Some recent results concerning the theoretical and numerical controllability of PDEs

Enrique FERNÁNDEZ-CARA
Dpto. E.D.A.N. - Univ. of Sevilla

several joint works with

A. MÜNCH
Lab. Mathématiques - Univ. Clermont-Ferrand, France

D.A. SOUZA
Dpto. E.D.A.N. - Univ. of Sevilla

F.D. ARARUNA, M.C. SANTOS
Dpto. Matemática - UFPB - Brazil

J. LIMACO
Dpto. Matemática - UFF - Brazil

S. DE MENEZES
Dpto. Matemática - UFCE - Brazil
1. Background

2. Hierarchical control
   - The system and the controls. Meaning
   - The Stackelberg-Nash strategy
   - The result. Idea of the proof
   - Additional results and comments

3. Turbulence control (I)
   - Background: turbulence, $\alpha$-models and control
   - Main results
   - The boundary controllability case
   - Additional results and comments

4. Turbulence control (II)
   - The Ladyzhenskaya-Smagorinsky model
   - Additional results and comments

5. A nonlinear-nonlocal parabolic system
   - The system
   - A numerical experiment
CONTROL PROBLEMS

What is usual: analysis and (numerical) resolution of

\[
\begin{aligned}
E(U) &= F \\
+ \ldots
\end{aligned}
\]

Beyond: control, i.e. acting to get good (or the best) results . . .

What is easier? Solving? Controlling?
The (general) optimal control problem; an Euler's sentence: "Everything in the world obeys to a maximum or minimum principle"

Minimize $J(v, y)$
Subject to $v \in V_{ad}$, $y \in Y_{ad}$, $(v, y)$ satisfies (S)

with

$E(y) = F(v) + \ldots$ (S)

Main questions: $\exists$, uniqueness/multiplicity, characterization, computation, \ldots
MODELLING AND OPTIMIZING RADIOTHERAPY STRATEGIES
(glioblastoma, results by R Echevarría and others, 2007)

- Brain ≈ a two-dimensional crown section
- 2 subdomains
Control oriented to therapy and tumor growth
Optimal radioterapy strategies

The state equation (a simplified description of the phenomenon):

\[
\begin{align*}
    & c_t - \partial_i(D(x)\partial_i c) = (\rho - v \mathbf{1}_\omega) c, \quad (x, t) \in \Omega \times (0, T) \\
    & c|_{t=0} = c_0, \quad x \in \Omega \\
    & + \ldots
\end{align*}
\]

\(c = c(x, t)\) is the state: a cancer cell population density
\(v = v(x, t)\) is the control: a radiotherapy administration dose
Glioblastoma [Murray-Swanson, 90's], \(D(x) = D_w\) or \(D_g\) (white and grey matters)

The optimal control problem:

\[
\begin{align*}
    & \text{Minimize} \quad J(v, y) = \frac{1}{2} \int_\Omega |c(x, T)|^2 + \frac{1}{2} \int_{\omega \times (0, T)} |v|^2 \\
    & \text{Subject to} \quad 0 \leq v \leq M, \quad \int_{\omega \times (0, T)} v \leq R, \ldots, \quad (v, y) \text{ satisfies } (E)
\end{align*}
\]
CONTROLLABILITY

A null controllability problem

Find \((v, y)\)
Such that \(v \in V_{ad}\), \((v, y)\) satisfies \((ES)\), \(y(T) = 0\)

with

\[ E(y) \equiv y_t + A(y) = F(v) + \ldots \]

\((ES)\)

Main questions: \(\exists\), uniqueness/multiplicity, characterization, computation, \ldots
FIRST EXAMPLE:
1D heat:

\[
(H_1) \quad \begin{cases}
    y_t - y_{xx} = v^1 \omega, & (x, t) \in (0, 1) \times (0, T) \\
    y(0, t) = y(1, t) = 0, & t \in (0, T) \\
    y(x, 0) = y^0(x), & x \in (0, 1)
\end{cases}
\]

We assume: \( \omega = (a, b), 0 < a < b < 1 \)

Null controllability problem: For all \( y^0 \) find \( v \) such that \( y(T) = 0 \)
NC? Yes, for all \( \omega \) and \( T \)

Applications: Heating and cooling, controlling a population, etc.
A HIERARCHICAL CONTROL PROBLEM

Three controls: one leader, two followers

\[
\begin{align*}
(y_t - y_{xx}) &= f_1 \mathcal{O} + v_1 \mathcal{O}_1 + v_2 \mathcal{O}_2, \quad (x, t) \in (0, 1) \times (0, T) \\
y(0, t) &= y(1, t) = 0, \quad t \in (0, T) \\
y(x, 0) &= y^0(x), \quad x \in (0, 1)
\end{align*}
\]

