

A COMPARISON BETWEEN RANDOM AND STOCHASTIC MODELING FOR A SIR MODEL

TOMÁS CARABALLO* AND RENATO COLUCCI

Dpto. Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla, Apdo. de Correos 1160
41080–Sevilla, Spain

(Communicated by the associate editor name)

ABSTRACT. In this article, a random and a stochastic version of a SIR nonautonomous model previously introduced in [19] is considered. In particular, the existence of a random attractor is proved for the random model and the persistence of the disease is analyzed as well. In the stochastic case, we consider some environmental effect on the model, in fact, we assume that one of the coefficients of the system is affected by some stochastic perturbation, and analyze the asymptotic behavior of the solutions. The paper is concluded with a comparison between the two different modeling strategies.

1. Introduction. The study of biological models in nonautonomous and nondeterministic frameworks have attracted the attention of many researcher over the last decades (see [8] [9], [10] and the references therein). In the nondeterministic case, the most used strategies consist in considering stochastic or random perturbation. These two different approaches lead to different modeling, for which different tools are available. In order to describe that, in this article we consider random and stochastic perturbation of the following deterministic model (see [3] for a detailed discussion):

$$\begin{cases} \dot{S}(t, \omega) = q - aS(t) + bI(t) - \gamma \frac{S(t) I(t)}{N(t)}, \\ \dot{I}(t, \omega) = -(a + b + c)I(t) + \gamma \frac{S(t) I(t)}{N(t)}, \\ \dot{R}(t, \omega) = cI(t) - aR(t). \end{cases} \quad (1)$$

The above model has been considered in [19] in a nonautonomous framework (see also [20] for a bifurcation scenario of a similar model with two time-dependent parameters). In particular it deals with the case in which the per capita/capita infection rate varies in time (see Thieme [25] for a more detailed discussion). This can be modeled by introducing a forcing term which can be either time dependent (see [19]) or random.

Here we consider a case in which the forcing term is nondeterministic and can be modeled in two different ways: in a first model (Section 3) we consider a random coefficient and study the problem in the framework of Random Dynamical Systems

2010 *Mathematics Subject Classification.* Primary: 34C11, 34F05; Secondary: 60H10.

Key words and phrases. Epidemiology, stochastic differential equations, random dynamical systems.

Partially supported by FEDER and Ministerio de Economía y Competitividad under grant MTM2015-63723-P and Junta de Andalucía under Proyecto de Excelencia P12-FQM-1492.

* Corresponding author: caraball@us.es.

(RDS for short). This allows not only to introduce a random counterpart of the concept of deterministic global attractor, but also the useful definition of random equilibria (also called random fixed points in [24]). In particular, we will use these tools to study the asymptotic and qualitative behavior of the solutions of the system under investigation. The main definitions and results concerning RDS are briefly recalled in Section 2. In the second case (which is analyzed in Section 4), we consider the system in a stochastic context. First, we consider the situation in which the system is subjected to some environmental effect, in the sense that one of the coefficients of the model is perturbed by an additive noise. This yields a stochastic system with multiplicative noise of the same intensity in all the equations of the system. In the last section of conclusions, we emphasize that the stochastic case may exhibit more modeling problems than the random one because, depending on which coefficient is perturbed by noise, the modeling technique can provide us with a more or less appropriate model to describe the real system.

In this stochastic case, we use the technique based on the construction of conjugated random dynamical systems thanks to an appropriate change of variables involving the Ornstein-Uhlenbeck process. A final comparison between these two strategies is discussed in the last section.

2. Some preliminaries definitions. In this section we review on some basic concepts from the theory of random dynamical systems (for more details see [1, 8, 13] amongst others).

Let $(X, \|\cdot\|_X)$ be a separable Banach space and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where \mathcal{F} is the σ -algebra of measurable subsets of Ω and \mathbb{P} is the probability measure.

We define a flow $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ on the probability space Ω with each θ_t being a mapping $\theta_t : \Omega \rightarrow \Omega$ satisfying

- (1) $\theta_0 = \text{Id}_\Omega$,
- (2) $\theta_s \circ \theta_t = \theta_{s+t}$ for all $s, t \in \mathbb{R}$,
- (3) the mapping $(t, \omega) \mapsto \theta_t \omega$ is measurable,
- (4) the probability measure \mathbb{P} is preserved by θ_t , i.e., $\mathbb{P}(\theta_t^{-1}A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$.

Finally, $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a *metric dynamical system* [1].

Definition 2.1. A stochastic process $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ is said to be a continuous RDS over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ with state space X if $\varphi : [0, +\infty) \times \Omega \times X \rightarrow X$ is $(\mathcal{B}[0, +\infty) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$ -measurable, and for each $\omega \in \Omega$,

- (i) the mapping $\varphi(t, \omega) : X \rightarrow X$, $x \mapsto \varphi(t, \omega)x$ is continuous for every $t \geq 0$;
- (ii) $\varphi(0, \omega)$ is the identity operator on X ;
- (iii) (cocycle property) $\varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \varphi(s, \omega)$ for all $s, t \geq 0$.

