Some remarks on the exact controllability to trajectories for the nonlinear heat equation

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Workshop on Control and Inverse Problems

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1 Introduction. Statement of the problem

2 Null Controllability of the linear problem with regular controls
   - First approach
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3 The “best” null control
1. Introduction. Statement of the problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial \Omega$ of class $C^2$. Let $\omega \subseteq \Omega$ be an open subset and let us fix $T > 0$.

In (1) and (2), $1_\omega$ represents the characteristic function of the set $\omega$, $y(x,t)$ is the state, $y_0$ is the initial datum and is given in an appropriate space, and $v$ is the control function (which is localized in $\omega$-distributed control). In (1), $a \in L^\infty(Q)$ is given. We will assume that $F : \mathbb{R} \to \mathbb{R}$ is a given function.
1. Introduction. Statement of the problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial \Omega$ of class $C^2$. Let $\omega \subseteq \Omega$ be an open subset and let us fix $T > 0$.

We consider the linear and nonlinear problems for the heat equation:

\begin{align*}
\begin{cases} 
\partial_t y - \Delta y + ay = v1_\omega & \text{in } Q = \Omega \times (0, T), \\
y = 0 & \text{on } \Sigma = \partial \Omega \times (0, T), \\
y(\cdot, 0) = y_0 & \text{in } \Omega,
\end{cases}
\end{align*}

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\partial_t y - \Delta y + F(y) = v1_\omega & \text{in } Q, \\
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\end{align*}

\begin{align*}
(2) \quad & \begin{cases}
\partial_t y - \Delta y + F(y) = v1_\omega & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y(\cdot, 0) = y_0 & \text{in } \Omega.
\end{cases}
\end{align*}

In (1) and (2), $1_\omega$ represents the characteristic function of the set $\omega$, $y(x, t)$ is the state, $y_0$ is the initial datum and is given in an appropriate space, and $v$ is the control function (which is localized in $\omega$ -distributed control-). In (1), $a \in L^\infty(Q)$ is given. We will assume that $F : \mathbb{R} \rightarrow \mathbb{R}$ is a given function.
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In this talk we are interested in studying the controllability properties of systems (1) and (2) (controllability to trajectories).
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**Linear Problem**: For every $\omega$ and $T$ system (1) is null controllable (equivalently exactly controllable to trajectories): For every $y_0 \in L^2(\Omega)$ there is $v \in L^2(Q)$ s.t. the solution $y$ to (1) satisfies $y(T) \equiv 0$ in $\Omega$.
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Remarks on the controllability for the nonlinear heat equation
1. Introduction. Statement of the problem

**Nonlinear Problem:** Under appropriate assumptions on the function $F$ (which has a **superlinear growth** at infinity) system (2) is exactly controllable to trajectories at time $T$:
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   $$F(s) \sim |s| \log(1 + |s|).$$


   $$F(s) \sim |s| \log^p(1 + |s|), \quad p \in [0, 3/2].$$


   $$F(s) \sim |s| \log^p(1 + |s|) \quad (p \in [0, 3/2)), \quad 1 \leq N < 6 \quad \text{and a dissipativity condition on the the nonlinearity:} \quad sF(s) \geq -\mu_0 |s|^2 \quad (\mu_0 \geq 0).$$
1. Introduction. Statement of the problem

**Remark**

**Common Point:** The linear problem (1) is solved with a control \( v \) in \( L^p(Q) \) \((p > \frac{N}{2} + 1)\) with estimates of its norm with respect to \( T \), \( \|a\|_\infty \) and \( y_0 \).
Remark

**Common Point:** The linear problem (1) is solved with a control $v$ in $L^p(Q)$ ($p > \frac{N}{2} + 1$) with estimates of its norm with respect to $T$, $\|a\|_\infty$ and $y_0$.

**DIFFERENT TECHNIQUES**

**GOAL:**

Revisit the main known techniques which allow to prove the null controllability result for system (1) with a control $v \in L^p(Q)$, $p \in (2, \infty]$ (with estimates).
1. Introduction. Statement of the problem

THREE BASIC REFERENCES


1. Introduction. Statement of the problem

THREE BASIC REFERENCES


ANOTHER REFERENCE

2. Linear null controllability result with regular controls

We consider the distributed controllability problem for the linear system:

\[
\begin{aligned}
\partial_t y - \Delta y + ay &= v^1\omega \quad \text{in } Q, \\
y &= 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0 \quad \text{in } \Omega,
\end{aligned}
\]

where \( \omega \subset \Omega \) is an open subset, \( v \in L^2(Q) \) is the control and \( y_0 \) is given in \( L^2(\Omega) \).
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where \( \omega \subset \Omega \) is an open subset, \( v \in L^2(Q) \) is the control and \( y_0 \) is given in \( L^2(\Omega) \).

Let us fix \( \varphi_0 \in L^2(\Omega) \) and consider the adjoint problem

\[
\begin{cases}
-\partial_t \varphi - \Delta \varphi + a\varphi = 0 & \text{in } Q, \\
\varphi = 0 \text{ on } \Sigma, \quad \varphi(T) = \varphi_0 & \text{in } \Omega.
\end{cases}
\]

It is well known:
2. Linear null controllability result with regular controls

Theorem

The following conditions are equivalent:

1. There exists $C$ s.t. $\forall y_0 \in L^2(\Omega)$, there is $v \in L^2(Q)$, with

\[
\|v\|_{L^2(Q)}^2 \leq C\|y_0\|_{L^2(\Omega)}^2,
\]

s.t. the solution $y_v$ to (1) associated to $y_0$ and $v$ satisfies

\[
y_v(T) = 0 \text{ in } L^2(\Omega).
\]

2. There exists $C > 0$ s.t. (observability inequality)

\[
\|\varphi(0)\|_{L^2(\Omega)}^2 \leq C \iint_{\omega \times (0,T)} |\varphi(x, t)|^2 \, dx \, dt,
\]

holds for every solution $\varphi$ to the adjoint problem (3) associated to $\varphi_0 \in L^2(\Omega)$.  