Different intervals \( \mathcal{O}, \mathcal{O}_i \)

Three objectives:
- Get \( y(T) = 0 \) — Null controllability
- At the same time, \( y \approx y_{i,d} \) in \( \mathcal{O}_{i,d} \times (0, T) \), \( i = 1, 2 \), reasonable effort:

\[
\text{Minimize } \alpha_i \int_{\mathcal{O}_{i,d} \times (0, T)} |y - y_{i,d}|^2 + \mu_i \int_{\mathcal{O}_i \times (0, T)} |v_i|^2, \quad i = 1, 2
\]

Bi-objective optimal control

What can we do?
Hierarchical control
The system and the controls. Meaning

\[
\begin{aligned}
(H) \quad & \left\{ \begin{array}{l}
y_t - y_{xx} = f^{1\mathcal{O}} + v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2}, \quad (x, t) \in (0, 1) \times (0, T) \\
y(0, t) = y(1, t) = 0, \quad t \in (0, T) \\
y(x, 0) = y^0(x), \quad x \in (0, 1)
\end{array} \right.
\end{aligned}
\]

Goal: drive \( y \) to rest and keep \( y \) close to \( y_{i,d} \) in \( \mathcal{O}_i \times (0, T) \) for \( i = 1, 2 \)

Many applications:

- **Heating**: Controlling temperatures
  Various heat sources at different locations
  Heat PDE (linear, semilinear, etc.)

- **Tumor growth**: Controlling tumor cell densities
  Radiotherapy strategies
  Reaction-diffusion systems (linear, semilinear, etc.), bilinear control

- **Fluid mechanics**: Controlling fluid velocity fields
  Several mechanical actions
  Stokes, Navier-Stokes or similar

- **Finance**: Controlling the price of an option
  Several agents at different stock prices, etc.
  Backwards in time heat-like PDE
THE STACKELBERG-NASH STRATEGY

Step 1: \( f \) is fixed

\[
J_i(v_1, v_2) := \alpha_i \int_0^T \int_{\Omega_i \times (0, T)} |y - y_{i,d}|^2 + \mu_i \int_0^T \int_{\Omega \times (0, T)} |v_i|^2, \quad i = 1, 2
\]

Find a Nash equilibrium \((v_1(f), v_2(f))\) with \(v_i(f) \in L^2(\Omega_i \times (0, T))\):

\[
\begin{align*}
J_1(v_1(f), v_2(f)) &\leq J_1(v_1, v_2(f)) \quad \forall v_1 \in L^2(\Omega_1 \times (0, T)) \\
J_2(v_1(f), v_2(f)) &\leq J_2(v_1(f), v_2) \quad \forall v_2 \in L^2(\Omega_2 \times (0, T))
\end{align*}
\]

Equivalent to:

\[
\begin{cases}
\dot{y} - y_{xx} = f^1_{\Omega} - \frac{1}{\mu_1} \phi_1^{1_{\Omega_1}} - \frac{1}{\mu_2} \phi_2^{1_{\Omega_2}} \\
-\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d})_1_{\Omega_i} , \quad i = 1, 2 \\
\phi_i(0, t) = \phi_i(1, t) = 0, \quad y(0, t) = y(1, t) = 0, \quad t \in (0, T) \\
y(x, 0) = y^0(x), \quad \phi_i(x, T) = 0, \quad x \in (0, 1)
\end{cases}
\]

\((HN)\)

Then: \(v_i(f) = -\frac{1}{\mu_i} \phi_i|_{\Omega_i \times (0, T)}\) (Pontryagin)
THE STACKELBERG-NASH STRATEGY

Step 2: Find $f$ such that

$$(HSN)_1 \begin{cases} y_t - y_{xx} = f 1_\varphi - \frac{1}{\mu_1} \phi_1 1_\varphi_1 - \frac{1}{\mu_2} \phi_2 1_\varphi_2 \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d}) 1_\varphi_i, & i = 1, 2 \\ \phi_i(0, t) = \phi_i(1, t) = 0, \ y(0, t) = y(1, t) = 0, & t \in (0, T) \\ y(x, 0) = y^0(x), \ \phi_i(x, T) = 0, & x \in (0, 1) \end{cases}$$

$$(HSN)_2 \quad y(x, T) = 0, \quad x \in (0, 1)$$

with $\|f\|_{L^2(\varphi \times (0, T))} \leq C \|y^0\|_{L^2}$

For instance, for $y_{i,d} \equiv 0$, equivalent to:

$R(L) \hookrightarrow R(M)$, with $Ly^0 := y(\cdot, T)$, $Mf := y(\cdot, T)$...

