LOCAL CLASSIFICATION AND EXAMPLES OF AN IMPORTANT CLASS OF PARACONTACT METRIC MANIFOLDS

VERÓNICA MARTÍN-MOLINA

Abstract. We study paracontact metric \((\kappa, \mu)\)-spaces with \(\kappa = -1\), equivalent to \(h^2 = 0\) but not \(h = 0\). In particular, we will give an alternative proof of Theorem 3.2 of [11] and present examples of paracontact metric \((-1, 2)\)-spaces and \((-1, 0)\)-spaces of arbitrary dimension with tensor \(h\) of every possible constant rank. We will also show explicit examples of paracontact metric \((-1, \mu)\)-spaces with tensor \(h\) of non-constant rank, which were not known to exist until now.

1. Introduction

Paracontact metric manifolds, the odd-dimensional analogue of paraHermitian manifolds, were first introduced in [10] and they have been the object of intense study recently, particularly since the publication of [14]. An important class among paracontact metric manifolds is that of the \((\kappa, \mu)\)-spaces, which satisfy the nullity condition [5]
\[
R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),
\]
for all \(X, Y\) vector fields on \(M\), where \(\kappa\) and \(\mu\) are constants and \(h = \frac{1}{2}L_\xi \varphi\).

This class includes the paraSasakian manifolds [10, 14], the paracontact metric manifolds satisfying \(R(X, Y)\xi = 0\) for all \(X, Y\) [15], certain \(g\)-natural paracontact metric structures constructed on unit tangent sphere bundles [7], etc.

The definition of a paracontact metric \((\kappa, \mu)\)-space was motivated by the relationship between contact metric and paracontact geometry. More precisely, it was proved in [4] that any non-Sasakian contact metric \((\kappa, \mu)\)-space accepts two paracontact metric \((\tilde{\kappa}, \tilde{\mu})\)-structures with the same contact form. On the other hand, under certain natural conditions, every non-paraSasakian paracontact \((\tilde{\kappa}, \tilde{\mu})\)-space admits a contact metric \((\kappa, \mu)\)-structure compatible with the same contact form [5].

Paracontact metric \((\kappa, \mu)\)-spaces satisfy that \(h^2 = (\kappa + 1)\varphi^2\) but this condition does not give any type of restriction over the value of \(\kappa\), unlike in contact metric geometry, because the metric of a paracontact metric manifold is not positive definite. However, it is useful to distinguish the cases \(\kappa > -1\), \(\kappa < -1\) and \(\kappa = -1\). In the first two, equation (1) determines the curvature completely and either the tensor \(h\) or \(\varphi h\) are diagonalisable [5]. The case \(\kappa = -1\) is equivalent to \(h^2 = 0\) but not to \(h = 0\). Indeed, there are examples of paracontact metric \((\kappa, \mu)\)-spaces with \(h^2 = 0\) but \(h \neq 0\), as was first shown in [2, 5, 8, 12].

However, only some particular examples were given of this type of space and no effort had been made to understand the general behaviour of the tensor \(h\) of a paracontact metric \((-1, \mu)\)-space until the author published [11], where a local classification depending on the rank of \(h\) was given in Theorem 3.2. The author also provided explicit examples of all the possible constant values of the rank of \(h\) when \(\mu = 2\). She explained why the values \(\mu = 0\) and \(\mu = 2\) are special and studying them is enough. Finally, she showed some paracontact metric \((-1, 0)\)-spaces of any dimension with
rank($h$) = 1 and of paracontact metric $(-1, 0)$-spaces of dimension 5 and 7 for any possible constant rank of $h$. These were the first examples of this type with $\mu \neq 2$ and dimension greater than 3.

In the present paper, after the preliminaries section, we will give an alternative proof of Theorem 3.2 of [11] that does not use [13] and we will complete the examples of all the possible cases of constant rank of $h$ by presenting $(2n + 1)$-dimensional paracontact metric $(-1, 0)$-spaces with rank($h$) = 2, . . . , n. Lastly, we will also show the first explicit examples ever known of paracontact metric $(-1, 2)$-spaces and $(-1, 0)$-spaces with $h$ of non-constant rank.

2. Preliminaries

An almost paracontact structure on a $(2n + 1)$-dimensional smooth manifold $M$ is given by a (1, 1)-tensor field $\varphi$, a vector field $\xi$ and a 1-form $\eta$ satisfying the following conditions [10]:

(i) $\eta(\xi) = 1, \varphi^2 = I - \eta \otimes \xi$,

(ii) the eigendistributions $D^+$ and $D^-$ of $\varphi$ corresponding to the eigenvalues 1 and $-1$, respectively, have equal dimension $n$.

