Convergence to equilibrium for smectic-A liquid crystals in 3D domains

F. Guillén-González,
Depto. EDAN & IMUS, Universidad de Sevilla. Spain.
AIMS, 07-11 July 2014, Madrid.

In collaboration: B. Climent-Ezquerra (Univ. Sevilla),
. [NA-TMA’14].
Liquid Crystals

- **Liquid crystals (LC)** exhibit properties between liquids and solids. A LC can behave like a liquid (macroscopically), but their molecules have a preferential orientation (microscopically), due to **elasticity** effects (**anisotropic liquids**).

- The time-dynamics interaction between macroscopic and microscopic is modeled by nonlinear parabolic PDE + **gradient flow**, involving:
  - Macroscopic: fluid dynamics (Navier-Stokes)
  - Microscopic: order parameter

- Different phases in LC, for instance:
  - **nematic** phases, with orientational order of molecules,
  - **smectic** phases, with moreover positional order (arranged in layers).
Liquid crystals (LC) exhibit properties between liquids and solids. A LC can behave like a liquid (macroscopically), but their molecules have a preferential orientation (microscopically), due to elasticity effects (anisotropic liquids).

The time-dynamics interaction between macroscopic and microscopic is modeled by nonlinear parabolic PDE + gradient flow, involving:

- Macroscopic: fluid dynamics (Navier-Stokes)
- Microscopic: order parameter

Different phases in LC, for instance:

- nematic phases, with orientational order of molecules,
- smectic phases, with moreover positional order (arranged in layers).
Liquid Crystals

- **Liquid crystals (LC)** exhibit properties between liquids and solids. A LC can behave like a liquid (macroscopically), but their molecules have a preferential orientation (microscopically), due to elasticity effects (anisotropic liquids).

- The time-dynamics interaction between **macroscopic** and **microscopic** is modeled by nonlinear parabolic PDE + **gradient flow**, involving:
  - Macroscopic: fluid dynamics (Navier-Stokes)
  - Microscopic: **order parameter**

- Different phases in LC, for instance:
  - **nematic** phases, with orientational order of molecules,
  - **smectic** phases, with moreover positional order (arranged in layers).
thermotropic liquid crystals

- Crystal
  - 3-D lattice
  - orientation
  - solid
  - anisotropic

- Liquid crystal (mesophases)
  - 1- (2-)D lattice
  - orientation
  - fluid
  - anisotropic

- Liquid
  - no lattice
  - no orientation
  - fluid
  - isotropic
Static Osseen-Frank’s theory for nematic LC

- \( \Omega \subset \mathbb{R}^d \ (d = 2, 3) \): domain filled by the LC, with boundary \( \partial \Omega \)
- Equilibrium states: minimum of a free energy
- Oseen-Frank free energy derive to the Static (Minimization) problem:

\[
\min_{|d|=1} E_{ela}(d)
\]

where \( E_{ela} \) is the elastic energy functional:

\[
E_{ela}(d) = \int_{\Omega} \left( \frac{k_1}{2} (\nabla \cdot d)^2 + \frac{k_2}{2} (d \cdot (\nabla \times d))^2 + \frac{k_3}{2} |d \times (\nabla \times d)|^2 \right)
\]

\( k_i > 0 \) elastic constants.

**Remark:** Problem with convex functional but non-convex constraints.
Static Osseen-Frank’s theory for nematic LC

- $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$): domain filled by the LC, with boundary $\partial \Omega$
- Equilibrium states: minimum of a free energy
- **Oseen-Frank** free energy derive to the Static (Minimization) problem:

$$\min_{|d|=1} E_{ela}(d)$$

where $E_{ela}$ is the **elastic** energy functional:

$$E_{ela}(d) = \int_{\Omega} \left( \frac{k_1}{2} (\nabla \cdot d)^2 + \frac{k_2}{2} (d \cdot (\nabla \times d))^2 + \frac{k_3}{2} |d \times (\nabla \times d)|^2 \right)$$

$k_i > 0$ elastic constants.

**Remark:** Problem with convex functional but non-convex constraints.
Static Osseen-Frank’s theory for nematic LC

- $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$): domain filled by the LC, with boundary $\partial \Omega$
- Equilibrium states: minimum of a free energy
- **Oseen-Frank** free energy derive to the Static (Minimization) problem:

$$\min_{|d|=1} E_{ela}(d)$$

where $E_{ela}$ is the **elastic** energy functional:

$$E_{ela}(d) = \int_{\Omega} \left( \frac{k_1}{2} (\nabla \cdot d)^2 + \frac{k_2}{2} (d \cdot (\nabla \times d))^2 + \frac{k_3}{2} |d \times (\nabla \times d)|^2 \right)$$

$k_i > 0$ elastic constants.

**Remark:** Problem with convex functional but non-convex constraints.
**Smectic-A LC**

- **Smectic LC**: the molecules have an orientational order vector $d$ and a positional order (layer structure of normal vector $n$). Assume:
  - $\exists$ a potential function $\varphi$ s.t. layers = level sets of $\varphi$
    
    $$\nabla \times n = 0 \iff n = \nabla \varphi$$
  - Incompressibility of the layers: $|n| = 1$.

- **Smectic-A LC**: Assume $d = n$.

- Since $d = n$ and $n = \nabla \varphi$, the elastic energy can be rewritten (reduced form) as

  $$E_{ela} = \frac{1}{2} \int_{\Omega} |\nabla \cdot n|^2 = \frac{1}{2} \int_{\Omega} |\Delta \varphi|^2$$
Smectic-A LC

- **Smectic LC**: the molecules have an orientational order vector $d$ and a positional order (layer structure of normal vector $n$). Assume:
  - $\exists$ a potential function $\varphi$ s.t. layers $= \text{level sets of } \varphi$:
    \[
    \nabla \times n = 0 \iff n = \nabla \varphi
    \]
  - Incompressibility of the layers: $|n| = 1$.

- **Smectic-A LC**: Assume $d = n$.

  Since $d = n$ and $n = \nabla \varphi$, the elastic energy can be rewritten (reduced form) as

  \[
  E_{\text{ela}} = \frac{1}{2} \int_{\Omega} |\nabla \cdot n|^2 = \frac{1}{2} \int_{\Omega} |\Delta \varphi|^2
  \]
• **Smectic LC**: the molecules have an orientational order vector $d$ and a positional order (layer structure of normal vector $n$). Assume:
  
  • ∃ a potential function $\varphi$ s.t. layers = level sets of $\varphi$:
    \[ \nabla \times n = 0 \iff n = \nabla \varphi \]
  
  • Incompressibility of the layers: $|n| = 1$.

• **Smectic-A LC**: Assume $d = n$.