In turn, equivalent to: $\|L^* \psi_T \| \leq \|M^* \psi_T \| \quad \forall \psi_T \in L^2(0, 1)$

(classical, functional analysis; [Russell, 1973])
Hierarchical control
The result. Idea of the proof

Theorem [Araruna-EFC-Santos]

Assume: \( \mathcal{O}_{1,d} = \mathcal{O}_{2,d}, \mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset \), large \( \mu_i \)

\[ \exists \hat{\rho} \text{ such that, if } \int_{\mathcal{O}_d \times (0,T)} \hat{\rho}^2 |y_{i,d}|^2 \, dx \, dt < +\infty, \, i = 1, 2, \text{ then:} \]

\[ \forall y^0 \in L^2(\Omega) \exists \text{ null controls } f \in L^2(\mathcal{O} \times (0, T)) \& \text{ Nash pairs } (v_1(f), v_2(f)) \]

Idea of the proof:

1 - Large \( \mu_i \) \( \Rightarrow \) \( \forall f \in L^2(\mathcal{O} \times (0, T)) \exists! \) Nash equilibrium \( (v_1(f), v_2(f)) \)

2 - \( \| L^* \psi^T \| \leq \| M^* \psi^T \| \) \( \forall \psi^T \in L^2(0, 1) \) means observability:

\[ \| \psi \big|_{t=0} \|^2 + \sum_{i=1}^{2} \int_{Q} \hat{\rho}^{-2} |\gamma^i|^2 \, dx \, dt \leq C \int_{\mathcal{O} \times (0, T)} |\psi|^2 \, dx \, dt \]

for all \( \psi^T \), with

\[ \begin{cases} 
- \psi_t - \psi_{xx} = \sum_{i=1}^{2} \alpha_i \gamma^i 1_{\mathcal{O}_d}, & \gamma^i_t - \gamma^i_{xx} = - \frac{1}{\mu_i} \psi 1_{\mathcal{O}_i} \\
\psi \big|_{t=T} = \psi^T(x), & \gamma^i \big|_{t=0} = 0, \text{ etc.} 
\end{cases} \]

Observability \( \iff \) Carleman estimates for \( \psi, \gamma^i \)

\[ \int_{Q} \hat{\rho}^{-2} |\psi|^2 \, dx \, dt + \sum_{i=1}^{2} \int_{Q} \hat{\rho}^{-2} |\gamma^i|^2 \, dx \, dt \leq C \int_{\mathcal{O} \times (0, T)} \hat{\rho}^{-2} |\psi|^2 \, dx \, dt \]
EXTENSIONS

- More followers, coefficients, non-scalar parabolic systems, other functionals, boundary controls, higher dimensions, etc.
- **Semilinear** systems, for instance:

\[
\begin{aligned}
  y_t - y_{xx} &= F(x, t; y) + f1_{\mathcal{O}} + \sum_{i=1}^{m} v_i 1_{\mathcal{O}_i} \\
y(0, t) &= y(1, t) = 0, \quad t \in (0, T), \quad \text{etc.}
\end{aligned}
\]

OK for Lipschitz-continuous \( F \)

- **Constraints**, for instance:

\[
\begin{aligned}
  y_t - y_{xx} &= f1_{\mathcal{O}} + \sum_{i=1}^{m} v_i 1_{\mathcal{O}_i} \\
y(0, t) &= y(1, t) = 0, \quad t \in (0, T), \quad \text{etc.}
\end{aligned}
\]

Find a constrained Nash equilibrium \((v_1(f), v_2(f))\) with \(v_i(f) \in \mathcal{U}_{i, ad} \subset L^2(\mathcal{O}_i \times (0, T))\):

\[
\begin{aligned}
  J_1(v_1(f), v_2(f)) &\leq J_1(v_1, v_2(f)) \quad \forall v_1 \in \mathcal{U}_{1, ad} \\
  J_2(v_1(f), v_2(f)) &\leq J_2(v_1(f), v_2) \quad \forall v_2 \in \mathcal{U}_{2, ad}
\end{aligned}
\]

Then, find \( f \) such that \( y|_{t=T} = 0 \)

OK for local constraints, i.e. \( \mathcal{U}_{i, ad} = \{ v_i \in L^2(\mathcal{O}_i \times (0, T)) : v_i(x, t) \in L_i \} \)
MORE COMMENTS:

- Previous work: [Guillén et al. 2013]
- The previous proof → a method to compute $f$ and $(v_1(f), v_2(f))$
- $O_{1,d} \neq O_{2,d}$?
- Other strategies? Stackelberg-Pareto controllability?
- Numerical results?

