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Interpolation by Tamed Entire Functions [1]

Luis Bernal-González

Presented by V. Kiryakova

In this note it is constructed an entire function that interpolates a prescribed pair of sequences in the complex plane, and with the property that its values are controlled in some sense on a given compact subset by those that it takes on finitely many prescribed nodes on the boundary.

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Throughout this paper we will use the following standard notations: $\mathbb{N}$ is the set of positive integers, $\mathbb{R}$ is the real line, $\mathbb{C}$ is the complex plane, $B(a,r)$ ($\overline{B}(a,r)$) is the euclidean open (closed, respectively) ball with center $a \in \mathbb{C}$ and radius $r > 0$. By a domain we mean a nonempty connected open subset of $\mathbb{C}$. A subset $A \subset \mathbb{C}$ is said to be convex if and only if the segment $[a,b]$ joining a to b lies on A whenever $a, b \in A$. Finally, if $A \subset \mathbb{C}$ then ∂A denotes its boundary in $\mathbb{C}$.

A well-known interpolation theorem due to Weierstrass (see [2, Chapter 15]) asserts that if a sequence of distinct points $(a_n) \subset \mathbb{C}$ with $\lim_{n\to\infty} a_n = \infty$ and an arbitrary sequence $(b_n) \subset \mathbb{C}$ are prescribed, then there exists an entire function f –that is, f is a complex-valued holomorphic function on $\mathbb{C}$– such that $f(a_n) = b_n$ for all $n \in \mathbb{N}$.

An interesting question is whether additional conditions –for instance, boundedness by a prescribed quantity on a prescribed subset of $\mathbb{C}$– can be imposed on our interpolating function f.

In this short note we obtain a positive answer to the latter question in a concrete direction. Specifically, we get that we can assign by f arbitrary values to finitely many nodes on the boundary of a compact subset K, so that $|f|$ is

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'almost controlled' on G by the maximum of such values. This result is more precisely stated in the theorem below, but before this we need to establish the following geometrical notion.

If $A \subset \mathbb{C}$ and $z_0 \in A$, then z_0 is said to be a *strictly extremal point* of A whenever there is a straight line Λ such that $z_0 \in \Lambda$ and $A \setminus \{z_0\}$ is contained in one of the open halfplanes determined by Λ. Note that this is a more restrictive notion than the one of extremal point. Recall that if $A \subset \mathbb{C}$ (or even $A \subset X$, where X is a linear space) then a point $z_0 \in A$ is called an *extremal point* of A if and only if [$a, b \in A$ and $ta + (1-t)b \in A$ for all $t \in (0,1)$] implies $a = b$, see [3, Chapter 3]. Of course, if z_0 is extremal for A then $z_0 \in \partial A$. Let us remark that even if A is convex we may have that a point $z_0 \in A$ is extremal but not strictly extremal. For instance, if $A = \{z = x+iy \in \mathbb{C} : 0 \le x \le 1, 0 \le y \le 1\} \cup \{z = x+iy \in \mathbb{C} : x \ge 1, y \ge 0, (x-1)^2 + y^2 \le 1\}$ then its set of extremal points is $\{0, i, 2, 1+i\} \cup \{1 + e^{i\theta} : 0 < \theta < \pi/2\}$, while the set of its strictly extremal points is $\{0, i, 2\} \cup \{1 + e^{i\theta} : 0 < \theta < \pi/2\}$.

We are now ready to state our theorem.

Theorem. *Let $K \subset \mathbb{C}$ be a compact subset with connected complement. Assume that $N \in \mathbb{N}$ and that $z_1, \dots, z_N$ are distinct strictly extremal points of K and that (a_n) is a sequence of distinct points of $\mathbb{C} \setminus K$ with $\lim_{n\to\infty} a_n = \infty$. Suppose also that $w_1, \dots, w_N, b_1, b_2, \dots, b_n, \dots$ are complex values. Let us fix a number $\alpha > \max_{1 \le j \le N} |w_j|$. Then there exists an entire function f satisfying the following properties:*

(a) $f(z_j) = w_j$ *for all* $j = 1, \dots, N$,

(b) $f(a_n) = b_n$ *for all* $n \in \mathbb{N}$, *and*

(c) $|f(z)| < \alpha$ *for all* $z \in K$.

P r o o f. We define our entire function f as

$$f := \varphi + F,$$

where φ and F are adequate entire functions to be constructed.

