let \((i, j, l_{i, j}) \in L\) be such that \( i > n_{\alpha} \) and \( r \in [n_{\alpha}] \) and \( u \in [\lambda_{\alpha}^{n}] \) be such that \( c_{i, u}^{\alpha} = i \). Then, \((\alpha_{1}^{-1}(i), \beta_{1}^{-1}(j)) \in S_{\Theta}\), and, therefore, \( l_{\alpha_{1}^{-1}(i), \beta_{1}^{-1}(j)} \) is known. Thus, \( l_{i, j} = \gamma_{m_{i, j}}(l_{\alpha_{1}^{-1}(i), \beta_{1}^{-1}(j)}). \)

Now, let \((i, j, l_{i, j}) \in L\) be such that \( i \in \text{Fix}(\alpha) \) and \( j > n_{\beta} \). Let \( s \in [n_{\beta}] \) and \( v \in [\lambda_{\beta}^{s}] \) be such that \( c_{s, v}^{\beta} = j \). From the hypothesis, the triple \((i, c_{s, v}^{\beta}, l_{i, c_{s, v}^{\beta}})\) is known. Thus, \( l_{i, j} = \gamma_{m_{s, v}}(l_{i, c_{s, v}^{\beta}}). \)

The final assertion is therefore an immediate consequence of the election of the cyclic decomposition of \( \Theta \). Specifically, it is verified that \((i_{\Theta}, j_{\Theta}) = (\alpha_{m_{i, j}}(i), \beta_{m_{i, j}}(j))\), where \( m_{i, j} = \min\{t \geq 0 \mid (\alpha^{t}(i), \beta^{t}(j)) \in S_{\Theta}\} \).

Given \( i, j, k \in [n] \), let us define \( k_{\Theta} = \gamma^{m}(k) \), where \( m \in [n] \) is such that \((i_{\Theta}, j_{\Theta}) = (\alpha^{m}(i), \beta^{m}(j)) \in S_{\Theta}\). Thus, from the cyclic decomposition of \( \Theta \), let us observe that \((i_{\Theta}, j_{\Theta}, k_{\Theta}) = (\alpha^{t}(i), \beta^{t}(j), \gamma^{t}(k))_{\Theta}\), for all \( i, j \in [n] \) and for all \( t \in [n] \). The following result holds:

**Theorem 2.4** There exists a 1-1 correspondence between \( FS(\Theta) \) and the set of feasible solutions \( FS'(\Theta) \) of the following system of equations in \( d_{\Theta} = ((n_{\alpha} - l_{\alpha}^{1}) \cdot n^{2} + l_{\alpha}^{1} \cdot n_{\beta} \cdot n - (l_{\alpha}^{1} \cdot l_{\beta}^{1} + l_{\alpha}^{1} \cdot l_{\beta}^{1} + l_{\beta}^{1} \cdot l_{\alpha}^{1} + (n_{\alpha} - l_{\alpha}^{1}))) \) variables:

\[
\begin{align*}
\sum_{i \in [n]} x_{i(i_{\Theta}, j_{\Theta})k_{\Theta}} &= 1, \forall j, k \in [n]. \quad (2.1)_{\Theta} \\
\sum_{j \in [n]} x_{i(j_{\Theta}, j_{\Theta})k_{\Theta}} &= 1, \forall i, k \in [n]. \quad (2.2)_{\Theta} \\
\sum_{k \in [n]} x_{i(i_{\Theta}, j_{\Theta})k_{\Theta}} &= 1, \forall i, j \in [n]. \quad (2.3)_{\Theta} \\
\sum_{i \notin [n]} x_{ijk} &= 1, \forall (i, j, k) \in S_{\Theta} \times [n] \setminus S_{F_{\text{fix}}(\Theta)}. \quad (2.4)_{\Theta}
\end{align*}
\]

**Proof.** Let us define the map \( \Psi_{\Theta} : FS'(\Theta) \subseteq \mathbb{R}^{d_{\Theta}} \rightarrow FS(\Theta) \subseteq \mathbb{R}^{n^{3}} \), such that

\[
\Psi_{\Theta}((x_{ijk})_{(i,j,k) \in S_{\Theta} \times [n] \setminus S_{F_{\text{fix}}(\Theta) \times [n]}}) = (X_{uvw})_{(u,v,w) \in [n]^{3}} = \begin{cases} 0, & \text{if } (u,v,w) \in S_{F_{\text{fix}}(\Theta)}, \\
X_{uvw}, & \text{otherwise.} \end{cases}
\]