It follows that $\varphi \xi = 0$, $\eta \circ \varphi = 0$ and rank($\varphi$) = 2$n$. If an almost paracontact manifold admits a semi-Riemannian metric $g$ such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

for all $X, Y$ on $M$, then $(M, \varphi, \xi, \eta, g)$ is called an almost paracontact metric manifold. Then $g$ is necessarily of signature $(n + 1, n)$ and satisfies $\eta = g(\cdot, \xi)$ and $g(\cdot, \varphi \cdot) = -g(\varphi \cdot, \cdot)$.

We can now define the fundamental 2-form of the almost paracontact metric manifold by $\Phi(X, Y) = g(X, \varphi Y)$. If $\Phi = 0$, then $\eta$ becomes a contact form (i.e. $\eta \wedge (d\eta)^n \neq 0$) and $(M, \varphi, \xi, \eta, g)$ is said to be a paracontact metric manifold.

We can also define on a paracontact metric manifold the tensor field $h := \frac{1}{2}L_{\xi} \varphi$, which is symmetric with respect to $g$ (i.e. $g(hX, Y) = g(X, hY)$, for all $X, Y$), anti-commutes with $\varphi$ and satisfies $h\xi = trh = 0$ and the identity $\nabla \xi = -\varphi + \varphi b$ (14). Moreover, it vanishes identically if and only if $\xi$ is a Killing vector field, in which case $(M, \varphi, \xi, \eta, g)$ is called a K-paracontact manifold.

An almost paracontact structure is said to be normal if and only if the tensor $[\varphi, \varphi] - 2d\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of $\varphi$ [14]:

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal paracontact metric manifold is said to be a paraSasakian manifold and is in particular $K$-paracontact. The converse holds in dimension 3 [8] but not in general in higher dimensions. However, it was proved in Theorem 3.1 of [11] that it also holds for $(-1, \mu)$-spaces. Every paraSasakian manifold satisfies

$$R(X, Y)\xi = -\eta(Y)X - \eta(X)Y,$$

for every $X, Y$ on $M$. The converse is not true, since Examples 3.8–3.11 of [11] and Examples 4.1 and 4.5 of the present one show that there are examples of paracontact metric manifolds satisfying equation (2) but with $h \neq 0$ (and therefore not $K$-paracontact or paraSasakian). Moreover, it is also clear in Example 4.5 that the rank of $h$ does not need to be constant either, since $h$ can be zero at some points and non-zero in others.

The main result of [11] is the following local classification of paracontact metric $(-1, \mu)$-spaces:

**Theorem 2.1** ([11]). Let $M$ be a $(2n + 1)$-dimensional paracontact metric $(-1, \mu)$-space. Then we have one of the following possibilities:

1. either $h = 0$ and $M$ is paraSasakian,
2. or $h \neq 0$ and rank($h_p$) $\in \{1, \ldots, n\}$ at every $p \in M$ where $h_p \neq 0$. Moreover, there exists a basis $\{\xi_p, X_1, Y_1, \ldots, X_n, Y_n\}$ of $T_p(M)$ such that the only non-vanishing components of $g$ are

$$g_p(\xi_p, \xi_p) = 1, \quad g_p(X_i, Y_i) = \pm 1,$$
and

\[ h_p(x, y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad h_p(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \]

where obviously there are exactly rank\( (h_p) \) submatrices of the first type.

If \( n = 1 \), such a basis \( \{\xi_p, X_1, Y_1\} \) also satisfies that

\[ \varphi_p X_1 = \pm X_1, \quad \varphi_p Y_1 = \mp Y_1, \]

and the tensor \( h \) can be written as

\[ h_p(\xi_p, X_1, Y_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \]

Many examples of paraSasakian manifolds are known. For instance, hyperboloids \( \mathbb{H}^{2n+1} \) and the hyperbolic Heisenberg group \( \mathcal{H}^{2n+1} = \mathbb{R}^{2n} \times \mathbb{R} \). We can also obtain \((\eta\text{-Einstein})\) paraSasakian manifolds from contact \((\kappa, \mu)\)-spaces with \( |1 - \frac{\kappa}{\mu}| < \sqrt{1 - \kappa} \). In particular, the tangent sphere bundle \( T_1N \) of any space form \( N(c) \) with \( c < 0 \) admits a canonical \( \eta\text{-Einstein} \) paraSasakian structure. Finally, we can see how to construct explicitly a paraSasakian structure on a Lie group (see Example 3.4 of \( \mathbb{H}^{2n+1} \)) or on the unit tangent sphere bundle, \( \mathbb{S}^n \).