• Since $d = n$ and $n = \nabla \varphi$, the elastic energy can be rewritten (reduced form) as
  \[ E_{ela} = \frac{1}{2} \int_\Omega |\nabla \cdot n|^2 = \frac{1}{2} \int_\Omega |\Delta \varphi|^2 \]
Nematic (N)  Smectic A (SmA)  Smectic C (SmC)
Regularized Oseen-Frank energy (by penalization of Ginzburg-Landau type):

\[ E_e := \int_{\Omega} \left( \frac{1}{2} (\Delta \varphi)^2 + \frac{1}{\varepsilon^2} F(\nabla \varphi) \right), \quad F(n) = \frac{1}{4}(|n|^2 - 1)^2 \]

Static (Minimization) problem:

\[ \min_{\varphi} E_e \]

**Remark:** Problem without constraints but non-convex functional.

Optimality system:

\[ \left\langle \frac{\delta E_e(\varphi)}{\delta \varphi}, \varphi' \right\rangle = \int_{\Omega} \Delta \varphi \Delta \varphi' + \frac{1}{\varepsilon^2} f(\nabla \varphi) \cdot \nabla \varphi' = 0 \quad \forall \varphi \]

where \( f(n) = \nabla_n F(n) = (|n|^2 - 1)n \).
A static Smectic-A LC problem

- Regularized Oseen-Frank energy (by penalization of Ginzburg-Landau type):

\[
E_e := \int_\Omega \left( \frac{1}{2} (\Delta \varphi)^2 + \frac{1}{\varepsilon^2} F(\nabla \varphi) \right), \quad F(n) = \frac{1}{4} (|n|^2 - 1)^2
\]

Static (Minimization) problem:

\[
\min_{\varphi} E_e
\]

Remark: Problem without constraints but non-convex functional.

- Optimality system:

\[
\left\langle \frac{\delta E_e(\varphi)}{\delta \varphi}, \varphi \right\rangle = \int_\Omega \Delta \varphi \Delta \varphi + \frac{1}{\varepsilon^2} f(\nabla \varphi) \cdot \nabla \varphi = 0 \quad \forall \varphi
\]

where \( f(n) = \nabla_n F(n) = (|n|^2 - 1)n \).
\( L^2(\Omega) \)-Identification of the variational derivative

1. Assuming \( \Delta \varphi|_{\partial \Omega} = 0 \) (N1) or \( \nabla \varphi \cdot m|_{\partial \Omega} = 0 \) (D2) (\( m \) is the normal vector):

\[
\left\langle \frac{\delta E_e(\varphi)}{\delta \varphi}, \varphi \right\rangle = - \int_{\Omega} w \cdot \nabla \varphi, \quad w := \nabla \Delta \varphi - \frac{1}{\varepsilon^2} f(\nabla \varphi).
\]

2. Assuming \( w \cdot m|_{\partial \Omega} = 0 \) (N2) or \( \varphi|_{\partial \Omega} = 0 \) (D1):

\[
\frac{\delta E_e}{\delta \varphi} \equiv \nabla \cdot w = \Delta^2 \varphi - \frac{1}{\varepsilon^2} \nabla \cdot f(\nabla \varphi) \quad \text{in} \ \Omega.
\]
1. Assuming $\Delta \varphi |_{\partial \Omega} = 0$ (N1) or $\nabla \varphi \cdot m |_{\partial \Omega} = 0$ (D2) ($m$ is the normal vector):

$$
\left\langle \frac{\delta E_e(\varphi)}{\delta \varphi}, \varphi \right\rangle = - \int_{\Omega} w \cdot \nabla \varphi, \quad w := \nabla \Delta \varphi - \frac{1}{\varepsilon^2} f(\nabla \varphi).
$$

2. Assuming $w \cdot m |_{\partial \Omega} = 0$ (N2) or $\varphi |_{\partial \Omega} = 0$ (D1):

$$
\frac{\delta E_e}{\delta \varphi} \equiv \nabla \cdot w = \Delta^2 \varphi - \frac{1}{\varepsilon^2} \nabla \cdot f(\nabla \varphi) \quad \text{in} \quad \Omega.
$$
Admissible B.C. to fourth-order Euler-Lagrange $\nabla \cdot \mathbf{w} = 0$

\[
[D1 - D2] \quad \varphi|_{\partial \Omega} = \varphi_1, \quad \nabla \varphi \cdot \mathbf{m}|_{\partial \Omega} = \varphi_2,
\]

\[
[D1 - N1] \quad \varphi|_{\partial \Omega} = \varphi_1, \quad \Delta \varphi|_{\partial \Omega} = 0,
\]

\[
[D2 - N2] \quad \nabla \varphi \cdot \mathbf{m}|_{\partial \Omega} = \varphi_2, \quad \mathbf{w} \cdot \mathbf{m}|_{\partial \Omega} = 0,
\]

\[
[N1 - N2] \quad \Delta \varphi|_{\partial \Omega} = 0, \quad \mathbf{w} \cdot \mathbf{m}|_{\partial \Omega} = 0.
\]

Non admissible B.C.: [D1-N2] and [D2-N1]
Admissible B.C. to fourth-order Euler-Lagrange $\nabla \cdot \mathbf{w} = 0$

\[
[D1 - D2] \quad \varphi|_{\partial \Omega} = \varphi_1, \quad \nabla \varphi \cdot \mathbf{m}|_{\partial \Omega} = \varphi_2,
\]

\[
[D1 - N1] \quad \varphi|_{\partial \Omega} = \varphi_1, \quad \Delta \varphi|_{\partial \Omega} = 0,
\]

\[
[D2 - N2] \quad \nabla \varphi \cdot \mathbf{m}|_{\partial \Omega} = \varphi_2, \quad \mathbf{w} \cdot \mathbf{m}|_{\partial \Omega} = 0,
\]

\[
[N1 - N2] \quad \Delta \varphi|_{\partial \Omega} = 0, \quad \mathbf{w} \cdot \mathbf{m}|_{\partial \Omega} = 0.
\]

Non admissible B.C.: [D1-N2] and [D2-N1]
Admissible B.C. to fourth-order Euler-Lagrange $\nabla \cdot \mathbf{w} = 0$

$[D1 - D2] \quad \varphi|_{\partial \Omega} = \varphi_1, \quad \nabla \varphi \cdot \mathbf{m}|_{\partial \Omega} = \varphi_2,$

$[D1 - N1] \quad \varphi|_{\partial \Omega} = \varphi_1, \quad \Delta \varphi|_{\partial \Omega} = 0,$

$[D2 - N2] \quad \nabla \varphi \cdot \mathbf{m}|_{\partial \Omega} = \varphi_2, \quad \mathbf{w} \cdot \mathbf{m}|_{\partial \Omega} = 0,$

$[N1 - N2] \quad \Delta \varphi|_{\partial \Omega} = 0, \quad \mathbf{w} \cdot \mathbf{m}|_{\partial \Omega} = 0.$