Definition 2.2. (i) A random set K is a measurable subset of $X \times \Omega$ with respect to the product σ -algebra $\mathcal{B}(X) \times \mathcal{F}$.

- (ii) The ω -section of a random set K is defined by

$$K(\omega) = \{x : (x, \omega) \in K\}, \quad \omega \in \Omega.$$

In the case that a set $K \subset X \times \Omega$ has closed or compact ω -sections it is a random set as soon as the mapping $\omega \mapsto d(x, K(\omega))$ is measurable (from Ω to $[0, \infty)$) for every $x \in X$, see [13]. Then K will be said to be a closed or

a compact, respectively, random set. It will be assumed that closed random sets satisfy $K(\omega) \neq \emptyset$ for all or at least for \mathbb{P} -almost all $\omega \in \Omega$.

- (iii) A bounded random set $K(\omega) \subset X$ is said to be tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} e^{-\beta t} \sup_{x \in K(\theta_{-t}\omega)} \|x\|_X = 0, \quad \text{for all } \beta > 0;$$

a random variable $\omega \mapsto r(\omega) \in \mathbb{R}$ is said to be tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} e^{-\beta t} \sup_{t \in \mathbb{R}} |r(\theta_{-t}\omega)| = 0, \quad \text{for all } \beta > 0.$$

We regard $\mathcal{D}(X)$ as the set of all tempered random sets of X .

Definition 2.3. A random set $\Gamma(\omega) \subset X$ is called a random absorbing set in $\mathcal{D}(X)$ if for any $K \in \mathcal{D}(X)$ and a.e. $\omega \in \Omega$, there exists $T_K(\omega) > 0$ such that

$$\varphi(t, \theta_{-t}\omega)K(\theta_{-t}\omega) \subset \Gamma(\omega), \quad \forall t \geq T_K(\omega).$$

Definition 2.4. Let $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ be an RDS over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ with state space X and let $\mathcal{A}(\omega) \subset X$ be a random set. Then $\mathcal{A}(\omega)$ is called a global random \mathcal{D} attractor (or pullback \mathcal{D} attractor) for $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ if $\omega \mapsto \mathcal{A}(\omega)$ satisfies

- (i) (random compactness) $\mathcal{A}(\omega)$ is a compact set of X for a.e. $\omega \in \Omega$;
- (ii) (invariance) for a.e. $\omega \in \Omega$ and all $t \geq 0$, it holds

$$\varphi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t\omega);$$

- (iii) (attracting property) for any $K \in \mathcal{D}(X)$ and a.e. $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} \text{dist}_X(\varphi(t, \theta_{-t}\omega)K(\theta_{-t}\omega), \mathcal{A}(\omega)) = 0,$$

where

$$\text{dist}_X(G, H) = \sup_{g \in G} \inf_{h \in H} \|g - h\|_X$$

is the Hausdorff semi-metric for $G, H \subseteq X$.

Proposition 1. [12, 15] *Let $\Gamma \in \mathcal{D}(X)$ be an absorbing set for the continuous random dynamical system $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ which is closed and satisfies the asymptotic compactness condition for a.e. $\omega \in \Omega$, i.e., each sequence $x_n \in \varphi(t_n, \theta_{-t_n}\omega)\Gamma(\theta_{-t_n}\omega)$ has a convergent subsequence in X when $t_n \rightarrow \infty$. Then the cocycle φ has a unique global random attractor with component subsets*

$$\mathcal{A}(\omega) = \bigcap_{\tau \geq T_\Gamma(\omega)} \overline{\bigcup_{t \geq \tau} \varphi(t, \theta_{-t}\omega)\Gamma(\theta_{-t}\omega)}.$$

If the pullback absorbing set is positively invariant, i.e., $\varphi(t, \omega)\Gamma(\omega) \subset \Gamma(\theta_t\omega)$ for all $t \geq 0$, then

$$\mathcal{A}(\omega) = \bigcap_{t \geq 0} \overline{\varphi(t, \theta_{-t}\omega)\Gamma(\theta_{-t}\omega)}.$$

Remark 1. When the state space $X = \mathbb{R}^d$, the asymptotic compactness follows trivially. Note that the random attractor is path-wise attracting in the pullback sense, but does not need to be path-wise attracting in the forward sense, although it is forward attracting in probability, due to some possible large deviations, see e.g., Arnold [1].

