Partial Latin rectangle graphs and autoparatopism groups of partial Latin rectangles with trivial autotopism groups

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A B S T R A C T

An $r \times s$ partial Latin rectangle (l_{ij}) is an $r \times s$ matrix containing elements of $\{1, 2, \ldots, n\} \cup \{\cdot\}$ such that each row and each column contain at most one copy of any symbol in $\{1, 2, \ldots, n\}$. An entry is a triple (i, j, l_{ij}) with $l_{ij} \neq \cdot$. Partial Latin rectangles are operated on by permuting the rows, columns, and symbols, and by uniformly permuting the coordinates of the set of entries. The stabilizers under these operations are called the autotopism group and the autoparatopism group, respectively.

We develop the theory of symmetries of partial Latin rectangles, introducing the concept of a partial Latin rectangle graph. We give constructions of m-entry partial Latin rectangles with trivial autotopism groups for all possible autoparatopism groups (up to isomorphism) when: (a) $r = s = n$, i.e., partial Latin squares, (b) $r = 2$ and $s = n$, and (c) $r = 2$ and $s \neq n$.

1. Introduction

Let $[n] = \{1, 2, \ldots, n\}$. An $r \times s$ *partial Latin rectangle* is an $r \times s$ matrix containing symbols from $[n] \cup \{\cdot\}$ such that each row and each column contain at most one copy of any symbol in $[n]$. The cells containing the symbol $\cdot$ are considered *empty*. We define the *weight* of a partial Latin rectangle as the number of non-empty cells, which we will typically denote m. Let $\mathrm{PLR}(r, s, n; m)$ denote the set of $r \times s$ partial Latin rectangles on the symbol set $[n] \cup \{\cdot\}$ of weight m. A partial Latin rectangle with dimension parameters satisfying $r = s = n$ is called a *partial Latin square*. We define $\mathrm{PLS}(n; m) = \mathrm{PLR}(n, n, n; m)$.

Throughout this paper, partial Latin rectangles will be assumed to have (a) at least one entry in each row, (b) at least one entry in each column, and (c) at least one copy of each symbol.

1.1. Symmetries

An *entry* of a partial Latin rectangle $L = (l_{ij}) \in \mathrm{PLR}(r, s, n; m)$ is any triple (i, j, l_{ij}) with $i \in [r]$ and $j \in [s]$ and $l_{ij} \neq \cdot$. The set of all entries of L is called the *entry set*, denoted $E(L)$.

Let S_t denote the symmetric group on $[t]$. The group $\mathcal{P} = \mathcal{P}_{r,s,n} := (S_r \times S_s \times S_n) \rtimes S_3$ operates on Latin rectangles in $\mathrm{PLR}(r, s, n; m)$ with $\theta = (\alpha, \beta, \gamma; \delta) \in \mathcal{P}$ mapping the partial Latin rectangle $L = (l_{ij})$ to the partial Latin rectangle L^θ in such a way that each entry $(i, j, k) \in E(L)$ is uniquely related to an entry $(i, j, k)^\theta \in E(L^\theta)$. We obtain L^θ by the following process:

- First, we permute the rows of L according to α, the columns according to β, and the symbols according to γ, giving the intermediate partial Latin square $L' = (l'_{ij})$. That is, if $(i, j, k) \in E(L)$, then $(\alpha(i), \beta(j), \gamma(k)) \in E(L')$.

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- Then we permute the coordinates of the entry set of L' according to δ, to give the entry set $E(L^\theta)$, i.e., if $(e_1, e_2, e_3) \in E(L')$, then it maps to $(e_{\delta(1)}, e_{\delta(2)}, e_{\delta(3)}) \in E(L^\theta)$.

If $(i, j, k) = (i, j, k)^\theta$, then we say that the entry (i, j, k) is *fixed* by θ. An element of $\mathcal{P}$ is called a *paratopism* and we say L and L^θ are *paratopic*. It is possible that $L^\theta = L$, in which case we call θ an *autoparatopism* of L. For example, the partial Latin rectangle

$$\begin{bmatrix} 1 & \cdot & 2 \\ \cdot & 1 & \cdot \end{bmatrix} \in \mathrm{PLR}(2, 3, 2; 3)$$

admits the autoparatopism $\theta = (\mathrm{id}, (23), \mathrm{id}; (13))$, since we have the same entry set after applying θ:

original entry	after applying α, β, and γ	after applying δ
(1, 1, 1)	(1, 1, 1)	(1, 1, 1)
(1, 3, 2)	(1, 2, 2)	(2, 2, 1)
(2, 2, 1)	(2, 3, 1)	(1, 3, 2)

Paratopisms $\theta = (\alpha, \beta, \gamma; \delta)$ with $\delta = \mathrm{id}$ are called *isotopisms* and we say that the partial Latin rectangles L and L^θ are *isotopic*. If θ is an isotopism and an autoparatopism of L, then we say that θ is an *autotopism* of L.

If $\theta = (\alpha, \beta, \gamma; \delta) \in \mathcal{P}_{a_1, a_2, a_3}$ and $L \in \mathrm{PLR}(a_1, a_2, a_3; m)$, then $L^\theta \in \mathrm{PLR}(a_{\delta(1)}, a_{\delta(2)}, a_{\delta(3)}; m)$. The set $\mathrm{PLR}(r, s, n; m)$ is therefore not closed under $\mathcal{P}_{r,s,n}$ unless $r = s = n$. Consequently, the operations defined by paratopisms cannot be defined in terms of group actions in full generality.

The sets of autotopisms and autoparatopisms in $\mathcal{P}_{r,s,n}$ of a partial Latin rectangle $L \in \mathrm{PLR}(r, s, n; m)$ form groups, which we call the *autotopism group*, denoted $\mathrm{atop}(L)$, and *autoparatopism group*, denoted $\mathrm{apar}(L)$, respectively. For given r, s, and n, they are respectively the stabilizer subgroups of L of the groups $(S_r \times S_s \times S_n) \rtimes \langle \mathrm{id} \rangle$ and $(S_r \times S_s \times S_n) \rtimes \mathrm{Stab}_{S_3}(r, s, n)$, where $\mathrm{Stab}_{S_3}(r, s, n)$ is the subgroup of S_3 which stabilizes the triple (r, s, n) under coordinate-wise permutation.

Lemma 1. *Let L be a partial Latin rectangle. The autotopism group $\mathrm{atop}(L)$ is a normal subgroup of the autoparatopism group $\mathrm{apar}(L)$, with the quotient group $\mathrm{apar}(L)/\mathrm{atop}(L)$ isomorphic to*

$$Q(L) := \{\delta \in S_3 : \text{there exists some } (\alpha, \beta, \gamma; \delta) \in \mathrm{apar}(L)\}.$$

Hence, $\mathrm{apar}(L)/\mathrm{atop}(L)$ is isomorphic to $\langle \mathrm{id} \rangle$, C_2, C_3, or S_3, and

$$[\mathrm{apar}(L) : \mathrm{atop}(L)] = \frac{|\mathrm{apar}(L)|}{|\mathrm{atop}(L)|} \in \{1, 2, 3, 6\},$$

and if $\mathrm{atop}(L) = \langle \mathrm{id} \rangle$, then $\mathrm{apar}(L)$ is isomorphic to $\langle \mathrm{id} \rangle$, C_2, C_3, or S_3.

Proof. If $\theta' \in \mathrm{atop}(L)$ and $\theta \in \mathrm{apar}(L)$, then $\theta^{-1}\theta'\theta$ has a trivial fourth component and thus belongs to $\mathrm{atop}(L)$. Hence $\mathrm{atop}(L)$ is a normal subgroup of $\mathrm{apar}(L)$. By definition, the cosets of $\mathrm{atop}(L)$ are $\{\theta\psi : \theta \in \mathrm{atop}(L)\}$ for $\psi \in \mathrm{apar}(L)$, and thus all elements within a coset agree at the fourth coordinate. If two autoparatopisms ψ and φ agree at the fourth coordinate, then $\psi^{-1}\varphi$ has a trivial fourth coordinate, so $\psi^{-1}\varphi \in \mathrm{atop}(L)$. Hence $\varphi = (\varphi\psi^{-1})\psi$ belongs to the same coset as ψ. Thus cosets of $\mathrm{atop}(L)$ are determined by their fourth coordinate. This proves the claims in the first sentence in the statement of the lemma. Since S_3 only has subgroups isomorphic to $\langle \mathrm{id} \rangle$, C_2, C_3 or S_3, the second and third sentences follow. $\square$

The topic of study in this paper is when $\mathrm{atop}(L)$ is trivial, so we will have $\mathrm{apar}(L)$ isomorphic to $\langle \mathrm{id} \rangle$, C_2, C_3, or S_3.

1.2. The question of symmetry groups

The paper aims to be a step towards answering the following general research question:

Question 1. *For which finite groups, H_1 and H_2, and weights m, does there exist a weight-m partial Latin rectangle L with $\mathrm{atop}(L) \cong H_1$ and $\mathrm{apar}(L) \cong H_2$?*

Given a simple graph $\Gamma = (V, E)$ with automorphism group G, Stones [30] gave a construction for a partial Latin square of weight $|V| + 3|E| + 49$ with $\mathrm{atop}(L) = \mathrm{apar}(L) \cong G$. Frucht's Theorem states that, for any finite group G, there exists some finite graph with automorphism group isomorphic to G. Thus we have the following result.

Theorem 1. *For all finite groups G, there exists a partial Latin square L with $\mathrm{atop}(L) = \mathrm{apar}(L) \cong G$.*

A comparable result to Theorem 1 was given by Phelps [25] where L was a Latin square (i.e., no empty cells). Phelps [26] also showed that for all $n \geq 7$ there is a Latin square of order n with no non-trivial autoparatopisms (another proof was given in [30]).

To the authors' knowledge, [30] is the only prior paper studying symmetry groups of partial Latin squares in their own right. Typically, prior research into partial Latin squares involved completion or avoidability problems. In relation

to symmetries, Falcón [8] identified which isotopisms can be autotopisms of partial Latin squares for $n \leq 4$, Falcón and Núñez [11] identified partial Latin squares of order $n \leq 5$ which admit a so-called Santilli autotopism, and Falcón [9] enumerated self-orthogonal partial Latin rectangles. More recently, the authors [12] have dealt with the distribution into isomorphism and isotopism classes of $r \times s$ partial Latin rectangles based on the set $[n]$, for all $r, s, n \leq 6$. This involves counting partial Latin rectangles that admit certain symmetries.

There is growing research into the symmetries of Latin squares in different directions (e.g., the number of symmetries [3,18], near-autoparatopisms [4,24], diagonally cyclic Latin squares [15,36], and symmetry certification [7,23,31] and enumeration [10]). Symmetries of Latin squares and rectangles are also used in enumeration [16,20] (see also [28]), are used to identify divisors of their numbers [21,29,32,33], and give insights into quasigroups [22,27], 1-factorizations [37], polytopes [6], and the Alon–Tarsi Conjecture [17,34]. Autotopisms of Latin squares are also being investigated for applications in cryptography [13,14,35].

2. Partial Latin rectangle properties

2.1. Invariants

In this section, we describe a series of lemmas that give conditions on when a paratopism can be an autoparatopism of a given partial Latin rectangle, and when a group of paratopisms can be an autoparatopism group. They all follow from the definition of an autoparatopism.

Definition 1. Given $L \in \text{PLR}(r, s, n; m)$, we define:

- the row weight w_i^{row} of a row i of L to be the number of non-empty cells in row i,
- the column weight w_j^{col} of a column j of L to be the number of non-empty cells in column j, and
- the symbol weight w_k^{sym} of a symbol $k \neq \cdot$ in L to be the number of copies of the symbol k.

We define the row partition, column partition, and the symbol partition of L as the multisets of row weights $\{w_i^{\text{row}}\}_{i=1}^r$, column weights $\{w_j^{\text{col}}\}_{j=1}^s$, and symbol weights $\{w_k^{\text{sym}}\}_{k=1}^n$, respectively. The sum of the elements of any of these three multisets is the weight of L.

The following lemma will be useful for eliminating the possibility of autoparatopisms with non-trivial δ in subsequent constructions.