Non admissible B.C.: [D1-N2] and [D2-N1]
Admissible B.C. to fourth-order Euler-Lagrange \( \nabla \cdot \mathbf{w} = 0 \)

\[
[D1 - D2] \quad \phi|_{\partial \Omega} = \phi_1, \quad \nabla \phi \cdot \mathbf{m}|_{\partial \Omega} = \phi_2,
\]

\[
[D1 - N1] \quad \phi|_{\partial \Omega} = \phi_1, \quad \Delta \phi|_{\partial \Omega} = 0,
\]

\[
[D2 - N2] \quad \nabla \phi \cdot \mathbf{m}|_{\partial \Omega} = \phi_2, \quad \mathbf{w} \cdot \mathbf{m}|_{\partial \Omega} = 0,
\]

\[
[N1 - N2] \quad \Delta \phi|_{\partial \Omega} = 0, \quad \mathbf{w} \cdot \mathbf{m}|_{\partial \Omega} = 0.
\]

Non admissible B.C.: [D1-N2] and [D2-N1]
Admissible B.C. to fourth-order Euler-Lagrange $\nabla \cdot \mathbf{w} = 0$

\[ D1 - D2 \quad \varphi|_{\partial \Omega} = \varphi_1, \quad \nabla \varphi \cdot \mathbf{m}|_{\partial \Omega} = \varphi_2, \]

\[ D1 - N1 \quad \varphi|_{\partial \Omega} = \varphi_1, \quad \Delta \varphi|_{\partial \Omega} = 0, \]

\[ D2 - N2 \quad \nabla \varphi \cdot \mathbf{m}|_{\partial \Omega} = \varphi_2, \quad \mathbf{w} \cdot \mathbf{m}|_{\partial \Omega} = 0, \]

\[ N1 - N2 \quad \Delta \varphi|_{\partial \Omega} = 0, \quad \mathbf{w} \cdot \mathbf{m}|_{\partial \Omega} = 0. \]

Non admissible B.C.: [D1-N2] and [D2-N1]
Smectic-A Liquid Crystal Model [E, ARMA’97]

- \( \Omega \subset \mathbb{R}^3, \ Q = \Omega \times (0, \ T) \) and \( \Sigma = \partial \Omega \times (0, \ T) \).

- **Linear momentum**, \((u, p)\)-system in \(Q\):

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nabla \cdot (\sigma^d + \lambda \sigma^e) = 0, \quad \nabla \cdot u = 0,
\]

- \( \sigma^d = \sigma^d(D(u), n) \) : dissipative (symmetric) stress tensor:

\[
\sigma^d = \mu_1(n^t Dn)n \otimes n + \mu_4 D + \mu_5(Dn \otimes n + n \otimes Dn), \quad D = \frac{\nabla u + \nabla u^t}{2}
\]

- \( \sigma^e = \sigma^e(\varphi) \) : elastic (non-symmetric) stress tensor, s.t. \( \nabla \cdot \sigma^e = \frac{\delta E}{\delta \varphi} \nabla \varphi + \nabla E_E \)

- **Angular momentum**, \(\varphi\)-equation in \(Q\) (of Allen-Cahn’s type):

\[
\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi + \gamma \frac{\delta E}{\delta \varphi} = 0,
\]

\(\gamma > 0\) constant (elastic relaxation time).
Smectic-A Liquid Crystal Model [E, ARMA’97]

- \( \Omega \subset \mathbb{R}^3, Q = \Omega \times (0, T) \) and \( \Sigma = \partial \Omega \times (0, T) \).

- **Linear momentum**, \((u, p)\)-system in \(Q\):

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nabla \cdot (\sigma^d + \lambda \sigma^e) = 0, \quad \nabla \cdot u = 0,
\]

\( \sigma^d = \sigma^d(D(u), n) \): dissipative (symmetric) stress tensor:

\[
\sigma^d = \mu_1(n^t Dn)n \otimes n + \mu_4 D + \mu_5(Dn \otimes n + n \otimes Dn), \quad D = \frac{\nabla u + \nabla u^t}{2}
\]

\( \sigma^e = \sigma^e(\varphi) \): elastic (non-symmetric) stress tensor, s.t.

\[
\nabla \cdot \sigma^e = \frac{\delta E_e}{\delta \varphi} \nabla \varphi + \nabla E_e
\]

- **Angular momentum**, \(\varphi\)-equation in \(Q\) (of Allen-Cahn’s type):

\[
\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi + \gamma \frac{\delta E_e}{\delta \varphi} = 0,
\]

\(\gamma > 0\) constant (elastic relaxation time).
\( \Omega \subset \mathbb{R}^3, \ Q = \Omega \times (0, T) \) and \( \Sigma = \partial \Omega \times (0, T) \).

**Linear momentum**, \((u, p)\)-system in \(Q\):

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nabla \cdot (\sigma^d + \lambda \sigma^e) = 0, \quad \nabla \cdot u = 0,
\]

\( \sigma^d = \sigma^d(D(u), n) \) : dissipative (symmetric) stress tensor:

\[
\sigma^d = \mu_1(n^t Dn)n \otimes n + \mu_4 D + \mu_5(Dn \otimes n + n \otimes Dn), \quad D = \frac{\nabla u + \nabla u^t}{2}
\]

\( \sigma^e = \sigma^e(\varphi) \) : elastic (non-symmetric) stress tensor, s.t. \( \nabla \cdot \sigma^e = \frac{\delta E_e}{\delta \varphi} \nabla \varphi + \nabla E_e \)

**Angular momentum**, \( \varphi \)-equation in \(Q\) (of Allen-Cahn’s type):

\[
\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi + \gamma \frac{\delta E_e}{\delta \varphi} = 0,
\]

\( \gamma > 0 \) constant (elastic relaxation time).
Initial-boundary problem (with Dirichlet [D1-D2] B.C. for $\varphi$) in $Q$:

\[
\begin{align*}
\frac{Du}{Dt} - \nabla \cdot \sigma^d + \nabla \tilde{p} - \lambda \frac{\delta E_e}{\delta \varphi} \nabla \varphi &= 0, & u \\
\nabla \cdot u &= 0, & p \\
\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi + \gamma \frac{\delta E_e}{\delta \varphi} &= 0, & \frac{\delta E_e}{\delta \varphi} \\
\Delta^2 \varphi - \frac{1}{\varepsilon^2} \nabla \cdot f(\nabla \varphi) &= \frac{\delta E_e}{\delta \varphi}, & \frac{\partial \varphi}{\partial t} \\
u|_{\Sigma} &= 0, \quad \varphi|_{\Sigma} = \varphi_1, \quad \partial_n \varphi|_{\Sigma} = \varphi_2, \\
u|_{t=0} &= u_0, \quad \varphi|_{t=0} = \varphi_0
\end{align*}
\]

$\tilde{p} = p + \lambda E_e$ (Lagrange multiplier $\sim$ constraint $\nabla \cdot u = 0$)