On the other hand, until \( \mathbb{S}^n \) only some types of non-paraSasakian paracontact metric \((-1, \mu)\)-spaces were known:

- \((2n + 1)\)-dimensional paracontact metric \((-1, 2)\)-space with rank\( (h) = n \), \( \mathbb{S}^n \).
- \(3\)-dimensional paracontact metric \((-1, 2)\)-space with rank\( (h) = 1 \), \( \mathbb{H}^2 \).
- \(3\)-dimensional paracontact metric \((-1, 0)\)-space with rank\( (h) = 1 \). This example is not paraSasakian but it satisfies \( \mathbb{H}^2 \).

The answer to why there seems to be only examples of paracontact metric \((-1, \mu)\)-spaces with \( \mu = 2 \) or \( \mu = 0 \) is a \( \mathcal{D}_c \)-homothetic deformation, i.e. the following change of a paracontact metric structure \((M, \varphi, \xi, \eta, g)\) \( \mathbb{S}^n \):

\[ \varphi' := \varphi, \quad \xi' := \frac{1}{c} \xi, \quad \eta' := c \eta, \quad g' := cg + c(c - 1) \eta \otimes \eta, \]

for some \( c \neq 0 \).

It is known that \((\varphi', \xi', \eta', g')\) is again a paracontact metric structure on \( M \) and that \( K\)-paracontact and paraSasakian structures are also preserved. However, curvature conditions like \( R(X, Y) \xi = 0 \) are destroyed, since paracontact metric \((\kappa, \mu)\)-spaces become other paracontact metric \((\kappa', \mu')\)-spaces with

\[ \kappa' = \frac{\kappa + 1 - \frac{c^2}{e}}{e}, \quad \mu' = \frac{\mu - 2 + 2c}{e}. \]

In particular, if \((M, \varphi, \xi, \eta, g)\) is a paracontact metric \((-1, \mu)\)-space, then the deformed manifold is another paracontact metric \((-1, \mu')\)-space with \( \mu' = \frac{\mu - 2 + 2c}{e} \).

Therefore, given a \((-1, 2)\)-space, a \( \mathcal{D}_c \)-homothetic deformation with arbitrary \( c \neq 0 \) will give us another paracontact metric \((-1, 2)\)-space. Given a paracontact metric \((-1, 0)\)-space, if we \( \mathcal{D}_c \)-homothetically deform it with \( c = \frac{2}{e} \neq 0 \) for some \( e \neq 2 \), we will obtain a paracontact metric \((-1, \mu)\)-space with \( \mu = 2 \). A sort of converse is also possible: given a \((-1, \mu)\)-space with \( \mu \neq 2 \), a \( \mathcal{D}_c \)-homothetic deformation with \( c = 1 - \frac{\mu}{e} \neq 0 \) will give us a paracontact metric \((-1, 0)\)-space. The case \( \mu = 0, h \neq 0 \) is also special because the manifold satisfies \( \mathbb{H}^2 \) but it is not paraSasakian.

Examples of non-paraSasakian paracontact metric \((-1, 2)\)-spaces of any possible dimension and constant rank of \( h \) were presented in \( \mathbb{H}^{2n+1} \):

Example 2.2 ((2n+1)-dimensional paracontact metric \((-1, 2)\)-space with rank\( (h) = m \in \{1, \ldots, n\}\). Let \( g \) be the \((2n+1)\)-dimensional Lie algebra with basis \( \{\xi, X_1, Y_1, \ldots, X_n, Y_n\} \) such that the only
non-zero Lie brackets are:

\[ [\xi, X_i] = Y_i, \quad i = 1, \ldots, m, \]
\[ [X_i, Y_j] = \begin{cases} 
\delta_{ij}(2 \xi + \sqrt{2}(1 + \delta_{im})Y_m) \\
+ (1 - \delta_{ij})\sqrt{2}(\delta_{im}Y_j + \delta_{jm}Y_i), & i, j = 1, \ldots, m, \\
\delta_{ij}(2 \xi + \sqrt{2}Y_i), & i = 1, \ldots, m, \quad j = m + 1, \ldots, n, \\
\sqrt{2}Y_i, & i = 1, \ldots, m, \quad j = m + 1, \ldots, n.
\end{cases} \]

If we denote by \( G \) the Lie group whose Lie algebra is \( \mathfrak{g} \), we can define a left-invariant paracontact metric structure on \( G \) the following way:

\[ \varphi \xi = 0, \quad \varphi X_i = X_i, \quad \varphi Y_i = -Y_i, \quad i = 1, \ldots, n, \]
\[ \eta(\xi) = 1, \quad \eta(X_i) = \eta(Y_i) = 0, \quad i = 1, \ldots, n. \]

The only non-vanishing components of the metric are

\[ g(\xi, \xi) = g(X_i, Y_i) = 1, \quad i = 1, \ldots, n. \]

A straightforward computation gives that \( hX_i = Y_i \) if \( i = 1, \ldots, m \), \( hX_i = 0 \) if \( i = m + 1, \ldots, n \) and \( hY_j = 0 \) if \( j = 1, \ldots, n \), so \( h^2 = 0 \) and \( \text{rank}(h) = m \). Furthermore, the manifold is a \((-1, 2)-\text{space}.