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Remarks on the controllability for the nonlinear heat equation
2. Linear null controllability result with regular controls

The observability inequality for the adjoint problem with an explicit expression of $C$ with respect to the data can be obtained from a global Carleman inequalities for the linear parabolic problem:

\[
\begin{align*}
-\partial_t \varphi - \Delta \varphi &= F_0 \quad \text{in } Q, \\
\varphi &= 0 \text{ on } \Sigma, \quad \varphi(\cdot, T) = \varphi_0 \quad \text{in } \Omega,
\end{align*}
\]

with $F_0 \in L^2(Q)$ and $\varphi_0 \in L^2(\Omega)$ are given.
2. Linear null controllability result with regular controls

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\end{aligned} \]

with $F_0 \in L^2(Q)$ and $\varphi_0 \in L^2(\Omega)$ are given.

In


it is proved:
2. Linear null controllability result with regular controls

Lemma

There exist a regular and strictly positive function, \( \alpha_0 \), and two constants \( C_0 \) and \( \sigma_0 \) (only depending on \( \Omega \) and \( \omega \)) s.t.

\[
\begin{align*}
I(\varphi) &\equiv s^{-1} \iint_{Q} e^{-2s\alpha} t(T - t) \left( |\partial_t \varphi|^2 + |\Delta \varphi|^2 \right) \\
&+ s \iint_{Q} e^{-2s\alpha} t^{-1} (T - t)^{-1} |\nabla \varphi|^2 + s^3 \iint_{Q} e^{-2s\alpha} t^{-3} (T - t)^{-3} |\varphi|^2 \\
&\leq C_0 \left( s^3 \iint_{\omega \times (0, T)} e^{-2s\alpha} t^{-3} (T - t)^{-3} |\varphi|^2 + \iint_{Q} e^{-2s\alpha} |F_0|^2 \right),
\end{align*}
\]

\( \forall s \geq s_0 = \sigma_0(\Omega, \omega)(T + T^2) \), \( (\varphi \text{ is the solution to (4) associated to } \varphi_0 \in L^2(\Omega)) \). The function \( \alpha = \alpha(x, t) \) is given by

\[
\alpha(x, t) = \alpha_0(x)/t(T - t).
\]
2. Linear null controllability result with regular controls

Coming back to the adjoint problem

(3) \[
\begin{align*}
- \partial_t \varphi - \Delta \varphi + a \varphi &= 0 \quad \text{in} \quad Q, \\
\varphi &= 0 \quad \text{on} \quad \Sigma, \\
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\end{align*}
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Coming back to the adjoint problem

\begin{align}
\left\{ \begin{array}{l}
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\varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot, T) = \varphi_0 & \text{in} \quad \Omega.
\end{array} \right.
\end{align}

Lemma

There exist $C_1 > 0$ and $\sigma_1 > 0$ (only depending on $\Omega$ and $\omega$) s.t.

\begin{align}
\mathcal{I}(\varphi) &= s^{-1} \iint_Q e^{-2s\alpha} t(T-t) \left( |\partial_t \varphi|^2 + |\Delta \varphi|^2 \right) \\
&\quad + s \iint_Q e^{-2s\alpha} t^{-1}(T-t)^{-1} |\nabla \varphi|^2 + s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3} |\varphi|^2 \\
&\quad \leq C_1 s^3 \iint_{\omega \times (0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3} |\varphi|^2,
\end{align}

\forall s \geq s_1 = \sigma_1(\Omega, \omega) \left( T + T^2 + T^2 \|a\|_{\infty}^{2/3} \right).
2.1. First approach

We follow:

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From the previous global Carleman inequality one has:

**Theorem**

*For every* $a \in L^\infty(Q)$ *and* $\varphi_0 \in L^2(\Omega)$ *one has (observability inequality)*

$$\|\varphi(0)\|_{L^2(\Omega)}^2 \leq \exp \left[ C M(T, \|a\|_\infty) \right] \int_0^T \int_{\omega \times (0,T)} |\varphi|^2,$$

(*$\varphi$ solution to (3)) with $C = C(\Omega, \omega) > 0$ *and* $M$ *given by:*

$$M(T, \|a\|_\infty) = 1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3}.$$
2.1. First approach

Remark

This inequality shows the null controllability result for the linear system (1) with a control $v$ in $L^2(Q)$ (in fact, $\text{Supp } v \subset \omega \times (0, T)$) and provides the following estimate for $\|v\|_{L^2(Q)}$:

$$\|v\|_{L^2(Q)}^2 \leq \exp \left[CM(T, \|a\|_{\infty})\right] \|y_0\|^2,$$

with $M$ given as before.

Is it possible to solve this problem with a control $v \in L^\infty(Q)$? YES. The key point is a better observability inequality with a weaker norm on the right hand-side:
2.1. First approach

A refined observability inequality:

**Proposition**

There exists \( C = C(\Omega, \omega) > 0 \) such that

\[
\| \varphi(0) \|_{L^2(\Omega)}^2 \leq \exp \left[ C \tilde{M}(T, \| a \|_\infty) \right] \left( \int_0^T \int_{\omega \times (0,T)} |\varphi| \right)^2,
\]

with \( C = C(\Omega, \omega) > 0 \) and \( \tilde{M} \) given by:

\[
\tilde{M}(T, \| a \|_\infty) = 1 + \frac{1}{T} + T + \left( T^{1/2} + T \right) \| a \|_\infty + \| a \|_\infty^{2/3},
\]

for any \( \varphi_0 \in L^2(\Omega) \) and \( T > 0 \).
2.1. First approach

Sketch of the proof:

1. We fix $\omega_0 \subset \subset \omega$ and we apply the previous observability inequality with $\omega_0$ and $[T/4, 3T/4]$ instead of $\omega$ and $[0, T]$. Using the energy inequality we get

$$\|\varphi(0)\|_{L^2(\Omega)}^2 \leq \exp [C \, M(T, \|a\|_{\infty})] \int\int_{\omega_0 \times (T/4, 3T/4)} |\varphi|^2,$$

for a new constant $C = C(\Omega, \omega_0)$ and $M$ as before.
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for a new constant $C = C(\Omega, \omega_0)$ and $M$ as before.