3. Random System. The random system we are concerned with is the following:

$$\begin{cases} \dot{S}(t, \omega) = q(\theta_t \omega) - aS(t) + bI(t) - \gamma \frac{S(t)I(t)}{N(t)}, \\ \dot{I}(t, \omega) = -(a + b + c)I(t) + \gamma \frac{S(t)I(t)}{N(t)}, \\ \dot{R}(t, \omega) = cI(t) - aR(t), \end{cases} \quad (2)$$

where a, b, c, γ are positive constants, $N(t) = S(t) + R(t) + I(t)$ is the total population and where

$$q(\theta_t \omega) \in q_0[1 - \varepsilon, 1 + \varepsilon], \quad q_0 > 0, \quad \varepsilon \in (0, 1). \quad (3)$$

Examples of this kind of noise can be found, for instance, in [2]. Given an Ornstein-Uhlenbeck process Z_t (see [8, 11]), the following processes

$$\xi_1(t) = q_0 \left(1 - 2\varepsilon \frac{Z_t}{1 + Z_t^2} \right),$$

and

$$\xi_2(t) = q_0 \left(1 - \frac{2\varepsilon}{\pi} \arctan Z_t \right),$$

take values in $q_0[1 - \varepsilon, 1 + \varepsilon]$, the first one peaking around $q_0[1 \pm \varepsilon]$ and the other centering at q_0 .

Although it is possible to consider a more general model in which some other coefficients can be also random, for simplicity in our analysis, we have preferred to consider just one of them random because this is enough to show how the technique works.

First of all we prove that solutions corresponding to nonnegative initial conditions remain nonnegative, that is:

Lemma 3.1. *The set*

$$\mathbb{R}_+^3 = \{(S, I, R) \in \mathbb{R}^3 : S \geq 0, I \geq 0, R \geq 0\},$$

is positively invariant for the system (2), for each fixed $\omega \in \Omega$.

Proof. We quickly verify that the vector field, at the boundary of \mathbb{R}_+^3 , points inwards.

On the plane $S = 0$ we have that $\dot{S} > 0$, the plane $I = 0$ is invariant since on it we have $\dot{I} = 0$ while on $R = 0$ we have $\dot{R} \geq 0$.

The positive S -semi axes is invariant, in fact we have:

$$\dot{S}(t, \omega) = q(\theta_t \omega) - aS(t),$$

that is

$$S(t, \omega) = S_0 e^{-a(t-t_0)} + e^{-at} \int_{t_0}^t q(\theta_s \omega) e^{as} ds. \quad (4)$$

The last term is bounded, in fact

$$q_0(1 - \varepsilon) e^{-at} \int_{t_0}^t e^{as} ds \leq e^{-at} \int_{t_0}^t q(\theta_s \omega) e^{as} ds \leq q_0(1 + \varepsilon) e^{-at} \int_{t_0}^t e^{as} ds,$$

that is

$$q_0(1 - \varepsilon)(1 - e^{-a(t-t_0)}) \leq a e^{-at} \int_{t_0}^t q(\theta_s \omega) e^{as} ds \leq q_0(1 + \varepsilon)(1 - e^{-a(t-t_0)}).$$

If we start on the positive R -semi axes we have that the solution enters the plane $I = 0$, while on the positive I -semi axes we have $\dot{S}, \dot{R} > 0$. \square

If we replace ω by $\theta_{-t}\omega$ in (4) and set $t_0 = 0$, we obtain

$$S(t; \omega, S_0) = S_0 e^{-at} + \int_{-t}^0 q(\theta_p \omega) e^{ap} dp, \quad (5)$$

which, for $t \rightarrow \infty$, pullback converges to

$$S_*(\omega) := \int_{-\infty}^0 q(\theta_p \omega) e^{ap} dp. \quad (6)$$

We observe that $S_*(\omega)$ is positive and bounded. In fact, for any $\omega \in \Omega$, we have:

$$q_0(1 - \varepsilon) \leq S_*(\omega) \leq q_0(1 + \varepsilon). \quad (7)$$

For simplicity we set $u(t) = (S(t), I(t), R(t)) \in \mathbb{R}_+^3$. Then we have:

Lemma 3.2. *For any $\omega \in \Omega$, $t_0 \in \mathbb{R}$ and any initial data $u_0 = (S(t_0), I(t_0), R(t_0)) \in \mathbb{R}_+^3$, system (2) admits a unique bounded solution $u(\cdot; t_0, \omega, u_0) \in C([t_0, +\infty), \mathbb{R}_+^3)$ with $u(t_0; t_0, \omega, u_0) = u_0$ provided that (3) is fulfilled.*

Moreover the solution generates a random dynamical system $\varphi(t, \omega)(\cdot)$ defined as

$$\varphi(t, \omega)u_0 = u(t; 0, \omega, u_0), \quad \forall t \geq 0, u_0 \in \mathbb{R}_+^3, \omega \in \Omega.$$

Proof. The system can be rewritten in the following form:

$$\dot{u}(t) = F(u, \theta_t \omega),$$

where $F(u, \theta_t \omega)$ is the right hand side of (2). Since $q(\theta_t \omega)$ is continuous with respect to t , the function $F(\cdot, \theta_t \omega) \in C(\mathbb{R}_+^3 \times [t_0, +\infty), \mathbb{R}_+^3)$ and is continuously differentiable with respect to (S, I, R) . Then, by classical results about ordinary differential equations we have that system (2) possesses a unique local solution.