Examples of non-paraSasakian paracontact metric \((-1, 0)\)-spaces of any possible dimension and \( \text{rank}(h) = 1 \) were also given in [11]:

**Example 2.3** \((2n + 1)\)-dimensional paracontact metric \((-1, 0)\)-space with \( \text{rank}(h) = 1 \). Let \( \mathfrak{g} \) be the \((2n + 1)\)-dimensional Lie algebra with basis \( \{\xi, X_1, Y_1, \ldots, X_n, Y_n\} \) such that the only non-zero Lie brackets are:

\[ [\xi, X_i] = X_1 + Y_1, \quad [\xi, Y_i] = -Y_1, \quad [X_1, Y_i] = 2\xi, \]
\[ [X_i, Y_i] = 2(\xi + Y_i), \quad [X_1, Y_i] = X_1 + Y_1, \quad [Y_1, Y_i] = -Y_1, \quad i = 2, \ldots, n. \]

If we denote by \( G \) the Lie group whose Lie algebra is \( \mathfrak{g} \), we can define a left-invariant paracontact metric structure on \( G \) the following way:

\[ \varphi \xi = 0, \quad \varphi X_1 = X_1, \quad \varphi Y_1 = -Y_1, \quad \varphi X_i = -X_i, \quad \varphi Y_i = Y_i, \quad i = 2, \ldots, n, \]
\[ \eta(\xi) = 1, \quad \eta(X_i) = \eta(Y_i) = 0, \quad i = 1, \ldots, n. \]

The only non-vanishing components of the metric are

\[ g(\xi, \xi) = g(X_1, Y_1) = 1, \quad g(X_i, Y_i) = -1, \quad i = 2, \ldots, n. \]

A straightforward computation gives that \( hX_1 = Y_1, hY_1 = 0 \) and \( hX_i = hY_i = 0, \quad i = 2, \ldots, n \), so \( h^2 = 0 \) and \( \text{rank}(h) = 1 \).

Moreover, by basic paracontact metric properties and Koszul’s formula we obtain that

\[ \nabla_\xi X_1 = 0, \quad \nabla_\xi Y_1 = 0, \quad \nabla_\xi X_i = X_i, \quad \nabla_\xi Y_i = -Y_i, \quad i = 2, \ldots, n, \]
\[ \nabla_{X_1} Y_1 = \delta_{11} \xi, \quad \nabla_{X_1} Y_j = \delta_{ij}(\xi + 2Y_i), \quad \nabla_{Y_1} X_1 = -\xi, \quad \nabla_{Y_1} X_j = -\delta_{ij} \xi, \quad i, j = 2, \ldots, n, \]
\[ \nabla_{X_i} X_j = 0, \quad \nabla_{Y_i} Y_1 = \nabla_{Y_1} Y_j = 0, \quad \nabla_{Y_j} Y_i = Y_1, \quad i = 2, \ldots, n, \]

and thus

\[ R(X_i, \xi) \xi = -X_i, \quad i = 1, \ldots, n, \]
\[ R(Y_i, \xi) \xi = -Y_i, \quad i = 1, \ldots, n, \]
\[ R(X_1, X_j) \xi = R(X_1, Y_j) \xi = R(Y_1, Y_j) \xi = 0, \quad i, j = 1, \ldots, n. \]

Therefore, the manifold is also a \((-1, 0)\)-space.
To our knowledge, the previous example is the first paracontact metric \((-1, \mu)\)-space with \(h^2 = 0, h \neq 0\) and \(\mu \neq 2\) that was constructed in dimensions greater than 3. For dimension 3, Example 4.6 of [8] was already known.

In dimension 5, there also exist examples of paracontact metric \((-1, 0)\)-space with \(\text{rank}(h) = 2\) and in dimension 7 of \(\text{rank}(h) = 2, 3\), as shown in [11]. Higher-dimensional examples of paracontact metric \((-1, 0)\)-spaces with \(\text{rank}(h) \geq 2\) were not included, which will be remedied in Example 4.1.

We will also see how to construct a 3-dimensional paracontact metric \((-1, 0)\)-space and \((-1, 2)\)-space where the rank of \(h\) is not constant.

3. New Proof of Theorem 2.1

We will now present a revised proof of Theorem 2.1 that does not use [13] when \(h \neq 0\) but constructs the basis explicitly.

