Some recent results on tempered pullback attractors for non-autonomous variants of Navier-Stokes equations

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Outline of the talk

Motivation

Abstract results on attractors theory
   Existence of minimal pullback attractors
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Navier-Stokes-Voigt
Motivation

- Non-autonomous dynamical systems
• Random dynamical systems (unbounded time-dependent terms)
Deterministic non-autonomous dynamical systems with the pullback approach with fixed bounded sets

• Deterministic non-autonomous dynamical systems with tempered universes:

★ Physical and mathematical questions: big-bang-bang–past, present, future; dissipative world
Abstract results on attractors theory. Existence of minimal pullback attractors

Consider given a metric space $(X, d_X)$, and let us denote
\[ \mathbb{R}^2_d = \{(t, \tau) \in \mathbb{R}^2 : \tau \leq t\}. \]
A process on $X$ is a mapping $U$ such that
\[ \mathbb{R}^2_d \times X \ni (t, \tau, x) \mapsto U(t, \tau)x \in X \text{ with } U(\tau, \tau)x = x \text{ for any } \]
\[ (\tau, x) \in \mathbb{R} \times X, \text{ and } U(t, r)(U(r, \tau)x) = U(t, \tau)x \text{ for any } \]
\[ \tau \leq r \leq t \text{ and all } x \in X. \]

**Definition**

A process $U$ on $X$ is said to be **closed** if for any $\tau \leq t$, and any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ and $U(t, \tau)x_n \to y \in X$, then $U(t, \tau)x = y$.

**Remark**

$U$ continuous

\[ \Rightarrow \text{ strong-weak (also known as norm-to weak)} \]
\[ \Rightarrow \text{ closed} \]

This more relaxed concepts are useful in some situations (e.g., dyn. syst. and attractors for strong sols. for RD eqns).
\( \mathcal{P}(X) \) the family of all nonempty subsets of \( X \), and consider a family of nonempty sets \( \hat{D}_0 = \{ D_0(t) : t \in \mathbb{R} \} \subset \mathcal{P}(X) \) [not required compactness or boundedness on these sets]

Definition

\( U \) is pullback \( \hat{D}_0 \)-asymptotically compact if for any \( t \in \mathbb{R} \) and any sequences \( \{ \tau_n \} \subset (-\infty, t] \) and \( \{ x_n \} \subset X \) satisfying \( \tau_n \to -\infty \) and \( x_n \in D_0(\tau_n) \) for all \( n \), the sequence \( \{ U(t, \tau_n)x_n \} \) is relatively compact in \( X \).

Denote

\[
\Lambda(\hat{D}_0, t) := \bigcap_{s \leq t} \bigcup_{\tau \leq s} U(t, \tau)D_0(\tau) \quad \forall t \in \mathbb{R}.
\]

Proposition

\( U \) pullback \( \hat{D}_0 \)-asymptotically compact \( \Rightarrow \) for all \( t \in \mathbb{R} \), the set \( \Lambda(\hat{D}_0, t) \) given by (8) is a nonempty compact subset of \( X \), and (attracts pullback)

\[
\lim_{\tau \to -\infty} \text{dist}_X(U(t, \tau)D_0(\tau), \Lambda(\hat{D}_0, t)) = 0.
\]

Moreover, this is the minimal family of closed sets satisfying (1).
Let be given $\mathcal{D}$ a nonempty class of families parameterized in time $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

**Definition**
It is said that $\hat{\mathcal{D}}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$—absorbing for the process $U$ on $X$ if for any $t \in \mathbb{R}$ and any $\hat{\mathcal{D}} \in \mathcal{D}$, there exists a $\tau_0(t, \hat{\mathcal{D}}) \leq t$ such that

$$U(t, \tau)D(\tau) \subset D_0(t) \quad \text{for all } \tau \leq \tau_0(t, \hat{\mathcal{D}}).$$

Observe that in the definition above $\hat{\mathcal{D}}_0$ does not belong necessarily to the class $\mathcal{D}$.

**Definition**
$U$ pullback $\mathcal{D}$—asymptotically compact if it is $\hat{\mathcal{D}}$-asymptotically compact for any $\hat{\mathcal{D}} \in \mathcal{D}$. 

Proposition
\[ \hat{D}_0 = \{ D_0(t) : t \in \mathbb{R} \} \subset \mathcal{P}(X) \text{ pullback } \mathcal{D} - \text{absorbing for a process } U \text{ on } X, \text{ which is pullback } \hat{D}_0 - \text{asymptotically compact. Then, } U \text{ is also pullback } \mathcal{D} - \text{asymptotically compact.} \]

Proposition
\[ U \text{ closed and pullback } \mathcal{D} - \text{asymptotically compact } \Rightarrow \text{ for each } \hat{D} \in \mathcal{D} \text{ and any } t \in \mathbb{R}, \text{ the set } \Lambda(\hat{D}, t) \text{ is a nonempty compact subset of } X, \text{ invariant for } U, \text{ that attracts } \hat{D} \text{ in the pullback sense, i.e. } \lim_{\tau \to -\infty} \text{dist}_X(U(t, \tau)D(\tau), \Lambda(\hat{D}, t)) = 0. \] (1)
Moreover, it is the minimal family of closed sets satisfying (1).
Theorem

$U : \mathbb{R}^2_d \times X \to X$ closed, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\hat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ pullback $\mathcal{D} -$absorbing for $U$, and $U$ pullback $\hat{D}_0 -$asymptotically compact.

Then, the family $\mathcal{A}_D = \{A_D(t) : t \in \mathbb{R}\}$ defined by

$$A_D(t) = \bigcup_{\hat{D} \in \mathcal{D}} \Lambda(\hat{D}, t)^X \quad t \in \mathbb{R},$$

(a) for any $t \in \mathbb{R}$, $A_D(t)$ is a nonempty compact subset of $X$, and $A_D(t) \subset \Lambda(\hat{D}_0, t)$,

(b) $A_D$ is pullback $\mathcal{D} -$attracting

(c) $A_D$ is invariant, i.e. $U(t, \tau)A_D(\tau) = A_D(t)$ for all $\tau \leq t$,

(d) if $\hat{D}_0 \in \mathcal{D}$, then $A_D(t) = \Lambda(\hat{D}_0, t) \subset \overline{D_0(t)}^X$, for all $t \in \mathbb{R}$.

The family $\mathcal{A}_D$ is minimal in the sense that if $\hat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets and $\mathcal{D} -$attracting, then $A_D(t) \subset C(t)$. 
Remark

Under the assumptions of Theorem 5, the family $A_D$ is called the minimal pullback $D-$attractor for the process $U$.

If $A_D \in D$, then it is the unique family of closed subsets in $D$ that satisfies (b)–(c).

A sufficient condition for $A_D \in D$ is to have that $\hat{D}_0 \in D$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family $\hat{D}$ is inclusion-closed (i.e. if $\hat{D} \in D$, and $\hat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all $t$, then $\hat{D}' \in D$).
Denote $\mathcal{D}_F^X$ the universe of fixed nonempty bounded subsets of $X$, i.e. the class of all families $\hat{D}$ of the form $\hat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with $D$ a fixed nonempty bounded subset of $X$.

For $\mathcal{D}_F^X$, the corresponding minimal pullback $\mathcal{D}_F^X$–attractor $A_{\mathcal{D}_F^X}$ is the one defined by Crauel, Debussche, and Flandoli.

**Corollary**

*Under the assumptions of Theorem 5, if the universe $\mathcal{D}$ contains the universe $\mathcal{D}_F^X$, then both attractors, $A_{\mathcal{D}_F^X}$ and $A_{\mathcal{D}}$, exist, and the following relation holds:*

$$A_{\mathcal{D}_F^X}(t) \subset A_{\mathcal{D}}(t) \quad \forall t \in \mathbb{R}.$$  

**Remark**

*Under the above assumptions, if, moreover, $\hat{D}_0 \in \mathcal{D}$, and for some $T \in \mathbb{R}$ the set $\cup_{t \leq T} D_0(t)$ is a bounded subset of $X$, then*

$$A_{\mathcal{D}_F^X}(t) = A_{\mathcal{D}}(t) \quad \forall t \leq T.$$
Comparison of pullback $\mathcal{D}_i$—attractors

**Theorem**

Let $\{(X_i, d_{X_i})\}_{i=1,2}$ be metric spaces, $X_1 \subset X_2$ contin. injected, and for $i = 1, 2$, let $\mathcal{D}_i$ be a universe in $\mathcal{P}(X_i)$, with $\mathcal{D}_1 \subset \mathcal{D}_2$. $U$ acts as a process in both cases, $U : \mathbb{R}^2_d \times X_i \to X_i$ for $i = 1, 2$.

$$\mathcal{A}_i(t) = \bigcup_{\hat{D}_i \in \mathcal{D}_i} \Lambda_i(\hat{D}_i, t)^{X_i}, \quad i = 1, 2.$$  

Then, $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$. 
Suppose moreover that the two following conditions are satisfied:

(i) $A_1(t)$ is a compact subset of $X_1$ for all $t \in \mathbb{R}$,

(ii) for any $\hat{D}_2 \in \mathcal{D}_2$ and any $t \in \mathbb{R}$, there exist a family $\hat{D}_1 \in \mathcal{D}_1$ and a $t^*_{\hat{D}_1} \leq t$ (both possibly depending on $t$ and $\hat{D}_2$), such that $U$ is pullback $\hat{D}_1$-asymptotically compact, and for any $s \leq t^*_{\hat{D}_1}$ there exists a $\tau_s \leq s$ such that

$$U(s, \tau)D_2(\tau) \subset D_1(s) \quad \text{for all } \tau \leq \tau_s.$$

Then, under all the conditions above, $A_1(t) = A_2(t)$ for all $t \in \mathbb{R}$. 
Remark

In the preceding theorem, if instead of assumption (ii) we consider the following condition:

(ii') for any $\hat{D}_2 \in \mathcal{D}_2$ and any sequence $\tau_n \to -\infty$ there exist another family $\hat{D}_1 \in \mathcal{D}_1$ and another sequence $\tau'_n \to -\infty$ with $\tau'_n \geq \tau_n$ for all $n$, such that $U$ is pullback $\hat{D}_1$-asymptotically compact, and

$$U(\tau'_n, \tau_n)D_2(\tau_n) \subset D_1(\tau'_n), \quad \text{for all } n,$$

then, with a similar proof, the equality $A_2(t) = A_1(t)$ for all $t \in \mathbb{R}$, also holds.

