The Riemann-Hilbert problem for matrix-valued orthogonal polynomials\textsuperscript{1}

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\textsuperscript{1}joint work with Andrei Martínez Finkelshtein
Outline

1. Preliminaries

2. The RH problem for OMP

3. An example
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1 Preliminaries

2 The RH problem for OMP

3 An example
Let $W$ be a $N \times N$ a weight matrix such that $dW(x) = W(x)dx$.

We can construct a family of OMP such that

$$
\int_{\mathbb{R}} P_n(x) W(x) P_m^*(x) dx = \delta_{n,m} I, \quad n, m \geq 0
$$

$$
P_n(x) = \gamma_n (x^n + a_{n,n-1}x^{n-1} + \cdots) = \gamma_n \hat{P}_n(x)
$$

The matrix-valued polynomials of the second kind, defined by

$$
Q_n(x) = \int_{\mathbb{R}} \frac{P_n(t) W(t)}{t - x} dt, \quad n \geq 0
$$

$(P_n)_n$ and $(Q_n)_n$ satisfy a three term recurrence relation

$$
tP_n(t) = A_{n+1} P_{n+1}(t) + B_n P_n(t) + A_n^* P_{n-1}(t), \quad n \geq 0
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$$
det(A_{n+1}) \neq 0, \quad B_n = B_n^*
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The coefficients of the TTRR satisfy

\[ A_n = \gamma_{n-1} \gamma_n^{-1}, \quad B_n = \gamma_n (a_{n,n-1} - a_{n+1,n}) \gamma_n^{-1} \]

The TTRR for monic OMP

\[ x \hat{P}_n(x) = \hat{P}_{n+1}(x) + \alpha_n \hat{P}_n(x) + \beta_n \hat{P}_{n-1}(x), \quad n \geq 0 \]
\[ \alpha_n = a_{n,n-1} - a_{n+1,n}, \quad \beta_n = (\gamma_n^* \gamma_n)^{-1} (\gamma_n^* \gamma_{n-1}) \gamma_{n-1} \]

Second-order differential equations of hypergeometric type

\[ P_n''(x) F_2(x) + P_n'(x) F_1(x) + P_n(x) F_0(x) = \Lambda_n P_n(x), \quad n \geq 0 \]
\[ \text{deg } F_i \leq i, \quad \Lambda_n \quad \text{Hermitian} \]
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Solution of the RH for OMP

\( Y^n : \mathbb{C} \rightarrow \mathbb{C}^{2N \times 2N} \) such that

1. **Analyticity.** \( Y^n \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \)

2. **Jump Condition.** \( Y^n_+(x) = Y^n_-(x) \begin{pmatrix} I & W(x) \\ 0 & I \end{pmatrix} \) when \( x \in \mathbb{R} \)

3. **Normalization.** \( Y^n(z) = (I + O(1/z)) \begin{pmatrix} z^nI & 0 \\ 0 & z^{-n}I \end{pmatrix} \) as \( z \to \infty \)

For \( n \geq 1 \) the unique solution of the RH problem above is given by

\[
Y^n(z) = \begin{pmatrix} \hat{P}_n(z) & \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\hat{P}_n(t)W(t)}{t-z} \, dt \\ -2\pi i \gamma^*_{n-1} \gamma_{n-1} \hat{P}_{n-1}(z) & -\gamma^*_{n-1} \gamma_{n-1} \int_{\mathbb{R}} \frac{\hat{P}_{n-1}(t)W(t)}{t-z} \, dt \end{pmatrix}
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For $n \geq 1$ the unique solution of the RH problem above is given by

$$Y^n(z) = \begin{pmatrix} \hat{P}_n(z) & 1 \int_{\mathbb{R}} \frac{\hat{P}_n(t)W(t)}{t-z} dt \\ -2\pi i \gamma^*_n \gamma_n \hat{P}_{n-1}(z) & -\gamma^*_n \gamma_n \int_{\mathbb{R}} \frac{\hat{P}_{n-1}(t)W(t)}{t-z} dt \end{pmatrix}$$
Also we find a solution of the inverse

\[
(Y^n)^{-1} = \left( - \left( \int_{\mathbb{R}} \frac{W(t) \hat{P}_{n-1}^*(t)}{t - z} \, dt \right) \gamma_{n-1}^* \gamma_{n-1} - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{W(t) \hat{P}_n^*(t)}{t - z} \, dt \right)
\]

- The Liouville-Ostrogradski formula

\[
Q_n(z) P_{n-1}^*(z) - P_n(z) Q_{n-1}^*(z) = A_n^{-1}
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- The Hermitian property

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2\pi i \hat{P}_{n-1}^*(z) \gamma_{n-1}^* \gamma_{n-1}
\end{pmatrix}
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The three-term recurrence relation