Proof. Since \(\kappa = -1\), we know from [9] that \(h^2 = 0\). If \(h = 0\), then \(R(X, Y)\xi = -\langle \eta(Y)X - \eta(X)Y, \xi \rangle\), for all \(X, Y\) on \(M\) and \(\xi\) is a Killing vector field, so Theorem 3.1 of [11] gives us that the manifold is paraSasakian.

If \(h \neq 0\), then let us take a point \(p \in M\) such that \(h_p \neq 0\). We know that \(\xi\) is a global vector field such that \(g(\xi, \xi) = 1\), that \(h\xi = 0\) and that \(h\) is self-adjoint, so \(\text{Ker}\eta \neq 0\)-invariant and \(h_p : \text{Ker}\eta \mapsto \text{Ker}\eta_p\) is a non-zero linear map such that \(h_p^2 = 0\). We will now construct a basis \(\{X_1, Y_1, \ldots, X_n, Y_n\}\) of \(\text{Ker}\eta\) that satisfies all of our requirements.

Take a non-zero vector \(v \in \text{Ker}\eta_p\) such that \(h_p v \neq 0\), which we know exists because \(h_p \neq 0\). Then we write \(\text{Ker}\eta_p = L_1 \oplus L_1^\perp\), where \(L_1 = \langle v, h_p v \rangle\). Both \(L_1\) and \(L_1^\perp\) are \(h_p\)-invariant because \(h_p\) is self-adjoint. Moreover, \(g_p(v, h_p v) = 0 = g_p(h_p v, w)\) for all \(w \in L_1^\perp\), \(h_p v \neq 0\) and \(g\) is a non-degenerate metric. We now distinguish two cases:

1. If \(g_p(v, v) = 0\), then we can take \(X_1 = \frac{1}{\sqrt{|g_p(v, h_p v)|}} v\) and \(Y_1 = \frac{1}{\sqrt{|g_p(v, h_p v)|}} h_p v\), which satisfy that \(g_p(X_1, X_1) = 0 = g_p(Y_1, Y_1), g_p(X_1, Y_1) = \pm 1\) and \(h_p X_1 = Y_1, h_p Y_1 = X_1\).

2. If \(g_p(v, v) \neq 0\), then \(v' = v - \frac{g_p(v, v)}{g_p(v, h_p v)} h_p v\) satisfies that \(g_p(v', v') = 0\), so we can take \(X_1 = \frac{\sqrt{|g_p(v, h_p v)|}}{g_p(v, v)} v', Y_1 = \frac{1}{\sqrt{|g_p(v, h_p v)|}} h_p v'.\) We have again that \(g_p(X_1, X_1) = 0 = g_p(Y_1, Y_1), g_p(X_1, Y_1) = \pm 1\) and \(h_p X_1 = Y_1, h_p Y_1 = X_1\).

In both cases, \(L_1 = \langle X_1, Y_1 \rangle\), so we now take a non-zero vector \(v \in L_1^\perp\) and check if \(h_p v \neq 0\). We know that we can take \(v\) such that \(h_p v \neq 0\) in this step as many times as the rank of \(h_p\), which is at minimum 1 (since \(h_p \neq 0\)) and at most \(n\) because \(\text{dim} \text{Ker}\eta_p = 2n\) and the spaces \(L_1\) have dimension 2.

If we denote by \(m\) the rank of \(h_p\), then we can write \(\text{Ker}\eta_p\) as the following direct sum of mutually orthogonal subspaces:

\[
\text{Ker}\eta_p = L_1 \oplus L_2 \oplus \cdots \oplus L_m \oplus V = \langle X_1, Y_1, \ldots, X_m, Y_m \rangle \oplus V,
\]

where \(h_p v = 0\) for all \(v \in V\). Each \(L_i\) is of signature \((1, 1)\) because \(\{\tilde{X}_i = \frac{1}{\sqrt{2}}(X_i + Y_i), \tilde{Y}_i = \frac{1}{\sqrt{2}}(X_i - Y_i)\}\) is a pseudo-orthonormal basis such that \(g_p(\tilde{X}_i, \tilde{X}_i) = -g_p(\tilde{Y}_i, \tilde{Y}_i) = g_p(X_i, Y_i) = \pm 1, g_p(\tilde{X}_i, \tilde{Y}_i) = 0\). Therefore, \(\langle X_1, Y_1, \ldots, X_m, Y_m \rangle\) is of signature \((m, m)\) and, since \(\text{Ker}\eta_p\) is of signature \((n, n)\), we can take a pseudo-orthonormal basis \(\{v_1, \ldots, v_{n-m}, w_1, \ldots, w_{n-m}\}\) of \(V\) such that \(g_p(v_i, v_j) = \delta_{ij}\) and \(g_p(w_i, w_j) = -\delta_{ij}\). Therefore, it suffices to define \(X_{m+i} = \frac{1}{\sqrt{2}}(v_i + w_i), Y_{m+i} = \frac{1}{\sqrt{2}}(v_i - w_i)\) to have \(g_p(X_i, X_i) = 0 = g_p(Y_i, Y_i), g_p(X_i, Y_i) = 1\) and \(h_p X_i = h_p Y_i = 0, i = m + 1, \ldots, n\).