If we sum the equations of the system we obtain

$$\dot{N}(t, \omega) = q(\theta_t \omega) - aN(t),$$

whose solution satisfying $N(t_0) = N_0$ is given by

$$N(t; t_0, \omega, N_0) = N_0 e^{-a(t-t_0)} + e^{-at} \int_{t_0}^t q(\theta_s \omega) e^{as} ds. \quad (8)$$

As in the proof of the previous lemma we have

$$q_0(1-\varepsilon) + [N_0 - q_0(1-\varepsilon)]e^{-a(t-t_0)} \leq N(t; t_0, \omega, N_0) \leq q_0(1+\varepsilon) + [N_0 - q_0(1+\varepsilon)]e^{-a(t-t_0)}, \quad (9)$$

from which we have that the solutions are bounded. Moreover, both forward and backward limits of $N(t; t_0, \omega, N_0)$ satisfy:

$$\begin{aligned} \lim_{t \rightarrow +\infty} N(t; t_0, \omega, N_0) &\in [q_0(1 - \varepsilon), q_0(1 + \varepsilon)], & \forall t_0 \in \mathbb{R}, \\ \lim_{t_0 \rightarrow -\infty} N(t; t_0, \omega, N_0) &\in [q_0(1 - \varepsilon), q_0(1 + \varepsilon)], & \forall t \in \mathbb{R}. \end{aligned}$$

Then the local solution can be extended to a global one $u(\cdot; t_0, \omega, u_0) \in C^1([t_0, \infty), \mathbb{R}_+^3)$.

It is easy to see that

$$u(t + t_0; t_0, \omega, u_0) = u(t; 0, \theta_{t_0} \omega, u_0),$$

for all $t_0 \in \mathbb{R}$, $t \geq 0$, $\omega \in \Omega$, $u_0 \in \mathbb{R}_+^3$. Then we can define a map $\varphi(t, \omega)(\cdot)$ which is a random dynamical system:

$$\varphi(t, \omega)u_0 = u(t; 0, \omega, u_0), \quad \forall t \geq 0, u_0 \in \mathbb{R}_+^3, \omega \in \Omega.$$

□

Proposition 2. *For each $\omega \in \Omega$ there exists a tempered bounded closed random absorbing set $\Gamma(\omega) \in \mathcal{D}(\mathbb{R}_+^3)$ of the random dynamical system $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ such that for any $K \in \mathcal{D}(\mathbb{R}_+^3)$ and each $\omega \in \Omega$ there exists a $T_K(\omega) > 0$ such that*

$$\varphi(t, \theta_{-t}\omega)K(\theta_{-t}\omega) \subset \Gamma(\omega), \quad \forall t \geq T_K(\omega).$$

In detail for any $0 < \eta < q_0(1 - \varepsilon)$, the set $\Gamma(\omega)$ can be chosen as the deterministic set

$$\Gamma_\eta := \{(S, I, R) \in \mathbb{R}_+^3 : q_0(1 - \varepsilon) - \eta \leq S + I + R \leq q_0(1 + \varepsilon) + \eta\},$$

for all $\omega \in \Omega$.

Proof. Using (8) and (9) we deduce that $\dot{N}(t, \omega) \leq 0$ on $N = q_0(1 + \varepsilon) + \eta$ while $\dot{N}(t, \omega) \geq 0$ on $N = q_0(1 - \varepsilon) - \eta$ for all $\eta \in [0, q_0(1 - \varepsilon)]$. Then Γ_η is positively invariant for $\eta \in [0, q_0(1 - \varepsilon)]$.

Now suppose that $N_0 \geq q_0(1 + \varepsilon) + \eta$ (the other case is similar), then

$$N(t; t_0, \omega, N_0) \leq N_0(\omega)e^{-a(t-t_0)} + q_0(1 + \varepsilon)[1 - e^{-a(t-t_0)}], \quad (10)$$

we replace ω by $\theta_{-t}\omega$ and obtain

$$N(t; \theta_{-t}\omega, N_0(\theta_{-t}\omega)) \leq \sup_{N_0 \in K(\theta_{-t}\omega)} N_0 e^{-a(t-t_0)} + q_0(1 + \varepsilon)[1 - e^{-a(t-t_0)}]. \quad (11)$$

Thanks to the previous inequality, there exists a time $T_K(\omega)$ such that for $t > T_K(\omega)$, $\varphi(t, \theta_{-t}\omega)u_0 \in \Gamma_\eta$ for all $u_0 \in K(\theta_{-t}\omega)$. That is, the set Γ_η is compact and absorbing for all $\eta \in (0, q_0(1 - \varepsilon))$, and absorbs all tempered random sets of \mathbb{R}_+^3 and in particular its bounded sets. \square

Using the above results we deduce:

Theorem 3.3. *The random dynamical system generated by system (2) possesses a global random attractor.*