Observe that a sufficient condition for (2) is that there exists $T > 0$ such that for any $\hat{D}_2 \in \mathcal{D}_2$, there exists a $\hat{D}_1 \in \mathcal{D}_1$ satisfying $U(\tau + T, \tau)D_2(\tau) \subset D_1(\tau + T)$, for all $\tau \in \mathbb{R}$. 
Application to a 2D-Navier-Stokes model

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f(t) \text{ in } (\tau, +\infty) \times \Omega, \\
\text{div } u &= 0 \text{ in } (\tau, +\infty) \times \Omega, \\
u > 0 \text{ is the kinematic viscosity,} \\
\text{u is the velocity field of the fluid,} \\
p \text{ is the pressure,} \\
uT \text{ is the initial velocity field, and} \\
f \text{ the external force (time-dep.) term (Ex.: Arctic sea, control, etc)}
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^2 \) is open and bounded with smooth enough \( \partial \Omega^1 \),

\(^1\text{Not for the results in } H \text{ but in } V.\)
\[ \mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^2 : \text{div} \ u = 0 \right\}, \]

\( H = \) the closure of \( \mathcal{V} \) in \( (L^2(\Omega))^2 \) with the norm \( |\cdot| \), and inner product \( (\cdot, \cdot) \), where for \( u, v \in (L^2(\Omega))^2 \),

\[
(u, v) = \sum_{j=1}^{2} \int_{\Omega} u_j(x)v_j(x)dx,
\]

\( \mathcal{V} = \) the closure of \( \mathcal{V} \) in \( (H^1_0(\Omega))^2 \) with the norm \( \|\cdot\| \) associated to the inner product \( ((\cdot, \cdot)) \), where for \( u, v \in (H^1_0(\Omega))^2 \),

\[
((u, v)) = \sum_{i,j=1}^{2} \int_{\Omega} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx.
\]
Definition (Weak solution)

A weak solution is a function $u$ that belongs to $L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$ for all $T > \tau$, with $u(\tau) = u_\tau$, such that for all $v \in V$,

$$
\frac{d}{dt}(u(t), v) + \nu \langle Au(t), v \rangle + b(u(t), u(t), v) = \langle f(t), v \rangle,
$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau, +\infty)$.

Remark

If $u$ is a weak solution, then we deduce that for any $T > \tau$, one has $u' \in L^2(\tau, T; V')$, and so $u \in C([\tau, +\infty); H)$, whence the initial datum has full sense. Moreover, in this case the following energy equality holds for all $\tau \leq s \leq t$:

$$
|u(t)|^2 + 2\nu \int_s^t \langle Au(r), u(r) \rangle dr = |u(s)|^2 + 2 \int_s^t \langle f(r), u(r) \rangle dr.
$$
Definition (Strong solution)

A strong solution is a weak solution $u$ of (17) such that $u \in L^2(\tau, T; D(A)) \cap L^\infty(\tau, T; V)$ for all $T > \tau$.

Remark

If $f \in L^2_{\text{loc}}(\mathbb{R}; H)$ and $u$ is a strong solution, then $u' \in L^2(\tau, T; H)$ for all $T > \tau$, and so $u \in C([\tau, +\infty); V)$. In this case the following energy equality holds:

\[
\|u(t)\|^2 + 2\nu \int_s^t |Au(r)|^2 \, dr + 2 \int_s^t b(u(r), u(r), Au(r)) \, dr \\
= \|u(s)\|^2 + 2 \int_s^t (f(r), Au(r)) \, dr, \quad \forall \tau \leq s \leq t.
\]
Theorem (Weak and strong solutions)

\( f \in L^2_{\text{loc}}(\mathbb{R}; V') \) and \( u_\tau \in H \Rightarrow \exists! \) weak solution \( u(\cdot) = u(\cdot; \tau, u_\tau). \)
\( f \in L^2_{\text{loc}}(\mathbb{R}; H) \Rightarrow u \in C((\tau, T]; V) \cap L^2(\tau + \varepsilon, T; (H^2(\Omega))^2) \) for every \( \varepsilon > 0 \) and \( T > \tau + \varepsilon. \)

If \( u_\tau \in V, \) then \( u \in C([\tau, T]; V) \cap L^2(\tau, T; (H^2(\Omega))^2) \) for every \( T > \tau, \) i.e. \( u \) is a strong solution.
Theorem (Weak and strong solutions)

\( f \in L^2_{\text{loc}}(\mathbb{R}; V') \) and \( u_\tau \in H \Rightarrow \exists! \) weak solution \( u(\cdot) = u(\cdot; \tau, u_\tau). \)

\( f \in L^2_{\text{loc}}(\mathbb{R}; H) \Rightarrow u \in C((\tau, T]; V) \cap L^2(\tau + \varepsilon, T; (H^2(\Omega))^2) \) for every \( \varepsilon > 0 \) and \( T > \tau + \varepsilon. \)

If \( u_\tau \in V, \) then \( u \in C([\tau, T]; V) \cap L^2(\tau, T; (H^2(\Omega))^2) \) for every \( T > \tau, \) i.e. \( u \) is a strong solution.

Therefore, when \( f \in L^2_{\text{loc}}(\mathbb{R}; V') \), we can define a process

\[ U : \mathbb{R}_d^2 \times H \to H \]

as

\[ U(t, \tau)u_\tau = u(t; \tau, u_\tau) \quad \forall u_\tau \in H, \quad \forall \tau \leq t, \]

and if \( f \in L^2_{\text{loc}}(\mathbb{R}; H) \), the restriction of this process to \( \mathbb{R}_d^2 \times V \) is a process in \( V. \)
Pullback $\mathcal{D}$-attractors in $H$

**Proposition (Continuity of the process)**

If $f \in L^2_{\text{loc}}(\mathbb{R}; V')$, for any pair $(t, \tau) \in \mathbb{R}^2$, the map $U(t, \tau)$ is continuous from $H$ into $H$.

Moreover, if $f \in L^2_{\text{loc}}(\mathbb{R}; H)$, then $U(t, \tau)$ is also continuous from $V$ into $V$. 

Pullback $\mathcal{D}$-attractors in $H$

**Proposition (Continuity of the process)**

If $f \in L^2_{\text{loc}}(\mathbb{R}; V')$, for any pair $(t, \tau) \in \mathbb{R}^2_d$, the map $U(t, \tau)$ is continuous from $H$ into $H$.

Moreover, if $f \in L^2_{\text{loc}}(\mathbb{R}; H)$, then $U(t, \tau)$ is also continuous from $V$ into $V$.

**Lemma**

Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ and $u_\tau \in H$. Consider any $\mu \in (0, 2\nu \lambda_1)$ fixed. Then, the solution $u$ satisfies for all $t \geq \tau$:

$$
|u(t)|^2 \leq e^{-\mu(t-\tau)}|u_\tau|^2 + \frac{e^{-\mu t}}{2\nu - \mu \lambda_1^{-1}} \int^t_{\tau} e^{\mu s} \|f(s)\|^2_* ds.
$$
Lemma
Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ and $u_\tau \in H$. Consider any $\mu \in (0, 2\nu \lambda_1)$ fixed. Then, the solution $u$ satisfies for all $t \geq \tau$:

$$|u(t)|^2 \leq e^{-\mu(t-\tau)}|u_\tau|^2 + \frac{e^{-\mu t}}{2\nu - \mu \lambda_1^{-1}} \int_\tau^t e^{\mu s} \|f(s)\|^2_* ds.$$ 

Definition (Universe)
We will denote by $\mathcal{D}^H_\mu$ the class of all families of nonempty subsets $\hat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H)$ such that

$$\lim_{\tau \to -\infty} \left( e^{\mu \tau} \sup_{v \in D(\tau)} |v|^2 \right) = 0.$$ 

Remark
$\mathcal{D}^H_F \subset \mathcal{D}^H_\mu$ and that $\mathcal{D}^H_\mu$ is inclusion-closed (tempered condition).
Corollary ($D^H_\mu -$absorbing family)

Assume that there exists some $\mu \in (0, 2\nu \lambda_1)$ such that

$$\int_{-\infty}^{0} e^{\mu s} \|f(s)\|_2^2 ds < +\infty.$$ 

Then, $\hat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_H(0, R^{1/2}_H(t))$,

$$R_H(t) = 1 + \frac{e^{-\mu t}}{2\nu - \mu \lambda_1^{-1}} \int_{-\infty}^{t} e^{\mu s} \|f(s)\|_2^2 ds,$$

is pullback $D^H_\mu -$absorbing for the process $U : \mathbb{R}_d^2 \times H \to H$ (and therefore $D^H_F -$absorbing too), and $\hat{D}_0 \in D^H_\mu$.

Lemma ($D^H_\mu -$asymptotic compactness)

The process $U$ is pullback $D^H_\mu -$asymptotically compact.
Corollary ($\mathcal{D}_\mu^H$—absorbing family)

Assume that there exists some $\mu \in (0, 2\nu \lambda_1)$ such that

$$\int_{-\infty}^{0} e^{\mu s} \| f(s) \|_2^* ds < +\infty.$$ 

Then, $\hat{D}_0 = \{ D_0(t) : t \in \mathbb{R} \}$ defined by $D_0(t) = \overline{B}_H(0, R_H^{1/2}(t))$,

$$R_H(t) = 1 + \frac{e^{-\mu t}}{2\nu - \mu \lambda_1^{-1}} \int_{-\infty}^{t} e^{\mu s} \| f(s) \|_2^* ds,$$

is pullback $\mathcal{D}_\mu^H$—absorbing for the process $U : \mathbb{R}^d_2 \times H \to H$ (and therefore $\mathcal{D}_F^H$—absorbing too), and $\hat{D}_0 \in \mathcal{D}_\mu^H$.

Lemma ($\mathcal{D}_\mu^H$—asymptotic compactness)

The process $U$ is pullback $\mathcal{D}_\mu^H$—asymptotically compact.

Proof (energy method based on non-increasing continuous functionals) omitted, see V case below.
Theorem (Pullback $\mathcal{D}^H_\mu$-attractor)

Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ satisfies for some $\mu \in (0, 2\nu \lambda_1)$ the above condition. Then, $\exists$ the minimal pullback $\mathcal{D}^H_F$-attractor

$$\mathcal{A}_{\mathcal{D}^H_F} = \{\mathcal{A}_{\mathcal{D}^H_F}(t) : t \in \mathbb{R}\}$$

and the minimal pullback $\mathcal{D}^H_\mu$-attractor

$$\mathcal{A}_{\mathcal{D}^H_\mu} = \{\mathcal{A}_{\mathcal{D}^H_\mu}(t) : t \in \mathbb{R}\},$$

for the process $U$. The family $\mathcal{A}_{\mathcal{D}^H_\mu}$ belongs to $\mathcal{D}^H_\mu$, and the following relation holds:

$$\mathcal{A}_{\mathcal{D}^H_F}(t) \subset \mathcal{A}_{\mathcal{D}^H_\mu}(t) \subset \overline{B}_H(0, R_H^{1/2}(t)) \quad \forall t \in \mathbb{R}.$$ 

Remark

Useful in unbounded “Poincaré”-domains to obtain $\mathcal{A}_{\mathcal{D}^H_F}$. 
Regularity: pullback $\mathcal{D}$-attractors in $V$

From now on we assume that $f \in L^2_{\text{loc}}(\mathbb{R}; H)$, and satisfies

$$\int_{-\infty}^{0} e^{\mu s} |f(s)|^2 \, ds < +\infty,$$

for some $\mu \in (0, 2\nu \lambda_1)$.