If \(n = 1\), then \(\varphi_p X_1 = \pm X_1\) and \(\varphi_p Y_1 = \mp Y_1\) follow directly from paracontact metric properties and the definition of the basis \(\{X_1, Y_1, \ldots, X_n, Y_n\}\). □
It is worth mentioning that Theorem 2.1 is true only pointwise, i.e. \( \text{rank}(h_p) \) does not need to be the same for every \( p \in M \). Indeed, we will see in Examples 4.3 and 4.5 that we can construct paracontact metric \((-1, \mu)\)-spaces such that \( h \) is zero in some points and non-zero in others.

4. New examples

We will first present an example of \((2n+1)\)-dimensional paracontact metric \((-1, 0)\)-space with rank of \( h \) greater than 1. This means that, together with Examples 2.2 and 2.3, we have examples of paracontact metric \((-1, \mu)\)-spaces of every possible dimension and constant rank of \( h \) when \( \mu = 0 \) and \( \mu = 2 \).

Example 4.1 ((2n+1)-dimensional paracontact metric \((-1, 0)\)-space with rank(h) = \( m \in \{2, \ldots, n\} \)).

Let \( g \) be the \((2n+1)\)-dimensional Lie algebra with basis \( \{\xi, X_1, \ldots, X_n, Y_n\} \) such that the only non-zero Lie brackets are:

\[
\begin{align*}
[\xi, X_1] &= X_1 + X_2 + Y_1, & [\xi, Y_1] &= -Y_1 + Y_2, \\
[\xi, X_2] &= X_1 + X_2 + Y_2, & [\xi, Y_2] &= Y_1 - Y_2, \\
[\xi, X_i] &= X_i + Y_i, & i &= 3, \ldots, m, & [\xi, Y_i] &= -Y_i, & i &= 3, \ldots, m,
\end{align*}
\]

\[
\begin{align*}
[X_i, X_j] &= \begin{cases} 
\sqrt{2}X_1, & \text{if } i = 1, j = 2, \\
-\sqrt{2}X_j, & \text{if } i = 2, j = 3, \ldots, m, \\
\sqrt{2}[\xi, X_i], & \text{if } i = 1, \ldots, m, j = m + 1, \ldots, n,
\end{cases} \\
[Y_i, Y_j] &= \begin{cases} 
\sqrt{2}(-Y_1 + Y_2), & \text{if } i = 1, j = 2, \\
\sqrt{2}Y_j, & \text{if } i = 1, 2, j = 3, \ldots, m,
\end{cases} \\
[X_i, Y_i] &= \begin{cases} 
2\xi + \sqrt{2}(X_2 + Y_2), & \text{if } i = 1, \\
-2\xi + \sqrt{2}X_1, & \text{if } i = 2, \\
-2\xi + \sqrt{2}(X_1 - X_2 - Y_2), & \text{if } i = 3, \ldots, m, \\
-2\xi - \sqrt{2}X_i, & \text{if } i = m + 1, \ldots, n,
\end{cases}
\]

\[
\begin{align*}
[X_i, Y_j] &= \begin{cases} 
\sqrt{2}(Y_1 + X_2), & \text{if } i = 1, j = 2, \\
\sqrt{2}X_1, & \text{if } i = 2, j = 1, \\
\sqrt{2}X_j, & \text{if } i = 1, 2, j = 3, \ldots, m, \\
\sqrt{2}Y_i, & \text{if } i = 3, \ldots, m, j = 2, \\
-\sqrt{2}[\xi, Y_j], & \text{if } i = m + 1, \ldots, n, j = 1, \ldots, m.
\end{cases}
\]

If we denote by \( G \) the Lie group whose Lie algebra is \( g \), we can define a left-invariant paracontact metric structure on \( G \) the following way:

\[
\begin{align*}
\varphi \xi &= 0, & \varphi X_i &= X_i, & \varphi Y_i &= -Y_i, & i &= 1, \ldots, n, \\
\eta(\xi) &= 1, & \eta(X_i) &= \eta(Y_i) &= 0, & i &= 1, \ldots, n.
\end{align*}
\]

The only non-vanishing components of the metric are

\[
g(\xi, \xi) = g(X_1, Y_1) = 1, \quad g(X_i, Y_i) = -1, & \quad i = 2, \ldots, n.
\]

A straightforward computation gives that \( hX_i = Y_i, \quad i = 1, \ldots, m, \quad hX_i = 0, \quad i = m + 1, \ldots, n \) and \( hY_i = 0, \quad i = 1, \ldots, n \), so \( h^2 = 0 \) and \( \text{rank}(h) = m \).