2. We use the inequality

$$\int_{\omega_0} \int_{T/4}^{3T/4} |\varphi|^2 \leq CT^\alpha (1 + T^{1/2}(1 + \|a\|_{\infty}))^\beta \left( \iint_{\omega \times (0, T)} |\varphi| \right)^2$$

valid for every solution $\varphi$ to the adjoint problem (3) ($\alpha, \beta > 0$).
2.1. First approach

**Corollary**

There exists $C = C(\Omega, \omega) > 0$ s.t. $\forall y_0 \in L^2(\Omega)$, there is $v \in L^\infty(Q)$, with

$$\|v\|_{L^\infty(Q)}^2 \leq \exp\left[C \tilde{M}(T, \|a\|_{\infty})\right] \|y_0\|_{L^2(\Omega)}^2,$$

s.t. the solution $y_v$ to (1) associated to $y_0$ and $v$ satisfies

$$y_v(T) = 0 \text{ in } L^2(\Omega).$$

($\tilde{M}$ is given by

$$\tilde{M}(T, \|a\|_{\infty}) = 1 + \frac{1}{T} + T + \left(T^{1/2} + T\right) \|a\|_{\infty} + \|a\|_{\infty}^{2/3}.$$
2.1. First approach

Remark

1. The previous technique uses the **local regularizing effect** of the heat equation. The result is independent of the **initial condition** $y_0$ and the **boundary condition**.

2. This technique can be applied to linear parabolic problems with first order terms $B \cdot \nabla y$:

3. The existence of the bounded control is deduced from the **observability inequality**: “If system (1) is exactly controllable to trajectories at time $T$ with controls in $L^2(Q)$, then system (1) is exactly controllable to trajectories at time $T$ with controls in $L^\infty(Q)$”.

Remarks on the controllability for the nonlinear heat equation
Remark

1. **More regularity**? For example, \( v \in L^2(0, T; H^2(\Omega) \times H^1_0(\Omega)) \) and \( \partial_t v \in L^2(Q) \) or \( v \in C^\infty(Q) \) when \( a \equiv 0 \) (as in the work of Lebeau-Robbiano).

2. What happens if \( \Omega \) and \( \omega \) are **unbounded** sets???

3. What happens if we consider coupled parabolic systems???
2.1. First approach

Coupled parabolic systems: Let us consider a “simple” coupled parabolic system

\[
\begin{align*}
\partial_t y - \Delta y &= Ay + Bv_1 \omega & \text{in } Q, \\
y &= 0 \text{ on } \Sigma, \ y(0) = y_0 & \text{in } \Omega,
\end{align*}
\]

\[
\begin{align*}
\partial_t \varphi + \Delta \varphi &= -A^* \varphi & \text{in } Q, \\
\varphi &= 0 \text{ on } \Sigma, \ \varphi(T) = \varphi_0 & \text{in } \Omega,
\end{align*}
\]

with \( A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \), \( B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) (one control force) and \( y_0 \in L^2(\Omega)^2 \).
2.1. First approach

Coupled parabolic systems: Let us consider a “simple” coupled parabolic system

\[
\begin{align*}
\frac{\partial t}{\partial t} y - \Delta y &= Ay + Bv_1 \omega \quad \text{in } Q, \\
y &= 0 \text{ on } \Sigma, \quad y(0) = y_0 \quad \text{in } \Omega,
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\end{align*}
\]

with \( A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \), \( B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) (one control force) and \( y_0 \in L^2(\Omega)^2 \).

The particular structure of \( A \) and \( B \) (cascade system) gives:

\[
\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \iint_{\omega_0 \times (T/4, 3T/4)} |\varphi_1|^2,
\]

for a constant \( C > 0 \). Then, there is \( v \in L^2(Q) \) s.t. \( y_v(T) = 0 \) in \( \Omega \) and

\[
\| v \|^2_{L^2(\Omega)} \leq C \| y_0 \|^2_{L^2(\Omega)^2}.
\]

Control in \( L^\infty(Q) \)??
2.1. First approach

Following this technique, does the following inequality hold?

\[
\iint_{\omega_0 \times (T/4, 3T/4)} |\varphi_1|^2 \leq C \left( \iint_{\omega \times (0, T)} |\varphi_1| \right)^2
\]

\[\text{hold?? NO.}\]

Remark

This first approach cannot be applied to the previous coupled system since the local regularizing effect of the linear adjoint problem involves the functions \(\varphi_1\) and \(\varphi_2\) while the corresponding “refined” observability inequality should only involve \(\varphi_1\) (recall that the control \(v\) only appears in first equation of the direct problem).
2.2. Second approach

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We recall the **global Carleman inequality** ($\partial \Omega \in C^2$):

\[
\begin{align*}
\mathcal{I}(\varphi) &= s^{-1} \iint_Q e^{-2s\alpha} t(T - t) \left( |\partial_t \varphi|^2 + |\Delta \varphi|^2 \right) \\
&\quad + s \iint_Q e^{-2s\alpha} t^{-1} (T - t)^{-1} |\nabla \varphi|^2 + s^3 \iint_Q e^{-2s\alpha} t^{-3} (T - t)^{-3} |\varphi|^2 \\
&\leq C_1 s^3 \iint_{\omega \times (0,T)} e^{-2s\alpha} t^{-3} (T - t)^{-3} |\varphi|^2,
\end{align*}
\]