If we set $t_0 = 0$ and replace ω by $\theta_{-t}\omega$ in (8) we have

$$N(t; \theta_{-t}\omega, N_0) = N_0 e^{-at} + \int_{-t}^0 q(\theta_p\omega) e^{ap} dp. \quad (12)$$

It is easy to see that the solution (8) both forward and pullback converges to

$$N_*(\omega) = \int_{-\infty}^0 q(\theta_p\omega) e^{ap} dp. \quad (13)$$

In details: the pullback limit reads as

$$|N(t; \theta_{-t}\omega, N_0) - N_*(\omega)| \rightarrow 0 \quad \text{for } t \rightarrow \infty,$$

while the forward limit is defined as

$$|N(t; \omega, N_0) - N_*(\theta_t\omega)| \rightarrow 0 \quad \text{for } t \rightarrow \infty.$$

We observe that the expression of (13) coincides with that of (7) on the S -axes. Since the computation is done with respect to the total population $N = S + I + R$, it is natural to wonder what happens to the three populations individually. In order to qualitatively describe the asymptotic behavior of (S, I, R) , we replace, in system (2), $N(t)$ by its forward limit $N_*(\theta_t\omega)$. To make the computations clearer to the reader, we proceed in this informal way, however the rigorous proof by taking

approximations of $N_*(\theta_t\omega)$ for large values of t can also be carried out. Then, we obtain the following SI random model:

$$\begin{cases} \dot{S}(t, \omega) = q(\theta_t\omega) - aS(t) + bI(t) - \alpha(\theta_t\omega)S(t) I(t), \\ \dot{I}(t, \omega) = -(a + b + c)I(t) + \alpha(\theta_t\omega)S(t) I(t), \end{cases} \quad (14)$$

where

$$\alpha(\omega) = \frac{\gamma}{N_*(\omega)}.$$

We observe that the random parameter $\alpha(\theta_t\omega)$ is bounded:

$$\frac{\gamma}{q_0(1 + \varepsilon)} \leq \alpha(\theta_t\omega) \leq \frac{\gamma}{q_0(1 - \varepsilon)}, \quad \text{for any } \omega \in \Omega, \quad (15)$$

and that \mathbb{R}_+^2 is positively invariant. We set $V = S + I$, then

$$\dot{V} = q(\theta_t\omega) - aV - cI(t),$$

from which

$$\dot{V} \leq q(\theta_t\omega) - aV,$$

and then the forward and pullback limit $v_*(\omega)$ of $V(t)$ satisfies

$$0 < v_*(\omega) \leq V_*(\omega) = \int_{-\infty}^0 e^{ap} q(\theta_p\omega) dp.$$

Moreover

$$V(t; t_0, \omega, V_0) \leq q_0(1 + \varepsilon) + e^{-at}(V_0 - 1).$$

By a similar argument to that used for the SIR model we conclude that the sets

$$B_\eta = \{(S, I) \in \mathbb{R}_+^2 : S + I \leq q_0(1 + \varepsilon) + \eta\}$$

are pullback absorbing for $\eta > 0$ and positively invariant for $\eta \geq 0$. In order to see what happens to the population I in the asymptotic behavior, we can obtain a sufficient condition for the disappearance of the disease.

We consider solutions starting in B_0 . We first observe that the S-axis is invariant and that solutions both forward and pullback converges to S^* on it. Then, by the second equation of the system we have

$$\begin{aligned} \dot{I}(t, \omega) &= -(a + b + c)I(t) + \alpha(\theta_t\omega)S(t) I(t) \\ &\leq I(t)[\alpha(\theta_t\omega)q_0(1 + \varepsilon) - (a + b + c)] \\ &= I(t) \left[\gamma \frac{1 + \varepsilon}{1 - \varepsilon} - (a + b + c) \right], \end{aligned}$$

then, if

$$\gamma \frac{1 + \varepsilon}{1 - \varepsilon} - (a + b + c) < 0, \quad (16)$$

we have that

$$\lim_{t \rightarrow \infty} I(t) = 0.$$

Then the systems tend to a disease free configuration if (16) is satisfied.

By the first equation of the SI system we have

$$\dot{S}(t, \omega) \geq q_0(1 - \varepsilon) - aS + I(t) \left[b - \frac{\gamma}{q_0(1 - \varepsilon)} S \right].$$

Then the set

$$\tilde{B} = \{(S, I) \in B_0 : S \geq q_0(1 - \varepsilon) \max\{1/a, b/\gamma\}\},$$

is positively invariant. Then if we restrict on \tilde{B} we have

$$\dot{I}(t, \omega) = -(a + b + c)I(t) + \alpha(\theta_t \omega)S(t) I(t) \quad (17)$$

$$\geq -(a + b + c)I(t) + \frac{\gamma}{q_0(1 + \varepsilon)}S(t)I(t) \quad (18)$$

$$\geq I(t) \left[\frac{1 - \varepsilon}{1 + \varepsilon} \max\{b, \gamma\} - (a + b + c) \right]. \quad (19)$$

We conclude that, if

$$\gamma > ab, \quad \text{and} \quad \frac{1 - \varepsilon}{1 + \varepsilon} \gamma > a + b + c, \quad (20)$$

the disease persists.

