LINEABILITY IN SEQUENCE SPACES

Pablo José Gerlach Mena

Dpto. Análisis Matemático

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1. Lineability

2. Some Known Results

3. New Results
Let $X$ be a topological vector space (t.v.s.) and $A \subset X$. 

### Definition

Let $X$ be a topological vector space (t.v.s.) and $A \subset X$. 

- **Lineability**: $A$ is lineable if $\exists M \subset A \cup \{0\}$, where $M$ is a vector subspace of infinite dimension. 
- **Spaceability**: $A$ is spaceable if $\exists M \subset A \cup \{0\}$, where $M$ is a closed vector subspace of infinite dimension. 
- **Dense-Lineability**: $A$ is dense-lineable if $M$ can be chosen dense in $X$. 
- **Maximal-Dense-Lineability**: $A$ is maximal-(dense)-lineable if $\dim(M) = \dim(X)$. 

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**Lineability in sequence spaces**
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- $A$ is lineable if $\exists M \subset A \cup \{0\}$ v.s. of infinite dimension.

- $A$ is spaceable if $\exists M \subset A \cup \{0\}$ closed v.s. of infinite dimension.

- $A$ is dense-lineable if $M$ can be chosen dense in $X$.

- $A$ is maximal-(dense)-lineable if $\dim(M) = \dim(X)$.
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**Definition**

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- **A is spaceable** if $\exists M \subset A \cup \{0\}$ closed v.s. of infinite dimension.
- **A is dense-lineable** if $M$ can be chosen dense in $X$.
- **A is maximal-(dense)-lineable** if $\dim(M) = \dim(X)$.
Recall that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is an everywhere surjective function if \( f(I) = \mathbb{R} \) for all interval \( I \subset \mathbb{R} \).
Recall that $f : \mathbb{R} \rightarrow \mathbb{R}$ is an everywhere surjective function if $f(I) = \mathbb{R}$ for all interval $I \subset \mathbb{R}$.

**Example**

- Let $\{l_n\}_{n \in \mathbb{N}} = \{(a_n, b_n)\}_{n \in \mathbb{N}}$ where $a_n, b_n \in \mathbb{Q}$ $\forall n \in \mathbb{N}$. 
Recall that $f : \mathbb{R} \rightarrow \mathbb{R}$ is an everywhere surjective function if $f(I) = \mathbb{R}$ for all interval $I \subset \mathbb{R}$.

**Example**

- Let $\{I_n\}_{n \in \mathbb{N}} = \{(a_n, b_n)\}_{n \in \mathbb{N}}$ where $a_n, b_n \in \mathbb{Q} \forall n \in \mathbb{N}$.
- $I_1$ contents a Cantor type subset, denote it $C_1$. 

\[ f(x) := \begin{cases} \Phi_n(x) & \text{if } x \in C_n, \\ 0 & \text{in other case} \end{cases} \]
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- $I_1$ contents a Cantor type subset, denote it $C_1$.
- We construct $\{C_n\}_{n \in \mathbb{N}}$ such that $C_n \subset I_n \setminus \left( \bigcup_{k=1}^{n-1} C_k \right)$. 
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- $l_1$ contains a Cantor type subset, denote it $C_1$.
- We construct $\{C_n\}_{n \in \mathbb{N}}$ such that $C_n \subset l_n \backslash \left( \bigcup_{k=1}^{n-1} C_k \right)$.
- Take any bijection $\Phi_n : C_n \rightarrow \mathbb{R}$.
Recall that $f : \mathbb{R} \rightarrow \mathbb{R}$ is an everywhere surjective function if $f(I) = \mathbb{R}$ for all interval $I \subset \mathbb{R}$.

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- Take any bijection $\Phi_n : C_n \rightarrow \mathbb{R}$.
- Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by
  $$f(x) := \begin{cases} 
\Phi_n(x) & \text{if } x \in C_n, \\
0 & \text{in other case.}
\end{cases}$$
**Theorem** (Araújo, Bernal, Muñoz, Prado and Seoane, 2017)

*The set of measureable everywhere surjective functions $\mathcal{MES}$ is $c$-lineable.*
Everywhere Surjective Functions

**Theorem (Araújo, Bernal, Muñoz, Prado and Seoane, 2017)**

The set of measureable everywhere surjective functions $\mathcal{MES}$ is $\mathfrak{c}$-lineable.