\(\forall s \geq s_1 = \sigma_1(\Omega, \omega) \left( T + T^2 + T^2 \|a\|_{2/3}^2 \right)\), where \(C_1 = C_1(\Omega, \omega) > 0\) and \(\varphi\) the solution to

\[
\begin{align*}
-\partial_t \varphi - \Delta \varphi + a \varphi &= 0 \quad \text{in} \quad Q, \\
\varphi &= 0 \quad \text{on} \quad \Sigma, \quad \varphi(\cdot, T) = \varphi_0(\cdot) \quad \text{in} \quad \Omega.
\end{align*}
\]
2.2. Second approach

In this work, a control in $L^p(Q)$, with $p = p(N)$, is obtained from the previous global Carleman inequality (we fix

\[ s = s_1 = \sigma_1(\Omega, \omega) \left( T + T^2 + T^2 \| a \|_\infty^{2/3} \right). \]
2.2. Second approach

In this work, a control in $L^p(Q)$, with $p = p(N)$, is obtained from the previous global Carleman inequality (we fix

$$s = s_1 = \sigma_1(\Omega, \omega) \left( T + T^2 + T^2 \|a\|_{\infty}^{2/3} \right).$$

First Step:

Lemma

For every $a \in L^\infty(Q)$ and $\varphi_0 \in L^2(\Omega)$ one has (observability inequality)

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq \exp \left[ C M(T, \|a\|_{\infty}) \right] \int_{\omega \times (0,T)} e^{-2s_1 \alpha t - 3(T - t)^{-3}} |\varphi|^2,$$

($\varphi$ solution to (3)) with $C = C(\Omega, \omega) > 0$ and $M$ given by:

$$M(T, \|a\|_{\infty}) = 1 + \frac{1}{T} + T \|a\|_{\infty} + \|a\|_{\infty}^{2/3}.$$
2.2. Second approach

Second Step: From this **observability inequality** we deduce

**Proposition**

\[ \forall y_0 \in L^2(\Omega), \text{ there is } v \in L^{p(N)}(Q), \text{ with } p(N) < \infty \text{ if } N = 2 \text{ and } \]
\[ p(N) = \frac{2(N + 2)}{N - 2} \text{ if } N \geq 3, \text{ and } \]
\[ \| v \|^2_{L^{p(N)}(Q)} \leq e^{[C M(T, \| a \|_\infty)]} \| y_0 \|^2_{L^2(\Omega)}, \]

s.t. the solution \( y_v \) to (1) associated to \( y_0 \) and \( v \) satisfies

\( y_v(T) = 0 \) in \( L^2(\Omega) \).
2.2. Second approach

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\[ p(N) = \frac{2(N + 2)}{N - 2} \text{ if } N \geq 3, \text{ and } \]

\[ \| v \|_{L^{p(N)}(Q)}^2 \leq e^{[C^M(T, \| a \|_\infty)]} \| y_0 \|_{L^2(\Omega)}^2, \]

s.t. the solution \( y_v \) to (1) associated to \( y_0 \) and \( v \) satisfies

\[ y_v(T) = 0 \text{ in } L^2(\Omega). \]

**Sketch of the proof:** 1.- We consider the optimal control problem

\[ \min_{v \in L^2(Q)} \left( \frac{1}{2} \int \int_Q e^{2s_1 \alpha t^3(T - t)^3} |v(x, t)|^2 \, dx \, dt + \frac{1}{2\varepsilon} \| y_v(T) \|_{L^2(\Omega)}^2 \right), \]

\( (y_v \in L^2(Q)^2 \text{ is the solution of (1) associated to } y_0 \text{ and } v). \)
2.2. Second approach

This problem has a unique solution $v_\varepsilon \in L^2(Q)$ and, using the optimality system, it is characterized:

$$v_\varepsilon = e^{-2s_1\alpha} t^{-3} (T - t)^{-3} \varphi_\varepsilon \mathbf{1}_\omega$$

and

$$\begin{cases}
\partial_t y_\varepsilon - \Delta y_\varepsilon + a y_\varepsilon = v_\varepsilon \mathbf{1}_\omega & \text{in } Q, \\
y_\varepsilon = 0 \text{ sobre } \Sigma, \quad y_\varepsilon(\cdot, 0) = y_0 & \text{in } \Omega,
\end{cases}$$

$$\begin{cases}
-\partial_t \varphi_\varepsilon - \Delta \varphi_\varepsilon + a \varphi_\varepsilon = 0 & \text{in } Q, \\
\varphi_\varepsilon = 0 \text{ on } \Sigma, \quad \varphi_\varepsilon(\cdot, T) = -\frac{1}{\varepsilon} y_\varepsilon(\cdot, T) & \text{in } \Omega.
\end{cases}$$

The previous observability inequality (Lemma 7) gives:

$$\iint_{\omega \times (0, T)} e^{-2s_1\alpha} t^{-3} (T - t)^{-3} |\varphi_\varepsilon|^2 + \frac{1}{\varepsilon} \|y_\varepsilon(T)\|_{L^2(\Omega)}^2 \leq e^{[CM(T, \|a\|_\infty)]} \|y_0\|_{L^2(\Omega)}^2$$
2.2. Second approach

Combining the last inequality and the global Carleman inequality (5) we get

\[
\int \int_{Q} e^{-2s_1 \alpha t(T-t)} \left( |\partial_t \varphi_\varepsilon|^2 + |\Delta \varphi_\varepsilon|^2 \right) \leq e^{CM(T,\|a\|_\infty)} \|y_0\|^2_{L^2(\Omega)}.
\]