In progress ...
CONTROLLING TURBULENCE (I)

The Leray-$\alpha$ model - distributed controls:

\[ \begin{align*}
    y_t + (z \cdot \nabla)y - \nu_0 \Delta y + \nabla p &= \nu 1_\omega, \quad \nabla \cdot y = 0 \\
    z - \alpha^2 \Delta z + \nabla \pi &= y, \quad \nabla \cdot z = 0 \\
    y(x, t) = z(x, t) &= 0, \quad (x, t) \in \partial\Omega \times (0, T) \\
    y(x, 0) &= y^0(x)
\end{align*} \]

AC? NC? ECT? OPEN
Fluid regimes: Laminar or turbulent
[Reynolds 1895], [Kolmogorov 1941], [Batchelor 1953]

Main characteristics of turbulence:
- Fast variations in space and time, wide range of length scales (eddy motion)
- Well behavior of (appropriately) averaged variables

Typically: small (resp. large) Re := $UL/\nu \Rightarrow$ laminar (resp. turbulent) flow
Background: turbulence, $\alpha$-models and control

Turbulence

Turbulent flows in waves and tornados
Background: turbulence, $\alpha$-models and control

Turbulent smoke rings

© Peter Emmett
http://www.emmett-photography.com
To understand something on turbulence: [Schlichting 1968], [Temam 1988], [Lesieur 1997], [Matthieu-Scott 2000]

**Turbulence modelling**

1 - Start from Navier-Stokes:

\[ y_t + (y \cdot \nabla)y - \nu_0 \Delta y + \nabla p = f, \quad \nabla \cdot y = 0 \]

2 - Averages:

\[ y = \bar{y} + y', \quad p = \bar{p} + p' \]

For instance, \( \bar{y}(x, t) := \lim_{\varepsilon \to 0^+} \int_{|(x', t') - (x, t)| \leq \varepsilon} y(x', t') \, dx \, dt \)

Reynolds (PDE's for \( \bar{y} \) and \( \bar{p} \)?):

\[ \bar{y}_t + \nabla \cdot (\bar{y} \otimes \bar{y}) - \nu_0 \Delta \bar{y} + \nabla \bar{p} = \bar{f}, \quad \nabla \cdot \bar{y} = 0 \]

3 - **Closure hypotheses**: assumptions relating \( \bar{y} \otimes \bar{y} \) and \( \bar{y} \)
Reynolds:
\[ \overline{y}_t + \nabla \cdot (\overline{y} \otimes \overline{y}) - \nu_0 \Delta \overline{y} + \nabla \overline{p} = \overline{f}, \quad \nabla \cdot \overline{y} = 0 \]

A particular closure hypothesis:
\[ \overline{y} \otimes \overline{y} \approx z_\alpha \otimes \overline{y}, \text{ with } z_\alpha = (\text{Id.} + \alpha^2 A)^{-1} \overline{y}, \quad \alpha \to 0^+ \]

Leray-$\alpha$ model:
\[
\begin{cases}
\overline{y}_t + (z_\alpha \cdot \nabla)\overline{y} - \nu_0 \Delta \overline{y} + \nabla \overline{p} = \overline{f}, \quad \nabla \cdot \overline{y} = 0 \\
z_\alpha - \alpha^2 \Delta z_\alpha + \nabla \pi_\alpha = \overline{y}, \quad \nabla \cdot z_\alpha = 0
\end{cases}
\]

[LeRay 1934], [Cheskidov-Holm-Olson-Titi 2005]
The significance of controlling a turbulence model:

\[ \bar{y}_t + \nabla \cdot S - \nu_0 \Delta \bar{y} + \nabla \bar{p} = \nu 1_\omega, \quad \nabla \cdot \bar{y} = 0 \]

with \( S = S(\bar{y}(\cdot, \cdot)) \) (an approximation of \( \bar{y} \otimes \bar{y} \))

- We control averaged states
- With averages depending on \( \alpha \), are controls uniformly bounded? Do averaged controls converge?
  If yes: controlling the Navier-Stokes system in the limit
Background: turbulence, $\alpha$-models and control

Basic results

Navier-Stokes:

\[
\begin{aligned}
y_t + (y \cdot \nabla)y - \nu_0 \Delta y + \nabla p &= v_1, \\
\nabla \cdot y &= 0,
\end{aligned}
\]

\[
\begin{aligned}
y(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T) \\
y(x, 0) &= y^0(x)
\end{aligned}
\]

AC? NC? ECT? OPEN

What we know: Local ECT

Theorem [EFC-Guerrero-Imnulov-Puel 2004]

Fix a solution $(\bar{y}, \bar{p})$, with $\bar{y} \in L^\infty$

$\exists \varepsilon > 0$ such that $\|y^0 - \bar{y}(0)\|_{H^1_0} \leq \varepsilon \Rightarrow \exists$ controls such that $y(T) = \bar{y}(T)$

For the proof:

1. Reduce ECT to NC, (NC) $\cong \"F(y, \nu) = 0\"$ in an appropriate space
2. Then: apply Liusternik’s Theorem (linearized at zero is NC)

Other results, among them:
- Global AC for when $N = 2$, Navier boundary conditions [Coron 1996]
- Global NC with periodicity [Fursikov-Imanuvilov 1999], without boundary
  [Coron-Fursikov 1996], . . .
The Leray-\(\alpha\) model - distributed controls:

\[
\begin{align*}
    y_t + (z \cdot \nabla) y - \nu_0 \Delta y + \nabla p &= \nu 1_\Omega, \\ z - \alpha^2 \Delta z + \nabla \pi &= y, \\ y(x, t) &= z(x, t) = 0, \\ y(x, 0) &= y^0(x) 
\end{align*}
\]

AC? NC? ECT? OPEN

What we know: local NC, controls converge as \(\alpha \to 0^+\):

**Theorem [Araruna, EFC, Souza 2014]**

\[\exists \varepsilon > 0 \text{ such that } y^0 \in H, \quad \|y^0\|_{L^2} \leq \varepsilon \Rightarrow \exists \text{ controls } v_\alpha \text{ such that } y(T) = 0\]

Furthermore, \(\|v_\alpha\|_{L^2} \leq C\)

\[H = \{ w \in L^2(\Omega)^N : \nabla \cdot w = 0 \text{ in } \Omega, \ w \cdot n = 0 \text{ on } \partial\Omega \}\]
Main results
Local uniform NC for Leray-$\alpha$

Idea of the proof (I):

Lemma (regularizing effect)

$\exists \phi = \phi(s) > 0$, with $\phi(s) \to 0$ as $s \to 0^+$:

a) $\exists$ arbitrarily small $t^* \in (0, \frac{T}{2})$ with $\|y(t^*)\|_{D(A)} \leq \phi(\|y_0\|_{L^2})$

b) The set of these $t^*$ has positive measure

This lemma $\Rightarrow$ we can assume that $\|y_0\|_{D(A)} << 1$

Idea of the proof (II):

- Fixed-Point formulation:

\[
\left\{\begin{array}{l}
z - \alpha^2 \Delta z + \nabla \pi = \bar{y}, \quad \nabla \cdot z = 0 \\
i.e. \quad z = (\text{Id.} + \alpha^2 A)^{-1} \bar{y} \\
y_t + (z \cdot \nabla) y - \nu_0 \Delta y + \nabla \rho = \nu_1 \omega, \quad \nabla \cdot y = 0, \quad \text{etc.}
\end{array}\right.
\]

- $\bar{y} \in L^\infty(0, T; D(A^{s/2})), s > N/2 \Rightarrow z \in L^\infty$ and NC for Oseen uniformly

- $\|v_\alpha\|_{L^\infty(L^2)} \leq C, \forall \alpha > 0$

- $y \in$ compact set of $L^\infty(0, T; D(A^{s/2}))$

- $\|y_0\|_{H^2}$ small $\Rightarrow$ $\|y\|_{L^\infty(0, T; D(A^{s/2})} \leq C$ if $\|\bar{y}\|_{L^\infty(0, T; D(A^{s/2})} \leq C$
Assume $y^0 \in H$, $\|y^0\|_{L^2} \leq \varepsilon$

\[
\begin{align*}
y_\alpha t + (z_\alpha \cdot \nabla)y_\alpha - \nu_0 \Delta y_\alpha + \nabla p &= v_\alpha 1_\omega, \quad \nabla \cdot y_\alpha = 0 \\
z_\alpha - \alpha^2 \Delta z_\alpha + \nabla \pi_\alpha &= y_\alpha, \quad \nabla \cdot z = 0 \\
y_\alpha(x, t) = z_\alpha(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T) \\
y_\alpha(x, 0) = y^0(x), \quad y_\alpha(x, T) &= 0
\end{align*}
\]

Then, at least for a subsequence

- $v_\alpha \to v$ weakly in $L^2(\omega \times (0, T))$
- $y_\alpha \to y$ and $z_\alpha \to y$ strongly in $L^2(\Omega \times (0, T))$ etc.

\[
\begin{align*}
y_t + (y \cdot \nabla)y - \nu_0 \Delta y + \nabla p &= v 1_\omega, \quad \nabla \cdot y = 0 \\
y(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T) \\
y(x, 0) &= y^0(x), \quad y(x, T) = 0
\end{align*}
\]
The Leray-\(\alpha\) model - boundary controls:
More natural, but how?