**Lemma**

*For any $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}^H_{\mu}$, there exists $\tau_1(\hat{D}, t) < t - 3$, such that for any $\tau \leq \tau_1(\hat{D}, t)$ and any $u_{\tau} \in D(\tau)$, it holds*

$$
\begin{align*}
|u(r; \tau, u_{\tau})|^2 &\leq \rho_1(t) \quad \text{for all } r \in [t - 3, t], \\
\|u(r; \tau, u_{\tau})\|^2 &\leq \rho_2(t) \quad \text{for all } r \in [t - 2, t], \\
\int_{r-1}^{r} |Au(\theta; \tau, u_{\tau})|^2 \, d\theta &\leq \rho_3(t) \quad \text{for all } r \in [t - 1, t], \\
\int_{r-1}^{r} |u'(\theta; \tau, u_{\tau})|^2 \, d\theta &\leq \rho_4(t) \quad \text{for all } r \in [t - 1, t],
\end{align*}
$$
where

\[
\rho_1(t) = 1 + \frac{e^{\mu(3-t)}}{2\nu\lambda_1 - \mu} \int_{-\infty}^t e^{\mu\theta} |f(\theta)|^2 \, d\theta,
\]

\[
\rho_2(t) = \max_{r \in [t-2, t]} \left\{ \left( \frac{1}{\nu} \rho_1(r) + \left( \frac{1}{\nu^2\lambda_1} + \frac{2}{\nu} \right) \int_{r-1}^r |f(\theta)|^2 \, d\theta \right) \times \exp \left[ 2C^{(\nu)} \rho_1(r) \left( \frac{1}{\nu} \rho_1(r) + \frac{1}{\nu^2\lambda_1} \int_{r-1}^r |f(\theta)|^2 \, d\theta \right) \right\} \right\},
\]

\[
\rho_3(t) = \frac{1}{\nu} \left( \rho_2(t) + \frac{2}{\nu} \int_{t-2}^t |f(\theta)|^2 \, d\theta + 2C^{(\nu)} \rho_1(t) \rho_2^2(t) \right),
\]

\[
\rho_4(t) = \nu \rho_2(t) + 2 \int_{t-2}^t |f(\theta)|^2 \, d\theta + 2C_1^2 \rho_2(t) \rho_3(t),
\]

and \( C^{(\nu)} = 27C_1^4(4\nu^3)^{-1} \).
Remark

\[ \lim_{t \to -\infty} e^{\mu t} \rho_1(t) = 0. \]

So \( \{ \overline{B}_H(0, \rho_{1}^{1/2}(t)) : t \in \mathbb{R} \} \in \mathcal{D}^H_\mu. \)

We will denote by \( \mathcal{D}^{H,V}_\mu \) the class of all families \( \hat{\mathcal{D}}_V \) of elements of \( \mathcal{P}(V) \) of the form \( \hat{\mathcal{D}}_V = \{ D(t) \cap V : t \in \mathbb{R} \} \), where \( \hat{\mathcal{D}} = \{ D(t) : t \in \mathbb{R} \} \in \mathcal{D}^H_\mu. \)

\( \mathcal{D}^V_F \) the universe of families (parameterized in time but constant for all \( t \in \mathbb{R} \)) of nonempty fixed bounded subsets of \( V. \)

\( \mathcal{D}^{H,V}_\mu \subset \mathcal{P}(V) \) is inclusion-closed, and evidently \( \mathcal{D}^V_F \subset \mathcal{D}^{H,V}_\mu. \)
Corollary (Absorbing in $H$-regularizing+tempered)

The family

$$\hat{D}_{0,V} = \{\overline{B}_H(0, \rho_1^{1/2}(t)) \cap V : t \in \mathbb{R}\}$$

belongs to $\mathcal{D}_{H,V}^{\mu}$ and satisfies that for any $t \in \mathbb{R}$ and any $\hat{D} \in \mathcal{D}_{H}^{\mu}$, there exists a $\tau(\hat{D}, t) < t$ such that

$$U(t, \tau)D(\tau) \subset D_{0,V}(t) \quad \text{for all } \tau \leq \tau(\hat{D}, t).$$

In particular, the family $\hat{D}_{0,V}$ is pullback $\mathcal{D}_{H,V}^{\mu}$-absorbing for the process $U : \mathbb{R}^2_d \times V \to V$. 
Lemma (Asymptotic compactness in $V$ norm)

The process $U : \mathbb{R}_d^2 \times V \to V$ is pullback $\mathcal{D}_{\mu}^{H,V}$ — asymptotically compact.

Sketch of the proof:

\[
\begin{aligned}
&u^n \overset{\ast}{\rightharpoonup} u \quad \text{weak-star in } L^\infty(t - 2, t; V), \\
&u^n \rightharpoonup u \quad \text{weakly in } L^2(t - 2, t; D(A)), \\
&(u^n)' \rightharpoonup u' \quad \text{weakly in } L^2(t - 2, t; H), \\
&u^n \to u \quad \text{strongly in } L^2(t - 2, t; V), \\
&u^n(s) \to u(s) \quad \text{strongly in } V, \ a.e. \ s \in (t - 2, t).
\end{aligned}
\]
Lemma (Asymptotic compactness in \( V \) norm)

The process \( U : \mathbb{R}^2_d \times V \to V \) is pullback \( \mathcal{D}^{H,V}_\mu \) - asymptotically compact.

Sketch of the proof:

\[
\begin{align*}
    u^n \stackrel{*}{\rightharpoonup} u & \quad \text{weak-star in } L^\infty(t - 2, t; V), \\
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    u^n \to u & \quad \text{strongly in } L^2(t - 2, t; V), \\
    u^n(s) \to u(s) & \quad \text{strongly in } V, \ a.e. \ s \in (t - 2, t).
\end{align*}
\]

From above \( u \in C([t - 2, t]; V) \) and \( u \) satisfies the eqn in \( (t - 2, t) \).
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\end{align*}
\]

From above $u \in C([t - 2, t]; V)$ and $u$ satisfies the eqn in $(t - 2, t)$.

\[
\{u^n\} \text{ is equi-continuous in } H, \text{ on } [t - 2, t].
\]
Lemma (Asymptotic compactness in $V$ norm)

The process $U : \mathbb{R}^2_d \times V \to V$ is pullback $\mathcal{D}^{H,V}_\mu$ – asymptotically compact.

Sketch of the proof:

\[
\begin{align*}
    u^n & \rightharpoonup^* u \quad \text{weak-star in } L^\infty(t-2, t; V), \\
    u^n & \rightharpoonup u \quad \text{weakly in } L^2(t-2, t; D(A)), \\
    (u^n)' & \rightharpoonup u' \quad \text{weakly in } L^2(t-2, t; H), \\
    u^n & \to u \quad \text{strongly in } L^2(t-2, t; V), \\
    u^n(s) & \to u(s) \quad \text{strongly in } V, \ a.e. \ s \in (t-2, t).
\end{align*}
\]

From above $u \in C([t-2, t]; V)$ and $u$ satisfies the eqn in $(t-2, t)$.

$\{u^n\}$ is equi-continuous in $H$, on $[t-2, t]$. Since $\{u^n\}$ is bounded in $C([t-2, t]; V)$,
Lemma (Asymptotic compactness in $V$ norm)

The process $U : \mathbb{R}_d^2 \times V \to V$ is pullback $D_{\mu}^{H/V}$ — asymptotically compact.

Sketch of the proof:

\[
\begin{align*}
&\quad u^n \rightharpoonup u \quad \text{weak-star in } L^\infty(t - 2, t; V), \\
&u^n \rightarrow u \quad \text{weakly in } L^2(t - 2, t; D(A)), \\
&(u^n)' \rightarrow u' \quad \text{weakly in } L^2(t - 2, t; H), \\
&u^n \rightarrow u \quad \text{strongly in } L^2(t - 2, t; V), \\
&u^n(s) \rightarrow u(s) \quad \text{strongly in } V, \text{ a.e. } s \in (t - 2, t).
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Lemma (Asymptotic compactness in $V$ norm)

The process $U : \mathbb{R}_d^2 \times V \to V$ is pullback $D_{\mu}^{H,V} -$ asymptotically compact.

Sketch of the proof:

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  u^n(s) & \to u(s) \quad \text{strongly in } V, \text{ a.e. } s \in (t - 2, t).
\end{align*}
\]

From above $u \in C([t - 2, t]; V)$ and $u$ satisfies the eqn in $(t - 2, t)$.

\{$u^n$\} is equi-continuous in $H$, on $[t - 2, t]$. Since \{$u^n$\} is bounded in $C([t - 2, t]; V)$, by $V \subset\subset H$ + Ascoli-Arzelà Th., $\exists$ subseq.

\[ u^n \to u \quad \text{strongly in } C([t - 2, t]; H). \]
For all sequence \( \{s_n\} \subset [t - 2, t] \) with \( s_n \to s_* \), it holds that

\[
u^n(s_n) \rightharpoonup u(s_*) \quad \text{weakly in } V,\]

Claim:

\[
u^n \to u \quad \text{strongly in } C([t - 1, t]; V),\]

If not, \( \{t_n\} \subset [t - 1, t] \), \( t_n \to t_* \geq t - 1 \)

\[
\|u^n(t_n) - u(t_*)\| \geq \varepsilon \quad \forall n \geq 1.
\]
For all sequence \( \{s_n\} \subset [t - 2, t] \) with \( s_n \to s_* \), it holds that

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\[
\|u^n(t_n) - u(t_*)\| \geq \varepsilon \quad \forall n \geq 1.
\]

\[
\|u(t_*)\| \leq \lim \inf_{n \to \infty} \|u^n(t_n)\|.
\]
for all $t - 2 \leq s_1 \leq s_2 \leq t$

$$\|u^n(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au^n(r)|^2 dr$$

$$\leq \|u^n(s_1)\|^2 + 2C(\nu) \int_{s_1}^{s_2} |u^n(r)|^2 \|u^n(r)\|^4 dr + \frac{2}{\nu} \int_{s_1}^{s_2} |f(r)|^2 dr,$$

and

$$\|u(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au(r)|^2 dr$$

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and

$$\|u(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au(r)|^2 dr$$

$$\leq \|u(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u(r)|^2 \|u(r)\|^4 dr + \frac{2}{\nu} \int_{s_1}^{s_2} |f(r)|^2 dr.$$  