Taking into account the expression \( v_\varepsilon = e^{-2s_1 \alpha t^{-3}} (T-t)^{-3} \varphi_\varepsilon 1_\Omega \), we deduce

\[
\begin{align*}
\nu_\varepsilon &\in H^{2,1}(Q) = \{ q : q \in L^2(0,T;D(-\Delta)), \partial_t q \in L^2(Q) \}, \\
\|\nu_\varepsilon\|^2_{H^{1,2}(Q)} + \frac{1}{\varepsilon} \|y_\varepsilon(T)\|^2_{L^2(\Omega)} &\leq e^{CM(T,\|a\|_\infty)} \|y_0\|^2_{L^2(\Omega)},
\end{align*}
\]

for a new constant \( C(\Omega,\omega) > 0 \). Thus, \( \{ v_\varepsilon \}_{\varepsilon > 0} \) is bounded in \( H^{2,1}(Q) \). We can extract a subsequence that converges to \( v \) weakly in \( H^{2,1}(Q) \). Clearly,

\[
\|v\|^2_{H^{1,2}(Q)} \leq e^{CM(T,\|a\|_\infty)} \|y_0\|^2_{L^2(\Omega)} \text{ and } y_v(T) = 0 \text{ in } \Omega.
\]
Finally, using the continuous embedding
\[ H^{2,1}(Q) \hookrightarrow L^{p(N)}(Q) \]
we deduce the proof.
2.2. Second approach

3.- Finally, using the continuous embedding

\[ H^{2,1}(Q) \hookrightarrow L^{p(N)}(Q) \]

we deduce the proof.

Remark

Observe that the previous control \( v \in L^{p(N)}(Q) \) provides a solution \( y_v \in W^{2,1,p(N)} = \{ q \in L^{p(N)}(0, T; W^{2,p(N)}(\Omega) \cap W_0^{1,p(N)}(\Omega)) : \partial_t q \in L^{p(N)}(Q) \} \). Thus, using again the continuous embedding of this space, if \( p(N) > N/2 + 1 \), i.e., if \( 1 \leq N < 6 \), the solution \( y_v \in L^\infty(Q) \). In Barbu's work, the nonlinear null controllability problem is treated with this constraint on the dimension \( N \).
2.2. Second approach. Remarks I

1. This technique uses the global regularizing effect of the heat equation. Then, the result depends on the boundary conditions but is independent of the initial condition $y_0$.

2. This technique cannot be directly applied if we consider a linear parabolic problem with a first order term $B \cdot \nabla y$. Observe that in the global Carleman inequality for the corresponding adjoint system the terms $\partial_t \varphi$ and $\Delta \varphi$ do not appear.

3. In fact, the control $v$ provided by this approach lies in $H^{2,1}(Q)$, but, more regularity? For example, when $a \equiv 0$, $v \in C^\infty(Q)$ (as in the work of Lebeau-Robbiano)??
2.2. Second approach. Remarks II

The existence of a control $v$ in $L^{p(N)}$ is deduced from the **global Carleman inequality** satisfied by the **adjoint system**. When $\Omega$ and $\omega$ are unbounded open sets and under some geometric conditions on $(\Omega, \omega)$, it is possible to establish a **global Carleman inequality** for the **adjoint system**:


In this situation it is possible to obtain a control $v$ with the same regularity as before.
This approach also works in the case of coupled parabolic systems (if we have proved a global Carleman inequality for the corresponding adjoint system).

Following the same approach, it is possible to solve the null controllability result for system (1) with controls in $W^{2,1}_p(Q)$, for every $p \in [1, \infty)$: Last section.
2.3. Third approach

We follow

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**ASSUMPTION**
Given $y_0 \in L^2(\Omega)$, there is $\tilde{v} \in L^2(Q)$, with $\text{Supp} \tilde{v} \subset \omega_0$ and $\omega_0 \subset \subset \omega$, such that the solution to (1) $\tilde{y}$ satisfies $\tilde{y}(\cdot, T) \equiv 0$ in $\Omega$.

One has

$\tilde{y} \in W(0, T) = \{y \in L^2(0, T; H_0^1(\Omega)) : \partial_t y \in L^2(0, T; H^{-1}(\Omega))\}$ and an explicit estimate $\|\tilde{y}\|_{W(0,T)} \leq \exp\left(C(1 + T)\|a\|_\infty\right) \left(\|y_0\|_2 + \|\tilde{v}\|_2\right)$. 

2.3. Third approach

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**ASSUMPTION**

Given $y_0 \in L^2(\Omega)$, there is $\tilde{v} \in L^2(Q)$, with $\text{Supp} \tilde{v} \subset \omega_0$ and $\omega_0 \subset \subset \omega$, such that the solution to (1) $\tilde{y}$ satisfies $\tilde{y}(\cdot, T) \equiv 0$ in $\Omega$.

One has

$\tilde{y} \in W(0, T) = \{ y \in L^2(0, T; H^1_0(\Omega)) : \partial_t y \in L^2(0, T; H^{-1}(\Omega)) \}$ and an explicit estimate $\|\tilde{y}\|_{W(0, T)} \leq \exp(C(1 + T)\|a\|_{\infty}) (\|y_0\|_2 + \|\tilde{v}\|_2)$.

The function $\tilde{y}$ is regular except near $t = 0$ and near $\omega_0$. The idea is to eliminate these irregular parts of $\tilde{y}$. 
2.3. Third approach

Let us now introduce two cut-off functions $\eta \in C^\infty([0, T])$ and $\theta \in C^\infty(\overline{\Omega})$ such that

\[
\begin{align*}
\eta &\equiv 1 \text{ in } [0, \frac{T}{4}], \quad \eta \equiv 0 \text{ in } [\frac{3T}{4}, T], \quad 0 \leq \eta \leq 1 \text{ in } [0, T], \quad |\eta'(t)| \leq C/T, \quad \forall t; \\
\theta &\equiv 1 \text{ in } \overline{\omega}_0, \quad 0 \leq \theta \leq 1 \text{ in } \Omega \text{ and } \text{Supp } \theta \subset \omega.
\end{align*}
\]