The good boundary control problem:

\[
\begin{aligned}
&y_t + (z \cdot \nabla)y - \nu_0 \Delta y + \nabla p = 0, \quad \nabla \cdot y = 0 \\
z - \alpha^2 \Delta z + \nabla \pi = y, \quad \nabla \cdot z = 0 \\
y(x, t) = z(x, t) = h_1\gamma, \quad (x, t) \in \partial \Omega \times (0, T) \\
y(x, 0) = y^0(x)
\end{aligned}
\]

Again, AC, NC, ECT are open and we get uniform local NC:

**Theorem [Araruna, EFC, Souza 2014]**

\(\exists \varepsilon > 0\) such that \(y^0 \in V, \|y^0\|_{H^1_0} \leq \varepsilon \Rightarrow \exists h_\alpha\) with \(\int_\gamma h_\alpha \cdot n \, d\Gamma = 0, y(T) = 0\)

Furthermore, \(\|h_\alpha\|_{L^\infty(0,T;H^{1/2}(\gamma))} \leq C\)

\(V = \{ w \in H^1_0(\Omega)^N : \nabla \cdot w = 0 \text{ in } \Omega \}\)
Idea of the proof (I): An auxiliary extension $\tilde{\Omega}$, a fictitious $\omega$

Lemma (modified regularizing effect)

$\exists \psi = \psi(s) > 0$, with $\psi(s) \to 0$ as $s \to 0^+$:

a) $\exists T_0 \in (0, T)$, $h_\alpha \in L^\infty(0, T_0; H^{1/2}(\gamma))$, $(y_\alpha, p_\alpha, z_\alpha, \pi_\alpha)$ and arbitrarily small $t^*$ such that

$y_\alpha$ can be extended to $\tilde{\Omega} \times (0, T_0)$, with $\|\tilde{y}_\alpha(t^*)\|_{D(\tilde{A})} \leq \psi(\|y_0\|_{H^1})$

b) The set of these $t^*$ has positive measure

c) $h_\alpha$ is uniformly bounded in $L^\infty(0, T_0; H^{1/2}(\gamma))$

This lemma $\Rightarrow$ we can work in $\tilde{\Omega} \times (0, T)$ assuming $\|\tilde{y}_0\|_{D(\tilde{A})} << 1$

Idea of the proof (II): Solve

$$\begin{align*}
\tilde{y}_t + (\tilde{z} \cdot \nabla)\tilde{y} - \nu_0 \Delta \tilde{y} + \nabla \tilde{p} &= v_1 \omega, & \nabla \cdot \tilde{y} &= 0, & \tilde{\Omega} \times (0, T) \\
z - \alpha^2 \Delta z + \nabla \pi &= \tilde{y}, & \nabla \cdot z &= 0, & \Omega \times (0, T) \\
\tilde{y}(x, t) &= 0, & \partial\tilde{\Omega} \times (0, T) \\
z(x, t) &= \tilde{y}(x, t), & \partial\Omega \times (0, T) \\
\tilde{y}(x, 0) &= \tilde{y}_0(x), & \tilde{y}(x, T) &= 0, & \tilde{\Omega}
\end{align*}$$

Again: Fixed-Point argument works...
Main results
Local uniform NC for Leray-$\alpha$

The extended domain and the fictitious control region
Main results
Local uniform NC for Leray-$\alpha$

Assume $y^0 \in V$, $\|y^0\|_{H^1_0} \leq \varepsilon$

\[
\begin{align*}
  y_{\alpha t} + (z_{\alpha} \cdot \nabla)y_{\alpha} - \nu_0 \Delta y_{\alpha} + \nabla p &= 0, \quad \nabla \cdot y_{\alpha} = 0 \\
  z_{\alpha} - \alpha^2 \Delta z_{\alpha} + \nabla \pi_{\alpha} &= y_{\alpha}, \quad \nabla \cdot z_{\alpha} = 0 \\
  y_{\alpha}(x, t) = z_{\alpha}(x, t) &= h_\alpha 1_\gamma, \quad (x, t) \in \partial \Omega \times (0, T) \\
  y_{\alpha}(x, 0) = y^0(x), \quad y_{\alpha}(x, T) &= 0
\end{align*}
\]

Then, at least for a subsequence

- $h_\alpha \to h$ weakly-$\star$ in $L^\infty(0, T; H^{1/2}(\gamma))$
- $y_{\alpha} \to y$ and $z_{\alpha} \to y$ strongly in $L^2(\Omega \times (0, T))$ etc.

\[
\begin{align*}
  y_t + (y \cdot \nabla)y - \nu_0 \Delta y + \nabla p &= 0, \quad \nabla \cdot y = 0 \\
  y(x, t) &= z(x, t) = h 1_\gamma, \quad (x, t) \in \partial \Omega \times (0, T) \\
  y(x, 0) = y^0(x), \quad y(x, T) &= 0
\end{align*}
\]
Additional results and comments
Other nonlinear systems