Lemma 2. *If R, S, and N respectively denote the row, column, and symbol partitions of $L \in \text{PLR}(r, s, n; m)$, and $\theta = (\alpha, \beta, \gamma; \delta) \in \text{apar}(L)$, then*

- *if $\delta = (12)$, then $R = S$ (and hence $r = s$),*
- *if $\delta = (13)$, then $R = N$ (and hence $r = n$),*
- *if $\delta = (23)$, then $S = N$ (and hence $s = n$), and*
- *if $\delta \in \{(123), (132)\}$, then $R = S = N$ (and hence $r = s = n$).*

Proof. If $\delta \in \{(12), (123)\}$, row i with row weight w_i^{row} in L maps to column $\alpha(i)$ in $L^\theta = L$ with column weight $w_{\alpha(i)}^{\text{col}} = w_i^{\text{row}}$. Thus $R = \{w_i^{\text{row}}\}_{i=1}^r = \{w_{\alpha(i)}^{\text{col}}\}_{i=1}^s = S$. Likewise, (a) if $\delta \in \{(13), (132)\}$, then $R = \{w_i^{\text{row}}\}_{i=1}^r = \{w_{\alpha(i)}^{\text{sym}}\}_{i=1}^n = N$, (b) if $\delta \in \{(23), (123)\}$, then $S = \{w_j^{\text{col}}\}_{j=1}^r = \{w_{\beta(j)}^{\text{sym}}\}_{j=1}^n = N$, and (c) if $\delta = (132)$, then $S = \{w_j^{\text{col}}\}_{j=1}^s = \{w_{\beta(j)}^{\text{row}}\}_{j=1}^r = R$. □

For example, the partial Latin square

$$\begin{bmatrix} 1 & 2 & \cdot & 4 \\ 3 & \cdot & 4 & 1 \\ 4 & \cdot & \cdot & \cdot \\ \cdot & 4 & \cdot & \cdot \end{bmatrix} \in \text{PLS}(4; 8) \tag{1}$$

has row, column, and symbol partitions $\{1, 1, 3, 3\}$, $\{1, 2, 2, 3\}$, and $\{1, 1, 2, 4\}$, respectively. Lemma 2 therefore implies that any autoparatopism $(\alpha, \beta, \gamma; \delta)$ of this partial Latin rectangle has a trivial fourth component ($\delta = \text{id}$), or, in other words, any autoparatopism is an autotopism.

Lemma 3. *If $L \in \text{PLR}(r, s, n; m)$, then*

- *if $\text{apar}(L)/\text{atop}(L)$ is isomorphic to C_3 or S_3, then $r = s = n$, i.e., L is a partial Latin square,*
- *if $\text{apar}(L)/\text{atop}(L)$ is isomorphic to C_2, then at least two of $r, s,$ or n are equal, and*
- *if $\theta \in \mathcal{P}_{r,s,n}$, then $\text{atop}(L) \cong \text{atop}(L^\theta)$, $\text{apar}(L) \cong \text{apar}(L^\theta)$, and $\text{apar}(L)/\text{atop}(L) \cong \text{apar}(L^\theta)/\text{atop}(L^\theta)$.*

Proof. The first two claims follow from Lemmas 1 and 2, whereas the last claim follows from the fact that $\theta' \in \mathcal{P}_{r,s,n}$ is an auto(para)topism of L if and only if $\theta\theta'\theta^{-1}$ is an auto(para)topism of L^θ. □

In studying autoparatopism groups of partial Latin rectangles L with trivial autotopism groups, Lemma 3 motivates us to consider three cases: (a) partial Latin squares with $\operatorname{apar}(L)$ isomorphic to $\langle \operatorname{id} \rangle$, C_2, C_3, or S_3, (b) partial Latin rectangles with $r < s = n$ with $\operatorname{apar}(L)$ isomorphic to $\langle \operatorname{id} \rangle$ or C_2, and (c) partial Latin rectangles with $r < s < n$ with $\operatorname{apar}(L)$ isomorphic to $\langle \operatorname{id} \rangle$. The following lemma restricts the possibilities on when an isotopism can be an autotopism.

Lemma 4. *Let $L \in \operatorname{PLR}(r, s, n; m)$ with the autotopism $\theta = (\alpha, \beta, \gamma; \operatorname{id})$. Then*

- $w_i^{row} = w_{\alpha(i)}^{row}$ *for all $i \in [r]$,*
- $w_j^{col} = w_{\beta(j)}^{col}$ *for all $j \in [s]$, and*
- $w_k^{sym} = w_{\gamma(k)}^{sym}$ *for all $k \in [n]$.*

Proof. We prove the first claim; the other cases follow analogously. The autotopism θ maps the entries in row i to entries in row $\alpha(i)$; the order in which they appear in the row and the symbol they contain (which might be modified by β and γ, respectively) does not alter the row weight. Hence $w_i^{row} = w_{\alpha(i)}^{row}$. $\square$

In the example (1), Lemma 4 implies that columns 1 and 3 must be fixed by the column permutation of any autotopism (since they have unique column weights), and the symbols 1 and 4 are fixed by the symbol permutation of any autotopism (since they have unique symbol weights).

For a partial Latin rectangle L with autoparatopism θ, the group $\langle \theta \rangle$ acts on the entry set $E(L)$, so each θ permutes $E(L)$.

Lemma 5. *Let L be a partial Latin rectangle with entry (i, j, k) and autoparatopism $\theta = (\alpha, \beta, \gamma; \delta)$.*

- *If $(i, j, k)^\theta = (i', j', k')$, then $\{w_i^{row}, w_j^{col}, w_k^{sym}\} = \{w_{i'}^{row}, w_{j'}^{col}, w_{k'}^{sym}\}$ as multisets.*
- *If $(i, j, k)^\theta = (i', j', k')$ and θ is an autotopism, then $(w_i^{row}, w_j^{col}, w_k^{sym}) = (w_{i'}^{row}, w_{j'}^{col}, w_{k'}^{sym})$.*

Proof. The isotopism $(\alpha, \beta, \gamma; \operatorname{id})$ preserves the row, column, and symbol weights, while δ permutes the order of these weights. $\square$

Definition 2. We call $\{w_i^{row}, w_j^{col}, w_k^{sym}\}$ and $(w_i^{row}, w_j^{col}, w_k^{sym})$ the *entry invariant* and *strong entry invariant* of the entry (i, j, k), respectively.

Continuing our example (1), the four entries with symbol 4 have strong entry invariants $(3, 2, 4)$, $(3, 1, 4)$, $(1, 3, 4)$, and $(1, 2, 4)$, as we go down the rows. Thus, since symbol 4 is fixed, all four of the entries with symbol 4 must be fixed by any autotopism. This implies every row and column is fixed, and thus the partial Latin rectangle in (1) has no non-trivial autotopisms, and hence has a trivial autoparatopism group.

Lemma 6. *Suppose L is a partial Latin rectangle with autotopism $\theta = (\alpha, \beta, \gamma; \operatorname{id})$. If $i \in [r]$, then the multiset of entry invariants (resp. strong entry invariants) of the entries in row i is equal to the multiset of entry invariants (resp. strong entry invariants) of the entries in row $\alpha(i)$. The same is true for β for $[s]$, and for γ for $[n]$.*

Proof. The result follows straightforwardly as a setwise version of the second bullet point in Lemma 5. $\square$

2.2. Partial Latin rectangle graphs

A Latin square has a natural representation as a graph with the vertex set being its entries, and edges between distinct entries whenever those two entries share a row, column, or symbol. This graph is known as a *Latin square graph*.[1] However, despite being a large family of strongly regular graphs, Latin square graphs have attracted little research attention in their own right. Recently, Besharati, Goddyn, Mahmoodian, and Mortezaeefar [1] have studied the chromatic number of Latin square graphs for special types of Latin squares, who attributed them to Bose [2]. A partial Latin rectangle similarly has a natural representation as a graph.

Definition 3. Any partial Latin rectangle $L = (l_{ij})$ in $\operatorname{PLR}(r, s, n; m)$ corresponds to an m-vertex graph, called its *partial Latin rectangle graph*, denoted $\Gamma = \Gamma(L)$, with

- vertex set equal to the entry set $E(L)$ of L, and
- an edge between distinct vertices (i, j, k) and (i', j', k') whenever $i = i'$, $j = j'$, or $k = k'$.

Throughout the paper, since many of the symmetry properties of partial Latin rectangles are more apparent from a graphical interpretation, it is beneficial to color the edges of Γ in some way.

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[1] Not to be confused with another notion of Latin square graphs [5], the rook's graph (i.e., the Cartesian product of K_n with K_n), or with orthogonal Latin square graphs [19].
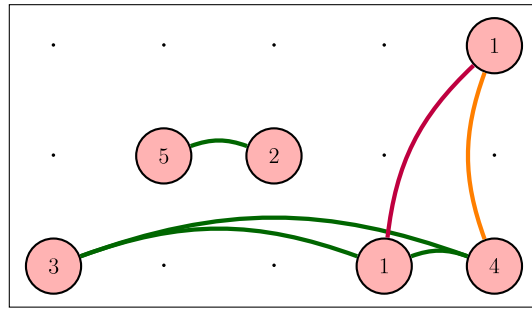
Fig. 1. Natural edge coloring of a partial Latin rectangle graph.

Definition 4. Let L be a partial Latin rectangle. We define the *natural edge coloring* $\Gamma^{(e)}(L)$ of the graph $\Gamma(L)$ as follows: An edge between two distinct vertices (i, j, k) and (i', j', k') is green (respectively, orange or purple) if $i = i'$ (respectively, $j = j'$ or $k = k'$).

Fig. 1 gives an example of the natural edge coloring of the partial Latin rectangle graph corresponding to the partial Latin rectangle

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 5 & 2 & \cdot & \cdot \\ 3 & \cdot & \cdot & 1 & 4 \end{bmatrix} \in \mathrm{PLR}(3, 5, 5; 6).$$

The following two lemmas relate the autoparatopisms of a partial Latin rectangle L to the automorphisms of its partial Latin rectangle graph $\Gamma(L)$.

Lemma 7. *Any autoparatopism θ of a partial Latin rectangle L induces an automorphism of the partial Latin rectangle graph $\Gamma(L)$ where a vertex (i, j, k) maps to the vertex $(i, j, k)^\theta$. Moreover, if $\Gamma(L)$ is given its natural edge coloring, then θ induces an automorphism of $\Gamma^{(e)}(L)$ that preserves the edge color classes, and if θ is an autotopism, then the automorphism preserves the edge colors of $\Gamma^{(e)}(L)$.*

Proof. Adjacent vertices in $\Gamma(L)$ correspond to distinct entries (i, j, k) and (i', j', k') in $E(L)$ that agree at exactly one coordinate. This occurs if and only if $(i, j, k)^\theta$ and $(i', j', k')^\theta$ in $E(L^\theta)$ agree at exactly one coordinate, corresponding to adjacent vertices of $\Gamma(L^\theta)$, thus proving the first claim.

For the second claim, if (i, j, k) and (i', j', k') agree at exactly one coordinate, the fourth component of θ determines which coordinate $(i, j, k)^\theta$ and $(i', j', k')^\theta$ agree at. This implies an edge-color-class preserving automorphism of the graph $\Gamma^{(e)}(L)$ endowed with the natural edge coloring. In the case where θ has a trivial fourth component, i.e., θ is an autotopism, we have (i, j, k) and (i', j', k') agreeing at a coordinate if and only if $(i, j, k)^\theta$ and $(i', j', k')^\theta$ agree at that coordinate. So, the automorphism induced by θ preserves the edge colors of $\Gamma^{(e)}(L)$. □

Lemma 8. *The group* $\mathrm{atop}(L)$ *is isomorphic to some subgroup of* $\mathrm{aut}(\Gamma(L))$. *Moreover, if L has a row of weight ≥ 2, a column of weight ≥ 2, and a symbol of weight ≥ 2, then* $\mathrm{apar}(L)$ *is isomorphic to some subgroup of* $\mathrm{aut}(\Gamma(L))$. *Both of these subgroups are identifiable, with $\theta \in \mathrm{atop}(L)$ (or $\theta \in \mathrm{apar}(L)$) corresponding to the automorphism of $\Gamma(L)$ which maps (i, j, k) to $(i, j, k)^\theta$.*

Proof. Non-trivial autotopisms do not fix all of the entries in $E(L)$, and thus induce non-trivial automorphisms of Γ. If θ and θ' are two distinct autotopisms of L, then $\theta^{-1}\theta'$ is a non-trivial autotopism of L, and thus induces a non-trivial automorphism of Γ. Hence, θ and θ' must induce two distinct automorphisms of Γ. Consequently, $\mathrm{atop}(L)$ is isomorphic to a subgroup of $\mathrm{aut}(\Gamma)$.