Let $Y$ be the solution to system (1) corresponding to $v \equiv 0$:

\[
\begin{align*}
\begin{cases}
\partial_t Y - \Delta Y + aY = 0 & \text{in } Q, \\
Y = 0 \text{ on } \Sigma, \quad Y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega,
\end{cases}
\end{align*}
\]
2.3. Third approach

Let us now introduce two cut-off functions $\eta \in C^\infty([0, T])$ and $\theta \in C^\infty(\overline{\Omega})$ such that

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\theta &\equiv 1 \text{ in } \overline{\omega}_0, \quad 0 \leq \theta \leq 1 \text{ in } \Omega \text{ and } \text{Supp } \theta \subset \omega.
\end{align*}
\]

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\partial_t Y - \Delta Y + aY &= 0 \quad \text{in } Q, \\
Y &= 0 \text{ on } \Sigma, \quad Y(\cdot, 0) = y_0(\cdot) \quad \text{in } \Omega,
\end{align*}
\]

We now take

\[
\begin{align*}
y &= (1 - \theta)\tilde{y} + \eta\theta Y \quad \text{in } Q, \\
v &= (\partial_t - \Delta + a)y.
\end{align*}
\]

It is clear that $\text{Supp } v(\cdot, t) \subseteq \text{Supp } \theta \subset \omega$, $y$ is the solution to (1) corresponding to the control $v$ and, taking into account that $\tilde{y}(T) \equiv 0$ in $\Omega$, we get $y(\cdot, T) \equiv 0$ in $\Omega$. 

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Remarks on the controllability for the nonlinear heat equation
In fact \( v \) is a regular control and its regularity properties are independent of \( y_0 \) and \( \tilde{v} \). Indeed, we can express \( y \) and \( v \) as

\[
y \equiv (1 - \theta)q + \eta(t) Y, \quad v \equiv \theta \eta' Y + 2\nabla \theta \cdot \nabla q + (\Delta \theta)q,
\]

where \( q \) is given by \( q = \tilde{y} - \eta Y \) and, therefore, satisfies

\[
\begin{align*}
\partial_t q - \Delta q + aq &= \tilde{v}1_\omega - \eta' Y \text{ in } Q, \\
q &= 0 \text{ on } \Sigma, \quad q(\cdot, 0) = 0 \text{ in } \Omega.
\end{align*}
\]
2.3. Third approach

In fact $v$ is a regular control and its regularity properties are independent of $y_0$ and $\tilde{v}$. Indeed, we can express $y$ and $v$ as

$$y \equiv (1 - \theta)q + \eta(t)Y, \quad v \equiv \theta \eta'Y + 2\nabla \theta \cdot \nabla q + (\Delta \theta)q,$$

where $q$ is given by $q = \tilde{y} - \eta Y$ and, therefore, satisfies

$$\begin{align*}
\partial_t q - \Delta q + aq &= \tilde{v}1_\omega - \eta'Y \quad \text{in } Q, \\
q &= 0 \quad \text{on } \Sigma, \quad q(\cdot, 0) = 0 \quad \text{in } \Omega.
\end{align*}$$

Let us fix $\delta \in (0, T/4)$, $p \in [2, \infty)$ and $O_0, O_1 \subseteq \Omega$ such that $O_1 \subseteq \overline{\Omega \setminus \omega_0}$ (and, in particular, $\overline{O_1 \cap \text{Supp } \tilde{v}} = \emptyset$). If we denote by

$$\begin{align*}
X_0^p &= \{y \in L^p(\delta, T; W^{2,p}(O_0)) : \partial_t y \in L^p(O_0 \times (\delta, T))\}, \\
X_1^p &= \{y \in L^p(0, T; W^{2,p}(O_1)) : \partial_t y \in L^p(O_i \times (0, T))\}
\end{align*}$$

then, $Y \in X_0^p$, $q \in X_1^p$ and $v \in L^p(0, T; W^{1,p}_0(\Omega))$.  

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Remarks on the controllability for the nonlinear heat equation
2.3. Third approach

In fact, we can obtain something better: if $p > N + 2$, one has $X_0^p \hookrightarrow C^{1+\alpha,(1+\alpha)/2}(\Omega_0 \times [\delta, T])$ and $X_1^p \hookrightarrow C^{1+\alpha,(1+\alpha)/2}(\Omega_1 \times [0, T])$ with $\alpha = 1 - (N + 2)/p$. Thus, $v \in C_0(Q)$ and

$$\|v\|_{C_0} \leq e^{C(1+T+T\|a\|_\infty)} \|\tilde{Y}\|_{W(0,T)}$$

with $C = C(\Omega, T) > 0$. 

M. González-Burgos  
Remarks on the controllability for the nonlinear heat equation
The previous regularity result for $v$ is independent of the initial datum $y_0$, the control $\tilde{v}$ and the regularity of the boundary $\partial \Omega$. We have only used the **local regularity** properties of the operator $L \equiv \partial_t - \Delta + a$. In the case in which $a \equiv 0$, we obtain $v \in C^\infty(\overline{Q})$ (as in the paper of Lebeau-Robbiano).

In fact we have proved: “Let us fix $y_0 \in L^2(\Omega)$ and assume that there exists $\tilde{v} \in L^2(Q)$ such that the solution $\tilde{y}$ to the linear problem (1) satisfies $\tilde{y}(T) \equiv 0$ in $\Omega$. Then, there exists $\tilde{v} \in C^0_0(\overline{Q})$ s.t. the solution $y_v$ of (1) also satisfies $y_v(T) \equiv 0$ in $\Omega$.”

This technique can be applied if we consider a linear parabolic problem with a first order term $B \cdot \nabla y$ obtaining the same regularity result.
2.2. Third approach. Remarks II

4 When $\Omega$ and $\omega$ are unbounded open sets we can obtain the same result:


5 This approach also works in the case of systems of two coupled parabolic equations.
3. The “best” null control

We consider once again the linear problem

\[
\begin{aligned}
\partial_t y - \Delta y + ay &= v_1\omega & \text{in } Q, \\
y &= 0 & \text{on } \Sigma, \\
y(\cdot, 0) &= y_0(\cdot) & \text{in } \Omega.
\end{aligned}
\]

Question

Fix \( p \in [1, \infty) \). Given \( y_0 \in L^2(\Omega) \), does there exist \( v \in W^{2,1}_p(Q) \) s.t. the solution to (1) satisfies \( y(T) = 0 \) in \( \Omega \)?