Moreover, very long but direct computations give that

\[
\begin{align*}
R(X_i, \xi)\xi &= -X_i, & i &= 1, \ldots, n, \\
R(Y_i, \xi)\xi &= -Y_i, & i &= 1, \ldots, n, \\
R(X_i, X_j)\xi &= R(X_i, Y_j)\xi = R(Y_i, Y_j)\xi = 0, & i, j &= 1, \ldots, n.
\end{align*}
\]
Therefore, the manifold is also a \((-1,0)\)-space.

**Remark 4.2.** Note that the previous example is only possible when \(n \geq 2\). If \(n = 1\), then we can only construct examples of \(\text{rank}(h) = 1\), as in Example 2.3.

In the definition of the Lie algebra of the previous example, some values of \(i\) and \(j\) are not possible for \(m = 2\) or \(m = n\). In that case, removing the affected Lie brackets from the definition will give us valid examples nonetheless.

We will present now an example of 3-dimensional paracontact metric \((-1,2)\)-space and one of 3-dimensional paracontact metric \((-1,0)\)-space, such that \(\text{rank}(h_p) = 0\) or 1 depending on the point \(p\) of the manifold. These are the first examples of paracontact metric \((\kappa, \mu)\)-spaces with \(h\) of non-constant rank that are known.

**Example 4.3** (3-dimensional paracontact metric \((-1,2)\)-space with \(\text{rank}(h_p)\) not constant). We consider the manifold \(M = \mathbb{R}^3\) with the usual cartesian coordinates \((x, y, z)\). The vector fields
\[
e_1 = \frac{\partial}{\partial x} + xz \frac{\partial}{\partial y} - 2y \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad \xi = \frac{\partial}{\partial z}
\]
are linearly independent at each point of \(M\). We can compute
\[
[e_1, e_2] = 2\xi, \quad [e_1, \xi] = -xe_2, \quad [e_2, \xi] = 0.
\]

We define the semi-Riemannian metric \(g\) as the non-degenerate one whose only non-vanishing components are \(g(e_1, e_2) = g(\xi, \xi) = 1\), and the 1-form \(\eta\) as \(\eta = 2ydx + dz\), which satisfies \(\eta(e_1) = \eta(e_2) = 0, \eta(\xi) = 1\). Let \(\varphi\) be the \(1,1\)-tensor field defined by \(\varphi e_1 = e_1, \varphi e_2 = -e_2, \varphi \xi = 0\). Then
\[
d\eta(e_1, e_2) = \frac{1}{2}(e_1(\eta(e_2)) - e_2(\eta(e_1))) = -1 = -g(e_1, e_2) = g(e_1, \varphi e_2),
\]
\[
d\eta(e_1, \xi) = \frac{1}{2}(e_1(\eta(\xi)) - \xi(\eta(e_1))) = 0 = g(e_1, \varphi \xi),
\]
\[
d\eta(e_2, \xi) = \frac{1}{2}(e_2(\eta(\xi)) - \xi(\eta(e_2))) = 0 = g(e_2, \varphi \xi).
\]

Therefore, \((\varphi, \xi, \eta, g)\) is a paracontact metric structure on \(M\).

Moreover, \(h\xi = 0, he_1 = xe_2, he_2 = 0\). Hence, \(h^2 = 0\) and, given \(p = (x, y, z) \in \mathbb{R}^3\), \(\text{rank}(h_p) = 0\) if \(x = 0\) and \(\text{rank}(h_p) = 1\) if \(x \neq 0\).

Let \(\nabla\) be the Levi-Civita connection. Using the properties of a paracontact metric structure and Koszul’s formula
\[
2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),
\]
we can compute
\[
\nabla_\xi \xi = 0, \quad \nabla_x \xi = -e_1 - xe_2, \quad \nabla_{e_2} \xi = e_2, \quad \nabla_\xi e_1 = -e_1, \quad \nabla_\xi e_2 = e_2,
\]
\[
\nabla_{e_1} e_1 = x\xi, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_1} e_2 = \xi, \quad \nabla_{e_2} e_1 = -\xi.
\]

Using the following definition of the Riemannian curvature
\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\{X, Y\}} Z,
\]
we obtain
\[
R(e_1, \xi)\xi = -e_1 + 2he_1, \quad R(e_2, \xi)\xi = -e_2 + 2he_2, \quad R(e_1, e_2)\xi = 0,
\]
so the paracontact metric manifold \(M\) is also a \((-1,2)\)-space.