If we call $R = Y^{n+1} (Y^n)^{-1}$ and denoting

$$Y^n(z) = \begin{pmatrix} I + \frac{1}{z} Y_1^n + \mathcal{O}_n(1/z^2) \end{pmatrix} \begin{pmatrix} z^n/l & 0 \\ 0 & z^{-n/l} \end{pmatrix}, \quad z \to \infty$$

then

$$Y^{n+1}(z) = \begin{pmatrix} zl + (Y_{11}^{n+1}) - (Y_1^n)_{11} & -(Y_1^n)_{12} \\ (Y_{11}^{n+1})_{21} & 0 \end{pmatrix} Y^n(z)$$

- $\gamma_{n-1} \gamma_{n-1} = -\frac{1}{2\pi i} (Y_1^n)_{21} = -\frac{1}{2\pi i} (Y_1^{n-1})_{12}^{-1}$
- $\alpha_n = (Y_1^n)_{11} - (Y_{11}^{n+1})_{11}, \quad \beta_n = (Y_1^n)_{12} (Y_1^n)_{21}$
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- $(Y_{11}^{n+1})_{21} (Y_1^n)_{12} = (Y_1^n)_{12} (Y_{11}^{n+1})_{21} = I, \quad (Y_1^n)_{11} + (Y_1^n)^*_{22} = 0$
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If we call $R = Y^{n+1}(Y^n)^{-1}$ and denoting

$$Y^n(z) = \left( I + \frac{1}{z} Y^n_1 + O_n(1/z^2) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \to \infty$$

then

$$Y^{n+1}(z) = \left( zI + (Y^{n+1}_1)_{11} - (Y^n_1)_{11} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y^n(z)$$

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- \( \gamma^*_n Y_{n-1} = -\frac{1}{2\pi i} (Y_1^n)_{21} = -\frac{1}{2\pi i} (Y_1^{n-1})^{-1}_{12} \)
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- \( (Y_1^{n+1})_{21} (Y_1^n)_{12} = (Y_1^n)_{12} (Y_1^{n+1})_{21} = I, \quad (Y_1^n)_{11} + (Y_1^n)^*_{22} = 0 \)
Let \((P_n)_n\) an orthonormal family. Then we have

\[
K_n(x, y) = \sum_{j=0}^{n-1} P_j^*(y) P_j(x) = \frac{P_{n-1}^*(y) A_n P_n(x) - P_n^*(y) A_n^* P_{n-1}(x)}{x - y}
\]

This kernel has the following properties

1. \(K_n(x, y) = K_n^*(y, x)\)
2. \(K_n(x, y) = \int_{\mathbb{R}} K_n(s, y) W(s) K_n(x, s) ds\)

We also have that

\[
K_n(x, y) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} 0 & I \end{pmatrix} (Y^n)^{-1}(y)(Y^n)^{+}(x) \begin{pmatrix} I \\ 0 \end{pmatrix}
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The kernel

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\]
A differential equation

We consider weight matrices of the form

\[ W(x) = \rho(x) T(x) T^*(x), \]

where \( T \) satisfies \( T'(x) = G(x) T(x) \).

Consider

\[ X^n(z) = Y^n(z) J(z) = Y^n(z) \begin{pmatrix} \rho(z)^{1/2} T(z) & 0 \\ 0 & \rho(z)^{-1/2} (T(z))^{-*} \end{pmatrix} \]

Then \( \frac{d}{dz} X^n(z) X^n(z)^{-1} \) is entire and near infinity it behaves like

\[ \left( I + \frac{Y_1}{z} + O(z^{-2}) \right) \begin{pmatrix} \frac{1}{2} \frac{\rho'(z)}{\rho(z)} + G(z) & 0 \\ 0 & -\frac{1}{2} \frac{\rho'(z)}{\rho(z)} - G(z)^* \end{pmatrix} \left( I - \frac{Y_1}{z} + O(z^{-2}) \right) \]
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We will study in detail the RH problem for the weight matrix

\[ W(x) = e^{-x^2} e^{Ax} e^{A^*x}, \quad x \in \mathbb{R}, \]

where

\[
A = \begin{pmatrix}
0 & \nu_1 & 0 & \cdots & 0 \\
0 & 0 & \nu_2 & \cdots & 0 \\
0 & 0 & 0 & \cdots & \nu_i \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \nu_{N-1} \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad \nu_i \in \mathbb{C} \setminus \{0\}
\]

Therefore, \( \rho(z) = e^{-z^2} \) and \( T(z) = e^{Az} \) with \( T'(z) = AT(z) \).