Since the points z_j $(j = 1, \dots, N)$ are strictly extremal for K, we can select open halfplanes Π_j $(j = 1, \dots, N)$ such that

$$z_j \in \partial \Pi_j \quad \text{and} \quad K \setminus \{z_j\} \subset \Pi_j \quad (j = 1, \dots, N). \tag{1}$$

For each $j \in \{1, \dots, N\}$, let t_j be a suitable unimodular constant –which carries a rotation– such that the mapping $z \mapsto t_j(z - z_j)$ takes Π_j isomorphically onto the open left halfplane $\Pi := \{\operatorname{Re} z < 0\}$. Since $|e^z| < 1$ on Π we obtain that if

$$e_j(z) := \exp(t_j(z - z_j)) \quad (h = 1, \dots, N)$$

then $e_j(z_j) = 1$ and

$$|e_j(z)| < 1 \quad \text{for all} \quad z \in \Pi_j \ (j = 1, \dots, N). \tag{2}$$

Now, consider the Lagrange interpolation polynomials

$$L_j(z) := \prod_{k \in \{1,\dots,N\} \setminus \{j\}} \frac{z - z_k}{z_j - z_k} \quad (j = 1, \dots, N).$$

Observe that

$$L_j(z_k) = \begin{cases} 1 & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k. \end{cases}$$

Next, we define the function

$$\varphi(z) := \sum_{j=1}^{N} w_j L_j(z) (e_j(z))^m, \tag{4}$$

where m is a positive integer to be defined later. It is clear that φ is entire and that, from (3),

$$\varphi(z_j) = w_j \quad \text{for all } j = 1, \dots, N. \tag{5}$$

Let us specify the natural number m. Choose any

$$\beta \in \big(\max_{1 \le j \le N} |w_j|, \alpha \big) .$$

From (1) and (2) we obtain for each $j \in \{1, \dots, N\}$ that

$$|e_j(z)| < 1 \quad \text{for all } z \in K \setminus \{z_j\}. \tag{6}$$

On the other hand, we can fix $\varepsilon > 0$ so small that

$$(1 + N\varepsilon) \max_{1 \le j \le N} |w_j| < \beta .$$

Therefore (3) and the continuity of each polynomial L_j allows us to select a radius $r > 0$ such that the balls $B(z_j, r)$ are mutually disjoint and, for every $j \in \{1, \dots, N\}$,

$$|L_j(z)| < \varepsilon \quad \text{for all } z \in \bigcup_{\substack{k=1 \\ k \neq j}}^{N} B(z_k, r) \tag{7}$$

and

$$|L_j(z) - 1| < \varepsilon \quad \text{for all } z \in B(z_j, r). \tag{8}$$

Let us define the set $\widetilde{K} := K \setminus B$, where $B := \bigcup_{j=1}^{N} B(z_j, r)$. Then $\widetilde{K}$ is compact and, from (6), there exists $\mu \in (0,1)$ such that

$$|e_j(z)| \le \mu \quad \left(z \in \widetilde{K},\ j \in \{1, \dots, N\}\right). \tag{9}$$

Hence we can choose $m \in \mathbb{N}$ satisfying

$$\mu^m < \frac{1}{N \max\limits_{1\le j\le N} \sup\limits_{z\in K} |L_j(z)|}. \tag{10}$$

Next, we estimate $|\varphi|$ on K. If $z \in K \cap B$ then z belongs to exactly one ball $B(z_j, r)$. Consequently, by (4), (7) and (8) we get

$$|\varphi(z)| \le |w_j|(1+\varepsilon) + \sum_{\substack{k=1\\ k\ne j}}^{N} |w_k|\varepsilon \le (1 + N\varepsilon) \max_{1\le j\le N} |w_j| < \beta.$$

And if $z \in \widetilde{K}$ then (9) together with (10) apply to yield

$$|\varphi(z)| \le \sum_{j=1}^{N} |w_j| \sup_K |L_j| \mu^m \le \max_{1\le j\le N} |w_j| < \beta.$$

Summarizingly,

$$|\varphi(z)| < \beta \quad \text{for all } z \in K. \tag{11}$$

The next step is to define F. Such function will be constructed by modifying suitably the standard proof of Weierstrass' interpolation theorem in order to control its size on K. Firstly, Weierstrass' factorization theorem guarantees the existence of an entire function h having zeros at the points $z_1, \dots, z_N, a_1, \dots, a_n, \dots$, such that the zeros a_n are simple. Therefore $h'(a_n) \ne 0$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we define

$$c_n := b_n - \varphi(a_n) \quad \text{and} \quad A_n = \frac{c_n}{h'(a_n)}.$$

Pick any point $a \in \mathbb{C} \setminus \{a_n : n \in \mathbb{N}\}$ (for instance, $a \in K$). Since $a_n \to \infty$ $(n \to \infty)$, there exists $n_0 \in \mathbb{N}$ with $K \subset \overline{B}(a, |a_n - a|/2)$ for all $n > n_0$. Let us denote

$$K_n = \begin{cases} K & \text{if } n \in \{1, \dots, n_0\} \\ \overline{B}(a, |a_n - a|/2) & \text{if } n > n_0. \end{cases}$$