In particular we can define the functions

$$J_n(s) = \|u^n(s)\|^2 - 2C^{(\nu)} \int_{t-2}^{s} |u^n(r)|^2 \|u^n(r)\|^4 dr - \frac{2}{\nu} \int_{t-2}^{s} |f(r)|^2 dr,$$

$$J(s) = \|u(s)\|^2 - 2C^{(\nu)} \int_{t-2}^{s} |u(r)|^2 \|u(r)\|^4 dr - \frac{2}{\nu} \int_{t-2}^{s} |f(r)|^2 dr.$$
$J_n(s) \to J(s)$ a.e. $s \in (t - 2, t)$. 
\( J_n(s) \to J(s) \quad \text{a.e. } s \in (t - 2, t). \)

\( \exists \{\tilde{t}_k\} \subset (t - 2, t_*) \) such that \( \tilde{t}_k \to t_* \), and

\[
\lim_{n \to +\infty} J_n(\tilde{t}_k) = J(\tilde{t}_k) \quad \text{for all } k.
\]
\[ J_n(s) \to J(s) \quad \text{a.e. } s \in (t - 2, t). \]

\[ \exists \{\tilde{t}_k\} \subset (t - 2, t_*) \text{ such that } \tilde{t}_k \to t_*, \text{ and} \]

\[ \lim_{n \to +\infty} J_n(\tilde{t}_k) = J(\tilde{t}_k) \quad \text{for all } k. \]

\( J_n \) are non-increasing, so

\[ J_n(t_n) - J(t_*) \leq J_n(\tilde{t}_{k\delta}) - J(t_*) \]

\[ \leq |J_n(\tilde{t}_{k\delta}) - J(t_*)| \]

\[ \leq |J_n(\tilde{t}_{k\delta}) - J(\tilde{t}_{k\delta})| + |J(\tilde{t}_{k\delta}) - J(t_*)| < \delta. \]
\[ J_n(s) \to J(s) \quad \text{a.e. } s \in (t - 2, t). \]

\[ \exists \{\tilde{t}_k\} \subset (t - 2, t_*) \text{ such that } \tilde{t}_k \to t_*, \text{ and} \]

\[
\lim_{n \to +\infty} J_n(\tilde{t}_k) = J(\tilde{t}_k) \quad \text{for all } k.
\]

\(J_n\) are non-increasing, so

\[
J_n(t_n) - J(t_*) \leq J_n(\tilde{t}_{k_\delta}) - J(t_*) \\
\leq |J_n(\tilde{t}_{k_\delta}) - J(t_*)| \\
\leq |J_n(\tilde{t}_{k_\delta}) - J(\tilde{t}_{k_\delta})| + |J(\tilde{t}_{k_\delta}) - J(t_*)| < \delta.
\]

This yields that

\[
\limsup_{n \to \infty} J_n(t_n) \leq J(t_*),
\]
\[ J_n(s) \to J(s) \quad \text{a.e. } s \in (t - 2, t). \]

\[ \exists \{\tilde{t}_k\} \subset (t - 2, t_*) \text{ such that } \tilde{t}_k \to t_*, \text{ and} \]

\[ \lim_{n \to +\infty} J_n(\tilde{t}_k) = J(\tilde{t}_k) \quad \text{for all } k. \]

\( J_n \) are non-increasing, so

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\[ \leq |J_n(\tilde{t}_{k_\delta}) - J(t_*)| \]

\[ \leq |J_n(\tilde{t}_{k_\delta}) - J(\tilde{t}_{k_\delta})| + |J(\tilde{t}_{k_\delta}) - J(t_*)| < \delta. \]

This yields that

\[ \limsup_{n \to \infty} J_n(t_n) \leq J(t_*), \]

and therefore,

\[ \limsup_{n \to \infty} \|u^n(t_n)\| \leq \|u(t_*)\|. \]

Thus, \( u^n(t_n) \to u(t_*) \) strongly in \( V \).
Theorem

There exist the minimal pullback $D^V_F$-attractor

$$\mathcal{A}_{D^V_F} = \{ \mathcal{A}_{D^V_F}(t) : t \in \mathbb{R} \},$$

and the minimal pullback $D^H_{\mu,V}$-attractor

$$\mathcal{A}_{D^H_{\mu,V}} = \{ \mathcal{A}_{D^H_{\mu,V}}(t) : t \in \mathbb{R} \}$$

for the process $U : \mathbb{R}^2_d \times V \to V$, and

$$\mathcal{A}_{D^V_F}(t) \subset \mathcal{A}_{D^H_F}(t) \subset \mathcal{A}_{D^H_{\mu}}(t) = \mathcal{A}_{D^H_{\mu,V}}(t) \quad \text{for all } t \in \mathbb{R},$$

In particular, the following pullback attraction result in $V$ holds:

$$\lim_{\tau \to -\infty} \text{dist}_V(U(t, \tau)D(\tau), \mathcal{A}_{D^H_{\mu}}(t)) = 0 \quad \text{for all } t \in \mathbb{R} \text{ and any } \hat{D} \in D^H_{\mu}.$$
Finally, if moreover $f$ satisfies

$$\sup_{s \leq 0} \left( e^{-\mu s} \int_{-\infty}^{s} e^{\mu \theta} |f(\theta)|^2 \, d\theta \right) < +\infty,$$

then (from $\rho_i$, $i = 1, 2$)

$$\mathcal{A}_{D_F}(t) = \mathcal{A}_{D_H}(t) = \mathcal{A}_{D_H}(t) = \mathcal{A}_{D_H}(t) = \mathcal{A}_{D_H}(t) \quad \text{for all } t \in \mathbb{R},$$

and for any bounded subset $B$ of $H$

$$\lim_{\tau \to -\infty} \text{dist}_V(U(t, \tau)B, \mathcal{A}_{D_H}(t)) = 0 \quad \text{for all } t \in \mathbb{R}.$$
Remark (Infinitely many bigger universes)

If \( f \in L^2_{\text{loc}}(\mathbb{R}; H) \) satisfies \( \int_{-\infty}^{0} e^{\mu s} |f(s)|^2 \, ds < +\infty \), then

\[
\int_{-\infty}^{0} e^{\sigma s} |f(s)|^2 \, ds < +\infty, \quad \text{for all } \sigma \in (\mu, 2\nu \lambda_1).
\]

Thus, for any \( \sigma \in (\mu, 2\nu \lambda_1) \), \( \exists \mathcal{D}^H_\sigma \)-pullback attractor, \( A_{\mathcal{D}^H_\sigma} \).
Remark (Infinitely many bigger universes)

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\]

Thus, for any \( \sigma \in (\mu, 2\nu \lambda_1) \), \( \exists \mathcal{D}^H_\sigma \)-pullback attractor, \( \mathcal{A}_{\mathcal{D}^H_\sigma} \).

Since \( \mathcal{D}^H_\mu \subset \mathcal{D}^H_\sigma \), by comparison, for any \( t \in \mathbb{R} \),

\[
\mathcal{A}_{\mathcal{D}^H_\mu}(t) \subset \mathcal{A}_{\mathcal{D}^H_\sigma}(t) \quad \text{for all } \sigma \in (\mu, 2\nu \lambda_1).
\]

Moreover, if \( f \) satisfies \( \sup_{s \leq 0} \left( e^{-\mu s} \int_{-\infty}^{s} e^{\mu \theta} |f(\theta)|^2 \, d\theta \right) < +\infty \), then, comparing with the \( \mathcal{D}^H_\mathcal{F} \) attractor,

\[
\mathcal{A}_{\mathcal{D}^H_\mathcal{F}}(t) = \mathcal{A}_{\mathcal{D}^H_\mu}(t) = \mathcal{A}_{\mathcal{D}^H_\sigma}(t) \quad \text{for all } t \in \mathbb{R}, \text{ and any } \sigma \in (\mu, 2\nu \lambda_1).
\]
Tempered behaviour of the pullback attractors

The pullback attractor $A_{D^H} \in D^H$, i.e. one has that

$$\lim_{t \to -\infty} \left( e^{\mu t} \sup_{v \in A_{D^H}(t)} |v|^2 \right) = 0.$$

Proposition

$f \in L^2_{\text{loc}}(\mathbb{R}; H)$: $\sup_{s \leq 0} \left( e^{-\mu s} \int_{-\infty}^{s} e^{\mu \theta} |f(\theta)|^2 d\theta \right) < +\infty$,

$\hat{D} \in D^H_{\mu} \text{ invariant w.r.t. } U: D(t) = U(t, \tau)D(\tau) \text{ for all } \tau \leq t$.

Then,

$$\lim_{t \to -\infty} \left( e^{\mu t} \sup_{v \in D(t)} \|v\|^2 \right) = 0.$$
Proposition (More a-priori + derivating eqn.)

\( f \in W_{loc}^{1,2}(\mathbb{R}; H): \int_{-\infty}^{0} e^{\mu s} |f(s)|^2 \, ds < +\infty, \) then for each \( t \in \mathbb{R} \) and \( \hat{D} \in D^H_\mu \) there exists \( \tau_1(\hat{D}, t) < t - 3 \) such that

\[ |AU(r, \tau)u_\tau|^2 \leq \rho_6(t) \quad \text{for all} \quad r \in [t-1, t], \, \tau \leq \tau_1(\hat{D}, t), \, u_\tau \in D(\tau), \]

where

\[ \rho_6(t) = \frac{4}{\nu^2}(\rho_5(t) + \max_{r \in [t-1, t]} |f(r)|^2) + \frac{2C^{(\nu)}}{\nu} \rho_1(t) \rho_2(t)^2, \]

with \( \rho_5(t) \) defined by

\[ \rho_5(t) = \left( \rho_4(t) + \frac{1}{\nu \lambda_1} \int_{t-2}^{t} |f'(\theta)|^2 \, d\theta \right) \exp \left( \frac{C_1^2}{\nu} \rho_2(t) \right). \]
Proposition (Above result + estimating $f$)

$f \in W_{loc}^{1,2}(\mathbb{R}; H)$: \( \sup_{s \leq 0} \left( e^{-\mu s} \int_{-\infty}^{s} e^{\mu \theta} |f(\theta)|^2 \, d\theta \right) < +\infty, \)

\[
\lim_{t \to -\infty} \left( e^{\mu t} \int_{t-1}^{t} |f'(\theta)|^2 \, d\theta \right) = 0, \quad \lim_{t \to -\infty} \left( e^{\mu t} |f(t)|^2 \right) = 0.
\]

Then, for every invariant family $\hat{D} \in \mathcal{D}_\mu^H$:

\[
\lim_{t \to -\infty} \left( e^{\mu t} \sup_{v \in D(t)} \|v\|_{(H^2(\Omega))^2}^2 \right) = 0.
\]
Proposition (Above result + estimating $f$)