It was shown by Durán-Grünbaum that

\[
\hat{P}_{n}''(z) + \hat{P}_{n}'(z)(2A - 2zI) + \hat{P}_{n}(z)(A^2 - 2J) = (-2nI + A^2 - 2J)^n P_n(z),
\]

where \( J \) is the diagonal matrix \( J = \sum_{i=1}^{N}(N - i)E_{i,i} \).
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where \( J \) is the diagonal matrix \( J = \sum_{i=1}^{N} (N - i)E_{i,i} \).
The Lax pair

Let $Y^n$ be the solution of the RH for $W$ and consider

$$X^n(z) = Y^n(z) \begin{pmatrix} e^{-z^2/2}e^{Az} & 0 \\ 0 & e^{z^2/2}e^{-A^*z} \end{pmatrix}$$

$$X^{n+1}(z) = \begin{pmatrix} zI + \left( Y_1^{n+1} \right)_{11} - \left( Y_1^n \right)_{11} - \left( Y_1^n \right)_{12} \\ \left( Y_1^{n+1} \right)_{21} \end{pmatrix} X^n(z)$$

$$\frac{d}{dz}X^n(z) = \begin{pmatrix} -zI + A & 2\left( Y_1^n \right)_{12} \\ -2\left( Y_1^n \right)_{21} & zI - A^* \end{pmatrix} X^n(z)$$

Compatibility conditions

$$2(\beta_{n+1} - \beta_n) = A\alpha_n - \alpha_n A + I$$

$$\alpha_n = \frac{1}{2} \left( A + (\gamma_n^*\gamma_n)^{-1}A^*(\gamma_n^*\gamma_n) \right)$$
The Lax pair

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Ladder operators

**Lowering operator**

\[ \hat{P}'_n(z) = A\hat{P}_n(z) - \hat{P}_n(z)A + 2\beta_n\hat{P}_{n-1}(z) \]

Therefore

\[ \beta_n = \frac{1}{2} (nl + a_{n,n-1}A - Aa_{n,n-1}) \]

**Raising operator**

\[ \hat{P}'_n(z) = -2\hat{P}_{n+1}(z) + 2z\hat{P}_n(z) + A\hat{P}_n(z) - \hat{P}_n(z)A - 2\alpha_n\hat{P}_n(z) \]
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Second-order differential equations

Introduce the following differential/difference operators

\[
\begin{align*}
R_1 &= \partial^1 + \partial^0 A, \quad L_1 = 2\beta_n E^{-1} + AE^0 \\
R_2 &= \partial^1 + \partial^0 (A - 2zI), \quad L_2 = -2E^1 + (A - 2\alpha_n)E^0 \\
\partial^k &= \frac{d^k}{dz^k}, \quad E^k f(n) = f(n + k)
\end{align*}
\]

The ladder operators are equivalent to

\[
\hat{P}_n(z)R_1 = L_1 \hat{P}_n(z), \quad \text{and} \quad \hat{P}_n(z)R_2 = L_2 \hat{P}_n(z)
\]

Therefore, the OMP \( \hat{P}_n \) satisfy two second-order differential equations

\[
\begin{align*}
\hat{P}_n(z)R_1 R_2 &= L_1 \hat{P}_n(z)R_2 = L_1 L_2 \hat{P}_n(z), \\
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Preliminaries
The RH problem for OMP
An example

Second-order differential equations

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\]
We will proof the following:

- Both equations are equivalent
- They are also equivalent to the second-order differential equation of hypergeometric type

\[
\hat{P}_n''(z) + \hat{P}_n'(z)(2A - 2zI) + \hat{P}_n(z)(A^2 - 2J) = (-2nI + A^2 - 2J)\hat{P}_n(z)
\]

The first one is

\[
\hat{P}_n''(z) + 2\hat{P}_n'(z)(A - zI) + \hat{P}_n(z)A(A - 2zI) =
- 4\beta_n\hat{P}_n(z) + 2\beta_n(A - 2\alpha_{n-1})\hat{P}_{n-1}(z) - 2A\hat{P}_{n+1}(z) + A(A - 2\alpha_n)\hat{P}_n(z)
\]

And the second

\[
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Subtracting two last equations we get an important relation

\[ \beta_n(A - 2\alpha_{n-1}) = (A - 2\alpha_n)\beta_n \]

Using ladder operators we get that both equations are

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\hat{P}''(z) + 2\hat{P}'(z)(A - zI) + \hat{P}(z)A(A - 2zI) =
2(A - \alpha_n)(\hat{P}'(z) + \hat{P}(z)A - A\hat{P}(z)) + (A^2 - 2zA - 4\beta_n)\hat{P}(z)
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\[
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To proof that the equations above are of hypergeometric type we use that

\[(A - \alpha_n)\hat{P}_n'(z) + (A - \alpha_n + zI)(\hat{P}_n(z)A - A\hat{P}_n(z)) - 2\beta_n\hat{P}_n(z) = \hat{P}_n(z)J - J\hat{P}_n(z) - n\hat{P}_n(z)\]

Remarks

- The OMP \(\hat{P}_n\) satisfy a first-order differential equation (not of hypergeometric type), something that is not possible in the scalar case.
- These results are consistent if we compare them with the scalar situation (Hermite polynomials with \(\alpha_n = 0\) and \(\beta_n = \frac{1}{2}n\)).
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