Similarly, the fourth component of any non-trivial autoparatopism $\theta \in \mathrm{apar}(L) \setminus \mathrm{atop}(L)$ gives rise to a transformation in $\Gamma(L)$ that maps the collinear vertices related to a given row, column or symbol of weight ≥ 2 to a distinct set of collinear vertices of the same cardinality. This implies that the automorphism of Γ is non-trivial. Analogous to the autotopism case, two distinct autoparatopisms θ and θ' of L induce two distinct automorphisms of Γ, and so $\mathrm{apar}(L)$ is isomorphic to some subgroup of $\mathrm{aut}(\Gamma)$. □

It is possible that a non-trivial autoparatopism induces the trivial automorphism of the partial Latin rectangle graph. For example, the autoparatopism $(\mathrm{id}, \mathrm{id}, \mathrm{id}; (123))$ of the partial Latin rectangle

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & 3 & \cdot \\ \cdot & \cdot & \cdot & 4 \end{bmatrix}$$
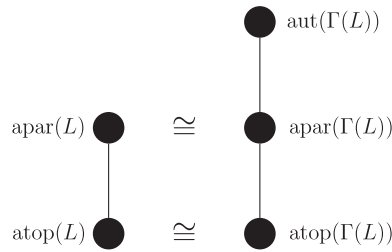
maps each entry to itself.

Fig. 2. (Partial) subgroup lattices for a safe partial Latin rectangle *L*.

Definition 5. If *L* is a partial Latin rectangle in which some row has weight ≥ 2, some column has weight ≥ 2, and some symbol has weight ≥ 2, then we call *L* *safe*.

Lemma 8 implies that, if *L* is safe, any two distinct autoparatopisms of *L* induce two distinct automorphisms of the partial Latin rectangle graph $\Gamma(L)$. Actually, Definition 5 does not give the weakest condition necessary to ensure that distinct autoparatopisms of *L* map to distinct automorphism of $\Gamma(L)$, but for the constructions in this paper, it is not a burden to meet the conditions of Definition 5 as all the partial Latin rectangles we work with will be safe (and we will usually not explicitly mention this).

Definition 6. If *L* is a safe partial Latin rectangle, we define the *autotopism group* and *autoparatopism group* of the partial Latin rectangle graph $\Gamma(L)$ to be the subgroups of $\mathrm{aut}(\Gamma(L))$ corresponding to $\mathrm{atop}(L)$ and $\mathrm{apar}(L)$, respectively, as described in Lemma 8. We denote these subgroups $\mathrm{atop}(\Gamma(L))$ and $\mathrm{apar}(\Gamma(L))$, respectively.

Fig. 2 illustrates the (partial) subgroup lattices of $\mathrm{apar}(L)$ and $\mathrm{aut}(\Gamma(L))$ for a safe partial Latin rectangle *L*.

It is possible that $\mathrm{apar}(L)$ corresponds to a proper subgroup of $\mathrm{aut}(\Gamma(L))$, and, in fact, this is often the case. For example, the partial Latin rectangle

$$L = \begin{bmatrix} 1 & 2 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 3 & 4 \\ \cdot & \cdot & \cdot & 5 \end{bmatrix}$$

has $|\mathrm{atop}(L)| = 1$, $|\mathrm{apar}(L)| = 2$, and $|\mathrm{aut}(\Gamma(L))| = 12$.

2.3. Graph properties and blockwise construction

Graph properties transfer to the partial Latin rectangle from its corresponding graph. For example, we can talk about the *connected components* of the partial Latin rectangle, i.e., they are the sets of entries of partial Latin rectangles that correspond to the vertex sets of connected components of the corresponding partial Latin rectangle graph. We define the *weight* of a connected component as the number of non-empty cells in it. We say a partial Latin rectangle is *connected* if it has one connected component.

It is possible to "glue" two partial Latin rectangles together in a blockwise fashion and study its symmetries.

Lemma 9. Let $A \in \mathrm{PLR}(r, s, n; m)$ and $B \in \mathrm{PLR}(r', s', n'; m')$ be two connected, non-paratopic partial Latin rectangles. Define another (disconnected) partial Latin rectangle *L* by the block structure

$$L = \begin{array}{|c|c|} \hline A & \cdot \\ \hline \cdot & B+n \\ \hline \end{array}.$$

Then

1. $\mathrm{apar}(L) \cong \{(\alpha, \beta, \gamma; \delta) \in \mathrm{apar}(A) : \delta \in Q(B)\} \times \{(\alpha, \beta, \gamma; \delta) \in \mathrm{apar}(B) : \delta \in Q(A)\}$.
2. $\mathrm{atop}(L) \cong \mathrm{atop}(A) \times \mathrm{atop}(B)$,
3. if $Q(A) \cap Q(B) = \langle \mathrm{id} \rangle$, then $\mathrm{apar}(L) = \mathrm{atop}(L)$, and
4. if $\mathrm{atop}(B) \cong \langle \mathrm{id} \rangle$ and $\mathrm{apar}(B) \cong S_3$, then $\mathrm{apar}(L) \cong \mathrm{apar}(A)$.

Proof. Since *A* and *B* are the two connected components of *L*, under any autoparatopism of *L*, we must either have *A* and *B* mapping to themselves, or swapping with one another. However, since *A* and *B* are, by assumption, non-paratopic, they must map to themselves.

Any autoparatopism of *L* induces an autoparatopism of *A*, by restricting to the rows $[r]$, columns $[s]$, and symbols $[n]$, and an autoparatopism of *B* by restricting to the rows $[r']+r$, columns $[s']+s$, and symbols $[n']+n$; these two autoparatopisms
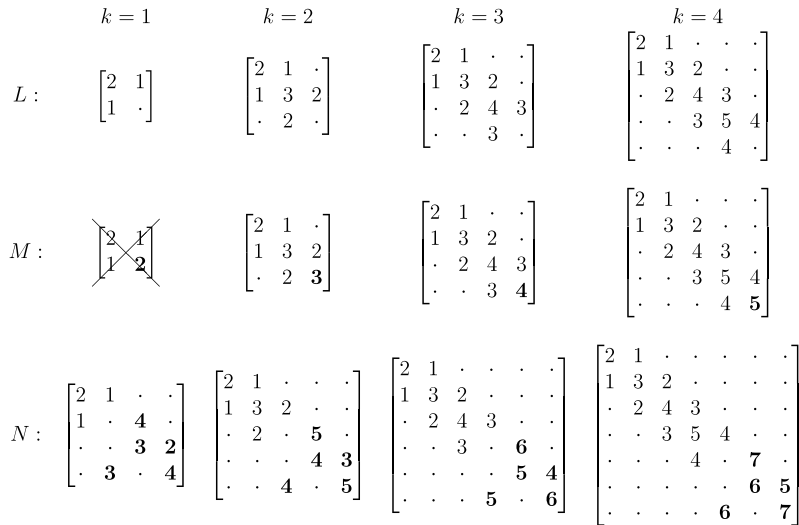
Fig. 3. The smallest partial Latin squares described by Construction 1. Entries outside of $E(L)$ are highlighted in bold. (Construction 1 for M when $k = 1$ has non-trivial autotopisms, and is thus excluded.)

have the same fourth component. Conversely, given autoparatopisms θ_A and θ_B of A and B, respectively, if they agree at the fourth coordinate, they can be combined to give an autoparatopism of L with the same fourth coordinate. This proves the first claim. The second claim follows from a similar argument, but restricting to a trivial fourth component. The third claim follows since

$$\{(\alpha, \beta, \gamma; \delta) \in \mathrm{apar}(A) : \delta \in Q(B)\} = \{(\alpha, \beta, \gamma; \delta) \in \mathrm{apar}(A) : \delta \in Q(A) \cap Q(B)\}$$
$$= \mathrm{atop}(A),$$

and

$$\{(\alpha, \beta, \gamma; \delta) \in \mathrm{apar}(B) : \delta \in Q(A)\} = \{(\alpha, \beta, \gamma; \delta) \in \mathrm{apar}(B) : \delta \in Q(A) \cap Q(B)\}$$
$$= \mathrm{atop}(B),$$

as we have assumed $Q(A) \cap Q(B)$ to be trivial. For the last claim, if $\theta_A \in \mathrm{apar}(A)$ and $\theta_B \in \mathrm{apar}(B)$ combine to give an autoparatopism of L, then θ_B is determined from the fourth component of θ_A, as there is a unique autoparatopism of B with that fourth component. □

3. Partial Latin squares

Here we describe constructions of partial Latin squares (i.e., with dimensions satisfying $r = s = n$) with trivial autotopism groups for all possible weights m and all possible autoparatopism groups (up to isomorphism). The constructions in this section follow the same theme. We begin with the weight-$3k$ partial Latin square L of order $k + 1$ defined by the entry set

$$E(L) = \bigcup_{1 \le i \le k} \big\{ (i, i, i + 1), (i, i + 1, i), (i + 1, i, i) \big\}. \tag{2}$$

For small k, this partial Latin square is given in the first row in Fig. 3. We will see in Construction 1 that L has a trivial autotopism group and autoparatopism group isomorphic to S_3. We can construct other partial Latin squares with varying weight and autoparatopism group isomorphism class by editing this construction. We will make use of the following lemma.

Lemma 10. *Suppose M is a partial Latin square in which $E(M) \supseteq E(L)$, as defined in (2). Suppose further that $\theta = (\alpha, \beta, \gamma; \mathrm{id})$ is an autotopism of M in which $\alpha, \beta,$ and γ fix 1. Then $\alpha, \beta,$ and γ fix $\{1, 2, \ldots, k + 1\}$ pointwise.*

Proof. We proceed by induction where the base case is a condition of the lemma. If $i \in \{1, 2, \ldots, k\}$ is fixed by $\alpha, \beta,$ and γ, then the entries $(i, i, i + 1), (i, i + 1, i),$ and $(i + 1, i, i)$ must map to entries of the form $(i, i, j_1), (i, j_2, i),$ and (j_3, i, i) for some $j_1 \in [n], j_2 \in [s],$ and $j_3 \in [r]$, respectively, which, since θ is an autotopism and M is a partial Latin square, implies that $i + 1$ is also fixed by $\alpha, \beta,$ and γ. □

The case of a trivial autotopism group and autoparatopism group isomorphic to S_3 is of particular importance because of its presence in Lemma 9.
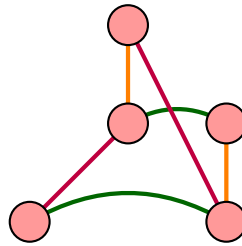
Fig. 4. The graph induced by $E(N) \setminus E(L)$ in the proof of Construction 1.

Construction 1 (*Autoparatopism Group S_3*)**.** *For $k \geq 1$, define the partial Latin square $L \in \mathrm{PLS}(k+1; 3k)$ by the entry set defined in* (2). *Further, define the partial Latin square $M \in \mathrm{PLS}(k+1; 3k+1)$ by*

$$E(M) = E(L) \cup \big\{(k+1, k+1, k+1)\big\}$$

whenever $k \geq 2$, and the partial Latin square $N \in \mathrm{PLS}(k+3; 3k+5)$ by

$$
\begin{aligned}
E(N) = E(L) \\
\cup \big\{(k+2, k+2, k+2), (k+3, k+3, k+3)\big\} \\
\cup \big\{(k+1, k+2, k+3), (k+3, k+1, k+2), (k+2, k+3, k+1)\big\}.
\end{aligned}
$$

Then, L, M, and N have trivial autotopism groups, and autoparatopism groups isomorphic to S_3. The smallest such partial Latin squares are given in Fig. 3.

Proof. We have designed L, M, and N so that we can readily observe that $L^{(\mathrm{id},\mathrm{id},\mathrm{id};\delta)} = L$ and $M^{(\mathrm{id},\mathrm{id},\mathrm{id};\delta)} = M$ for all $\delta \in S_3$ and $N^{(\mathrm{id},\mathrm{id},\mathrm{id};\delta)} = N$ when $\delta \in \langle(123)\rangle \subseteq S_3$. We can also observe that $N^{(\alpha,\alpha,\alpha;(12))} = N$ where α is the permutation that swaps $k+2$ and $k+3$. Hence, it is sufficient to prove that L, M, and N have trivial autotopism groups.

Suppose $(\alpha, \beta, \gamma; \mathrm{id})$ is an autotopism of L or M. The entries $(1, 1, 2)$, $(1, 2, 1)$, and $(2, 1, 1)$ have unique strong entry invariants in L and M, for $k \geq 1$ and $k \geq 2$, respectively. Lemma 5 implies they are fixed entries, so 1 is fixed by α, β, and γ. Lemma 10 now implies α, β, and γ fix $\{1, 2, \ldots, k+1\}$ pointwise. This is sufficient to prove that $\mathrm{atop}(L)$ and $\mathrm{atop}(M)$ are trivial.