Remark 2. We observe that the conditions for persistence/extinction of the disease are not complementary like in [19] since in the SI model under consideration we have two (randomly) perturbed coefficients (q and α) instead of one (q) as in [19]. In this case the persistence/extinction conditions are based on the upper and lower bound of $\alpha(\cdot)$ (see (15)).

4. Stochastic System. The stochastic model we will consider now is in complete agreement with a largely used approach in the published literature on this topic (see, for instance, [18], [4], [6] and references cited therein) which consists in considering white noise that is directly proportional to the quantities $S(t)$, $I(t)$, $R(t)$ in each equation respectively. Moreover, it can be considered as the result of the environmental noise effect on some of the parameters in the model. For example, if we assume that one of the parameters, say a , is affected by a noisy perturbation of the type $\sigma \dot{W}(t)$, in other words, if we replace in the deterministic model the parameter a by $a - \sigma \dot{W}(t)$, then the model becomes:

$$\begin{cases} dS = \left[q - aS(t) + bI(t) - \gamma \frac{S(t) I(t)}{N(t)} \right] dt + \sigma S \circ dW(t), \\ dI = \left[-(a + b + c)I(t) + \gamma \frac{S(t) I(t)}{N(t)} \right] dt + \sigma I \circ dW(t), \\ dR = [cI(t) - aR(t)] dt + \sigma R \circ dW(t). \end{cases} \quad (21)$$

We remark that this kind of perturbations describing environmental noise has been used in several applied situations as can be seen, for example, in [16], [17], [26]. In fact, if we sum the three equations of the system and set $N = S + I + R$, then we obtain the following perturbed equation for the total population of the system:

$$dN = (q - aN(t)) + \sigma N \circ dW(t),$$

that has been obtained, as we mentioned above, by perturbing coefficient a in each equation by the same noise (see for example [22] or [27]).

Notice that the noise considered for three populations is correlated (following the approach of [22]). Moreover, for simplicity, we have considered that the noise intensity is the same in all the equations but it does not make a substantial difference if we consider different ones in each equation, as well as different mutually independent Wiener processes in each equation. Only the computations will be more complicated, but the technique is the same.

It is easy to see that solutions, corresponding to nonnegative initial data, remain nonnegative and, as a consequence, the model is well defined.

The strategy consists in transforming the stochastic system into a random one with

random coefficients and without white noise. To this end, we introduce the following Ornstein-Uhlenbeck process on $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$

$$z^*(\theta_t \omega) = - \int_{-\infty}^0 e^s \theta_t \omega(s) ds, \quad t \in \mathbb{R}, \quad \omega \in \Omega_0, \quad (22)$$

which solves the following Langevin equation [1, 11]

$$dz = -zdt + dW(t), \quad t \in \mathbb{R}, \quad (23)$$

where $W(t)(\omega) = W(t, \omega) = \omega(t)$ for $\omega \in \Omega$, $t \in \mathbb{R}$, is a two sided Wiener process (see [11] for a more detailed description). Let us now recall some basic properties of z^* which will be used in our analysis ([1, 11]) :

Proposition 3. *There exists a θ_t -invariant set $\tilde{\Omega} \in \mathcal{F}$ of Ω of full \mathbb{P} measure such that for $\omega \in \tilde{\Omega}$, we have*

(i) *the random variable $|z^*(\omega)|$ is tempered, i.e., for $\omega \in \tilde{\Omega}$,*

$$\lim_{t \rightarrow +\infty} e^{-\beta t} \sup_{t \in \mathbb{R}} |z^*(\theta_{-t} \omega)| = 0, \quad \forall \beta > 0;$$

(ii) *the mapping*

$$(t, \omega) \rightarrow z^*(\theta_t \omega) = - \int_{-\infty}^0 e^s \omega(t+s) ds + \omega(t)$$

is a stationary solution of Ornstein-Uhlenbeck equation (23) with continuous trajectories;

(iii) *In addition, for any $\omega \in \tilde{\Omega}$:*

$$\lim_{t \rightarrow \pm\infty} \frac{|z^*(\theta_t \omega)|}{t} = 0; \quad (24)$$

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t z^*(\theta_s \omega) ds = 0; \quad (25)$$

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t |z^*(\theta_s \omega)| ds = \mathbb{E}[z^*] < \infty. \quad (26)$$

Then, in order to obtain the associate random system, we consider the following change of variable:

$$\tilde{S}(t) = S(t)e^{-\sigma z^*(\theta_t \omega)}, \quad \tilde{I}(t) = I(t)e^{-\sigma z^*(\theta_t \omega)}, \quad \tilde{R}(t) = R(t)e^{-\sigma z^*(\theta_t \omega)}.$$