Other B.C. are possible. For instance, $w \cdot m|_{\partial \Omega} = 0$ implies the conservation property $\frac{d}{dt} \int_{\Omega} \varphi = 0$. 
Dissipative Energy law and weak solutions

Testing by $u, p, \frac{\delta E_e}{\delta \varphi}, \frac{\partial \varphi}{\partial t}$:

\[
\frac{d}{dt} \left( E_{\text{kin}}(u) + \lambda E_e(\varphi) \right) + \int_\Omega \sigma^d : D + \lambda \gamma \int_\Omega \left| \frac{\delta E_e}{\delta \varphi} \right|^2 = 0
\]

where

\[
E_{\text{kin}}(u) := \frac{1}{2} \int_\Omega |u|^2, \quad E_e := \int_\Omega \left( \frac{1}{2} (\Delta \varphi)^2 + \frac{1}{\varepsilon^2} F(\nabla \varphi) \right),
\]

\[
\sigma^d : D = \mu_1 (n^t Dn)^2 + \mu_4 |D|^2 + \mu_5 |Dn|^2 \geq \mu_4 |D|^2.
\]

Weak solutions (variational $\sim (u, p)$, point-wise $\sim \varphi$):

\[
u \in L^\infty(0, +\infty; L^2(\Omega)) \cap L^2(0, +\infty; H^1(\Omega)), \quad \frac{\delta E_e}{\delta \varphi} \in L^2(0, +\infty; L^2(\Omega)).
\]

\[
\varphi \in L^\infty(0, +\infty; H^2(\Omega)) \cap L^2_{\text{loc}}(0, +\infty; H^4(\Omega)), \quad \frac{\partial \varphi}{\partial t} \in L^2(0, +\infty; L^2(\Omega)).
\]
Dissipative Energy law and weak solutions

Testing by $u, p, \frac{\delta E_e}{\delta \varphi}, \frac{\partial \varphi}{\partial t}$:

\[(EL) \quad \frac{d}{dt}\left( E_{\text{kin}}(u) + \lambda E_e(\varphi) \right) + \int_{\Omega} \sigma^d : D + \lambda \gamma \int_{\Omega} \left| \frac{\delta E_e}{\delta \varphi} \right|^2 = 0 \]

where

\[E_{\text{kin}}(u) := \frac{1}{2} \int_{\Omega} |u|^2, \quad E_e := \int_{\Omega} \left( \frac{1}{2} (\Delta \varphi)^2 + \frac{1}{\varepsilon^2} F(\nabla \varphi) \right),\]

\[\sigma^d : D = \mu_1 (n^t Dn)^2 + \mu_4 |D|^2 + \mu_5 |Dn|^2 \geq \mu_4 |D|^2.\]

Weak solutions (variational $\sim (u, p)$, point-wise $\sim \varphi$):

\[u \in L^\infty(0, +\infty; L^2(\Omega)) \cap L^2(0, +\infty; H^1(\Omega)), \quad \frac{\delta E_e}{\delta \varphi} \in L^2(0, +\infty; L^2(\Omega)).\]

\[\varphi \in L^\infty(0, +\infty; H^2(\Omega)) \cap L^2_{\text{loc}}(0, +\infty; H^4(\Omega)), \quad \frac{\partial \varphi}{\partial t} \in L^2(0, +\infty; L^2(\Omega)).\]
Local strong estimates and strong solutions

- Combining adequately $u$-system by $Au + \partial_t u$ and $\partial_t (\varphi\text{-eq})$ by $\partial_t \varphi$:

  $$\Phi' + \Psi \leq C(\Phi^3 + 1),$$

  $$\Phi(t) := \|u\|^2_{H^1} + \|\partial_t \varphi\|^2_{L^2}, \quad \Psi(t) := \|u\|^2_{H^2} + \|\partial_t u\|^2_{L^2} + \|\partial_t \varphi\|^2_{H^2}$$

- **Strong (or regular) solutions** \((\text{full point-wise})\):

  $$u \in L^\infty(0, +\infty; H^1(\Omega)) \cap L^2_{\text{loc}}(0, +\infty; H^2(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2_{\text{loc}}(0, +\infty; L^2(\Omega)),$$

  $$\frac{\partial \varphi}{\partial t} \in L^\infty(0, +\infty; L^2(\Omega)) \cap L^2_{\text{loc}}(0, +\infty; H^2(\Omega)).$$

  $$\varphi \in L^\infty(0, +\infty; H^4) \cap L^2_{\text{loc}}(0, +\infty; H^6),$$
Local strong estimates and strong solutions

Combining adequately \( u \)-system by \( Au + \partial_t u \) and \( \partial_t (\varphi \text{-eq}) \) by \( \partial_t \varphi \):

\[
\Phi' + \Psi \leq C(\Phi^3 + 1),
\]

\[
\Phi(t) := \| u \|^2_{H^1} + \| \partial_t \varphi \|^2_{L^2}, \quad \Psi(t) := \| u \|^2_{H^2} + \| \partial_t u \|^2_{L^2} + \| \partial_t \varphi \|^2_{H^2}
\]

Strong (or regular) solutions (full point-wise):

\[
u \in L^\infty(0, +\infty; H^1(\Omega)) \cap L^2_{loc}(0, +\infty; H^2(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2_{loc}(0, +\infty; L^2(\Omega)),
\]

\[
\frac{\partial \varphi}{\partial t} \in L^\infty(0, +\infty; L^2(\Omega)) \cap L^2_{loc}(0, +\infty; H^2(\Omega)).
\]

\[
\varphi \in L^\infty(0, +\infty; H^4) \cap L^2_{loc}(0, +\infty; H^6),
\]
Previous results

- [Liu, DCDS’00] Time-independent [D1-D2] B.C.:
  Existence of global in time weak solutions.
  Existence (and uniqueness) of global in time regular solutions for large viscosity \(\mu_4 \gg\), and its \(\omega\)-limit is a nonempty set furnished by “equilibrium points”. Global minimizers of the elastic energy \(E_e\) are stable.

- [Climent-Ezquerra & GG, CPAA’10] Extension for time-dependent [D1-D2] B.C. Moreover, Existence of time-periodic weak solutions and regularity for \(\mu_4 \gg\).

- [Segatti & Wu, SIMA’11] Long-time behaviour with periodic B.C.:
  Finite dimensional attractor in 2D domains.
  Convergence to a single equilibrium (and convergence rate)
  In particular, \(\varphi(t) \rightarrow \varphi_{\infty}\) as \(t \rightarrow \infty\), with \(\frac{\delta E_e(\varphi_{\infty})}{\delta \varphi} = 0\).
  Moreover, local minimizers of the elastic energy \(E_e\) are stable.