$f \in W^{1,2}_{loc}(\mathbb{R}; H) : \sup_{s \leq 0} \left( e^{-\mu s} \int_{-\infty}^{s} e^{\mu \theta} |f(\theta)|^2 \, d\theta \right) < +\infty,$

$$\lim_{t \to -\infty} \left( e^{\mu t} \int_{t-1}^{t} |f'(\theta)|^2 \, d\theta \right) = 0, \quad \lim_{t \to -\infty} (e^{\mu t} |f(t)|^2) = 0.$$

Then, for every invariant family $\hat{D} \in \mathcal{D}_{\mu}^H$:

$$\lim_{t \to -\infty} \left( e^{\mu t} \sup_{v \in D(t)} \|v\|_{(H^2(\Omega))^2}^2 \right) = 0.$$ 

Proof: $|f(r)| \leq |f(t - 1)| + \left( \int_{t-1}^{t} |f'(\theta)|^2 \, d\theta \right)^{1/2} \forall r \in [t - 1, t]$. 
A splitting of the solutions into high and low components

A very common technique in the study of the qualitative behaviour of solutions for PDE problems (long-time dynamics):
A splitting of the solutions into high and low components

A very common technique in the study of the qualitative behaviour of solutions for PDE problems (long-time dynamics):

- In the construction of invariant manifolds:
Flattening property: shorter proof of asymp.compact in $V$

A splitting of the solutions into high and low components

A very common technique in the study of the qualitative behaviour of solutions for PDE problems (long-time dynamics):

- In the construction of invariant manifolds:

- Inertial manifolds:
The squeezing property:

The squeezing property:

Determining modes:
• D. A. Jones and E. S. Titi, Upper bounds on the number of determining modes, nodes, and volume elements for the Navier-Stokes equations, *Indiana Univ. Math. J.* 42 (1993), 875–887.
Existence of attractors
Existence of attractors

Existence of attractors


**Definition (Pullback $\hat{D}_0$-flattening property)**

$U$ satisfies the pullback $\hat{D}_0$-flattening property if for any $t \in \mathbb{R}$ and $\epsilon > 0$, there exist $\tau_\epsilon < t$, a finite dimensional subspace $X_\epsilon$ of $X$, and a mapping $P_\epsilon : X \rightarrow X_\epsilon$ such that

$$\bigcup_{\tau \leq \tau_\epsilon} P_\epsilon U(t, \tau)D_0(\tau)$$

is bounded in $X$

$$\| (Id_X - P_\epsilon)U(t, \tau)u^\tau \|_X < \epsilon$$

for any $\tau \leq \tau_\epsilon$, $u^\tau \in D_0(\tau)$. 
Proposition (Flattening implies asymp. compact)

$t \in \mathbb{R}$, sequences $(t \geq) \tau_n \to -\infty$, $x_n \in D_0(\tau_n)$. Then

$\{U(t, \tau_n)x_n : n \geq 1\}$ is relatively compact in $X$ (Banach space).
Proposition (Flattening implies asymp. compact)

$t \in \mathbb{R}$, sequences $(t \geq) \tau_n \to -\infty$, $x_n \in D_0(\tau_n)$. Then
\[ \{U(t, \tau_n)x_n : n \geq 1\} \text{ is relatively compact in } X \text{ (Banach space)}. \]

Proof. Fix $k \geq 1$ (integer), $\exists P_k : X \to X_k$ (fin.dim.subspace of $X$)
\[ \{P_k U(t, \tau_n)x_n\}_{n \geq N_k} \text{ bounded in } X_k \text{ (therefore relatively compact)} \]
Proposition (Flattening implies asymptotic compactness)

$t \in \mathbb{R},$ sequences $(t \geq) \tau_n \to -\infty,$ $x_n \in D_0(\tau_n).$ Then

$\{U(t, \tau_n)x_n : n \geq 1\}$ is relatively compact in $X$ (Banach space).

Proof. Fix $k \geq 1$ (integer), $\exists P_k : X \to X_k$ (fin.dim.subspace of $X$)

$\{P_k U(t, \tau_n)x_n\}_{n \geq N_k}$ bounded in $X_k$ (therefore relatively compact)

$\|(I - P_k)U(t, \tau_n)x_n\|_X \leq 1/(3k)$ for all $n \geq N_k.$
Proposition (Flattening implies asymp. compact)

$t \in \mathbb{R}$, sequences $(t \geq)\tau_n \to -\infty$, $x_n \in D_0(\tau_n)$. Then

\[
\{U(t, \tau_n)x_n : n \geq 1\} \text{ is relatively compact in } X \text{ (Banach space)}.
\]

Proof. Fix $k \geq 1$ (integer), $\exists P_k : X \to X_k$ (fin.dim.subspace of $X$)

\[
\{P_k U(t, \tau_n)x_n\}_{n \geq N_k} \text{ bounded in } X_k \text{ (therefore relatively compact)}
\]

\[
\|(I - P_k)U(t, \tau_n)x_n\|_X \leq 1/(3k) \text{ for all } n \geq N_k.
\]

Thus, $\{P Ux_n\} \subset \bigcup_{i=1}^M B_{X_k}(P Ux_i, 1/(3k))$ (reordering)
Pullback $\hat{D}_0$-flattening $\Rightarrow$ pullback $\hat{D}_0$-asymptotic compact

Proposition (Flattening implies asymp.compact)

$t \in \mathbb{R}$, sequences $(t \geq)\tau_n \to -\infty$, $x_n \in D_0(\tau_n)$. Then
\{ $U(t, \tau_n)x_n : n \geq 1$ \} is relatively compact in $X$ (Banach space).

Proof. Fix $k \geq 1$ (integer), $\exists P_k : X \to X_k$ (fin.dim.subspace of $X$)
\{ $P_k U(t, \tau_n)x_n$ \}_{n \geq N_k} bounded in $X_k$ (therefore relatively compact)
$\| (I - P_k) U(t, \tau_n)x_n \|_X \leq 1/(3k)$ for all $n \geq N_k$.

Thus, \{ $P U x_n$ \} $\subset \bigcup_{i=1}^M B_{X_k}(P U x_i, 1/(3k))$ (reordering)
$\Rightarrow \| U x_n - U x_i \| \leq \| P U x_n - P U x_i \| + \| Q U x_n \| + \| Q U x_i \| \leq 1/k$
Pullback $\hat{D}_0$-flattening $\Rightarrow$ pullback $\hat{D}_0$-asymptotic compact

**Proposition (Flattening implies asymp. compact)**

$t \in \mathbb{R}$, sequences $(t \geq) \tau_n \to -\infty$, $x_n \in D_0(\tau_n)$. Then

$\{U(t, \tau_n)x_n : n \geq 1\}$ is relatively compact in $X$ (Banach space).

**Proof.** Fix $k \geq 1$ (integer), $\exists P_k : X \to X_k$ (fin.dim.subspace of $X$)

$\{P_k U(t, \tau_n)x_n\}_{n \geq N_k}$ bounded in $X_k$ (therefore relatively compact)

$\| (I - P_k) U(t, \tau_n)x_n \|_X \leq 1/(3k)$ for all $n \geq N_k$.

Thus, $\{P U x_n\} \subset \bigcup_{i=1}^M B_{X_k}(P U x_i, 1/(3k))$ (reordering)

$\Rightarrow \| U x_n - U x_i \| \leq \| P U x_n - P U x_i \| + \| Q U x_n \| + \| Q U x_i \| \leq 1/k$

$\{U x_n\} \subset \bigcup_{i=1}^M B_X(U x_i, 1/k)$ (get a ball with infinite elements)
Pullback $\hat{D}_0$-flattening $\Rightarrow$ pullback $\hat{D}_0$-asymptotic compact

Proposition (Flattening implies asymp.compact)

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$\{P_k U(t, \tau_n)x_n\}_{n \geq N_k}$ bounded in $X_k$ (therefore relatively compact)

$\|(I - P_k)U(t, \tau_n)x_n\|_X \leq 1/(3k)$ for all $n \geq N_k$.

Thus, $\{PUx_n\} \subset \bigcup_{i=1}^{M} B_{X_k}(PUx_i, 1/(3k))$ (reordering)

$\Rightarrow \|Ux_n - Ux_i\| \leq \|PUx_n - PUx_i\| + \|QUx_n\| + \|QUx_i\| \leq 1/k$

$\{Ux_n\} \subset \bigcup_{i=1}^{M} B_X(Ux_i, 1/k)$ (get a ball with infinite elements)

$\{U(t, \tau_n)x_n : n \geq 1\}$ possesses a Cauchy subseq. in $X$ (Banach)
If \( f \in L^2_{\text{loc}}(\mathbb{R}; H) \) satisfies \( \int_{-\infty}^{0} e^{\mu s} |f(s)|^2 ds < \infty \) for some \( \mu \in (0, 2\nu \lambda_1) \), then, for any \( t \in \mathbb{R} \),

\[
\lim_{\rho \to \infty} e^{-\rho t} \int_{-\infty}^{t} e^{\rho s} |f(s)|^2 \, ds = 0.
\]

**Proposition**

For any \( \varepsilon > 0 \) and \( t \in \mathbb{R} \), there exists \( m = m(\varepsilon, t) \in \mathbb{N} \) such that for any \( \hat{D} \in D_H^\mu \), the projection \( P_m : V \to V_m := \text{span}[w_1, \ldots, w_m] \) satisfies the following properties:

\[
\{ P_m U(t, \tau) D(\tau) : \tau \leq \tau_1(\hat{D}, t) \}
\]

is bounded in \( V \),

and

\[
\|(I - P_m) U(t, \tau) u_{\tau}\| < \varepsilon \quad \text{for any} \ \tau \leq \tau_1(\hat{D}, t), \ u_{\tau} \in D(\tau),
\]