Then the system can be written as

$$\begin{cases} \dot{\tilde{S}}(t, \omega) = qe^{-\sigma z^*} - a\tilde{S}(t) + b\tilde{I}(t) - \gamma \frac{\tilde{S}(t)\tilde{I}(t)}{\tilde{N}(t)} + \sigma \tilde{S}z^*, \\ \dot{\tilde{I}}(t, \omega) = -(a+b+c)\tilde{I}(t) + \gamma \frac{\tilde{S}(t)\tilde{I}(t)}{\tilde{N}(t)} + \sigma \tilde{I}z^*, \\ \dot{\tilde{R}}(t, \omega) = c\tilde{I}(t) - a\tilde{R}(t) + \sigma \tilde{R}z^*. \end{cases} \quad (27)$$

Adding the three equation we obtain

$$\dot{\tilde{N}}(t, \omega) = qe^{-\sigma z^*} - a\tilde{N}(t) + \sigma \tilde{N}z^* = qe^{-\sigma z^*} - (a - \sigma z^*)\tilde{N}. \quad (28)$$

The previous equation has a nontrivial random solution that is both forward and pullback attracting. In fact, for any initial datum N_0 we have:

$$N(t; \omega, N_0) = N_0 e^{-\int_0^t [a - \sigma z^*(\theta_s \omega)] ds} + \int_0^t q e^{-\sigma z^*(\theta_s \omega)} e^{\int_s^t [a - \sigma z^*(\theta_\tau \omega)] d\tau} ds,$$

and replacing ω by $\theta_{-t}\omega$ we obtain

$$N(t; \theta_{-t}\omega, N_0) = N_0 e^{-\int_{-t}^0 [a - \sigma z^*(\theta_p\omega)] dp} + \int_{-t}^0 q e^{-\sigma z^*(\theta_p\omega)} e^{-\int_p^0 [a - \sigma z^*(\theta_q\omega)] dq} dp,$$

which pullback converges for $t \rightarrow +\infty$ to

$$N^*(\omega) = q \int_{-\infty}^0 e^{-\sigma z^*(\theta_p\omega)} e^{-\int_p^0 [a - \sigma z^*(\theta_q\omega)] dq} dp. \quad (29)$$

We observe that $N^*(\omega)$ is well defined since, using ergodicity of z^* , the integrand behaves like e^{ap} for $p \rightarrow -\infty$ due to the following properties:

$$\frac{z^*(\theta_p\omega)}{p}, \quad \frac{1}{p} \int_p^0 z^*(\theta_q\omega) dq \rightarrow 0, \quad \text{for } p \rightarrow -\infty.$$

5. Conclusions and discussions. In this paper we have considered two non-deterministic versions of a non-autonomous system proposed in [19]. In the first case we have considered a random system by replacing the non-autonomous and bounded parameter $q(t)$ by its random and bounded counterpart $q(\theta_t\omega)$. Another possibility analyzed is the stochastic version of the model. We have considered an environmental effect produced on one of the parameters of the systems following the direction of recent published literature on this topic. We have only considered the noisy perturbation on one of the parameters although it may be interesting to consider the same perturbation in other coefficients what can provide different stochastic systems which may yield to a different qualitative behavior. We plan to analyse this aspect in our future investigation.

It is worth noticing that, in both cases under examination we have obtained a random stationary solution which both forward and pullback attracts the solutions. From this point of view we can regard the random system proposed as a good nondeterministic version of the nonautonomous system studied in [19].

From a more general point of view it is also worth emphasizing the following:

- ◊ The stochastic case gives rise to an equivalent random differential system with unbounded random coefficients while the first case of random coefficients is concerned with bounded ones.
- ◊ The differential equation describing the behavior of the total population is similar in both cases and provides us with a stationary process pullback and forward attracting any other solution. However, an important difference is that in the random case of bounded noise, this equilibrium can be bounded (see (13)), what allows for a posteriori control of this special solution in order to validate the model under study. In the stochastic case of unbounded noise (see (29)) we are not able to obtain such estimates and the corresponding additional information.
- ◊ Another important difference is related to the fact that, in the case of bounded noise, the random system preserves its form and structure, in fact, we only replace the time dependent parameter $q(t)$ by a random parameter with a special structure, and consequently, the modeling procedure is not a priori affected, while in the stochastic case, one has to choose carefully which coefficient can be perturbed stochastically in order to preserve the positiveness of solutions. Notice that if the perturbation is imposed on the coefficient q , then the preservation of the positiveness cannot be guaranteed.