Idea

We are going to add “better” terms on the left hand-side of the global Carleman inequality for the adjoint problem and then apply again the approach of Barbu.
3. The “best” null control

We consider once again the linear problem

\[
\begin{align*}
\partial_t y - \Delta y + ay &= v1_\omega & \text{in } Q, \\
y &= 0 \text{ on } \Sigma, & y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega.
\end{align*}
\]

Question

Fix \( p \in [1, \infty) \). Given \( y_0 \in L^2(\Omega) \), does there exist \( v \in W^{2,1}_p(Q) \) s.t. the solution to (1) satisfies \( y(T) = 0 \) in \( \Omega \)?? Estimates of \( v \)??

\[
W^{2,1}_p(Q) = \{ u \in L^p(0, T; W^{2,p}(\Omega)) : \partial_t u \in L^p(Q) \}.
\]
3. The “best” null control

We consider once again the linear problem

\[
\begin{cases}
\partial_t y - \Delta y + ay = v1_\omega & \text{in } Q, \\
y = 0 \text{ on } \Sigma, & y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega.
\end{cases}
\]

Question

Fix \( p \in [1, \infty) \). Given \( y_0 \in L^2(\Omega) \), does there exist \( v \in W^{2,1}_p(Q) \) s.t. the solution to (1) satisfies \( y(T) = 0 \) in \( \Omega \)? Estimates of \( v \)?

\[
W^{2,1}_p(Q) = \{ u \in L^p(0, T; W^{2,p}(\Omega)) : \partial_t u \in L^p(Q) \}.
\]

Idea

We are going to add “better” terms on the left hand-side of the global Carleman inequality for the adjoint problem and then apply again the approach of Barbu.
3. The “best” null control

The adjoint problem:

(3) \[
\begin{cases}
-\partial_t \varphi - \Delta \varphi + a \varphi = 0 & \text{in } Q, \\
\varphi = 0 & \text{on } \Sigma, \\
\varphi(\cdot, T) = \varphi_0(\cdot) & \text{in } \Omega.
\end{cases}
\]

From the Carleman inequality, we deduce,

\[
\begin{cases}
s^{-1} \iint_Q e^{-2s\alpha} t(T - t) \left( |\partial_t \varphi|^2 + |\Delta \varphi|^2 \right) \\
\leq C_1 s^3 \iint_{\omega \times (0, T)} e^{-2s\alpha} t^{-3} (T - t)^{-3} |\varphi|^2,
\end{cases}
\]

\[\forall s \geq s_1 = \sigma_1(\Omega, \omega) \left( T + T^2 + T^2 \|a\|_\infty^{2/3} \right), \text{ where } C_1 = C_1(\Omega, \omega) > 0.\]

We take:

\[\alpha^*_0 = \max_{x \in \Omega} \alpha_0(x), \quad \alpha^*(t) = \frac{\alpha^*_0}{t(T - t)}.\]
The function $\alpha_0$ is given by

$$\alpha_0(x) = e^{2Cm||\eta_0||_{\infty}} - e^{C(m||\eta_0||_{\infty} + \eta_0(x))},$$

with $m > 1$ an arbitrary constant, $\eta_0$, a function only depending on $\Omega$ and $\omega$, and $C = C(\Omega, \omega) > 0$. The construction of $\eta_0 = \eta_0(x)$ is given in [Fursikov-Imanuvilov]. This function satisfies:

$$\eta_0 \in C^2(\overline{\Omega}), \quad \eta_0 \geq 0 \text{ in } \Omega, \quad \frac{\partial \eta_0}{\partial n} \leq 0 \text{ on } \partial \Omega \quad \text{and} \quad \nabla \eta_0 \neq 0 \text{ in } \overline{\Omega} \setminus \omega.$$  

($n = n(x)$: the outward unit normal to $\Omega$ at point $x \in \partial \Omega$).
3. The “best” null control

We take

\[ \psi = s^{-5/2}e^{-s\alpha^*(t)}t^{5/2}(T - t)^{5/2}\varphi = \rho_0(t)\varphi. \]

Then,

\[ \begin{cases} 
\partial_t \psi + \Delta \psi = a\rho_0(t)\varphi + \partial_t \rho_0(t)\varphi & \text{in } Q, \\
\psi = 0 \text{ on } \Sigma, \quad \psi(\cdot, T) = 0 & \text{in } \Omega.
\end{cases} \]
3. The “best” null control

We take

$\psi = s^{-5/2}e^{-s\alpha^*(t)}t^{5/2}(T - t)^{5/2}\varphi = \rho_0(t)\varphi.$

Then,

$$\begin{cases}
\partial_t \psi + \Delta \psi = a\rho_0(t)\varphi + \partial_t \rho_0(t)\varphi & \text{in } Q, \\
\psi = 0 \text{ on } \Sigma, \quad \psi(\cdot, T) = 0 & \text{in } \Omega.
\end{cases}$$

If $s \geq s_1 = \sigma_1 \left( T + T^2 + T^2\|a\|_\infty^{2/3} \right)$, we have $\partial_t \rho_0(t)\varphi \in H^{2,1}(Q)$ and

$$||\partial_t \rho_0(t)\varphi||_{H^{2,1}}^2 \leq Cs^{-1} \int \int_Q e^{-2s\alpha} t(T - t) \left( |\partial_t \varphi|^2 + |\Delta \varphi|^2 \right).$$