**Proof:** Recall the strong estimates we had...
\( \forall t \in \mathbb{R}, \hat{D} \in D_{\mu}^H, \exists \tau_1(\hat{D}, t) < t - 2 \) s. t. \( \forall \tau \leq \tau_1(\hat{D}, t), u_\tau \in D(\tau) \)

\[
|u(r; \tau, u_\tau)|^2 \leq R_1^2(t) \quad \forall r \in [t - 2, t],
\]

\[
\|u(r; \tau, u_\tau)\|^2 \leq R_2^2(t) \quad \forall r \in [t - 1, t],
\]

\[
\nu \int_{t-1}^{t} |Au(\theta; \tau, u_\tau)|^2 d\theta \leq R_3^2(t),
\]
\( \forall t \in \mathbb{R}, \hat{D} \in \mathcal{D}_H^{\mu}, \exists \tau_1(\hat{D}, t) < t - 2 \text{ s. t. } \forall \tau \leq \tau_1(\hat{D}, t), u_\tau \in D(\tau) \)

\[
|u(r; \tau, u_\tau)|^2 \leq R_1^2(t) \ \forall r \in [t - 2, t],
\]

\[
\|u(r; \tau, u_\tau)\|^2 \leq R_2^2(t) \ \forall r \in [t - 1, t],
\]

\[
\nu \int_{t-1}^{t} |Au(\theta; \tau, u_\tau)|^2 d\theta \leq R_3^2(t),
\]

where

\[
R_1^2(t) = 1 + e^{-\mu(t-2)}(2\nu \lambda_1 - \mu)^{-1} \int_{-\infty}^{t} e^{\mu\theta} |f(\theta)|^2 d\theta,
\]

\[
R_2^2(t) = \nu^{-1} \left( R_1^2(t) + (\nu^{-1} \lambda_1^{-1} + 2) \int_{t-2}^{t} |f(\theta)|^2 d\theta \right) \times \exp \left[ 2\nu^{-1} C^{(\nu)} R_1^2(t) \left( R_1^2(t) + \nu^{-1} \lambda_1^{-1} \int_{t-2}^{t} |f(\theta)|^2 d\theta \right) \right],
\]

\[
R_3^2(t) = R_2^2(t) + 2\nu^{-1} \int_{t-1}^{t} |f(\theta)|^2 d\theta + 2C^{(\nu)} R_1^2(t) R_2^4(t).
\]
\( \{w_j\}_{j \geq 1} \) special basis \( \Rightarrow \) \( P_m \) non-expansive in \( V \)
\( \Rightarrow \) \( \{P_m U(t, \tau) D(\tau) : \tau \leq \tau_1(\hat{D}, t)\} \) bounded in \( V \) \( \forall m \geq 1. \)
\{w_j\}_{j \geq 1} \text{ special basis} \Rightarrow P_m \text{ non-expansive in } V
\Rightarrow \{P_m U(t, \tau)D(\tau) : \tau \leq \tau_1(\hat{D}, t)\} \text{ bounded in } V \quad \forall m \geq 1.

q_m(r) = u(r) - P_m u(r) \quad \text{and the second energy equality}

\frac{1}{2} \frac{d}{dr} \|q_m(r)\|^2 + \nu |Aq_m(r)|^2 = -b(u(r), u(r), Aq_m(r)) + (f(r), Aq_m(r))

\leq \frac{\nu}{2} |Aq_m(r)|^2 + \frac{1}{\nu} |f(r)|^2 + \frac{C_1^2}{\nu} R_1(t)R_2^2(t)|Au(r)| \quad \text{a.e. } t - 1 < r < t.
\{w_j\}_{j \geq 1} \text{ special basis } \Rightarrow P_m \text{ non-expansive in } V \\
\Rightarrow \{P_m U(t, \tau) D(\tau) : \tau \leq \tau_1(\hat{D}, t)\} \text{ bounded in } V \ \forall m \geq 1.

q_m(r) = u(r) - P_m u(r) \quad \text{and the second energy equality}

\frac{1}{2} \frac{d}{dr} \|q_m(r)\|^2 + \nu |Aq_m(r)|^2 = -b(u(r), u(r), Aq_m(r)) + (f(r), Aq_m(r))

\leq \frac{\nu}{2} |Aq_m(r)|^2 + \frac{1}{\nu} |f(r)|^2 + \frac{C_1^2}{\nu} R_1(t) R_2^2(t) |Au(r)| \text{ a.e. } t - 1 < r < t.

|Aq_m(r)|^2 \geq \lambda_{m+1} \|q_m(r)\|^2, \text{ implies that (a.e. } t - 1 < r < t)

\frac{d}{dr} \|q_m(r)\|^2 + \nu \lambda_{m+1} \|q_m(r)\|^2 \leq 2 \nu^{-1} |f(r)|^2 + 2 C_1^2 \nu^{-1} R_1(t) R_2^2(t) |Au(r)|
Multiplying by $e^{\nu \lambda m + 1 r}$, integrating from $t - 1$ to $t$, 

\[
\left\| q_m(t) \right\|_2 \leq e^{\nu \lambda m + 1 (t - 1)} \left\| u(t - 1) \right\|_2 + 2 \nu - 1 \int_{t - 1}^{t} e^{\nu \lambda m + 1 r} \left| f(r) \right|_2 \, dr + 2 C^2 \frac{1}{\nu - 3/2} R_1(t) R_2^2(t) \left( \int_{t - 1}^{t} e^{2 \nu \lambda m + 1 r} \, dr \right)^{1/2} \left( \int_{t - 1}^{t} \left| Au(r) \right|_2^2 \, dr \right)^{1/2}.
\]
Multiplying by $e^{\nu \lambda_m + 1 r}$, integrating from $t - 1$ to $t$,

\[
e^{\nu \lambda_m + 1 t} \| q_m(t) \|^2 \leq e^{\nu \lambda_m + 1 (t-1)} \| q_m(t - 1) \|^2 + 2\nu^{-1} \int_{t-1}^{t} e^{\nu \lambda_m + 1 r} |f(r)|^2 \, dr \]

\[
+ 2C_2^2 \nu^{-1} R_1(t) R_2^2(t) \int_{t-1}^{t} e^{\nu \lambda_m + 1 r} |Au(r)| \, dr \]

\[
\leq e^{\nu \lambda_m + 1 (t-1)} \| u(t - 1) \|^2 + 2\nu^{-1} \int_{t-1}^{t} e^{\nu \lambda_m + 1 r} |f(r)|^2 \, dr \]

\[
+ 2C_2^2 \nu^{-1} R_1(t) R_2^2(t) \left( \int_{t-1}^{t} e^{2\nu \lambda_m + 1 r} \, dr \right)^{1/2} \left( \int_{t-1}^{t} |Au(r)|^2 \, dr \right)^{1/2} \]

\[
\leq e^{\nu \lambda_m + 1 (t-1)} R_2^2(t) + 2\nu^{-1} \int_{t-1}^{t} e^{\nu \lambda_m + 1 r} |f(r)|^2 \, dr \]

\[
+ 2C_2^2 \nu^{-3/2} R_1(t) R_2^2(t) R_3(t) (2\nu \lambda_m + 1)^{-1/2} e^{\nu \lambda_m + 1 t} .
\]
Multiplying by $e^{\nu \lambda_{m+1} r}$, integrating from $t - 1$ to $t$,

\[
e^{\nu \lambda_{m+1} t} \| q_m(t) \|^2 \leq e^{\nu \lambda_{m+1} (t-1)} \| q_m(t - 1) \|^2 + 2 \nu^{-1} \int_{t-1}^{t} e^{\nu \lambda_{m+1} r} |f(r)|^2 \, dr \\
+ 2 C_2^2 \nu^{-1} R_1(t) R_2^2(t) \int_{t-1}^{t} e^{\nu \lambda_{m+1} r} |Au(r)| \, dr \\
\leq e^{\nu \lambda_{m+1} (t-1)} \| u(t - 1) \|^2 + 2 \nu^{-1} \int_{t-1}^{t} e^{\nu \lambda_{m+1} r} |f(r)|^2 \, dr \\
+ 2 C_2^2 \nu^{-1} R_1(t) R_2^2(t) \left( \int_{t-1}^{t} e^{2 \nu \lambda_{m+1} r} \, dr \right)^{1/2} \left( \int_{t-1}^{t} |Au(r)|^2 \, dr \right)^{1/2} \\
\leq e^{\nu \lambda_{m+1} (t-1)} R_2^2(t) + 2 \nu^{-1} \int_{t-1}^{t} e^{\nu \lambda_{m+1} r} |f(r)|^2 \, dr \\
+ 2 C_2^2 \nu^{-3/2} R_1(t) R_2^2(t) R_3(t) (2 \nu \lambda_{m+1})^{-1/2} e^{\nu \lambda_{m+1} t}.
\]

Since $\lambda_m \to \infty$ as $m \to \infty$, $\exists m = m(\varepsilon, t) \in \mathbb{N}$ s.t. \\
\[
\|(I - P_m)U(t, \tau)u_\tau\| < \varepsilon \forall \tau \leq \tau_1(\hat{D}, t), \, u_\tau \in D(\tau).
\]
Navier-Stokes eqns with delay terms

The functional Navier-Stokes problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f(t) + g(t, u_t) \quad \text{in} \quad \Omega \times (\tau, \infty), \\
\mathrm{div} \ u &= 0 \quad \text{in} \quad \Omega \times (\tau, \infty), \\
u \Delta u + (u \cdot \nabla) u + \nabla p &= f(t) + g(t, u_t) \quad \text{in} \quad \Omega \times (\tau, \infty), \\
\end{align*}
\]

\[
\begin{align*}
u = 0 \quad \text{on} \quad \partial \Omega \times (\tau, \infty), \\
u(x, \tau) &= u^\tau(x), \quad x \in \Omega, \\
u(x, \tau + s) &= \phi(x, s), \quad x \in \Omega, s \in (-h, 0), \\
u_t \quad \text{the function defined a.e. on} \ (-h, 0) \ \text{by the relation} \\
u_t(s) &= u(t + s), \ a.e. \ s \in (-h, 0). \\
\end{align*}
\]
\( C_H = C([-h, 0]; H) \) with norm \( |\varphi|_{C_H} = \max_{s \in [-h, 0]} |\varphi(s)| \),

\( L^2_X = L^2(-h, 0; X) \) for \( X = H, V \).

\( g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2 \) satisfies

(I) \( \forall \xi \in C_H, \mathbb{R} \ni t \mapsto g(t, \xi) \in (L^2(\Omega))^2 \) is measurable,

(II) \( g(t, 0) = 0 \), for all \( t \in \mathbb{R} \),

(III) \( \exists L_g > 0 \) s.t. \( \forall t \in \mathbb{R}, \xi, \eta \in C_H \),

\[ |g(t, \xi) - g(t, \eta)| \leq L_g |\xi - \eta|_{C_H}, \]

(IV) \( \exists C_g > 0 \) s.t. \( \forall \tau \leq t, u, v \in C([\tau - h, t]; H) \),

\[ \int_{\tau}^{t} |g(s, u_s) - g(s, v_s)|^2 ds \leq C_g^2 \int_{\tau - h}^{t} |u(s) - v(s)|^2 ds. \]
Observe that (I) – (III) imply that given $T > \tau$ and $u \in C([\tau - h, T]; H)$, the function $g_u : [\tau, T] \rightarrow (L^2(\Omega))^2$ defined by $g_u(t) = g(t, u_t)$ for all $t \in [\tau, T]$, is measurable and, in fact, belongs to $L^\infty(\tau, T; (L^2(\Omega))^2)$. Then, thanks to (IV), the mapping $G : u \in C([\tau - h, T]; H) \rightarrow g_u \in L^2(\tau, T; (L^2(\Omega))^2)$ has a unique extension to a mapping $\tilde{G}$ which is uniformly continuous from $L^2(\tau - h, T; H)$ into $L^2(\tau, T; (L^2(\Omega))^2)$. From now on, we will denote $g(t, u_t) = \tilde{G}(u)(t)$ for each $u \in L^2(\tau - h, T; H)$, and thus property (IV) will also hold for all $u, v \in L^2(\tau - h, T; H)$. 
Observe that (I) — (III) imply that given $T > \tau$ and $u \in C([\tau - h, T]; H)$, the function $g_u : [\tau, T] \rightarrow (L^2(\Omega))^2$ defined by $g_u(t) = g(t, u_t)$ for all $t \in [\tau, T]$, is measurable and, in fact, belongs to $L^\infty(\tau, T; (L^2(\Omega))^2)$.