Thus, \( \Psi_{\Theta} \) is a 1-1 correspondence between \( FS'(\Theta) \) and \( FS(\Theta) \). Specifically, from Corollary 2.2 and Proposition 2.3, equations (1.1), (1.2) and (1.3) and conditions (1.4) in \( FS(\Theta) \) are
equivalent to (2.1)_{\Theta}, (2.2)_{\Theta}, (2.3)_{\Theta} and (2.4) in FS'(\Theta), respectively. Now, let us consider
\((x_{ijk})_{(i,j,k)\in S_\Theta\times [n]\setminus S_{\text{Fix}(\Theta)}} \in FS'(\Theta)\) and \((X_{uvw})_{(u,v,w)\in [n]^3} = \Psi_{\Theta}((x_{ijk})_{(i,j,k)\in S_\Theta\times [n]\setminus S_{\text{Fix}(\Theta)}})\).

Given \(u, v, w \in [n]\), it is verified that \(X_{uvw} = \begin{cases} 
0 = X_{\alpha(u)\beta(v)\gamma(w)}, & \text{if } (u,v,w) \in S_{\text{Fix}(\Theta)}; \\
x_{u\Phi_{\Theta}v\Phi_{\Theta}} = X_{\alpha(u)\beta(v)\gamma(w)}, & \text{otherwise}. 
\end{cases}\)

Therefore equations (1.5)_{\Theta} are also verified. \(\square\)

In general, many of the expressions of (2)_{\Theta} are the same equation and so, they are redundant. An immediate consequence of Theorem 2.4 is the following:

**Corollary 2.5** \(\Psi_{\Theta}^{-1} \circ \Phi_{|\text{LS}(\Theta)}\) is a 1-1 correspondence between LS(\Theta) and FS'(\Theta). \(\square\)

### 3 0/1-polytopes related to a Latin square autotopism

Given a autotopism \(\Theta \in \mathfrak{A}_n\), let \(A_{\Theta}\) and \(A'_{\Theta}\) be the constraint matrices of (1)_{\Theta} and (2)_{\Theta}, respectively. Let us define the following 0/1-polytopes:

\[P_{\text{LS}(\Theta)} = \text{conv}(FS(\Theta)) = \text{conv}(x \in \{0, 1\}^n \mid A_{\Theta} \cdot x = e_{\Theta}) \subseteq \mathbb{R}^n,\]

\[P'_{\text{LS}(\Theta)} = \text{conv}(FS'(\Theta)) = \text{conv}(x \in \{0, 1\}^n \mid A'_{\Theta} \cdot x = e'_{\Theta}) \subseteq \mathbb{R}^d,\]

where \(e_{\Theta} = (1, ..., 1)^t\) and \(e'_{\Theta} = (1, ..., 1)^t\) have \(3 \cdot n^2 + n^3\) and \(3 \cdot n^2\) entries, respectively. The following results hold:

**Corollary 3.1** Both 0/1-polytopes, \(P_{\text{LS}(\Theta)}\) and \(P'_{\text{LS}(\Theta)}\), have \(\Delta(\Theta)\) vertices.

**Proof.** It is enough to consider the 1-1 correspondences of Theorem 2.1 and Corollary 2.5. \(\square\)

**Theorem 3.2** \(\dim(P_{\text{LS}(\Theta)}) = \dim(P'_{\text{LS}(\Theta)}) \leq d_{\Theta} - \text{rank}(A'_{\Theta}).\)

**Proof.** The inequality is an immediate consequence of the definition of \(P'_{\text{LS}(\Theta)}\). Besides, from the definition of \(\Psi_{\Theta}\) given in the proof of Theorem 2.4, it is immediate to see that a set of \(m\) affinely vertices of \(P'_{\text{LS}(\Theta)}\) induces a set of \(m\) affinely vertices of \(P_{\text{LS}(\Theta)}\), because we can identify all the coordinates of the first ones in the second ones. So, \(\dim(P'_{\text{LS}(\Theta)}) \leq \dim(P_{\text{LS}(\Theta)})\).

