Isotopisms of Lie algebras.

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ABSTRACT. The distribution of algebras into equivalence classes is usually done according to the concept of isomorphism. However, such a distribution can also be done into isotopy classes. The concept of isotopy was explicitly introduced in 1942 by Abraham Adrian Albert [1] to classify non-associative algebras. In this paper we deal with the study of isotopisms of Lie algebras. The reasons for using both criteria, isotopisms and isomorphisms, to classify Lie algebras is due to that classifications by isotopisms are different from those by isomorphisms, which involves obtaining new information about these algebras. On a sake of example, we indicate some recent results obtained by ourselves, which are related to the distribution into isotopism and isomorphism classes of filiform Lie algebras over finite fields.

INTRODUCTION. Two algebras \( g \) and \( h \) are isotopic [1] if there exist three non-singular linear transformations \( f, g \) and \( h \) from \( g \) to \( h \) such that

\[
[f(u), g(v)] = h([u, v]), \quad \text{for all } u, v \in g. \tag{1}
\]

It is also said \( g \) to be an isotope of \( h \). The tuple \((f, g, h)\) is called an isotopism of algebras. It is said to be principal if \( h \) is the identity transformation. To be isotopic is an equivalence relation among algebras. If \( f = g = h \), then the algebras \( g \) and \( h \) are isomorphic. Isotopisms are therefore a generalization of isomorphisms that can be used to gather together non-isomorphic algebras. Since the original paper of Albert [1], it has been analyzed the isotopisms of a wide variety of types of algebras like division, Jordan, alternative or structural algebras. Nevertheless, there barely exists any result about isotopisms of Lie algebras, apart from the next two results of Albert and Bruck.

Preliminaries on Lie algebras.

An \( n \)-dimensional algebra \( g \) is a Lie algebra if its second inner law is bilinear and anti-commutative and satisfies the Jacobi identity

\[
J(u, v, w) = [u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0.
\]

for all \( u, v, w \in g \). The centralizer of a subset \( S \subseteq g \) is the set \( \text{Cent}_g(S) = \{ h \in g \mid [u, h] = 0 \text{ for all } u \in S \} \). Given \( m \leq n \), we define \( d_m(g) = \max(\dim \text{Cent}_g(S) \mid S \text{ is } m \text{-dimensional ideal}) \). The sequence \( d_m(g) = \{ d_1(g), \ldots, d_n(g) \} \) is an isotopism invariant of Lie algebras.

The lower central series of a Lie algebra \( g \) is defined as \( g^1 = g, g^2 = [g, g], \ldots, g^n = [g^{n-1}, g] \). A basis \( \{ e_1, \ldots, e_n \} \) of \( g \) is compatible with respect to its lower central series if \( g^2 = \langle e_2, \ldots, e_{n-1} \rangle, g^3 = \langle e_2, \ldots, e_{n-2}, e_1 \rangle, \ldots, g^n = 0 \).

The Lie algebra \( g \) is filiform if \( d_m(g) = n - m \), for all \( k \in \{ 2, \ldots, n \} \). We define a filiform basis of \( g \) as a compatible basis \( \{ e_1, \ldots, e_n \} \) respect to its lower central series such that either \( [e_1, e_i] = e_1 - e_i \), or \( [e_1, e_i] = e_{i-1} \), for \( 3 \leq i \leq n \). Every finite-dimensional filiform Lie algebra has a filiform basis. If the only brackets direct of zero are those of the form \( [e_1, e_i] = e_{i-1} \), then \( g \) is called model and is not isotopic to any other filiform Lie algebra of the same dimension. In fact, the model algebra is the only isomorphism (isotopism) class of filiform Lie algebras of dimension \( n \leq 4 \). For \( n = 5 \), there exist two isomorphism (isotopism) classes of filiform Lie algebras: the model algebra and that having an abelian basis satisfying the brackets \( [e_1, e_5] = e_2 \).

ISOTOPISMS OF FILIFORM LIE ALGEBRAS OVER FINITE FIELDS.

It is verified that

a) Given a six-dimensional filiform Lie algebra \( g \) over \( k \), there exist \( a, b, c \in k \) such that

\[
g \cong \overline{g}_{abc} \equiv \langle [e_1, e_1], [e_1, e_5] = e_i, \quad \text{for all } i > 1, [e_1, e_2] = a e_2, [e_1, e_4] = e_3, [e_2, e_3] = e_2 + e_3, [e_2, e_4] = e_3 + a e_4 \rangle.
\]

b) Given a seven-dimensional filiform Lie algebra \( g \) over a field \( K \) of characteristic distinct of two, there exist \( a, b, c, d \in k \) and a filiform basis such that

\[
g \cong \overline{g}_{abcd} \equiv \langle [e_1, e_2] = e_1, \quad \text{for all } i > 1, [e_1, e_3] = e_1, [e_1, e_4] = e_3, [e_2, e_3] = a e_2, [e_2, e_4] = e_3 + a e_4 \rangle.
\]

If the characteristic is two, then \( g \) can also be isomorphic to either

\[
\overline{g}_7 \equiv \langle [e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_2, e_3] = e_2, [e_2, e_4] = e_3, [e_3, e_4] = e_3 \rangle.
\]

or

\[
\overline{g}_7 \equiv \langle [e_1, e_2] = e_4, [e_2, e_3] = e_5, [e_3, e_4] = e_3 \rangle.
\]

References


In the last years, the concept of isotopy of Lie algebras has reappeared in the literature. In 2008, Jiménez-Gestal and Pérez-Izquierdo [6] study the relationship that exists between the isotopisms of a finite-dimensional real division algebra and the Lie algebra of its ternary derivations. Shortly after, Alli-son et al. [2, 3] study isotopes of a class of graded Lie algebras called Lie tori, but the notion of isotopism that they use is quite different from the conventional one. More recently, in 2014, Falcón et al. [5] return to the conventional notion of isotopy in order to study the distribution of filiform Lie algebras over finite fields into isotopism classes. In the rest of the paper, we expose precisely some results in this regard.