But, $H^{2,1}(Q) \hookrightarrow L^{p(N)}(Q)$ with $p(N) = \frac{2(N+2)}{N-2}$. Thus,

$$||\partial_t \rho_0(t)\varphi||_{L^{p(N)}(Q)} \leq C ||\partial_t \rho_0(t)\varphi||_{H^{2,1}}.$$
3. The “best” null control

We take

\[ \psi = s^{-5/2} e^{-s \alpha^*(t)} t^{5/2} (T - t)^{5/2} \varphi = \rho_0(t) \varphi. \]

Then,

\[
\begin{cases}
\partial_t \psi + \Delta \psi = a \rho_0(t) \varphi + \partial_t \rho_0(t) \varphi & \text{in} \quad Q, \\
\psi = 0 \text{ on } \Sigma, \quad \psi(\cdot, T) = 0 & \text{in} \quad \Omega.
\end{cases}
\]

If \( s \geq s_1 = \sigma_1 \left( T + T^2 + T^2 \| a \|_\infty^{2/3} \right) \), we have \( \partial_t \rho_0(t) \varphi \in H^{2,1}(Q) \) and

\[
\| \partial_t \rho_0(t) \varphi \|_{H^{2,1}}^2 \leq C s^{-1} \int \int_Q e^{-2s \alpha} t(T - t) \left( |\partial_t \varphi|^2 + |\Delta \varphi|^2 \right) \]

But, \( H^{2,1}(Q) \hookrightarrow L^{p(N)}(Q) \) with \( p(N) = \frac{2(N+2)}{N-2} \). Thus,

\[
\| \partial_t \rho_0(t) \varphi \|_{L^{p(N)}(Q)} \leq C \| \partial_t \rho_0(t) \varphi \|_{H^{2,1}}
\]

We can also prove that \( a \rho_0(t) \varphi \in L^{p(N)}(Q) \) and

\[
\| a \rho_0(t) \varphi \|^2_{L^{p(N)}(Q)} \leq C s^{-1} \int \int_Q e^{-2s \alpha} t(T - t) \left( |\partial_t \varphi|^2 + |\Delta \varphi|^2 \right) \]

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Remarks on the controllability for the nonlinear heat equation
3. The “best” null control

The maximal parabolic regularity for the heat equation ($\partial \Omega \in C^2$) gives

$$\psi = s^{-5/2} e^{-s \alpha^* (t)} t^{5/2} (T - t)^{5/2} \varphi \in W^{2,1}_{p(N)(Q)}$$

and

$$\begin{cases} 
\| \psi \|_{W^{2,1}_{p(N)(Q)}}^2 \leq C s^{-1} \iint_Q e^{-2s \alpha} t (T - t) \left( |\partial_t \varphi|^2 + |\Delta \varphi|^2 \right) \\
\leq C_2 s^3 \iint_{\omega \times (0,T)} e^{-2s \alpha} t^{-3} (T - t)^{-3} |\varphi|^2.
\end{cases}$$
3. The “best” null control

The maximal parabolic regularity for the heat equation \((\partial \Omega \in C^2)\) gives

\[ \psi = s^{-5/2} e^{-s\alpha^*(t)} t^{5/2} (T - t)^{5/2} \varphi \in W^{2,1}_{p(N)}(Q) \]

and

\[
\begin{cases}
\|\psi\|^2_{W^{2,1}_{p(N)}(Q)} \leq C s^{-1} \iint_Q e^{-2s\alpha} t(T - t) \left( |\partial_t \varphi|^2 + |\Delta \varphi|^2 \right)

\leq C_2 s^3 \iint_{\omega \times (0, T)} e^{-2s\alpha} t^{-3} (T - t)^{-3} |\varphi|^2.
\end{cases}
\]

Conclusion

We have obtained a new Carleman inequality for the problem (3)

\[
\begin{cases}
\|s^{-5/2} e^{-s\alpha^*(t)} t^{5/2} (T - t)^{5/2} \varphi\|^2_{W^{2,1}_{p(N)}(Q)} + I(\varphi)

\leq C_2 s^3 \iint_{\omega \times (0, T)} e^{-2s\alpha} t^{-3} (T - t)^{-3} |\varphi|^2,

\forall s \geq s_1 = \sigma_1 \left( T + T^2 + T^2 \|a\|_{\infty}^{2/3} \right).
\end{cases}
\]
Corollary

∀y₀ ∈ L²(Ω), there is v ∈ W²¹_p(N)(Q), with p(N) < ∞ if N = 2 and

\[ p(N) = \frac{2(N + 2)}{N - 2} \] if N ≥ 3, and

\[ \| v \|_{W^{2,1}_{p(N)}}^2 ≤ e^{[C M(T, \| a \|_\infty)]} \| y_0 \|_{L^2(\Omega)}^2, \]

s.t. the solution yᵥ to (1) associated to y₀ and v satisfies

\[ yᵥ(T) = 0 \text{ in } L^2(\Omega). \]
3. The “best” null control

**Corollary**

\[ \forall y_0 \in L^2(\Omega), \text{ there is } v \in W^{2,1}_{p(N)}(Q), \text{ with } p(N) < \infty \text{ if } N = 2 \text{ and } \]

\[ p(N) = \frac{2(N + 2)}{N - 2} \text{ if } N \geq 3, \text{ and } \]

\[ \| v \|_{W^{2,1}_{p(N)}}^2 \leq e^{CM(T,\|a\|_{\infty})} \| y_0 \|_{L^2(\Omega)}^2, \]

s.t. the solution \( y_v \) to (1) associated to \( y_0 \) and \( v \) satisfies

\[ y_v(T) = 0 \text{ in } L^2(\Omega). \]

**Remark**

We can apply a boot-strap argument and deduce that the previous result is valid for every \( p \in [2, \infty) \). In this case the constant \( C \) also depends on \( p \).
3. The “best” null control

Reference


- M. González-Burgos, *Remarks on the controllability for the nonlinear heat equation*
3. The “best” null control

Reference