**Remark 4.4.** The previous example does not contradict Theorem 2.4 as we will see by constructing explicitly the basis of the theorem on each point \(p\) where \(h_p \neq 0\), i.e., on every point \(p = (x, y, z)\) such that \(x \neq 0\).

Indeed, let us take a point \(p = (x, y, z) \in \mathbb{R}^3\). If \(x \neq 0\), then we define \(X_1 = \frac{e_1}{\sqrt{|x|}}, Y_1 = \frac{h_p e_1}{\sqrt{|x|}}\).

We obtain that \(\{\xi_p, X_1, Y_1\}\) is a basis of \(T_p(\mathbb{R}^3)\) that satisfies that:
the only non-vanishing components of \( g \) are \( g_p(\xi_p, \xi_p) = 1 \), \( g_p(X_1, Y_1) = \text{sign}(x), \)

- the tensor \( h \) can be written as \( h_p|_{(\xi_p, X_1, Y_1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \)

- \( \varphi_p \xi = 0 \), \( \varphi_p X_1 = X_1 \), \( \varphi_p Y_1 = -Y_1 \).

**Example 4.5** (3-dimensional paracontact metric \((-1,0)\)-space with rank \((h_p)\) not constant). We consider the manifold \( M = \mathbb{R}^3 \) with the usual cartesian coordinates \((x, y, z)\). The vector fields

\[
e_1 = \frac{\partial}{\partial x} + xe^{-2z} \frac{\partial}{\partial y} - 2y \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad \xi = \frac{\partial}{\partial z}
\]

are linearly independent at each point of \( M \). We can compute

\[
[e_1, e_2] = 2\xi, \quad [e_1, \xi] = 2xe^{-2z}e_2, \quad [e_2, \xi] = 0.
\]

We define the semi-Riemannian metric \( g \) as the non-degenerate one whose only non-vanishing components are \( g(e_1, e_2) = g(\xi, \xi) = 1 \), and the 1-form \( \eta \) as \( \eta = 2ydx + dz \), which satisfies \( \eta(e_1) = \eta(e_2) = 0 \), \( \eta(\xi) = 1 \). Let \( \varphi \) be the \((1,1)\)-tensor field defined by \( \varphi e_1 = e_1, \varphi e_2 = -e_2, \varphi \xi = 0 \). Then

\[
d\eta(e_1, e_2) = \frac{1}{2}(\eta(e_2)) - e_2(\eta(e_1)) - \eta([e_1, e_2])) = -1 = -g(e_1, e_2) = g(e_1, \varphi e_2),
\]

\[
d\eta(e_1, \xi) = \frac{1}{2}(\eta(\xi)) - \xi(\eta(e_1)) - \eta([e_1, \xi])) = 0 = g(e_1, \varphi \xi),
\]

\[
d\eta(e_2, \xi) = \frac{1}{2}(\eta(\xi)) - \xi(\eta(e_2)) - \eta([e_2, \xi])) = 0 = g(e_2, \varphi \xi).
\]

Therefore, \((\varphi, \xi, \eta, g)\) is a paracontact metric structure on \( M \).

Moreover, \( h\xi = 0 \), \( he_1 = -2xe^{-2z}e_2 \), \( he_2 = 0 \). Hence, \( h^2 = 0 \) and, given \( p = (x, y, z) \in \mathbb{R}^3 \), rank \((h_p)\) = 0 if \( x = 0 \) and rank \((h_p)\) = 1 if \( x \neq 0 \).

Let \( \nabla \) be the Levi-Civita connection. Using the properties of a paracontact metric structure and Koszul’s formula \((4)\), we can compute

\[
\nabla_\xi \xi = 0, \quad \nabla_{e_1} \xi = -e_1 + 2xe^{-2z}e_2, \quad \nabla_{e_2} \xi = e_2, \quad \nabla_\xi e_1 = -e_1, \quad \nabla_\xi e_2 = e_2, \quad \nabla_{e_1} e_1 = -2xe^{-2z}\xi, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_1} e_2 = \xi, \quad \nabla_{e_2} e_1 = -\xi.
\]

Using now \((5)\), we obtain

\[
R(e_1, \xi)\xi = -e_1, \quad R(e_2, \xi)\xi = -e_2, \quad R(e_1, e_2)\xi = 0,
\]

so the paracontact metric manifold \( M \) is also a \((-1,0)\)-space.

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**References**


E-mail address: vmartin@unizar.es

Centro Universitario de la Defensa de Zaragoza, Academia General Militar, Ctra. de Huesca s/n, 50090 Zaragoza, SPAIN, and I.U.M.A, Universidad de Zaragoza, SPAIN