**Simplified models:** the Burgers and Burgers-\(\alpha\) systems

\(L > 0, \ T > 0\)

**Burgers:**

\[
\begin{aligned}
&y_t - \nu_0 y_{xx} + yy_x = f, \quad (x, t) \in (0, L) \times (0, T) \\
y(0, \cdot) = y(L, \cdot) = 0, \quad t \in (0, T) \\
y(\cdot, 0) = y_0, \quad x \in (0, L)
\end{aligned}
\]

**Burgers-\(\alpha\):**

\[
\begin{aligned}
&y_t - \nu_0 y_{xx} + z_\alpha y_x = f, \quad (x, t) \in (0, L) \times (0, T) \\
z_\alpha - \alpha^2 (z_\alpha)_{xx} = y, \\
y(0, \cdot) = y(L, \cdot) = z_\alpha(0, \cdot) = z_\alpha(L, \cdot) = 0, \\
y(\cdot, 0) = y_0, \\
z_\alpha(0, \cdot) = z_\alpha(L, \cdot) = 0, \quad t \in (0, T) \\
x \in (0, L)
\end{aligned}
\]

**Motivations:**

- A “toy model” for Leray-\(\alpha\)
- Applications to the description of 1D motion

**Similar results**
1D motion in a neon tube
Traffic motion
For small $y_0$, again:

- NC
- $\|v_\alpha\|_{L^\infty(\omega \times (0, T))}$ is uniformly bounded

Remarks:

- Comparison (maximum) principle, easier to get $z_\alpha$ bounded in $L^\infty$
- Burgers is not globally NC.
  Therefore: for large $y^0$, at most, $\|v_\alpha\|_{L^\infty(\omega \times (0, T))}$ is unbounded
CONTROLLING TURBULENCE (II)

The Ladyzhenskaya-Smagorinsky model:
Coming back to turbulence modelling - Reynolds:

$$\bar{y}_t + \nabla \cdot (\bar{y} \otimes \bar{y}) - \nu_0 \Delta \bar{y} + \nabla \bar{p} = \bar{f}, \quad \nabla \cdot \bar{y} = 0$$

How to relate $\bar{y} \otimes \bar{y}$ and $\bar{y}$?
Boussinesq-like closure hypotheses:

$$\bar{y} \otimes \bar{y} \approx \bar{y} \otimes \bar{y} - R, \text{ with } R = \nu_T(\nabla \bar{y}(\cdot, \cdot))D\bar{y}$$

$R$ is the Reynolds tensor, $\nu_T$ is the turbulent viscosity
[Launder-Spalding 1972], [Cebeci-Smith 1974]

A simple assumption: $\nu_T = \nu_1(\| \nabla \bar{y}(\cdot, t) \|^2)$

$$\bar{y}_t + (\bar{y} \cdot \nabla)\bar{y} - \nu(\int_\Omega |\nabla \bar{y}|^2) \Delta \bar{y} + \nabla \bar{p} = \bar{y}, \quad \nabla \cdot \bar{y} = 0$$

[Ladyzhenskaya 1961], [Smagorinsky 1963]
Turbulence control (II)
Another model

The Ladyzhenskaya-Smagorinsky model:

\[
\begin{aligned}
    y_t + (y \cdot \nabla)y - \nu(\mathcal{Q} \, |\nabla y|^2) \Delta y + \nabla p &= \nu_1, \\
    \nabla \cdot y &= 0, \\
    y(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
    y(x, 0) &= y^0(x)
\end{aligned}
\]

We assume: \( \nu_T \in C^1_b, \nu_T \geq \nu_0 > 0 \)

AC? NC? ECT? OPEN - What we know: local NC

Theorem [EFC-Limaco-Menezes 2014]

\( \exists \varepsilon > 0 \) such that \( \|y^0\|_{L^2} \leq \varepsilon \Rightarrow \exists \) null controls

Arguments similar to those for Navier-Stokes:

1. Rewrite NC in the form \( (NC) \cong \langle F(y, \nu) = 0 \rangle \) in an appropriate space \( X \)

   Key point: Choose \( X \) to have
   - \( F : X \to Z \) well defined and \( C^1 \) (small)
   - \( F'(0, 0) \in L(X; Z) \) onto (large)

2. Then: apply Liusternik's Theorem (linearized at zero is Stokes, NC)

Attention: local ECT is also open!
ADDITIONAL COMMENTS:

- Many open questions remain:
  - Other similar $\alpha$-models (LANS-$\alpha$, Cannasa-Holm model, etc.). NC?
  - Global control results?
  - Reducing the number of controls? Specially difficult in the boundary case!
  - Control results of other kinds? In particular, Lagrangian controllability?
    [Glass-Horsin 2010 . . .]