By Lemma 7, a non-trivial autotopism of N would induce a non-trivial edge-color-preserving automorphism of the partial Latin rectangle graph for N with the natural edge coloring. And since $E(L)$ is fixed pointwise, we get such an automorphism of the graph induced by $E(N) \setminus E(L)$ (see Fig. 4), but no such automorphism exists. □

The subsequent two constructions give partial Latin squares with trivial autotopism groups and autoparatopism groups isomorphic to C_2 and C_3, respectively.

Construction 2 (*Autoparatopism Group C_2*)**.** *For $k \geq 1$, define the partial Latin square L by the entry set defined in* (2) *and define three partial Latin squares $M \in \mathrm{PLS}(k+3; 3k+4)$, $M' \in \mathrm{PLS}(k+3; 3k+5)$, and $N \in \mathrm{PLS}(k+3; 3k+6)$ by*

$$
\begin{aligned}
E(M) &= E(L) \cup \big\{(k+2, k+2, k+2)\big\} \cup \big\{(k+2, k+3, k+3), (k+3, k+2, k+3)\big\} \cup \big\{(k+3, k+3, 1)\big\}, \\
E(M') &= E(M) \cup \big\{(k+1, k+1, k+2)\big\},
\end{aligned}
$$

and

$$E(N) = E(M) \cup \big\{(k+2, k+1, k+1), (k+1, k+2, k+1)\big\}.$$

Then M, M', and N have trivial autotopism groups, and autoparatopism groups isomorphic to C_2. The smallest such partial Latin squares are given in Fig. 5.

Proof. We check the claim is true for the partial Latin squares with $k \leq 2$ computationally, so now assume $k \geq 3$. We first show M, M', and N have trivial autotopism groups; suppose $\theta = (\alpha, \beta, \gamma; \mathrm{id})$ is an autotopism of one of these partial Latin

$$
\begin{array}{cccc}
k=1 & k=2 & k=3 & k=4
\end{array}
$$

$$
M:\quad
\begin{bmatrix}
2&1&\cdot&\cdot\\
1&\cdot&\cdot&\cdot\\
\cdot&\cdot&3&4\\
\cdot&\cdot&4&1
\end{bmatrix}
\quad
\begin{bmatrix}
2&1&\cdot&\cdot&\cdot\\
1&3&2&\cdot&\cdot\\
\cdot&2&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&4&5\\
\cdot&\cdot&\cdot&5&1
\end{bmatrix}
\quad
\begin{bmatrix}
2&1&\cdot&\cdot&\cdot&\cdot\\
1&3&2&\cdot&\cdot&\cdot\\
\cdot&2&4&3&\cdot&\cdot\\
\cdot&\cdot&3&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&5&6\\
\cdot&\cdot&\cdot&\cdot&6&1
\end{bmatrix}
\quad
\begin{bmatrix}
2&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&3&2&\cdot&\cdot&\cdot&\cdot\\
\cdot&2&4&3&\cdot&\cdot&\cdot\\
\cdot&\cdot&3&5&4&\cdot&\cdot\\
\cdot&\cdot&\cdot&4&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&6&7\\
\cdot&\cdot&\cdot&\cdot&\cdot&7&1
\end{bmatrix}
$$

$$
M':\quad
\begin{bmatrix}
2&1&\cdot&\cdot\\
1&\mathbf{3}&\cdot&\cdot\\
\cdot&\cdot&3&4\\
\cdot&\cdot&4&1
\end{bmatrix}
\quad
\begin{bmatrix}
2&1&\cdot&\cdot&\cdot\\
1&3&2&\cdot&\cdot\\
\cdot&2&\mathbf{4}&\cdot&\cdot\\
\cdot&\cdot&\cdot&4&5\\
\cdot&\cdot&\cdot&5&1
\end{bmatrix}
\quad
\begin{bmatrix}
2&1&\cdot&\cdot&\cdot&\cdot\\
1&3&2&\cdot&\cdot&\cdot\\
\cdot&2&4&3&\cdot&\cdot\\
\cdot&\cdot&3&\mathbf{5}&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&5&6\\
\cdot&\cdot&\cdot&\cdot&6&1
\end{bmatrix}
\quad
\begin{bmatrix}
2&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&3&2&\cdot&\cdot&\cdot&\cdot\\
\cdot&2&4&3&\cdot&\cdot&\cdot\\
\cdot&\cdot&3&5&4&\cdot&\cdot\\
\cdot&\cdot&\cdot&4&\mathbf{6}&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&6&7\\
\cdot&\cdot&\cdot&\cdot&\cdot&7&1
\end{bmatrix}
$$

$$
N:\quad
\begin{bmatrix}
2&1&\cdot&\cdot\\
1&\cdot&\mathbf{2}&\cdot\\
\cdot&\mathbf{2}&3&4\\
\cdot&\cdot&4&1
\end{bmatrix}
\quad
\begin{bmatrix}
2&1&\cdot&\cdot&\cdot\\
1&3&2&\cdot&\cdot\\
\cdot&2&\cdot&\mathbf{3}&\cdot\\
\cdot&\cdot&\mathbf{3}&4&5\\
\cdot&\cdot&\cdot&5&1
\end{bmatrix}
\quad
\begin{bmatrix}
2&1&\cdot&\cdot&\cdot&\cdot\\
1&3&2&\cdot&\cdot&\cdot\\
\cdot&2&4&3&\cdot&\cdot\\
\cdot&\cdot&3&\cdot&\mathbf{4}&\cdot\\
\cdot&\cdot&\cdot&\mathbf{4}&5&6\\
\cdot&\cdot&\cdot&\cdot&6&1
\end{bmatrix}
\quad
\begin{bmatrix}
2&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&3&2&\cdot&\cdot&\cdot&\cdot\\
\cdot&2&4&3&\cdot&\cdot&\cdot\\
\cdot&\cdot&3&5&4&\cdot&\cdot\\
\cdot&\cdot&\cdot&4&\cdot&\mathbf{5}&\cdot\\
\cdot&\cdot&\cdot&\cdot&\mathbf{5}&6&7\\
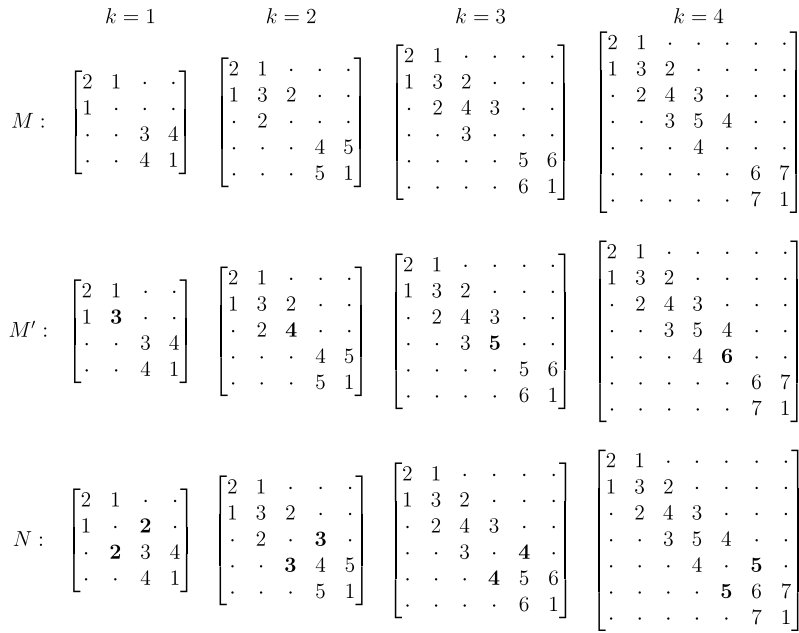\cdot&\cdot&\cdot&\cdot&\cdot&7&1
\end{bmatrix}
$$

Fig. 5. The smallest partial Latin squares described by Construction 2. Entries outside of $E(M)$ are highlighted in bold.

squares. The multisets of entry invariants in a given row are tabulated below:

row	multiset of entry invariants in a given row		
	M	*M'*	*N*
1	$\{\{2,2,1\},\{2,3,3\}\}$	$\{\{2,2,1\},\{2,3,3\}\}$	$\{\{2,2,1\},\{2,3,3\}\}$
2	$\{\{3,2,3\},\{3,3,3\},\{3,3,3\}\}$	$\{\{3,2,3\},\{3,3,3\},\{3,3,3\}\}$	$\{\{3,2,3\},\{3,3,3\},\{3,3,3\}\}$
k	$\{\{3,3,3\},\{3,3,1\},\{3,1,3\}\}$	$\{\{3,3,3\},\{3,3,1\},\{3,2,3\}\}$	$\{\{3,3,3\},\{3,3,3\},\{3,2,3\}\}$
k+1	$\{\{1,3,3\}\}$	$\{\{2,3,3\},\{2,2,2\}\}$	$\{\{2,3,3\},\{2,3,3\}\}$
k+2	$\{\{2,2,1\},\{2,2,2\}\}$	$\{\{2,2,2\},\{2,2,2\}\}$	$\{\{3,2,3\},\{3,3,1\},\{3,2,2\}\}$
k+3	$\{\{2,2,2\},\{2,2,3\}\}$	$\{\{2,2,2\},\{2,2,3\}\}$	$\{\{2,3,2\},\{2,2,3\}\}$
all others	$\{\{3,3,3\},\{3,3,3\},\{3,3,3\}\}$	$\{\{3,3,3\},\{3,3,3\},\{3,3,3\}\}$	$\{\{3,3,3\},\{3,3,3\},\{3,3,3\}\}$.

In any case, we observe that at least rows $1, k+1, k+2$, and $k+3$ have unique multisets of entry invariants. Lemma 6 thus implies α fixes $\{1, k+1, k+2, k+3\}$ pointwise. Since M, M', and N are symmetric matrices, the same argument implies β fixes $\{1, k+1, k+2, k+3\}$ pointwise. Since $k+3$ is fixed by α and β, the entry $(k+3, k+3, 1)$ is fixed by θ, so 1 is fixed by γ. Lemma 10 now implies α, β, and γ fix $\{1, 2, \ldots, k+1\}$ pointwise, and hence θ is trivial. So M, M', and N have trivial autotopism groups.

By definition, M, M', and N are symmetric, i.e., they admit the autoparatopism $(\text{id}, \text{id}, \text{id}; (12))$. Since M, M', and N have trivial autotopism groups, if they admit any other non-trivial autoparatopism, Lemma 2 would imply they have identical column and symbol partitions, but they do not:

	row/column partition	symbol partition
M :	$\{1, 2, 2, 2, \overbrace{3, 3, \ldots, 3}^{k-1}\}$	$\{1, 1, 2, \overbrace{3, 3, \ldots, 3}^{k}\}$
M' :	$\{2, 2, 2, 2, \overbrace{3, 3, \ldots, 3}^{k-1}\}$	$\{1, 2, 2, \overbrace{3, 3, \ldots, 3}^{k}\}$
N :	$\{2, 2, 2, \overbrace{3, 3, \ldots, 3}^{k}\}$	$\{1, 2, \overbrace{3, 3, \ldots, 3}^{k+1}\}$.