Then the function $A_n/(z-a_n)$ is holomorphic in the open set $\mathbb{C}\setminus\{a_n\}$, which contains the compact set K_n. But K_n has connected complement. Thus, Runge's approximation theorem (see [1] or [2, Chapter 13]) guarantees the existence of a polynomial P_n such that

$$(12). \qquad \left|\frac{A_n}{z-a_n} - P_n(z)\right| < \frac{\alpha-\beta}{2^n(1+\sup_K |h|)} \quad \text{for all } z \in K_n.$$

Since any compact set $L \subset \mathbb{C}$ is contained in K_n for all n large enough, a standard argument using Weierstrass' M-test and Weierstrass' convergence theorem reveals that the series

$$(13) \qquad g(z) := \sum_{n=1}^{\infty} \left(\frac{A_n}{z-a_n} - P_n(z) \right)$$

defines a function which is holomorphic in $\mathbb{C} \setminus \{a_n : n \in \mathbb{N}\}$ and has at most (simple) poles at the points a_n.

Let us define F as $F = gh$. Since h has zeros at the points a_n, we have that F is an entire function. We now study its properties:

- For every $j \in \{1, \dots, N\}$, $F(z_j) = g(z_j)h(z_j) = 0$.
- For every $n \in \mathbb{N}$, $F(a_n) = \lim_{z\to a_n} F(z) = \lim_{z\to a_n}(z-a_n)g(z)\dfrac{h(z)-h(a_n)}{z-a_n}$ $= (\mathrm{Res}_{a_n} g)h'(a_n) = A_n h'(a_n) = c_n$.
- For every $z \in K$ we obtain from (12) and (13) that

$$(14) \qquad |F(z)| \le \sup_K |h| \sum_{n=1}^{\infty} \frac{\alpha-\beta}{2^n(1+\sup_K |h|)} < \alpha - \beta.$$

Finally, we had defined our function f as $f = \varphi + F$. Then f is entire and satisfies: $f(z_j) = \varphi(z_j) + F(z_j) = w_j + 0 = w_j$ $(j = 1, \dots, N)$ by (5); $f(a_n) = \varphi(a_n) + F(a_n) = \varphi(a_n) + c_n = b_n$ $(n \in \mathbb{N})$; for all $z \in K$, $|f(z)| \le |\varphi(z)| + |F(z)| < \beta + \alpha - \beta = \alpha$, due to (11) and (14). This concludes the proof. ■

We remark that if we do not impose the interpolation on the nodes z_j then an argument similar to the construction of F in the last part of the proof shows the following: Let $K \subset \mathbb{C}$ be a compact subset with connected complement. Assume that (a_n) is a sequence of distinct points of $\mathbb{C}\setminus K$ with $\lim_{n\to\infty} a_n = \infty$. Suppose also that $b_1, b_2, \dots, b_n, \dots$ are complex values. Then there exists a sequence (f_k) of entire functions such that $f_k \to 0$ $(k \to \infty)$ uniformly on K and $f_k(a_n) = b_n$ for all $k, n \in \mathbb{N}$.

To finish, the next consequence is obtained just by considering the real part of an adequate entire function.

Corollary. *Let $K \subset \mathbb{R}^2$ be a compact subset with connected complement. Assume that $N \in \mathbb{N}$ and that $z_1, \dots, z_N$ are distinct strictly extremal points of K and that (a_n) is a sequence of distinct points of $\mathbb{R}^2 \setminus K$ with $\lim_{n\to\infty} a_n = \infty$. Suppose also that $w_1, \dots, w_N, b_1, b_2, \dots, b_n, \dots$ are real values. Let us fix a number $\alpha > \max_{1\le j\le N} |w_j|$. Then there exists a harmonic function $u : \mathbb{R}^2 \to \mathbb{R}$ satisfying the following properties:*

(a) $u(z_j) = w_j$ *for all* $j = 1, \dots, N$,

(b) $u(a_n) = b_n$ *for all* $n \in \mathbb{N}$, *and*

(c) $|u(z)| < \alpha$ *for all* $z \in K$.

Under a physical point of view, if one takes into account that the temperature on a plain domain behaves as a harmonic function, one can interprete the corollary as follows: It is possible to establish on the two-dimensional space a temperature scalar field T with prefixed values on infinitely many points tending to infinity such that, in addition, T is controlled as much as one desires on any prefixed 'reasonable' bounded region.

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Luis Bernal-González,
Departamento de Análisis Matemático
Facultad de Matemáticas, Apdo. 1160
Avenida Reina Mercedes
41080 Sevilla, Spain
e-mail: lbernal@us.es

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