RK: It may exists a “continuum” of equilibrium solutions with the same energy.
Previous results

- **[Liu, DCDS’00]** Time-independent [D1-D2] B.C.:
  Existence of global in time weak solutions.
  Existence (and uniqueness) of global in time regular solutions for large viscosity ($\mu_4 >>$), and its $\omega$-limit is a nonempty set furnished by “equilibrium points”. Global minimizers of the elastic energy $E_e$ are stable.

- **[Climent-Ezquerra & GG, CPAA’10]** Extension for time-dependent [D1-D2] B.C.:
  Moreover, Existence of time-periodic weak solutions and regularity for $\mu_4 >>$.

- **[Segatti & Wu, SIMA’11]** Long-time behaviour with periodic B.C.:
  Finite dimensional attractor in 2D domains.
  Convergence to a single equilibrium (and convergence rate)
  In particular, $\varphi(t) \to \varphi_\infty$ as $t \to \infty$, with $\frac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0$.
  Moreover, local minimizers of the elastic energy $E_e$ are stable.

**RK**: It may exists a “continuum” of equilibrium solutions with the same energy.
Previous results

- **[Liu, DCDS’00]** Time-independent [D1-D2] B.C.:
  Existence of global in time weak solutions.
  Existence (and uniqueness) of global in time regular solutions for large viscosity \((\mu_4 \gg)\), and its \(\omega\)-limit is a nonempty set furnished by “equilibrium points”. Global minimizers of the elastic energy \(E_e\) are stable.

- **[Climent-Ezquerra & GG, CPAA’10]** Extension for time-dependent [D1-D2] B.C.
  Moreover, Existence of time-periodic weak solutions and regularity for \(\mu_4 \gg\).

- **[Segatti & Wu, SIMA’11]** Long-time behaviour with periodic B.C.:
  Finite dimensional attractor in 2D domains.
  Convergence to a single equilibrium (and convergence rate)
  In particular, \(\varphi(t) \to \varphi_{\infty}\) as \(t \to \infty\), with \(\frac{\delta E_e(\varphi_{\infty})}{\delta \varphi} = 0\).
  Moreover, local minimizers of the elastic energy \(E_e\) are stable.

RK: It may exists a “continuum” of equilibrium solutions with the same energy.
Previous results

- **[Liu, DCDS’00]** Time-independent [D1-D2] B.C.:
  Existence of global in time weak solutions.
  Existence (and uniqueness) of global in time regular solutions for large viscosity ($\mu_4 \gg$), and its $\omega$-limit is a nonempty set furnished by “equilibrium points”. Global minimizers of the elastic energy $E_e$ are stable.

- **[Climent-Ezquerra & GG, CPAA’10]** Extension for time-dependent [D1-D2] B.C. Moreover, Existence of time-periodic weak solutions and regularity for $\mu_4 \gg$.

- **[Segatti & Wu, SIMA’11]** Long-time behaviour with periodic B.C.:
  Finite dimensional attractor in 2D domains.
  Convergence to a single equilibrium (and convergence rate)
  In particular, $\varphi(t) \to \varphi_\infty$ as $t \to \infty$, with $\frac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0$.
  Moreover, local minimizers of the elastic energy $E_e$ are stable.

**RK:** It may exists a “continuum" of equilibrium solutions with the same energy.
Main result

Let $S$ be the set of equilibrium points:

$$S = \{(0, \varphi) : \varphi \in H^4(\Omega), \Delta^2 \varphi - \nabla \cdot \frac{1}{\varepsilon^2} f(\varphi) = 0, \varphi|_{\partial \Omega} = \varphi_1, \partial_n \varphi|_{\partial \Omega} = \varphi_2\}.$$

and the $\omega$-limit set of a global weak solution:

$$\omega(u, \varphi) = \{(u_\infty, \varphi_\infty) \in V \times H^4 : \exists \{t_n\} \uparrow \infty \text{ s.t. } (u(t_n), \varphi(t_n)) \to (u_\infty, \varphi_\infty) \text{ in } H^1 \times H^4\}.$$

**Theorem**

- $\omega(u, \varphi) \subset S$
- $\omega(u, \varphi) = \{(0, \varphi_\infty)\}$
Goal: Long-time behavior with admissible B.C.

**Generic situation:** Let \( E(t), D(t) \geq 0 \) satisfying the “weak estimates”:

\[(WE) \quad E'(t) + D(t) \leq 0, \quad \text{a.e. } t \in (0, +\infty).\]

Then, \( E \in L^\infty(0, +\infty) \) and \( E(t) \searrow E_\infty \geq 0 \). Moreover, integrating (WE), one has \( D \in L^1(0, +\infty) \).

**Lemma ([Climent-Ezquerra, Rodriguez-Bellido & GG, JBC’10])**

Let \( D(t) \geq 0, D \in L^1(0, +\infty) \) satisfying the “strong estimate”:

\[(SE) \quad D'(t) \leq K(D(t)^3 + 1) \quad (K > 0).\]

Then, there exists \( T^* \geq 0 \) (large enough) such that \( D \in L^\infty(T^*, +\infty) \) and

\[\exists \lim_{t \to +\infty} D(t) = 0.\]
Goal: Long-time behavior with admissible B.C.

Generic situation: Let $E(t), D(t) \geq 0$ satisfying the "weak estimates":

\[(WE) \quad E'(t) + D(t) \leq 0, \quad \text{a.e. } t \in (0, +\infty).\]

Then, $E \in L^\infty(0, +\infty)$ and $E(t) \searrow E_\infty \geq 0$. Moreover, integrating (WE), one has $D \in L^1(0, +\infty)$.

Lemma ([Climent-Ezquerra, Rodriguez-Bellido & GG, JBC’10])

Let $D(t) \geq 0, D \in L^1(0, +\infty)$ satisfying the “strong estimate”:

\[(SE) \quad D'(t) \leq K(D(t)^3 + 1) \quad (K > 0).\]

Then, there exists $T^* \geq 0$ (large enough) such that $D \in L^\infty(T^*, +\infty)$ and

\[\exists \lim_{t \to +\infty} D(t) = 0.\]
Extension for sequences

**Goal:** Convergence uniformly with respect to the Galerkin sequence’s index.

**Theorem ([Climent-Ezquerra & GG, NA-TMA’14])**

Let $E_m(t)$, $D_m(t) \geq 0$, satisfying

\[(WE) \quad (E^m)'(t) + D^m(t) \leq 0,\]

\[(SE) \quad (D^m)'(t) \leq K(D^m(t)^3 + 1) \quad K > 0 \text{ independent of } m.\]

Then, $\forall \varepsilon < 1, \exists T^* = T^*(\varepsilon) \geq 0 \text{ (large enough), independent of } m$, such that

$$\|D^m\|_{L^\infty(T^*, +\infty)} \leq \varepsilon.$$
Main steps to prove convergence to a single steady equilibrium

1. To obtain (WE) and (SE) for $E(t) = E_{\text{kin}}(u(t)) + \lambda E_e(\varphi(t))$ and

$$D(t) := \|u(t)\|_{H^1}^2 + \left\| \frac{\delta E_e}{\delta \varphi} \right\|_{L^2}^2 \sim \|u\|_{H^1}^2 + \|\partial_t \varphi\|_{L^2}^2$$

2. Existence of weak solutions $(u, \varphi)$ in $(0, +\infty)$ which are strong solutions in $(T^{*}_{\text{reg}}, +\infty)$ for some $T^{*}_{\text{reg}} > 0$.