Thus, they have trivial autotopism groups and autoparatopism groups $\langle(\text{id}, \text{id}, \text{id}; (12))\rangle \cong C_2$. $\square$

$$
\begin{array}{ccccc}
 & k=1 & k=2 & k=3 & k=4 \\
M: &
\begin{bmatrix}
2 & 1 & \cdot & \cdot \\
1 & \cdot & 4 & \cdot \\
\cdot & \cdot & 3 & 2 \\
\cdot & 3 & \cdot & \cdot
\end{bmatrix}
&
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot \\
1 & 3 & 2 & 5 & \cdot \\
\cdot & 2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 4 & 2 \\
\cdot & 4 & \cdot & \cdot & \cdot
\end{bmatrix}
&
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & 6 & \cdot \\
\cdot & 2 & 4 & 3 & \cdot & \cdot \\
\cdot & \cdot & 3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 5 & 2 \\
\cdot & 5 & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
&
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & \cdot & 7 & \cdot \\
\cdot & 2 & 4 & 3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 3 & 5 & 4 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 4 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 6 & 2 \\
\cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{ccccc}
M': &
\begin{bmatrix}
2 & 1 & \cdot & \cdot \\
1 & \mathbf{2} & 4 & \cdot \\
\cdot & \cdot & 3 & 2 \\
\cdot & 3 & \cdot & \cdot
\end{bmatrix}
&
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot \\
1 & 3 & 2 & 5 & \cdot \\
\cdot & 2 & \mathbf{3} & \cdot & \cdot \\
\cdot & \cdot & \cdot & 4 & 2 \\
\cdot & 4 & \cdot & \cdot & \cdot
\end{bmatrix}
&
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & 6 & \cdot \\
\cdot & 2 & 4 & 3 & \cdot & \cdot \\
\cdot & \cdot & 3 & \mathbf{4} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 5 & 2 \\
\cdot & 5 & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
&
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & \cdot & 7 & \cdot \\
\cdot & 2 & 4 & 3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 3 & 5 & 4 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 4 & \mathbf{5} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 6 & 2 \\
\cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{ccccc}
N: &
\begin{bmatrix}
2 & 1 & \cdot & \mathbf{4} \\
1 & \cdot & 4 & \cdot \\
\cdot & \cdot & \cdot & 2 \\
\mathbf{4} & 3 & \cdot & \mathbf{1}
\end{bmatrix}
&
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \mathbf{5} \\
1 & 3 & 2 & 5 & \cdot \\
\cdot & 2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 2 \\
\mathbf{5} & 4 & \cdot & \cdot & \mathbf{1}
\end{bmatrix}
&
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot & \mathbf{6} \\
1 & 3 & 2 & \cdot & 6 & \cdot \\
\cdot & 2 & 4 & 3 & \cdot & \cdot \\
\cdot & \cdot & 3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 2 \\
\mathbf{6} & 5 & \cdot & \cdot & \cdot & \mathbf{1}
\end{bmatrix}
&
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot & \cdot & \mathbf{7} \\
1 & 3 & 2 & \cdot & \cdot & 7 & \cdot \\
\cdot & 2 & 4 & 3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 3 & 5 & 4 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 4 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 \\
\mathbf{7} & 6 & \cdot & \cdot & \cdot & \cdot & \mathbf{1}
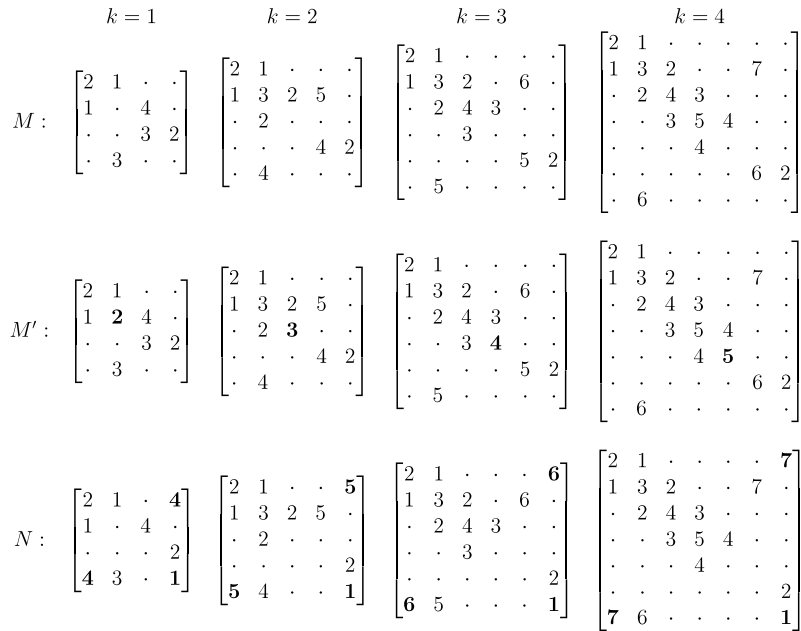\end{bmatrix}
\end{array}
$$

Fig. 6. The smallest partial Latin squares described by Construction 3. Entries outside of $E(M)$ are highlighted in bold.

Construction 3 (*Autoparatopism Group C_3*). *For $k \geq 1$, define the partial Latin square L by the entry set defined in* (2) *and define three partial Latin squares $M \in \mathrm{PLS}(k+3; 3k+4)$, $M' \in \mathrm{PLS}(k+3; 3k+5)$, and $N \in \mathrm{PLS}(k+3; 3k+6)$ by*

$$
\begin{aligned}
E(M) = {}& E(L) \cup \big\{(k+2, k+2, k+2)\big\} \\
& \cup \big\{(2, k+2, k+3), (k+2, k+3, 2), (k+3, 2, k+2)\big\}, \\
E(M') = {}& E(M) \cup \big\{(k+1, k+1, k+1)\big\},
\end{aligned}
$$

and

$$
\begin{aligned}
E(N) = {}& \big(E(M) \setminus \big\{(k+2, k+2, k+2)\big\}\big) \\
& \cup \big\{(1, k+3, k+3), (k+3, 1, k+3), (k+3, k+3, 1)\big\}.
\end{aligned}
$$

Then M, M', and N have trivial autotopism groups, and autoparatopism groups isomorphic to C_3. The smallest such partial Latin squares are given in Fig. 6.

Proof. We check the claim is true for the partial Latin squares with $k \leq 3$ computationally, so now assume $k \geq 4$. We first show M, M', and N have trivial autotopism groups; suppose $\theta = (\alpha, \beta, \gamma; \mathrm{id})$ is an autotopism of one of these partial Latin squares. The multisets of entry invariants in a given row are tabulated below:

row/col./sym.	multiset of entry invariants in a given row, column, or for a given symbol		
	M	M'	N
1	$\{\{2,2,4\}, \{2,4,2\}\}$	$\{\{2,2,4\}, \{2,4,2\}\}$	$\{\{3,3,4\}, \{3,4,3\}\}$
2	$\{\{4,2,2\}, \{4,4,3\}, \{4,3,4\}, \{4,2,1\}\}$	$\{\{4,2,2\}, \{4,4,3\}, \{4,3,4\}, \{4,2,1\}\}$	$\{\{4,3,3\}, \{4,4,3\}, \{4,3,4\}, \{4,1,3\}\}$
3	$\{\{3,4,4\}, \{3,3,3\}, \{3,3,3\}\}$	$\{\{3,4,4\}, \{3,3,3\}, \{3,3,3\}\}$	$\{\{3,4,4\}, \{3,3,3\}, \{3,3,3\}\}$
k	$\{\{3,3,3\}, \{3,3,1\}, \{3,1,3\}\}$	$\{\{3,3,3\}, \{3,3,2\}, \{3,2,3\}\}$	$\{\{3,3,3\}, \{3,3,1\}, \{3,1,3\}\}$
$k+1$	$\{\{1,3,3\}\}$	$\{\{2,3,3\}, \{2,2,2\}\}$	$\{\{1,3,3\}\}$
$k+2$	$\{\{2,2,2\}, \{2,1,4\}\}$	$\{\{2,2,2\}, \{2,1,4\}\}$	$\{\{1,3,4\}\}$
$k+3$	$\{\{1,4,2\}\}$	$\{\{1,4,2\}\}$	$\{\{3,3,3\}, \{3,4,1\}, \{3,3,3\}\}$
all others	$\{\{3,3,3\}, \{3,3,3\}, \{3,3,3\}\}$	$\{\{3,3,3\}, \{3,3,3\}, \{3,3,3\}\}$	$\{\{3,3,3\}, \{3,3,3\}, \{3,3,3\}\}$.

In any case, we observe that rows 1, $k+2$, and $k+3$ have unique multisets of entry invariants. So Lemma 6 implies that α fixes $\{1, k+2, k+3\}$ pointwise. Since M, M', and N admit the autoparatopism $(\mathrm{id}, \mathrm{id}, \mathrm{id}; (123))$, the same argument implies β and γ also fix 1, $k+2$, and $k+3$. Lemma 10 now implies θ is trivial. So M, M', and N have trivial autotopism groups.

Since M, M', and N have trivial autotopism groups and admit the autoparatopism $(\mathrm{id}, \mathrm{id}, \mathrm{id}; (123))$, the existence of any other non-trivial autoparatopism would imply one of M, M', or N is isotopic to its transpose. We exclude this possibility by

observing the strong entry invariants of entries in rows of row weight 1, tabulated below:

	entries in rows of weight 1	corresponding strong entry invariant(s)
M :	$(k+1, k, k)$; $(k+3, 2, k+2)$	$(1, 3, 3)$; $(1, 4, 2)$
M' :	$(k+3, 2, k+2)$	$(1, 4, 2)$
N :	$(k+1, k, k)$; $(k+2, k+3, 2)$	$(1, 3, 3)$; $(1, 3, 4)$
M^T :	$(k+1, k, k)$; $(k+3, k+2, 2)$	$(1, 3, 3)$; $(1, 2, 4)$
$(M')^T$:	$(k+3, k+2, 2)$	$(1, 2, 4)$
N^T :	$(k+1, k, k)$; $(k+2, 2, k+3)$	$(1, 3, 3)$; $(1, 4, 3)$.

Since isotopisms preserve strong entry invariants, we conclude that M and M^T are not isotopic, M' and $(M')^T$ are not isotopic, and N and N^T are not isotopic. We conclude that M, M', and N have autoparatopism groups $\langle(\mathrm{id}, \mathrm{id}, \mathrm{id}; (123))\rangle \cong C_3$. □

Partial Latin squares with trivial autoparatopism groups were studied in [30] (described as "asymmetric"), where a construction was given for a member of PLS$(n; 3n/2)$ for even $n \geq 4$, and a member of PLS$(n; (3n − 1)/2)$ for odd $n \geq 5$. Construction 4 gives different partial Latin squares with trivial autoparatopism groups following the theme of this section. We note, however, that a lengthy proof of asymmetry was given for the partial Latin squares in [30], but another proof arises simply by inspecting the corresponding partial Latin rectangle graphs (a tool not used in [30]).

Construction 4 (*Trivial Autoparatopism Group*). For $k \geq 3$, define the partial Latin square L by the entry set defined in (2) and define three partial Latin squares $M \in$ PLS$(k + 2; 3k + 3)$, $M' \in$ PLS$(k + 2; 3k + 4)$, and $N \in$ PLS$(k + 2; 3k + 5)$ by

$$E(M) = E(L) \cup \{(k+1, k+2, k+2), (k+2, k+1, k+2), (3, k+2, 1)\},$$
$$E(M') = E(M) \cup \{(k+2, k+2, 2)\},$$

and

$$E(N) = E(M') \cup \{(k+1, k+1, 1)\}.$$

Then M, M', and N have trivial autoparatopism groups (and hence trivial autotopism groups). The smallest such partial Latin squares are given in Fig. 7.

Proof. The row, column, and symbol partitions of M, M', and N are tabulated below:

	row partition	column partition	symbol partition
M :	$\{1, 2, 2, \overbrace{3, \ldots, 3}^{k-2}, 4\}$	$\{2, 2, 2, \overbrace{3, \ldots, 3}^{k-1}\}$	$\{1, 2, \overbrace{3, \ldots, 3}^{k}\}$
M' :	$\{2, 2, 2, \overbrace{3, \ldots, 3}^{k-2}, 4\}$	$\{2, 2, \overbrace{3, \ldots, 3}^{k}\}$	$\{1, 2, \overbrace{3, \ldots, 3}^{k-1}, 4\}$
N :	$\{2, 2, \overbrace{3, \ldots, 3}^{k-1}, 4\}$	$\{2, \overbrace{3, \ldots, 3}^{k+1}\}$	$\{1, 2, \overbrace{3, \ldots, 3}^{k-2}, 4, 4\}$.

In each case, the row, column, and symbol partitions are distinct, so Lemma 2 implies that apar$(M) =$ atop(M), apar$(M') =$ atop(M'), and apar$(N) =$ atop(N). Suppose $\theta = (\alpha, \beta, \gamma; \mathrm{id})$ is an autotopism of one of M, M', or N. The multisets of strong entry invariants for rows containing two entries are tabulated below:

	multiset of strong entry invariants in a given row		
row	M	M'	N
1	$\{(2, 2, 3), (2, 3, 3)\}$	$\{(2, 2, 4), (2, 3, 3)\}$	$\{(2, 2, 4), (2, 3, 4)\}$
$k+1$	$\{(2, 3, 3), (2, 2, 2)\}$	$\{(2, 3, 3), (2, 3, 2)\}$	$-$
$k+2$	$-$	$\{(2, 2, 2), (2, 3, 4)\}$	$\{(2, 3, 2), (2, 3, 4)\}$.

In any case, the multiset of strong entry invariants in row 1 is unique, so Lemma 6 implies 1 is fixed by α. Moreover, the entries in row 1, namely $(1, 1, 2)$ and $(1, 2, 1)$, have distinct strong entry invariants, so those entries map to themselves. Hence 1 is fixed by β and γ. Lemma 10 thus implies θ is trivial. □

The previous constructions are summarized in the following theorem.