- Numerical analysis and convergence results for these and other problems: in progress . . .
Similar results for nonlinear-nonlocal parabolic systems:

\[
\begin{align*}
 y_t - a(\int_{\Omega} y, \int_{\Omega} z)\Delta y &= f(y, z) + v1_{\omega}, \\
 z_t - b(\int_{\Omega} y, \int_{\Omega} z)\Delta y &= g(y, z),
\end{align*}
\]

\(\text{(NN)}\)

\(\left\{
\begin{array}{l}
 y(x, t) = z(x, t) = 0, \\
 y(x, 0) = y^0(x), 
\end{array}
\right.
\)

Several difficulties, mainly:

- **Nonlinear** \(a, b, f, g\)
- Only one control

[EFC-Límaco-Menezes 2013]

**Applications:** Controlling reacting media, interacting populations, among others
A nonlinear-nonlocal parabolic system

The system

An experiment, nonlinear-nonlocal system:

\[ \begin{aligned}
   y_t - a(\int_{\Omega} y, \int_{\Omega} z) \Delta y &= f(y, z) + v_1 \omega, \\ 
   z_t - b(\int_{\Omega} y, \int_{\Omega} z) \Delta y &= g(y, z), \\ 
   y(x, t) &= z(x, t) = 0, \\
   y(x, 0) &= y^0(x), \\ 
   z(x, 0) &= z^0(x), \quad x \in \Omega
\end{aligned} \]

\((NN)\)

\[ a, b, f, g \in C^1_b, a \geq a_0 > 0, b \geq b_0 > 0, \partial_y g(0, 0) \neq 0 \]

\[ \Omega = (0, 1), \omega = (0.2, 0.8), T = 0.5, y_0(x) \equiv \sin(\pi x), z_0(x) \equiv \sin(2\pi x), \]

\[ f \equiv A_1(1 + \sin y) y + B_1(1 + \sin z) z, \quad g \equiv A_2(1 + \sin y) y + B_2(1 + \sin z) z \]

\[ a \equiv a_0(1 + (1 + r^2 + s^2)^{-1}), \quad b \equiv b_0(1 + (1 + r^2 + s^2)^{-1}). \]

Formulation \( F(y, z, v) = 0 \) + Quasi-Newton method — Only \( F'(0, 0, 0) \)!

Convergence is ensured

At every step: NC for a linear parabolic system (1 control)

Approximation: \( P_1 \) in \((x, t)\) + multipliers (mixed formulation), \( C^0 \) in \((x, t)\)

freeFem++ & mesh adaptation
A nonlinear-nonlocal parabolic system

Figure: The initial mesh. Number of vertices: 232. Number of triangles: 402. Total number of unknowns: \(6 \times 232 = 1392\).
A nonlinear-nonlocal parabolic system

The mesh

MESH ADAPTATION

Figure: The final adapted mesh. Number of vertices: 2903. Number of triangles: 5594. Total number of unknowns: $6 \times 2903 = 17418$. 
A nonlinear-nonlocal parabolic system

The control

Figure: The computed null control.
A nonlinear-nonlocal parabolic system

The state

Figure: The computed state $y$. 
A nonlinear-nonlocal parabolic system
The state

**Figure:** The computed state $z$. 
REFERENCES:

ARARUNA, F.D., FERNÁNDEZ-CARA, E. AND SANTOS, M.C.  
Stackelberg-Nash exact controllability for linear and semilinear parabolic equations.  

ARARUNA, F.D., FERNÁNDEZ-CARA, E. AND SOUZA, D.A.  
On the control of the Burgers-alpha model.  

ARARUNA, F.D., FERNÁNDEZ-CARA, E. AND SOUZA, D.A.  
Uniform local null control of the Leray-$\alpha$ model.  

CLARK, H., FERNÁNDEZ-CARA, E., LÍMACO, J. AND MEDEIROS, L.A.  
Theoretical and numerical local null controllability for a parabolic system with local and nonlocal nonlinearities.  

FERNÁNDEZ-CARA, E., LÍMACO, J. AND MENEZES, S.B.  
On the theoretical and numerical control of a Ladyzhenskaya-Smagorinsky model of turbulence.  
*Submitted.*
REFERENCES (Cont.):

Fernández-Cara, E. and Münch, A.
Numerical null controllability of semi-linear 1D heat equations: fixed point, least squares and Newton methods.
*MCRI* Volume 2, Number 3, September 2012.

Fernández-Cara, E. and Münch, A.
Strong convergent approximations of null controls for the 1D heat equation.

Fernández-Cara, E. and Münch, A.
Numerical Exact controllability of the 1D heat equation: duality and Carleman weights.
THANK YOU VERY MUCH ...