3. $E(u(t), \varphi(t)) \downarrow E_{\infty}$ in $\mathbb{R}$, $u(t) \to 0$ in $H^1_0$ and $\frac{\delta E_e(\varphi(t))}{\delta \varphi} \to 0$ in $L^2$ (as $t \uparrow +\infty$).

4. Recall the $\omega$-limit set:

$$\omega = \{(u_{\infty}, \varphi_{\infty}) \in H^1 \times H^4 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (u(t_n), \varphi(t_n)) \to (u_{\infty}, \varphi_{\infty}) \text{ in } H^1 \times H^4\}$$

Then, $u_{\infty} = 0$ and $E_e(\varphi_{\infty}) = E_{\infty}$. Moreover:

- $\omega \neq \emptyset$.
- If $(0, \varphi_{\infty}) \in \omega$ then $\frac{\delta E_e(\varphi_{\infty})}{\delta \varphi} = 0$ in $\Omega$, + B.C. for $\varphi_{\infty}$ (i.e. $\varphi_{\infty}$ is an “equilibrium point”).

5. (Application of Lojasiewicz-Simon). $\varphi(t) \to \varphi_{\infty}$ in $H^4$ as $t \uparrow +\infty$. 

F. Guillén-González (AIMS, 2014)
Main steps to prove convergence to a single steady equilibrium

1. To obtain (WE) and (SE) for \( E(t) = E_{\text{kin}}(u(t)) + \lambda E_e(\varphi(t)) \) and

\[
D(t) := \|u(t)\|^2_{H^1} + \left\| \frac{\delta E_e}{\delta \varphi} \right\|^2_{L^2} \sim \|u\|^2_{H^1} + \|\partial_t \varphi\|^2_{L^2}
\]

2. Existence of weak solutions \((u, \varphi)\) in \((0, +\infty)\) which are strong solutions in \((T^*_\text{reg}, +\infty)\) for some \(T^*_\text{reg} > 0\).

3. \(E(u(t), \varphi(t)) \downarrow E_\infty\) in \(\mathbb{R}\), \(u(t) \rightarrow 0\) in \(H^1_0\) and \(\frac{\delta E_e(\varphi(t))}{\delta \varphi} \rightarrow 0\) in \(L^2\) (as \(t \uparrow +\infty\)).

4. Recall the \(\omega\)-limit set:

\[
\omega = \{(u_\infty, \varphi_\infty) \in H^1 \times H^4 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (u(t_n), \varphi(t_n)) \rightarrow (u_\infty, \varphi_\infty) \text{ in } H^1 \times H^4\}
\]

Then, \(u_\infty = 0\) and \(E_e(\varphi_\infty) = E_\infty\). Moreover:

- \(\omega \neq \emptyset\).
- If \((0, \varphi_\infty) \in \omega\) then \(\frac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0\) in \(\Omega\), + B.C. for \(\varphi_\infty\) (i.e. \(\varphi_\infty\) is an “equilibrium point”).

5. (Application of Lojasiewicz-Simon). \(\varphi(t) \rightarrow \varphi_\infty\) in \(H^4\) as \(t \uparrow +\infty\).
Main steps to prove convergence to a single steady equilibrium

1. To obtain (WE) and (SE) for $E(t) = E_{\text{kin}}(u(t)) + \lambda E_e(\varphi(t))$ and
   $$D(t) := \|u(t)\|_{H^1}^2 + \left\|\frac{\delta E_e}{\delta \varphi}\right\|_{L^2}^2 \sim \|u\|_{H^1}^2 + \|\partial_t \varphi\|_{L^2}^2$$

2. Existence of weak solutions $(u, \varphi)$ in $(0, +\infty)$ which are strong solutions in $(T_{\text{reg}}^*, +\infty)$ for some $T_{\text{reg}}^* > 0$.

3. $E(u(t), \varphi(t)) \searrow E_\infty$ in $\mathbb{R}$, $u(t) \to 0$ in $H^1_0$ and $\frac{\delta E_e(\varphi(t))}{\delta \varphi} \to 0$ in $L^2$ (as $t \uparrow +\infty$).

4. Recall the $\omega$-limit set:
   $$\omega = \{(u_\infty, \varphi_\infty) \in H^1 \times H^4 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (u(t_n), \varphi(t_n)) \to (u_\infty, \varphi_\infty) \text{ in } H^1 \times H^4\}$$
   Then, $u_\infty = 0$ and $E_e(\varphi_\infty) = E_\infty$. Moreover:
   - $\omega \neq \emptyset$.
   - If $(0, \varphi_\infty) \in \omega$ then $\frac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0$ in $\Omega$, + B.C. for $\varphi_\infty$ (i.e. $\varphi_\infty$ is an “equilibrium point”).

5. (Application of Lojasiewicz-Simon). $\varphi(t) \to \varphi_\infty$ in $H^4$ as $t \uparrow +\infty$. 
Main steps to prove convergence to a single steady equilibrium

1. To obtain (WE) and (SE) for \( E(t) = E_{\text{kin}}(u(t)) + \lambda E_e(\varphi(t)) \) and
\[
D(t) := \|u(t)\|_{H^1}^2 + \left\| \frac{\delta E_e}{\delta \varphi} \right\|_{L^2}^2 \sim \|u\|_{H^1}^2 + \|\partial_t \varphi\|_{L^2}^2
\]

2. Existence of weak solutions \((u, \varphi)\) in \((0, +\infty)\) which are strong solutions in \((T_{reg}^*, +\infty)\) for some \(T_{reg}^* > 0\).

3. \(E(u(t), \varphi(t)) \downarrow E_\infty\) in \(\mathbb{R}\), \(u(t) \to 0\) in \(H^1_0\) and \(\frac{\delta E_e(\varphi(t))}{\delta \varphi} \to 0\) in \(L^2\) (as \(t \uparrow +\infty\)).

4. Recall the \(\omega\)-limit set:
\[
\omega = \{(u_\infty, \varphi_\infty) \in H^1 \times H^4 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (u(t_n), \varphi(t_n)) \to (u_\infty, \varphi_\infty) \text{ in } H^1 \times H^4\}
\]
Then, \(u_\infty = 0\) and \(E_e(\varphi_\infty) = E_\infty\). Moreover:
- \(\omega \neq \emptyset\).
- If \((0, \varphi_\infty) \in \omega\) then \(\frac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0\) in \(\Omega\), + B.C. for \(\varphi_\infty\) (i.e. \(\varphi_\infty\) is an “equilibrium point”).