Theorem 2. *For $m = 1$ and all $m \geq 3$, there exists a weight-m, partial Latin square with a trivial autotopism group, and autoparatopism group isomorphic to S_3. No partial Latin rectangle with weight 2 satisfies this condition.*

For all $m \geq 5$, there exists a weight-m, partial Latin square with a trivial autotopism group, and autoparatopism group isomorphic to C_2. No partial Latin rectangle with weight $m \in \{1, 2, 3, 4\}$ satisfies this condition.

For all $m \geq 7$, there exists a weight-m, partial Latin square with a trivial autotopism group, and autoparatopism group isomorphic to C_3. No partial Latin rectangle with weight $m \in \{1, 2, \ldots, 6\}$ satisfies this condition.

For all $m \geq 6$, there exists a weight-m, partial Latin square with a trivial autoparatopism group (and hence trivial autotopism group). No partial Latin square with weight $m \in \{1, 2, 3, 4, 5\}$ satisfies this condition. The weight-5 partial Latin rectangle

$$\begin{bmatrix} 1 & \cdot & 2 & \cdot \\ 2 & 1 & \cdot & 3 \end{bmatrix}$$

has a trivial autoparatopism group. No partial Latin rectangle with weight $m \in \{1, 2, 3, 4\}$ satisfies this condition.

All of the claims of this theorem remain true if we restrict to connected partial Latin rectangles.

Proof. The non-existence claims in the theorem are verified computationally. The remaining cases are resolved by Construction 1 (for S_3), Construction 2 (for C_2), Construction 3 (for C_3), and Construction 4 (for $\langle \mathrm{id} \rangle$) with some additional small cases listed below:

$$\begin{bmatrix} 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & \cdot & 3 \\ \cdot & 1 & \cdot \\ 2 & \cdot & \cdot \end{bmatrix} \qquad \begin{bmatrix} \cdot & 3 & \cdot \\ \cdot & 2 & 1 \\ 2 & \cdot & 3 \end{bmatrix} \qquad \begin{bmatrix} \cdot & 1 & 2 \\ 1 & 2 & \cdot \\ 2 & \cdot & \cdot \end{bmatrix} \qquad \begin{bmatrix} \cdot & 1 & 2 \\ 1 & 2 & \cdot \\ 2 & \cdot & 3 \end{bmatrix}$$

$$\begin{matrix} \mathrm{apar} \cong S_3 & \mathrm{apar} \cong S_3 & \mathrm{apar} \cong S_3 & \mathrm{apar} \cong C_2 & \mathrm{apar} \cong C_2 \\ \text{weight 1} & \text{weight 4} & \text{weight 5} & \text{weight 5} & \text{weight 6} \end{matrix}$$

$$\begin{bmatrix} 1 & \cdot & \cdot & 4 \\ \cdot & 1 & 3 & \cdot \\ 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & \cdot & \cdot \\ 3 & 1 & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & \cdot \\ 3 & 1 & \cdot & \cdot \\ \cdot & \cdot & 2 & 1 \\ 4 & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\begin{matrix} \mathrm{apar} \cong \langle \mathrm{id} \rangle & \mathrm{apar} \cong \langle \mathrm{id} \rangle & \mathrm{apar} \cong \langle \mathrm{id} \rangle \\ \text{weight 6} & \text{weight 7} & \text{weight 8} \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & \cdot & \cdot \\ \cdot & \cdot & 2 & 1 \\ 4 & \cdot & \cdot & \cdot \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & \cdot \\ \cdot & \cdot & 2 & 1 \\ 4 & \cdot & \cdot & \cdot \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & \cdot \\ \cdot & 4 & 2 & 1 \\ 4 & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\begin{matrix} \mathrm{apar} \cong \langle \mathrm{id} \rangle & \mathrm{apar} \cong \langle \mathrm{id} \rangle & \mathrm{apar} \cong \langle \mathrm{id} \rangle \\ \text{weight 9} & \text{weight 10} & \text{weight 11.} \end{matrix}$$

The constructions, and the above examples, are all connected partial Latin rectangles, which shows the claims of the theorem hold even when restricting to connected partial Latin rectangles. □

Lemma 9 and Theorem 2 give the following corollaries.

Corollary 1. *For all pairs of finite groups (H_1, H_2), if there exists a connected partial Latin rectangle L with $\mathrm{atop}(L) \cong H_1$ and $\mathrm{apar}(L) \cong H_2$, then there exists a minimum constant $c \leq \mathrm{weight}(L) + 3$ such that for all $m \geq c$, there exists a (possibly disconnected) weight-m partial Latin rectangle M with $\mathrm{atop}(M) \cong H_1$ and $\mathrm{apar}(M) \cong H_2$.*

Proof. From Theorem 2, there exists a connected, weight-m partial Latin rectangle L' with trivial autotopism group and whose autoparatopism group is isomorphic to S_3, for all $m \geq 3$. This suffices to prove the claim when $H_1 \cong \langle \mathrm{id} \rangle$ and $H_2 \cong S_3$. Now assume $H_1 \not\cong \langle \mathrm{id} \rangle$ and/or $H_2 \not\cong S_3$.

We observe that (a) L and L' are connected, and (b) L and L' are non-paratopic since they have different autotopism and/or autoparatopism groups. The result then follows from the blockwise construction in Lemma 9 for L and L'. □

In the particular case that $H_1 = H_2 = G$, we have the following.

Corollary 2. *For all finite groups G, there exists a minimum constant c such that for all $m \geq c$, there exists a (possibly disconnected) weight-m partial Latin square with $\mathrm{atop}(L) = \mathrm{apar}(L) \cong G$.*

Proof. Theorem 1 implies that a partial Latin square L with $\mathrm{atop}(L) = \mathrm{apar}(L) \cong G$ exists. Theorem 2 implies that a weight-m, connected partial Latin square L' with $\mathrm{atop}(L') = \langle \mathrm{id} \rangle$ and $\mathrm{apar}(L') \cong S_3$ exists for all $m \geq 3$. Let n be the order of L' and assume $m > \mathrm{weight}(L)$. Then any autoparatopism of the partial Latin square

$$M := \begin{array}{|c|c|} \hline L' & \cdot \\ \hline \cdot & L + n \\ \hline \end{array}$$

maps L' to itself (as it is the unique weight-m component in the partial Latin rectangle graph). Like in Lemma 9, we have $\mathrm{atop}(M) \cong \mathrm{atop}(L') \times \mathrm{atop}(L) = \langle \mathrm{id} \rangle \times \mathrm{atop}(L) \cong G$ and $\mathrm{apar}(M) \cong \mathrm{apar}(L)$ (since $\mathrm{atop}(L') = \langle \mathrm{id} \rangle$ and $\mathrm{apar}(L') = S_3$). □

$$
\begin{array}{ccc}
k = 3 & k = 4 & k = 5
\end{array}
$$

$$
M: \quad
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & \cdot \\
\cdot & 2 & 4 & 3 & 1 \\
\cdot & \cdot & 3 & \cdot & 5 \\
\cdot & \cdot & \cdot & 5 & \cdot
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & \cdot & \cdot \\
\cdot & 2 & 4 & 3 & \cdot & 1 \\
\cdot & \cdot & 3 & 5 & 4 & \cdot \\
\cdot & \cdot & \cdot & 4 & \cdot & 6 \\
\cdot & \cdot & \cdot & \cdot & 6 & \cdot
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 2 & 4 & 3 & \cdot & \cdot & 1 \\
\cdot & \cdot & 3 & 5 & 4 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 4 & 6 & 5 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 5 & \cdot & 7 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 7 & \cdot
\end{bmatrix}
$$

$$
M': \quad
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & \cdot \\
\cdot & 2 & 4 & 3 & 1 \\
\cdot & \cdot & 3 & \cdot & 5 \\
\cdot & \cdot & \cdot & 5 & \mathbf{2}
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & \cdot & \cdot \\
\cdot & 2 & 4 & 3 & \cdot & 1 \\
\cdot & \cdot & 3 & 5 & 4 & \cdot \\
\cdot & \cdot & \cdot & 4 & \cdot & 6 \\
\cdot & \cdot & \cdot & \cdot & 6 & \mathbf{2}
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 2 & 4 & 3 & \cdot & \cdot & 1 \\
\cdot & \cdot & 3 & 5 & 4 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 4 & 6 & 5 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 5 & \cdot & 7 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 7 & \mathbf{2}
\end{bmatrix}
$$

$$
N: \quad
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & \cdot \\
\cdot & 2 & 4 & 3 & 1 \\
\cdot & \cdot & 3 & \mathbf{1} & 5 \\
\cdot & \cdot & \cdot & 5 & \mathbf{2}
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & \cdot & \cdot \\
\cdot & 2 & 4 & 3 & \cdot & 1 \\
\cdot & \cdot & 3 & 5 & 4 & \cdot \\
\cdot & \cdot & \cdot & 4 & \mathbf{1} & 6 \\
\cdot & \cdot & \cdot & \cdot & 6 & \mathbf{2}
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 2 & 4 & 3 & \cdot & \cdot & 1 \\
\cdot & \cdot & 3 & 5 & 4 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 4 & 6 & 5 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 5 & \mathbf{1} & 7 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 7 & \mathbf{2}
\end{bmatrix}
$$
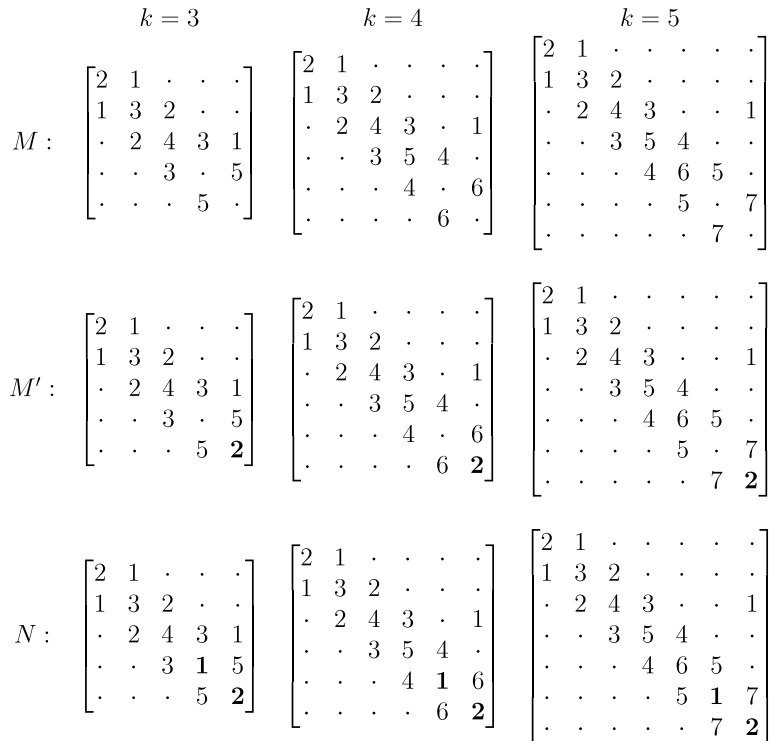
Fig. 7. The smallest partial Latin squares described by Construction 4. Entries outside of $E(M)$ are highlighted in bold.

4. Two-row partial Latin rectangles

Here we describe constructions of 2-row partial Latin rectangles (i.e., with dimensions satisfying $r = 2$) with trivial autotopism groups for all possible weights m and all possible autoparatopism groups (up to isomorphism). We include three situations: (a) $s = n$ and autoparatopism group C_2, (b) $s = n$ and autoparatopism group $\langle \text{id} \rangle$, and (c) $s \neq n$ and autoparatopism group $\langle \text{id} \rangle$. Provided $s > 2$ and $n > 2$, Lemma 3 implies these are the only possible scenarios up to paratopism (if $s = n > 2$, then we can have an autoparatopism with fourth component (23), and if r, s, and n are all distinct then the autotopism group and autoparatopism group will be both trivial).

There are two non-paratopic partial Latin rectangles with trivial autotopism groups with at least two dimension parameters (i.e., r, s, or n) equal to 2:

$$
\begin{bmatrix}
1 & 2 \\
\cdot & 1
\end{bmatrix}
\qquad
\begin{bmatrix}
1 & 2 \\
3 & \cdot
\end{bmatrix}
$$

$$
\begin{array}{cc}
\text{apar} \cong S_3 & \text{apar} \cong C_2 \\
\text{PLS}(2; 3) & \text{PLR}(2, 2, 3; 3).
\end{array}
$$

Henceforth we will ensure $r = 2$ and $s > 2$ and $n > 2$.