Now, let \(\{V_1, ..., V_m\}\) be a set of \(m\) affinely independent vertices of \(P_{\text{LS}(\Theta)}\), where \(V_i = (v_{i,1}, ..., v_{i,n})\), for all \(i \in [m]\). From Theorem 2.4, \(V'_i = \Psi_{\Theta}^{-1}(V_i) = (v'_{i,1}, ..., v'_{i,d_{\Theta}})\) is a vertex of \(P'_{\text{LS}(\Theta)}\), for all \(i \in [m]\). Let us suppose that there exist \(\lambda_1, ..., \lambda_m \in \mathbb{R}\), such that \(\sum_{i=1}^{m} \lambda_i = 1\) and \(\sum_{i=1}^{m} \lambda_i \cdot v'_i = 0\). From the definition of \(\Psi_{\Theta}\), given \(j \in [n]\) non corresponding to a triple of \(S_{\text{Fix}(\Theta)}\), there exists \(k \in [d_{\Theta}]\), such that \(v_{i,j} = v'_{i,k}\), for all \(i \in [m]\). Thus, \(\sum_{i=1}^{m} \lambda_i \cdot V_i = 0\), which is a contradiction. Therefore, \(\dim(P_{\text{LS}(\Theta)}) \leq \dim(P'_{\text{LS}(\Theta)})\). \(\square\)

**Theorem 3.3** Let \((l_0, l_1, l_1)\) be the cycle structure of a Latin square autotopism and let us consider \(\Theta_1 = (\alpha_1, \beta_1, \gamma_1), \Theta_2 = (\alpha_2, \beta_2, \gamma_2) \in \mathfrak{A}_n(l_0, l_1, l_1)\). Then, \(P_{\text{LS}(\Theta_1)}\) and \(P_{\text{LS}(\Theta_2)}\) are 0/1-equivalents. Analogously, \(P'_{\text{LS}(\Theta_1)}\) and \(P'_{\text{LS}(\Theta_2)}\) are 0/1-equivalents.
Proof. Let us prove the first assertion, the other case follows analogously. So, since $\Theta_1$ and $\Theta_2$ have the same cycle structure, we can consider the isotopism $\Theta = (\sigma_1, \sigma_2, \sigma_3) \in \mathcal{I}_n$, where:

i) $\sigma_1(\ell_{ij}^{\alpha_1}) = \ell_{ij}^{\beta_1}$, for all $i \in [k_{\alpha_1}]$ and $j \in [\lambda_{\alpha_1}^1]$,

ii) $\sigma_2(\ell_{ij}^{\beta_1}) = \ell_{ij}^{\gamma_1}$, for all $i \in [k_{\beta_1}]$ and $j \in [\lambda_{\beta_1}^1]$,

iii) $\sigma_3(\ell_{ij}^{\gamma_1}) = \ell_{ij}^{\delta_1}$, for all $i \in [k_{\gamma_1}]$ and $j \in [\lambda_{\gamma_1}^1]$.

Let $L \in LS(\Theta_1)$ and $(x_{ijk})_{i,j,k \in [n]} = \Phi(L)$. From [5], we know that $L \in LS(\Theta_1)$ if and only if $L^\Theta \in LS(\Theta_2)$. Thus, if $(X_{ijk})_{i,j,k \in [n]} = \Phi(L^\Theta)$, then it must be $x_{ijk} = x_{\sigma_1(i)\sigma_2(j)\sigma_3(k)}$, for all $i, j, k \in [n]$. So, the permutation of coordinates $\pi(x_{ijk}) = x_{\sigma_1(i)\sigma_2(j)\sigma_3(k)}$ is a 1-1 correspondence between $FS(\Theta_1)$ and $FS(\Theta_2)$, which are the set of vertices of $P_{LS(\Theta_1)}$ and $P_{LS(\Theta_2)}$, respectively. Thus, $\pi$ transforms $P_{LS(\Theta_1)}$ into $P_{LS(\Theta_2)}$. \qed

From Theorem 3.3, the dimension of $P_{LS(\Theta)}$ and $P'_{LS(\Theta)}$ only depends on the cycle structure of $\Theta$. Moreover, since rows, columns and symbols have an interchangeable role in Latin squares and since affine independence does not depend on these interchanges, we can suppose that the cycles $\alpha, \beta$ and $\gamma$ of $\Theta$ verify that $n_\alpha \leq n_\beta \leq n_\gamma$. Thus, let us finish this paper by following the classification of all possible cycle structures given in [4], in order to show in Tables 1 and 2 the dimensions of all possible polytopes related to any autotopisms of order up to 9. Specifically, the exact dimension is shown when the set $LS(\Theta)$ is known. As an upper bound we show the difference between $d_\Theta$ and $rank(A_\Theta^*)$, which indeed can not be reached, as we can observe in Table 1. As a lower bound, we study the subsets of $LS(\Theta)$ given in [6].

References


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Table 1: Number of vertices and dimensions of polytopes related to \( \mathcal{A}_n \), for \( n \leq 7 \).


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Table 2: Number of vertices and dimensions of polytopes related to A_8 and A_9.