5. (Application of Lojasiewicz-Simon). \(\varphi(t) \to \varphi_\infty\) in \(H^4\) as \(t \uparrow +\infty\).
Main steps to prove convergence to a single steady equilibrium

1. To obtain (WE) and (SE) for \( E(t) = E_{\text{kin}}(u(t)) + \lambda E_e(\varphi(t)) \) and
   \[
   D(t) := \|u(t)\|_{H^1}^2 + \left\| \frac{\delta E_e}{\delta \varphi} \right\|_{L^2}^2 \sim \|u\|_{H^1}^2 + \|\partial_t \varphi\|_{L^2}^2
   \]

2. Existence of weak solutions \((u, \varphi)\) in \((0, +\infty)\) which are strong solutions in \((T_{\text{reg}}^*, +\infty)\) for some \(T_{\text{reg}}^* > 0\).

3. \(E(u(t), \varphi(t)) \downarrow E_\infty\) in \(\mathbb{R}\), \(u(t) \to 0\) in \(H_0^1\) and \(\frac{\delta E_e(\varphi(t))}{\delta \varphi} \to 0\) in \(L^2\) (as \(t \uparrow +\infty\)).

4. Recall the \(\omega\)-limit set:
   \[
   \omega = \{(u_\infty, \varphi_\infty) \in H^1 \times H^4 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (u(t_n), \varphi(t_n)) \to (u_\infty, \varphi_\infty) \text{ in } H^1 \times H^4\}
   \]
   Then, \(u_\infty = 0\) and \(E_e(\varphi_\infty) = E_\infty\). Moreover:
   - \(\omega \neq \emptyset\).
   - If \((0, \varphi_\infty) \in \omega\) then \(\frac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0\) in \(\Omega\), + B.C. for \(\varphi_\infty\) (i.e. \(\varphi_\infty\) is an “equilibrium point”).

5. (Application of Lojasiewicz-Simon). \(\varphi(t) \to \varphi_\infty\) in \(H^4\) as \(t \uparrow +\infty\).
Main steps to prove convergence to a single steady equilibrium

1. To obtain (WE) and (SE) for

\[ E(t) = E_{\text{kin}}(u(t)) + \lambda E_e(\varphi(t)) \]

and

\[ D(t) := \|u(t)\|^2_{H^1} + \left\| \frac{\delta E_e}{\delta \varphi} \right\|^2_{L^2} \sim \|u\|^2_{H^1} + \|\partial_t \varphi\|^2_{L^2} \]

2. Existence of weak solutions \((u, \varphi)\) in \((0, +\infty)\) which are strong solutions in \((T_{\text{reg}}^*, +\infty)\) for some \(T_{\text{reg}}^* > 0\).

3. \(E(u(t), \varphi(t)) \downarrow E_\infty \text{ in } \mathbb{R}, u(t) \to 0 \text{ in } H^1_0\) and \(\frac{\delta E_e(\varphi(t))}{\delta \varphi} \to 0 \text{ in } L^2 \text{ (as } t \uparrow +\infty).\)

4. Recall the \(\omega\)-limit set:

\[
\omega = \{(u_\infty, \varphi_\infty) \in H^1 \times H^4 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (u(t_n), \varphi(t_n)) \to (u_\infty, \varphi_\infty) \text{ in } H^1 \times H^4\}
\]

Then, \(u_\infty = 0\) and \(E_e(\varphi_\infty) = E_\infty\). Moreover:

- \(\omega \neq \emptyset\).
- If \((0, \varphi_\infty) \in \omega\) then \(\frac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0 \text{ in } \Omega, + \text{ B.C. for } \varphi_\infty \) (i.e. \(\varphi_\infty\) is an “equilibrium point”).

5. (Application of Lojasiewicz-Simon). \(\varphi(t) \to \varphi_\infty \text{ in } H^4 \text{ as } t \uparrow +\infty\).
Theorem (“Strong” Lojasiewicz-Simon inequality for smectic-A problems)

If \( \varphi \in S \), then there exists \( C > 0, \beta > 0 \) and \( \theta \in (0, 1/2) \) depending on \( \varphi \) such that for all \( \varphi \in H^4 \) with \( \varphi|_{\partial \Omega} = \varphi_1, \partial_n \varphi|_{\partial \Omega} = \varphi_2 \) and \( \| \varphi - \varphi \|_{H^4} \leq \beta \), it holds

\[
|E_e(\varphi) - E_e(\varphi)|^{1-\theta} \leq C \left\| \frac{\delta E_e(\varphi)}{\delta \varphi} \right\|_{L^2}
\]


Corollary ([Segatti,Wu, SIMA’11])

It’s possible to relax the local hypothesis \( \| \varphi - \varphi \|_{H^4} \leq \beta \) by \( \| \varphi - \varphi \|_{H^3} \leq \tilde{\beta} \)
Numerical Approximation

**Scheme:** Finite Difference in time and Mixed Finite Element in space.

For simplicity,

- Uniform partition in time $t_n = n k$ of $[0, T]$ with $k = T/N$ (time step).
- inf-sup stable pair of FE for velocity-pressure.

**Difficulties:**

- Approximate $\varphi \in H^4(\Omega)$ by only continuous Finite Elements.
- Strongly nonlinear coupling between fluids and phase-field models.
- Decreasing discrete energy law $\Rightarrow$ energy-stability.
**Numerical Approximation**

**Scheme:** Finite Difference in time and Mixed Finite Element in space.

For simplicity,

- Uniform partition in time $t_n = n k$ of $[0, T]$ with $k = T / N$ (time step).
- inf-sup stable pair of FE for velocity-pressure.

**Difficulties:**

- Approximate $\varphi \in H^4(\Omega)$ by only continuous Finite Elements.
- Strongly nonlinear coupling between fluids and phase-field models.
- Decreasing discrete energy law $\Rightarrow$ energy-stability.
Mixed second-order reformulation of (P)

By using the auxiliary variable \( \psi = -\Delta \varphi \):

\[
\begin{align*}
\left( \frac{D\mathbf{u}}{Dt}, \mathbf{u} \right) + \left( \sigma^d, D(\mathbf{u}) \right) - (p, \nabla \cdot \mathbf{u}) + \frac{\lambda}{\gamma} \left( \frac{\partial \varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi \right) \nabla \varphi, \mathbf{u} \right) &= 0, \\
\left( \nabla \cdot \mathbf{u}, p \right) &= 0, \\
\frac{\lambda}{\gamma} \left( \frac{\partial \varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi, \varphi \right) + \lambda \left( \nabla \psi, \nabla \varphi \right) + \frac{\lambda}{\varepsilon^2} \left( f(\nabla \varphi), \nabla \varphi \right) &= 0, \\
\left( \partial_t \psi, \bar{\psi} \right) - \left( \nabla \partial_t \varphi, \nabla \bar{\psi} \right) &= 0.
\end{align*}
\]