Construction 5. For $k \geq 3$, the partial Latin rectangles $L \in \text{PLR}(2, k + 1, k + 1; 2k - 1)$ with entry set

$$
E(L) = \{(1, j, j) : 1 \leq j \leq k\} \cup \{(2, j, j - 1) : 3 \leq j \leq k\} \cup \{(2, k + 1, k + 1)\}
$$

and $M \in \text{PLR}(2, k + 1, k + 1; 2k)$ with entry set

$$
E(M) = E(L) \cup \{(2, 2, 1)\}
$$

have trivial autotopism groups, and autoparatopism groups isomorphic to C_2. The smallest such partial Latin rectangles are given in Fig. 8.

Proof. The partial Latin rectangle graphs $\Gamma^{(e)}(L)$ and $\Gamma^{(e)}(M)$ are drawn in Fig. 9 when $k = 5$, and are given their natural edge colorings. Since $k \geq 3$, the green edges of $\Gamma^{(e)}(L)$ and $\Gamma^{(e)}(M)$ contain a triangle, whereas the orange and purple edges do not. Thus, green edges map to green edges in any edge-color-class preserving automorphism. We can see that the only

$$L : \quad \begin{array}{ccc} k=3 & k=4 & k=5 \end{array}$$

$$L : \quad \begin{bmatrix} 1 & 2 & 3 & \cdot \\ \cdot & \cdot & 2 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & \cdot \\ \cdot & \cdot & 2 & 3 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \cdot \\ \cdot & \cdot & 2 & 3 & 4 & 6 \end{bmatrix}$$

$$M : \quad \begin{bmatrix} 1 & 2 & \cdot \\ \cdot & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & \cdot \\ \cdot & 1 & 2 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & \cdot \\ \cdot & 1 & 2 & 3 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \cdot \\ \cdot & 1 & 2 & 3 & 4 & 6 \end{bmatrix}$$

Fig. 8. The smallest partial Latin squares described by Construction 5. We also list a PLR(2, 3, 3; 4) which fits the pattern in the even-weight case (i.e., it has a trivial autotopism group, and autoparatopism group isomorphic to C_2).
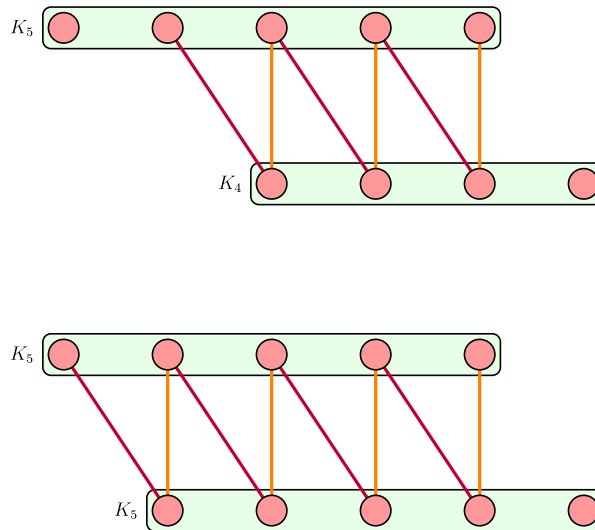
Fig. 9. Graphs $\Gamma^{(e)}(L)$ and $\Gamma^{(e)}(M)$ in Construction 5, for $k = 5$. The rectangles indicate cliques with green edges.

non-trivial edge-color-class preserving automorphism of these graphs flips the path formed by the orange and purple edges, which swaps the orange and purple edge color classes.

The non-trivial edge-color-class preserving automorphism of $\Gamma^{(e)}(L)$ corresponds to the autoparatopism $\theta := (\mathrm{id}, \beta, \beta; (23))$ where β is the permutation that reverses the sequence $(2, 3, \ldots, k)$. To verify this, we observe

$$\{(1, j, j) : 1 \leq j \leq k\} \xmapsto{\theta} \{(1, \beta(j), \beta(j)) : 1 \leq j \leq k\},$$

$$\{(2, j, j-1) : 3 \leq j \leq k\} \xmapsto{\theta} \{(2, \beta(j-1), \beta(j)) : 3 \leq j \leq k\}, \text{ and}$$

$$(2, k+1, k+1) \xmapsto{\theta} (2, k+1, k+1)$$

and we can check that the left-hand side actually equals the right hand side in all three cases, implying L and L^θ have the same entry sets, i.e., $\theta \in \mathrm{apar}(L)$.

The non-trivial edge-color-class preserving automorphism of $\Gamma^{(e)}(M)$ corresponds to the autoparatopism $\theta' := (\mathrm{id}, \beta', \beta'; (23))$ where β' is the permutation that reverses the sequence $(1, 2, \ldots, k)$. To verify this, we observe

$$\{(1, j, j) : 1 \leq j \leq k\} \xmapsto{\theta} \{(1, \beta'(j), \beta'(j)) : 1 \leq j \leq k\},$$

$$\{(2, j, j-1) : 2 \leq j \leq k\} \xmapsto{\theta} \{(2, \beta'(j-1), \beta'(j)) : 2 \leq j \leq k\}, \text{ and}$$

$$(2, k+1, k+1) \xmapsto{\theta} (2, k+1, k+1)$$

and again we can check that the left-hand side actually equals the right hand side in all three cases, so $\theta' \in \mathrm{apar}(M)$.

Lemma 7 therefore implies that $\mathrm{apar}(L) = \langle (\mathrm{id}, \beta, \beta; (23)) \rangle \cong C_2$ and $\mathrm{apar}(M) = \langle (\mathrm{id}, \beta', \beta'; (23)) \rangle \cong C_2$, and their autotopism groups are trivial. $\square$

Construction 6. *For $k \geq 2$, the partial Latin rectangles $L \in \mathrm{PLR}(2, k+3, k+3; 2k+3)$ with entry set*

$$E(L) = \{(1, j, j) : 1 \leq j \leq k\} \cup \{(2, j+1, j) : 1 \leq j \leq k-1\}$$
$$\cup \{(1, k+2, k+2), (1, k+3, k+3), (2, 1, k+1), (2, k+1, k+2)\}$$

$$
\begin{array}{ccc}
k=2 & k=3 & k=4 \\
L: \begin{bmatrix} 1 & 2 & \cdot & 4 & 5 \\ 3 & 1 & 4 & \cdot & \cdot \end{bmatrix} &
\begin{bmatrix} 1 & 2 & 3 & \cdot & 5 & 6 \\ 4 & 1 & 2 & 5 & \cdot & \cdot \end{bmatrix} &
\begin{bmatrix} 1 & 2 & 3 & 4 & \cdot & 6 & 7 \\ 5 & 1 & 2 & 3 & 6 & \cdot & \cdot \end{bmatrix}
\end{array}
$$

$$
M: \begin{bmatrix} 1 & 2 & \cdot & 4 & 5 & \cdot & 7 \\ 3 & 1 & 4 & \cdot & \cdot & 7 & 6 \end{bmatrix}
\begin{bmatrix} 1 & 2 & 3 & \cdot & 5 & 6 & \cdot & 8 \\ 4 & 1 & 2 & 5 & \cdot & \cdot & 8 & 7 \end{bmatrix}
\begin{bmatrix} 1 & 2 & 3 & 4 & \cdot & 6 & 7 & \cdot & 9 \\ 5 & 1 & 2 & 3 & 6 & \cdot & \cdot & 9 & 8 \end{bmatrix}.
$$

Fig. 10. The smallest partial Latin squares described by Construction 6.
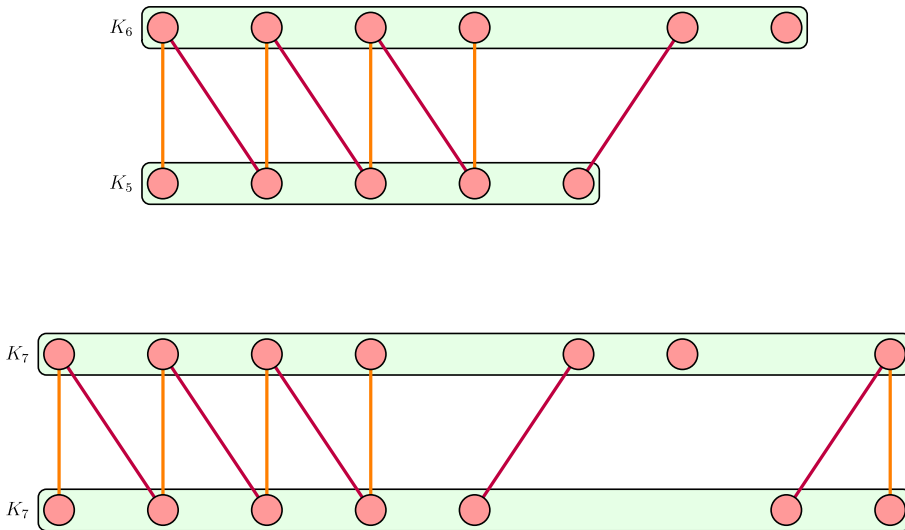
Fig. 11. Graphs $\Gamma^{(e)}(L)$ and $\Gamma^{(e)}(M)$ in Construction 6, for $k = 4$.

and $M \in \mathrm{PLR}(2, k+5, k+5; 2k+6)$ with entry set

$$E(M) = E(L) \cup \{(1, k+5, k+5), (2, k+4, k+5), (2, k+5, k+4)\}$$

have trivial autoparatopism groups (and hence trivial autotopism groups). The smallest such partial Latin rectangles are given in Fig. 10.

Proof. The partial Latin rectangle graphs are drawn in Fig. 11 when $k = 4$. By Lemma 7, it is sufficient to prove that $\Gamma^{(e)}(L)$ and $\Gamma^{(e)}(M)$ have no non-trivial edge-color-class preserving automorphisms.

In both graphs $\Gamma^{(e)}(L)$ and $\Gamma^{(e)}(M)$, the green edges contain a triangle, whereas the orange and purple edges do not, so green edges map to green edges. Therefore, any edge-color-class preserving automorphism of $\Gamma^{(e)}(L)$ and $\Gamma^{(e)}(M)$ is also an edge-color-class preserving automorphism of $\Gamma^{(e)}(L) \setminus \{\text{green edges}\}$ and $\Gamma^{(e)}(M) \setminus \{\text{green edges}\}$, respectively.

In $\Gamma^{(e)}(L) \setminus \{\text{green edges}\}$ and $\Gamma^{(e)}(L) \setminus \{\text{green edges}\}$, the only non-trivial edge-color-class preserving automorphisms swap the endpoints of the isolated purple edge and/or flip the orange–purple alternating path, but this is not an automorphism of $\Gamma^{(e)}(L)$ nor $\Gamma^{(e)}(M)$, respectively. So there are no non-trivial edge-color-class preserving automorphisms of $\Gamma^{(e)}(L)$ or $\Gamma^{(e)}(M)$. □

Construction 7. *For $k \geq 4$, the partial Latin rectangles $L \in \mathrm{PLR}(2, k, k+1; 2k-2)$ with entry set*

$$E(L) = \{(1, j, j+1) : 1 \leq j \leq k\} \cup \{(2, j, j-1) : 2 \leq j \leq k-1\}$$

and $M \in \mathrm{PLR}(2, k, k+1; 2k-1)$ with entry set

$$E(M) = E(L) \cup \{(2, 1, k+1)\}$$

have trivial autoparatopism groups (and hence a trivial autotopism groups). The smallest such partial Latin rectangles are given in Fig. 12.

Proof. The dimensions 2, k, and $k+1$ are all distinct (as $k \geq 4$), so Lemma 2 implies that $\mathrm{apar}(L) = \mathrm{atop}(L)$ and $\mathrm{apar}(M) = \mathrm{atop}(M)$.