Then, thanks to (IV), the mapping

$$
\mathcal{G} : u \in C([\tau - h, T]; H) \rightarrow g_u \in L^2(\tau, T; (L^2(\Omega))^2)
$$

has a unique extension to a mapping $\tilde{\mathcal{G}}$ which is uniformly continuous from $L^2(\tau - h, T; H)$ into $L^2(\tau, T; (L^2(\Omega))^2)$. From now on, we will denote $g(t, u_t) = \tilde{\mathcal{G}}(u)(t)$ for each $u \in L^2(\tau - h, T; H)$, and thus property (IV) will also hold for all $u$, $v \in L^2(\tau - h, T; H)$. 
Definition
A weak solution \( u \in L^2(\tau - h, T; H) \cap L^2(\tau, T; V) \cap L^\infty(\tau, T; H) \) for all \( T > \tau \), with \( u(\tau) = u^\tau \), \( u(t) = \phi(t - \tau) \) a.e. \( t \in (\tau - h, \tau) \),
Definition
A weak solution \( u \in L^2(\tau - h, T; H) \cap L^2(\tau, T; V) \cap L^\infty(\tau, T; H) \) for all \( T > \tau \), with \( u(\tau) = u^\tau \), \( u(t) = \phi(t - \tau) \) a.e. \( t \in (\tau - h, \tau) \), and \( \forall v \in V \), it holds (in \( D'(\tau, \infty) \))\)

\[
\frac{d}{dt}(u(t), v) + \nu \langle Au(t), v \rangle + b(u(t), u(t), v) = \langle f(t), v \rangle + (g(t, u_t), v).
\]

Remark
u weak solution, then \( u' \in L^2(\tau, T; V') \), so \( u \in C([\tau, \infty); H) \).

Energy equality:

\[
|u(t)|^2 + 2\nu \int^t_s \|u(r)\|^2 dr = |u(s)|^2 + 2 \int^t_s [\langle f(r), u(r) \rangle + (g(r, u_r), u(r))] dr
\]

for all \( \tau \leq s \leq t \).
Definition
A strong solution is a weak solution $u$ such that $u \in L^2(\tau, T; D(A)) \cap L^\infty(\tau, T; V)$ for all $T > \tau$.

Remark
If $f \in L^2_{\text{loc}}(\mathbb{R}; (L^2(\Omega))^2)$ and $u$ is a strong solution, then $u' \in L^2(\tau, T; H)$ for all $T > \tau$, and so $u \in C([\tau, \infty); V)$.

Second energy equality:

$$
\|u(t)\|^2 + 2\nu \int_{s}^{t} |Au(r)|^2 \, dr + 2 \int_{s}^{t} b(u(r), u(r), Au(r)) \, dr \\
= \|u(s)\|^2 + 2 \int_{s}^{t} (f(r) + g(r, u_r), Au(r)) \, dr \quad \forall \tau \leq s \leq t.
$$
Theorem

Let us consider $u^\tau \in H$, $\phi \in L^2_H$, $f \in L^2_{loc}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying (I)–(IV).

Then, for each $\tau \in \mathbb{R}$, there exists a unique weak solution $u$.

Moreover, if $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$, then

(a) $u \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; D(A))$ for all $T > \tau + \varepsilon > \tau$.

(b) If $u^\tau \in V$, $u$ is in fact a strong solution.
We may consider the Banach space $C_H$, and the Hilbert space $M_H^2 = H \times L_H^2$ with associated norm
\[
\|(u^\tau, \phi)\|_{M_H^2}^2 = |u^\tau|^2 + \int_{-h}^0 |\phi(s)|^2 \, ds \quad \text{for } (u^\tau, \phi) \in M_H^2.
\]

A fifth assumption on $g$ and $f$ for asymptotic estimates:

(\text{V}) Assume that $\nu \lambda_1 > C_g$, and $\exists \eta \in (0, 2(\nu \lambda_1 - C_g))$ s.t. for any $u \in L^2(\tau - h, t; H)$,
\[
\int_{\tau}^{t} e^{\eta s} |g(s, u_s)|^2 \, ds \leq C_g^2 \int_{\tau-h}^{t} e^{\eta s} |u(s)|^2 \, ds \quad \forall \tau \leq t,
\]
\[
\int_{-\infty}^{0} e^{\eta s} \|f(s)\|^2_* \, ds < \infty.
\]
Definition
For any $\eta > 0$, we will denote by $\mathcal{D}_\eta(C_H)$ the class of all families of nonempty subsets $\hat{D} = \{ D(t) : t \in \mathbb{R} \} \subset \mathcal{P}(C_H)$ such that

$$\lim_{\tau \to -\infty} \left( e^{\eta \tau} \sup_{\varphi \in D(\tau)} |\varphi|_{C_H}^2 \right) = 0.$$ 

Analogously, we will denote by $\mathcal{D}_\eta(M^2_H)$ the class of all families of nonempty subsets $\hat{D} = \{ D(t) : t \in \mathbb{R} \} \subset \mathcal{P}(M^2_H)$ such that

$$\lim_{\tau \to -\infty} \left( e^{\eta \tau} \sup_{(w, \varphi) \in D(\tau)} \|(w, \varphi)\|_{M^2_H}^2 \right) = 0.$$
Theorem

\( f \in L^2_{\text{loc}}(\mathbb{R}; V') \) and \( g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2 \) satisfy (I)–(V).

Then, \( \exists \{A_{DF}(C_H)(t)\}_{t \in \mathbb{R}}, \{A_{D\eta}(C_H)(t)\}_{t \in \mathbb{R}}, \{A_{DF}(M^2_H)(t)\}_{t \in \mathbb{R}}, \) and \( \{A_{D\eta}(M^2_H)(t)\}_{t \in \mathbb{R}}, \) in \( C_H \) and \( M^2_H \) respectively.

\[ j(\{A_{DF}(C_H)(t)\}) \subset \{A_{DF}(M^2_H)(t)\}, \forall t \in \mathbb{R}, \]

and \( j(\{A_{DF}(C_H)(t)\}) = \{A_{DF}(M^2_H)(t)\}, \forall t \in \mathbb{R}, \]

\( j(\{A_{D\eta}(C_H)(t)\}) = \{A_{D\eta}(M^2_H)(t)\}, \forall t \in \mathbb{R}, \]

\( \sup_{s \leq 0} (e^{-\eta s} \int_{s-\infty}^{s} e^{\eta \theta} \|f(\theta)\|^2 \rangle d\theta) < \infty, \)

the inclusions are in fact equalities.
Theorem

$f \in L^2_{\text{loc}}(\mathbb{R}; V')$ and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfy (I)--(V).

Then, $\exists \{A_{DF}(C_H)(t)\}_{t \in \mathbb{R}}, \{A_{D\eta}(C_H)(t)\}_{t \in \mathbb{R}}, \{A_{DF}(M^2_H)(t)\}_{t \in \mathbb{R}},$ and $\{A_{D\eta}(M^2_H)(t)\}_{t \in \mathbb{R}},$ in $C_H$ and $M^2_H$ respectively.

$$A_{DF}(C_H)(t) \subset A_{D\eta}(C_H)(t), \text{ and } A_{DF}(M^2_H)(t) \subset A_{D\eta}(M^2_H)(t) \quad \forall \ t \in \mathbb{R},$$

$$j(A_{DF}(C_H)(t)) \subset A_{DF}(M^2_H)(t) \quad \forall \ t \in \mathbb{R}, \quad \text{and}$$

$$j(A_{D\eta}(C_H)(t)) = A_{D\eta}(M^2_H)(t) \quad \forall \ t \in \mathbb{R},$$

[j the canonical injection of $C_H$ into $M^2_H : j(\varphi) = (\varphi(0), \varphi).$]

If $f$ also satisfies $\sup_{s \leq 0} \left( e^{-\eta s} \int_{-\infty}^{s} e^{\eta \theta} \|f(\theta)\|_{L^*}^2 \, d\theta \right) < \infty,$ the inclusions are in fact equalities.
A modification of Navier-Stokes eqns:

A time-delayed term in the Burgers’ equation was considered

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + (u(t - \rho(t)) \cdot \nabla)u + \nabla p &= f(t) + g(t, u_t) \quad \text{in } \Omega \times (\tau, \infty), \\
\text{div } u &= 0 \quad \text{in } \Omega \times (\tau, \infty), \\
u(t - \rho(t)) \cdot \nabla)u + \nabla p &= f(t) + g(t, u_t) \quad \text{in } \Omega \times (\tau, \infty), \\
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + (u(t - \rho(t)) \cdot \nabla)u + \nabla p &= f(t) + g(t, u_t) \quad \text{in } \Omega \times (\tau, \infty), \\
\text{div } u &= 0 \quad \text{in } \Omega \times (\tau, \infty), \\
u(t - \rho(t)) \cdot \nabla)u + \nabla p &= f(t) + g(t, u_t) \quad \text{in } \Omega \times (\tau, \infty), \\
\end{align*}
\]

where $\Omega \subset \mathbb{R}^2$, $\tau \in \mathbb{R}$, $h > 0$

$u_t$ denotes the delay function $u_t(s) = u(t + s)$

$\rho \in C^1(\mathbb{R}; [0, h])$ with $\rho'(t) \leq \rho^* < 1 \forall t \in \mathbb{R}.$
Interesting features and goal:

("Small delays don’t matter" ... unless in the nonlinearity)

- $u' \in L^{4/3}(V')$ even in 2D
- Lack of uniqueness and more troubles for dynamical systems: see Ball (1997), Kapustyan & Valero (2007), MR & Robinson (2003)...
- Goal here: under slightly better conditions, uniqueness, and (pullback) attractors
- Remarkable fact: special type of (tempered) universes
TRILINEAR TERM AND WEAK SOLUTION:

$|b(u, v, w)| \leq C|u|^{1/2}||u||^{1/2}|v||w|^{1/2}||w||^{1/2} \quad \forall u, v, w \in V.$

Suppose that $u^\tau \in H$, $\phi \in L^2_V$, and $f \in L^2_{loc}(\mathbb{R}; V').$

Remark

$|b(u(t-\rho(t)), u(t), v)| \leq \tilde{C}|u(t-\rho(t)||u(t)||^{1/2}|u(t)||^{1/2}|v||, \quad \forall v \in V$

$1/2 + 1/4 = 3/4 \quad \Rightarrow \quad B(u(\cdot - \rho(\cdot)), u(\cdot)) \in L^{4/3}(\tau, T; V').$

$u' \in L^{4/3}(\tau, T; V') \Rightarrow$

$u \in C([\tau, T]; V') \quad \text{and} \quad u \in C_w([\tau, T]; H) \quad \forall T > \tau$

(whence initial datum $u^\tau \in H$ meaningful).
Existence and uniqueness:

Theorem
*(Existence of weak solution by compactness method)* $u^{\tau} \in H$, $\phi \in L^2_V$, $f \in L^2_{loc}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying assumptions (H1)–(H4). Then, there exists at least one weak solution $u(\cdot; \tau, u^{\tau}, \phi)$. 