REFERENCES

- [1] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, Berlin 1998.
- [2] Y. Asai and P.E. Kloeden, Numerical schemes for random ODEs via stochastic differential equations, *Commun. Appl. Analysis*, vol. **17**, no. 3–4 (2013), 521–528.
- [3] F. Brauer and C. Castillo-Chavez, *Mathematical Models for Communicable Diseases*, Society for Industrial and Applied Mathematics, Series: CBMS-NSF Regional Conference Series in Applied Mathematics (Book 84), December 28, 2012.
- [4] T. Britton, Stochastic epidemic models: A survey, *Mathematical Biosciences*, vol. **225**, no. 1, (2010), 24–35.
- [5] J. L. Brooks and I. D. Stanley, Predation, body size, and composition of plankton, *Science*, 150.3692 (1965), 23–35.
- [6] Y. Cai, X. Wang, W. Wang and M. Zhao, Stochastic Dynamics of an SIRS Epidemic Model with Ratio-Dependent Incidence Rate, *Abstract and Applied Analysis*, vol. 2013, Article ID 172631, 11 pages <http://dx.doi.org/10.1155/2013/172631>.
- [7] T. Caraballo, M.J. Garrido-Atienza, B. Schmalfuß and J. Valero, Non-autonomous and random attractors for delay random semilinear equations without uniqueness, *Discrete Contin. Dyn. Syst.*, vol. **21**, no. 2, (2008), 415–443.
- [8] T. Caraballo, R. Colucci and X. Han, Predation with indirect effects in fluctuating environments, *Nonlinear Dynamics*, vol. **84**, no. 1, (2016), 115–126.
- [9] T. Caraballo, X. Han and P.E. Kloeden, Chemostats with time-dependent inputs and wall growth, *Applied Mathematics and Information Sciences*, vol. **9**, no. 5, (2015), 2283–2296.
- [10] T. Caraballo, X. Han and P.E. Kloeden, Chemostats with random inputs and wall growth, *Mathematical Methods in the Applied Sciences*, vol. **38**, no. 16, (2015), 3538–3550.
- [11] T. Caraballo and K. Lu, Attractors for stochastic lattice dynamical systems with a multiplicative noise, *Front. Math. China*, vol. **3**, no. 3, (2008), 317–335.
- [12] T. Caraballo, G. Lukaszewicz and J. Real, Pullback attractors for asymptotically compact nonautonomous dynamical systems, *Nonlinear Anal.*, vol. **6** (2006), 484–498.
- [13] H. Crauel and P.E. Kloeden, Nonautonomous and random attractors, *Jahresber. Dtsch. Math.-Ver.*, vol. **117**, no. 3, (2015), 173–206.
- [14] J. Cresson, M. Efendiev and S. Sonner, *On the positivity of solutions of systems of stochastic PDEs*, ZAMM, vol. **93**, no. 6–7, (2013), 414–422.
- [15] F. Flandoli and B. Schmalfuß, Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise, *Stochastics Stochastics Rep.*, vol. **59**, no. 1–2, (1996), 21–45.
- [16] C. Jia, D. Jianga, Q. Yanga and N. Shia, Dynamics of a multigroup SIR epidemic model with stochastic perturbation, *Automatica*, vol. **48**, no. 1, (2012), 121–131.
- [17] D. Jiang, J. Yua, C. Ji and N. Shi, Asymptotic behavior of global positive solution to a stochastic SIR model, *Mathematical and Computer Modelling*, vol. **54** (2011), 221–232.
- [18] D. Jiang, C. Ji, N. Shi and J. Yu, The long time behavior of DI SIR epidemic model with stochastic perturbation, *J. Math. Anal. Appl.*, vol. **372** (2010), 162–180.
- [19] P. E. Kloeden and V. S. Kozyakin, The dynamics of epidemiological systems with nonautonomous and random coefficients, *MESA: Mathematics in Engineering, Science and Aerospace*, vol. **2**, no. 2 (2011).
- [20] P.E. Kloeden and C. Pötzsche, Nonautonomous bifurcation scenarios in SIR models, *Mathematical Methods in the Applied Sciences*, vol. **38** (2015), 3495–3518.
- [21] P. E. Kloeden and M. Rasmussen, *Nonautonomous Dynamical Systems*, American Mathematical Society, Providence, RI, 2011.
- [22] Y. Lin, D. Jiang and P. Xia, Long-time behavior of a stochastic SIR model, *Applied Mathematics and Computation*, vol. **236** (2014), 1–9.
- [23] K. Rohde, *Nonequilibrium Ecology*, Cambridge University Press, 2005.
- [24] B. Schmalfuß, A random fixed point theorem and the random graph transformation, *J. Math. Anal. Appl.*, vol. **225**, no. 1, (1998), 91–113.
- [25] H.R. Thieme, *Mathematics in population biology*, Princeton Series in Theoretical and Computational Biology, Princeton University Press, Princeton, NJ, 2003.
- [26] K. Xu, Z. Zhou and H. Zhao, Dynamical Analysis of a Parasite-Host Model within Fluctuating Environment, *Mathematical Problems in Engineering*, vol. 2016, (2016), Article ID 2972956, 12 pages.

- [27] Q. Yang, D. Jiang, N. Shi, C. Ji, The ergodicity and extinction of stochastically perturbed SIR and SEIR epidemic models with saturated incidence, *J. Math. Anal. Appl.*, vol. **388** (2012) 248–271.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: caraball@us.es

E-mail address: renatocolucci@hotmail.com