Energy law: Choosing \((\bar{\mathbf{u}}, \bar{p}, \bar{\varphi}, \bar{\psi}) = (\mathbf{u}, p, \partial_t \varphi, \lambda \psi)\)

\[
\frac{d}{dt} E_{\text{tot}}(\mathbf{u}, \psi, \nabla \varphi) + \left( \sigma^d, D(\mathbf{u}) \right) + \frac{\lambda}{\gamma} \left\| \frac{\partial \varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi \right\|_{L^2}^2 = 0,
\]

where \( E_{\text{tot}}(\mathbf{u}, \psi, \nabla \varphi) := E_{\text{kin}}(\mathbf{u}) + \lambda \left( E_{\text{ela}}(\psi) + E_{\text{pen}}(\nabla \varphi) \right), \)

\[
E_{\text{kin}}(\mathbf{u}) = \frac{1}{2} \left\| \mathbf{u} \right\|_{L^2}^2, \quad E_{\text{ela}}(\psi) = \frac{1}{2} \left\| \psi \right\|_{L^2}^2, \quad E_{\text{pen}}(\nabla \varphi) = \frac{1}{\varepsilon^2} \int_{\Omega} F(\nabla \varphi).
\]
Mixed second-order reformulation of (P)

By using the auxiliary variable $\psi = -\Delta \varphi$:

$$
\begin{cases}
\left( \frac{Du}{Dt}, u \right) + \left( \sigma^d, D(u) \right) - \left( p, \nabla \cdot u \right) + \frac{\lambda}{\gamma} \left( \frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi \right) \nabla \varphi, \bar{u} \right) = 0, \\
\left( \nabla \cdot u, \bar{p} \right) = 0, \\
\frac{\lambda}{\gamma} \left( \frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi, \overline{\varphi} \right) + \lambda \left( \nabla \psi, \nabla \overline{\varphi} \right) + \frac{\lambda}{\varepsilon^2} \left( f(\nabla \varphi), \nabla \overline{\varphi} \right) = 0 \\
\left( \partial_t \psi, \overline{\psi} \right) - \left( \nabla \partial_t \varphi, \nabla \overline{\psi} \right) = 0.
\end{cases}
$$

Energy law: Choosing $(\bar{u}, \bar{p}, \overline{\varphi}, \overline{\psi}) = (u, p, \partial_t \varphi, \lambda \psi)$

$$
\frac{d}{dt} E_{tot}(u, \psi, \nabla \varphi) + \left( \sigma^d, D(u) \right) + \frac{\lambda}{\gamma} \| \frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi \|_{L^2}^2 = 0,
$$

where $E_{tot}(u, \psi, \nabla \varphi) := E_{kin}(u) + \lambda \left( E_{ela}(\psi) + E_{pen}(\nabla \varphi) \right)$,

$$
E_{kin}(u) = \frac{1}{2} \| u \|_{L^2}^2, \quad E_{ela}(\psi) = \frac{1}{2} \| \psi \|_{L^2}^2, \quad E_{pen}(\nabla \varphi) = \frac{1}{\varepsilon^2} \int_{\Omega} F(\nabla \varphi).
$$
Generic second order scheme

Given \((u^n, \varphi^n, \psi^n)\), compute \((u^{n+1}, p^{n+\frac{1}{2}}, \varphi^{n+1}, \psi^{n+1}) \in \mathbf{U}_h \times P_h \times \Phi_h \times \Psi_h\) such that for any \((\bar{u}, \bar{p}, \bar{\varphi}, \bar{\psi}) \in \mathbf{U}_h \times P_h \times \Phi_h \times \Psi_h:\)

\[
\begin{align*}
(\delta_t u^{n+1}, \bar{u}) + c(\bar{u}, u^{n+\frac{1}{2}}, \bar{u}) + \left( \sigma^d(D(u^{n+\frac{1}{2}}), \nabla \bar{\varphi}), D(\bar{u}) \right) \\
- \left( p^{n+\frac{1}{2}}, \nabla \cdot \bar{u} \right) + \frac{\lambda}{\gamma} \left( (\delta_t \varphi^{n+1} + u^{n+\frac{1}{2}} \cdot \nabla \bar{\varphi}) \nabla \bar{\varphi}, \bar{u} \right) &= 0, \\
(\nabla \cdot u^{n+\frac{1}{2}}, \bar{p}) &= 0, \\
\frac{\lambda}{\gamma} \left( \delta_t \varphi^{n+1} + u^{n+\frac{1}{2}} \cdot \nabla \bar{\varphi}, \bar{\varphi} \right) + \lambda \left( \nabla \psi^{n+\frac{1}{2}}, \nabla \bar{\varphi} \right) + \frac{\lambda}{\varepsilon^2} \left( f^k(\nabla \varphi^{n+1}, \nabla \varphi^n), \nabla \bar{\varphi} \right) &= 0, \\
(\delta_t \psi^{n+1}, \bar{\psi}) - (\delta_t \nabla \varphi^{n+1}, \nabla \bar{\psi}) &= 0
\end{align*}
\]

where \(u^{n+\frac{1}{2}} = (u^{n+1} + u^n)/2\) etc.
Lemma

The following discrete energy inequality holds:

\[
\delta_t E_{\text{tot}}(u^{n+1}, \nabla \varphi^{n+1}, w^{n+1}) + \mu_4 \|D(u^{n+\frac{1}{2}})\|_{L^2(\Omega)}^2 \\
+ \frac{\lambda}{\gamma} \|\delta_t \varphi^{n+1} + u^{n+\frac{1}{2}} \cdot \nabla \tilde{\varphi}\|_{L^2(\Omega)}^2 + \frac{\lambda}{\varepsilon^2} ND \leq 0,
\]

where

\[
ND = \int_{\Omega} f^k(\nabla \varphi^{n+1}, \nabla \varphi^n) \cdot \delta_t \nabla \varphi^{n+1} - \delta_t \left( \int_{\Omega} F(\nabla \varphi^{n+1}) \right).
\]
Conclusions and open problems

Conclusions:

1. Regularity and uniqueness for large times
2. Convergence to single equilibrium steady solutions
3. Unconditional fully discrete energy-stable schemes

Open problems:

1. Long-time behaviour for other B.C.
2. Splitting energy-stable schemes
3. Numerical simulations
Conclusions and open problems

Conclusions:

1. Regularity and uniqueness for large times
2. Convergence to single equilibrium steady solutions
3. Unconditional fully discrete energy-stable schemes

Open problems:

1. Long-time behaviour for other B.C.
2. Splitting energy-stable schemes
3. Numerical simulations
THANK YOU FOR YOUR ATTENTION