Remark *(Uniqueness improving the initial data)* $u^{\tau} \in H$ and $\phi \in L^2_V \cap L^\infty_H$. Then

$$|b(u(t - \rho(t), u(t), v)| \leq C |u(t - \rho(t))|^{1/2} \|u(t - \rho(t))\|^{1/2} \|v\| \times |u(t)|^{1/2} \|u(t)\|^{1/2} \Rightarrow B(u(\cdot - \rho(\cdot), u(\cdot), u(\cdot) \in L^2(\tau, T; V') \text{ for all } T > \tau, \text{ and so } u^t \in L^2(\tau, T; V')$$
Existence and uniqueness:

Theorem
(Existence of weak solution by compactness method) \( u^\tau \in H, \phi \in L^2_V, f \in L^2_{loc}(\mathbb{R}; V'), \) and \( g : \mathbb{R} \times C_H \to (L^2(\Omega))^2 \) satisfying assumptions \((H1)-(H4)\). Then, there exists at least one weak solution \( u(\cdot; \tau, u^\tau, \phi) \).

Remark
(Uniqueness improving the initial data) \( u^\tau \in H \) and \( \phi \in L^2_V \cap L^\infty_H \). Then

\[
|b(u(t - \rho(t)), u(t), v)| \leq C|u(t - \rho(t))|^{1/2}\|u(t - \rho(t))\|^{1/2}\|v\|
\times |u(t)|^{1/2}\|u(t)\|^{1/2} \Rightarrow
\]

\( B(u(\cdot - \rho(\cdot)), u(\cdot)) \in L^2(\tau, T; V') \) for all \( T > \tau \), and so \( u' \in L^2(\tau, T; V') \)
\Rightarrow uniqueness + energy equality
An appropriate concept of (tempered) universe

Definition

We will denote by $D_{\eta}^{H,L^2_H}(H \times (L^2_V \cap L^\infty_H))$ the class of all families of nonempty subsets $\hat{D} = \{ D(t) : t \in \mathbb{R} \} \subset \mathcal{P}(H \times (L^2_V \cap L^\infty_H))$ such that

$$\lim_{\tau \to -\infty} \left( e^{\eta \tau} \sup_{(\zeta,\varphi) \in D(\tau)} (|\zeta|^2 + \|\varphi\|_{L^2_H}^2) \right) = 0.$$ 

Observe that the above definition does not make the most use of the natural norm of $(\zeta,\varphi)$ in $H \times (L^2_V \cap L^\infty_H)$, but just in $H \times L^2_H$. 
Navier-Stokes-Voigt

\[ \Omega \subset \mathbb{R}^3 \text{ bounded domain with smooth (e.g., } C^2) \partial \Omega. \]

\[
\begin{align*}
\frac{\partial}{\partial t} (u - \alpha^2 \Delta u) - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f(t) \text{ in } \Omega \times (\tau, \infty), \\
\text{div } u &= 0 \text{ in } \Omega \times (\tau, \infty), \\
u \Delta u + (u \cdot \nabla) u + \nabla p &= f(t) \text{ in } \Omega \times (\tau, \infty), \\
u \Delta u + (u \cdot \nabla) u + \nabla p &= f(t) \text{ in } \Omega \times (\tau, \infty), \\
\end{align*}
\]

a length scale parameter \( \alpha > 0 \), characterizing the elasticity of the fluid (in the sense that the ratio \( \alpha^2/\nu \) describes the reaction time that is required for the fluid to respond to the applied force)
Motivation NSV

-The Navier-Stokes-Voigt (NSV) model of viscoelastic incompressible fluid was introduced by Oskolkov [LOMI 1973]
-gives an approximate description of the Kelvin-Voigt fluid, [Oskolkov, 1985]
-proposed as a regularization of the 3D-Navier-Stokes with purpose of direct numerical simulations [Cao, Lunasin, Titi, 2006]
-The extra regularizing term $-\alpha^2 \Delta u_t$ changes the parabolic character of the equation, which makes it so that in 3D the problem is well-posed (forward and backward), but one does not observe any immediate smoothing of the solution
-the inviscid equation is the simplified Bardina subgrid scale model of turbulence (relation studied in [Cao, Lunasin, Titi, 2006]
-global compact attractor and estimates on fractal and Hausdorff dim by Kalantarov and Titi [LOMI, 1988; J. Nonlinear Sci. 2009]
-uniform attractors by Yue and Zhong [DCDS-B, 2011]
The autonomous equation \( u + \alpha^2 Au = g \)

For \( g \in V' \), \( \exists! \) solution \( u_g \) (Lax-Milgram)

The mapping \( C : u \in V \mapsto u + \alpha^2 Au \in V' \) is linear and bijective.

\( C^{-1}(H) = D(A) \)

Definition

\( u \) is a weak solution if \( u \) belongs to \( L^2(\tau, T; V) \) for all \( T > \tau \), and

\[
\frac{d}{dt}(u(t) + \alpha^2 Au(t)) + \nu Au(t) + B(u(t)) = f(t), \quad \text{in } D'(\tau, \infty; V'),
\]

\( u(\tau) = u_\tau. \)
Remark
If \( u \in L^2(\tau, T; V) \) for all \( T > \tau \) and satisfies the eqn, then

\[
\nu(\cdot) = u(\cdot) + \alpha^2 A u(\cdot) \in L^2(\tau, T; V') \quad \text{and} \quad \nu' = \frac{dv}{dt} \in L^1(\tau, T; V').
\]

So, \( \nu \in C([\tau, \infty); V') \), and \( u \in C([\tau, \infty); V) \).
In particular, \( u(\tau) = u_\tau \) has a sense.
Moreover, then, \( \nu' \in L^2(\tau, T; V') \), and \( u' \in L^2(\tau, T; V) \).
Thus, \( u \) is a weak solution iff \( u \in C([\tau, \infty); V) \), \( u' \in L^2(\tau, T; V) \) for all \( T > \tau \), and

\[
u(t) = u(t) + \alpha^2 A u(t) + \int_{\tau}^{t} (\nu A u(s) + B(u(s))) \, ds = u_\tau + \alpha^2 A u_\tau + \int_{\tau}^{t} f(s) \, ds.
\]

Lemma
If \( u \) is a weak solution, then

\[
\frac{1}{2} \frac{d}{dt} (|u(t)|^2 + \alpha^2 \|u(t)\|^2) + \nu \|u(t)\|^2 = \langle f(t), u(t) \rangle, \quad \text{a.e.} \ t > \tau.
\]
Theorem
Let $f \in L^2_{loc}(\mathbb{R}; V')$ be given. Then, for each $\tau \in \mathbb{R}$ and $u_\tau \in V$, there exists a unique weak solution.
Moreover, if $f \in L^2_{loc}(\mathbb{R}; H)$ and $u_\tau \in D(A)$, then

$$u \in C([\tau, \infty); D(A)), \quad u' \in L^2(\tau, T; D(A)) \text{ for all } T > \tau,$$

and

$$\frac{1}{2} \frac{d}{dt} (\|u(t)\|^2 + \alpha^2 |Au(t)|^2) + \nu |Au(t)|^2 + (B(u(t)), Au(t)) = (f(t), Au(t)), $$
Existence of minimal pullback attractors in $V$ norm

**Lemma**
Assume that $f \in L^2_{loc}(\mathbb{R}; V')$ and $u_\tau \in V$. Then, for any

$$0 < \sigma < 2\nu(\lambda^{-1}_1 + \alpha^2)^{-1},$$

$$\|u(t)\|^2 + \varepsilon\alpha^{-2} \int^t_{\tau} e^{\sigma(s-t)}\|u(s)\|^2 \, ds$$

$$\leq (1 + \alpha^{-2}\lambda^{-1}_1)e^{\sigma(\tau-t)}\|u_\tau\|^2 + \alpha^{-2}\varepsilon^{-1} \int^t_{\tau} e^{\sigma(s-t)}\|f(s)\|_*^2 \, ds$$

for all $t \geq \tau$, where $\varepsilon = \nu - \frac{\sigma}{2}(\lambda^{-1}_1 + \alpha^2)$.

**Definition**
For $\sigma \in (0, 2\nu(\lambda^{-1}_1 + \alpha^2)^{-1})$ s.t. $\int_{-\infty}^{0} e^{\sigma s}\|f(s)\|_*^2 \, ds < \infty$, we will denote by $\mathcal{D}_\sigma^V$ the class of all families of nonempty subsets

$$\hat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(V) \text{ s.t. } \lim_{\tau \to -\infty} (e^{\sigma\tau} \sup_{v \in D(\tau)} \|v\|^2) = 0.$$
Attraction in $D(A)$ norm

Lemma

Assume that $f \in L^2_{loc}(\mathbb{R}; H)$ s.t. $\sup_{r \leq 0} \int_{r-1}^{r} \|f(s)\|^2 ds$. Then, if

$$0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}, \quad \text{and} \quad 0 < \underline{\sigma} < \sigma/3,$$

$$\|u(t)\|^2 + \alpha^2|Au(t)|^2 \leq e^{\sigma(t-t)}(\|u_{\tau}\|^2 + \alpha^2|Au_{\tau}|^2) + 2\varepsilon^{-1} \times \int_{\tau}^{t} e^{\sigma(s-t)}|f(s)|^2 ds + 4C_{\varepsilon} C_{\underline{\sigma}}^3(\sigma - 3\underline{\sigma})^{-1} \left(e^{-3\sigma(t-\tau)}\|u_{\tau}\|^6 + M_{t,\underline{\sigma}}^3\right)$$

for all $t \geq \tau$, where $M_{t,\underline{\sigma}} = \sup_{r \leq t} \int_{-\infty}^{r} e^{\sigma(s-r)}\|f(s)\|^2 ds$.

Definition

For any $\sigma, \underline{\sigma} > 0$, consider the universe $D^{D(A)}_\sigma \cap D^{V}_{\underline{\sigma}}$ formed by

$$\hat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(D(A))$$

such that

$$\lim_{\tau \to -\infty} \left(e^{\sigma\tau} \sup_{v \in D(\tau)} |Av|^2 \right) = \lim_{\tau \to -\infty} \left(e^{\sigma\tau} \sup_{v \in D(\tau)} \|v\|^2 \right) = 0.$$