Fig. 13 illustrates the partial Latin rectangle graphs for $k = 7$. In both $\Gamma^{(e)}(L)$ and $\Gamma^{(e)}(M)$, the green edges contain a triangle, whereas the orange and purple edges do not, so green edges map to green edges. Therefore, any edge-color

$$
L: \quad
\begin{array}{c} k=4 \\ \begin{bmatrix} 2 & 3 & 4 & 5 \\ \cdot & 1 & 2 & \cdot \end{bmatrix} \end{array}
\quad
\begin{array}{c} k=5 \\ \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ \cdot & 1 & 2 & 3 & \cdot \end{bmatrix} \end{array}
\quad
\begin{array}{c} k=6 \\ \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ \cdot & 1 & 2 & 3 & 4 & \cdot \end{bmatrix} \end{array}
$$

$$
M: \quad
\begin{bmatrix} 2 & 3 & 4 \\ 4 & 1 & \cdot \end{bmatrix}
\quad
\begin{bmatrix} 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & \cdot \end{bmatrix}
\quad
\begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & \cdot \end{bmatrix}
\quad
\begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 1 & 2 & 3 & 4 & \cdot \end{bmatrix}
$$

Fig. 12. The smallest partial Latin squares described by Construction 7. We also list a PLR(2, 3, 4; 5) which fits the pattern in the odd-weight case (i.e., it has a trivial autoparatopism group).
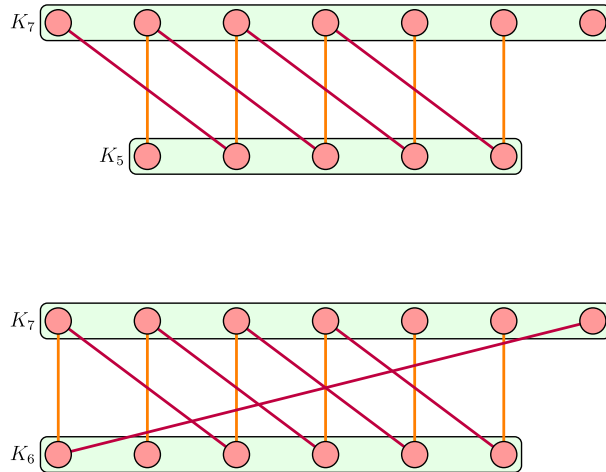
Fig. 13. Graphs $\Gamma^{(e)}(L)$ and $\Gamma^{(e)}(M)$ in Construction 7, for $k=7$.

preserving automorphism of $\Gamma^{(e)}(L)$ and $\Gamma^{(e)}(M)$ is also an edge-color preserving automorphism of $\Gamma^{(e)}(L) \setminus \{\text{green edges}\}$ and $\Gamma^{(e)}(M) \setminus \{\text{green edges}\}$, respectively. However, the only non-trivial edge-color preserving automorphisms of $\Gamma^{(e)}(L) \setminus \{\text{green edges}\}$ and $\Gamma^{(e)}(M) \setminus \{\text{green edges}\}$ are not automorphisms of $\Gamma^{(e)}(L)$ nor $\Gamma^{(e)}(M)$. Lemma 7 therefore implies that $\mathrm{apar}(L) = \mathrm{atop}(L)$ and $\mathrm{apar}(M) = \mathrm{atop}(M)$ are trivial. $\square$

The previous constructions combine to give the following theorem.[2]

Theorem 3. *There exists a weight-m, 2-row partial Latin rectangle with an equal number of columns and symbols and a trivial autotopism group and autoparatopism group isomorphic to C_2 for all $m \geq 4$. No partial Latin rectangle with weight $m \in \{1, 2, 3\}$ satisfies this condition.*

There exists a weight-m, 2-row partial Latin rectangle with an equal number of columns and symbols and a trivial autoparatopism group when $m = 7$ and for all $m \geq 9$. No partial Latin rectangle with weight $m \in \{1, 2, \ldots, 6\} \cup \{8\}$ satisfies this condition.

There exists a weight-m, 2-row partial Latin rectangle with an unequal number of columns and symbols and a trivial autoparatopism group for all $m \geq 5$. No partial Latin rectangle with weight $m \in \{1, 2, 3, 4\}$ satisfies this condition.

5. Final remarks and further research

The concept of partial Latin rectangle graph has been introduced in this paper to deal with the theory of symmetries of partial Latin rectangles. This tool has helped us to determine when there exists a weight-m partial Latin rectangle with a trivial autotopism group and one of the possible autoparatopism groups up to isomorphism (namely, $\langle \mathrm{id} \rangle$, C_2, C_3, or S_3). We have given the construction of such partial Latin rectangles for all possible weights when (a) $r = s = n$, (b) $r = 2$ and $s = n$, and (c) $r = 2$ and $s \neq n$. In order to resolve the last two cases, we give constructions of partial Latin rectangles which happen to large weight. In fact, all the proposed examples have at most four empty cells. This motivates us to ask the following question:

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[2] The non-existence of members of PLR(2, n, n; 6) and PLR(2, n, n; 8) with trivial autoparatopism groups was resolved computationally. The same code used for this search also constructed members of PLR(2, 5, 5; 7), PLR(2, 6, 6; 9), and PLR(2, 7, 7; 10) with trivial autoparatopism groups.
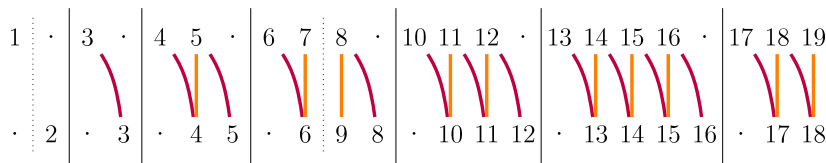
Fig. 14. A two-row partial Latin rectangle L with $s \neq n$ and atop$(L) = $ apar$(L) = \langle\text{id}\rangle$ overlaying its partial Latin rectangle graph. Green edges are not shown. The vertical lines highlight component boundaries.

Question 2. *What is the smallest weight of an $r \times s$ partial Latin rectangle containing symbols from $[n] \cup \{\cdot\}$ with given autotopism and autoparatopism groups?*

We will consider the two-row case. Under our initial assumptions, partial Latin rectangles have at least one entry per row, at least one entry per column, and at least one copy of each symbol. The weight of a partial Latin rectangle satisfying this assumption must be at least max$\{r, s, n\}$. In the particular case that is considered in Section 4 (i.e., $r = 2$ and trivial autotopism group), this bound is tight for the paratopic partial Latin rectangles

$$\begin{bmatrix} 1 & 2 \\ \cdot & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & \cdot \\ \cdot & \cdot & 2 \end{bmatrix}$$
$$\text{PLR}(2, 2, 3; 3) \qquad \text{PLR}(2, 3, 2; 3).$$
$$\text{atop} \cong \langle\text{id}\rangle \qquad \text{atop} \cong \langle\text{id}\rangle$$
$$\text{apar} \cong C_2 \qquad \text{apar} \cong C_2$$

The proportion of non-empty cells, $m/(rs)$, for these partial Latin rectangles is $3/4$ and $1/2$, respectively, and any two-row partial Latin rectangle with an entry in each column will have a proportion of at least $1/2$.

For any two-row partial Latin rectangle L, its graph $\Gamma^{(e)}(L)$ comprises of two green-edge cliques $K_{w_1^{\text{row}}} \cup K_{w_2^{\text{row}}}$ connected to one another by an orange matching and a purple matching. So $G := \Gamma^{(e)}(L) \setminus \{\text{green edges}\}$ consists of the union of alternating-color paths and alternating-color even-length cycles involving orange and purple edges. Fig. 14 gives an example with row weights 17 and 16.

An even-length cycle in G gives rise to a non-trivial color-preserving automorphism where we cycle the vertices by 2, and a corresponding non-trivial autotopism of the partial Latin rectangle which cyclically permutes the columns involved. Since we wish to consider the case of trivial autotopism groups, we exclude cycles from G in our analysis.

Let a, b, and c respectively be the number of paths in G with (a) both ends purple (necessarily odd-length), (b) no edges, or opposite-color edges at its ends (necessarily even-length), and (c) both ends orange (necessarily odd-length). For the example in Fig. 14, $a = 4$, $b = 5$, and $c = 0$. The number of empty cells in L is equal to the number purple pendant edges in G, which is equal to $2a + b$. The number of non-empty cells in L is the number of vertices in G.

If $w_1^{\text{row}} \neq w_2^{\text{row}}$, a non-trivial autotopism of L can only arise if G contains: (a) two odd-length paths of equal lengths, with both paths having the same end edge colors, (b) two even-length paths of equal lengths with endpoints in the same green-edge clique in $\Gamma^{(e)}(L)$. Thus, if atop$(L) = \langle\text{id}\rangle$, the number of vertices in G is at least $a(a+1) + \lceil b^2/2 \rceil + c(c+1)$, with equality when each path is of the minimum length that does not result in an autotopism. Thus, with a, b, and/or c tending to infinity, the number of cells is $\Theta(a^2 + b^2 + c^2)$, so the number of columns $s = \Theta(a^2 + b^2 + c^2)$, and the number of empty cells is $O(\sqrt{s})$ implying almost all cells are non-empty.

If $w_1^{\text{row}} = w_2^{\text{row}}$, the autotopisms (a)–(c) are all still possibilities, as are additional autotopisms where the two green cliques are swapped, making the analysis more difficult. In our subsequent constructions, we ensure $w_1^{\text{row}} \neq w_2^{\text{row}}$ (by assuming b is odd, and taking minimum length paths) to avoid this issue.

Suppose we choose $a = 2d$ and $b = 2d + 1$, for some $d \to \infty$, and $c = 0$, with the minimum path lengths. Fig. 14 is a $d = 2$ example. In this setting, we have $6d^2 + 10d + 2$ cells, so $s = \frac{1}{2}(6d^2 + 10d + 2) = 3d^2 + 5d + 1$. Thus there are $6d + 1 = \sqrt{12s} + o(\sqrt{s})$ empty cells (where $\sqrt{12} \simeq 3.464$). This example will also have a trivial autoparatopism group as $r < n < s$.

For the trivial autoparatopism group case with $r = 2$ and $s = n > 2$, we take $a = c = d - 1$ and $b = 2d + 1$ as $d \to \infty$, with the minimum path lengths excluding the purple path of length 1 and the orange–purple–orange path of length 3. Fig. 15 gives a $d = 3$ example. Any non-trivial autoparatopism would swap the orange and purple edges, but since there is no purple path of length 1 while there is an orange path of length 1, this is not possible. So any autoparatopism is an autotopism, and all autotopisms are trivial. In this setting, we have $s = 2d^2 + 4d - 3$. Thus there are $4d - 1 = \sqrt{8s} + o(\sqrt{s})$ empty cells (where $\sqrt{8} \simeq 2.828$).

For the trivial autotopism group and C_2 autoparatopism group case, we must have $a = c$ (as we have a non-trivial automorphism of the graph which swaps the purple and orange color classes). Here, we choose $a = c = d$ and $b = 2d + 1$, for some $d \to \infty$, with the minimum path lengths. Fig. 16 is a $d = 2$ example. In this setting, we have $s = 2d^2 + 4d + 1$. Thus there are $4d + 1 = \sqrt{8s} + o(\sqrt{s})$ empty cells.

The choices for a, b, and c above ensure that the least dense blocks are added to the partial Latin rectangles as d grows (subject to the constraints, and barring a finite number of small exceptions). We thus expect that the asymptotic constants ($\sqrt{12}$ and $\sqrt{8}$) are the smallest possible or close to it.
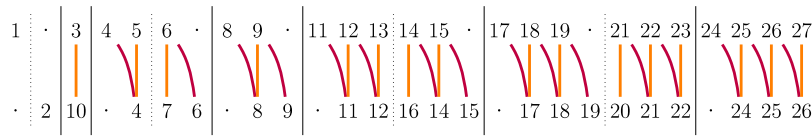
Fig. 15. A two-row partial Latin rectangle L with $s = n$ and $\text{atop}(L) = \text{apar}(L) = \langle \text{id} \rangle$ overlaying its partial Latin rectangle graph.
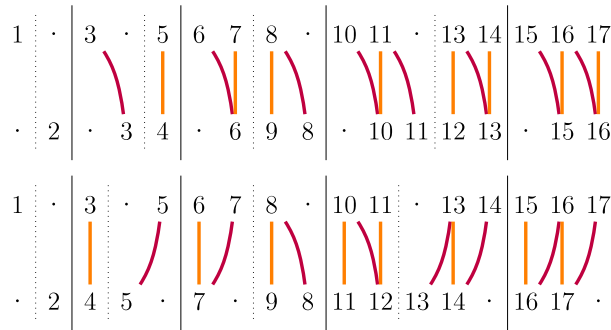

Fig. 16. A two-row partial Latin rectangle L with $\text{atop}(L) = \langle \text{id} \rangle$ and $\text{apar}(L) \cong C_2$ overlaying its partial Latin rectangle graph. We also show L^θ where $\theta = (\text{id}, \text{id}, \text{id}; (23))$ so the reader can confirm there is a unique isotopism between L and L^θ.

Further studies are required in order to deal with other cases, the most general question being:

Question 3. *For which finite groups, H_1 and H_2, does there exist a partial Latin rectangle $L \in \text{PLR}(r, s, n; m)$ with $\text{atop}(L) \cong H_1$ and $\text{apar}(L) \cong H_2$?*

The tools developed here would likely continue to be useful in extending this research, and resolving more cases of Question 3. However, Question 3 has six parameters (H_1, H_2, r, s, n, and m) and a complete solution would likely be an enormous task.

Acknowledgments

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