**Theorem (A, B, M, P and S, 2017)**

The family of sequences $(f_n)_{n \in \mathbb{N}}$ of Lebesgue measurable functions such that $f_n \rightarrow 0$ pointwise and $f_n \in \mathcal{MES}$ is $\mathfrak{c}$-lineable.
Recall that $f_n \rightarrow f$ in measure if $\forall \, \varepsilon > 0$ we have

$$
\mu \left( \{ x \in X : |f_n(x) - f(x)| \geq \varepsilon \} \right) \rightarrow 0, \quad (n \rightarrow \infty).
$$

Theorem (Riesz)

$f_n \rightarrow f$ in measure $\Rightarrow \exists$ $(f_n^k) \subset (f_n)$ such that $f_n^k \rightarrow f$ pointwise a.e.

Theorem (A, B, M, P and S, 2017)

The family of sequences of functions $(f_n) \subset L^0[0,1]$ such that $f_n \rightarrow 0$ in measure $f_n \not\rightarrow 0$ pointwise almost everywhere is maximal-dense-lineable.
Recall that $f_n \longrightarrow f$ in measure if $\forall \, \varepsilon > 0$ we have
$$\mu \left( \{ x \in X : |f_n(x) - f(x)| \geq \varepsilon \} \right) \longrightarrow 0, \quad (n \to \infty).$$

**Theorem (Riesz)**

$f_n \longrightarrow f$ in measure $\implies \exists (f_{n_k}) \subset (f_n)$ such that $f_{n_k} \longrightarrow f$ pointwise a.e.
Measure versus Almost Convergence

Recall that $f_n \to f$ in measure if $\forall \varepsilon > 0$ we have

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**Theorem (A, B, M, P and S, 2017)**

The family of sequences of functions $(f_n) \subset L_0[0, 1]$ such that

- $f_n \to 0$ in measure
- $f_n \not\to 0$ pointwise almost everywhere
Measure versus Almost Convergence

Recall that \( f_n \xrightarrow{} f \) in measure if \( \forall \varepsilon > 0 \) we have

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\( f_n \xrightarrow{} f \) in measure \( \iff \exists (f_{n_k}) \subset (f_n) \) such that \( f_{n_k} \xrightarrow{} f \) pointwise a.e.

**Theorem (A, B, M, P and S, 2017)**

The family of sequences of functions \( (f_n) \subset L_0[0, 1] \) such that

- \( f_n \xrightarrow{} 0 \) in measure
- \( f_n \not\xrightarrow{} 0 \) pointwise almost everywhere

is maximal-dense-lineable.
The family of sequences of functions \((f_n) \subset L^0[0,1]\) such that \(f_n \to 0\) pointwise \(f_n \not\to 0\) uniformly is maximal-dense-lineable.
Theorem (Calderón, G.M. and Prado)

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Theorem (Calderón, G.M. and Prado)

The family of sequences of functions \((f_n) \subset L_0[0, 1]\) such that

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Theorem (Calderón, G.M. and Prado)

The family of sequences of functions \((f_n) \subset L_0[0, 1]\) such that

\[ f_n \longrightarrow 0 \text{ pointwise} \]
\[ f_n \not\longrightarrow 0 \text{ uniformly} \]

is maximal-dense-lineable.
Let \( f_n(x) = \chi_{\left[1/n+1,1/n\right]}(x) \).

Consider now \( f_n,t(x) = \chi_{\left[1/n+1,1/n\right]}(1/2(x-t)) = \chi_{\left[2/n+1+t,2/n+t\right]}(x), \quad t \in (-1,0) \).

Let \( M := \text{span}\{ (f_n,t) : t \in (-1,0) \} \).

Then \( \dim(M) = c \), so \( A \) is maximal-lineable.

Take \( X = L^N_0, B = \tilde{L}^N : = \{ \Phi = (f_n) \in L^N_0 : \exists N = N(\Phi) \in \mathbb{N} | f_n = 0 \forall n \geq N \} \) and \( A \) the family of sequences.

Thus, \( A \) is maximal-dense-lineable.
Sketch of the Proof

- Let $f_n(x) = \chi_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}(x)$. 

Pointwise versus Uniformly Convergence

Sketch of the Proof

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- Consider now
  \[ f_{n,t}(x) = \chi_{\left[\frac{1}{n+1}, \frac{1}{n}\right]} \left( \frac{1}{2}(x - t) \right) = \chi_{\left[\frac{2}{n+1}+t, \frac{2}{n}+t\right]}(x), \quad t \in (-1, 0). \]
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- Let $f_n(x) = \chi_{[\frac{1}{n+1}, \frac{1}{n}]}(x)$.
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- Let $M := \text{span}\{(f_{n,t}) : \ t \in (-1, 0)\}$. Then $\dim(M) = c$.
**Pointwise versus Uniformly Convergence**

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- Let \( M := \text{span}\{(f_{n,t}) : t \in (-1, 0)\} \). Then \( \dim(M) = c \), so \( A \) is maximal-lineable.
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- Let $M := \text{span}\{(f_{n,t}) : t \in (-1, 0)\}$. Then $\dim(M) = \mathfrak{c}$, so $A$ is maximal-lineable.
- Take $X = L_0^\mathbb{N}$,
Pointwise versus Uniformly Convergence

**Sketch of the Proof**

- Let $f_n(x) = \chi_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}(x)$.

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- Let $M := \text{span}\{(f_{n,t}) : t \in (-1, 0)\}$. Then $\dim(M) = \mathfrak{c}$, so $A$ is maximal-lineable.

- Take $X = L_0^\mathbb{N}$, $B = \tilde{L} := \{ \Phi = (f_n) \in L_0^\mathbb{N} : \exists N = N(\Phi) \in \mathbb{N} \mid f_n = 0 \ \forall n \geq N \}$.
Pointwise versus Uniformly Convergence

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- Let $f_n(x) = \chi_{[\frac{1}{n+1}, \frac{1}{n}]}(x)$.
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- Let $M := \text{span}\{(f_{n,t}) : t \in (-1, 0)\}$. Then $\dim(M) = c$, so $A$ is maximal-lineable.
- Take $X = L_0^N$, $B = \widetilde{L} := \{\Phi = (f_n) \in L_0^\mathbb{N} : \exists N = N(\Phi) \in \mathbb{N} \mid f_n = 0 \ \forall n \geq N\}$ and $A$ the family of sequences.
Sketch of the Proof

Let \( f_n(x) = \chi_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}(x) \).

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Let \( M := \text{span}\{(f_{n,t}) : t \in (-1,0)\} \). Then \( \dim(M) = \aleph_0 \), so \( A \) is maximal-lineable.

Take \( X = L_0^\mathbb{N} \), \( B = \tilde{L} := \{ \Phi = (f_n) \in L_0^\mathbb{N} : \exists N = N(\Phi) \in \mathbb{N} \mid f_n = 0 \ \forall n \geq N \} \) and \( A \) the family of sequences.

Thus, \( A \) is maximal-dense-lineable.
The family of sequences of functions $(f_n) \subset L^0[0, +\infty)$ such that $f_n \to 0$ uniformly $f_n \not\to 0$ in $\|\cdot\|_{L^1}$ norm is $c$-lineable.
Theorem (Calderón, G.M. and Prado)

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The family of sequences of functions \((f_n) \subset L_0[0, +\infty)\) such that

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\begin{align*}
  f_n &\to 0 \text{ uniformly} \\
  f_n &\not\to 0 \text{ in } \| \cdot \|_{L^1} \text{ norm}
\end{align*}
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**Theorem (Calderón, G.M. and Prado)**

The family of sequences of functions \((f_n) \subset L_0[0, +\infty)\) such that

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\begin{align*}
f_n & \to 0 \text{ uniformly} \\
\|f_n\|_{L^1} & \not\to 0
\end{align*}
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is \(c\)-lineable.
Some Known Results

New Results

Uniformly versus $L^1$ Norm Convergence

Sketch of the Proof

Let $f_n = \frac{1}{n} \chi_{[n,2n]}$. Consider now $f_n(x - nt) = \frac{1}{n} \chi_{[n(t+1),n(t+2)]}(x)$, $t \in [0,1)$.

Let $M := \text{span} \{ (f_n, t) : t \in [0,1) \}$. Then $\dim(M) = c$, so $A$ is $c$-lineable.
Let $f_n = \frac{1}{n} \chi_{[n,2n]}$. 
Uniformly versus $L^1$ Norm Convergence

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  \]
- Let \( M := \text{span}\{(f_n,t) : t \in [0,1]\} \). Then \( \dim(M) = c \).
Uniformly versus $L^1$ Norm Convergence

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- Let $M := \text{span}\{(f_n,t) : t \in [0,1]\}$. Then $\dim(M) = c$, so $A$ is $c$-lineable.
Thank you very much for your attention